Three Way Duels

Infinite Games on the Unit Square

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Preface

This bachelor thesis is made as part of my bachelor in Applied Mathematics.

During the course of the project I experienced a different and fun side of mathematics. Many people were helpful during the course of the project, which is why I would like to thank the following people: Dr. ir. R.J. Fokkink for his guidance, explanation of material unknown to me and for reading and pointing out how to improve my thesis; Drs. E. M. van Elderen for helping me prepare and present a mathematical presentation; Damy Ha for being my guinea pig when testing several games and for thinking about clever ways to solve certain problems; Li Yong Pan for help with analysis related questions.

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Abstract

With the growing wealth and economy of a country, there are an increasing amount of small and big businesses. Every company has its own marketing strategy that it uses in order to lure customers away from their competition and increase their sales. Choosing the perfect time to advertise or discount several products is of essence for a company to gain more money than their competition. These type of marketing games are all slight variations of duels. The purpose of this report is to research how this duel is played most optimal when there are two or more participants. Several types of two-player duels shall be analysed first in order to understand and analyse a three-player duel.

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Introduction

In this report two types of three-player duels, a noisy and silent duel, will be analysed and an optimal strategy or Nash equilibrium will be sought. The information used is mainly from Karlin [5] with some elements of Ferguson [2]. First several definitions of finite and infinite games shall be given in chapter 2 to understand the basics of a game. In chapter 3 the definition of a duel and several examples of two-player duels shall be given. The examples will be solved by calculating optimal strategies for both players and we will verify that the calculated strategies are indeed optimal. In chapter 4 a general method of calculating optimal strategies for two-player silent duels shall be given, which is more complex than that of the noisy duel. Finally in chapter 5 the three-player noisy and silent duel shall be analysed for several cases. This will be done in a similar fashion to chapter 3, but now for three players. This report will build up towards the main question of the report:

How do we calculate the optimal strategies or Nash equilibrium in a three-player noisy and silent duel?

To that end, the following sub-questions shall be answered implicitly:

1. What is an optimal strategy and a Nash equilibrium?
2. What is a noisy/silent duel and what are its rules?
3. How do we solve a noisy/silent two-player duel?

The first sub-question will be answered in chapter 2 and chapter 5. The second sub-questions will be answered in chapter 3 and the last sub-question will be answered in chapter 3 and chapter 4.

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Finite and Infinite Games

2.1. Finite Game

In game theory, a game can be played between two or more players. For two-player games, player one and two shall be named Alice and Bob respectively. All games can be divided into different types. In this thesis, we shall focus on finite and infinite zero-sum games and we will try to analyse a three-player noisy and silent duel. First, finite games shall be defined, because their definition is closely akin to that of infinite games. It also helps to explain certain definitions that also apply to infinite games and to show that infinite games are more complicated than finite games. Definitions can be understood more easily by looking at examples of games, hence we shall give enough examples to support the theory. First we shall define what a move is, consider the following game.

Example 2.1.1 (Tic-tac-toe) *Tic-tac-toe is played on an empty* $3 \times 3-$*grid. Alice and Bob take turns marking one of the empty spots with an x and o respectively. The game ends when there is a horizontal, vertical or diagonal line of length 3 marked by one of the players or when there are no empty spots left. If there is a line as described, the player that marked the line is the winner. If there is no such line and there are no empty spots left, the game ends in a tie.*

Assume Alice and Bob start a game of tic-tac-toe. In the first turn, Alice has to mark any empty spot in the grid. She could for example mark the center. This is a move Alice makes. After Alice marks a spot, it is Bob's turn to make a move. He can choose to mark any spot that is not marked yet. After Bob made his move, it is Alice her turn again. This repeats until the game ends.

Definition 2.1.1 *A move is an action taken by a player at some point during the game.*

In this game, both players take turns to make a move. This is not always the case. In some games, moves need to be made simultaneously. Consider the following game.

Example 2.1.2 (Rock-paper-scissors) *A game of rock-paper-scissors played between Alice and Bob. In this game, both players seek to pick the right hand to best their opponent. Both players pick a hand and show them at the same time. Rock beats scissors, scissors beats paper and paper beats rock. The losing player pays the winning player one unit.*

In this game, both players need to make a move at the same time. Alice and Bob can both play either: rock, paper or scissors. These are the pure strategies of both players. There are finite many pure strategies, hence the rock-paper-scissors game is a finite game.

Definition 2.1.2 *A pure strategy is a strategy in which a certain move is played with probability 1.*

The rock-paper-scissors game is a zero-sum game. The losing player has to pay the winning player one unit, therefore the gains and losses of both players add up to zero. If Alice chooses strategy x and Bob strategy y, the pay-off for Alice is given by $A(x, y)$. The pay-off denotes how much a player gains or loses in a certain situation. This is equal to $+1$ if Alice wins, -1 if Alice loses and 0 in a tie.

Definition 2.1.3 *A zero-sum game is a game in which the net gain of all players sum up to zero.*

Definition 2.1.4 *The (pay-off) kernel $K(x, y)$ for any strategy x and y of Alice and Bob respectively denotes the pay-off to Alice when she plays strategy x and Bob y.*

For every game, the pay-off shall always be given for Alice unless stated otherwise. If $K(x, y)$ is the pay-off for Alice in a two-player zero-sum game, then $-K(x, y)$ is the pay-off for Bob. For finite games, we use $A(x, y)$ instead of $K(x, y)$ to denote that A is a matrix.

The rock-paper-scissors game defined in example 2.1.2 has three pure strategies. An example of a pure strategy is playing paper with probability 1. Assume that Alice plays the pure strategy paper. If Bob plays the pure strategy rock, then Alice will win from Bob. However, the pure strategy that Alice uses does not always work in her favour. If Bob plays the pure strategy scissors, he will win against Alice her strategy. Alice can also opt to play the pure strategy rock. This will make her win if Bob plays the pure strategy scissors, but lose when Bob plays the pure strategy paper.

Definition 2.1.5 *All possible pure strategies of a player form the strategy space of that player. Denote X and Y as the strategy space of Alice and Bob respectively.*

A player can also play a combination of several pure strategies. Alice can assign the probabilities r, p and s towards playing the respective pure strategies rock, paper and scissors. If she does so, she is playing a probability distribution on the strategy space, a mixed strategy. Any mixed strategy x for the rock-paper-scissors game can be seen as a vector $x = (r, p, s)$ in which r, p and s are non-negative, sum up to 1 and denote the probability of playing the pure strategies rock, paper and scissors respectively.

Definition 2.1.6 *A mixed strategy x or y is any probability distribution on the strategy space X or Y respectively.*

Note that any pure strategy can be seen as a mixed strategy by choosing the probability distribution such that the matching pure strategy is played with probability 1. From this point onward, pure strategies will be denoted by ξ and η and mixed strategies by x and y for Alice and Bob respectively.

The rock-paper-scissors game has a total of three different pure strategies for both players. Thus, there are a total of nine different scenarios of how the game can end. This can be represented as a $3 \times 3-$matrix in which the pay-off is given for every scenario. This matrix, denoted by A, will tell us what Alice will gain or lose in every scenario and is equal to the pay-off kernel defined in definition 2.1.4.

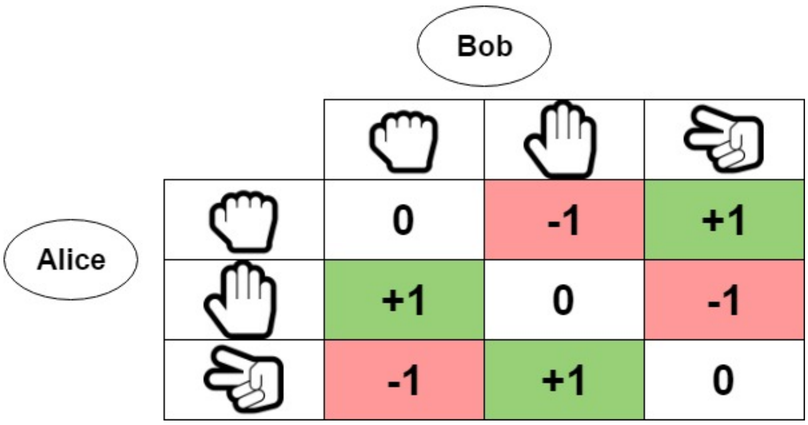

Figure 2.1: Rock-paper-scissors pay-off kernel for Alice

If Alice and Bob play a pure strategy ξ and η respectively, the pay-off is equal to $A(\xi, \eta)$. For example if Alice plays rock and Bob scissors, $A(\xi, \eta) = 1$. If one or both players play a mixed strategy they will play several pure strategies with a certain probability. In this case, we are unable to assign a pay-off to Alice because it is not always the same. For example if Alice plays paper and scissors both with probability $\frac{1}{2}$ and Bob plays rock. Alice will win half of the time, but she will also lose half of the time. If one or both players play a mixed strategy, the pay-off is equal to the expected value.

Using the pay-off matrix A in fig. 2.1, the pay-off can be calculated for any pure or mixed strategy played by Alice and Bob. First the strategy of Alice and Bob is written as a vector x and y as defined before. Solving the matrix equation $x^T Ay$ yields the pay-off for any pure or mixed strategy played. A finite game can now be defined as followed.

Definition 2.1.7 *A finite game between Alice and Bob can be defined as a triplet $\{X, Y, K\}$ in which X is the strategy space of Alice, Y is the strategy space of Bob, X and Y are finite and K is the pay-off kernel for Alice.*

2.2. Optimal and Dominating Strategies

Consider the rock-paper-scissors game defined in example 2.1.2. Alice and Bob can play any pure or mixed strategy and the respective pay-off can be calculated. Alice wants the pay-off to be as high as possible, whereas Bob wants it to be as low as possible. Let us look for these strategies that maximize or minimize the pay-off.

Alice wants to play a strategy that can guarantee her at least a certain pay-off no matter what Bob plays and this pay-off needs to be as high as possible. She seeks a highest lower bound on the pay-off. To further explain this, assume Alice plays the pure strategy rock. Given a strategy of Alice, the best counter strategy of Bob is the best possible strategy Bob can play to minimize the pay-off. In this example, the best counter strategy available to Bob is the pure strategy paper. If Bob plays this counter strategy, he will always win. Therefore, the pay-off is equal to -1. This is a lower bound to the pay-off. When Alice uses the pure strategy rock, her pay-off is always equal to -1 or higher. If Bob plays any other pure or mixed strategy, the pay-off can change in favor of Alice.

The pure strategy rock played by Alice is not very effective, because the lower bound is as low as possible (the maximum a player can lose is 1). So assume Alice plays the mixed strategy $x = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. The best possible counter strategy of Bob is the pure strategy paper. If Bob plays this pure strategy, the pay-off is equal to $-\frac{1}{4}$. If Bob plays any strategy other than the pure strategy paper, be it pure or mixed, the pay-off can be higher in favor of Alice. But the pay-off will never be less than $-\frac{1}{4}$. So with the mixed strategy x of Alice, the lower bound on the pay-off is equal to $-\frac{1}{4}$. This lower bound is higher than the lower bound when using the pure strategy rock. Therefore, the mixed strategy is less risky to use as it guarantees the pay-off (expected value) not to go below $-\frac{1}{4}$ instead of -1.

Alice is in fact looking for a strategy x_0, in this case a mixed strategy, such that for the best counter strategy η of Bob, the pay-off $K(x_0, \eta) = l$ is as high as possible. The counter strategy η of Bob is a pure strategy and the value l is a lower bound on the pay-off. In other words, Alice picks a strategy x_0 such that the lower bound l on the pay-off is maximized. If Alice plays the strategy x_0, her pay-off is equal to l or higher depending on Bob's strategy (pure or mixed). However, if Bob plays it smart and uses his best counter strategy η, Alice will only get a pay-off equal to l. The following inequality and equality show this relation.

$$K(x_0, y) \geq l \tag{2.1}$$

$$\min_{\eta \in Y} K(x_0, \eta) = \max_{x \in X} \min_{\eta \in Y} K(x, \eta) = l \tag{2.2}$$

The same can be said for Bob, who seeks to minimize the pay-off. Bob is looking for a strategy that holds the pay-off down as much as possible no matter what Alice plays, a lowest upper bound to the pay-off. Assume Bob plays the pure strategy rock. The best counter strategy of Alice is the pure strategy paper. The pay-off corresponding to these pure strategies is $+1$. This is an upper bound. If Alice plays any other pure or mixed strategy, the pay-off can change in favor of Bob.

Bob needs to pick a strategy that minimizes the upper bound. Assume Bob plays the mixed strategy $y = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ instead of the pure strategy rock. The best possible counter strategy of Alice is the pure strategy paper. For this mixed strategy of Bob and pure strategy of Alice, the pay-off is equal to $\frac{1}{4}$. This means that if Bob uses the mixed strategy y, Alice can only get a maximum pay-off equal to $\frac{1}{4}$. Therefore, the upper bound of the mixed strategy y is equal to $\frac{1}{4}$. This mixed strategy provides a lower upper bound to the pay-off compared to the pure strategy rock. That's Bob will choose the mixed strategy y over the pure strategy rock, as it guarantees the pay-off to be equal or smaller than $\frac{1}{4}$ instead of 1.

Bob in fact chooses the best strategy y_0, in this case a mixed strategy, such that for the best counter strategy ξ of Alice, the pay-off $K(\xi, y_0) = h$ is as low as possible. Bob is minimizing the upper bound h on the pay-off with his best strategy y_0. If Bob plays y_0, the pay-off is equal h or lower depending on Alice her strategy (pure or mixed). If Alice plays it smart and uses her best counter strategy ξ, she can get a pay-off equal to h. The following inequality and equality show this relation.

$$K(x, y_0) \leq h \tag{2.3}$$

$$\max_{\xi \in X} K(\xi, y_0) = \min_{y \in Y} \max_{\xi \in X} K(\xi, y) = h \tag{2.4}$$

In short, to get an optimal strategy for Alice, we look at the strategies of Alice and let Bob minimize the pay-off for every strategy. Then we pick the strategy with the highest minimum pay-off for Alice and that is her optimal strategy. For Bob, we look at the strategies of Bob and let Alice maximize the pay-off for every strategy. Then we pick the strategy with the lowest maximum pay-off and that is Bob's optimal strategy. These strategies are also called minimax strategies.

Inequalities 2.1 and 2.3 tell us that when both players play their minimized and maximized strategies, Alice can guarantee herself of at least l, whereas Bob can hold Alice down to at most h. There is a theorem that tells us that the lower and upper bounds have to be equal which means that $l = h$. This theorem is called the *Min-Max Theorem* and is proved by John von Neumann in 1928. We are only interested in the theorem and assume it as true, a proof can be found in Karlin [5] page 13.

Theorem 1 (Min-Max Theorem [5]) *If x and y range over X^n and Y^m, respectively, then*

$$\min_{y \in Y} \max_{x \in X} K(x, y) = \max_{x \in X} \min_{y \in Y} K(x, y) = v \tag{2.5}$$

Definition 2.2.1 *Strategies x_0 satisfying inequality 2.1 and y_0 satisfying inequality 2.3 are called optimal strategies or minimax strategies for Alice and Bob respectively. The variable v is called the value of the game to Alice and -v for Bob.*

For two-player zero-sum games, optimal strategies are a solution to a game in the sense that it tells us what strategies will be used by the players and the outcome of the game. Theorem 1 tells us that optimal strategies exist for finite games, which determine the outcome of the game (Alice her pay-off is equal to v). Because every game is different, finding optimal strategies is different for every game. It usually involves guessing what the optimal strategy would look like followed by trial and error.

Theorem 1 also tells us that the inequalities $l \leq h$ and $l \geq h$ need to hold. It is obvious that $l \leq h$, this follows from the fact that if there are x_0 and y_0 such that

$$K(x_0, y) \geq l \quad \forall y \in Y \quad \text{and} \quad K(x, y_0) \leq h \quad \forall x \in X$$

then necessarily

$$l \leq K(x_0, y_0) \leq h$$

It is never possible that $l > h$, otherwise there is an x_0 and y_0 such that

$$K(x_0, y_0) \geq l > h \quad \text{and} \quad K(x_0, y_0) \leq h < l$$

which leads to a contradiction. It follows from theorem 1 that the rock-paper-scissors game defined in example 2.1.2 has optimal strategies for both players and that the game has a value. Because the pay-off matrix in fig. 2.1 satisfies $A = -A^T$, the game is a symmetric matrix game. Symmetric matrix games are a special type of games that have certain properties. Two of these properties are:

1. The value of the game $v = 0$.

2. Both players share the same optimal strategy, i.e. a strategy that is optimal for Alice is also optimal for Bob and vice versa.

Therefore any strategy of the rock-paper-scissors game is optimal if it guarantees a pay-off that is at least 0 or holds the pay-off down to 0 (because $l = h = v = 0$).

Consider the optimal strategy $z = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ for any player. Let us show that this strategy is indeed optimal. It is sufficient to show that the strategy z is optimal for Alice, because Alice and Bob share the same optimal strategy. Assume Alice plays the strategy z and Bob plays any arbitrary strategy $y = (a, b, c)$. Recall that for this game, the pay-off can be calculated by solving the matrix-equation $x^T Ay$. Filling in this equation with the strategy z and y for Alice and Bob respectively yields the following matrix-equation.

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.6}$$

It follows that with the optimal strategy z, Alice will always get a pay-off that is equal to 0. Therefore, the strategy z is indeed an optimal optimal strategy for Alice and because of symmetry, it is also optimal for Bob. This can also be shown by solving a similar equation where Bob uses the strategy z and Alice uses y. Furthermore, when a player plays the optimal strategy z, it does not matter what the opponent plays. The pay-off will always be equal to 0. To illustrate this, assume Alice plays the optimal strategy z and Bob plays a strategy $y = (a, b, c)$. Bob will play rock with probability a. When playing rock, Bob will win, lose or tie with probability $\frac{1}{3}$. Thus the expected value is equal to: $a\left(\frac{1}{3}\cdot(-1) + \frac{1}{3}\cdot(0) + \frac{1}{3}\cdot(+1)\right)$ and this is equal to 0 for any a. The same applies to the probability b and c of Bob playing paper and scissors respectively. Therefore, the pay-off is always equal to 0. Of course, this could've also been derived from eq. (2.6) as $z^T A$ equals the zero-vector. The strategy z has the following property.

$$K(z, y) = c \quad \forall y \in Y \quad \text{and} \quad c \in \mathbb{R} \tag{2.7}$$

Definition 2.2.2 *A strategy z satisfying equation 2.7 is called an equalizer for Alice*

A similar definition of an equalizer can be given for Bob if it satisfies the following.

$$K(x, z) = d \quad \forall x \in X \quad \text{and} \quad d \in \mathbb{R} \tag{2.8}$$

In the rock-paper-scissors game, the optimal strategy z is also an equalizer. When a player plays the strategy z, it does not matter what the opponent plays. Anything the opponent plays is a best response and the pay-off is always the same. If two strategies x and y for Alice and Bob satisfy eq. (2.7) and eq. (2.8) respectively, then necessarily $c = d$ and the strategy set (x, y) form an equilibrium.

In some games there are strategies which are always better to play than others. Consider the following finite game.

Example 2.2.1 (Burglar-Police) *A sneaky burglar B is trying to break into a house. There are a total of four houses. We call them house* 1,2,3 *and* 4. *All houses contain jewelry and money valued at* 100, 200, 500 *and* 1000 *euro respectively. The burglar can only break into one house and if he successfully breaks in, he will gladly take it all. There is a police officer P patrolling the area who is looking for the burglar B. B and P pick a house at the start of the game. When both players pick a different house, the burglar escapes with full loot. When both players pick the same house, the criminal will be apprehended and gets nothing.*

In this finite game, the pay-off for B can be given by the following matrix.

$$\begin{array}{cc} & P \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ B \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 100 & 100 & 100 \\ 200 & 0 & 200 & 200 \\ 500 & 500 & 0 & 500 \\ 1000 & 1000 & 1000 & 0 \end{pmatrix} \end{array} \tag{2.9}$$

In eq. (2.9) the pure strategies of B are the rows and the pure strategies of P are the columns. So when B plays the pure strategy 1 (row 1), he will go to house 1 and his pay-off is equal to 100 unless P also goes to house 1.

Keep in mind that B is trying to maximize his pay-off. Consider the mixed strategy x for B in which B plays the pure strategies 3 and 4 with probability $\frac{1}{2}$ each, i.e. B will go to house 3 or 4 with probability $\frac{1}{2}$ each. The pay-off for this strategy is given by the expected value of going to house 3 and 4 with probability $\frac{1}{2}$. This is equal to the following equation.

$$\frac{1}{2}\begin{pmatrix}500 & 500 & 0 & 500\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1000 & 1000 & 1000 & 1000\end{pmatrix}=\begin{pmatrix}750 & 750 & 500 & 250\end{pmatrix}$$

This means that when B plays the mixed strategy x, he is expected to get 750, 750, 500 and 250 when P plays his pure strategy 1, 2, 3 and 4 respectively.

Note that the pay-off when B uses the mixed strategy x is always equal or greater than when he uses the pure strategy 1 or 2. B will get a pay-off equal to $(0,100,100,100)$ if he plays the pure strategy 1, but if he plays the mixed strategy he is expected to get $(750,750,500,250)$ which is more no matter where P goes. Every entry of the pure strategy 1 and 2 are equal or smaller than the respective entry of the mixed strategy x. So if B uses the pure strategy 1 or 2, he is better off using the mixed strategy x as it always gives him an equal or higher pay-off. Therefore, the mixed strategy x of Bob is dominating his pure strategy 1 and 2 (Hurtado [3] and Ferguson [2]).

Definition 2.2.3 *A strategy x of Alice dominates a strategy x' if* $K(x,y)\geq K(x',y)\ \ \forall y\in Y$. *Strategy x is called the dominating strategy and strategy x' is called the dominated strategy.*

Definition 2.2.4 *A strategy y of Bob dominates a strategy y' if* $K(x,y)\leq K(x,y')\ \ \forall x\in X$. *Strategy y is called the dominating strategy and strategy y' is called the dominated strategy.*

Because strategy 1 and 2 are being dominated, they can be dropped from the matrix and an optimal strategy for B will not consist of going to house 1 or 2. Of course, playing the strategy x always gives B an equal or higher pay-off than when playing pure strategy 1 or 2. So there is no reason to play the pure strategy 1 and 2. Removing these strategies from the matrix results in the following matrix.

$$\begin{array}{ccc} & & P \\ & & \begin{array}{cccc}1 & 2 & 3 & 4\end{array} \\ B & \begin{array}{c}3\\4\end{array} & \begin{pmatrix}500 & 500 & 0 & 500\\ 1000 & 1000 & 1000 & 0\end{pmatrix}\end{array}$$

We can now do the same as before, but now for P. P is minimizing the pay-off for B. Consider the mixed strategy y for P that plays the pure strategy 3 and 4 both with probability $\frac{1}{2}$. The pay-off for this mixed strategy is given by the vector: $(250,500)^T$. This pay-off is equal or lower than when P uses the pure strategies 1 or 2, which are both $(500,1000)^T$. The entries of the mixed strategy are equal or lower than the respective entry in the pure strategies. So if P plays the mixed strategy y, B will gain less than when P uses pure strategy 1 or 2. Therefore, the mixed strategy y is dominating the pure strategies 1 and 2 of P and these strategies can be removed from the matrix.

Removing these columns lead to a 2×2-matrix that can easily be solved, by maximizing or minimizing the pay-off. Both players will either go to house 3 or 4. Let B go to house 3 and 4 with probability a and b respectively. Then we solve the matrix-equation

$$\begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}0 & 500\\ 1000 & 0\end{pmatrix}$$

which gives the vector $(1000b, 500a)$. This vector denotes the pay-off for B depending on whether P goes to house 3 or 4. If P goes to house 3, B will gain $1000b$. If P goes to house 4, B will gain $500a$. B would like his pay-off to be as high as possible in both cases, he can do so choosing the probabilities a and b. So, B is in fact trying to maximize the following system.

$$\begin{cases}1000b\\ 500a\\ a+b=1\end{cases}$$

Solving this system yields $a=\frac{2}{3}$ and $b=\frac{1}{3}$. Thus, an optimal strategy for B is to go to house 3 and 4 with probability $\frac{2}{3}$ and $\frac{1}{3}$ respectively. By doing so, he has guaranteed himself a pay-off that is at least

$$\min(1000b,500a)=\frac{1000}{3}=l$$

no matter what P plays.

Now assume that P will go to house 3 and 4 with probability a and b respectively. We solve the matrix-equation

$$\begin{pmatrix} 0 & 500 \\ 1000 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

which yields the vector $(500b, 1000a)^T$. This means that B will get $500b$ and $1000a$ when B goes to house 3 and 4 respectively. P wants to minimize the values $500b$ and $1000a$. This problem can be written as the following system that needs to be minimized.

$$\begin{cases} 500b \\ 1000a \\ a+b=1 \end{cases}$$

Solving this system yields $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Thus, an optimal strategy for P is to go to house 3 and 4 with probability $\frac{1}{3}$ and $\frac{2}{3}$ respectively. If he does so, he can guarantee that the pay-off for B is not greater than

$$\max(500b, 1000a) = \frac{1000}{3} = h$$

Because the values for the lower and upper bound agree, both strategies are optimal and the value of the game is equal to $v = \frac{1000}{3}$.

2.3. Infinite Game

In the next chapters, the main focus is on games of timing which is a class of infinite games. To be able to understand games of timing, we first need to define and understand infinite games. An example of an infinite game shall be given to understand the difference between finite and infinite games. The solution to this game shall also be given. First a definition of an infinite game.

Definition 2.3.1 *An infinite game between Alice and Bob is defined as a triplet* $\{X, Y, K\}$ *where X is the strategy space of Alice, Y is the strategy space of Bob, both X and Y are infinite and K is the pay-off kernel for Alice.*

Example 2.3.1 *Alice and Bob pick their* ξ *and* η *respectively in the interval* $[0,1]$. *Bob has to pay Alice* $(\xi-\eta)^2$.

The game defined in example 2.3.1 is a relative easy game to start with. Alice and Bob are able to play infinite many pure strategies in the interval $[0,1]$, hence the game is an infinite game. In this game, Bob wants to pick his η as close as possible to Alice her ξ, to minimize his loss. Alice on the other hand wants to pick her ξ as far away as possible from Bob's η.

If Alice plays any pure strategy ξ, she is at risk of Bob playing the exact same value for his η. If this happens, Alice gains nothing. Therefore, it is more likely that Alice will play a mixed strategy. A mixed strategy for Alice is any probability distribution on $[0,1]$. Consider the mixed strategy x of Alice that plays the pure strategies $\xi = 0$ or $\xi = 1$ both with probability $\frac{1}{2}$. The best counter strategy available to Bob is the pure strategy $\eta = \frac{1}{2}$. If Alice plays x and Bob plays $\eta = \frac{1}{2}$, then Bob has to pay Alice $\left(1-\frac{1}{2}\right)^2 = \frac{1}{4}$ half of the time and $\left(0-\frac{1}{2}\right)^2 = \frac{1}{4}$ in the other half. So with the mixed strategy x of Alice, she can guarantee herself a pay-off that is at least $\frac{1}{4}$. This is the lower bound on the pay-off. Any other η Bob plays results in a higher pay-off for Alice.

Looking from Bob's perspective, if he plays the pure strategies $\eta = 0$ or $\xi = 1$, he is at risk of paying the full price. Of course, if Bob plays $\eta = 1$ and Alice plays $\eta = 0$, he will have to pay up. Bob wants to minimize the pay-off for anything Alice can play. If he plays the pure strategy $\eta = \frac{1}{2}$, the best counter strategy of Alice is any mixed strategy consisting of the pure strategies $\xi = 0$ and $\xi = 1$. In this case, the pay-off is always equal to $\frac{1}{4}$. So with the pure strategy $\eta = \frac{1}{2}$ of Bob, he can hold the pay-off down to $\frac{1}{4}$. This is the upper bound on the pay-off. If Alice plays any other strategy, her pay-off can be lower.

Because the lower and upper bound agree, the game has a value and optimal strategies for both players exist. An optimal strategy for Alice is the mixed strategy x and for Bob it's the pure strategy $\eta = \frac{1}{2}$. The value of the game is equal to $v = \frac{1}{4}$.

However, not all infinite games have a value. Sometimes the lower and upper bound do not agree. Theorem 1 holds for infinite games only when the kernel $K(x,y)$ is continuous in both variables. An infinite game can have no value if the kernel is discontinuous. We can still find lower and upper bounds to the pay-off and the inequality $l < h$ will still hold, but the equality does not necessarily need to hold.

2.4. Infinite Game without Value

The game of *Sion and Wolfe* is an example of an infinite game without value. If the lower and upper bound on the pay-off do not agree, the game is said to have no value. A game without value can still have optimal strategies for both players which we will derive in this section. Consider the following game of *Sion and Wolfe* taken from Boudreau and Schwartz [1].

Example 2.4.1 (Sion and Wolfe) *Alice and Bob both pick their respective ξ and η in the interval [0,1]. If Bob manages to pick his η in the open interval $\left(\xi, \xi+\frac{1}{2}\right)$, Alice loses the game (-1). If Alice and Bob both pick the same number or Bob's number is equal $\xi+\frac{1}{2}$, the game ends in a tie $(+0)$. Alice will win the game in all other cases $(+1)$. The pay-off kernel for this game is given by the following function.*

$$K(\xi,\eta)=\begin{cases}-1, & \xi < \eta < \xi+\frac{1}{2}\\ 0, & \eta = \xi \quad or \quad \eta = \xi+\frac{1}{2}\\ 1, & otherwise\end{cases} \tag{2.10}$$

The kernel is clearly discontinuous. Therefore, theorem 1 does not hold and we cannot say for certain that the game has a value. To solve this game and derive an optimal strategy for both players, the behaviour of both players shall be analyzed first. This might help us understand how a possible optimal strategy might look like.

To show how the strategy of both players change, we pretend that the game is played multiple times, both players play only pure strategies and only the losing party is allowed to change their strategy. For starters, assume Alice picks $\xi=\frac{1}{4}$ and Bob picks $\eta=\frac{1}{2}$. Alice will lose the game in this case, so she will change her strategy. If Alice picks a lower ξ, she will either lose or tie. Therefore, she is forced to pick a ξ greater than $\eta=\frac{1}{2}$.

Assume Alice plays $\xi=\frac{3}{4}$ and Bob plays $\eta=\frac{1}{2}$. In this case, Alice wins the game and Bob will have to change his strategy. If Bob plays a lower η, he will only lose. Therefore, he will pick his η greater than $\xi=\frac{3}{4}$.

Assume Alice plays $\xi=\frac{3}{4}$ and Bob plays $\eta=1$. In this case, Alice loses the game and she will have to change her strategy. She can either play $\xi=1$, to tie the game, or play an ξ small such that $\xi+\frac{1}{2}$ is smaller than η. Alice would of course prefer a win over a tie, hence she chooses the second option. Assume Alice plays $\xi=\frac{1}{4}$ and Bob plays $\eta=1$. This will result in a win for Alice and Bob will have to change his strategy. Bob will obviously lower his η such that it is slightly bigger than ξ, but not too big. He could pick $\eta=\frac{1}{2}$, which would result in a win for him.

To summarize the behaviour of both players, Alice wants to pick her ξ greater than η if possible. If that is not possible, she will pick her ξ small such that $\left(\xi,\xi+\frac{1}{2}\right)$ does not contain η. Playing $\xi=1$ will never lose her the game and playing only a pure strategy can never be optimal. Therefore, an optimal strategy for Alice is a mixed strategy that might consists of the point $\xi=1$ and a ξ' small.

Bob on the other hand wants to pick his η such that it is contained in the open interval $\left(\xi,\xi+\frac{1}{2}\right)$. He has to guess where Alice will play her ξ and play his η slightly bigger. If Bob plays his η slightly smaller than $\frac{1}{2}$, any ξ Alice plays on the left side of the unit interval, $\left[0,\frac{1}{2}\right)$, will be won by Bob. If Bob plays his η slightly smaller than 1, the right side of the unit interval, $\left[\frac{1}{2},1\right)$, will be covered. Therefore, an optimal strategy for Bob might consist of η that are slightly smaller than $\frac{1}{2}$ and 1.

We shall first calculate an optimal strategy for Alice. Since the game is played over the unit interval, $[0,1]$, we divide the unit interval into several parts.

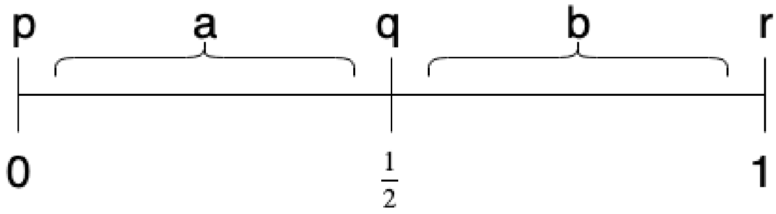

Figure 2.2: Unit interval divided into different parts

The unit interval in fig. 2.2 is split into five parts. The points $0, \frac{1}{2}, 1$ and the open intervals $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$. Any mixed strategy of Alice is now a probability distribution over the interval $[0,1]$. We assume that Alice plays a mixed strategy x and that she distributes her probability over these five parts, say p on 0, q on $\frac{1}{2}$, r on 1, a over $\left(0, \frac{1}{2}\right)$ and b over $\left(\frac{1}{2}, 1\right)$. So Alice will play $\xi = 0$ with probability p, $\xi = \frac{1}{2}$ with probability q etc.. When Alice plays a ξ_1 in $\left(0, \frac{1}{2}\right)$ with probability a, we assume that this ξ_1 is always the same. So Alice will play ξ_1 with probability a. We make a similar assumption for all ξ_2 Alice plays in the interval $\left(\frac{1}{2}, 1\right)$ with probability b. This reduces the probability distribution Alice plays over the interval $[0,1]$ to a probability distribution over five points. A mixed strategy for Alice can now be written as the following vector: $x = (p, q, r, a, b)$.

For every pure strategy η of Bob, the pay-off can be estimated for Alice. For example, if Bob plays $\eta = 1$, the pay-off is equal to: $(q+r)\cdot 0 + b\cdot(-1) + (p+a)\cdot 1 = p + a - b$. Alice wants to maximize her pay-off and she can do so by changing the probabilities p, q, r, a and b. However, the pay-off just mentioned is only when Bob plays $\eta = 1$. Therefore, all possible pure strategies η of Bob need to be considered and maximized. First we define the following notation.

Definition 2.4.1 *Define x^+ and x^- $\forall x \in \mathbb{R}$ such that*

$$x^+ = \lim_{c \downarrow x} c \quad \textit{and} \quad x^- = \lim_{c \uparrow x} c \tag{2.11}$$

We assume that Bob will only play the following pure strategies: $0, \frac{1}{2}^-, \frac{1}{2}, \frac{1}{2}^+, 1^-$ and 1. Bob is now limited to finite many pure strategies instead of infinitely many. All other pure strategies Bob can play are being dropped. We are able to make this assumption, because we claim that the pure strategies $0, \frac{1}{2}^-, \frac{1}{2}, \frac{1}{2}^+, 1^-$ and 1 are dominating all other pure strategies in $[0,1]$, i.e. all other pure strategies in the interval $[0,1]$ result in a higher pay-off for Alice than the six mentioned pure strategies. To prove this, we need to show that any pure strategy in $\left(0, \frac{1}{2}\right)$ or $\left(\frac{1}{2}, 1\right)$ that is not one of the six mentioned pure strategies, is being dominated by these six pure strategies. So we are reducing infinitely many pure strategies of Bob to only six pure strategies.

Consider the interval $\left(0, \frac{1}{2}\right)$. We shall prove that any pure strategy in this interval is dominated by $\eta = \frac{1}{2}^-$. Take any arbitrary pure strategy $\eta' \in \left(0, \frac{1}{2}\right)$ such that $\eta' \neq \eta$. Alice uses her probability distribution x, Bob uses his pure strategy η or η' and the pay-off is given as follows.

$$K(x,\eta) = K\left(x, \frac{1}{2}^-\right) = \lim_{c \uparrow \frac{1}{2}} K(x,c) = -p - a + q + b + r \tag{2.12}$$

$$K(x,\eta') = -p + a + q + b + r \tag{2.13}$$

The only difference in pay-off between the strategies η and η' is in the term a. It is subtracted when Bob uses η and added when he uses η'. This means that when Bob uses the strategy η, he will win when Alice plays her ξ in the interval $\left(0, \frac{1}{2}\right)$ with probability a. Of course, if Bob plays $\eta = \frac{1}{2}^-$, Alice can only win with her ξ in the interval $\left(0, \frac{1}{2}\right)$ if $\eta < \xi < \frac{1}{2}$. But $\frac{1}{2}^-$ converges to $\frac{1}{2}$, therefore Alice can only win if her ξ is also a converging strategy to $\frac{1}{2}$ and converges faster than that of Bob. We assume that it is impossible for both Alice and Bob to play a converging strategy simultaneously. In this case Alice can not play a converging strategy.

However, if Bob plays η' it is possible for Alice to play her ξ in the interval $\left(\eta', \frac{1}{2}\right)$. In this case, Alice wins when she plays her ξ in the interval $\left(0, \frac{1}{2}\right)$ with probability a. With the pay-offs in eq. (2.12) and eq. (2.13), we can see that the pay-off when using strategy η is always equal or lower than when using η'. Thus, we conclude that the strategy η is dominating η' and because η' is taken arbitrary, it holds for any pure strategy η' in $\left(0, \frac{1}{2}\right)$

such that $\eta' \neq \eta$.

Now that any pure strategy in the interval $\left(0, \frac{1}{2}\right)$ is being dominated by the strategy $\eta = \frac{1}{2}^-$, we shall prove a similar statement for the interval $\left(\frac{1}{2}, 1\right)$. We claim that any pure strategy in the interval $\left(\frac{1}{2}, 1\right)$ is dominated by either $\eta_0 = \frac{1}{2}^+$ or $\eta_1 = 1^-$. Take any arbitrary pure strategy $\eta' \in \left(\frac{1}{2}, 1\right)$ such that $\eta' \neq \eta_0$ and $\eta' \neq \eta_1$. The pay-offs are given as follows.

$$K(x, \eta_0) = K\left(x, \frac{1}{2}^+\right) = \lim_{c \downarrow \frac{1}{2}} K(x, c) = p - a - q + b + r \tag{2.14}$$

$$K(x, \eta_1) = K(x, 1^-) = \lim_{c \uparrow 1} K(x, c) = p + a - q - b + r \tag{2.15}$$

$$K(x, \eta') = p + a - q + b + r \tag{2.16}$$

The difference in pay-off between the strategies η_0, η_1 and η' are in the terms a and b. When Bob uses the strategy η', Alice can play her ξ in the interval the $(0, \frac{1}{2})$ with probability a or in the interval $(\frac{1}{2}, 1)$ with probability b such that she will always win from Bob's η'. To show this, assume Bob plays η' and Alice plays her ξ in the interval $\left(0, \frac{1}{2}\right)$ or $\left(\frac{1}{2}, 1\right)$. There are two cases.

1. ($\xi \in \left(0, \frac{1}{2}\right)$ with probability a): In this case, Alice plays her ξ such that the interval $(\xi, \xi + \frac{1}{2})$ does not contain η'. This will make Alice win. To prevent this from happening, Bob needs to play as close as possible to the point $\frac{1}{2}$. This is exactly the strategy η_0. This strategy converges to $\frac{1}{2}$ from above, so any ξ Alice plays in the interval $\left(0, \frac{1}{2}\right)$ creates an interval $\left(\xi, \xi + \frac{1}{2}\right)$ that must contain η_0. A consequence of the strategy η_0 is that any ξ played by Alice in the interval $\left(\frac{1}{2}, 1\right)$ with probability b is greater than η_0, so it is won by Alice. This explains the terms $-a + b$ in the pay-off.

2. ($\xi \in \left(\frac{1}{2}, 1\right)$ with probability b): In this case, Alice plays her ξ such that $\eta' < \xi$. This will make Alice win. To prevent this, Bob needs to play as close as possible to the point 1. This is exactly the strategy η_1, because it converges to 1 from below. If Bob plays η_1, any ξ Alice plays in the interval $\left(\frac{1}{2}, 1\right)$ creates an interval $\left(\xi, \xi + \frac{1}{2}\right)$ that must contain η_1. A consequence of the strategy η_1 is that any ξ played in the interval $\left(0, \frac{1}{2}\right)$ with probability a creates an interval $\left(\xi, \xi + \frac{1}{2}\right)$ that does not contain η_1. Therefore, anything Alice plays in the interval $\left(0, \frac{1}{2}\right)$ is won by her. This explains the terms $+a - b$ in the pay-off.

If we look at the pay-offs in eq. (2.14), eq. (2.15) and eq. (2.16), we can see that when Bob uses η_0 instead of η', the pay-off is lowered by $2a$. If Bob uses η_1 instead of η', the pay-off is lowered by $2b$. Therefore, both η_0 and η_1 are dominating η' and because η' is taken arbitrary it holds for any pure strategy η' in $\left(\frac{1}{2}, 1\right)$ such that $\eta' \neq \eta_0$ and $\eta' \neq \eta_1$. Neither η_0 and η_1 are dominating each others, because one strategy results in Alice winning a but losing b and the other strategy results in the reverse.

All pure strategies of Bob have now been reduced to only six pure strategies. All other pure strategies are being dominated by these six strategies. Our next step is to calculate the pay-off when Alice uses her mixed strategy and Bob uses one of the six pure strategies.

If Bob plays $\eta = 0$, then he will only lose or tie the game. This is a very bad strategy, because there is no ξ such that he will win and it is very likely that this strategy is being dominated by the others. Therefore, this strategy shall be dropped. If Bob plays $\eta = \frac{1}{2}$, the pay-off is given by: $-a + b + r$. For $\eta = 1$, the pay-off given by: $p + a - b$. The pay-off for the remaining strategies have already been calculated before. The following table lists the pay-off when Alice plays her mixed strategy and Bob plays one of the six mentioned pure strategies η.

Strategy	Pay-off
$\eta = \frac{1}{2}^-$	$-p - a + q + b + r$
$\eta = \frac{1}{2}$	$-a + b + r$
$\eta = \frac{1}{2}^+$	$p - a - q + b + r$
$\eta = 1^-$	$p + a - q - b + r$
$\eta = 1$	$p + a - b$

Table 2.1: Pay-off corresponding to pure strategies of Bob

Alice wants to maximize her pay-off given all the possible pure strategies of Bob. The pay-off in table 2.1 can be simplified using the fact that $p + a + q + b + r = 1$. This leads to the following equivalent table.

Strategy	Pay-off
$\eta = \frac{1}{2}^-$	$1 - 2(p + a)$
$\eta = \frac{1}{2}$	$1 - p - 2a - q$
$\eta = \frac{1}{2}^+$	$1 - 2(a + q)$
$\eta = 1^-$	$1 - 2(b + q)$
$\eta = 1$	$1 - q - 2b - r$

Table 2.2: Simplified pay-off corresponding to pure strategies of Bob

The pay-off in table 2.2 is in a more convenient form, it can now written as a matrix equation. See the following equation.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ a \\ b \end{pmatrix} \tag{2.17}$$

Our goal is to maximize the pay-off for Alice in eq. (2.17). This will be done by using dominated strategies in a matrix similar to example 2.2.1. Thus, our focus shall be on the 5×5-matrix.

Alice her strategies are the columns and Bob's strategies are the rows. Alice wants to minimize the matrix, whereas Bob wants to maximize it. By observation, we can see that the entries of the third column, the column corresponding to $\xi = 1$, are always equal or smaller than the corresponding entries in the second column, which belongs to $\xi = \frac{1}{2}$. This means that the strategy of the third column gives a pay-off equal or higher than the strategy of the second column. Thus, a column x is dominating a column x' if all entries of x are equal or smaller than their respective entries in x'.

As a result, the third column is also dominating the fifth column and the first column is dominating the fourth column. Therefore, only the first and the third column are of importance. The second, fourth and fifth column are being dominated, hence instead of playing the mixed strategy $x = (p, q, r, a.b)$, Alice is better off playing a mixed strategy $x' = (p', 0, r', 0, 0)$. Removing the columns that are being dominated from eq. (2.17) gives us the following matrix-equation.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix} \tag{2.18}$$

Every row in eq. (2.18) corresponds to a pure strategy η of Bob and he wants to minimize the pay-off. The first row gives a pay-off equal to $1 - 2p$, whereas the second, third and forth row give a pay-off equal to $1 - p$, 1 and 1 respectively. Therefore, the first row is dominating the second, third and fourth row and these rows can be removed. Only the first and last row are taken into consideration. After removing these rows, the minimization problem can be translated to a system of function that need to be minimized for every function.

$$\begin{cases} 2p \\ r \\ p+r=1 \end{cases} \tag{2.19}$$

This is a relative easy system of functions that can be minimized by hand. It follows that $p=\frac{1}{3}$ and $r=\frac{2}{3}$ is a solution that is minimized, hence $p=\frac{1}{3}$ and $r=\frac{2}{3}$ is a solution to the maximization problem in eq. (2.17). An optimal strategy $x_0=\left(\frac{1}{3},0,\frac{2}{3},0,0\right)$ of Alice consists of playing $\xi=0$ with probability $p=\frac{1}{3}$ and $\xi=1$ with probability $r=\frac{2}{3}$. If she does so, a lower bound for her pay-off can be calculated by filling in x_0 in eq. (2.17). This gives the following vector.

$$\frac{1}{3}\begin{pmatrix}1\\2\\3\\3\\1\end{pmatrix} \tag{2.20}$$

The best counter strategy available to Bob corresponds to the strategy with the lowest pay-off. In this case it is either the first row $\eta=\frac{1}{2}^-$ (which is any pure strategy η slightly smaller than $\frac{1}{2}$) or the last row $\eta=1$. When Bob plays one of his best counter strategies, the pay-off to Alice is equal to $\frac{1}{3}$. Therefore, Alice can guarantee herself a pay-off that is at least $\frac{1}{3}$ with her optimal strategy x_0, this value is the lower bound on the pay-off. The following equality holds.

$$\min_{y\in Y}\max_{x\in X} K(x,y)=\frac{1}{3} \tag{2.21}$$

Now that an optimal strategy for Alice has been calculated, we shall look for an optimal strategy for Bob. This goes analogue to before. The unit interval shall once again be divided as in fig. 2.2. Any mixed strategy of Bob is now a probability distribution over the interval $[0,1]$. We assume that Bob plays a mixed strategy y and that he distribution his probability over the five parts, say p on 0, q on $\frac{1}{2}$, r on 1, a over $\left(0,\frac{1}{2}\right)$ and b over $\left(\frac{1}{2},1\right)$. When Bob plays a η_1 in $\left(0,\frac{1}{2}\right)$ with probability a, we assume that this η_1 is always the same. So Bob will play η_1 with probability a. We make a similar assumption for all η_2 Bob plays in the interval $\left(\frac{1}{2},1\right)$ with probability b. This reduces the probability distribution Bob plays over the interval $[0,1]$ to a probability distribution over five points. Any mixed strategy for Bob can now be written as a vector $y=(p,q,r,a,b)$.

We will now assume that Alice only plays the following pure strategies: 0, 0^+, $\frac{1}{2}^-$, $\frac{1}{2}$, 1^- and 1. Alice is now limited to finite many pure strategies. We will show that all other pure strategies are being dominated by these six pure strategies. This is equivalent to showing that all pure strategies in $\left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$, that are not one of the mentioned six pure strategies, are being dominated by these six pure strategies.

Consider the interval $\left(0,\frac{1}{2}\right)$. We will prove that any pure strategy ξ' Alice plays in this interval is being dominated by the strategies $\xi_0=0^+$ and $\xi_1=\frac{1}{2}^-$ with $\xi'\neq\xi_0$ and $\xi'\neq\xi_1$. Take any arbitrary pure strategy $\xi'\in\left(0,\frac{1}{2}\right)$ that is not ξ_0 or ξ_1. The pay-offs for these strategies are given as follows.

$$K(\xi_0,y)=K(0^+,y)=\lim_{c\downarrow 0}K(c,y)=p-a-q+b+r \tag{2.22}$$

$$K(\xi_1,y)=K\left(\frac{1}{2}^-,y\right)=\lim_{c\uparrow\frac{1}{2}}K(c,y)=p+a-q-b+r \tag{2.23}$$

$$K(\xi',y)=p-a-q-b+r \tag{2.24}$$

The pay-off for the strategies ξ_0, ξ_1 and ξ' differ in the terms a and b. When Alice plays ξ', Bob can play his η in the interval $\left(0,\frac{1}{2}\right)$ with probability a or in the interval $\left(\frac{1}{2},1\right)$ with probability b such that he will always win from Alice her ξ'. To show this, assume Alice plays ξ' and Bob plays his η in the interval $\left(0,\frac{1}{2}\right)$ or $\left(\frac{1}{2},1\right)$. There are two cases.

1. ($\eta \in \left(0, \frac{1}{2}\right)$ with probability a): In this case, Bob plays his η such that $\xi' < \eta$. This will make Bob win. To prevent this, Alice needs to play her ξ as close as possible to the point $\frac{1}{2}$ such that it is always greater than η. This is exactly the strategy ξ_1, because it converges to $\frac{1}{2}$ from below. The assumption that two players can't play a converging strategy simultaneously still holds for this analysis, so Bob can't play a converging strategy η. A consequence of this strategy for Alice is that anything Bob plays in the interval $\left(\frac{1}{2}, 1\right)$ with probability b is won by Bob. This explains the terms $+a - b$ in the pay-off.

2. ($\eta \in \left(\frac{1}{2}, 1\right)$ with probability b): In this case, Bob plays his η such that the interval $(\xi', \xi' + \frac{1}{2})$ contains this η. This will make Bob win. To prevent this, Alice needs to play as close as possible to the point 0. This is exactly the strategy ξ_0. This strategy converges to 0 from above, hence the smallest upper bound of the created interval $\left(\xi_0, \xi_0 + \frac{1}{2}\right)$ will converge to $\frac{1}{2}$ from above. So, when Bob plays his η and Alice plays ξ_0, the created interval will eventually not contain the η played by Bob. A consequence of playing ξ_0 is that any $\eta \in \left(0, \frac{1}{2}\right)$ played with probability a will be won by Bob. This explains the terms $-a + b$ in the pay-off.

If we look at the pay-offs in eq. (2.22), eq. (2.23) and eq. (2.24), we can see that if Alice uses ξ_0 instead of ξ', the pay-off increases with $2b$. If Alice uses ξ_1 instead of ξ', the pay-off increases with $2a$. Hence, both strategies ξ_0 and ξ_1 are dominating ξ' and because ξ' is taken arbitrary in the interval $\left(0, \frac{1}{2}\right)$ not equal to ξ_0 and ξ_1, it holds for any ξ' in the interval.

Now we need to consider the interval $\left(\frac{1}{2}, 1\right)$. We claim that the strategy $\xi = 1^-$ is dominating any strategy $\xi' \in \left(\frac{1}{2}, 1\right)$ such that $\xi' \neq \xi_1$. Take any arbitrary pure strategy $\xi' \in \left(\frac{1}{2}, 1\right)$ such that $\xi' \neq \xi$. The pay-offs are given as follows.

$$K(\xi, y) = K(1^-, y) = \lim_{c \uparrow 1} K(c, y) = p + a + q + b - r \tag{2.25}$$

$$K(\xi', y) = p + a + q - b - r \tag{2.26}$$

The pay-off when using ξ and ξ' only differ in the term b. It is added when using the strategy ξ and subtracted when using ξ'. This means that when Alice uses the strategy ξ, she will win whenever Bob plays his η in the interval $\left(\frac{1}{2}, 1\right)$. This is because ξ will converge to 1 from below. Therefore, any η in the interval $\left(\frac{1}{2}, 1\right)$ is smaller than this ξ and won by Alice.

However, when Alice plays ξ', it is possible from Bob to play an $\eta > \xi'$. In this case, Alice will lose whenever Bob plays his η in the interval $\left(\frac{1}{2}, 1\right)$ with probability b. Therefore, we can conclude that the strategy ξ is dominating the strategy ξ'.

So now all pure strategies of Alice have now been reduced to six pure strategies. From these six pure strategies we need to calculate the pay-off. The following table denotes the pay-off when Alice plays one of the six pure strategies and Bob plays his mixed strategy y.

Strategy	Pay-off
$\xi = 0$	$-a + b + r$
$\xi = 0^+$	$p - a - q + b + r$
$\xi = \frac{1}{2}^-$	$p + a - q - b + r$
$\xi = \frac{1}{2}$	$p + a - b$
$\xi = 1^-$	$p + a + q + b - r$
$\xi = 1$	$p + a + q + b$

Table 2.3: Pay-off for Alice corresponding to strategies of Alice

The pay-off in table 2.3 can be simplified using the fact that $p + q + r + a + b = 1$.

Strategy	Pay-off
$\xi = 0$	$1 - p - q - 2a$
$\xi = 0^+$	$1 - 2(a + q)$
$\xi = \frac{1}{2}^-$	$1 - 2(q + b)$
$\xi = \frac{1}{2}$	$1 - q - 2b - r$
$\xi = 1^-$	$1 - 2r$
$\xi = 1$	$1 - r$

Table 2.4: Simplified pay-off for Alice corresponding to strategies of Alice

Bob wants to minimize the pay-off in table 2.4 for all pure strategies of Alice. The pay-off in table 2.4 can be written as the following matrix-equation which needs to be minimized.

$$\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} - \begin{pmatrix}1 & 1 & 0 & 2 & 0\\0 & 2 & 0 & 2 & 0\\0 & 2 & 0 & 0 & 2\\0 & 1 & 1 & 0 & 2\\0 & 0 & 2 & 0 & 0\\0 & 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}p\\q\\r\\a\\b\end{pmatrix} \tag{2.27}$$

In eq. (2.27), the terms that are subtracted from the all-ones vector needs to be maximized. We will once again make use of dominated strategies in a matrix. We have to take into account that Bob's strategies are the columns and Alice her strategies are the rows. So now a column x is dominating a column x' if all entries of x are equal or greater than the respective entries in x' and a row y is dominating a row y' if all entries of y are equal or smaller than the respective entries in y'.

With this in mind, it follows from observation that the second column dominates the first column and the sixth row dominates the fourth and fifth row. These columns and rows can be dropped, resulting in the following matrix-equation.

$$\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} - \begin{pmatrix}1 & 0 & 2 & 0\\2 & 0 & 2 & 0\\2 & 0 & 0 & 2\\0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}q\\r\\a\\b\end{pmatrix} \tag{2.28}$$

By removing columns and rows of a matrix that were being dominated, the new matrix can have new dominated and dominating columns and rows. Therefore, the matrix in eq. (2.28) needs to be check for new dominated and dominating columns and rows. It follows from observation that the first column is dominating the fourth column and the first row is dominating the second row. By dropping these columns and rows, eq. (2.28) changes to the following matrix-equation.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1 & 0 & 2\\2 & 0 & 0\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}q\\r\\a\end{pmatrix} \tag{2.29}$$

The matrix in eq. (2.29) has no columns and rows that are being dominated, therefore we translate it into a system of functions that need to be maximized for every function. See the following system of functions.

$$\begin{cases} q + 2a \\ 2q \\ r \\ q + r + a = 1 \end{cases} \tag{2.30}$$

This system can also be solved by hand, it follows that $q = \frac{2}{7}$, $r = \frac{4}{7}$, $a = \frac{1}{7}$ is a solution that maximizes every function. Thus, an optimal strategy $y_0 = \left(0, \frac{2}{7}, \frac{4}{7}, \frac{1}{7}, 0\right)$ of Bob consists of playing $\eta = \frac{1}{2}$, $\eta = 1$ and any $\eta \in \left(0, \frac{1}{2}\right)$

with probability $q = \frac{2}{7}$, $r = \frac{4}{7}$ and $a = \frac{1}{7}$ respectively. If Bob plays his optimal strategy y_0, an upper bound to the pay-off can found using eq. (2.27). Filling in the optimal strategy, we obtain the following vector.

$$\frac{1}{7}\begin{pmatrix} 3 \\ 1 \\ 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} \tag{2.31}$$

The best counter strategy available to Alice corresponds to the strategy with the highest pay-off. In this case, the best counter strategy of Alice is any of the pure strategies $\xi = 0$, $\xi = \frac{1}{2}^-$ (which is any pure strategy ξ slightly smaller than $\frac{1}{2}$) and $\xi = 1$. Alice can get a maximum pay-off equal to $\frac{3}{7}$ if Bob plays the optimal strategy y_0. This is the upper bound on the pay-off. The following equality holds.

$$\max_{x\in X} \min_{y\in Y} K(x, y) = \frac{4}{7} \tag{2.32}$$

To conclude this game, both players have a different optimal strategy x_0 and y_0. Alice can guarantee herself a pay-off that is at least $\frac{1}{3}$, whereas Bob can hold the pay-off down to at most $\frac{4}{7}$ with their respective optimal strategies. Because the values of eq. (2.21) and eq. (2.32) are not equal, the game does not have a value.

3
Duels

3.1. Games of Timing

In this chapter, two-player games of timing shall be discussed. Games of timing are a certain type of infinite zero-sum game. The primary focus of this chapter is to derive optimal strategies for specific cases and to validate that these strategies are indeed optimal. There are several types of games of timing which are often called duels. Our focus is mainly on the noisy and silent duel. Finding optimal strategies for a two-player silent duel can become very complex. In the next chapter we shall give a general method of how to do so. First a general description of a duel.

A duel between Alice and Bob is an infinite zero-sum game defined on the unit square. Alice and Bob both have one unit of firepower, i.e. they can both fire at each other exactly once at any time in the interval $[0,1]$. Alice and Bob pick their ξ and η respectively in the interval $[0,1]$. These pure strategies represent the time a player fires at the other. The accuracy function, i.e. the chance of a player successfully hitting their opponent, is a continuous monotone-increasing function from 0 to 1. If a player fires at $t = 0$, it will always miss, whereas firing at $t = 1$ will always hit. The player that hits the opponent first wins. The pay-off kernel for these type of games depend on the order of firing. They have the following structure.

$$K(\xi,\eta) = \begin{cases} L(\xi,\eta), & \xi < \eta \\ \Phi(\xi), & \xi = \eta \\ M(\xi,\eta), & \xi > \eta \end{cases} \tag{3.1}$$

The function $L(\xi,\eta)$ and $M(\xi,\eta)$ in eq. (3.1) are monotone-increasing in ξ for fixed η and monotone-decreasing in η for fixed ξ. The defined kernel is also discontinuous. However, the functions functions $L(\xi,\eta)$ and $M(\xi,\eta)$ are jointly continuous in ξ and η. This is different than in the game of *Sion and Wolfe* in example 2.4.1, hence duels do have a value.

Both players want to increase their chance of success by waiting for as long as possible, but they do not want to wait too long or else their opponent will precede them. The optimal strategy for these type games express a balance between the desire and danger of delay.

There are several types of duels that differ on the information both players have. We shall only focus on two types of duels. The first type is a noisy duel. In a noisy duel, both players know when the other has fired. If Alice fires before Bob at a time t and misses, Bob knows that Alice has fired and can not fire again. Therefore, Bob will wait until $t = 1$ before he fires for a guaranteed hit. The same applies to Alice when Bob fires before her and misses. If a player misses, he is almost surely done for and this is taken into the pay-off kernel.

The second type of duel is a silent duel. In a silent duel, both players do not know when their opponent has fired unless it hits. If a player fires and misses, their opponent will not know. Therefore, the opponent does not necessarily wait until $t = 1$ before firing. There is also a probability that both players fire and miss at different times, which results in a tie. Because both players have less information about each other, the terms in the pay-off kernel are slightly different than that of a noisy duel.

To end the chapter, we will also look at what happens in a silent duel that is not played over the unit interval $[0,1]$. Think about two duelists seeing each other and drawing their gun only to find out there is no bullet in the gun. Both duelists need to reload their gun and we assume that reloading takes a total time of b with $0 < b \leq 1$ for both players. The first time both players are able to fire at is at $t = b$ in which both players have non-zero accuracy. The game is now played on the square $[b,1]$ instead of $[0,1]$ and the optimal strategy for the normal silent duel might or might not be optimal in the new duel.

3.2. Noisy Duel

In this section, we shall analyse a two-player noisy duel for specific cases and calculate an optimal strategy for both players. We will also derive a method of how to calculate an optimal strategy for general cases. First, a more precise definition of a noisy duel.

Example 3.2.1 (Noisy duel) *Alice and Bob are allowed to fire at each other exactly once at a time $t \in [0,1]$. Both players are equipped with a noisy gun, so that each player knows when their opponent has fired. The accuracy functions for Alice and Bob are given by the continuous and monotone-increasing functions $p(\xi)$ and $q(\eta)$ respectively with $p(0) = q(0) = 0$ and $p(1) = q(1) = 1$. If Alice hits Bob first, Alice wins the game $(+1)$. If Bob hits Alice first, Bob wins the game (-1). If both players hit each other at the same time or no one has been hit after $t = 1$, the game ends in a tie $(+0)$.*

Alice and Bob pick their ξ and η in the interval $[0,1]$ respectively. The pay-off kernel consists of three functions that represent the order of firing. To calculate the pay-off kernel, we need to consider every order of firing.

1. $(\xi < \eta)$: In this case, Alice fires before Bob. The probability that Alice hits Bob is equal to $p(\xi)$ and the pay-off for hitting Bob is equal to $+1$. Alice will miss Bob with probability $(1 - p(\xi))$. If this happens, Bob knows that Alice has fired and will therefore fire at $t = 1$. This will guarantee Bob hitting Alice resulting in a pay-off equal to -1. Combining both cases leads to the following pay-off.
$$L(\xi,\eta) = 2p(\xi) - 1 \tag{3.2}$$
2. $(\xi = \eta)$: In this case, Alice and Bob fire at the same time. A player can either hit or miss their opponent. If Alice hits Bob and Bob misses Alice, the pay-off is $+1$. The probability of this happening is equal to $p(\xi)(1 - q(\eta))$. If Bob hits Alice and Alice misses Bob, the pay-off is -1. This will happen with probability $q(\eta)(1 - p(\xi))$. Finally if Alice and Bob both hit or miss each other, the pay-off is 0. For completeness, this will happen with probability $p(\xi)q(\eta)$ and $1 - p(\xi) - q(\eta) + p(\xi)q(\eta)$ respectively. All together leads to the following pay-off.
$$\Phi(\xi,\eta) = p(\xi) - q(\eta) \tag{3.3}$$
3. $(\xi > \eta)$: In the last case, Bob will fire before Alice. The probability that Alice will get hit is equal to $q(\eta)$ and the pay-off for getting hit by Bob is equal to -1. Bob will miss Alice with probability $(1 - q(\eta))$. If Bob misses, Alice will wait until $t = 1$ before firing for a guaranteed hit resulting in a pay-off equal to $+1$. This leads to the following pay-off.
$$K(\xi,\eta) = 1 - 2q(\eta) \tag{3.4}$$

Combining the calculated pay-off in eq. (3.2), eq. (3.3) and eq. (3.4) gives us the following pay-off kernel.

$$K(\xi,\eta) = \begin{cases} 2p(\xi) - 1, & \xi < \eta \\ p(\xi) - q(\eta), & \xi = \eta \\ 1 - 2q(\eta), & \xi > \eta \end{cases} \tag{3.5}$$

Note that in the pay-off in eq. (3.5), if a player fires first and misses, their opponent will fire at $t = 1$. Both players pick their ξ and η beforehand, but they are able to switch to $\xi = 1$ and $\eta = 1$ respectively when their opponent fires before them. This is important to remember. The pay-off kernel shows the desire of Alice to delay her shot, but also the danger of delaying. If Alice is the first to fire, she would want to wait for as long as possible to increase her pay-off given by $2\xi - 1$. However, if she waits too long and Bob precedes her, the pay-off is given by $1 - 2\eta$ which has decreased more by waiting. We will first take a look at how an optimal strategy might look like.

Assume that Alice and Bob play a noisy duel given two accuracy functions. Bob has some eye problems, so his accuracy function is lower than that of Alice until $t = 1$ of course. Alice and Bob see each other far away and both players walk towards each other with their gun pointed at the other. Assume that at a time t_0, the accuracy (probability to hit) for Alice and Bob are equal to $\frac{1}{5}$ and $\frac{1}{6}$ respectively. Alice has a slightly higher accuracy if she fires at t_0. However, it is not optimal for her to fire at t_0 even though her probability to hit is higher than that of Bob. If Alice is the first to fire at t_0, she will win with probability $\frac{1}{5}$. She can also win when Bob is the first to fire at t_0 and misses with probability $\frac{5}{6}$. The probability that Alice will win by firing first increases from 0 to 1, whereas the probability that she wins by letting Bob fire first decreases from 1 to 0. In this case, Alice would rather have that Bob fires at t_0, as it gives her a higher probability of winning. However, if Bob does not fire at t_0 it is best for Alice to fire at a later time to increase her probability of winning if she fires first.

Now at a later point, $t_1 > t_0$, assume the accuracy at t_1 for Alice and Bob are equal to $\frac{1}{2}$ and $\frac{3}{10}$ respectively. It is tempting to think that firing at t_1 is a good strategy for Alice, because if she is the first to fire at t_1 she will win or lose with probability $\frac{1}{2}$. However, this is not the case. Because if Bob is the first to fire at t_1, the probability that he will miss and Alice will win is equal to $\frac{7}{10}$. So, Alice would rather have Bob fire at t_1 instead of firing herself. She can still wait a little longer to increase her probability of winning when firing first.

Now assume that at t_2, the accuracy of Alice and Bob are equal to $\frac{3}{5}$ and $\frac{2}{5}$ respectively. If Alice is the first to fire at t_2, she will win with probability $\frac{3}{5}$. If Bob is the first to fire at t_2, the probability that Alice wins is also $\frac{3}{5}$. This is the moment where both players should fire. If Alice fires before t_2, then her accuracy is lower than at t_2. Therefore, her probability of hitting Bob and winning is also lower. If Alice fires after t_2, say at $t_3 > t_2$, then Bob can fire slightly before t_3 such that Bob will hit Alice with a probability that is higher than $\frac{2}{5}$. This means that Alice will win with a probability that is lower than $\frac{3}{5}$. But we have just seen that if Alice fires at t_2, she can win with probability $\frac{3}{5}$. So firing after t_2 leaves Alice at risk.

For the same reason mentioned before, Bob should also fire at t_2. We can see that the optimal strategy for both players consists of waiting for a time t_0 such that the probability of winning by firing first is equal to the probability of winning when the opponent fires first. Firing before this moment leads to a reduced probability of winning, whereas firing after this moment can lead to a reduced probability of winning. If the opponent has fired before this time t_0, then firing at $t = 1$ is optimal.

Let us assume that the accuracy functions are now given by: $p(\xi) = q(\xi) = \xi$. With this assumption, the kernel can be rewritten as the following function.

$$K(\xi,\eta) = \begin{cases} 2\xi - 1, & \xi < \eta \\ 0, & \xi = \eta \\ 1 - 2\eta, & \xi > \eta \end{cases} \tag{3.6}$$

The kernel given in eq. (3.6) has a special property. By observation we can see that $K(\xi,\eta) = -K(\eta,\xi)$, hence the game is a symmetric game as described on page 6. The concept of symmetric games is completely analogous to the concept of symmetric matrix games. Therefore, both players share an optimal strategy and the value of the game is equal to 0. When verifying that the game is indeed symmetric, we need to keep the following in mind. If $\xi < \eta$, then $K(\xi,\eta) = 2\xi - 1$. If we calculate $-K(\eta,\xi)$, then $\xi < \eta$ still holds. The first variable is now greater than the second variable. Therefore, $-K(\eta,\xi) = -(1-2\xi) = 2\xi - 1$. Something similar follows when $\xi > \eta$.

The optimal strategy we are looking for can guarantee a pay-off that is at least 0 or hold it down to 0. It is sufficient to look for an optimal strategy for Alice, because both players share the same optimal strategy. We are in fact looking for a time t such that the probability that Alice wins by firing first is equal to the probability of her winning when her opponent fires first. This is exactly the time t that satisfies the following equation.

$$t = 1 - t$$

If we solve this equation for t, we find that $t = \frac{1}{2}$ is a solution. Therefore, an optimal strategy x for both players is to fire at: $t = \frac{1}{2}$ when the opponent has not fired yet; $t = 1$ when the opponent has fired already. We will show that this strategy is indeed optimal for both players. It is sufficient to show that Alice can guarantee a pay-off that is at least equal to 0 when she uses the optimal strategy x. Assume Alice plays x. Bob can fire either before, at or after $t = \frac{1}{2}$. So, there are three cases.

1. (Bob fires at $t_0 < \frac{1}{2}$): In this case, the pay-off is given by $K(x, t_0) = 1 - 2t_0$. Because $t_0 < \frac{1}{2}$, it follows that $K(x, t_0) = 1 - 2t_0 > 0$. So if Bob fires before $t = \frac{1}{2}$, Alice her pay-off is greater than 0.

2. (Bob fires at $t_0 = \frac{1}{2}$): In this case, the pay-off is given by $K(x, t_0) = 0$. So if Bob fires at $t = \frac{1}{2}$, the pay-off to Alice is equal to 0.

3. (Bob fires at $t_0 > \frac{1}{2}$): In this case, Alice has already fired at $t = \frac{1}{2}$. So it is best for Bob to fire at $t = 1$, if he is still alive. The pay-off in this case is given by $K(x, 1) = 0$. So firing after $t = \frac{1}{2}$ gives Alice a pay-off equal to 0.

In all cases, the pay-off to Alice is at least 0. So with the strategy x of Alice, she can guarantee herself a pay-off that is at least 0. Hence, the strategy x is optimal for Alice and with the symmetric property of the game it is also optimal for Bob.

So far we have assumed that the accuracy functions were the same for both players and given by $p(\xi) = q(\xi) = \xi$. We have also calculated an optimal strategy for both players in this case. We are curious what were to happen to the optimal strategies when different accuracy functions are used. Suppose that Bob is nearsighted, he can see close objects clear, but objects far away are blurry. The accuracy function for Bob is now given by $q(\eta) = \eta^2$ whereas the accuracy function for Alice is still given by $p(\xi) = \xi$. Intuitively, we would think both players still fire at the same time t. The time t where both players fire at will shift to a later time compared to the previous game, because Bob does not pose a threat early on. Therefore, Alice can wait a little longer before firing. To show if this is true, we will use a slightly different approach than before. Both methods result in the same optimal strategy. We will first calculate the pay-off kernel using eq. (3.5).

$$K(\xi, \eta) = \begin{cases} 2\xi - 1, & \xi < \eta \\ \xi - \xi^2, & \xi = \eta \\ 1 - 2\eta^2, & \xi > \eta \end{cases} \tag{3.7}$$

Alice wants to maximize the pay-off in every case. If Alice is the first to fire at a time t, the pay-off is given by $2t - 1$. If Bob is the first to fire at t, the pay-off is given by $1 - 2t^2$. If both players fire at t, then the pay-off is given by $t - t^2$. These are the functions Alice wants to maximize. The following plot shows how the functions behave in the interval $[0, 1]$.

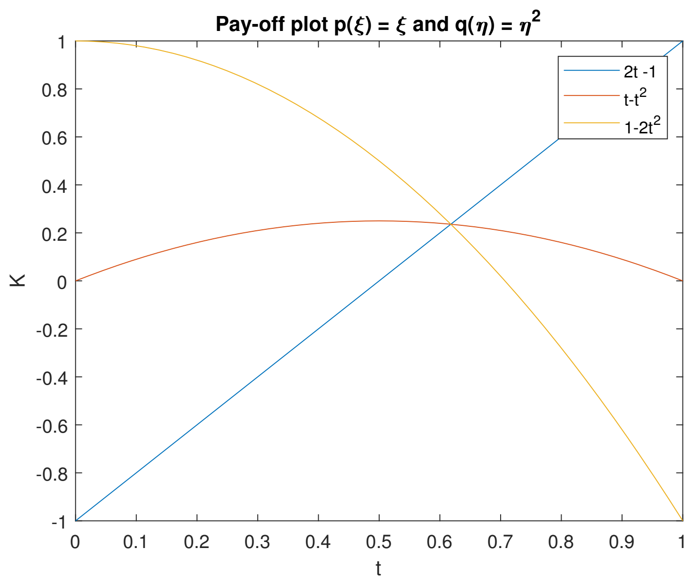

Figure 3.1: Functions of eq. (3.7) representing the pay-off kernel

In fig. 3.1 there are three plotted functions that each belong to a different order of firing. The yellow function belongs to the case in which Bob fires before Alice. So if we take $t = 1$ for example, the value of the yellow function is equal to -1. Of course, if Bob is the first to fire at $t = 1$ that means Alice is not firing. In this case, she will always lose resulting in a pay-off equal to -1. The blue function belongs to the case in which Alice fires before Bob and the red function belongs to the case in which both players fire at the same time.

We can see that the three functions all intersect at a single point for a certain time t, say at t_0. This is no surprise. We know that the pay-off when $\xi < \eta$ is monotone-increasing from -1 to 1 and it is monotone-decreasing from 1 to -1 when $\xi > \eta$ (this can be verified in eq. (3.7)). Therefore, they have to intersect at a time t_0. The pay-off when both players fire at the same time is the average of the pay-off when Alice fires before Bob and the other way around. Thus, the three functions intersect at exactly one point. This point is part of the optimal strategy for both players.

If Alice is the first to fire at a $t < t_0$ (blue function), her pay-off will be lower compared to firing at t_0. She will therefore not fire before t_0. If Alice fires at $t > t_0$, she is at risk of Bob firing slightly before t (yellow function) such that her pay-off is lower than firing at t_0. The same also applies to Bob. If Bob is the first to fire at a time $t < t_0$ (yellow function), he will increase the pay-off for Alice compared to firing at t_0. If Bob fires at $t > t_0$, he is at risk of Alice firing slightly before t (blue function) resulting in a higher pay-off than firing at t_0. The point where all functions intersect guarantees both players a certain pay-off.

Another way of verifying that this point is indeed part of the optimal strategy is by looking at pay-off functions in fig. 3.1. Alice wants a strategy that can guarantee her the highest pay-off when Bob plays the best counter strategy. Say for example that Alice fires at $t = 0.3$. The blue function (which belongs to $\xi < \eta$) is the lowest of the three functions at time t, thus the best counter strategy of Bob is to fire at $t = 1$. If Bob fires at the same time (red) or slightly before Alice (yellow), the pay-off is substantially higher in both cases. Now say that Alice fires at $t = 0.9$. The yellow function (which belongs to $\xi > \eta$) is the lowest in this case, thus the best counter strategy of Bob is to fire slightly before Alice. If Bob fires at the same time or after Alice, the pay-off is yet again higher in favor of Alice. So Alice defines a new function that is equal to the minimum value of the three functions at every t and then maximizes it. This is equal to the following.

$$\begin{aligned} f(t) &= \min(2t-1, t-t^2, 1-2t^2) \\ \max_{t\in[0,1]} f(t) &= t_0 \end{aligned} \tag{3.8}$$

Bob on the other hand wants to play a strategy that can hold the pay-off down as much as possible when Alice plays the best counter strategy. If Bob fires at $t = 0.3$, the yellow function is the highest. Therefore, the best counter strategy of Alice is to fire at $t = 1$. If Bob fires at $t = 0.9$, the blue function is the highest, thus the best counter strategy of Alice is firing slightly before Bob. So Bob defines a new function that is equal to the maximum value of the three functions at every t and then minimizes it. This is equal to the following.

$$\begin{aligned} g(t) &= \max(2t-1, t-t^2, 1-2t^2) \\ \min_{t\in[0,1]} g(t) &= t_0 \end{aligned} \tag{3.9}$$

The values of eq. (3.8) and eq. (3.9) are the same, this can also be seen in fig. 3.1. Now we need to find this point t_0 such that the functions intersect. This is equivalent to solving the following equation.

$$2t-1 = 1-2t^2 \tag{3.10}$$

It follows that $t_0 = \frac{1}{2}\sqrt{5} - \frac{1}{2} \approx 0.62$ is a solution to the equation. If both players fire at t_0, the pay-off is equal to $\sqrt{5}-2$. An optimal strategy for both players is to fire at: $t = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ when the opponent has not fired yet; $t = 1$ when the opponents has fired already. If Alice uses this optimal strategy, she can guarantee herself a pay-off that is at least $\sqrt{5}-2$. If Bob uses this optimal strategy, he can hold the pay-off down to $\sqrt{5}-2$. Hence, the value of the game is equal to $v = \sqrt{5}-2 \approx 0.24$. Thus, by giving Bob a slight disadvantage in his accuracy function early on, the value of the game increases in favor of Alice and the time where both players should fire shifts to a later time. It is remarkable that even though this game is not symmetric, both players still share the same optimal strategy.

For general accuracy functions $p(\xi)$ and $q(\eta)$ for Alice and Bob respectively, an optimal strategy can be found in two ways. The first way is to solve the equation $p(t) = 1 - q(t)$. The time t_0 that satisfies this equation is the time where the probability that Alice wins by firing first is equal to the probability that Alice wins when Bob fires first. This time t_0 in combination with firing at $t = 1$ forms an optimal strategy for both players.

The second way is by calculating the pay-off when Alice fires before Bob and the other way around. We will then have to look for a time t where both functions intersect (if we see the functions as functions of t). This is equivalent to solving the equation $2p(t)-1 = 1-2q(t)$. The time t_0 that satisfies this equation in combination with firing at $t = 1$ forms an optimal strategy. Both methods result in the same optimal strategy. When both players use this optimal strategy, the probability that Alice wins is equal to $p(t_0)$ and the probability that Bob wins is equal to $q(t_0)$.

3.3. Silent Duel

In this section, we shall discuss a general two-player silent duel. We shall calculate an optimal strategy for a specific case and verify that it is indeed optimal. First a more precise definition of a silent duel.

Example 3.3.1 (Silent duel) *Alice and Bob are allowed to fire at each other exactly once at a time $t \in [0,1]$. Both players are equipped with a silent gun, so that each player does not know whether their opponent has fired already or not. The accuracy function for Alice and Bob are given by the continuous and monotone-increasing functions $p(\xi)$ and $q(\eta)$ respectively with $p(0) = q(0) = 0$ and $p(1) = q(1) = 1$. If Alice hits Bob first, Alice wins the game $(+1)$. If Bob hits Alice first, Bob wins the game (-1). If both players hit each other at the same time or no one has been hit at $t = 1$, the game ends in a tie $(+0)$.*

Let us first assume that the accuracy functions are given by $p(\xi) = q(\xi) = \xi$. Note that the calculated optimal strategy of the noisy duel with the same accuracy functions in the previous section is not optimal for this silent duel. A player does not know whether the opponent has fired already or not, thus the optimal strategy of the noisy duel is now a mere pure strategy at $t = \frac{1}{2}$. Firing slightly before $t = \frac{1}{2}$ is a counter strategy to firing at $t = \frac{1}{2}$. Say Alice fires at $\xi = \frac{1}{2}$ and Bob at η slightly before $t = \frac{1}{2}$. With our chosen accuracy functions, the game is in fact symmetric (this will be shown later on). So if the strategy of Alice is optimal, it should guarantee her a pay-off that is at least 0. However, in this case Alice will win, lose and tie with probability $\approx \frac{1}{4}$, $\approx \frac{1}{2}$ and $\approx \frac{1}{4}$ respectively. So, the pay-off is approximately

$$\frac{1}{4}\cdot(+1)+\frac{1}{2}\cdot(-1)+\frac{1}{4}\cdot(0) = -\frac{1}{4} < 0$$

Hence, the pure strategy at $\xi = \frac{1}{2}$ can't be optimal.

The optimal strategy of this game is now a mixed strategy not on two points, but on infinitely many points in an interval. We have a probability distribution over an interval. This interval is not necessarily $[0,1]$, it can be any interval of the form $[a,1]$ with $a \geq 0$. Think about two duelers dueling in the dark with heavy snowfall and fog. They are only able to see each other when they are close to each other. The accuracy functions are then given by functions that are very flat at the start and steep at the end. A player will not consider firing too early, because his accuracy is just too bad. The player will only consider firing after a certain time a. If Alice and Bob pick their ξ and η in the interval $[0,1]$ respectively, the pay-off can be given for each order of firing.

1. $(\xi < \eta)$: In this case, Alice fires before Bob. The probability that Alice hits is equal to $p(\xi)$ and the pay-off for hitting Bob is equal to $(+1)$. The probability that Alice misses and Bob hits Alice at a time η is equal to $(1-p(\xi))q(\eta)$ and the pay-off is equal to (-1). The pay-off when both players miss is equal to 0 and shall therefore be dropped. This leads to the following pay-off.
$$L(\xi,\eta) = p(\xi)-(1-p(\xi))q(\eta) \tag{3.11}$$

2. $(\xi = \eta)$: In this case, both players fire at the same time. The probability that Alice hits and Bob misses is equal to $p(\xi)(1-q(\eta))$ with pay-off $(+1)$. The probability that Bob hits and Alice misses is equal to $q(\eta)(1-p(\xi))$ with pay-off (-1). The pay-off when both players hit or miss is equal to 0 and therefore dropped. Adding up all terms results in the following pay-off.
$$\Phi(\xi,\eta) = p(\xi)-q(\eta) \tag{3.12}$$

3. $(\xi > \eta)$: In this case, Bob fires before Alice. The probability that Bob will hit is equal to $q(\eta)$ and the pay-off is equal (-1). The probability that Bob misses and Alice hits is equal to $(1-q(\eta))p(\xi)$ with pay-off $(+1)$. The pay-off when both players miss is 0 and therefore dropped. This gives the following pay-off.
$$M(\xi,\eta) = -q(\eta)+(1-q(\eta))p(\xi) \tag{3.13}$$

Now we combine eq. (3.11), eq. (3.12) and eq. (3.13) into the following pay-off kernel.

$$K(\xi,\eta) = \begin{cases} p(\xi) - (1-p(\xi))q(\eta), & \xi < \eta \\ p(\xi) - q(\eta), & \xi = \eta \\ -q(\eta) + (1-q(\eta))p(\xi), & \xi > \eta \end{cases} \tag{3.14}$$

Using our assumption earlier for the accuracy function, the pay-off can then be written as followed.

$$K(\xi,\eta) = \begin{cases} \xi - (1-\xi)\eta, & \xi < \eta \\ 0, & \xi = \eta \\ -\eta + (1-\eta)\xi, & \xi > \eta \end{cases} \tag{3.15}$$

In eq. (3.15) we can observe that $K(\xi,\eta) = -K(\eta,\xi)$, hence the game is symmetric for our chosen accuracy function. Therefore, both players share the same optimal strategies and the value of the game is 0. We will look for an optimal strategy of Alice that is of the following form.

$$x(\xi) = \int_a^\xi f(t)dt \tag{3.16}$$

The optimal strategy x is a density f defined over the interval $[a,1]$. The probability that Alice fires before a time $t_0 \in [a,1]$ is given by $x(t_0)$ and similar for Bob. If $\xi = 1$, then $x(1)$ is equal to the density integrated over its interval. This is equal to the probability of firing before $t = 1$ (or in the interval $[a,1]$ to be more precise), which is of course equal to 1. Therefore, the following normalization holds.

$$x(1) = \int_a^1 f(t)dt = 1 \tag{3.17}$$

If a player plays with a mixed strategy, the pay-off is obtained by averaging. We also know that the value of the game is 0. Assume that Alice plays an optimal strategy in the form of a density f defined over the interval $[a,1]$. The following holds for an optimal strategy.

$$\int_a^1 K(\xi,\eta)f(\xi)d\xi = v = 0 \tag{3.18}$$

Equation (3.18) can be expanded using the kernel in eq. (3.15). Assume that Bob fires at a time η in $[a,1]$, because it is not beneficial for any player to fire before $t = a$. The integral is split in two depending on η.

$$\int_a^1 K(\xi,\eta)f(\xi)d\xi = \int_a^\eta (\xi-(1-\xi)\eta)f(\xi)d\xi + \int_\eta^1 (-\eta+(1-\eta)\xi)f(\xi)d\xi \equiv 0 \tag{3.19}$$

This integral can be rewritten by grouping terms and using the normalization in eq. (3.17). Rewriting gives the following equation.

$$\int_a^1 \xi f(\xi)d\xi - \eta + \eta\int_a^\eta \xi f(\xi)d\xi - \eta\int_\eta^1 \xi f(\xi)d\xi \equiv 0 \tag{3.20}$$

If we define $r(\xi) = \xi f(\xi)$, substitute this in eq. (3.20) and perform two differentiations with respect to η (using the Fundamental Theorem of Calculus), we obtain the following equation.

$$2\eta r'(\eta) + 4r(\eta) = 0 \tag{3.21}$$

A general solution for this equation yields: $r(\eta) = k\eta^{-2}$, therefore $f(\xi) = r(\xi)\xi^{-1} = k\xi^{-3}$. Now if we insert this solution in eq. (3.20) and simplify, we get the following equation.

$$\eta(-1+\frac{k}{a}+k) + k(-3+\frac{1}{a}) \equiv 0 \tag{3.22}$$

Observing eq. (3.22), we find that $a = \frac{1}{3}$ is necessary in the second term. Using this value for a in the first term yields the value $k = \frac{1}{4}$. So now we have the following density

$$f(\xi) = \begin{cases} 0, & 0 \le \xi < \frac{1}{3} \\ \frac{1}{4}\xi^{-3}, & \frac{1}{3} \le \xi \le 1 \end{cases} \tag{3.23}$$

which also satisfies the normalization

$$\int_{\frac{1}{3}}^{1} \frac{1}{4}\xi^{-3} d\xi = 1$$

This density is in fact our optimal strategy. We were looking for a density f such that the pay-off in eq. (3.18) is identically zero. So if Alice fires using this density and Bob fires at $\eta \in [\frac{1}{3}, 1]$, Alice is expected to get a pay-off equal to 0. It will be shown later what happens when Bob fires at $\eta < \frac{1}{3}$. For completeness we can write the optimal strategy as followed.

$$x(\xi) = \int_{\frac{1}{3}}^{\xi} \frac{1}{4}\xi^{-3} d\xi \tag{3.24}$$

This optimal strategy tells a player not to fire before $t = \frac{1}{3}$. Only after $t = \frac{1}{3}$ should the player consider firing and he should fire using the density $f(x)$. To validate that this strategy is indeed optimal for both players, assume Alice plays this optimal strategy x and Bob plays any $t \in [0, 1]$. We will show that it does not matter what the value for t is, because the pay-off will always be zero or higher.

Assume that $t \geq \frac{1}{3}$. First we will check when Alice gets a positive pay-off, which is when Alice will win. The cases in which Alice will win are either: Alice fires before t and hits; Bob fires at t and misses and Alice fires after t and hits. The pay-off for this is equal to the following.

$$\int_{\frac{1}{3}}^{t} \xi f(\xi) d\xi + (1-t) \int_{t}^{1} \xi f(\xi) d\xi \tag{3.25}$$

Note that the integral

$$\int_{\frac{1}{3}}^{t} \xi f(\xi) d\xi$$

denotes the probability of firing in the interval $[\frac{1}{3}, t]$ and hitting the opponent. Filling in eq. (3.25) with the density $f(\xi)$ and simplifying, we get the following pay-off.

$$\frac{1}{4} + \frac{t}{4} \tag{3.26}$$

Now we need to add the pay-off when Alice loses. The cases in which Alice will lose are either: Bob fires at t and hits and Alice fires after t; Alice fires before t and misses and Bob fires at t and hits. The pay-off for this is equal to the following.

$$-\left(t \int_{t}^{1} f(\xi) d\xi + t \int_{\frac{1}{3}}^{t} (1-\xi) f(\xi) d\xi \right) \tag{3.27}$$

Filling in eq. (3.27) with the density $f(\xi)$ and simplifying, we get the following pay-off.

$$-\frac{t}{4} - \frac{1}{4} \tag{3.28}$$

Both eq. (3.26) and eq. (3.28) add up to zero, thus the pay-off for any $t \geq \frac{1}{3}$ played by Bob is equal to 0. Now we need to look at when Bob plays $t < \frac{1}{3}$. The pay-off can be given by the following equation.

$$\int_{t}^{1} (-\eta + (1-\eta)\xi) f(\xi) d\xi \tag{3.29}$$

Note that the density $f(\xi)$ is equal to 0 outside of the interval $\left[\frac{1}{3}, 1\right]$. Filling in $f(\xi)$ and simplifying gives the following pay-off.

$$-\frac{3}{2}t + \frac{1}{2} \tag{3.30}$$

It follows that when $t < \frac{1}{3}$, the pay-off satisfies

$$-\frac{3}{2}t + \frac{1}{2} > 0$$

This means that Alice can guarantee herself a pay-off that is at least 0 if she uses the optimal strategy x. Hence, the strategy x is optimal for both players.

By taking arbitrary accuracy functions, the optimal strategies might no longer have the form of a density over an interval $[a,1]$ with $0 \le a < 1$. We will see in the next chapter that depending on the chosen accuracy functions, the interval $[a,1]$ changes and a player might even wait the full duration before firing.

3.4. Silent Duel over an arbitrary Square

Another way of making the silent duel more complicated is by changing the initial moment both players can fire at each other. Instead of saying that both Alice and Bob pick a ξ and η in $[0,1]$, they will now pick them from the interval $[b,1]$ with $0 < b \le 1$. The game is now played over the square $b \le \xi, \eta \le 1$, instead of over the unit square and both players start with an initial accuracy that is non-zero (if we see $t = b$ as the start of the game).

So consider the same silent duel as defined in section 3.3 with accuracy functions given by: $p(\xi) = q(\xi) = \xi$. Let Alice and Bob pick their respective ξ and η in the interval $[b,1]$ with $0 < b \le 1$. The pay-off for this new game is still given by eq. (3.15). Therefore, the silent duel over the square $[b,1]$ is also a symmetric game. Let us take a look at the optimal strategy for the old silent duel with density given by eq. (3.23), as it could still be optimal in this new duel.

If $b \le \frac{1}{3}$, then both players are able to fire at $t = \frac{1}{3}$. Because our previous optimal strategy is a density over $\left[\frac{1}{3}, 1\right]$, it is only logical that nothing changes to the optimal strategy for this choice of b. The optimal strategy tells us to consider firing after $t = \frac{1}{3}$ and the rules of the game tell us we can only fire after $b \le \frac{1}{3}$. So in this case, the optimal strategy is still optimal.

However, if $b > \frac{1}{3}$ the probability of firing in the interval $[b,1]$ is now smaller than 1. This follows from the normalization

$$1 = \int_{\frac{1}{3}}^{1} \frac{1}{4}\xi^{-3} d\xi > \int_{b}^{1} \frac{1}{4}\xi^{-3} d\xi$$

Therefore, the old optimal strategy is not optimal anymore. Both players should have already considered firing when they are allowed to fire. The optimal strategy changes from a density over the old interval to a density over a new interval $[a,1]$ with a discrete mass α at b. This mass α denotes the probability of firing instantly at $t = b$, because both players have a non-zero accuracy at $t = b$. It could also be that b is big enough such that the optimal strategy for both players is equal to firing at the start. In this case $\alpha = 1$. Let us first look for the optimal strategy consisting of a density and a mass α at b. The optimal strategy has the form

$$x = (\alpha I_b, f) \tag{3.31}$$

in which α is the probability of firing instantly at $t = b$ and f is the density defined over a new interval $[a,1]$ and satisfies

$$\int_a^1 f(\xi) d\xi = 1 - \alpha \tag{3.32}$$

We assume that Alice plays the strategy x and Bob plays his η in the new interval $[a,1]$, because firing before $t = a$ is not beneficial. Take any $b > \frac{1}{3}$. The pay-off for this game can be defined similar to eq. (3.19), we only need to add an extra term that denotes the pay-off for firing at $t = b$ with probability α. This pay-off is given as follows.

$$\int_a^1 K(\xi,\eta) f(\xi) d\xi = \int_a^{\eta} (\xi - (1-\xi)\eta) f(\xi) d\xi + \int_{\eta}^{1} (-\eta + (1-\eta)\xi) f(\xi) d\xi + \alpha(b - (1-b)\eta) \equiv 0 \tag{3.33}$$

The pay-off in eq. (3.33) differs from eq. (3.19) only by a linear term of η. So by substituting $r(\xi) = \xi f(\xi)$, differentiating twice with respect to η and solving the equation, we find the same density $f(\xi) = k\xi^{-3}$. Now filling in eq. (3.32) and using this to calculate eq. (3.33), we get the following two equations.

$$-\frac{k}{2} + \frac{k}{2a^2} = 1 - \alpha \tag{3.34}$$

$$\eta k\left(\frac{3}{2}-\frac{1}{2a^2}+\frac{1}{a}\right)+k\left(-3+\frac{1}{a}\right)+\alpha(b-\eta+b\eta)=0 \tag{3.35}$$

Equation (3.35) can be solved using eq. (3.34) for the values a, α and k. This is very troublesome and we will leave the calculation in the appendix, see appendix A.2. The solution that follows assuming $b > \frac{1}{3}$ is given by the following.

$$a=\frac{b}{2-3b}, \quad \alpha=\frac{3b-1}{2b^2}, \quad k=\frac{1}{4}$$

Since $0 \le \alpha \le 1$, it follows that $1-\alpha$ is also bounded. Therefore, the left-hand side of eq. (3.34) is also bounded. Because of this and our derived constants, it follows that $\frac{1}{3} \le b \le \frac{1}{2}$, this has been proven in appendix A.2. This means that an optimal strategy

$$x=\left(\frac{3b-1}{2b^2}I_b, \frac{1}{4}\xi^{-3}\right) \tag{3.36}$$

consisting of a density $f=\frac{1}{4}\xi^{-3}$ over an interval $\left[\frac{b}{2-3b},1\right]$ and a discrete mass $\alpha=\frac{3b-1}{2b^2}$ on $t=b$ only exists when $b\in\left[\frac{1}{3},\frac{1}{2}\right]$. Note that when $b=\frac{1}{3}$, the discrete mass $\alpha=0$ and the optimal strategy is only a density. We will show that the derived strategy is indeed optimal. Take any $b\in\left(\frac{1}{3},\frac{1}{2}\right]$. Assume Alice plays the optimal strategy x and Bob plays any pure strategy $\eta\in[b,1]$. There are three cases.

1. ($\eta > a$): This case follows immediately, because of how we chose the constants in eq. (3.33). Hence, the pay-off is always equal to 0.
2. ($b<\eta<a$): The pay-off is now given by

$$\alpha(b-\eta+b\eta)+\int_\eta^1(\xi-\eta-\xi\eta)f(\xi)d\xi \tag{3.37}$$

   which can be rewritten as

$$\frac{1}{2}+\eta\left(\frac{3}{2}-\frac{1}{b}\right)$$

   see appendix A.2. It follows that the term

$$\left(\frac{3}{2}-\frac{1}{b}\right)$$

   is always negative for $\frac{1}{3}<b\le\frac{1}{2}$. So, using the fact that $\eta<a$, it follows that

$$\frac{1}{2}+\eta\left(\frac{3}{2}-\frac{1}{b}\right)>\frac{1}{2}+a\left(\frac{3}{2}-\frac{1}{b}\right)=0$$

   Therefore, the pay-off when Bob fires at $b<\eta<a$ is greater than 0.
3. ($b=\eta<a$): The pay-off is now given by

$$\alpha\cdot 0+\int_\eta^1(\xi-\eta-\xi\eta)f(\xi)d\xi \tag{3.38}$$

   which is equal to 0, see appendix A.2.

So in all cases, the pay-off is at least 0. Hence, the strategy x is indeed optimal.

We now know that if $b\le\frac{1}{3}$, the mixed strategy with density given by eq. (3.23) is optimal and if $\frac{1}{3}<b\le\frac{1}{2}$, the optimal strategy is given by eq. (3.36). Therefore, we only need to look for an optimal strategy when $b>\frac{1}{2}$.

Assume that $b>\frac{1}{2}$ and that Alice fires at $\xi=b$. If Bob fires at $\eta=b$, the pay-off is equal to 0. Now if Bob fires at $\eta\neq b$, the pay-off is given by $b-(1-b)\eta$. It follows from $b>\frac{1}{2}$ that

$$b-(1-b)\eta>0$$

So the best possible counter strategy of Bob is to fire at $\eta=b$. This means that Alice can guarantee herself a pay-off that is equal or greater than 0, when she fires at $t=b$ and $b>\frac{1}{2}$. With the symmetric property of the game, this strategy has to be optimal. So when $b>\frac{1}{2}$, an optimal strategy for both players is to fire at $t=b$. Now we can finally give an optimal strategy for every $b\in[0,1]$, which is optimal for both players.

1. ($b \leq \frac{1}{3}$): The optimal strategy is a mixed strategy with density given by eq. (3.23).
2. ($\frac{1}{3} < b \leq \frac{1}{2}$): The optimal strategy is a mixed strategy given by eq. (3.36).
3. ($b > \frac{1}{2}$): The optimal strategy is a pure strategy $x = I_b$, which is instantly firing at $t = b$.

4

Solving Infinite Games by Operators

In this chapter, we will give a general algorithm of how to calculate optimal strategies in two-player silent duel for both players. The optimal strategy of any silent duel can be written in a general form. Depending on the accuracy functions, the optimal strategy takes on a more specific form. We shall first give the general form of a strategy and determine when this strategy exists. Afterwards we will determine what specific form it takes before calculating the exact optimal strategy. The analysis will mostly be done for regular silent duels. Regular silent duels are silent duels as described in section 3.3 in which the accuracy functions are monotone increasing from 0 to 1 for both players. In some variants of the silent duel, players can start with an initial accuracy that is non-zero or their accuracy can even be decreasing. The silent duel over an arbitrary square $[b,1]$ with $0 < b \leq 1$ as described in section 3.4 is also a variant of a regular silent duel. To that end we will restrict our analysis to a subclass of kernels. This will be done by imposing certain restrictions on the kernel. Consider a silent duel with a kernel of the following form.

$$K(\xi,\eta) = \begin{cases} L(\xi,\eta), & \xi < \eta \\ \Phi(\eta), & \xi = \eta \\ M(\xi,\eta), & \xi > \eta \end{cases} \tag{4.1}$$

We impose the following restrictions on this kernel:

1. The functions $L(\xi,\eta)$ and $M(\xi,\eta)$ are defined over their respective triangles $0 \leq \xi \leq \eta \leq 1$ and $0 \leq \eta \leq \xi \leq 1$. Both functions also have continuous second partial derivatives define in their respective closed triangles.

2. $\Phi(1)$ lies between $L(1,1)$ and $M(1,1)$ and $\Phi(0)$ lies between $L(0,0)$ and $M(0,0)$.

3. $L(\xi,\eta)$ and $M(\xi,\eta)$ are strictly increasing in ξ for fixed η in their respective closed triangle with the possible exception that $M_\xi(1,1) = 0$. $L(\xi,\eta)$ and $M(\xi,\eta)$ are strictly decreasing in η for fixed ξ in their respective closed triangle with the possible exception that $L_\eta(1,1) = 0$.

The kernels that satisfy these conditions form a subclass of kernels which we will focus on. The optimal strategies for kernels that satisfy these conditions can be written in a general form.

4.1. General Form of a Strategy

In the previous chapter, we have seen that a mixed strategy x is any probability distribution over the interval $[0,1]$. This probability distribution can be a density over an interval, one or more discrete masses or even a combination both. Because the game is played over the interval $[0,1]$, there are infinitely many possible discrete masses. We will assume that the general optimal strategy for any silent duel will consist of a density f over an interval $[a,b]$ with $0 \leq a < 1$ and two discrete masses α and β on $t = 0$ and $t = 1$ respectively. So every silent duel has optimal strategies that are of this form. The fact that a mixed strategy takes on this form is provable, but we will assume it as true. In a regular silent duel where players start with an initial accuracy equal to $p(0) = q(0) = 0$, the discrete mass $\alpha = 0$. Of course, firing with no probability of hitting can never be optimal. Because both players have no information about whether the opponent has fired already or not, a player might even wait the full duration of the game and fire at $t = 1$. If a player does this, he is hoping that his opponent will miss. This can in some games be optimal. We will use the following notation

$$x = (\alpha I_0, f_{ab}, \beta I_1) \tag{4.2}$$

in which α and β are the discrete masses at the pure strategies $I_0 = 0$ and $I_1 = 1$ respectively and f_{ab} is a density over the interval $[a, b]$. This strategy tells a player to fire at $t = 0$ with probability α, to fire at $t = 1$ with probability β and to fire in the interval $[a, b]$ using the density f_{ab}. In several variants of the silent duel, both players might start with an initial non-zero accuracy. In these duels, the discrete mass α can be non-zero as it might optimal to fire instantly. Think about the duel described in section 3.4. The discrete mass β and the density f can also be zero, in that case we will drop them. For example $x = (f_{ab})$ means that $\alpha = \beta = 0$. If the density is defined over an interval $[a, 1]$, we will denote it as f_a.

Now consider a silent duel with arbitrary accuracy functions in which Alice and Bob both have optimal strategies of the form $x = (\alpha I_0, f_{ab}, \beta I_1)$ and $(\gamma I_0, g_{cd}, \delta I_1)$ respectively. Let us forget about the discrete masses and only think about the densities f and g for now. The densities f and g of optimal strategies are defined over the intervals $[a, b]$ and $[c, d]$ respectively. The density of Alice tells her to consider firing only when $t > a$. Similarly for Bob, his density tells him to consider firing only when $t > c$. If $a \neq c$, then one of the two is bigger than the other. Say for example that $a > c$. This means that Bob will consider firing after $t = c < a$ and Alice will consider firing after $t = a$. Hence, there is an interval $[c, a]$ in which Bob will fire using his density g. However, Alice will not fire in this interval. Therefore, Bob is better off firing at $t = a$ instead of using his density over the interval $[c, a]$. This creates a new strategy that is more optimal, but this contradicts that y is an optimal strategy. Hence, it can't be that $a > c$. A similar argument proves that it can't be that $a < c$. Therefore, it follows that $a = c$. The following lemma tells us this is indeed true. Moreover, the lemma also tells us something about the relation between the values b and d.

Lemma 1 *If both players possess optimal strategies of the form $x = (\alpha I_0, f_{ab}, \beta I_1)$ and $y = (\gamma I_0, g_{cd}, \delta I_1)$, then $a = c$ and $b = d = 1$. The absolute continuous part of both distributions has the same support extending from a to 1.*

The prove of this lemma can be found in Karlin [5] page 111, we assume it as true. The lemma tells us if both players have an optimal strategy of the said form, then their densities need to be defined over the same interval $[a, 1]$. This raises the question when a silent duel has an optimal strategy of the said form. The answer to this question is very closely tied to the solutions of integral equations.

4.2. Integral operators

In order to use lemma 1 to solve general silent duels, we will need to show when optimal strategies of the desired form exist. In this section, we will outline the general method of proving the existence of optimal strategies of the desired form. The details of this method can be found in Karlin [5] page 110. The problem will be separated into two cases.

The first case is when $L(1, 1) \leq M(1, 1)$. Consider a duel with the following kernel (a variant of the silent duel).

$$K(\xi, \eta) = \begin{cases} \xi - \eta, & \xi < \eta \\ \frac{1}{2}\xi, & \xi = \eta \\ 2\xi - \eta, & \xi > \eta \end{cases} \tag{4.3}$$

Alice and Bob both pick their ξ and η respectively in the interval $[0, 1]$. This kernel satisfies the condition: $0 = L(1, 1) \leq M(1, 1) = 1$ and also conditions 1, 2 and 3 on page 31. By observation of eq. (4.3), we can see that there are only positive terms of ξ in every order of firing. Thus playing ξ as high as possible only has a positive effect for Alice. Similarly for Bob, all terms of η in the pay-off are negative. Playing η as high as possible is beneficial to Bob. For this reason we expect that the pure strategy I_1 is optimal for both players. We will show that this is indeed true.

Assume Bob plays $\eta = 1$ and Alice plays any ξ, the pay-off is now given as follows

$$K(\xi, 1) = \begin{cases} \xi - 1, & \xi < 1 \\ \frac{1}{2}\xi, & \xi = \eta \end{cases}$$

If Alice plays $\xi = 1$, her pay-off is equal to $\frac{1}{2}$. However, if Alice plays any other ξ, her pay-off will be given by $\xi - 1 < 0$. Thus, Bob can guarantee that the pay-off is not greater than $h = \frac{1}{2}$ with his pure strategy $\eta = 1$.

Now assume Alice plays $\xi = 1$ and Bob plays any η, the pay-off is now given as follows

$$K(1,\eta) = \begin{cases} \frac{1}{2}, & 1 = \eta \\ 2-\eta, & 1 > \eta \end{cases}$$

If Bob plays $\eta = 1$, the pay-off is equal to $\frac{1}{2}$. But if Bob plays any other η, then the pay-off will be given by $2-\eta > \frac{1}{2}$. Therefore, Alice can guarantee a pay-off that is at least $l = \frac{1}{2}$ with her pure strategy $\eta = 1$. Because $l = h$, both strategies have to be optimal. Hence, the strategies $x = y = (I_1)$ are optimal.

It is remarkable that it is optimal for both players to fire at the end. The reason why this happens is because of the assumed inequality

$$L(1,1) \le M(1,1) \tag{4.4}$$

at the start of this section together with properties 1, 2 and 3 on page 31. At first it might not be clear what the inequality tells us, but with the use of property 3 on page 31, we get the following inequality.

$$L(\xi,1) \le L(1,1) \le M(1,1) \le M(1,\eta) \tag{4.5}$$

This inequality follows from the fact that the functions $L(\xi,\eta)$ and $M(\xi,\eta)$ are strictly increasing and decreasing in ξ and η respectively. Now using property 2 on page 31, we get the following.

$$L(\xi,1) \le L(1,1) \le \Phi(1) \le M(1,1) \le M(1,\eta) \tag{4.6}$$

Hence, the following holds.

$$K(\xi,1) \le K(1,1) \le K(1,\eta) \tag{4.7}$$

Now it is clear that both players will not deviate from $\xi = \eta = 1$. If Bob fires at $\eta = 1$, Alice can maximize her pay-off $K(\xi,1)$ by firing at $\xi = 1$. Similarly, if Alice fires at $\xi = 1$, Bob can minimize the pay-off $K(1,\eta)$ by firing at $\eta = 1$. If any player deviates from firing at $t = 1$, their opponent can still fire at $t = 1$ such that the pay-off increases in their opponents favour. The pay-off is maximized and minimized when both players fire at $\xi = \eta = 1$. This proves that optimal strategies exist of the form $x = (\alpha I_0, f_{ab}, \beta I_1)$ when $L(1,1) \le M(1,1)$.

However, we are more interested in the case where $L(1,1) > M(1,1)$, because all regular silent duels belong to this case. Silent duels in which the kernel satisfies $L(1,1) \le M(1,1)$ can never have monotone increasing accuracy functions from 0 to 1. Therefore, we will only consider silent duels in which the kernel satisfies $L(1,1) > M(1,1)$.

Consider the case in which $L(1,1) > M(1,1)$. Because both functions are continuous, there is an interval $[a,1]$ such that $L(\xi,\xi) > M(\xi,\xi)$ for $\xi \in [a,1]$. If there exists optimal strategies of the form $x = (\alpha I_0, f_{ab}, \beta I_1)$ and $y = (\gamma I_0, g_{cd}, \delta I_1)$ for Alice and Bob respectively, consider the pay-off when Alice uses x and Bob uses a pure strategy η.

$$\int_0^1 K(\xi,\eta)dx(\xi) = \alpha L(0,\eta) + \int_a^\eta L(\xi,\eta) f_{ab}(\xi)d\xi + \int_\eta^b M(\xi,\eta) f_{ab}(\xi)d\xi + \beta M(1,\eta) \tag{4.8}$$

Of course if $\eta = 0$, we replace $L(0,0)$ by $\Phi(0)$. If $\eta = 1$, we replace $M(1,1)$ by $\Phi(1)$. If $\eta < a$, the first integral is dropped and if $\eta > b$ the second integral is dropped. A similar equation holds when Bob uses his optimal strategy y and Alice uses a pure strategy ξ.

Because the strategy x is optimal, we know that eq. (4.8) is identically v when $a \le \eta < 1$. Differentiation of eq. (4.8) with respect to η followed by a division of the term $L(\eta,\eta) - M(\eta,\eta)$ gives an equation we will consider over the interval $[a,1]$ (see appendix A.2 for a step-by-step derivation).

$$f(t)-\int_a^1 T(\xi,t)f(\xi)d\xi=\alpha p_0(t)+\beta p_1(t) \tag{4.9}$$

$$T(\xi,t)=\begin{cases}\frac{-L_\eta(\xi,t)}{L(t,t)-M(t,t)}, & a\le\xi<t\le 1\\ \frac{-M_\eta(\xi,t)}{L(t,t)-M(t,t)}, & a\le t\le\xi\le 1\end{cases}$$

$$p_0(t)=\frac{-L_\eta(0,t)}{L(t,t)-M(t,t)},\quad p_1(t)=\frac{-M_\eta(1,t)}{L(t,t)-M(t,t)}$$

A similar equation holds for the optimal strategy y of Bob.

$$g(u)-\int_a^1 U(u,\eta)g(\eta)d\eta=\gamma q_0(u)+\delta q_1(u) \tag{4.10}$$

$$U(u,\eta)=\begin{cases}\frac{M_\xi(u,\eta)}{L(u,u)-M(u,u)}, & a\le\eta<u\le 1\\ \frac{L_\xi(u,\eta)}{L(u,u)-M(u,u)}, & a\le u\le\eta\le 1\end{cases}$$

$$q_0(u)=\frac{M_\xi(u,0)}{L(u,u)-M(u,u)},\quad q_1(u)=\frac{L_\xi(u,1)}{L(u,u)-M(u,u)}$$

Let us denote T_a and U_a as the integral operator with kernel $T(\xi,t)$ and $U(u,\eta)$ respectively with lower limit a. The identity operator will be denoted by I. The integral equations in eq. (4.9) and eq. (4.10) can now be written as follows.

$$\begin{aligned}(I-T_a)f&=\alpha p_0+\beta p_1\\ (I-U_a)g&=\gamma q_0+\delta q_1\end{aligned} \tag{4.11}$$

So optimal strategies of the desired form exist when eq. (4.11) is satisfied. The integral operators have two important properties. The first property is that any function f that is non-negative and piecewise continuous will be transformed by the integral operation into strictly positive and continuous functions T_af and U_af on the interval $[a,1]$. This follows from the fact that $L(\xi,\xi)>M(\xi,\xi)$ on $[a,1]$ and L and M being monotone-decreasing in the second variable and monotone-increasing in the first variable (property 3 on page 31). So if f is a density, then T_af and U_af are strictly positive.

The second property is that the operators are completely continuous. Any function f that is uniformly bounded on the interval $[a,1]$ is transformed by the operator in a continuous function. This follows from the fact that f is bounded and $|T_a(t_1)-T_a(t_2)|\to 0$ when $|t_1\to t_2|$. For the full proof, see Karlin [5] page 119.

With the strict positivity and complete continuity of the integral operators, the spectral radius $\lambda(a)$ of T_a and $\mu(a)$ of U_a have certain properties. The spectral radius $\lambda(a)$ of T_a denotes the smallest radius of the circle in the complex plane centered at the origin that contains all eigenvalues of T_a. The eigenvalues of the integral operator T_a are the values λ that satisfy

$$T_af=\lambda f$$

with f the eigenfunctions. The eigenfunctions f have the property that when the integral operator T_a transforms the functions, it is only scaled by a certain constant (Sherrill [7]). A similar definition holds for the spectral radius $\mu(a)$ of U_a. The spectral radius has the following properties:

1. $\lambda(a)$ is an eigenvalue of T_a and has a positive eigenfunction f^a.

2. $\lambda(a)$ is continuous and strictly monotone function of a. If $a\to 1$, then $\lambda(a)\to 0$.

3. If $\lambda>\lambda(a)$, then

   $$\left(I-\frac{T_a}{\lambda}\right)^{-1}$$

   exists and can be evaluated by means of the series

   $$\sum_{n=0}^{\infty}\left(\frac{T_a}{\lambda}\right)^n \tag{4.12}$$

Similar properties apply to the spectral radius $\mu(a)$ of U_a. We are looking for positive functions f_a and g_a that satisfy eq. (4.11). If there are positive functions that satisfy eq. (4.11), then there exists optimal strategies of the desired form when $L(1,1) > M(1,1)$. There is a theorem (theorem 2) that tells us that the densities f_a are either solutions of the equation $T_a f_a = f_a$ or expressed as a Neumann series of the form

$$\sum_{n=0}^{\infty} T_a^n(\alpha p_0 + \beta p_1) \tag{4.13}$$

and similarly for g_a. If f_a satisfies $T_a f_a = f_a$, it follows from eq. (4.11) that

$$0 = \alpha p_0 + \beta p_1$$

Hence, the discrete masses α and β are zero and the optimal strategy is only a density f_a in this case. The theorem also tells us that the general optimal strategies take on a more specific form. The full theorem will be given later, we will only make use of part of the theorem now. With this theorem, we know that if f_a is a density of an optimal strategy that does not satisfy $T_a f_a = f_a$, then it can be expressed as the Neumann series in eq. (4.13). The Neumann series needs to converge, this will happen when the biggest eigenvalue $\lambda(a)$ of T_a is smaller than 1. Therefore, eq. (4.11) has solutions if $\lambda(a) < 1$ and similar for μ. Now if there is a value a such that $\lambda(a) = 1$, then we know with property 2 on page 34 that $\lambda(a^*) < 1$ holds for $a^* > a$. The following lemma determines whether there exists a value $a > 0$ such that $\lambda(a) = 1$.

Lemma 2 *If $L(1,1) > M(1,1)$ and there exists a value $\xi_0 \in [0,1)$ such that $L(\xi_0,\xi_0) = M(\xi_0,\xi_0)$, then there exists a value $a > \xi_0$ such that $\lambda(a) = 1$. Similarly there exists a value $a' > \xi_0$ such that $\mu(a') = 1$.*

Note that the value a and a' are not necessarily the same. If the conditions of lemma 2 are satisfied, optimal strategies of the desired form exist and can be found by inverting the operator in eq. (4.11).

$$\begin{aligned} f_a &= (I - T_a)^{-1}(\alpha p_0 + \beta p_1) \\ g_a &= (I - U_a)^{-1}(\gamma q_0 + \delta q_1) \end{aligned} \tag{4.14}$$

It is possible that there is no value $a \in [0,1]$ such that $\lambda(a) = 1$ or $\mu(a) = 1$. This is only possible when $L(\xi,\xi) > M(\xi,\xi)$ for all $\xi \in [0,1]$, $\lambda(0) < 1$ and $\mu(0) < 1$ (this will not happen for regular silent duels).

4.3. Specific Form of the Optimal Strategy

In the previous sections we have seen that the optimal strategies can be written as $x = (\alpha I_0, f_{ab}, \beta I_1)$ and $y = (\gamma I_0, g_{cd}, \delta I_1)$ for both players respectively. This is the general form of the optimal strategies and we have also seen when these strategies exist. In this section we determine a more specific form of the optimal strategy i.e. when the constants α, β, γ and δ are zero. This will mainly depend on the values $\lambda(a)$ and $\mu(a)$. The following table groups kernels based on certain properties and lists a more specific form of the optimal strategy for both players.

Kernel	Optimal F	Optimal G
Group 1: A) $L(1,1) \le M(1,1)$	(I_1)	(I_1)
Group 2: $L(\xi,\xi) > M(\xi,\xi) \quad a \le \xi \le 1$ B) $\lambda(a) = 1$, $\mu(a) < 1$ C) $\lambda(a) = 1$, $\mu(a) = 1$ D) $\lambda(a) < 1$, $\mu(a) = 1$ If $\exists \xi_0 \in [0,1] : L(\xi_0,\xi_0) = M(\xi_0,\xi_0)$, then either B, C or D occurs.	 (f_a) (f_a) $(f_a, \beta I_1)$	 $(g_a, \delta I_1)$ (g_a) (g_a)
Group 3: $L(\xi,\xi) > M(\xi,\xi) \quad 0 \le \xi \le 1 \quad \lambda(0) < 1$, $\mu(0) < 1 \quad L(0,1) < M(1,0)$ E) $\Phi(0) = L(0,0)$ F) $L(0,0) > \Phi(0) > S_0$ G) $\Phi(0) = S_0$ H) $S_0 > \Phi(0) > M(0,0)$ I) $\Phi(0) = M(0,0)$ S_0 is a value such that $M(0,0) < S_0 < L(0,0)$.	 $(\alpha I_0, f_0)$ $(\alpha I_0, f_a)$ $(\alpha I_0, f_a)$ $(\alpha I_0, f_a, \beta I_1)$ $(f_0, \beta I_1)$	 $(g_0, \delta I_1)$ $(\gamma I_0, g_a, \delta I_1)$ $(\gamma I_0, g_a)$ $(\gamma I_0, g_a)$ $(\gamma I_0, g_0)$
Group 4: $L(\xi,\xi) > M(\xi,\xi) \quad 0 \le \xi \le 1 \quad \lambda(0) < 1$, $\mu(0) < 1 \quad L(0,1) \ge M(1,0)$ J) $L(0,1) \ge \Phi(0) \ge M(1,0)$ K) $L(0,0) > \Phi(0) > L(0,1)$ L) $L(0,0) = \Phi(0)$ M) $M(1,0) > \Phi(0) > M(0,0)$ N) $\Phi(0) = M(0,0)$	 (I_0) $(\alpha I_0, f_a)$ $(\alpha I_0, f_0)$ $(\alpha I_0, f_a, \beta I_1)$ $(f_0, \beta I_1)$	 (I_0) $(\gamma I_0, g_a, \delta I_1)$ $(g_0, \delta I_1)$ $(\gamma I_0, g_a)$ $(\gamma I_0, g_0)$

Table 4.1: Specific form of optimal strategies for kernels with certain characteristics

The table is obtained by proving that the optimal strategy in every case must be of a specific form, this is done in Karlin [5] page 124. The following theorem tells us that the forms in table 4.1 are indeed optimal in every case.

Theorem 2 *The optimal strategies for the pay-off kernel $K(\xi,\eta)$ in silent duels are unique and take the forms indicated in table 4.1. Furthermore, the densities f_a are either solutions of the equation $T_a f_a = f_a$ (when $\lambda(a) = 1$) or expressible as Neumann series of the form*

$$\sum_{n=0}^{\infty} T_a^n(\alpha p_0 + \beta p_1)$$

and similarly for g_a.

Given two accuracy functions for Alice and Bob, we know with theorem 2 and table 4.1 how the optimal strategies for both players look like. All that is left now is to calculate the exact values of α, β, γ, δ, a and the densities f and g for a silent duel.

4.4. Silent Duel with general Accuracy Function

Consider a silent duel with accuracy functions gives by $p(\xi)$ and $q(\eta)$, which are continuous and monotone-increasing from 0 to 1 (so $\alpha = \gamma = 0$). We know that the pay-off kernel for silent duels can be given as in eq. (3.14). This kernel satisfies conditions 2 and 3 on page 31. The conditions of lemma 2 are satisfied by $\xi_0 = 0$, therefore there exists values a and a' such that $\lambda(a) = 1$ and $\mu(a') = 1$. So with table 4.1 we know that the optimal strategies are one of the cases in B,C or D in group 2. This is dependent on the value of a and a'. If both values agree, then case C of group 2 happens. In this case, both densities f_a and g_a satisfy $T_a f_a = f_a$ and $U_a g_a = g_a$. When the values do not agree, then we know that either $\lambda(a) = 1$ and $\mu(a) < 1$ or $\lambda(a) < 1$ and $\mu(a) = 1$ (property 2 on page2). In the first case, $U_a g_a \ne g_a$, hence the optimal strategy for Bob is found by solving eq. (4.14) and similar for the second case. We will first look for the values a and a' by solving $T_a f_a = f_a$ and $U_{a'} g_{a'} = g_{a'}$ respectively. Filling in both equations gives us the following.

$$\begin{aligned} f(t) &= \frac{q'(t)}{2p(t)q(t)}\Big(\int_a^1 f(\xi)d\xi - \int_a^t p(\xi)f(\xi)d\xi + \int_t^1 p(\xi)f(\xi)d\xi\Big) \\ g(u) &= \frac{p'(u)}{2p(u)q(u)}\Big(\int_{a'}^1 g(\eta)d\eta - \int_{a'}^u q(\eta)g(\eta)d\eta + \int_u^1 q(\eta)g(\eta)d\eta\Big) \end{aligned} \tag{4.15}$$

We define the functions $h(\xi) = p(\xi)f(\xi)$ and $l(\eta) = q(\eta)g(\eta)$ and normalize f and g such that

$$\int_a^1 f(\xi)d\xi = 1 = \int_{a'}^1 g(\eta)d\eta \tag{4.16}$$

holds. Equation (4.15) can be written as followed.

$$\begin{aligned} \frac{2h(t)}{\left(1 - \int_a^t h(\xi)d\xi + \int_t^1 h(\xi)d\xi\right)} &= \frac{q'(t)}{q(t)} \\ \frac{2l(u)}{\left(1 - \int_{a'}^u l(\eta)d\eta + \int_u^1 l(\eta)d\eta\right)} &= \frac{p'(u)}{p(u)} \end{aligned} \tag{4.17}$$

Integrating both sides of eq. (4.17) with respect to t for the first equation and with respect to u for the second equation and simplifying results in the following equations.

$$\begin{aligned} 1 - \int_a^t h(\xi)d\xi + \int_t^1 h(\xi)d\xi &= \frac{k}{q(t)} \\ 1 - \int_{a'}^u l(\eta)d\eta + \int_u^1 l(\eta)d\eta &= \frac{k_2}{p(u)} \end{aligned} \tag{4.18}$$

Differentiation of eq. (4.18) with respect to t for the first equation and with respect to u for the second equation yields

$$-2h(t) = -2p(t)f(t) = -\frac{kq'(t)}{(q(t))^2}$$

$$-2l(u) = -2q(u)g(u) = -\frac{k_2 p'(u)}{(p(u))^2}$$

hence

$$f(t) = \frac{k_1 q'(t)}{p(t)(q(t))^2} \tag{4.19}$$

and

$$g(u) = \frac{k_3 p'(u)}{q(u)(p(u))^2} \tag{4.20}$$

with $k_1 = \frac{k}{2}$ and $k_3 = \frac{k_2}{2}$. The functions $f(t)$ and $g(u)$ are solutions to the differential equations obtained by integrating and differentiating eq. (4.17). To determine the value a and a' that will make the derived functions a solution to eq. (4.17), we need to insert both functions in eq. (4.17) respectively. Filling in and simplifying the first equation, we find the following for $f(t)$.

$$\frac{1}{k_1} = \frac{1}{q(a)} + 1$$

Using the normalization in eq. (4.16), the equation can be expanded to the following.

$$\frac{1}{k_1} = \frac{1}{q(a)} + 1 = \int_a^1 \frac{q'(t)}{p(t)(q(t))^2}dt \tag{4.21}$$

For $g(u)$ we find a similar equation.

$$\frac{1}{k_3} = \frac{1}{p(a')} + 1$$

This equation can also be expanded using the normalization in eq. (4.16).

$$\frac{1}{k_3} = \frac{1}{p(a')} + 1 = \int_{a'}^{1} \frac{p'(u)}{q(u)(p(u))^2} du \tag{4.22}$$

For any two accuracy functions p and q, the values of a and a' can now be calculated by eq. (4.21) and eq. (4.22) respectively. These values are not necessarily identical and they tell us if the optimal strategy consists of only a density or a density with a discrete mass on I_1 (group 2 in table 4.1). Let us take a look at the values of a and a'.

If $a = a'$, then $\lambda(a) = \mu(a) = 1$. Therefore, $T_a f = \lambda f = f$ and $U_a g = \mu g = g$. Hence, the optimal strategy for both players is given by a density over $[a, 1]$ as depicted in table 4.1. The densities f_a and g_a are exactly the densities calculated in eq. (4.19) and eq. (4.20) with the constants k_1 and k_3 calculated in eq. (4.21) and eq. (4.22) respectively.

If $a \neq a'$, then one of the two is larger than the other. First consider the case where $a > a'$. Because of property 2 on page 34, we know that $\lambda(a) = 1$ and $\mu(a) < 1$. Table 4.1 tells us that Alice plays with only a density f_a and Bob plays with a density g_a and a discrete mass δ on I_1. Both densities are defined over the interval $[a, 1]$. The density of Alice is given by eq. (4.19) with the constant calculated in eq. (4.21). To calculate an optimal strategy for Bob, we need to solve the equation

$$g_a = (I - U_a)^{-1} \delta q_1 \tag{4.23}$$

with the following normalization

$$\int_a^1 g(\eta) d\eta = 1 - \delta \tag{4.24}$$

Writing eq. (4.23) out results in the following equation.

$$g(u) - \int_a^u \frac{p'(u)(1 - q(\eta))}{2p(u)q(u)} g(\eta) d\eta - \int_u^1 \frac{p'(u)(1 + q(\eta))}{2p(u)q(u)} g(\eta) d\eta = \frac{2\delta p'(u)}{2p(u)q(u)} \tag{4.25}$$

This equation can be solved by a substitution followed by an integration and differentiation with respect to u. This goes analogue to our previous calculation. We find the following density.

$$g(u) = \frac{k_5 p'(u)}{q(u)(p(u))^2} \tag{4.26}$$

Now we need to fill in eq. (4.25) with eq. (4.26) and use the normalization in eq. (4.24). This results in the following equation.

$$k_5 \left(\frac{1}{p(a)} + 1 \right) = 1 + \delta \tag{4.27}$$

The normalization in eq. (4.24) together with the density in eq. (4.26) give following equation.

$$\int_a^1 g(u) du = \int_a^1 \frac{k_5 p'(u)}{q(u)(p(u))^2} du = 1 - \delta \tag{4.28}$$

Now we can derive the values of the constants δ and k_5 using eq. (4.27) and eq. (4.28). The optimal strategy for Bob is now given by the density in eq. (4.26) over the interval $[a, 1]$ and a discrete mass δ on I_1.

Now consider the case where $a < a'$. This case goes analogue to the previous case. Because of property 2 on page 34 it follows that $\lambda(a') < 1$ and $\mu(a') = 1$. Table 4.1 tells us that Alice plays with a density and a discrete mass β on I_1, whereas Bob plays with only a density. Both densities are defined over the interval $[a', 1]$. The density calculated in eq. (4.20) is optimal for Bob with the constant calculated in eq. (4.22). For Alice we need to solve the equation

$$f_{a'} = (I - T_{a'})^{-1} \beta p_1 \tag{4.29}$$

with the normalization

$$\int_{a'}^{1} f(\xi)d\xi = 1-\beta \tag{4.30}$$

Equation (4.29) can be written out as followed.

$$f(t) - \int_{a}^{t} \frac{q'(t)(1-p(\xi))}{2p(t)q(t)} f(\xi)d\xi - \int_{t}^{1} \frac{q'(t)(1+p(\xi))}{2p(t)q(t)} f(\xi)d\xi = \frac{2\beta q'(t)}{2p(t)q(t)} \tag{4.31}$$

Rewriting eq. (4.31) followed by a substitution, integration and differentiation with respect to t gives the following equation.

$$f(t) = \frac{k_7 q'(t)}{p(t)(q(t))^2} \tag{4.32}$$

Filling in eq. (4.31) with eq. (4.32) and using the normalization in eq. (4.30) gives the following equations.

$$k_7\left(\frac{1}{q(a)}+1\right) = 1+\beta \tag{4.33}$$

$$\int_{a'}^{1} f(t)dt = \int_{a'}^{1} \frac{k_7 q'(t)}{p(t)(q(t))^2} dt = 1-\beta \tag{4.34}$$

The constants β and k_7 can now be found using eq. (4.33) and eq. (4.34). The optimal strategy for Alice is now given by the density in eq. (4.32) over the interval $[a', 1]$ and a discrete mass β on I_1.

The value for games that belong in group 2 of table 4.1 can be calculated using the following equation.

$$v = \int_{a}^{n} L(\xi,\eta) f_a(\xi)d\xi + \int_{\eta}^{1} M(\xi,\eta) f_a(\xi)d\xi \tag{4.35}$$

If the optimal strategy of Alice consists of a discrete mass α on I_1 that is non-zero, we need to add a term that denotes the pay-off when she fires at I_1 with probability α.

We finish this section with a summary of how to calculate optimal strategies for both players in a two-player silent duel:

1. Solve the equations eq. (4.21) and eq. (4.22) for the variables a and a' respectively.
2. Determine which of the variables a or a' is greater than the other and denote that variable as z. The densities of the optimal strategy for both players will be defined over the interval $[z, 1]$.
3. Determine the values of $\lambda(z)$ and $\mu(z)$ and use table 4.1 to determine the specific form of the optimal strategy for both players
    (a) If the optimal strategy of a player is only a density, then the densities given by eq. (4.19) and eq. (4.20) are optimal and the constant term in the density can be calculated with eq. (4.21) and eq. (4.22) respectively.
    (b) If the optimal strategy of a player is a density with a discrete mass on I_1, then the densities are given by eq. (4.32) and eq. (4.26) respectively. The constant in the densities and the discrete mass on I_1 can be calculated using eq. (4.33) and eq. (4.34) for f and eq. (4.27) and eq. (4.28) for g.

4.5. Silent duel Revisited

In section 3.3 we discussed a silent duel with accuracy functions given by $p(t) = q(t) = t$. The optimal strategy for both players was given by the density

$$f(t) = \frac{1}{4t^3}$$

over the interval $\left[\frac{1}{3}, 1\right]$ and the value of the game is equal to zero. We will verify that the algorithm derived in the previous section gives the same optimal strategy.

First we fill in eq. (4.21) and eq. (4.22).

$$\frac{1}{k_1} = \frac{1}{q(a)} + 1 = \frac{1}{a} + 1 = \int_a^1 \frac{1}{t^3}\,dt = -\frac{1}{2} + \frac{1}{2a^2}$$
$$\frac{1}{k_3} = \frac{1}{p(a')} + 1 = \frac{1}{a'} + 1 = \int_{a'}^1 \frac{1}{u^3}\,du = -\frac{1}{2} + \frac{1}{2(a')^2}$$

If we substitute $\frac{1}{a} = x$ and $\frac{1}{a'} = y$ and simplify the equation, we get the following polynomials.

$$x^2 - 2x - 3 = 0$$
$$y^2 - 2y - 3 = 0$$

Both polynomials have the same roots at $x = y = 3$ and $x = y = -1$. Substituting $\frac{1}{a}$ and $\frac{1}{a'}$ back, we find that $a = a' = \frac{1}{3}$ is the only valid solution. It follows that $\lambda(a) = \mu(a)$. Therefore, we know with table 4.1 that the optimal strategy for both players is given by a density over the interval $\left[\frac{1}{3}, 1\right]$.

The densities are given by filling in eq. (4.19) and eq. (4.20) with the constants k_1 and k_3 calculated in eq. (4.21) and eq. (4.22) respectively. We find the densities

$$f(t) = \frac{k_1 q'(t)}{p(t)(q(t))^2} = \frac{1}{4t^3}$$
$$g(u) = \frac{k_3 p'(u)}{q(u)(p(u))^2} = \frac{1}{4u^3}$$

which is exactly the density as calculated in section 3.3. The value of the game is obtained by filling in eq. (4.35) which gives

$$v = \int_{\frac{1}{3}}^{\eta} (\xi - \eta + \xi\eta)\frac{1}{4\xi^3}\,d\xi + \int_{\eta}^{1} (\xi - \eta - \xi\eta)\frac{1}{4\xi^3}\,d\xi$$

A quick calculation yields a value equal to zero, which agrees with the game being symmetric.

Now let us take a look at a silent duel in which both players have different accuracy functions. We will show that the strategies calculated with the algorithm are indeed optimal. We will take the following accuracy functions for Alice and Bob respectively: $p(t) = t$ and $q(t) = t^2$. The pay-off kernel for a silent duel with these accuracy functions is given by the following function.

$$K(\xi,\eta) = \begin{cases} \xi - \eta^2 + \xi\eta^2, & \xi < \eta \\ \xi - \eta^2, & \xi = \eta \\ \xi - \eta^2 - \xi\eta^2, & \xi > \eta \end{cases} \tag{4.36}$$

This kernel satisfies all conditions on page 31. The conditions of lemma 2 are satisfied by $\xi_0 = 0$, so we proceed to calculate the values a and a' using eq. (4.21) and eq. (4.22). We find the following equations

$$\frac{1}{a^2} + 1 = 2\int_a^1 t^{-4}\,dt = 2\left(-\frac{1}{3} + \frac{1}{3a^3}\right)$$
$$\frac{1}{a'} + 1 = \int_{a'}^1 u^{-4}\,du = -\frac{1}{3} + \frac{1}{3(a')^3}$$

in which we can substitute $x = \frac{1}{a}$ and $y = \frac{1}{a'}$ and rewrite to the following polynomials.

$$2x^3 - 3x^2 - 5 = 0$$
$$y^3 - 3y - 4 = 0$$

Because both polynomials are third degree polynomials, we let a program solve them for us. It follows that $a \approx 0.481$ and $a' \approx 0.455$. With property 2 on 34 it follows that $\lambda(a) = 1$ and $\mu(a) < 1$. So it follows from table 4.1 that the optimal strategy of Alice is a density over $[a, 1]$ and an optimal strategy for Bob is a density over $[a, 1]$ with a discrete mass on I_1. The density for Alice is calculated in eq. (4.19) and the constant in the density is calculated in eq. (4.21). We find the following constant

$$k_1 = \frac{a^2}{1+a^2} \approx 0.188$$

and thus the density is given by

$$f(t) = \frac{0.376t}{t^5} = \frac{0.376}{t^4}$$

So an optimal strategy for Alice is given by this density over the interval [0.481;1].

For Bob we rewrite eq. (4.27) to a single term of k_5 and substitute it in eq. (4.28). We get the following equation.

$$\frac{1+\delta}{3.079}\int_{0.481}^{1} \frac{1}{u^4} du \approx 1-\delta$$

If we solve this integral equation, we find that $\delta \approx 0.073$. It follows that $k_5 \approx 0.348$. Therefore, the density of Bob's optimal strategy is given by

$$g(u) = \frac{0.348}{u^4}$$

over the interval [0,481;1] and a discrete mass of 0.073 on I_1. To visualize at what time both players might fire, the following plot is made.

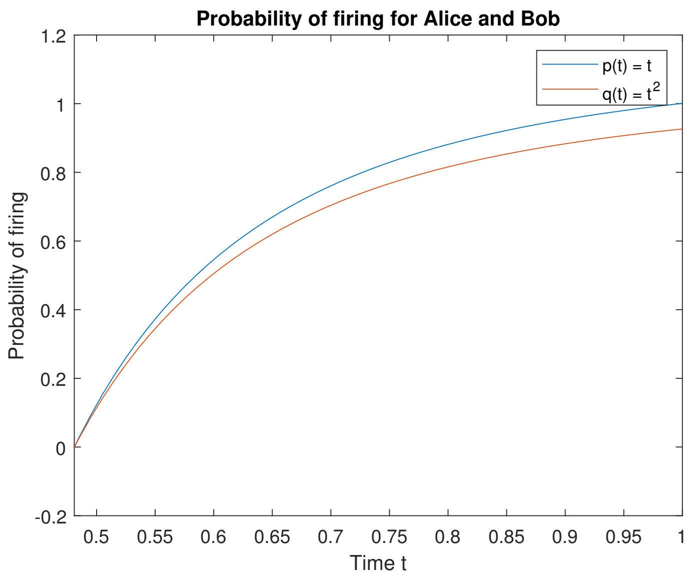

Figure 4.1: Plot of densities integrated over the interval [0.481;1]

In fig. 4.1 the value of the blue line at any time $t \in [0.481;1]$ denotes the probability of Alice firing in the interval $[0.481;t]$ and similar for the red line and Bob. Because the blue line is greater than the red line at any $t \in (0,1)$, the probability that Alice fires in any interval $[0.481;t]$ is always greater than that of Bob. The red line makes a jump at the end of the figure, because of the discrete mass δ of Bob.

It is expected that the value of this game is not equal to zero, because Alice has a slightly better accuracy function at the start. We will show that the optimal strategies derived of the algorithm are indeed optimal. To that end, we will show what pay-off Alice can guarantee herself of if she plays her optimal strategy and what pay-off Bob can hold Alice down to if he plays his optimal strategy. These two values should be the same for both players, because the game has a value.

Assume Alice plays her optimal strategy and Bob uses a pure strategy η. There are two cases.

1. ($\eta \leq a = 0.481$): In this case, Bob fires first and hits with probability η^2. He will miss with probability $(1-\eta^2)$. If that happens, the probability that Alice fires in the interval $[0.481;1]$ and hits is equal to the following integral.

$$\int_{0.481}^{1} \xi f(\xi) d\xi = 0.376 \int_{0.481}^{1} \frac{1}{\xi^3} d\xi \approx 0.625$$

So the pay-off is given by

$$-\eta^2 + (1-\eta^2)0.625 \leq -a^2 + (1-a^2)0.625 = -0.481^2 + (1-0.481^2)0.625 \approx 0.249$$

2. ($\eta > a = 0.481$): In this case, the pay-off is given as follows.

$$0.376 \int_{0.481}^{\eta} \frac{\xi - \eta^2 + \xi\eta^2}{\xi^4} d\xi + 0.376 \int_{\eta}^{1} \frac{\xi - \eta^2 - \xi\eta^2}{\xi^4} d\xi \tag{4.37}$$

This integral equation can be solved by hand (see appendix A.2). Because the values are not exact, we can only give a close approximation with error smaller than 10^{-2}. It follows that eq. (4.37) is equal to 0.249, which has no terms of η.

So in both cases, Alice can guarantee herself a pay-off that is at least 0.249. If we can show that Bob can hold the pay-off down to 0.249, it follows that the derived strategies are indeed optimal. Assume Bob plays his optimal strategy and Alice plays a pure strategy ξ. There are two cases.

1. ($\xi \leq a = 0.481$): In this case, Alice fires first and hits with probability ξ. Alice will miss with probability $(1-\xi)$. If Alice misses, the probability that Bob fires in the interval $[0.481;1]$ and hits is equal to the following integral.

$$\int_{0.481}^{1} \eta^2 g(\eta) d\eta = 0.348 \int_{0.481}^{1} \frac{1}{\eta^2} d\eta \approx 0.375$$

Note that the integral is a little different that before, because bob has accuracy function $q(\eta) = \eta^2$. Bob also fires at $t = 1$ with probability $\delta = 0.073$. Hence, the pay-off is given by

$$\xi - (1-\xi)(0.375 + 0.073) \geq a - (1-a)0.448 = 0.481 + (1-0.481) \approx 0.249$$

2. ($\xi > a = 0.481$): In this case, the pay-off is given as follows.

$$0.348 \int_{0.481}^{\xi} \frac{\xi - \eta^2 - \xi\eta^2}{\eta^4} d\eta + 0.348 \int_{\xi}^{1} \frac{\xi - \eta^2 + \xi\eta^2}{\eta^4} d\eta + 0.073(2\xi - 1) \tag{4.38}$$

This equation has been solved in appendix A.2. Some terms are rounded which makes our answer not exact, but it still is a close approximation with a small enough error. It follows that eq. (4.38) is equal to 0.249.

So in both cases, Bob can hold the pay-off down to at most 0.249. This value is the same as the pay-off Alice can guarantee herself of if she plays her optimal strategy. This means that the derived strategies are indeed optimal and that the value of the game is equal to 0.249.

It is quite interesting that when Bob is given a slightly lower accuracy function, both players should consider firing at a later time compared to the silent duel in which both players have the same accuracy function $p(t) = q(t) = t$. It is less favourable for Bob to fire early on, hence the first time Bob will consider firing at will change to a later time. Alice on the other hand does not think Bob poses a threat early on. Therefore, she can consider firing at a later time to increase her probability of hitting. For Bob to maintain a threat until the end, he has to fire at $t = 1$ with a certain non-zero probability.

5

Three-player Duels

In this chapter, we will analyse three-player duels. As the name suggests, a three-player duel is a duel played between three player. We will name the players Alice, Bob and Charlie. In this game, all participants fire at a common target. At the start of the game Alice, Bob and Charlie are positioned far away from a target such that when they fire instantly, they will always miss. When the game starts, all players walk towards the target and they are allowed to fire once at any time $t \in [0,1]$. The chances of hitting the target when firing increases over time. Firing at $t = 0$ will always miss, whereas firing at $t = 1$ will always hit. The first player that hits the target is the winner and obtains $+2$ units, one from each player. The game is ended when a player has hit the target or no one has hit the target at $t = 1$.

Due to unforeseen difficulties and the lack of time, we will only analyse a three-player noisy duel. We leave the three-player silent duel to our successor. Recall that in a two-player noisy duel, both players know when the other has fired. This is the same for a three-player duel, every player knows who has fired already and who can still fire at any time t.

An important assumption we make throughout this chapter is that it is impossible for two or more players to fire at the same time. When two or more players do fire at the same time, we flip a fair two sided coin (or three sided dice). Depending on this flip, one of the players will fire slightly earlier. Moreover, if the player that fires slightly earlier misses in a noisy duel, both opponents are able to react in time. This means that the two remaining players that have not fire yet, hear the shot and are able to fire at a different time they initially had in mind. With this assumption, the game has at most one winner.

In a three-player duel we say that Alice, Bob and Charlie play their pure strategies ξ, η and θ respectively. The pay-off kernel $K(\xi,\eta,\theta)$ is now a vector (p_1, p_2, p_3) in which p_1, p_2 and p_3 denote the pay-off for Alice, Bob and Charlie respectively. Given a kernel $K(\xi,\eta,\theta)$ we define the following notation which represents each player's pay-off.

$$\begin{aligned} K_1(\xi,\eta,\theta) &= p_1 \\ K_2(\xi,\eta,\theta) &= p_2 \\ K_3(\xi,\eta,\theta) &= p_3 \end{aligned} \tag{5.1}$$

So when Alice fires at $\xi = 1$ and Bob and Charlie fire at $\eta = \theta = 0$, the pay-off for Alice is given by $K_1(1,0,0) = +2$. Similarly, the pay-off for Bob and Charlie is given by $K_2(1,0,0) = -1$ and $K_3(1,0,0) = -1$ respectively. The pay-off is yet again dependent on the order of firing and there are a total of six orders of firing.

$$K(\xi,\eta,\theta) = \begin{cases} H(\xi,\eta,\theta), & \eta < \xi < \theta \\ J(\xi,\eta,\theta), & \theta < \xi < \eta \\ L(\xi,\eta,\theta), & \xi < \eta < \theta \\ M(\xi,\eta,\theta), & \theta < \eta < \xi \\ N(\xi,\eta,\theta), & \xi < \theta < \eta \\ O(\xi,\eta,\theta), & \eta < \theta < \xi \end{cases} \tag{5.2}$$

In two-player duels, a strategy that minimizes the maximum loss an opponent can inflict is defined as an optimal strategy. When Alice has an optimal strategy x, she can always guarantee herself a pay-off that is at least l. Bob's best response is a strategy η such that $K(x,\eta) = l$. In three-play duels a similar definition holds, but now the maximum loss the opponents can inflict is dependent on two players.

Definition 5.0.1 *In a three-player game, a strategy x, is optimal for Alice if*

$$\max_{x' \in X}\left(\min_{y' \in Y, z' \in Z}(K_1(x', y', z'))\right) = \min_{y' \in Y, z' \in Z} K_1(x, y', z') \tag{5.3}$$

Similar for Bob and Charlie, the strategies y and z are optimal for Bob and Charlie respectively if

$$\max_{y' \in Y}\left(\min_{x' \in X, z' \in Z}(K_2(x', y', z'))\right) = \min_{x' \in X, z' \in Z} K_2(x', y, z') \tag{5.4}$$

$$\max_{z' \in Z}\left(\min_{x' \in X, y' \in Y}(K_3(x', y', z'))\right) = \min_{x' \in X, y' \in Y} K_3(x', y', z) \tag{5.5}$$

However, in three-player duels an optimal strategy may not always be the best strategy to use. For any strategy player Alice plays, her pay-off is dependent on two opponents instead of one. The minimum pay-off l Alice can guarantee can be dependent on the values η and θ which are played by the opponents of Alice. Therefore, we will look for solutions of a different form, a Nash equilibrium.

Assume Alice, Bob and Charlie choose the strategy x, y and z respectively. With these strategies, the pay-off for Alice is given by $K_1(x, y, z)$. If Alice is given the strategies y and z of the others (i.e. Bob plays y and Charlie plays z) and there is a different strategy x' for Alice such that

$$K_1(x', y, z) > K_1(x, y, z)$$

then playing x' would be better for Alice. However, if there is no strategy x' such that this holds and no strategies y' and z' such that

$$K_2(x, y', z) > K_2(x, y, z)$$

and

$$K_3(x, y, z') > K_3(x, y, z)$$

holds, then the strategies (x, y, z) form a Nash equilibrium (Hémon et al. [4]).

Definition 5.0.2 *The strategies (x, y, z) form a Nash equilibrium when the following holds:*

$$\begin{aligned} \max_{x' \in X} K_1(x', y, z) &= K_1(x, y, z) \\ \max_{y' \in Y} K_2(x, y', z) &= K_2(x, y, z) \\ \max_{z' \in Z} K_3(x, y, z') &= K_3(x, y, z) \end{aligned} \tag{5.6}$$

In other words, if the strategy set (x, y, z) is a Nash equilibrium, then given the strategies of the opponents, no player can increase their own pay-off (and therefore also the player's probability of winning) by changing only his own strategy. A Nash equilibrium is a solution to a game in which each player knows the strategy of their opponents, but gains nothing by switching his own strategy.

A Nash equilibrium has similarities to optimal strategies (also called minimax strategies). In two-player zero-sum games, these two strategies are the same. For a three-player zero-sum game, the strategies can be the same. The difference between these strategies is in what a player maximizes against. In a Nash equilibrium, Alice knows that Bob and Charlie will play the strategies y and z respectively, so she maximizes her strategy x against these two strategies. The pay-off for any player does not improve if only one player changes his/her strategy, but it could change when two or more players change their strategies. In an optimal strategy, Alice maximizes her strategy against anything Bob and Charlie can play. The pay-off for Alice does not change irrespective of whether Bob and/or Charlie change their strategies. In the next few games, we will look for either optimal strategies or strategies that form a Nash equilibrium. Let us first look at several three-player noisy duels.

5.1. Three-player Noisy Duel Game 1

In section 3.2, we have discussed several two-player noisy duels and a method of how to calculate optimal strategies for general accuracy functions. An optimal strategy for any two-player noisy duel consist of firing at a specific time t_0. Firing before this time results in a lower pay-off, whereas firing after can result in a lower pay-off. If the opponent has fired before t_0 and misses, then the player should fire at $t = 1$. In a three-player noisy duel, we need to keep in mind that when a player fires first and misses, the remaining two players continue the game as a two-player game. Say Alice fires at a time t and misses, then Bob and Charlie continue the game as a two-player noisy duel. Therefore, the optimal strategy will most likely involve the optimal strategy of a two-player noisy duel between every pair of players.

Consider a three-player noisy duel in which the accuracy functions for Alice, Bob and Charlie are given by: $p(t) = q(t) = r(t) = t$. Every player has the same accuracy function and no player has an advantage over the others. Therefore, all players share the same optimal strategy and this strategy should guarantee a pay-off that is at least 0. This is equivalent to saying that the optimal strategy guarantees a player to win with at least probability $\frac{1}{3}$. We will look for a strategy x that has this property and verify that the strategy set (x, x, x) forms a Nash equilibrium.

In section 3.2, we found out that an optimal strategy for any player is found by equating the player's probability of winning when firing at time t against the player's probability of winning when the opponent fires at the same time t. To solve this three-player duel, a similar approach is used. If Alice fires at a time t, her probability of winning is equal to $p(t) = t$. Now if not Alice but one of her opponents fires at the same time t, say Bob (it does not matter who, because they have the same accuracy functions), then Bob will miss with probability $(1-q(t)) = (1-t)$. If Bob fires and misses, the game is not over yet, because Alice still needs to duel Charlie on the remaining interval $[t, 1]$. However, this is a two-player duel that has previously been solved in section 3.2. Therefore, it is known that firing at $t_0 = \frac{1}{2}$ is optimal for both Alice and Charlie. So when Bob misses at time t, Alice and Charlie will both win with equal probability $\frac{1}{2}$, i.e. Bob, Alice and Charlie win with probability t, $(1-t)\frac{1}{2}$ and $(1-t)\frac{1}{2}$ respectively. Thus, to find an optimal strategy for Alice, we solve the following equation.

$$t = (1-t)\frac{1}{2} \tag{5.7}$$

It follows that $t_0 = \frac{1}{3}$ is a solution to eq. (5.7). Therefore, it is optimal for Alice to fire at t_0 when no one has fired yet. When one of her opponents fires before t_0 and misses, the game changes in a two-player duel in which both players have the same accuracy function given by $p(t) = t$. We have seen in section 3.2 that it is optimal for the remaining two players to fire at $t_1 = \frac{1}{2}$. If one of the remaining two players fires before t_1 and misses, the last player should fire at $t_2 = 1$. Therefore, the strategy given by the vector $x = \left(\frac{1}{3}, \frac{1}{2}, 1\right)$ is an optimal strategy for Alice. This strategy tells Alice to fire at: $t = \frac{1}{3}$ when none of her opponents has fired; $t = \frac{1}{2}$ when one of her opponents has fired; $t = 1$ when both opponents have fired. The strategy x is also optimal for Bob and Charlie, because all players share the same optimal strategies.

To verify the strategy, we will show that every player wins with equal probability when they play with the strategy x and that no play can benefit from changing strategies. This is equivalent to proving that the strategy set (x, x, x) forms a Nash equilibrium.

Assume that all players play the strategy x. First, we will calculate each player's probability of winning. When Alice, Bob and Charlie all fire at $t_0 = \frac{1}{3}$ they roll a fair three-sided dice to determine who fires slightly earlier. There are three cases:

1. (Alice fires slightly earlier with probability $\frac{1}{3}$): The probability that Alice will hit and win when firing at $t_0 = \frac{1}{3}$ is equal to $\frac{1}{3}$. She will miss with probability $\left(1 - \frac{1}{3}\right) = \frac{2}{3}$. In this case, Bob and Charlie play a two-player noisy duel. We know that both players have equal probability of winning in this two-player duel when they play their optimal strategies, thus Bob and Charlie both win with probability $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$.

2. (Bob fires slightly earlier with probability $\frac{1}{3}$): This case is similar to the first case with a few names swapped. We find that all players have equal probability of $\frac{1}{3}$ to win.

3. (Charlie fires slightly earlier with probability $\frac{1}{3}$): This case has identical results as the previous cases.

In every case, all players have equal probability of winning. So it does not matter who fires first at $t_0 = \frac{1}{3}$, because all players will win with probability $\frac{1}{3}$. Now assume Bob and Charlie play the optimal strategy x.

Alice can fire either before or after $t_0 = \frac{1}{3}$. Firing at $t < t_0 = \frac{1}{3}$ obviously does not increase Alice her probability of winning. If Alice fires after t_0, then either Bob or Charlie has fired at t_0. There are two cases:

1. (Bob fires before Charlie with probability $\frac{1}{2}$): Bob will miss with probability $\frac{2}{3}$. In this case, Alice and Charlie continue as a two-player duel in which Charlie will fire at $t_1 = \frac{1}{2}$. In section 3.2 we have seen the exact same two-player noisy duel and we know that firing at $t_1 = \frac{1}{2}$ maximizes ones probability of winning. In this case, Alice and Charlie win with equal probability $\frac{1}{2}$. Therefore, Alice will win with probability $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

2. (Charlie fires before Bob with probability $\frac{1}{2}$): This case goes analogue to the previous case and we find that Alice also has a probability of $\frac{1}{3}$ to win.

Averaging and adding both cases, we find that Alice has a probability of $\frac{1}{3}$ to win when she fires after t_0. Therefore, firing after t_0 also does not improve Alice her probability of winning given that her opponents play strategy x. Because Alice can not increase her probability of winning by deviating from the optimal strategy x, the best thing she can do is play the optimal strategy x. Because of symmetry, this also holds for Bob and Charlie. Hence, the strategy set (x, x, x) forms a Nash equilibrium.

5.2. Three-player Noisy Duel Game 2

Let us look at a slightly more difficult version of the three-player noisy duel. We will assume that the accuracy functions are now given by: $p(t) = t$ and $q(t) = r(t) = t^2$ for Alice, Bob and Charlie respectively. Because Bob and Charlie have the same accuracy function in this three-player duel, both players have the same optimal strategy. So, we can assume that Bob and Charlie play the same strategy. Alice on the other hand, does not necessarily have the same optimal strategy. The optimal strategies are found similar to section 5.1. When calculating the optimal strategies, it is important for Bob and Charlie to know who of their opponents has fired already. If Alice is the first to fire and misses, Bob and Charlie play a two-player noisy duel with their accuracy functions $q(t) = r(t) = t^2$. In this case it is optimal for Bob and Charlie to fire at $t_A = \frac{1}{\sqrt{2}}$. However, if Charlie (or Bob) is the first to fire and misses, Alice and Bob play a two-player noisy duel with accuracy functions $p(t) = t$ and $q(t) = t^2$ respectively. Now, Alice and Bob should fire at $t_B = \frac{1}{2}\sqrt{5} - \frac{1}{2} \neq t_A$. The following table denotes the optimal time for both players to fire at in a two-player noisy duel.

Accuracy Player 1	Accuracy Player 2	t_0
t	t^2	$\frac{1}{2}\sqrt{5} - \frac{1}{2} \approx 0.618$
t^2	t^2	$\frac{1}{\sqrt{2}} \approx 0.707$

Table 5.1: Optimal firing times in two-player duels

We will start by looking for an optimal strategy for Alice. If Alice fires at a time t, her probability of winning is equal to $p(t) = t$. Now if not Alice but one of her opponents fires at the same time t, say Bob (it does not matter who, because they have the same accuracy function), then Bob will miss with probability $(1 - t^2)$. In this case, Alice and Charlie play two-player duel. With table 5.1, it follows that firing at $t_1 = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ is optimal for both players. So, Alice should fire at a time t that satisfies

$$t = (1 - t^2)\left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right) \tag{5.8}$$

It follows that

$$t_0 = \frac{-1 + \sqrt{7 - 2\sqrt{5}}}{-1 + \sqrt{5}} \approx 0.477 \tag{5.9}$$

is a solution to eq. (5.8). Hence, an optimal strategy for Alice is given by the following vector.

$$x = \left(\frac{-1 + \sqrt{7 - 2\sqrt{5}}}{-1 + \sqrt{5}}, \frac{1}{2}\sqrt{5} - \frac{1}{2}, 1\right) \approx (0.477; 0.618; 1) \tag{5.10}$$

This strategy tells Alice to fire at: $t \approx 0.477$ when none of her opponents have fired; $t \approx 0.618$ when one of her opponents has fired; $t = 1$ when both opponents have fired. Before validating that this strategy is indeed optimal, let us first look for an optimal strategy for Bob and Charlie.

We assumed earlier that Bob and Charlie play the same strategy. So if Bob fires at a time t, so will Charlie. When Bob and Charlie fire at the same time t, there are two cases.

1. (Bob fires before Charlie with probability $\frac{1}{2}$): The probability that Bob will hit and win is equal to t^2. Bob will miss with probability $(1-t^2)$. In this case, Alice and Charlie play a two-player duel. With table 5.1 it follows that Charlie should fire at $t_1 = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ if he wants to play it optimal. Charlie will win the two-player duel with probability $\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2$. Therefore, the total probability of Charlie winning is equal to $(1-t^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2$
2. (Charlie fires before Bob with probability $\frac{1}{2}$): This case goes analogue to the previous case and it follows that Bob's and Charlie's probability of winning is now swapped. Hence, Bob will win with probability $(1-t^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2$ and Charlie will win with probability t^2.

Averaging and adding both cases, it follows that both Bob and Charlie have a probability of

$$\frac{1}{2}t^2 + \frac{1}{2}(1-t^2)\cdot\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2 \tag{5.11}$$

to win, when they fire at the same time t.

Now if not Bob and Charlie but Alice fires at the same time t, she will miss with probability $(1-t)$. If this happens, Bob and Charlie play a two-player duel and according to table 5.1, both players should fire at $t_1 = \frac{1}{\sqrt{2}}$ to win with equal probability $t_1^2 = \frac{1}{2}$. So, when Alice fires at the same time t, Bob (and Charlie) will win with probability

$$\frac{1}{2}(1-t) \tag{5.12}$$

Equating this expression with eq. (5.11) gives the equation

$$\frac{1}{2}t^2 + \frac{1}{2}(1-t^2)\cdot\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2 = \frac{1}{2}(1-t) \tag{5.13}$$

which can be simplified to the quadratic equation

$$\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)t^2 + t - \left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right) = 0 \tag{5.14}$$

This equation has exactly one solution in the interval $[0,1]$, which is

$$t_0 = \frac{-1+\sqrt{7-2\sqrt{5}}}{-1+\sqrt{5}} \approx 0.477 \tag{5.15}$$

Note that the time t_0 denotes the first time Bob and Charlie should fire at and it is equal to that of Alice, see eq. (5.9). So it looks like, all three player should initially fire at the same time t_0. Depending on who fires first, Bob (and Charlie) should fire at either $t_1 = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ or $t_2 = \frac{1}{\sqrt{2}}$ (see table 5.1). An optimal strategy for Bob and Charlie is given by the following vector.

$$y = \left(\frac{-1+\sqrt{7-2\sqrt{5}}}{-1+\sqrt{5}}; (\frac{1}{2}\sqrt{5}-\frac{1}{2}; \frac{1}{\sqrt{2}}); 1\right) \approx (0.477; (0.618; 0.707); 1) \tag{5.16}$$

This strategy tells Bob (and Charlie) to fire at: $t \approx 0.477$ when none of his opponents have fired; $t \approx 0.618$ when only Alice has fired; $t \approx 0.707$ when only Bob has fired; $t = 1$ when both opponents have fired already.

So now we have two strategies, a strategy x for Alice and a strategy y for Bob and Charlie. We claim that the strategy set (x, y, y) forms a Nash equilibrium. To prove this, we first calculate each player's probability of winning when Alice plays x and Bob and Charlie y. We will show that no player can improve his probability of winning by changing only his strategy, given that the others play their respective strategy x and/or y.

Assume Alice, Bob and Charlie play the strategy x. y and y respectively. All players initially fire at $t_0 \approx 0.477$. There are three cases.

1. (Alice fires first with probability $\frac{1}{3}$): The probability that Alice will hit and win is equal to t_0. Alice will miss with probability $(1-t_0)$. In this case, Bob and Charlie both fire at $t_1 = \frac{1}{\sqrt{2}}$ and both players have equal probability of $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ to win. So, Alice, Bob and Charlie have the following probabilities of winning: t_0, $\frac{1}{2}(1-t_0)$ and $\frac{1}{2}(1-t_0)$ respectively.
2. (Bob fires first with probability $\frac{1}{3}$): The probability that Bob will hit and win is equal to t_0^2. Bob will miss with probability $(1-t_0^2)$. In this case, Alice and Charlie both fire at $t_1 = \frac{1}{2}\sqrt{5}-\frac{1}{2}$. Alice will win the two-player duel with probability $\frac{1}{2}\sqrt{5}-\frac{1}{2}$ and Charlie will win it with probability $\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2$. So, Alice, Bob and Charlie have the following probabilities of winning: $(1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)$, t_0^2 and $(1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2$ respectively.
3. (Charlie fires first with probability $\frac{1}{3}$): This case goes analogue to the case in which Bob fires first. The results are the same as in the previous case, with the exception that the probability that Bob and Charlie win are swapped. Hence, Alice, Bob and Charlie have the follopwing probabilities of winning: $(1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)$, $(1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2$ and t_0^2 respectively.

Averaging and adding all cases, we get each players probability of winning when everyone plays their optimal strategy. This is equal to

$$\frac{1}{3}\left(t_0 + (1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right) + (1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)\right) = t_0 \approx 0.477$$

for Alice,

$$\frac{1}{3}\left(\frac{1}{2}(1-t_0) + t_0^2 + (1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2\right) = \frac{1}{2}(1-t_0) \approx 0.261$$

for Bob and

$$\frac{1}{3}\left(\frac{1}{2}(1-t_0) + (1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2 + t_0^2\right) = \frac{1}{2}(1-t_0) \approx 0.261$$

for Charlie.

We will show that there is no strategy x' for Alice, such that her probability of winning is greater than $t_0 \approx 0.477$ given that Bob and Charlie play the strategy y. Alice can fire either before or after t_0, so there are two cases.

1. (Alice fires at $t < t_0$): In this case, Alice will win with a probability that is less than t_0. So firing before t_0 does not increase her probability of winning.
2. (Alice fires at $t > t_0$): In this case, either Bob or Charlie has fired at t_0. There are two identical cases.
   (a) (Bob fires before Charlie with probability $\frac{1}{2}$): The probability that Bob will miss is equal to $(1-t_0^2)$. When Bob misses, it is optimal for Alice to fire at $t_1 = \frac{1}{2}\sqrt{5}-\frac{1}{2}$ according to table 5.1. Charlie will also fire at the same time t_1, so it follows that Alice will win with probability $(1-t_0^2)\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right) = t_0 \approx 0.477$.
   (b) (Charlie fires before Bob with probability $\frac{1}{2}$): This case goes analogue to the previous case and Alice has the same probability of winning.

   Averaging and adding both cases, we find that Alice has a probability of

   $$\frac{1}{2}(t_0 + t_0) = t_0 \approx 0.477$$

   to win, when she fires after t_0 and Bob and Charlie play the strategy y.

So given that Bob and Charlie play the strategy y, there is no strategy x' such that Alice can win with a probability that is greater than with the strategy x. In this case, there is no reason for Alice to deviate from the strategy x, unless Bob and Charlie do not play the strategy y. We will now show that both Bob and Charlie will not deviate from the strategy y, given the strategy of the others. Because Bob and Charlie play the same strategy, we need to show that there is no strategy y' for Bob and Charlie such that their probability of winning is greater than $\frac{1}{2}(1-t_0) \approx 0.261$ given that Alice plays the strategy x.

Assume Alice plays the strategy x. Bob and Charlie can fire either before or after t_0. There are two cases.

1. (Bob and Charlie fire at $t < t_0$): This case has two sub cases depending on who fires first.
   (a) (Bob fires before Charlie with probability $\frac{1}{2}$): The probability that Bob will hit and win is equal to t^2. Bob will miss with probability $(1-t^2)$, in that case it is optimal for Charlie to fire at $t_1 = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ (see table 5.1). So Bob will win with probability t^2 and Charlie will win with probability $\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2(1-t^2)$.
   (b) (Charlie fires before Bob with probability $\frac{1}{2}$): This case goes analogue to the previous one and we find that Bob will win with probability $\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2(1-t^2)$ and Charlie will win with probability t^2.

   Averaging and adding both cases, we find that Bob and Charlie both have a probability of
   $$\frac{1}{2}\left(t^2+\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2(1-t^2)\right)$$
   to win when $t < t_0$. Rewriting this expression and using the fact that $t < t_0$, we find that
   $$\frac{1}{2}\left(t^2+\left(\frac{1}{2}\sqrt{5}-\frac{1}{2}\right)^2(1-t^2)\right) = \left(\frac{1}{4}\sqrt{5}-\frac{1}{4}\right)t^2+\frac{3}{4}-\frac{1}{4}\sqrt{5} < \left(\frac{1}{4}\sqrt{5}-\frac{1}{4}\right)t_0^2+\frac{3}{4}-\frac{1}{4}\sqrt{5} = \frac{1}{2}(1-t_0) \approx 0.261$$
   This means that when Bob and Charlie fire at $t < t_0$, they will reduce their probability of winning.
2. (Bob and Charlie fire at $t > t_0$): In this case, Alice has fired at t_0. The probability that she misses is equal to $(1-t_0)$. According to table 5.1, it is optimal for Bob and Charlie to fire at $t_1 = \frac{1}{\sqrt{2}}$ such that both players win with equal probability. So Bob and Charlie will both win with probability $\frac{1}{2}(1-t_0) \approx 0.261$. This means that firing after t_0 also does not increase Bob's and Charlie's probability of winning.

So given that Alice plays the strategy x, there is no strategy y' such that Bob and Charlie can increase their probability of winning compared to the strategy y. Therefore, the strategy set (x, y, y) forms a Nash equilibrium and the strategies x, y and y are solutions to the game for Alice, Bob and Charlie respectively.

What is remarkable about this Nash equilibrium is that if only Bob (or Charlie) changes his strategy to fire at a time $t > t_0$, he will actually increase his own probability of winning. The reason why this won't happen is because as soon as Bob changes his strategy and fires at a time $t > t_0$, Charlie can copy Bob's strategy. This will result in both players firing at a later time $t_1 > t_0$. However, when this happens Alice can also fire at a later time t_2 such that $t_0 < t_2 < t_1$. This will increase the pay-off for Alice and decrease the pay-off for Bob and Charlie. Bob and Charlie in their turn, will both change their strategies and fire before t_2 to increase their own probability of winning. Alice will react to this, by firing even before Bob and Charlie until we're at square one (everyone fires at t_0).

A solution to this game is found in a slightly different way than in section 5.1, because we assumed that Bob and Charlie play the same strategy. We were able to make this assumption, because both players have the same accuracy function. This basically changes the game into a two-player game, because there are two strategies that need to be chosen, one for Alice and one for Bob that is shared with Charlie. However, when all players have a different accuracy function, it becomes much harder to find an Nash equilibrium. Let us look at such a game that we leave unsolved.

5.3. Three-player Noisy Duel Game 3

Assume the accuracy functions are now given by $p(t) = t$, $q(t) = t^2$ and $r(t) = t^3$ for Alice, Bob and Charlie respectively. In this example, Alice has the best accuracy function, i.e. the accuracy of Alice is greater than the accuracy of Bob and Charlie at any time t in the interval $(0,1)$. Bob has the second best accuracy function, so it is expected that Alice and Bob will initially fire at the same time t_0, whereas Charlie will wait until one of the players has fired. This is because the accuracy of Charlie is just too bad to compete against Alice or Bob early on. Let us try to calculate the time t_0 where Alice and Bob should initially fire at using the method that has been used in the previous sections.

First, we will look for a time Alice should fire at. We assume that either Alice or Bob is the first to fire, because Charlie will not compete early on. So if Alice is the first to fire at a time t, she will win with probability t. Now if not Alice but Bob fires at the same time t, Bob will miss with probability $(1-t^2)$. In this case, Alice plays a two-player noisy duel against Charlie, who has accuracy function given by: $r(t) = t^3$. From section 3.2 we know that this two-player duel is played optimal when both players fire at a time s such that

$$s = (1 - s^3) \tag{5.17}$$

It follows that $s \approx 0.68233$ is a solution to eq. (5.17). Thus, when Bob fires at a time t, Alice will win with probability approximately $(1 - t^2)0.68233$. So, to find the time t_0 for Alice, we solve the following equation.

$$t = (1 - t^2)0.68233 \tag{5.18}$$

It follows that $t_A \approx 0.507$ is a solution to eq. (5.18), this means that Alice should initially fire at the time $t_A \approx 0.507$.

Let us now look for a time time Bob should fire at and verify whether it is the same as t_A or not. Here we also assume that either Alice or Bob fires first. If Bob is the first to fire at a time t, he will win with probability t^2. Now if not Bob but Alice fires at the same time t, she will miss with probability $(1 - t)$. In this case, Bob and Charlie play a two-player noisy duel. It follows from section 3.2 that both players play it optimal if they fire at time s such that

$$s^2 = (1 - s^3) \tag{5.19}$$

It follows that $s \approx 0.75488$ is a solution to eq. (5.19). Thus, when Alice fires at a time t, Bob will win with probability approximately $(1 - t)0.75488^2$. So, to find the time t_0 for Bob, we need to solve the following equation.

$$t^2 = (1 - t)0.75488^2 \tag{5.20}$$

It follows that $t_B \approx 0.522$ is a solution to eq. (5.20), this means that Bob should initially fire at the time $t_B \approx 0.570$. However, since $t_A \neq t_B$ this solution can't be optimal. If Alice fires at t_A and Bob fire at t_B, then Alice is better off firing slightly before t_B. So if x is a strategy of Alice that tells her to initially fire at t_A, y is a strategy of Bob that tells him to initially fire at t_B and z is an arbitrary strategy for Charlie, then any strategy set (x, y, z) can not form a Nash equilibrium.

To further explain this, if Alice uses a pure strategy x that tells her to fire at $t = 0.507$ and Bob fires at $t_B \approx 0.522$, then a better strategy for Alice would be the pure strategy x' which tells her to fire at $t = 0.508$. But if Alice plays x', then a better strategy for her would be the strategy x'' that tells her to fire at $t = 0.509$. This can continue forever and we keep finding slightly better strategies for Alice. So, a Nash equilibrium can not be reached by using only pure strategies.

Deriving a Nash equilibrium for this three-player duel is a very complex problem, which is very surprising. As soon as there is no symmetry in the game, i.e. every player has a different accuracy function, there is no Nash equilibrium consisting of pure strategies. This suggests that the Nash equilibrium either does not exist or it needs to be constructed of mixed strategies. The latter looks more credible, as there are theorems that suggest a mixed Nash equilibrium always exists for infinite n-player zero sum games under certain conditions (Reny [6]). We will leave this problem to our successor.

Conjecture 5.3.1 *In a three-player noisy duel in which all players have different accuracy functions, the Nash equilibrium is constructed of mixed strategies.*

6 Conclusion

In this report we have seen what duels are and how they are played. Duels are games that are classified as infinite zero-sum games. We have seen when a game belongs to the class of infinite games and why they are more complex than finite games. Two types of duels were analysed: a noisy and a silent duel. To solve duels, we have calculated either optimal strategies and/or strategies that constitute to a Nash equilibrium.

For two-player noisy duels in which the accuracy functions are given by $p(t)$ and $q(t)$, optimal strategies for both players consisted of firing at a certain time t_0 if the opponent has not fired yet. If the opponent has fired before t_0, then firing at $t = 1$ is optimal. The time t_0 is found by solving the following equation.

$$p(t) = 1 - q(t)$$

For two-player silent duels, an optimal strategy is either a density f over an interval $[a, 1]$ or a combination of this density f and a discrete mass β at $t = 1$. Both players can have different optimal strategies in contrast to two-player noisy duels. The optimal strategies are found by solving integral equations.

Furthermore, we have also looked at a three-player noisy duel. The strategies that constitute to a Nash equilibrium are found similar to the two-player noisy duel. Given that Alice, Bob and Charlie have accuracy functions given by $p(t)$, $q(t)$ and $r(t)$ respectively, we have analysed three different cases:

1. (All players have the same accuracy function): In this case, all players have the same strategy x such that the strategy set (x, x, x) forms a Nash equilibrium. The strategy x tells a player to fire at: t_0 when no other played has fired; t_1 when one of the opponents has fired; $t = 1$ when both opponents have fired. Hence, all players initially fire at the same time t_0. The time t_1 is a solution to the following equation.

   $$p(t) = 1 - p(t)$$

   The time t_0 is found by solving the following equation.

   $$p(t) = (1 - p(t))\, t_1$$

2. (Two players have the same accuracy function): In this case, the two players that have the same accuracy functions use the same strategy y, whereas the player with a different accuracy function uses a different strategy x. Assuming that Bob and Charlie have the same accuracy function, the strategy set (x, y, y) forms a Nash equilibrium. All players initially fire at a common time t_0. Depending on who fires first, the remaining two opponents will both fire at either t_1 or t_2. The time t_1 is found by solving

   $$p(t) = 1 - q(t)$$

   and the time t_2 is found by solving

   $$q(t) = 1 - q(t)$$

   Alice will fire at t_1 when one of her opponents has fired already. Bob will fire at t_1 if only Charlie has fired and he will fire at t_2 if only Alice has fired. Charlie will do the same as Bob. The time t_0 where all players initially fire at can be found solving the following equation.

   $$p(t) = (1 - q(t))\, t_1$$

3. (All players have a different accuracy function): In this case, a Nash equilibrium does not exist of pure strategies, i.e. there is no common time t_0 where two or more players fire at. Hence, there have to be mixed strategies x, y and z such that the strategy set (x, y, z) forms a Nash equilibrium. This case is left to our successor.

A

Appendix

A.1. Code

MATLAB code - Figure 3.1

```
clear all;
close;

xt = @(x) 2*x -1;
yt = @(x) x - x*x;
zt = @(x) 1 - 2*x*x

fplot(xt,[0 1])
hold on
fplot(yt,[0 1])
hold on
fplot(zt,[0 1])

legend('2t -1', 't-t^2', '1-2t^2')
title('Pay-off plot p(\xi) = \xi and q(\eta) = \eta^2')
xlabel('t')
ylabel('K')
hold off
```

MATLAB code - Figure 4.1

```
clear all;
close;

x = linspace(0,1);
f1 = 0.376 * ( -1*(1./(3*x.^3)) + 2.995);
plot(x,f1)
hold on
f2 = 0.348 * ( -1*(1./(3*x.^3)) + 2.995);
plot(x,f2)
xlim([0.481,1]);
xlabel("Time t")
ylabel("Probability of firing")
title("Probability of firing for Alice and Bob")
legend("p(t) = t", "q(t) = t^2")
```

A.2. Calculations

Rewriting eq. (3.19) to eq. (3.20)

$$
\begin{aligned}
\int_a^\eta (\xi - (1-\xi)\eta) f(\xi)d\xi + \int_\eta^1 (-\eta + (1-\eta)\xi) f(\xi)d\xi \equiv 0 \\
\int_a^\eta \xi f(\xi)d\xi - \eta\int_a^\eta f(\xi)d\xi + \eta\int_a^\eta \xi f(\xi)d\xi - \eta\int_\eta^1 f(\xi)d\xi + \int_\eta^1 \xi f(\xi)d\xi - \eta\int_\eta^1 \xi f(\xi)d\xi \equiv 0 \\
\int_a^1 \xi f(\xi)d\xi - \eta\int_a^1 f(\xi)d\xi + \eta\int_a^\eta \xi f(\xi)d\xi - \eta\int_\eta^1 \xi f(\xi)d\xi \equiv 0 \\
\int_a^1 \xi f(\xi)d\xi - \eta + \eta\int_a^\eta \xi f(\xi)d\xi - \eta\int_\eta^1 \xi f(\xi)d\xi \equiv 0
\end{aligned}
\tag{A.1}
$$

Substitution and differentiation of eq. (3.20) to eq. (3.21)

$$\begin{aligned}
\int_a^1 r(\xi)d\xi - \eta + \eta\int_a^\eta r(\xi)d\xi - \eta\int_\eta^1 r(\xi)d\xi &\equiv 0 \\
-1 + \int_a^\eta r(\xi)d\xi + \eta r(\eta) - \int_\eta^1 r(\xi)d\xi + \eta r(\eta) &= 0 \qquad \text{(differentiation)} \\
r(\eta) + r(\eta) + \eta r'(\eta) + r(\eta) + r(\eta) + \eta r'(\eta) &= 0 \qquad \text{(differentiation)} \\
2\eta r'(\eta) + 4r(\eta) &= 0
\end{aligned} \tag{A.2}$$

Filling in and simplifying eq. (3.20) to eq. (3.22)

$$\begin{aligned}
\int_a^1 k\xi^{-2}d\xi - \eta + \eta\int_a^\eta k\xi^{-2}d\xi - \eta\int_\eta^1 k\xi^{-2}d\xi &\equiv 0 \\
[-k\xi^{-1}]_a^1 - \eta + \eta[-k\xi^{-1}]_a^\eta - \eta[-k\xi^{-1}]_\eta^1 &\equiv 0 \\
-k + \frac{k}{a} - \eta + \eta\left(-\frac{k}{\eta} + \frac{k}{a}\right) - \eta\left(-k + \frac{k}{\eta}\right) &\equiv 0 \\
-3k + \frac{k}{a} - \eta + \frac{\eta k}{a} + \eta k &\equiv 0 \\
\eta\left(-1 + \frac{k}{a} + k\right) + k\left(-3 + \frac{1}{a}\right) &\equiv 0
\end{aligned} \tag{A.3}$$

Filling in and simplifying eq. (3.25) to eq. (3.26)

$$\begin{aligned}
\int_{\frac{1}{3}}^t \xi f(\xi)d\xi + (1-t)\int_t^1 \xi f(\xi)d\xi &= \\
\frac{1}{4}\int_{\frac{1}{3}}^t \xi^{-2}d\xi + \frac{1-t}{4}\int_t^1 \xi^{-2}d\xi &= \\
\frac{1}{4}[-\xi^{-1}]_{\frac{1}{3}}^t + \frac{1-t}{4}[-\xi^{-1}]_t^1 &= \\
\frac{1}{4}\left(-\frac{1}{t} + 3\right) + \frac{1-t}{4}\left(-1 + \frac{1}{t}\right) &= \\
-\frac{1}{4t} + \frac{3}{4} - \frac{1}{4} + \frac{t}{4} + \frac{1}{4t} - \frac{1}{4} &= \frac{1}{4} + \frac{t}{4}
\end{aligned} \tag{A.4}$$

Filling in and simplifying eq. (3.27) to eq. (3.28)

$$\begin{aligned}
-\left(t\int_t^1 f(\xi)d\xi + t\int_{\frac{1}{3}}^t (1-\xi)f(\xi)d\xi\right) &= \\
-\frac{t}{4}\int_t^1 \xi^{-3}d\xi - \frac{t}{4}\int_{\frac{1}{3}}^t \xi^{-3}d\xi + \frac{t}{4}\int_{\frac{1}{3}}^t \xi^{-2}d\xi &= \\
-\frac{t}{4}\left[-\frac{1}{2}\xi^{-2}\right]_t^1 - \frac{t}{4}\left[-\frac{1}{2}\xi^{-2}\right]_{\frac{1}{3}}^t + \frac{t}{4}[-\xi^{-1}]_{\frac{1}{3}}^t &= \\
-\frac{t}{4}\left(-\frac{1}{2} + \frac{1}{2t^2}\right) - \frac{t}{4}\left(-\frac{1}{2t^2} + \frac{9}{2}\right) + \frac{t}{4}\left(-\frac{1}{t} + 3\right) &= \\
\frac{t}{8} - \frac{1}{8t} + \frac{1}{8t} - \frac{9t}{8} - \frac{1}{4} + \frac{3t}{4} &= -\frac{1}{4} - \frac{t}{4}
\end{aligned} \tag{A.5}$$

Filling in and simplifying eq. (3.29) to eq. (3.30)

$$\begin{aligned}\int_t^1 (-t+(1-t)\xi)f(\xi)d\xi &= \\ -t+\frac{1}{4}\int_{\frac{1}{3}}^1 \xi^{-2}d\xi - \frac{t}{4}\int_{\frac{1}{3}}^1 \xi^{-2}d\xi &= \\ -t+\frac{1}{4}\left[-\xi^{-1}\right]_{\frac{1}{3}}^1 - \frac{t}{4}\left[-\xi^{-1}\right]_{\frac{1}{3}}^1 &= \\ -t-\frac{1}{4}+\frac{3}{4}+\frac{t}{4}-\frac{3t}{4} &= -\frac{3}{2}t+\frac{1}{2}\end{aligned} \tag{A.6}$$

Filling in and simplifying eq. (3.33) to eq. (3.35)

$$\begin{aligned}\int_a^\eta (\xi-(1-\xi)\eta)f(\xi)d\xi + \int_\eta^1 (-\eta+(1-\eta)\xi)f(\xi)d\xi + \alpha((b-(1-b)\eta)) &\equiv 0 \\ \int_a^1 k\xi^{-2}d\xi - \eta\int_a^1 k\xi^{-3}d\xi + \eta\int_a^\eta k\xi^{-2}d\xi - \eta\int_\eta^1 k\xi^{-2}d\xi + \alpha(b-\eta+b\eta) &\equiv \\ -k+\frac{k}{a}-\eta\left(-\frac{k}{2}+\frac{k}{2a^2}\right)-k+\frac{\eta k}{a}+\eta k-k+\alpha(b-\eta+b\eta) &= \\ \eta k\left(\frac{3}{2}-\frac{1}{2a^2}+\frac{1}{a}\right)+k\left(-3+\frac{1}{a}\right)+\alpha(b-\eta+b\eta) &\equiv 0\end{aligned} \tag{A.7}$$

Deriving constants of eq. (3.35)

We will show that $a=\frac{b}{2-3b}$, $\alpha=\frac{3b-1}{2b^2}$ and $k=\frac{1}{4}$ given the following two equations.

$$-\frac{k}{2}+\frac{k}{2a^2}=1-\alpha \tag{A.8}$$

$$\eta k\left(\frac{3}{2}-\frac{1}{2a^2}+\frac{1}{a}\right)+k\left(-3+\frac{1}{a}\right)+\alpha(b-\eta+b\eta)=0 \tag{A.9}$$

Rewriting eq. (A.8) gives the following for α.

$$\alpha=\frac{k}{2}-\frac{k}{2a^2}+1 \tag{A.10}$$

Filling in this α in eq. (A.9) gives the following.

$$\begin{aligned}\eta k\left(\frac{3}{2}-\frac{1}{2a^2}+\frac{1}{a}\right)+k\left(-3+\frac{1}{a}\right)+\left(\frac{k}{2}-\frac{k}{2a^2}+1\right)(b-\eta+b\eta) &= \\ \eta k\left(\frac{3}{2}-\frac{1}{2a^2}+\frac{1}{a}\right)+k\left(-3+\frac{1}{a}\right)+\frac{bk}{2}-\frac{\eta k}{2}+\frac{\eta bk}{2}-\frac{bk}{2a^2}+\frac{\eta k}{2a^2}-\frac{\eta bk}{2a^2} &= \\ \eta k\left(\frac{3}{2}-\frac{1}{2a^2}+\frac{1}{a}-\frac{1}{2}+\frac{b}{2}+\frac{1}{2a^2}-\frac{b}{2a^2}\right)+k\left(-3+\frac{1}{a}+\frac{b}{2}-\frac{b}{2a^2}\right)+\eta(b-1)+b &= \\ \eta k\left(1+\frac{1}{a}+\frac{b}{2}-\frac{b}{2a^2}\right)+k\left(-3+\frac{1}{a}+\frac{b}{2}-\frac{b}{2a^2}\right)+\eta(b-1)+b &= 0\end{aligned} \tag{A.11}$$

From eq. (A.11) the terms of η and the constants need to add up to zero, hence the following holds.

$$\begin{aligned}k\left(1+\frac{1}{a}+\frac{b}{2}-\frac{b}{2a^2}\right) &= -(b-1) \\ k\left(-3+\frac{1}{a}+\frac{b}{2}-\frac{b}{2a^2}\right) &= -b\end{aligned} \tag{A.12}$$

Rewriting the first equation of eq. (A.12) we get the following.

$$\begin{aligned}k+\frac{k}{a}+\frac{bk}{2}-\frac{bk}{2a^2} &= -b+1 \\ k+\frac{k}{a}-1 &= -b-\frac{bk}{2}+\frac{bk}{2a^2} \\ k+\frac{k}{a}-1 &= b\left(-1-\frac{k}{2}+\frac{k}{2a^2}\right)\end{aligned} \tag{A.13}$$

Rewriting the second equation of eq. (A.12) we get the following.

$$\begin{aligned} -3k+\frac{k}{a}+\frac{bk}{2}-\frac{bk}{2a^2} &= -b \\ -3k+\frac{k}{a} &= -b-\frac{bk}{2}+\frac{bk}{2a^2} \\ -3k+\frac{k}{a} &= b\Big(-1-\frac{k}{2}+\frac{k}{2a^2}\Big) \end{aligned} \tag{A.14}$$

Now we can see that the right-hand side of eq. (A.13) and eq. (A.14) are equal. So we can equate the left-hand of both equations. This gives the following.

$$\begin{aligned} k+\frac{k}{a}-1 &= -3k+\frac{k}{a} \\ 4k &= 1 \\ k &= \frac{1}{4} \end{aligned} \tag{A.15}$$

This value of k can now be used in eq. (A.13).

$$\begin{aligned} \frac{1}{4}+\frac{1}{4a}-1 &= b\Big(-1-\frac{1}{8}+\frac{1}{8a^2}\Big) \\ \frac{1}{4}+\frac{1}{4a}-\frac{b}{8a^2} &= -b-\frac{b}{8}+1 \end{aligned} \tag{A.16}$$

Substituting $z=\frac{1}{a}$ in eq. (A.16).

$$\begin{aligned} \frac{1}{4}+\frac{1}{4}z-\frac{b}{8}z^2 &= -b-\frac{b}{8}+1 \\ -\frac{b}{8}z^2+\frac{1}{4}z+\Big(\frac{9}{8}b-\frac{3}{4}\Big) &= 0 \end{aligned} \tag{A.17}$$

Using the Pythagorean theorem.

$$\begin{aligned} z &= \frac{-\frac{1}{4}\pm\sqrt{\left(\frac{1}{4}\right)^2-4\cdot\left(-\frac{b}{8}\right)\left(\frac{9}{8}b-\frac{3}{4}\right)}}{-\frac{b}{4}} \\ &= \frac{-\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{9}{16}b^2-\frac{3}{8}b}}{-\frac{b}{4}} \\ &= \frac{1}{b}\pm\sqrt{\frac{\frac{1}{16}+\frac{9}{16}b-\frac{3}{8}b^2}{\frac{b^2}{16}}} \\ &= \frac{1}{b}\pm\sqrt{\frac{1}{b^2}+9-\frac{6}{b}} \\ &= \frac{1}{b}\pm\sqrt{\Big(\frac{1}{b}-3\Big)^2} \\ &= \frac{1}{b}\pm\frac{1}{b}-3 \end{aligned} \tag{A.18}$$

It follows that $z=\frac{2}{b}-3$ or $z=-3$. If $z=-3$, then $a=-\frac{1}{3}$. This is not possible with the context of the game, because $a\in[0,1]$. Therefore $z=\frac{2}{b}-3$. We find the following value for a.

$$\begin{aligned} \frac{1}{a} &= \frac{2}{b}-3 \\ \frac{1}{a} &= \frac{2-3b}{b} \\ a &= \frac{b}{2-3b} \end{aligned} \tag{A.19}$$

Using this value for a and $k=\frac{1}{4}$, we can fill in eq. (A.10).

$$\begin{aligned}\alpha &= \frac{1}{8} - \frac{1}{8\left(\frac{b^2}{4-12b+9b^2}\right)} + 1 \\ &= \frac{1}{8}\left(1 - \frac{4-12b+9b^2}{b^2}\right) + 1 \\ &= \frac{9}{8} - \frac{4-12b}{8b^2} - \frac{9b^2}{8b^2} \\ &= -\frac{1-3b}{2b^2} = \frac{3b-1}{2b^2}\end{aligned} \tag{A.20}$$

So now we have what we wanted: $a = \frac{b}{2-3b}$, $\alpha = \frac{3b-1}{2b^2}$ and $k = \frac{1}{4}$.

Proof of inequality of constants in section 3.4

Given $0 \leq \alpha \leq 1$, $a = \frac{b}{2-3b}$ and eq. (3.34), we will show that $\frac{1}{3} \leq b \leq \frac{1}{2}$.

$$\begin{aligned} 0 &\leq 1-\alpha \leq 1 \\ 0 &\leq -\frac{k}{2} + \frac{k}{2a^2} \leq 1 \\ 0 &\leq -\frac{1}{8} + \frac{1}{8a^2} \leq 1 \\ \frac{1}{8} &\leq \frac{1}{8a^2} \leq \frac{9}{8} \\ 1 &\leq \frac{1}{a^2} \leq 9 \\ 1 &\leq \frac{(2-3b)^2}{b^2} \leq 9 \\ 1 &\leq \frac{4}{b^2} - \frac{12}{b} + 9 \leq 9 \\ -8 &\leq \frac{1}{b}(\frac{4}{b} - 12) \leq 0 \end{aligned} \tag{A.21}$$

Now it follows that $f(x) = \frac{1}{b}\left(\frac{4}{b} - 12\right) = 0$ when $b = \frac{1}{3}$. The derivative is given by $f'(x) = -\frac{8}{b^3} + \frac{12}{b^2}$. This derivative is negative in the interval $\left(0, \frac{2}{3}\right)$, thus the function $f(x)$ is monotone-decreasing in the interval $\left(0, \frac{2}{3}\right)$. Therefore the value b such that $f(b) = -8$ is the lowest b satisfying the bound inequality in A.21. So we look for the following.

$$\begin{aligned} \frac{1}{b}\left(\frac{4}{b} - 12\right) &= 8 \\ \frac{4}{b^2} - \frac{12}{b} - 8 &= 0 \end{aligned} \tag{A.22}$$

Taking $z = \frac{1}{b}$ gives the following.

$$4z^2 - 12z - 8 = 0 \tag{A.23}$$

Solving with the Pythagorean theorem gives $z = 1$ or $z = 2$. Translating back gives us $b = 1$ or $b = \frac{1}{2}$. Because $f(x)$ is monotone-decreasing in $(0, \frac{2}{3})$, any value of $f(x)$ with $x \in (\frac{1}{2}, \frac{2}{3})$ is smaller than -8. Hence it can't be $b = 1$. Therefore it has to be $b = \frac{1}{2}$. It follows that $\frac{1}{3} \leq b \leq \frac{1}{2}$.

Solving eq. (3.37)

$$
\begin{aligned}
\alpha(b-\eta+b\eta)+\int_\eta^1(\xi-\eta-\xi\eta)f(\xi)d\xi &= \frac{3b-1}{2b^2}(b-\eta+b\eta)+\int_a^1(\xi-\eta-\xi\eta)f(\xi)d\xi\\
&= \frac{3b-1}{2b^2}(b-\eta+b\eta)+(1-\eta)\int_a^1\xi f(\xi)d\xi-\eta\int_a^1 f(\xi)d\xi\\
&= \frac{3b-1}{2b^2}(b-\eta+b\eta)+(1-\eta)\int_a^1\xi f(\xi)d\xi-\eta(1-\alpha)\\
&= \frac{3b-1}{2b}-\eta\frac{3b-1}{2b^2}+\eta\frac{3b-1}{2b}+(1-\eta)\frac{1}{4}\left[-\xi^{-1}\right]_a^1-\eta\left(1-\frac{3b-1}{2b^2}\right)\\
&= \frac{3b-1}{2b}-\eta\frac{3b-1}{2b^2}+\eta\frac{3b-1}{2b}+(1-\eta)\frac{1}{4}\left(-1+\frac{2-3b}{b}\right)-\eta+\eta\frac{3b-1}{2b^2}\\
&= \frac{3b-1}{2b}+\eta\frac{3b-1}{2b}+\frac{1}{4}\left(-1+\frac{2-3b}{b}+\eta-\eta\frac{2-3b}{b}\right)-\eta\\
&= \frac{3}{2}-\frac{1}{2b}+\frac{3}{2}\eta-\frac{1}{2b}\eta+\frac{1}{4}\left(-1+\frac{2}{b}-3+\eta-\frac{2}{b}\eta+3\eta\right)-\eta\\
&= \frac{3}{2}-\frac{1}{2b}+\frac{3}{2}\eta-\frac{1}{2b}\eta-1+\eta+\frac{1}{2b}-\frac{1}{2b}\eta-\eta\\
&= \frac{3}{2}-\frac{1}{2b}+\frac{3}{2}\eta-\frac{1}{2b}\eta-1+\frac{1}{2b}-\frac{1}{2b}\eta\\
&= \frac{1}{2}+\eta\left(\frac{3}{2}-\frac{1}{b}\right)
\end{aligned}
\tag{A.24}
$$

Solving eq. (3.38)

$$
\begin{aligned}
\alpha\cdot 0+\int_b^1(\xi-b-\xi b)f(\xi)d\xi &= (1-b)\int_a^1\xi f(\xi)d\xi-b\int_a^1 f(\xi)d\xi\\
&= (1-b)\frac{1}{4}\left[-\xi^{-1}\right]_a^1-b(1-\alpha)\\
&= (1-b)\frac{1}{4}\left(-1+\frac{2-3b}{b}\right)-b\left(1-\frac{3b-1}{2b^2}\right)\\
&= \frac{1}{4}\left(-1+\frac{2-3b}{b}+b-2+3b\right)-b+\frac{3b-1}{2b}\\
&= \frac{1}{4}\left(-6+4b+\frac{2}{b}\right)-b+\frac{3}{2}-\frac{1}{2b}\\
&= -\frac{3}{2}+b+\frac{1}{2b}-b+\frac{3}{2}-\frac{1}{2b}\\
&= 0
\end{aligned}
\tag{A.25}
$$

Differentiation and simplifying eq. (4.8) to eq. (4.9)

$$
\begin{aligned}
v=\int_0^1 K(\xi,\eta)dx(\xi) &= \alpha L(0,\eta)+\int_a^\eta L(\xi,\eta)f_{ab}(\xi)d\xi+\int_\eta^b M(\xi,\eta)f_{ab}(\xi)d\xi+\beta M(1,\eta)\\
0 &= \alpha L_\eta(0,\eta)+\beta M_\eta(1,\eta)+\int_a^\eta L_\eta(\xi,\eta)f(\xi)d\xi+L(\eta,\eta)f(\eta)+\int_\eta^1 M_\eta(\xi,\eta)f(\xi)d\xi-M(\eta,\eta)f(\eta)\\
\frac{-\alpha L_\eta(0,\eta)-\beta M_\eta(1,\eta)}{L(\eta,\eta)-M(\eta,\eta)} &= \frac{L(\eta,\eta)f(\eta)-M(\eta,\eta)f(\eta)}{L(\eta,\eta)-M(\eta,\eta)}+\int_a^\eta\frac{L_\eta(\xi,\eta)}{L(\eta,\eta)-M(\eta,\eta)}f(\xi)d\xi+\int_\eta^1\frac{M_\eta(\xi,\eta)}{L(\eta,\eta)-M(\eta,\eta)}f(\xi)d\xi\\
\alpha p_0(\eta)+\beta p_1(\eta) &= f(\eta)-\int_a^1 T(\xi,\eta)f(\xi)d\xi\\
\alpha p_0(t)+\beta p_1(t) &= f(t)-\int_a^1 T(\xi,t)f(\xi)d\xi
\end{aligned}
\tag{A.26}
$$

Calculation of eq. (4.15)

$$
\begin{aligned}
f(t) &= \int_a^t \frac{-L_\eta(\xi,t)}{L(t,t)-M(t,t)} f(\xi)d\xi + \int_t^1 \frac{-M_\eta(\xi,t)}{L(t,t)-M(t,t)} f(\xi)d\xi \\
&= \int_a^t \frac{q'(t)-p(\xi)q'(t)}{2p(t)q(t)} f(\xi)d\xi + \int_t^1 \frac{q'(t)+p(\xi)q'(t)}{2p(t)q(t)} f(\xi)d\xi \\
&= \frac{q'(t)}{2p(t)q(t)}\Big(\int_a^1 f(\xi)d\xi - \int_a^t p(\xi)f(\xi)d\xi + \int_t^1 p(\xi)f(\xi)d\xi\Big) \\
g(u) &= \int_{a'}^u \frac{M_\xi(u,\eta)}{L(u,u)-M(u,u)} g(\eta)d\eta + \int_u^1 \frac{L_\xi(u,\eta)}{L(u,u)-M(u,u)} g(\eta)d\eta \\
&= \int_{a'}^u \frac{p'(u)-p'(u)q(\eta)}{2p(u)q(u)} g(\eta)d\eta + \int_u^1 \frac{p'(u)+p'(u)q(\eta)}{2p(u)q(u)} g(\eta)d\eta \\
&= \frac{p'(u)}{2p(u)q(u)}\Big(\int_{a'}^1 g(\eta)d\eta - \int_{a'}^u q(\eta)g(\eta)d\eta + \int_u^1 q(\eta)g(\eta)d\eta\Big)
\end{aligned}
\tag{A.27}
$$

Rewriting eq. (4.15) to eq. (4.17)

$$
\begin{aligned}
f(t) &= \frac{q'(t)}{2p(t)q(t)}\Big(\int_a^1 f(\xi)d\xi - \int_a^t p(\xi)f(\xi)d\xi + \int_t^1 p(\xi)f(\xi)d\xi\Big) \\
f(t) &= \frac{q'(t)}{2p(t)q(t)}\Big(1 - \int_a^t h(\xi)d\xi + \int_t^1 h(\xi)d\xi\Big) \\
\frac{f(t)}{\Big(1-\int_a^t h(\xi)d\xi + \int_t^1 h(\xi)d\xi\Big)} &= \frac{q'(t)}{2p(t)q(t)} \\
\frac{2h(t)}{\Big(1-\int_a^t h(\xi)d\xi + \int_t^1 h(\xi)d\xi\Big)} &= \frac{q'(t)}{q(t)} \\
g(u) &= \frac{p'(u)}{2p(u)q(u)}\Big(\int_{a'}^1 g(\eta)d\eta - \int_{a'}^u q(\eta)g(\eta)d\eta + \int_u^1 q(\eta)g(\eta)d\eta\Big) \\
g(u) &= \frac{p'(u)}{2p(u)q(u)}\Big(1 - \int_{a'}^u l(\eta)d\eta + \int_u^1 l(\eta)d\eta\Big) \\
\frac{g(u)}{\Big(1-\int_{a'}^u l(\eta)d\eta + \int_u^1 l(\eta)d\eta\Big)} &= \frac{p'(u)}{2p(u)q(u)} \\
\frac{2l(u)}{\Big(1-\int_{a'}^u l(\eta)d\eta + \int_u^1 l(\eta)d\eta\Big)} &= \frac{p'(u)}{p(u)}
\end{aligned}
\tag{A.28}
$$

Integrating and simplifying eq. (4.17) to eq. (4.18)

$$
\begin{aligned}
-\ln\left(1-\int_a^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi\right)+c_1 &= \ln(q(t))+c_2\\
\ln\left(1-\int_a^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi\right) &= -(\ln(q(t))+c_2-c_1)\\
\ln\left(1-\int_a^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi\right) &= \ln(q(t)^{-1})+c_3\\
1-\int_a^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi &= \frac{e^{c_3}}{q(t)}\\
1-\int_a^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi &= \frac{k}{q(t)}\\
-\ln\left(1-\int_{a'}^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta\right)+c_4 &= \ln(p(u))+c_5\\
\ln\left(1-\int_{a'}^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta\right) &= -(\ln(p(u))+c_5-c_4)\\
\ln\left(1-\int_{a'}^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta\right) &= \ln(p(u)^{-1})+c_6\\
1-\int_{a'}^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta &= \frac{e^{c_6}}{p(u)}\\
1-\int_{a'}^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta &= \frac{k_2}{p(u)}
\end{aligned}
\tag{A.29}
$$

Filling in eq. (4.17) with eq. (4.19) for f and simplifying to eq. (4.21)

$$
\begin{aligned}
\frac{\frac{2k_1 q'(t)}{(q(t))^2}}{\left(1-\int_a^t \frac{k_1 q'(\xi)}{(q(\xi))^2}d\xi+\int_t^1 \frac{k_1 q'(\xi)}{(q(\xi))^2}d\xi\right)} &= \frac{q'(t)}{q(t)}\\
\frac{2k_1}{1-k_1\left[-\frac{1}{q(\xi)}\right]_a^t+k_1\left[-\frac{1}{q(\xi)}\right]_t^1} &= q(t)\\
\frac{2k_1}{1+\frac{2k_1}{q(t)}-\frac{k_1}{q(a)}-k_1} &= q(t)\\
\frac{2}{\frac{1}{k_1}+\frac{2}{q(t)}-\frac{1}{q(a)}-1} &= q(t)\\
2 &= \frac{q(t)}{k_1}+\frac{2q(t)}{q(t)}-\frac{q(t)}{q(a)}-q(t)\\
2-\frac{q(t)}{k_1} &= 2-\frac{q(t)}{q(a)}-q(t)\\
\frac{q(t)}{k_1} &= \frac{q(t)}{q(a)}+q(t)\\
\frac{1}{k_1} &= \frac{1}{q(a)}+1
\end{aligned}
\tag{A.30}
$$

Filling in eq. (4.17) with eq. (4.20) for g and simplifying to eq. (4.22)

$$
\begin{aligned}
\frac{\frac{2k_3 p'(u)}{(p(u))^2}}{\left(1-\int_{a'}^{u}\frac{k_3 p'(\eta)}{(p(\eta))^2}d\eta+\int_u^1\frac{k_3 p'(\eta)}{(p(\eta))^2}d\eta\right)} &= \frac{p'(u)}{p(u)}\\
\frac{2k_3}{1-k_3\left[-\frac{1}{p(\eta)}\right]_{a'}^{u}+k_3\left[-\frac{1}{p(\eta)}\right]_u^1} &= p(u)\\
\frac{2k_3}{1+\frac{2k_3}{p(u)}-\frac{k_3}{p(a')}-k_3} &= p(u)\\
\frac{2}{\frac{1}{k_3}+\frac{2}{p(u)}-\frac{1}{p(a')}-1} &= p(u)\\
2 &= \frac{p(u)}{k_3}+\frac{2p(u)}{p(u)}-\frac{p(u)}{p(a')}-p(u)\\
2-\frac{p(u)}{k_3} &= 2-\frac{p(u)}{p(a')}-p(u)\\
\frac{p(u)}{k_3} &= \frac{p(u)}{p(a')}+p(u)\\
\frac{1}{k_3} &= \frac{1}{p(a')}+1
\end{aligned}
\tag{A.31}
$$

Integrating and differentiating eq. (4.25) to eq. (4.26)

$$
\begin{aligned}
g(u)-\int_a^u\frac{p'(u)(1-q(\eta))}{2p(u)q(u)}g(\eta)d\eta-\int_u^1\frac{p'(u)(1+q(\eta))}{2p(u)q(u)}g(\eta)d\eta &= \frac{2\delta p'(u)}{2p(u)q(u)}\\
g(u) = \int_a^u\frac{p'(u)(1-q(\eta))}{2p(u)q(u)}g(\eta)d\eta+\int_u^1\frac{p'(u)(1+q(\eta))}{2p(u)q(u)}g(\eta)d\eta &+ \frac{2\delta p'(u)}{2p(u)q(u)}\\
g(u) = \frac{p'(u)}{2p(u)q(u)}\left(\int_a^1 g(\eta)d\eta-\int_a^u q(\eta)g(\eta)d\eta+\int_u^1 q(\eta)g(\eta)d\eta+2\delta\right)\\
\frac{2l(u)}{\left(2\delta+1-\delta-\int_a^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta\right)} &= \frac{p'(u)}{p(u)}\\
-\ln\left(\delta+1-\int_a^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta\right)+c_7 &= \ln(p(u))+c_8 \qquad \text{(integrating)}\\
\ln\left(\delta+1-\int_a^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta\right) &= -\ln(p(u))-(c_8-c_7)\\
\ln\left(\delta+1-\int_a^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta\right) &= \ln(p(u)^{-1})+c_9\\
\delta+1-\int_a^u l(\eta)d\eta+\int_u^1 l(\eta)d\eta &= \frac{e^{c_9}}{p(u)}=\frac{k_4}{p(u)}\\
-2l(u) &= -\frac{k_4 p'(u)}{(p(u))^2} \qquad \text{(differentiating)}\\
-2q(u)g(u) &= -\frac{k_4 p'(u)}{(p(u))^2}\\
g(u) &= \frac{k_5 p'(u)}{q(u)(p(u))^2}
\end{aligned}
\tag{A.32}
$$

Filling in and simplifying eq. (4.25) to eq. (4.27)

$$
\begin{aligned}
g(u)-\int_a^u \frac{p'(u)(1-q(\eta))}{2p(u)q(u)}g(\eta)d\eta-\int_u^1 \frac{p'(u)(1+q(\eta))}{2p(u)q(u)}g(\eta)d\eta &= \frac{2\delta p'(u)}{2p(u)q(u)}\\
\frac{k_5 p'(u)}{q(u)(p(u))^2} &= \frac{p'(u)}{2p(u)q(u)}\Big(\int_a^1 g(\eta)d\eta-\int_a^u q(\eta)g(\eta)d\eta+\int_u^1 q(\eta)g(\eta)d\eta+2\delta\Big)\\
\frac{k_5}{p(u)} &= \frac{1}{2}\Big(1-\delta-\int_a^u \frac{k_5 p'(\eta)}{(p(\eta))^2}d\eta+\int_u^1 \frac{k_5 p'(\eta)}{(p(\eta))^2}d\eta+2\delta\Big)\\
\frac{k_5}{p(u)} &= \frac{1}{2}\Big(1+\delta-k_5\Big[-p(\eta)^{-1}\Big]_a^u+k_5\Big[-p(\eta)^{-1}\Big]_u^1\Big)\\
\frac{k_5}{p(u)} &= \frac{1}{2}+\frac{\delta}{2}+\frac{k_5}{p(u)}-\frac{k_5}{2p(a)}-\frac{k_5}{2}\\
\frac{k_5}{p(u)}-\frac{k_5}{p(u)}+\frac{k_5}{2p(a)}+\frac{k_5}{2} &= \frac{1}{2}+\frac{\delta}{2}\\
k_5\Big(\frac{1}{2p(a)}+\frac{1}{2}\Big) &= \frac{1}{2}+\frac{\delta}{2}\\
k_5\Big(\frac{1}{p(a)}+1\Big) &= 1+\delta
\end{aligned}
\tag{A.33}
$$

Integrating and differentiating eq. (4.31) to eq. (4.32)

$$
\begin{aligned}
f(t)-\int_{a'}^t \frac{q'(t)(1-p(\xi))}{2p(t)q(t)}f(\xi)d\xi-\int_t^1 \frac{q'(t)(1+p(\xi))}{2p(t)q(t)}f(\xi)d\xi &= \frac{2\beta q'(t)}{2p(t)q(t)}\\
f(t) &= \int_{a'}^t \frac{q'(t)(1-p(\xi))}{2p(t)q(t)}f(\xi)d\xi+\int_t^1 \frac{q'(t)(1+p(\xi))}{2p(t)q(t)}f(\xi)d\xi+\frac{2\beta q'(t)}{2p(t)q(t)}\\
f(t) &= \frac{q'(t)}{2p(t)q(t)}\Big(\int_{a'}^1 f(\xi)d\xi-\int_{a'}^t p(\xi)f(\xi)d\xi+\int_t^1 p(\xi)f(\xi)d\xi+2\beta\Big)\\
\frac{2h(t)}{\Big(2\beta+1-\beta-\int_{a'}^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi\Big)} &= \frac{q'(t)}{q(t)}\\
-\ln\Big(\beta+1-\int_{a'}^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi\Big)+c_{10} &= \ln(q(t))+c_{11} \qquad \text{(integrating)}\\
\ln\Big(\beta+1-\int_{a'}^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi\Big) &= -\ln(q(t))-(c_{11}-c_{10})\\
\ln\Big(\beta+1-\int_{a'}^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi\Big) &= \ln(q(t)^{-1})+c_{12}\\
\beta+1-\int_{a'}^t h(\xi)d\xi+\int_t^1 h(\xi)d\xi &= \frac{e^{c_{12}}}{q(t)}=\frac{k_6}{q(t)}\\
-2h(t) &= -\frac{k_6 q'(t)}{(q(t))^2} \qquad \text{(differentiating)}\\
-2p(t)f(t) &= -\frac{k_6 q'(t)}{(q(t))^2}\\
f(t) &= \frac{k_7 q'(t)}{p(t)(q(t))^2}
\end{aligned}
\tag{A.34}
$$

Filling in and simplifying eq. (4.31) to eq. (4.33)

$$\begin{aligned}
f(t) - \int_{a'}^{t} \frac{q'(t)(1-p(\xi))}{2p(t)q(t)} f(\xi)d\xi - \int_{t}^{1} \frac{q'(t)(1+p(\xi))}{2p(t)q(t)} f(\xi)d\xi &= \frac{2\beta q'(t)}{2p(t)q(t)} \\
\frac{k_7 q'(t)}{p(t)(q(t))^2} &= \frac{q'(t)}{2p(t)q(t)}\Big(\int_{a'}^{1} f(\xi)d\xi - \int_{a'}^{t} p(\xi)f(\xi)d\xi + \int_{t}^{1} p(\xi)f(\xi)d\xi + 2\beta\Big) \\
\frac{k_7}{q(t)} &= \frac{1}{2}\Big(1-\beta-\int_{a'}^{t} \frac{k_7 q'(\xi)}{(q(\xi))^2}d\xi + \int_{t}^{1} \frac{k_7 q'(\xi)}{(q(\xi))^2}d\xi + 2\beta\Big) \\
\frac{k_7}{q(t)} &= \frac{1}{2}\Big(1+\beta - k_7\Big[-q(\xi)^{-1}\Big]_{a'}^{t} + k_7\Big[-q(\xi)^{-1}\Big]_{t}^{1}\Big) \\
\frac{k_7}{q(t)} &= \frac{1}{2} + \frac{\beta}{2} + \frac{k_7}{q(t)} - \frac{k_7}{2q(a)} - \frac{k_7}{2} \\
\frac{k_7}{q(t)} - \frac{k_7}{q(t)} + \frac{k_7}{2q(a)} + \frac{k_7}{2} &= \frac{1}{2} + \frac{\beta}{2} \\
k_7\Big(\frac{1}{2q(a)} + \frac{1}{2}\Big) &= \frac{1}{2} + \frac{\beta}{2} \\
k_7\Big(\frac{1}{q(a)} + 1\Big) &= 1+\beta
\end{aligned} \tag{A.35}$$

Calculating value in eq. (4.35)

$$\begin{aligned}
v &= \int_{\frac{1}{3}}^{\eta} (\xi - \eta + \xi\eta)\frac{1}{4\xi^3}d\xi + \int_{\eta}^{1} (\xi-\eta-\xi\eta)\frac{1}{4\xi^3}d\xi \\
&= \int_{\frac{1}{3}}^{1} \frac{1}{4\xi^2}d\xi - \eta\int_{\frac{1}{3}}^{1}\frac{1}{4\xi^3}d\xi + \eta\int_{\frac{1}{3}}^{\eta}\frac{1}{4\xi^2}d\xi - \eta\int_{\eta}^{1}\frac{1}{4\xi^2}d\xi \\
&= \frac{1}{2} - \eta + \frac{\eta}{4}\Big(-\frac{1}{\eta}+3\Big) - \frac{\eta}{4}(-1+\frac{1}{\eta}) \\
&= \frac{1}{2} - \eta - \frac{1}{4} + \frac{3\eta}{4} + \frac{\eta}{4} - \frac{1}{4} \\
&= 0
\end{aligned} \tag{A.36}$$

Solving integral in eq. (4.37)

$$\begin{aligned}
0.376\int_{0.481}^{\eta} \frac{\xi-\eta^2+\xi\eta^2}{\xi^4}d\xi + 0.376\int_{\eta}^{1}\frac{\xi-\eta^2-\xi\eta^2}{\xi^4}d\xi &\approx \\
0.376\int_{0.481}^{1}\xi^{-3}d\xi - 0.376\eta^2\int_{0.481}^{1}\xi^{-4}d\xi + 0.376\eta^2\int_{0.481}^{\eta}\xi^{-3}d\xi - 0.376\eta^2\int_{\eta}^{1}\xi^{-3}d\xi &\approx \\
0.625 - \eta^2 + 0.376\eta^2\Big(-\frac{1}{2\eta^2} + \frac{1}{2(0.481)^2}\Big) - 0.376\eta^2\Big(-\frac{1}{2}+\frac{1}{2\eta^2}\Big) &\approx \\
0.625 - \eta^2 - 0.188 + 0.812\eta^2 + 0.188\eta^2 - 0.188 \approx 0.249
\end{aligned} \tag{A.37}$$

Solving integral in eq. (4.38)

$$\begin{aligned}
0.348\int_{0.481}^{\xi}\frac{\xi-\eta^2-\xi\eta^2}{\eta^4}d\eta + 0.348\int_{\xi}^{1}\frac{\xi-\eta^2+\xi\eta^2}{\eta^4}d\eta + 0.073(2\xi-1) &\approx \\
0.348\xi\int_{0.481}^{1}\eta^{-4}d\eta - 0.348\int_{0.481}^{1}\eta^{-2}d\eta - 0.348\xi\int_{0.481}^{\xi}\eta^{-2}d\eta + 0.348\xi\int_{\xi}^{1}\eta^{-2}d\eta + 0.073(2\xi-1) &\approx \\
1.072\xi - 0.073 - 0.348\Big(-1+\frac{1}{0.481}\Big) - 0.348\xi\Big(-\frac{1}{\xi}+\frac{1}{0.481}\Big) + 0.348\xi\Big(-1+\frac{1}{\xi}\Big) &\approx \\
1.072\xi - 0.073 + 0.348 - 0.723 + 0.348 - 0.723\xi - 0.348\xi + 0.348 \approx 0.249
\end{aligned} \tag{A.38}$$

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