ICASSP 2025 - 2025 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP) | 979-8-3503-6874-1/25/$31.00 ©2025 IEEE | DOI: 10.1109/ICASSP49660.2025.10888741

Noise-Resilient Unlimited Sampling and Recovery of Sparse Signals

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***Abstract*—In this paper, we investigate the use of modulo-ADCs in compressed sensing to handle the issue of the limited dynamic range of standard ADCs. The current state-of-the-art algorithm for modulo-compressed sensing uses an ℓ_1-norm-based approximation of the sparsity constraint, resulting in a computationally demanding mixed-integer linear optimization. Handling noisy measurements further complicates the problem, requiring mixed-integer quadratic programming, a problem known to be NP-hard. We present an alternative iterative hard-thresholding approach to address this issue. Our solution is computationally simpler and capable of handling noisy measurements. Additionally, we provide theoretical guarantees that the algorithm can successfully recover sparse vectors if the sampling operator satisfies the integer augmented-restricted isometry property, which holds when the number of measurements is sufficiently large.**

***Index Terms*—Modulo-compressed sensing, iterative hard thresholding, restricted isometry property, self-reset ADC**

I. Introduction

Compressed sensing is a powerful data acquisition technique that requires fewer measurements to represent signals by exploiting their sparsity in a suitable basis [1], [2], [3]. The standard compressed sensing problem deals with solving for a sparse $\boldsymbol{x} \in \mathbb{R}^n$ from a set of linear measurements $\boldsymbol{Ax}$ where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with m denoting the number of measurements. However, practical data acquisition systems often face clipping or saturation, which occurs when a signal's amplitude exceeds a certain threshold [4], [5], [6]. In this case, the m measurements can be represented as $\mathcal{S}(\boldsymbol{Ax})$. Here, $\mathcal{S}$ denotes the saturation function, $\mathcal{S}(t) = \operatorname{sign}(t)\mu$ if $|t| > \mu$ and $\mathcal{S}(t) = t$ if $|t| < \mu$, with μ representing the threshold. Recently, a new hardware solution, self-reset analog-to-digital converters (SR-ADCs), has been introduced to mitigate clipping by increasing the dynamic range [7], [8]. This approach, referred to as unlimited sampling, introduces a nonlinear sensing method that folds signal amplitudes back into the dynamic range using modulo arithmetic [9], [10], [11]. Here, the m measurements can be represented as $\mathcal{M}_{\lambda,\mu}(\boldsymbol{Ax})$. The non-linear function $\mathcal{M}_{\lambda,\mu}$ represents the folding architecture of SR-ADC, which maps the signal to the range $(-\lambda, \lambda)$, given by

$$\mathcal{M}_{\lambda,\mu}(t) = 2\lambda\left(\left[\!\left[\frac{t}{2\mu} + \frac{1}{2}\right]\!\right] - \frac{1}{2}\right), \tag{1}$$

where $[\![t]\!] = t - \lfloor t \rfloor$ represents the fractional part or modulo 1 operation. We address the compressed sensing problem of recovering the sparse vector $\boldsymbol{x}$ with unlimited sampling.

Compressed sensing with modulo measurements was first applied in a high dynamic range imaging system that used modulo measurements, though limited to the special case of noiseless modulo folding with two periods [12], [13]. Subsequent work extended this model to general noisy modulo folding, introducing approximate message-passing algorithms tailored to modulo-compressed sensing. However, these approaches often assume strong conditions on sparse signals, such as a Bernoulli-Gaussian distribution with known parameters [14], [15]. Other research has addressed the modulo-compressed sensing problem for line spectral estimation, presenting a two-stage recovery procedure that combines dynamic programming with orthogonal matching pursuit [16]. Nevertheless, these methods lack theoretical guarantees on their performance. Convex relaxation-based methods were also studied for compressed sensing with modulo folding, both with and without limited periods [17], [18]. These studies developed a mixed integer linear program (MILP) for recovering sparse signals from modulo measurements in the noiseless case. However, these approaches are computationally intensive. Also, handling noise with convex relaxation introduces a mixed integer quadratic program, which is even more challenging to solve.

To address these gaps in the literature, we present a novel approach to solving the signal recovery problem in the presence of noise. Our specific contributions are twofold:

- *Iterative hard thresholding (IHT) algorithm:* We introduce a new algorithm for signal recovery from noisy modulo measurements, with or without knowing the saturation status of each measurement, with linear runtime and memory complexity.
- *Theoretical Guarantees:* The notion of integer-augmented restricted isometry property (RIP) is introduced to prove that when the measurement matrix satisfies this property, the algorithm converges to the true solution within a finite error tolerance. We also derive a sufficient condition based on the number of measurements needed for the algorithm to meet the property.

Overall, our work provides a new fast modulo-compressed sensing algorithm with theoretical guarantees, which is useful in practical applications where noise is an inherent challenge.

II. Noisy Modulo Compressed Sensing

Let $\boldsymbol{x} \in \mathbb{R}^n$ be a sparse vector with at most s nonzero entries, i.e., $\|\boldsymbol{x}\|_0 \le s$. We consider m noisy linear measurements $\boldsymbol{y} \in \mathbb{R}^m$ of $\boldsymbol{x}$ passed through a SR-ADC. Without loss of generality, we assume the SR-ADC parameters are $\lambda = \mu = 1/2$, as λ and μ only serve to scale the measurements and sparse vectors, respectively [18]. In the general case, we consider estimating $\boldsymbol{x}/(2\mu)$ from $\boldsymbol{y}/(2\lambda)$. Now, the measurements are

$$\boldsymbol{y} = \mathcal{M}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{w}), \tag{2}$$

where $\boldsymbol{w} \in \mathbb{R}^m$ denotes the measurement noise and the non-linear function $\mathcal{M} = \mathcal{M}_{1/2,1/2}$ represents the folding operator in (1). Also, we let $\mathcal{I}$ denote the indices of measurements that exceed the dynamic range $(-\mu, \mu)$, i.e, for $i \notin \mathcal{I}$, we have

$$\boldsymbol{y}_i = \boldsymbol{A}_i^\top \boldsymbol{x} + \boldsymbol{w}_i,$$

where $\boldsymbol{A}_i$ denotes the ith row of $\boldsymbol{A}$. Our goal is to recover $\boldsymbol{x}$ from $\boldsymbol{y}$.

To solve the above problem, (2) can be rewritten as

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w} - \boldsymbol{v}, \tag{3}$$

where $\boldsymbol{v}_i = \lfloor \boldsymbol{A}_i^\top \boldsymbol{x} + \boldsymbol{w}_i + \frac{1}{2} \rfloor \in \mathbb{Z}$. Since $\boldsymbol{y}_i \in (-1/2, 1/2)$, any $\boldsymbol{x} \in \mathbb{R}^n$ satisfying (3) for some $\boldsymbol{v} \in \mathbb{Z}^m$ also satisfies (2). Therefore, in the following, we solve for $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{Z}^m$ such that

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w} - \boldsymbol{v} \quad \text{and} \quad \|\boldsymbol{x}\|_0 \le s. \tag{4}$$

We note that if $i \notin \mathcal{I}$, then $\boldsymbol{v}_i = 0$, leading to $\|\boldsymbol{v}\|_0 \le |\mathcal{I}|$. However, unlike sparse recovery from saturated measurements, where $\mathcal{I}$ represents the number of measurements equal to $\pm\mu$, we can not directly estimate $\mathcal{I}$ from the modulo measurements [1], [4]. Nonetheless, it is possible to incorporate side information about $\mathcal{I}$ using a simple amplitude comparator alongside the SR-ADC. Therefore, we consider three scenarios: one with known $\mathcal{I}$, a second with only the knowledge of $|\mathcal{I}|$, and a third without the knowledge of $|\mathcal{I}|$.

Solving for $\boldsymbol{x}$ and $\boldsymbol{v}$ from (4) is a hard problem because of the sparsity constraint on $\boldsymbol{x}$ and integer constraint on $\boldsymbol{v}$. So, we use an extension of the iterative hard thresholding algorithm to solve the problem [19]. The algorithm uses a projected gradient descent algorithm, where in every iteration, the algorithm projects the gradient descent iterate of $\boldsymbol{x}$ and $\boldsymbol{v}$ to the set of s-sparse vectors and integers, respectively. In the rth iteration, the algorithm computes

$$\boldsymbol{x}^{(r+1)} = \mathcal{P}_s(\boldsymbol{x}^{(r)} + \alpha^{(r)}\boldsymbol{A}^\top(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(r)} + \boldsymbol{v}^{(r)}) \tag{5}$$

$$\boldsymbol{v}^{(r+1)} = \mathcal{Q}(\boldsymbol{v}^{(r)} + \alpha^{(r)}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(r)} + \boldsymbol{v}^{(r)}), \tag{6}$$

where $\alpha^{(r)}$ is the step size in the rth iteration. Here, the nonlinear operator $\mathcal{P}_s(\cdot)$ projects its argument onto the set of s-sparse vectors by retaining the s largest (in magnitude) entries, functioning as a hard thresholding operator. Meanwhile, $\mathcal{Q}$ depends on the available side information on $\mathcal{I}$. If $\mathcal{I}$ is known, $\mathcal{Q} = \mathcal{Q}_{\mathcal{I}}$ rounds the entries indexed by $\mathcal{I}$ to the nearest nonzero integer and sets the remaining entries to 0. If only $|\mathcal{I}|$ or an upper bound on $|\mathcal{I}|$ is known, $\mathcal{Q} = \mathcal{Q}_{|\mathcal{I}|}$ projects the iterate onto the set of $|\mathcal{I}|$-sparse integer vectors by retaining the $|\mathcal{I}|$ largest entries and rounding them to the nearest integer. If $|\mathcal{I}|$ is unknown, a simple approach is to assume the upper bound $|\mathcal{I}| = m$ and set $\mathcal{Q} = \mathcal{Q}_m$. Alternatively, the algorithm considers multiple values of $|\mathcal{I}|$ from $\{1, 2, \ldots, m\}$, iterating between (5) and (6) with $\mathcal{Q} = \mathcal{Q}_{|\mathcal{I}|}$ for each choice in parallel and selecting the solution with the smallest residual error $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}\|$.

We conclude our discussion on algorithms by addressing their complexity. Recall that MILP in [18], which employs the branch-and-bound approach, can have an exponential complexity in system dimensions. In contrast, our modulo iterative hard thresholding method is computationally efficient and easy to implement, as it only involves multiplications with $\boldsymbol{A}$ and $\boldsymbol{A}^\top$. The per-iteration runtime of our algorithm is $\mathcal{O}(mn)$, with a memory complexity of $\mathcal{O}(Lmn)$, where L represents the set of possible choices for $|\mathcal{I}|$. When $\mathcal{I}$ or $|\mathcal{I}|$ is known, we have $L = 1$, simplifying the memory complexity to $\mathcal{O}(mn)$.

III. Theoretical Guarantees

To guarantee robust recovery of the sparse vectors, we need to ensure that the modulo compressed sensing operator in (3) is a one-to-one mapping from the set of sparse vectors to the measurement range $(-1/2, 1/2)$. This condition can be imposed using a RIP-type condition, as defined below:

Definition 1 (Integer augmented RIP)**.** *A given matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is said to satisfy integer augmented RIP with restricted isometric constant (RIC) $\delta_{s,q} < 1$ of order (s, q) if*

$$(1-\delta)(\|\boldsymbol{x}\|^2 + \|\boldsymbol{v}\|^2) \le \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}\|^2 \le (1+\delta)(\|\boldsymbol{x}\|^2 + \|\boldsymbol{v}\|^2),$$

for all s-sparse vectors $\boldsymbol{x} \in \mathbb{R}^n$, q-sparse vectors $\boldsymbol{v} \in \mathbb{Z}^m$, and $\delta \ge \delta_s$.

The above condition implies that $\|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}\| \ne 0$ for any nonzero s-sparse vector $\boldsymbol{x}$ and q-sparse integer vector $\boldsymbol{v}$, ensuring that the noiseless case yields a unique solution, i.e.,

$$|\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{y} = \mathcal{M}(\boldsymbol{A}\boldsymbol{x}),\ \|\boldsymbol{x}\|_0 \le s, \\ \|\lfloor \boldsymbol{A}\boldsymbol{x} + 1/2\ \mathbf{1} \rfloor\|_0 \le q\}| = 1.$$

We now state the main result, which ensures that our algorithm's estimation error converges to the true solution within a constant factor of the noise vector's norm.

Theorem 1. *Consider the sparse recovery problem in* (4) *where $\boldsymbol{x}$ is s-sparse and $|\mathcal{I}| \le q$. Suppose that $\|\boldsymbol{w}\|^2 \le \epsilon$ and $\boldsymbol{A}$ satisfies the integer augmented-RIC $\delta_{2s,2q} = \delta < 1/\sqrt{3}$ of order $(2s, 2q)$. For a given $\gamma > 0$, let*

$$r^* \le \frac{\ln \gamma/(\|\boldsymbol{x}\|^2 + \|\boldsymbol{v}\|^2)}{\ln 2\left(\frac{1}{\alpha(1-\delta)} - 1\right)} \tag{7}$$

$$\alpha^{(r)} = \alpha, \ \ \forall r \quad \textit{and} \quad 1 - \delta \le \frac{1}{\alpha} < \frac{1}{1+\delta}. \tag{8}$$

Then, for any $\gamma > 0$ and constant $c \le \frac{\alpha(3-\delta)}{3\alpha(1-\delta)-2}$, the output of our algorithm, i.e., (5)-(6) *with $\mathcal{Q} = \mathcal{Q}_q$, after r^* iterations satisfies*

$$\left\|\boldsymbol{x} - \boldsymbol{x}^{(r^*)}\right\|^2 + \left\|\boldsymbol{v} - \boldsymbol{v}^{(r^*)}\right\|^2 \le c\epsilon + \gamma. \tag{9}$$

Proof. See the appendix. □

We note that the result also applies to the case when $|\mathcal{I}|$ is unknown, where we use the bound $|\mathcal{I}| \leq m$. However, the RIP condition becomes more stringent as $|\mathcal{I}|$ increases. Further, when $\mathcal{I}$ is known, we can modify Definition 1 to the integer augmented-RIC $\delta_{2s,\mathcal{I}}$ by replacing $|\mathcal{I}|$-sparse vectors $\boldsymbol{v} \in \mathbb{Z}^m$ with $\boldsymbol{v} \in \mathbb{Z}^m$ supported on $\mathcal{I}$. Then, the error bound in Theorem 1 requires $\boldsymbol{A}$ to satisfy the modified integer augmented-RIC $\delta_{2s,\mathcal{I}} < 1/\sqrt{3}$.

Corollary 1. *Consider the sparse recovery problem in* (4) *where $\boldsymbol{x}$ is s-sparse and $|\mathcal{I}| \leq q$. Suppose that $\|\boldsymbol{w}\|^2 \leq \epsilon$, $\boldsymbol{A}$ is a random matrix with independent entries drawn from $\mathcal{N}(0, 1/m)$, and* (7) *and* (8) *hold. For any $\gamma > 0$, there exist universal constants $C_1, C_2 > 0$ such that the algorithm output after r^* iterations satisfies* (9) *with probability $1 - 3e^{-C_2 m}$ if*

$$m \geq C_1 (s+q) \log \left(\frac{n+m}{s+q} \right). \tag{10}$$

Proof. We note that $\boldsymbol{A}$ satisfies the integer augmented RIP of order (s,q) with RIC δ if

$$(1-\delta) \|\tilde{\boldsymbol{x}}\|^2 \leq \left\| \begin{bmatrix} \boldsymbol{A} & \boldsymbol{I} \end{bmatrix} \tilde{\boldsymbol{x}} \right\|^2 \leq (1+\delta) \|\tilde{\boldsymbol{x}}\|^2,$$

where $\tilde{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x}^\top & \boldsymbol{v}^\top \end{bmatrix}^\top \in \mathbb{R}^n \times \mathbb{Z}^m \subset \mathbb{R}^{n+m}$, for all s-sparse vectors $\boldsymbol{x}$ and q-sparse vectors $\boldsymbol{v}$. The above condition holds if the matrix $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{I} \end{bmatrix}$ has RIC of order $s+q$ less than δ. Further, [20, Theorem 1] guarantees that it holds with probability $1 - 3e^{-C_2 m}$ if (10) holds. $\square$

Also, the proof of Theorem 1 shows that the thresholding algorithm reduces the estimation error in each iteration (see (15)), proving that as the number of iterations goes to ∞, the error is bounded by $c\epsilon$. Also, (15) implies that the algorithm converges to the true solution in the noiseless case (i.e., $\epsilon = 0$). Furthermore, in the noiseless case, we derive a stronger bound by relaxing the bound on $|\mathcal{I}|$.

Corollary 2. *Consider the sparse recovery problem in* (4) *where $\boldsymbol{x}$ is s-sparse with $\|\boldsymbol{x}\|^2 \leq \beta$ in the noiseless setting. Suppose that $\boldsymbol{A}$ is a random matrix with independent entries drawn from $\mathcal{N}(0, 1/m)$, and* (7) *and* (8) *hold. Then, for any $0 < \gamma < 1$, there exist universal constants $C_1, C_2 > 0$ such that our algorithm output after r^* iterations satisfies* (9) *with $\epsilon = 0$ with probability $1 - 3e^{-C_2 m}$ if*

$$m \geq C_1 \left(s + 4\left(1 + 1/\sqrt{3}\right)\beta \right) \log \left(\frac{n+m}{s + 4\left(1+1/\sqrt{3}\right)\beta} \right).$$

Proof. From Corollary 1, it suffices to prove that $|\mathcal{I}| \leq 4\left(1+1/\sqrt{3}\right)\beta$ with $\delta = 1/\sqrt{3}$. For this, we note that the definition of $\mathcal{I}$ implies

$$\begin{aligned} |\mathcal{I}|\, \mu^2 &\leq \sum_{i \in \mathcal{I}} |\boldsymbol{A}_i^\top \boldsymbol{x}|^2 \leq \|\boldsymbol{A}\boldsymbol{x}\|^2 \\ &= \left\| \begin{bmatrix} \boldsymbol{A} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{bmatrix} \right\|^2 \leq \left(1 + 1/\sqrt{3}\right) \|\boldsymbol{x}\|^2, \end{aligned}$$

where the last step follows because the RIC of $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{I} \end{bmatrix}$, $\delta = 1/\sqrt{3}$ and since $\|\boldsymbol{x}\|^2 \leq \beta$ and $\mu = 1/2$, the result follows. $\square$

Moreover, we can extend the result when the noise $\boldsymbol{w}$ is additive zero-mean white Gaussian. When the entries of $\boldsymbol{w}$ are drawn from $\mathcal{N}(0, \sigma^2)$, it follows [21, Equation 8.89],

$$\mathbb{P}\left\{ \|\boldsymbol{w}\| / \sigma \geq \sqrt{m} + t \right\} \leq e^{-t^2/2}.$$

Therefore, (9) holds with probability exceeding $1 - e^{-t^2/2}$ when $\epsilon = \sigma^2(\sqrt{m} + t)^2$.

IV. Simulation Results

This section presents numerical results to demonstrate the performance of our algorithm. We use sparse vectors of length $n = 60$ and $s = 6$ nonzero entries, positioned uniformly at random with values drawn from a Gaussian distribution with zero mean and variance $P_x = 2$. The measurement matrix $\boldsymbol{A}$ has Gaussian-distributed entries with zero-mean and variance $1/m$. The noise $\boldsymbol{w}$ is white Gaussian with variance σ^2, which depends on the SNR value defined as $20\log(\mathbb{E}(\|\boldsymbol{A}\boldsymbol{x}|^2 / \|\boldsymbol{n}|^2) = 20\log(nP_x/(m\sigma^2))$. We compare three schemes: iterative hard thresholding with knowledge of $\mathcal{I}$, with knowledge of $|\mathcal{I}|$ only, and without knowledge of $|\mathcal{I}|$. When $|\mathcal{I}|$ is unknown, we use the worst-case bound $|\mathcal{I}| = m$. The results, averaged over 1×10^4 trials, are given in Fig. 1 (where IHT refers to iterative hard thresholding) and Table I.

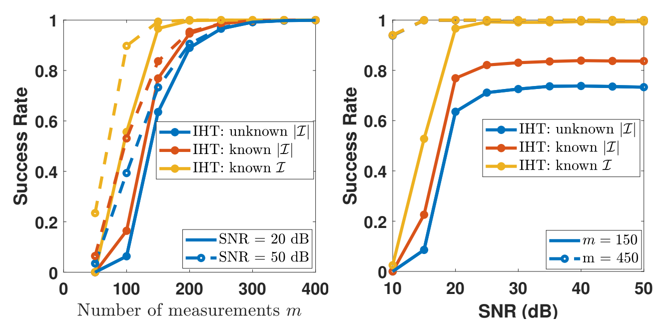

Fig. 1. The success rate of the different iterative hard thresholding schemes as a function of SNR and number of measurements m with the length of unknown sparse vector $n = 60$ and sparsity $s = 6$.

Fig. 1 presents the success rate, defined as the fraction of trials where the normalized mean squared error in estimating the sparse vector is less than 10^{-2}. All schemes succeed when both SNR and the number of measurements are high. However, for $m < 150$, all fail, indicating a minimum measurement threshold for successful recovery, as suggested by Corollary 1. In the low measurement regime with $m = 150$, only the known $\mathcal{I}$ scheme can achieve a (near-)perfect success rate, while others plateau below 1 across all SNR values. Also, as m increases, the success rates improve, with the known $\mathcal{I}$ case showing the best performance, followed by the known $|\mathcal{I}|$ case. This trend aligns with the expectation that more side information enhances recovery. Further, in the high measurement regime with $m = 450$, all the schemes exhibit similar performance, irrespective of SNR value. Furthermore, the difference between the performance at SNRs 20 and 50 dB is not significant compared to the gap between 10 and 20 dB. This observation suggests a minimum SNR threshold below which the algorithms fail, consistent with the noise bound in Theorem 1. This threshold decreases as more information about the sparse vector becomes available.

TABLE I

RUNTIME IN DIFFERENT MODULO ALGORITHMS IN THE NOISELESS SETTING WITH $n = 60$ AND $s = 6$

	MILP [18]	Iterative hard thresholding		
		unknown $\lvert\mathcal{I}\rvert$	known $\lvert\mathcal{I}\rvert$	known $\mathcal{I}$
$m = 450$	749 ms	13.9 ms	10.8 ms	10.5 ms
$m = 600$	844.8 ms	22.2 ms	16.8 ms	16.8 ms

Table I compares the runtime of our algorithms with MILP [18] in the noiseless setting. We note that MILP does not apply for the noisy case, so it is not included in Fig. 1. In this setting, all the algorithms have a success rate equal to one. The results demonstrate that our algorithm is at least one order of magnitude faster than MILP, supporting complexity analysis in Sec. II. While all the IHT-based algorithms have the same per-iteration complexity, IHT without the knowledge of $\mathcal{I}$ or $|\mathcal{I}|$ requires more runtime, indicating that it takes more iterations to converge.

V. CONCLUSION

We addressed the modulo-compressed sensing problem using an iterative hard-thresholding approach. Unlike existing methods, our approach is computationally simpler and effectively handles noise. Additionally, we provided theoretical guarantees that the algorithm can recover sparse vectors if the measurement matrix satisfies the integer augmented RIP. Exploring the exact measurement bounds for the algorithm based on this property and studying algorithms for other structured sparsity patterns could be promising future research directions.

APPENDIX

We need the following lemma to prove the result.

Lemma 1. *The modulo IHT iterates satisfy*

$$\boldsymbol{x}^{(r+1)} = \operatorname*{arg\,inf}_{\boldsymbol{z}\in\mathbb{R}^n:\|\boldsymbol{z}\|_0\le s} \frac{1}{\alpha}\|\boldsymbol{z}-\boldsymbol{x}^{(r)}\|^2 - 2\boldsymbol{f}^\top(\boldsymbol{z}-\boldsymbol{x}^{(r)})$$
$$\boldsymbol{v}^{(r+1)} = \operatorname*{arg\,inf}_{\substack{\boldsymbol{u}\in\mathbb{Z}^m\\ \|\boldsymbol{u}\|_0\le q}} \frac{1}{\alpha}\left\|\boldsymbol{u}-\boldsymbol{v}^{(r)}\right\|^2 - 2\boldsymbol{g}^\top(\boldsymbol{u}-\boldsymbol{v}^{(r)}), \quad (11)$$

where $\boldsymbol{f} = \boldsymbol{A}^\top\boldsymbol{g}$ *and* $\boldsymbol{g} = \boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}^{(r)}+\boldsymbol{v}^{(r)}$.

Proof. We start by noting that

$$\begin{aligned}
&\operatorname*{arg\,inf}_{\boldsymbol{z}\in\mathbb{R}^n:\|\boldsymbol{z}\|_0\le s} \frac{1}{\alpha}\left\|\boldsymbol{z}-\boldsymbol{x}^{(r)}\right\|^2 - 2\boldsymbol{f}^\top(\boldsymbol{z}-\boldsymbol{x}^{(r)})\\
&= \operatorname*{arg\,inf}_{\boldsymbol{z}\in\mathbb{R}^n:\|\boldsymbol{z}\|_0\le s} \frac{1}{\alpha}\left\|\boldsymbol{z}-\boldsymbol{x}^{(r)}-\alpha\boldsymbol{f}\right\|^2 - \alpha\|\boldsymbol{f}\|^2\\
&= \operatorname*{arg\,inf}_{\boldsymbol{z}\in\mathbb{R}^n:\|\boldsymbol{z}\|_0\le s} \left\|\boldsymbol{z}-\boldsymbol{x}^{(r)}-\alpha\boldsymbol{f}\right\|^2 = \boldsymbol{x}^{(r+1)},
\end{aligned}$$

which follows from (5). Using similar arguments, we can prove (11) from (6). □

Proof of Theorem 1. Let $\Delta^{(r)} = \left\|\boldsymbol{x}-\boldsymbol{x}^{(r)}\right\|^2 + \left\|\boldsymbol{v}-\boldsymbol{v}^{(r)}\right\|^2$. In the rth iteration of the algorithm, using the integer augmented RIP, we have

$$\begin{aligned}
\Delta^{(r+1)} &\le \frac{1}{1-\delta}\left\|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}^{(r+1)})-(\boldsymbol{v}-\boldsymbol{v}^{(r+1)})\right\|^2\\
&\le \frac{1}{1-\delta}\left\|\boldsymbol{y}-\boldsymbol{w}-\boldsymbol{A}\boldsymbol{x}^{(r+1)}+\boldsymbol{v}^{(r+1)}\right\|^2\\
&\le \frac{2}{1-\delta}\left\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}^{(r+1)}+\boldsymbol{v}^{(r+1)}\right\|^2+\epsilon, \quad (12)
\end{aligned}$$

where we use the fact that for any two vector $\boldsymbol{a},\boldsymbol{b}$,

$$\|\boldsymbol{a}+\boldsymbol{b}\|^2 \le \|\boldsymbol{a}\|^2+\|\boldsymbol{b}\|^2+2\|\boldsymbol{a}\|\|\boldsymbol{b}\| \le 2\|\boldsymbol{a}\|^2+2\|\boldsymbol{b}\|^2.$$

We rewrite the first term in (12) using $\boldsymbol{f}$ and $\boldsymbol{g}$ defined in Lemma 1 as

$$\begin{aligned}
&\left\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}^{(r+1)}+\boldsymbol{v}^{(r+1)}\right\|^2\\
&= \|\boldsymbol{g}\|^2 - 2\boldsymbol{f}^\top(\boldsymbol{x}^{(r+1)}-\boldsymbol{x}^{(r)}) - 2\boldsymbol{g}^\top(\boldsymbol{v}^{(r+1)}-\boldsymbol{v}^{(r)})\\
&\quad + \left\|\boldsymbol{A}(\boldsymbol{x}^{(r+1)}-\boldsymbol{x}^{(r)})-(\boldsymbol{v}^{(r+1)}-\boldsymbol{v}^{(r)})\right\|^2. \quad (13)
\end{aligned}$$

Here, using the integer augmented RIP, we derive

$$\begin{aligned}
&\left\|\boldsymbol{A}(\boldsymbol{x}^{(r+1)}-\boldsymbol{x}^{(r)})-(\boldsymbol{v}^{(r+1)}-\boldsymbol{v}^{(r)})\right\|^2\\
&\le (1+\delta)\left[\left\|\boldsymbol{x}^{(r+1)}-\boldsymbol{x}^{(r)}\right\|^2+\left\|\boldsymbol{v}^{(r+1)}-\boldsymbol{v}^{(r)}\right\|^2\right], \quad (14)
\end{aligned}$$

Since $(1+\delta)\le 1/\alpha$ by (8), combing (13) and (14), we deduce

$$\begin{aligned}
&\left\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}^{(r+1)}+\boldsymbol{v}^{(r+1)}\right\|^2\\
&\le \|\boldsymbol{g}\|^2 - 2\boldsymbol{f}^\top(\boldsymbol{x}^{(r+1)}-\boldsymbol{x}^{(r)}) - 2\boldsymbol{g}^\top(\boldsymbol{v}^{(r+1)}-\boldsymbol{v}^{(r)})\\
&\quad + \frac{1}{\alpha}\left[\left\|\boldsymbol{x}^{(r+1)}-\boldsymbol{x}^{(r)}\right\|^2+\left\|\boldsymbol{v}^{(r+1)}-\boldsymbol{v}^{(r)}\right\|^2\right]\\
&\le \|\boldsymbol{g}\|^2 - 2\boldsymbol{f}^\top(\boldsymbol{x}-\boldsymbol{x}^{(r)}) - 2\boldsymbol{g}^\top(\boldsymbol{v}-\boldsymbol{v}^{(r)}) + \frac{1}{\alpha}\Delta^{(r)}\\
&= \|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}+\boldsymbol{v}\|^2 - \left\|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}^{(r)})-(\boldsymbol{v}-\boldsymbol{v}^{(r)})\right\|^2 + \frac{1}{\alpha}\Delta^{(r)},
\end{aligned}$$

where the last inequality uses Lemma 1. Nonetheless, the integer augmented RIP further bounds the second term on the right-hand side to yield

$$\begin{aligned}
&\left\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}^{(r+1)}+\boldsymbol{v}^{(r+1)}\right\|^2\\
&\qquad\qquad \le \|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}+\boldsymbol{v}\|^2 + \left(\frac{1}{\alpha}-(1-\delta)\right)\Delta^{(r)}.
\end{aligned}$$

Hence, the assumption $\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}+\boldsymbol{v}\|^2\le\epsilon$ and (12) imply

$$\begin{aligned}
\Delta^{(r+1)} &\le 2\left(\frac{1}{\alpha(1-\delta)}-1\right)\Delta^{(r)} + \left(\frac{2}{1-\delta}+1\right)\epsilon\\
&\le 2^r\left(\frac{1}{\alpha(1-\delta)}-1\right)^r\Delta^{(1)} + \frac{\alpha(3-\delta)}{3\alpha(1-\delta)-2}\epsilon, \quad (15)
\end{aligned}$$

since $2\left(\frac{1}{\alpha(1-\delta)}-1\right)<1$ by (8). Hence, after r^* iterations, the error reduces to the desired value when $\boldsymbol{x}^{(1)}=\boldsymbol{0}$ and $\boldsymbol{v}^{(1)}=\boldsymbol{0}$, leading to $\Delta^{(1)}=\|\boldsymbol{x}\|^2+\|\boldsymbol{v}\|^2$. □

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