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VERY HIGH ORDER RESIDUAL DISTRIBUTION ON TRIANGULAR GRIDS

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Abstract. *We review some criteria allowing the systematic construction of arbitrary order non-oscillatory residual distribution schemes for the solution of hyperbolic conservation laws on unstructured triangulations. For simplicity, we present the main ideas for the case of scalar conservation laws, and then present results for some representative problems involving the solution of scalar models, of a Cauchy-Riemann system, and of the Euler equations.*

1 INTRODUCTION

We seek numerical approximations of weak solutions to the system of conservation laws

$$\frac{\partial \mathrm{u}}{\partial t} + \nabla \cdot \mathrm{f}(\mathrm{u}) = 0 \quad \text{on} \quad \Omega \times [0,\, t_f] \subset \mathbb{R}^2 \times \mathbb{R}^+ \tag{1}$$

where u is an array of conserved variables, and f(u) is the tensor of the convective fluxes. For simplicity, we focus on the two dimensional case $\mathrm{f} = (\mathrm{f}_1,\, \mathrm{f}_2)$. The extension to three space dimensions is trivial. System (1) is supposed to satisfy standard assumptions: existence of an entropy pair, hence symmetrizability and hyperbolic character.

The objective of this paper is to give a short review of a systematic way of producing approximations to weak solutions to (1) when $t_f \to \infty$, by means of residual distribution schemes. The main theoretical building blocks reviewed here have been given in[1,2,3,4,5]. In order to simplify the presentation, we will discuss the case in which u is a scalar and $\mathrm{f} = (\mathrm{f}_1,\, \mathrm{f}_2) \in \mathbb{R}^2$. The generalization of the theory to systems is quite simple and will be reported elsewhere.

Our main goal is to recall the main ideas underlying the residual distribution technique, and under which conditions we can produce stable, convergent, non-oscillatory, and arbitrary order approximations of solutions to (1).

1.1 Geometry and polynomial approximation

We denote by $\mathcal{T}_h$ an unstructured triangulation of the spatial domain Ω, with h a characteristic mesh size (*e.g.* largest element diameter). We are going to build polynomial approximations of solutions to (1) on $\mathcal{T}_h$, which we shall denote by u_h.

In particular, we are interested in piecewise polynomial continuous functions defined on $\mathcal{T}_h$. Here this is achieved using standard P^k Lagrangian finite elements basis functions to build continuous polynomials of degree k, interpolating values of the unknown in some mesh locations. Denoting by ψ_i the shape function associated to a node i, we will write

$$\mathrm{u}_h = \sum_{i\in\mathcal{T}_h} \mathrm{u}_i \psi_i \tag{2}$$

Examples of Lagrangian elements, with related position and number of nodes (per element), are shown in figure 1. Note that $\forall\, k$, u_h is a polynomial of degree at most k. The objective of this paper is to describe how to compute the (unknown) nodal coefficients u_i in (2), such that u_h converges to a weak solution of (1) as $h \to 0$, such that the convergence is as fast as possible (possibly of $\mathcal{O}(h^{k+1})$), and that the solution has also some non-oscillatory properties in correspondence of discontinuities.

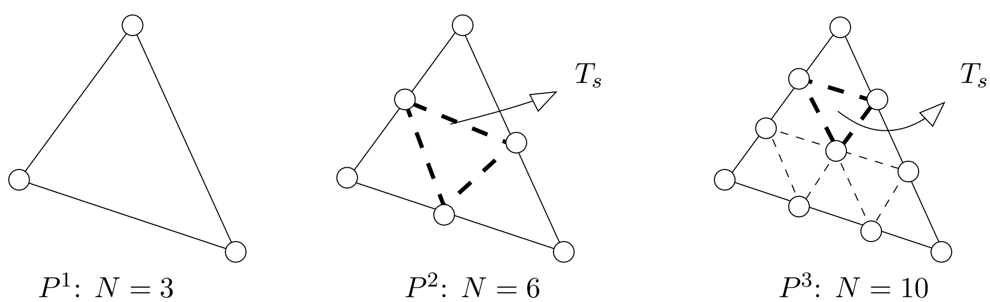


Figure 1: Lagrangian elements, sub-triangulation

2 HIGH ORDER RESIDUAL DISTRIBUTION

Denoting by T the generic triangle of $\mathcal{T}_h$, and by $\mathcal{D}_i$ the set of triangles containing node i, the schemes we will consider compute the nodal values $\{\mathrm{u}_i\}_{i\in\mathcal{T}_h}$ as the limit $n \to \infty$ of the discrete pseudo-time marching procedure

$$\begin{aligned}\mathrm{u}_i^{n+1} =& \mathrm{u}_i^n - \omega_i \sum_{T\in\mathcal{D}_i} \phi_i^T(\mathrm{u}_h^n)\\ \mathrm{u}_i^0 =& \mathrm{u}_0(x_i,\, y_i)\end{aligned} \tag{3}$$

with $\omega_i > 0$ an iteration parameter, with $\mathrm{u}_0(x, y)$ a given initial solution, and with the local nodal residuals $\phi_i^T(\mathrm{u}_h)$ respecting

$$\sum_{j\in T}\phi_j^T(\mathrm{u}_h) = \int_T \nabla\cdot \mathrm{f}_h(\mathrm{u}_h)\,dx\,dy = \phi^T(\mathrm{u}_h) \tag{4}$$

with $\mathrm{f}_h(\mathrm{u}_h)$ some discrete approximation of the flux f. The quantity $\phi^T(\mathrm{u}_h)$ is known as the *element residual* or *fluctuation*, while the ϕ_i^T are called the *local nodal residuals* or *split residuals*. Following[1,2] we remark that, in general, each ϕ_i^T in (3) can be further decomposed as

$$\phi_i^T(\mathrm{u}_h) = \sum_{T_s\subset T}\phi_i^{T_s}(\mathrm{u}_h) \tag{5}$$

having denoted by T_s the generic sub-triangle of T, constructed as shown in figure 1.

The question is how to define the ϕ_i^Ts (or equivalently the $\phi_i^{T_s}$s) such that the discrete solution converges to a weak solution of (1), and under which conditions we can ensure some additional stability properties which in turn allow to achieve optimal convergence rates, and a non-oscillatory solution.

We will review some accuracy and monotonicity criteria for the choice of the local residuals ϕ_i^T in the following paragraphs. Here we just recall that in[1] a Lax-Wendroff theorem for scheme (3)-(4) has been proved. In particular, in the reference it has been shown that, under some continuity assumptions on the ϕ_i^Ts and on the numerical approximation of the flux $\mathrm{f}_h(\mathrm{u}_h)$ in (4), if u_h is bounded and convergent, than the limit u of u_h as $h \to 0$, is a weak solution of (1).

2.1 Error analysis and accuracy condition

In this paragraph we recall how to characterize the accuracy of scheme (3)-(4). We report the early results presented in[1,6]. The extension to the non-homogeneous and time-dependent case can be found in[2,5].

The idea is to consider a smooth exact steady-state solution of the problem, say w, verifying (1) in a pointwise manner. Then we consider scheme (3)-(4) at steady state, when all the quantities are computed by using the $k-$th degree piecewise polynomial continuous approximation of w

$$\mathrm{w}_h = \sum_{i\in\mathcal{T}_h}\mathrm{w}_i\psi_i \tag{6}$$

In particular, one can define the following error

$$\mathcal{E}_h := \sum_{i\in\mathcal{T}_h}\varphi_i\sum_{T\in\mathcal{D}_i}\phi_i^T(\mathrm{w}_h) \tag{7}$$

with φ_i the nodal value of an *arbitrary* smooth compactly supported function, $\varphi\in C_0^1(\Omega)$. Steady-state discrete solutions of our scheme are obtained by requiring the last summation in (7) to vanish $\forall\, i \in \mathcal{T}_h$ (cf. equation (3)). However, since w_h *is not the discrete solution obtained with the residual distribution scheme but the interpolant of the exact solution*, then $\mathcal{E}_h$ is in general non-zero, and its magnitude gives an estimate of the accuracy of the approximation.

What one can easily prove is that[1,5] *provided that the condition*

$$\phi_i^T(\mathrm{w}_h) = \mathcal{O}(h^{k+2}) \tag{8}$$

is met, than the error (7) *verifies*

$$\mathcal{E}_h = -\int_{\mathbb{R}^2} \mathrm{f}_h \cdot \nabla\varphi_h \; dx\, dy + \mathcal{O}(h^{k+1}) \tag{9}$$

where φ_h is the $k-$th degree piecewise polynomial continuous approximation of φ.

Recalling that, neglecting boundary conditions, steady-state weak exact solutions of (1) are defined by

$$-\int_{\mathbb{R}^2} \mathrm{f} \cdot \nabla\varphi \; dx\, dy = 0 \tag{10}$$

for any given $\varphi \in C_0^1(\mathbb{R}^2)$, and using the smoothness of φ, and the properties of the continuous discrete approximation φ_h, one easily concludes that (see[1,5] for more): *provided that condition* (8) *is met, than* $\mathcal{E}_h = \mathcal{O}(h^{k+1})$. *In this case we say that the scheme is* $k+1$*-th order accurate.*

Our objective is hence to design schemes verifying (8). An easy way to do this is the following. Observe that if f_h is a $k+1$-order approximation of the flux (such as $\mathrm{f}(\mathrm{w}_h)$ for example), than since $\mathrm{f} - \mathrm{f}_h = \mathcal{O}(h^{k+1})$, and since w respects (1) in a pointwise manner, then

$$\phi^T(\mathrm{w}_h) = \int_T \nabla\cdot(\mathrm{f}_h - \mathrm{f}) \; dx\, dy = \oint_{\partial T} (\mathrm{f}_h - \mathrm{f})\, dl = \mathcal{O}(h^{k+1}) \times \mathcal{O}(h) = \mathcal{O}(h^{k+2}) \tag{11}$$

Last equation shows that a straightforward way of constructing schemes respecting (8) is to set

$$\phi_i^T = \beta_i^T \phi_T \tag{12}$$

with β_i^T *uniformly bounded.* note that the same analysis holds on every sub-triangle, so that schemes defined by

$$\phi_i^{T_s} = \beta_i^{T_s} \phi_{T_s} \tag{13}$$

also verify (8). Schemes defined by (11) (or equivalently (12)) are often referred to as *Linearity Preserving*. Linearity preserving schemes verify *by construction* the accuracy condition (8).

2.2 Non-oscillatory schemes: a general procedure

This section is devoted to the discussion of a systematic way of constructing schemes verifying (8), and also having some monotonicity preservation properties. In the framework of residual distribution schemes, the monotonicity of the discrete solution is characterized by means of the theory of *positive coefficient discretizations*.

Suppose to compute the element residual by integrating exactly the quasi-linear form of (1). Denoting the flux Jacobian by $\vec{a} = \partial \mathrm{f}/\partial \mathrm{u}$, we suppose to compute the residual as

$$\phi^T(\mathrm{u}_h) = \int_T \nabla \cdot \mathrm{f}_h(\mathrm{u}_h)\, dx\, dy = \int_T \vec{a}(\mathrm{u}_h) \cdot \nabla \mathrm{u}_h\, dx\, dy \tag{14}$$

In this case, using (2) one is always able to express the local nodal residuals as[7,8]

$$\phi_i^T(\mathrm{u}_h) = \sum_{\substack{j \in T \\ j \neq i}} c_{ij}(u_i - u_j) \tag{15}$$

Last expression allows to manipulate (3) as follows

$$u_i^{n+1} = (1 - \omega_i \sum_{T \in \mathcal{D}_i} \sum_{\substack{j \in T \\ j \neq i}} c_{ij}) u_i^n + \omega_i \sum_{T \in \mathcal{D}_i} \sum_{\substack{j \in T \\ j \neq i}} c_{ij} u_j^n = (1 - \omega_i \sum_{\substack{j \in \mathcal{D}_i \\ j \neq i}} \widetilde{c}_{ij}) u_i^n + \omega_i \sum_{\substack{j \in \mathcal{D}_i \\ j \neq i}} \widetilde{c}_{ij} u_j^n \tag{16}$$

having introduced the $\widetilde{c}_{ij}$ coefficients

$$\widetilde{c}_{ij} = \sum_{T \in \mathcal{D}_i \bigcap \mathcal{D}_j} c_{ij} \tag{17}$$

A scheme is said to be *positive if all the* $\widetilde{c}_{ij}$ *are positive, and if*

$$\omega_i \leq \frac{1}{\sum\limits_{\substack{j \in \mathcal{D}_i \\ j \neq i}} \widetilde{c}_{ij}} \tag{18}$$

A positive scheme respects the local discrete maximum principle

$$\min_{j \in \mathcal{D}_i} \mathrm{u}_j^n \leq \mathrm{u}_i^{n+1} \leq \max_{j \in \mathcal{D}_i} \mathrm{u}_j^n \tag{19}$$

Unfortunately, *due to an extension of Godunov's theorem to residual distribution*[9,10]*, linear positive schemes cannot respect the accuracy condition of proposition 8*, where a linear scheme is one for which (for a linear problem) the c_{ij}s are independent on the solution.

In order to combine the two properties (monotonicity and $k+1$-th order of accuracy at steady state), the following nonlinear construction is proposed[1]. Suppose to have a positive scheme, and denote by $\{\phi_i^{\mathcal{P}}\}_{i\in T}$ the corresponding local nodal residuals. The $\phi_i^{\mathcal{P}}$s do not respect (8), hence the scheme cannot yield optimal accuracy. In practice, such schemes are first order accurate. However, we can still use the sign of the $\phi_i^{\mathcal{P}}$s as a reference to construct a monotone scheme. In particular, consider the scheme defined by $\phi_i^T = 0$ if $\phi^T = 0$ or $\phi_i^{\mathcal{P}} = 0$, otherwise $\phi_i^T = \beta_i^T \phi^T$ with

$$\begin{cases} \sum\limits_{j\in T} \beta_j^T & = 1 & \text{(consistency)} \\ \beta_i^T \beta_i^{\mathcal{P}} & \geq 0 & \text{(monotonicity)} \\ |\beta_i^T| & < C & \text{(linearity preservation)} \end{cases} \tag{20}$$

where $\beta_i^{\mathcal{P}} = \phi_i^{\mathcal{P}}/\phi^T$. Such a scheme respects condition (8) by construction, and it is positive since one easily shows that[1,3]

$$\phi_i^T = \alpha_i \phi_i^{\mathcal{P}}, \qquad \alpha_i \geq 0 \tag{21}$$

The remaining task is to find a (necessarily) nonlinear mapping $\{\beta_i^{\mathcal{P}}\}_{i\in T} \mapsto \{\beta_i^T\}_{i\in T}$ respecting (20). Examples of such mappings are given in[1]. The most successful of all is the PSI mapping

$$\beta_i^T = \frac{\max(0,\, \beta_i^{\mathcal{P}})}{\sum\limits_{j\in T} \max(0,\, \beta_j^{\mathcal{P}})} \tag{22}$$

Schemes obtained using this procedure are referred to as limited schemes.

2.3 Upwinding, stability and convergence

Up to now we have given design criteria allowing to construct schemes which satisfy by construction the $k+1$-th order of accuracy condition (8), and which also have a monotonicity preserving character. This, provided that we are able to find positive lower order splittings allowing to apply the procedure of §2.2. The generality of this procedure, however, hides a catch. The problem has a subtle algebraic nature, even though can also be analyzed using arguments related to the L^2-stability and/or dissipative character of the nonlinear schemes.

The P^1 case has been discussed in some detail in[3]. The main problem is to understand under which conditions the steady problem

$$\sum_{T\in\mathcal{D}_i} \beta_i^T \phi^T(\mathrm{u}_h) = 0 \qquad \forall\, i \in \mathcal{T}_h \tag{23}$$

admits a solution, and a unique one. The analysis made in[3] shows that:

1. upwind schemes admit a solution, which is unique;

2. for dissipative schemes the iterative procedure (3) is convergent;

The first result is related to the structure of the matrices obtained when linearizing (23). A more general stability statement had already been proved in[1], showing that upwind schemes verify an *inf-sup* type stability criterion when recast in the appropriate variational form. Note that by upwind schemes, we refer to discretizations for which

$$\phi_i^T = \sum_{\substack{T_s \subset T \\ i \in T_s}} \phi_i^{T_s} \quad \text{with} \qquad \phi_i^{T_s} = 0 \quad \text{if} \quad \vec{a} \cdot \vec{n}_i \leq 0 \text{ in } T_s \tag{24}$$

with $\vec{n}_i$ the inward normal to the edge of T_s facing node i, as in figure 2.

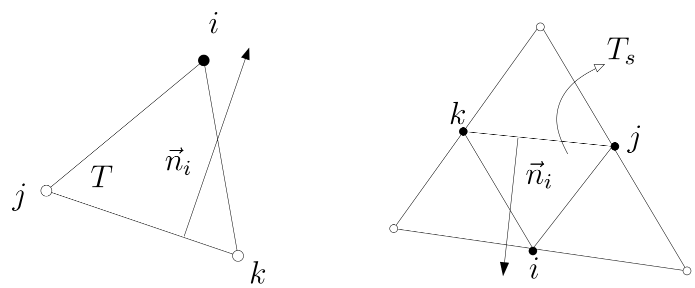


Figure 2: Nodal normal $\vec{n}_i$, P^1 and P^2 case

The second result is less general, in the sense that all dissipative schemes are known to respect some *inf-sup* type stability criterion. However, it allows to devise a strategy to stabilize limited nonlinear residual distribution schemes which are not upwind. Some information concerning this aspect can be found in[3,4], while a more detailed analysis will be reported elsewhere. Here we limit ourselves to observe that *non-upwind nonlinear limited schemes, which generally lead to ill-posed algebraic systems of type* (23), *can be stabilized by adding a properly scaled SUPG-type dissipation*:

$$\phi_i^T(\mathrm{u}_h) = \beta_i^T \phi^T(\mathrm{u}_h) + h\Theta(\mathrm{u}_h) \int_T \vec{a}(\mathrm{u}_h) \cdot \nabla\psi_i \nabla \cdot \mathrm{f}_h(\mathrm{u}_h)\, dx\, dy \tag{25}$$

where $\Theta(\mathrm{u}_h)$ is designed to reduce to $\Theta(\mathrm{u}_h) = \mathcal{O}(h^p)$, $p > 0$, across singularities, while $\Theta(\mathrm{u}_h) = \mathcal{O}(1)$ in smooth regions. This property leads to a discretization which retains the monotonicity preserving character of the limited scheme, up to $\mathcal{O}(h^p)$, reason for which they have been named essentially non-oscillatory in[3]. Examples of such schemes are given in the following. Theoretical details will be given in a forthcoming paper, while we refer once more to[3,4] for the analysis of the P^1 case and for a preliminary overview of the general P^k case.

2.4 Examples of schemes

In the following we will present some results obtained with two basic discretizations. The first is the one proposed in[1]. It is defined by local nodal residuals

$$\phi_i^T = \sum_{\substack{T_s \subset T \\ i \in T_s}} \beta_i^{T_s} \phi^{T_s}, \quad \beta_i^{T_s} = \frac{\max(0,\, \beta_i^{\mathrm{N}})}{\displaystyle\sum_{j \in T_s} \max(0,\, \beta_j^{\mathrm{N}})} \tag{26}$$

where on each T_s one has $\beta_j^{\mathrm{N}} = \phi_j^{\mathrm{N}} / \phi^{T_s}$ with the N scheme defined by

$$\phi_j^{\mathrm{N}} = k_j^+ (\mathrm{u}_j - \mathrm{u}_{\mathrm{in}}), \quad k_j = \frac{1}{2} \frac{\partial \mathrm{f}}{\partial \mathrm{u}} \cdot \vec{n}_j \tag{27}$$

As in[1], the state u_{in} is computed as

$$\mathrm{u}_{\mathrm{in}} = \left(\sum_{j \in T_s} k_j^- \right)^{-1} \sum_{j \in T_s} k_j^- \mathrm{u}_j \tag{28}$$

As remarked in[1] the N scheme is does not verify (4). However, it is positive, and the sign of its residuals constitutes a good reference for the construction of a nonlinear scheme. Depending on the type of polynomial approximation used to compute ϕ^{T_s}, in the following we refer to this limited variant of the N scheme as to the LN-P^k scheme.

Here we also show some preliminary results obtained by limiting the centered Lax-Frederich's (LF) like scheme defined by

$$\phi_i^{\mathrm{LF}} = \frac{1}{M} (\phi^T + \alpha \sum_{j \in T} (\mathrm{u}_i - \mathrm{u}_j)), \quad M = \frac{(k+1)(k+2)}{2} \tag{29}$$

where M is the number of nodes belonging to the generic P^k element, while the dissipation coefficient $\alpha > 0$ is computed to be big enough to guarantee the positivity of the scheme[4,3]. The scheme obtained by applying the limiter (22) to the LF scheme is referred to as the LLF-P^k scheme. The scheme obtained by adding to the LLF-P^k scheme the stabilization term (25) is instead referred to as the LLFs-P^k scheme.

3 RESULTS

3.1 Scalar advection

On the spatial domain $[-1, 1] \times [0, 1]$, we consider the scalar problem obtained with $\mathrm{f} = (y, -x)\mathrm{u}$, corresponding to scalar rotational advection. On the inlet boundary $[-1, 0]$ we set the inflow condition $\mathrm{u} = \sin(10x)$. Results are shown with the LN-P^k and LLFs-P^k schemes, for $k = 1, 2$.

In particular, contour plots of the solutions obtained with the second and third order schemes are reported on figures 3 and 4, while the outlet solutions are compared in figure 5. Note that all the computations have been run on the same number of degrees of freedom. The gain in accuracy brought by the higher order polynomial representation is evident.

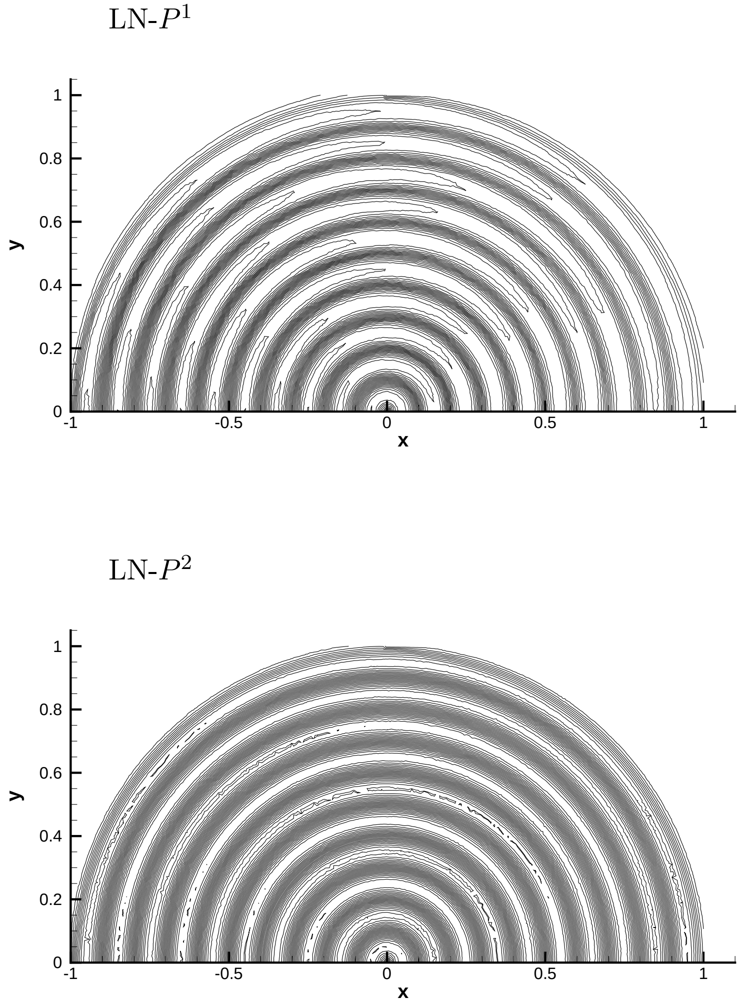
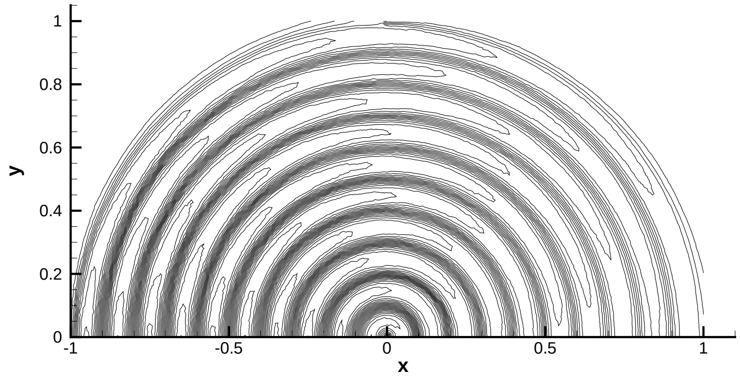


Figure 3: Rotation of the smooth profile: $u_{\text{in}} = \sin(10x)$. Top: limited N scheme, P^1 approximation (LN-P^1). Bottom: limited N scheme, P^2 approximation (LN-P^2). Computations run on the same number of degrees of freedom. Reference mesh size $h = 1/80$

LLF-P^1



LLF-P^2

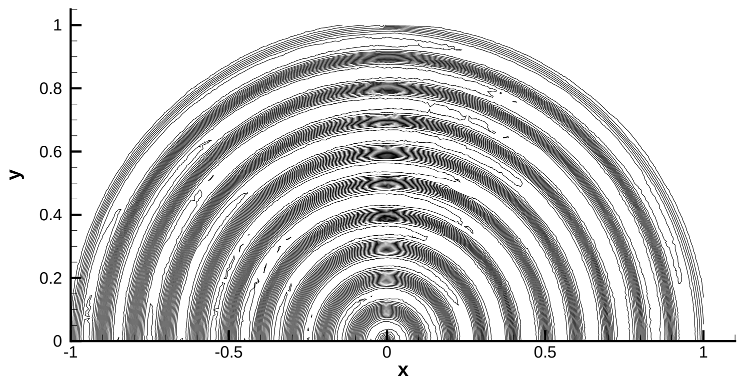


Figure 4: Rotation of the smooth profile: $u_{\text{in}} = \sin(10x)$. Top: limited and stabilized LF scheme, P^1 approximation (LLFs-P^1). Bottom: limited and stabilized LF scheme, P^2 approximation (LLFs-P^2). Computations run on the same number of degrees of freedom. Reference mesh size $h = 1/80$

3.2 Cauchy-Riemann equations

We consider now the steady Cauchy-Riemann equations

$$(A - x\,\mathrm{Id})\frac{\partial \mathrm{u}}{\partial x} + (B - y\,\mathrm{Id})\frac{\partial \mathrm{u}}{\partial y} = 0 \tag{30}$$

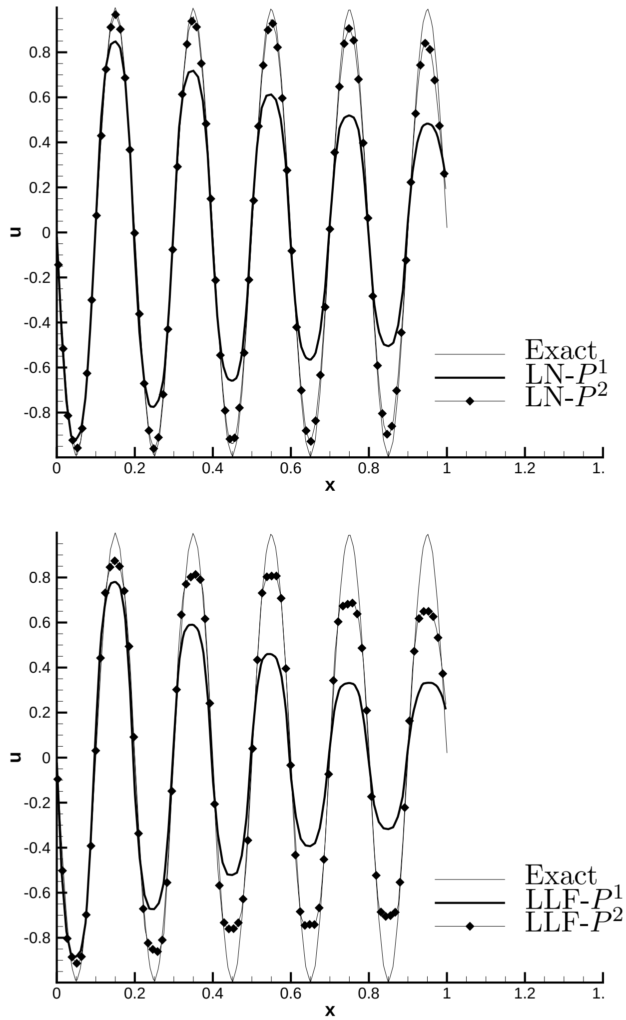


Figure 5: Rotation of the smooth profile: $u_{\text{in}} = \sin(10x)$. Computed outlet profile. Left: LN schemes. Right: LLFs schemes. All computations run on the same number of degrees of freedom. Reference mesh size $h = 1/80$

with $\mathrm{u} = (u, v)^t$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{31}$$

The problem considered is the same of[8] to which we refer for details. The final solution consists of 4 intermediate states separated by 2 shocks and by a smooth continuous region. In order to show the influence of the stabilization term (25), we report in figure 6 the contours of the solutions obtained with the LLF-P^3 and LLFs-P^3 schemes. While the shocks are kept quite monotone by the extra stabilization, thanks to the definition of

$\Theta(\mathrm{u}_h)$ given in[3,4] (cf. equation (25)), the improved smoothness of the solution in the regular region shows the important effect of the stabilization.

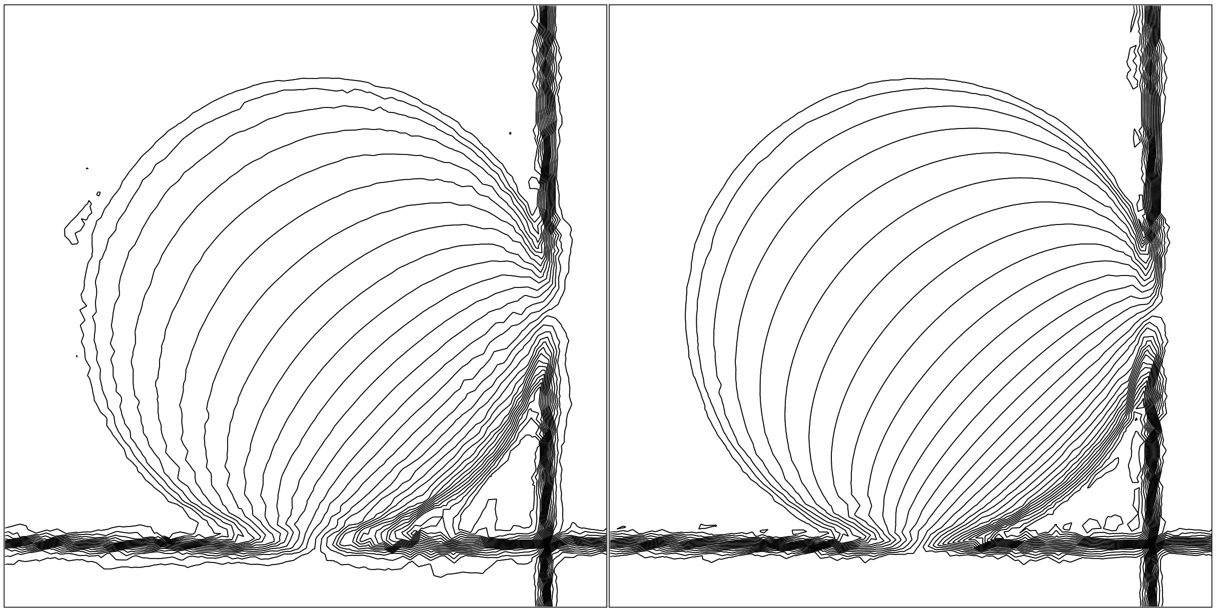


Figure 6: Cauchy-Riemann problem of[8]. LLF-P^3 (left) and LLFs-P^3 (right) scheme. u-variable contours. Zoom of the smooth region.

3.3 Euler equations

Lastly we present the results obtained for the interaction of two supersonic jets of perfect gas. The computations have been run with the N, LN-P^1, and LN-P^2 schemes on meshes containing the same number of degrees of freedom. The results, in terms of contours of the density, are displayed on figure 7.

The preservation of the monotonicity of the solution as well as the improved resolution of the flow features when passing from first, to second, to third order of accuracy are clear.

4 CONCLUSION

In this paper we have discussed the general construction of non-oscillatory residual distribution schemes of arbitrary accuracy, for the solution of conservation laws on unstructured triangulations.

We have given a formal condition guaranteeing that the discrete error of the schemes is of $\mathcal{O}(h^{k+1})$, and illustrated a possible construction guaranteeing the respect of this condition, while also ensuring a monotonicity preserving character of the discretization.

The preliminary results shown confirm our theoretical expectations. Thorough theoretical investigation of the approach and application to more complex cases are on-going

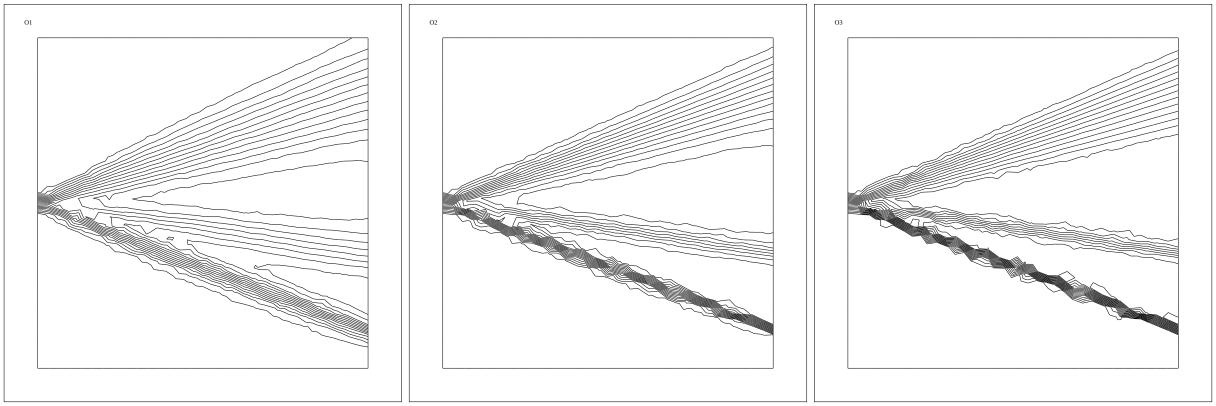


Figure 7: Interaction of two supersonic jets: density contours. From left to right: N scheme, LN-P^1 scheme, and LN-P^2 scheme. Computations using the same number of degrees of freedom.

and will be reported elsewhere.

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