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On Cloitre's hiccup sequences

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Abstract

In 2003, Benoit Cloitre entered a family of sequences in the OEIS that we call *hiccup* sequences. We collect the various claims, observations, and proofs of properties of these sequences that have been entered in the OEIS over the years, and present a unified approach, inspired by a remarkable theorem of Bosma, Dekking, and Steiner.

Keywords Hiccup sequence · Morphic sequence · Beatty sequence · Continued fraction

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Aronson's sequence

$$1, 4, 11, 16, 24, 29, 33, 35, 39, 45, 47, 51, 56, 58, 62, 64, \ldots,$$

is entry A005224 in the On-Line Encyclopedia of Integer Sequences. It comes from the self-referential sentence

T is The firsT, fourTh, elevenTh, sixTeenTh, ... letter in this sentence,

which keeps track of the positions of the T's. Spaces and punctuation are ignored. In the English language, ordinal numbers, with the notable exception of second, contain enough T's to keep the sentence going on forever. Hofstadter [19] attributed this sentence to the British clinical pharmacologist Jeffrey Aronson.

The words in Aronson's sentence generate numbers that generate words. It is a Munchausen manner to produce an integer sequence, and there are many different ways to do this. Cloitre et al. [11] considered various methods to generate self-referential sequences. We are interested in one specific type. Twenty years ago, Benoit Cloitre

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entered several sequences of this type in the OEIS. One of these is A086398, and its first entries are

$$1, 5, 7, 9, 11, 15, 17, 21, 23, 27, 29, 33, 35, 37, 39, 43, 45, 49, 51, 53, \ldots . \quad (1)$$

where $a(n) = a(n-1) + 4$ if $n-1$ is already in the sequence, and otherwise $a(n) = a(n-1) + 2$, starting from $a(1) = 1$. Actually, the definition in the OEIS is slightly different from ours, but it is not hard to check that it is equivalent. The differences $a(n) - a(n-1)$ take two values that appear in some sort of rhythm, just like bouts of hiccups [4, p. 96], and that is why we call them *hiccup sequences*. We have collected many hiccup sequences from the OEIS in Table 1 below.

If we vary the definition of A086398 a bit and put $a(n) = a(n-1) + 4$ if n is in the sequence and $a(n) = a(n-1) + 2$ otherwise, then we get A080903, which has a lot of overlap with A086398

$$1, 3, 7, 9, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 39, 41, 45, 47, 51, 53, \ldots . \quad (2)$$

This overlap of the two sequences will be cleared up in Sect. 3. Another hiccup sequence in the OEIS is A064437. Robert Israel empirically found a closed formula for this sequence:

$$a(n) = \left\lceil \left(1 + \sqrt{2}\right)(n-1) + \frac{1}{2+\sqrt{2}} \right\rceil \quad (3)$$

and checked its validity for the first hundred entries. The reader may recognize that this is a non-homogeneous Beatty sequence [16] with slope $1+\sqrt{2}$ and offset $\frac{1}{2+\sqrt{2}}$. Similar formulas can be found on the OEIS for other hiccup sequences.

It is possible to define hiccup sequences that are not increasing, or have first differences that take more than two values. There are plenty of such hiccup sequences in the OEIS as well. For example, the hiccup sequence A080782 produces a permutation of the natural numbers. Other examples are A079354, A080458, A080900, and A111202. A hiccup sequence with three gaps is A080574. These are all very interesting sequences, but we do not consider them in this paper. We only consider hiccup sequences that are strictly increasing.

In a recent paper, Cloitre [10] has revisited his sequences and also considers non-decreasing hiccup sequences.

Our paper is organized as follows. In Sect. 1, we define the (k, x, y, z)-hiccup sequences and give an overview of all the results and claims that were entered into the OEIS. We prove that hiccup sequences are morphic in Sect. 2. We use this result in Sect. 3 to retrieve Cloitre's recent theorem that hiccup sequences are Beatty sequences if $|y - z| = 1$ under certain further restrictions on the parameters. It turns out that $(0, x, 1, z)$-sequences are examples of morphic non-automatic sequences, which feature in a recent overview article by Allouche et al. [3]. We move beyond Beatty sequences and settle a conjecture of Kimberling, combining the automatic theorem prover `Walnut` [21] with results of Ollinger and others on Dumont–Thomas numeration [9, 23]. In Sect. 4 we generalize a conjecture of Bosma et al. [8] on a connection between continued fractions and Beatty sequences. We settle a specific instance of that generalized conjecture.

1 Definition, literature, and OEIS insights

We call an increasing sequence a (j, x, y, z)-*hiccup sequence* if it starts at x, and from then on, you add y if the number $n - j$ already showed up in the sequence, and z if it did not.

Definition 1 ((j, x, y, z)-hiccup sequence) Let $a(n)$ be a strictly increasing sequence of integers, and let $j, x \in \mathbb{Z}_{\geq 0}$, and $y, z \in \mathbb{Z}_{\geq 1}$ with $y \neq z$. We say that $a(n)$ is the (j, x, y, z)-*hiccup sequence* if it satisfies:

$$a(1) = x,$$
$$a(n) = \begin{cases} a(n-1) + y, & \text{if } n - j \in \{a(k)\}_{k<n}, \\ a(n-1) + z, & \text{otherwise,} \end{cases} \qquad \text{for } n \geq 2.$$

In a recent paper, Cloitre [10] considers $(0, x, y, z)$-hiccup sequences in which y or z can be zero. Our results partially overlap Cloitre's, but are obtained by a different approach.

A hiccup sequence is trivial if $x = z = 1$. Consider, for example, the $(0, 1, y, 1)$-hiccup sequence for an arbitrary $y > 1$. What is $a(2)$? At index 2, the only number already in the sequence is $a(1) = 1$ and the next number is $a(3) = 3$, etc. The y plays no role. This problem does not arise if $x > 1$, as is illustrated by the first entry in Table 1. It is the only hiccup sequence with $z = 1$ in the OEIS.

The literature on hiccup sequences is limited. Apart from Cloitre's recent paper, the only study is by Bosma et al. [8]. It concerns the $(1, 1, 3, 2)$-sequence. Its first few entries are

$$1, 4, 6, 8, 11, 13, 16, 18, 21, 23, 25, 28, 30, 33, 35, 37, 40, 42, 45, 47, \ldots$$

Bosma, Dekking, and Steiner proved that there are different ways to characterize this sequence, that at first sight seem to be unrelated. The aim of our paper is to generalize their remarkable result to hiccup sequences that are strictly increasing.

Theorem 2 (Bosma–Dekking–Steiner). *The sequence A086377 can be characterized in the following equivalent ways:*

(1) Hiccup sequence – *Defined recursively by*

$$a(n) = \begin{cases} a(n-1) + 2, & \text{if } n - 1 \notin \{a(k)\}_{k<n}, \\ a(n-1) + 3, & \text{if } n - 1 \in \{a(k)\}_{k<n}, \end{cases}$$

with initial condition $a(1) = 1$.

Table 1 Table of (j, x, y, z)-hiccup sequences that are recognized as such in the OEIS. Almost all parameters are $j = 0$, $x = 1$, and $|y - z| = 1$

OEIS	j	x	y	z	OEIS	j	x	y	z
A004956	0	2	2	1	A081841	0	0	3	2
A007066	0	1	2	3	A081840	0	0	3	4
A045412	0	3	1	3	A081842	0	0	4	3
A064437	0	1	3	2	A081839	0	0	4	5
A080578	0	1	1	3	A081843	0	0	5	4
A080579	0	1	1	4	A080578	0	1	1	3
A080580	0	1	2	4	A080579	0	1	1	4
A080590	0	1	3	4	A007066	0	1	2	3
A080600	0	4	4	3	A080580	0	1	2	4
A080652	0	2	3	2	A064437	0	1	3	2
A080667	0	3	4	3	A080590	0	1	3	4
A080903	0	1	4	2	A080903	0	1	4	2
A081834	0	1	4	3	A081834	0	1	4	3
A081835	0	1	5	4	A081835	0	1	5	4
A081839	0	0	4	5	A004956	0	2	2	1
A081840	0	0	3	4	A080652	0	2	3	2
A081841	0	0	3	2	A045412	0	3	1	3
A081842	0	0	4	3	A080667	0	3	4	3
A081843	0	0	5	4	A080600	0	4	4	3
A086377	1	1	3	2	A086377	1	1	3	2
A086398	1	1	4	2	A086398	1	1	4	2

The left-hand panel is ordered according to the OEIS number. The right-hand panel is lexicographic in the parameters. All sequences with the exception of the first two were entered by Benoit Cloitre over the course of 2003

(2) Morphic characterization – *The positions of* 1*'s in the fixed point of the morphism*

$$0 \mapsto 10, 1 \quad \mapsto 100.$$

(3) Beatty sequence representation –

$$a(n) = \left\lfloor (1+\sqrt{2})n - \frac{\sqrt{2}}{2} \right\rfloor.$$

(4) Iterated function sequence –

$$a(n) = \left\lfloor r_n + \frac{1}{2} \right\rfloor,$$

$$\textit{where } r_1 = \frac{4}{\pi} \textit{ and}$$

$$r_{n+1} = f_n(r_n), \quad \textit{with } f_n(x) = \frac{n^2}{x + 1 - 2n}.$$

We will refer to Theorem 2 as the *BDS theorem*. The sections in our paper are numbered according to the characterizations of this theorem. In each section we extend the BDS theorem to more general hiccup sequences.

We highlight some of the scattered comments on hiccup sequences that can be found on the OEIS. The (0, 1, 2, 3)-hiccup sequence is entry A007066 with first few numbers

$$1, 4, 7, 9, 12, 15, 17, 20, 22, 25, 28, 30, 33, 36, 38, 41, 43, \ldots$$

It is entered in the OEIS as a Beatty sequence

$$a(n) = \left\lceil (n-1) \cdot \phi^2 + 1 \right\rceil,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean. Except for its initial entry, it is equal to the upper Wythoff sequence A001950 plus two. It also corresponds to the positions of 1 of the fixed point of $0 \mapsto 010,\ 1 \mapsto 10$ that starts with 1. These facts follow from Fraenkel's characterizations of the upper Wythoff sequence [17].

The (1, 1, 4, 2)-hiccup sequence A086398 starts out with

$$1, 5, 7, 9, 11, 15, 17, 21, 23, 27, 29, 33, 35, 37, 39, 43, 45, 49, \ldots.$$

Kimberling asked if it corresponds to the positions of 1 in the unique fixed point of $0 \mapsto 10,\ 1 \mapsto 1000$. It is not so hard to see that this is true. The fixed point

$$\omega = 10001010101000101000 10\cdots$$

satisfies

$$\omega = \sigma(\omega) = \sigma(1)\sigma(0)\sigma(0)\sigma(0)\sigma(1)\sigma(0)\cdots.$$

The nth 1 in the sequence corresponds to the nth $\sigma(d)$ for $d = 0$ or $d = 1$ in $\sigma(\omega)$. It is 4 ahead of the previous 1 exactly if the $(n-1)$th digit of ω is a 1. In other words, it is 4 ahead if and only if $n - 1$ is in the sequence.

Kimberling also asked if $-1 < n\left(1 + \sqrt{3}\right) - a(n) < 4$, which we will confirm in Sect. 3. There is no Beatty sequence readily available, unlike in the previous examples. Since all numbers in the sequence are odd, we can convert it by $a(n) \mapsto b(n) = (a(n) + 1)/2$, which gives A026363. Here, Kimberling asks if $-1 < n\left(1 + \sqrt{3}\right)/2 - b(n) < 2$, which is a slightly weaker bound.

The (0, 1, 2, 4)-hiccup sequence is entry A080580 in the OEIS.

$$1, 5, 9, 13, 15, 19, 23, 27, 29, 33, 37, 41, 43, 47, 49, 53, 57, 61, \ldots$$

Joe Slater proves in a comment on this sequence that it coincides with the positions of 1 in the fixed point of $0 \mapsto 0010,\ 1 \mapsto 10$ that starts with 1. By the same argument, the nth 1 in this sequence is 4 ahead of the previous 1 exactly if n is not in the sequence. Otherwise, it is ahead by 2. We will call this *Slater's construction* in this paper.

If we change the x parameter from 2 to 1 we get the (0, 1, 1, 4)-hiccup sequence, which is the previous entry A080579

$$1, 5, 9, 13, 14, 18, 22, 26, 27, 31, 35, 39, 40, 41, 45, 49, 53, 54, \ldots$$

in the OEIS. Slater's construction works for the fixed point of $0 \mapsto 0001,\ 1 \mapsto 1$ that starts with 10. If we change z from 4 to 3 we get entry A080578, the (0, 1, 1, 3)-hiccup sequence, which by Slater's construction is the fixed point of $0 \mapsto 001,\ 1 \mapsto 1$ that starts with 10, see also [14]. Interestingly, this sequence is morphic but not automatic, as shown in example 16 of [3].

Most hiccup sequences are stand alone, but some have been linked to other seemingly unrelated sequences. In particular, A040412 has been linked to bottom-up search and A080578 has been linked to meta-Fibonacci sequences. More details on such connections can be found in Cloitre's recent paper [10].

2 Morphisms for hiccup sequences

An increasing sequence $a(n)$ corresponds to a binary sequence $b(n)$, such that $b(n) = 1$ if and only if $n = a(k)$ for some k. If we interpret the sequence as a set $A = \{a(n)\} \subset \mathbb{N}$, then the *characteristic sequence* is equal to 1 at position a if $a \in A$ and it is 0 otherwise. We will show that the characteristic sequence of a hiccup sequence is morphic. A *morphism* or *substitution* on an alphabet is a formal rule that replaces each letter of the alphabet with a finite word over that same alphabet.

Definition 3 Let Σ and Δ be finite alphabets. A sequence $(x_n)_{n \geq 1} \in \Delta^{\mathbb{N}}$ is called a *morphic sequence*, see [2, Chap. 7], if there exist:

- a morphism $\phi : \Sigma \to \Sigma^*$ such that σ has a fixed point $w = \phi^{\infty}(a)$ for some $a \in \Sigma$, and
- a coding $\pi : \Sigma \to \Delta$ (i.e., a letter-to-letter morphism),

such that

$$x_n = \pi(w_n) \quad \text{for all } n \geq 1.$$

We say that an increasing sequence $a(n)$ is morphic if its characteristic sequence is morphic.

Lemma 4 *A* $(0, 1, y, z)$*-hiccup sequence is morphic.*

Proof In the previous section we encountered Slater's construction, which shows that the $(0, 1, 2, 4)$-hiccup sequence is morphic. This construction can be extended. We first assume that $z > y$ and define the morphism

$$\phi(1) = 10^{y-1},\ \phi(0) = 0^{z-y}10^{y-1}. \tag{4}$$

Let (w_i) be the fixed point with $w_i \in \{0, 1\}$ such that $w_1 w_2 = 10$. Note that this second letter $w_2 = 0$ is needed in case $y = 1$. Since the sequence is a fixed point and each code word contains a single 1, the nth 1 is in $\phi(w_n)$. It is preceded by $y-1$ zeros if and only if the first letter of $\phi(w_n)$ is 1. Otherwise, it is preceded by $z-1$ zeros. Therefore, if $a(n)$ is the position of the nth zero, then $a(n) - a(n-1) = y$ if n is in the sequence and $a(n) - a(n-1) = z$ if it is not. This is the desired hiccup sequence, which is morphic.

Now suppose that $z < y$, and remember that we require $z > 1$. We add a letter b to the alphabet, and define a morphism on $\{b, 0, 1\}$ by

$$b \mapsto b0^{z-1},\ 0 \mapsto 10^{z-1},\ 1 \mapsto 0^{y-z}10^{z-1}. \tag{5}$$

Let (w_i) be the unique fixed point with initial letters $b0$. If we count b as a 1, i.e., apply a coding $b \mapsto 1$, then every code word contains a single 1. By the same argument as in the case $z > y$, the nth 1 is preceded by $y-1$ zeros if $w_n = 1$ and by $z-1$ zeros if $w_n = 0$. □

The additional letter b in the proof of the lemma, which counts as a special 1 so to speak, is needed to get the right initial condition $a(1) = 1$. The same trick works if $a(1) > 1$, as explained in the following lemma.

Lemma 5 *A $(0, x, y, z)$-hiccup sequence with $x > 1$ is morphic.*

Proof We add a letter b to the alphabet and consider the fixed point that starts with b. If $y < z$ we define the morphism

$$b \mapsto b0^{x-2}10^{y-1},\ 0 \mapsto 0^{z-y}10^{y-1},\ 1 \mapsto 10^{y-1}. \tag{6}$$

Note that b can be removed from this morphism if $x = z - y + 1$, when the fixed point that starts with 0 produces the hiccup sequence. If $y > z$ we define

$$b \mapsto b0^{x-2}10^{z-1},\ 0 \mapsto 10^{z-1},\ 1 \mapsto 0^{y-z}10^{z-1}. \tag{7}$$

Now code $b \mapsto 0$. □

Note that if we remove b from the morphism in Eq. 6, this gives the morphism in Eq. 4. Similarly, if we replace $b \mapsto b0^{x-2}10^{z-1}$ by $b \mapsto b0^{z-1}$ in Eq. 7, i.e., remove the factor $0^{x-2}1$ from the image of b, then we get Eq. 5. With these morphisms for $j = 0$ in hand, we can find sequences in the OEIS that have not yet been recognized as hiccup sequences. For instance, if $x = 2$, $y = 2$, $z = 3$ then we get A026356. This sequence is equal to A007066, except for the first entry, which is the hiccup sequence with $x = 1, y = 2, z = 3$. The morphism

$$0 \mapsto 010, \; 1 \mapsto 10 \tag{8}$$

generates both of these sequences. A026356 is the fixed point starting with 0, and A007066 is the one starting with 1, see [15, Exercise 6.1.25]. That explains why the two sequences are the same, except for their initial entries.

The morphisms in these lemmas are not unique, there are other morphisms that code the same sequence. For instance, our morphism for the (0, 2, 4, 2)-hiccup sequence in Eq. 7 is given by $b \mapsto b10, \; 0 \mapsto 10, \; 1 \mapsto 0010$. This morphism can be simplified to $0 \mapsto 01, \; 1 \mapsto 0001$, which by Slater's construction produces the same fixed point without using the special letter b. The (0, 2, 4, 2)-hiccup sequence is entry A284753 in the OEIS. It is not yet listed as a hiccup sequence.

There is a degree of freedom in the morphism, by a cyclic permutation of the image words of 0 and 1. If we replace the morphism in Eq. 6 by $b \mapsto b0^{x-2}10^{y-2}, \; 0 \mapsto 0^{z-y+1}10^{y-2}, \; 1 \mapsto 010^{y-2}$, assuming that $y \geq 2$, then we get the same fixed point. Similarly, we can replace the morphism in Eq. 7 by $b \mapsto b0^{x-2}10^{z-2}, \; 0 \mapsto 010^{z-2}, \; 1 \mapsto 0^{y-z+1}10^{z-2}$, if $z \geq 2$. This is exactly how we got that morphism $0 \mapsto 01, \; 1 \mapsto 0001$ above. If the parameters allow it, it is possible to get rid of the special letter b by iterating the cyclic permutation. A sequence is *purely morphic* if there is no need for the coding π in Definition 3.

Lemma 6 *A $(0, x, y, z)$-hiccup sequence is purely morphic if $z - y + 1 \leq x \leq z$.*

Proof A cyclic permutation of the code words for 0 and 1 gives the morphism

$$0 \mapsto 0^{x-1}10^{z-x}, \; 1 \mapsto 0^{x+y-z-1}10^{z-x}. \tag{9}$$

□

Lemma 7 *A $(1, x, y, z)$-hiccup sequence with $x \geq 1$ is morphic. It is purely morphic if $x = 1$ and $y > 1$.*

Proof First suppose $x > 1$. We add the letter b and we define

$$b \mapsto b0^{x-2}10^{z-1}, \; 0 \mapsto 10^{z-1}, \; 1 \mapsto 10^{y-1}.$$

Let (w_i) be the unique fixed point that starts with b, which is coded to 0. The nth 1 in the sequence is in $\sigma(w_n)$. It is preceded by $y - 1$ zeros if $w_{n-1} = 1$, i.e., if $n - 1$ is in the sequence, and otherwise by $z - 1$ zeros.

If $x = 1$ and $y = 1$, then we get the trivial sequence $a(n) = n$. If $x = 1$ and $y > 1$ then there is no need for the letter b, and we can take $0 \mapsto 10^{z-1}, \; 1 \mapsto 10^{y-1}$. □

Some of these morphisms correspond to entries in the OEIS that have not been recognized as hiccup sequences. For instance, the (1, 1, 2, 1)-hiccup sequence is the fixed point of the Fibonacci morphism $0 \mapsto 1, \; 1 \mapsto 10$, which gives the lower

Wythoff sequence A000201. The $(1, 1, 3, 1)$-hiccup sequence is equal to A003156, and the $(1, 1, 2, 3)$-hiccup sequence is A026352. The result is not sharp. There are more choices of parameters for which hiccup sequences are purely morphic. For instance, the morphism $0 \mapsto 010,\ 1 \mapsto 01$, which is the reversal of the morphism in 8, generates the $(1, 2, 2, 3)$-hiccup sequence by Slater's construction. It is the upper Wythoff sequence A001950.

Lemma 8 *Let $b(n)$ be the $(0, z-1, y, z)$-hiccup sequence for $y, z > 1$, and let*

$$a(n+1) = b(n) + 1,\ a(1) = 0.$$

Then $a(n)$ is the $(0, 0, y, z)$-hiccup sequence. In particular, $a(n+1)$ is morphic.

For instance, A081841 is equal to A064437 plus one.

Proof Since $y, z > 1$ we have that $b(n+1) \geq b(n)+2$ for all n and $b(1) = z-1 \geq 1$. It follows that $b(n) > n$ for $n > 1$. By definition, $a(2) = b(1) + 1 = z$, which is the correct value for the $(0, 0, y, z)$-hiccup sequence. For $n > 1$ we have

$$a(n+1) = b(n) + 1 = b(n-1) + y + 1 = a(n) + y$$

if $n \in \{b(k)\}_{k<n}$, and otherwise

$$a(n+1) = b(n) + 1 = b(n-1) + z + 1 = a(n) + z.$$

Since $a(k) = b(k-1) + 1$ and $a(1) = 0$, it follows that $a(n+1) = a(n) + y$ if and only if $n+1 \in \{a(k)\}_{k<n+1}$, and it is $a(n+1) = a(n) + z$ otherwise.

The sequence $b(n)$ corresponds to the positions of 1 in a coded fixed point of a morphism. This morphism has an alphabet of two or three letters, depending on the parameters. To convert this to a morphism for $a(n+1) = b(n) + 1$, add a letter c and add the substitution $c \mapsto cd$ to the morphism, where d is the initial letter of the fixed point for $b(n)$. This is the morphism for $a(n+1)$. Add $c \mapsto 0$ to the coding. □

Lemma 9 *Let $a(n)$ is the (j, x, y, z)-hiccup sequence for $j > 0$ and let*

$$b(n) = a(n) + j.$$

Then $b(n)$ is the $(0, x+j, y, z)$-hiccup sequence.

There are two entries in Table 1 that have $j = 1$. Hiccup sequence A086377, which was considered in [8], is equal to A080652 minus one. The other entry A086398 is equal to A284753 minus one.

Proof By definition, $b(n) = a(n) + j = a(n-1) + y + j$ if $n - j \in \{a(k)\}_{k<n}$ and it is $a(n-1) + z + j$ if not. Now $n - j \in \{a(k)\}_{k<n}$ is equal to $n \in \{b(k)\}_{k<n}$, which defines the (j, z, y, z)-hiccup sequence. □

Theorem 10 *Hiccup sequences are morphic. A $(0, x, y, z)$-sequence is purely morphic if $z - y + 1 \leq x \leq z$ or if $x = 1$.*

Proof This is a consequence of the preceding lemmas. The sequence is purely morphic if $z-y+1 \leq x \leq z$ according to Lemma 6. If $x = 1$ and $y > z$, then $z-y+1 \leq x \leq z$. If $x = 1$ and $y < z$, then the sequence is purely morphic according to Eq. 4. □

This extends the second characterization in the BDS theorem. We conclude with some remarks on the morphisms that we encountered in this section. They are defined on the alphabet $\{b, 0, 1\}$ and the letter b only occurs as the initial letter of the fixed point. The *adjacency matrix* of a morphism registers the number of letters in the substitution words. For instance, the morphism $0 \mapsto 0100,\ 1 \mapsto 100$ for the $(0, 1, 3, 4)$-hiccup sequence has adjacency matrix $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$. For the morphisms that we found in this section, the restriction to $\{0, 1\}$ has adjacency matrix equal to

$$\begin{bmatrix} z-1 & y-1 \\ 1 & 1 \end{bmatrix}. \tag{10}$$

If an iterate of the adjacency matrix has all entries > 0, then the morphism is *primitive*. In particular, our morphisms are *primitive* if $y > 1$, when restricted to the alphabet $\{0, 1\}$. If $y = 1$, then the morphism is not primitive.

3 Beatty sequences and beyond

We move on to the third characterization in the BDS theorem, which connects hiccup sequences to Beatty sequences. This only applies under certain restrictions on the parameters. By the results of the previous section, in particular Lemma 9, we may restrict our attention to $(0, x, y, z)$-hiccup sequences with x, y, z in $\mathbb{Z}_{\geq 1}$. It can only be a Beatty sequence if $|y - z| = 1$.

The semigroup of *Sturmian morphisms*, see [6, p. 72], is generated by the three substitutions L, R, and E:

$$E : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases} \qquad L : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases} \qquad R : \begin{cases} 0 \mapsto 10 \\ 1 \mapsto 0 \end{cases}$$

For instance, the square R^2 produces the morphism $0 \mapsto 010,\ 1 \mapsto 10$ in Eq. 8 which has two fixed points. One corresponds to A026356 and the other corresponds to A007066. The square L^2 produces the reversal of R^2 given by $0 \mapsto 010,\ 1 \mapsto 01$. Its fixed point is the $(1, 2, 2, 3)$-hiccup sequence, which is better known as the upper Wythoff sequence A001950.

It follows from Theorem 10 that a $(0, x, y, z)$-hiccup sequence is purely morphic if $x \leq z$ and $|z - y| = 1$. We prove that the morphisms that generate these sequences are Sturmian.

Lemma 11 *A $(0, x, y, z)$-hiccup sequence is Sturmian if $x \leq z$ and $|z - y| = 1$.*

Proof We use an elegant criterion that is due to Tan and Wen. We say that a morphism τ satisfies the uv-condition if there exist words u and v such that $\tau(01) = u10v$ and $\tau(10) = u01v$, or $\tau(01) = u01v$ and $\tau(10) = u10v$. It follows from [26, Prop. 2.1] that a sequence is Sturmian if it is the fixed point of a morphism that satisfies the uv-condition.

We verify that this condition holds for the $(0, x, z + 1, z)$ and the $(0, x, z - 1, z)$-hiccup sequence. According to Eq. 9, the morphism $0 \mapsto 0^{x-1}10^{z-x},\ 1 \mapsto 0^x 10^{z-x}$ generates the $(0, x, z + 1, z)$-hiccup sequence. It satisfies the uv-condition for $u = 0^{x-1}$ and $v = 0^{z-1}10^{z-x}$.

There are two cases for the $(0, x, z - 1, z)$-hiccup sequence: $x = 1$ and $x > 1$. If $x = 1$, then according to Eq. 4 the morphism is $0 \mapsto 010^{z-2},\ 1 \mapsto 10^{z-2}$. In this case u is the empty word and $v = 0^{z-2}10^{z-2}$. Finally, if $x > 1$, then the morphism is $0 \mapsto 0^{x-1}10^{z-x},\ 1 \mapsto 0^{x-2}10^{z-x}$. It satisfies the uv-condition with $u = 0^{x-2}$ and $v = 0^{z-2}10^{z-x}$. □

There is a well-known correspondence between Sturmian substitutions and sequences

$$S_{\alpha,\beta}(n) = \lfloor \alpha(n + 1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor,$$

which are known as *Sturmian words* or *mechanical words*, see [6, p. 53]. Here $0 < \alpha < 1$ is irrational and β is chosen such that $0 < \alpha + \beta \leq 1$. Note that rounding up or down only makes a difference in the special case that $\alpha m + \beta$ is an integer for some k. The values of α and β can be determined from the following transformations

$$T_E(x, y) = (1 - x, 1 - y)\,,\ T_L(x, y) = \left(\frac{1 - x}{1 - 2x}, \frac{1 - y}{1 - 2x}\right),$$
$$T_R(x, y) = \left(\frac{1 - x}{1 - 2x}, \frac{2 - x - y}{1 - 2x}\right).$$

A composition of E, L, R corresponds to a composition of these transformations, and (α, β) is a fixed point of the resulting composite transformation, see [6, p. 73] and [8]. As an example, we compute α and β for the fixed point of the morphism $0 \mapsto 010,\ 1 \mapsto 10$. The morphism has adjacency matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

which has largest eigenvalue $\lambda = \frac{3+\sqrt{5}}{2}$. Now α is equal to the density of $1's$ in the fixed point of the morphism, and a minor calculation gives $\alpha = \frac{3-\sqrt{5}}{2}$. The fixed point (α, β) for $T_{R^2} = T_R T_R$ satisfies $\beta = 1 - \alpha$. This is one of the special cases in which rounding up or down makes a difference for the mechanical word. The rounded-up sequence corresponds to the fixed point beginning with 0, while the rounded-down one begins with 1.

Mechanical words can be converted to Beatty sequences [16]. The following result and its proof is almost identical to [8, Lemma 1].

Lemma 12 *Let* $0 < \alpha < 1$ *be irrational, and let* $(s_n)_{n\geq 1}$ *be given by the rounded-up mechanical sequence*

$$s_n = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil,$$

for some real number β *with* $0 < \alpha + \beta \leq 1$. *Then the rounded-down Beatty sequence*

$$\left\lfloor \frac{k}{\alpha} - \frac{\beta}{\alpha} \right\rfloor \text{ for } k = 1, 2, \ldots$$

corresponds to the sequence of positions of 1 *in* (s_n). *Similarly, a rounded-down mechanical sequence corresponds to a rounded-up Beatty sequence.*

Proof Since $\alpha + \beta \leq 1$ we have that $\frac{k}{\alpha} - \frac{\beta}{\alpha} \geq 1$. It follows that the elements of the Beatty sequence are positive integers.

$$\begin{aligned} \exists k \geq 1 : n = \left\lfloor \frac{k}{\alpha} - \frac{\beta}{\alpha} \right\rfloor &\iff \exists k \geq 1 : n \leq \frac{k}{\alpha} - \frac{\beta}{\alpha} < n + 1 \\ &\iff \exists k \geq 1 : n\alpha + \beta \leq k < (n+1)\alpha + \beta \\ &\iff \exists k \geq 1 : \lceil n\alpha + \beta \rceil = k \text{ and } \lceil (n+1)\alpha + \beta \rceil = k + 1 \\ &\text{because } \alpha < 1 \\ &\iff \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil = 1 \\ &\iff s_n = 1. \end{aligned}$$

The same argument gives that $n = \left\lceil \frac{k}{\alpha} - \frac{\beta}{\alpha} \right\rceil$ if and only if $\lfloor n\alpha + \beta \rfloor - \lfloor (n-1)\alpha + \beta \rfloor = 1$. □

For instance, A007066 is given by

$$\left\lceil \frac{k}{\alpha} - \frac{1-\alpha}{\alpha} \right\rceil$$

while A026356 is given by

$$\left\lfloor \frac{k}{\alpha} - \frac{1-\alpha}{\alpha} + 1 \right\rfloor,$$

for $\alpha = \frac{3-\sqrt{5}}{2}$. These lemmas lead to the following generalization of the third characterization of the BDS theorem, which was recently proved by Cloitre. He provides many more concrete expressions for hiccup sequences that are Beatty sequences.

Theorem 13 (Cloitre, [10]) *A hiccup sequence is a Beatty sequence if* $x \leq z$ *and* $y > 1$ *and* $|y - z| = 1$.

Out of the 21 hiccup sequences in Table 1, there are 15 that satisfy this condition. Out of the remaining 6 sequences, 3 have parameter $y = 1$, namely (0, 1, 1, 3), (0, 1, 1, 4),

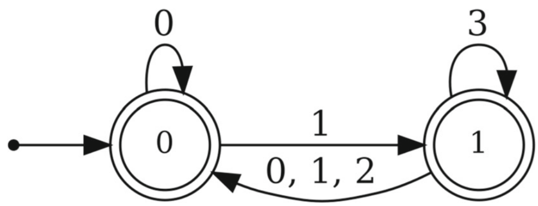

Fig. 1 The msd expressions that are accepted in the Dumont–Thomas numeration system for the morphism $0 \mapsto 01,\ 1 \mapsto 0001$ has digit set $\{0, 1, 2, 3\}$. This automaton has two accepting states. Transitions that are not included in the automaton are rejected. The expression that are accepted are of the form $\{0^*13^*\{0|1|2\}\}^*$, where $*$ is the Kleene star, which means that the expression can be repeated zero or more times

and (0, 3, 1, 3). The morphisms that generate these sequences are not primitive, as we saw in Eq. 10 above. As mentioned already, the (0, 1, 1, 3)-hiccup sequence A080578 is a morphic sequence that is not automatic. It is example 16 in [3]. The remaining 3 hiccup sequences are (0, 1, 2, 4), (0, 1, 4, 2), and (1, 1, 4, 2). We already pointed out the similarity between the (0, 1, 4, 2) and the (1, 1, 4, 2) sequences in the introduction.

By Lemma 9, the (1, 1, 4, 2)-sequence is a translate of the (0, 2, 4, 2)-sequence, which is entry A284753 in the OEIS. It is a relatively recent addition. In the comments on this sequence on the OEIS, Dekking proves that A284753 is equal to 2 times A026363. We denote A284753 by $a(n)$. Kimberling conjectures that $-2 < (1 + \sqrt{3})n - a(n) < 3$, and Dekking remarks that $(1 + \sqrt{3})n - a(n)$ is bounded by a result of Adamczewski [1]. We will settle Kimberling's conjecture with the help of the automatic theorem prover `Walnut` [21, 25]. We used Ollinger's licofage toolkit [23], which is described in Carton et al. [9], to convert the morphism to a Dumont–Thomas numeration system for `Walnut`.

The (0, 2, 4, 2)-hiccup sequence is generated by $0 \mapsto 01,\ 1 \mapsto 0001$. The lengths of the iterated substitution words

$$0 \mapsto 01 \mapsto 010001 \mapsto 0100010101010001 \mapsto \cdots,$$

are equal to 1, 2, 6, 16, 44, ..., and form the base B_n of a Dumont–Thomas numeration system. The base satisfies the linear recurrence $B_{n+1} = 2B_n + 2B_{n-1}$. In this base, every number can be represented by a unique word with digits $\{0, 1, 2, 3\}$ that is accepted by the automaton in Fig. 1. For instance, the number 39 is represented by 1321.

The first ten entries of the sequence are given in Table 2 in decimal and in Dumont–Thomas representation. It is now easy to guess that the Dumont–Thomas representation of the sequence is $(0w, w0)$ where w runs over the admissible words with most initial digit 1. This is in line with the results of Schaeffer, Shallit, and Zorcic, who proved that such expressions describe Beatty sequences in Ostrowski numeration [24]. It is a standard exercise to define an automaton in `Walnut` that recognizes such expressions, see [25, p. 106]. We implemented it as `shift0123` and we named the Dumont–Thomas numeration system as `msd_dumthomabaaab`. It is a standard task for `Walnut` to verify that the sequence is indeed of the form $(0w, w0)$ in this system.

Table 2 A table with the first 10 entries of $(n, a(n))$ where $a(n)$ is sequence A284753, in the standard decimal representation and in the Dumont–Thomas representation

Decimal		Dumont–Thomas	
n	$a(n)$	n	$a(n)$
1	2	1	10
2	6	10	100
3	8	11	110
4	10	12	120
5	12	13	130
6	16	100	1000
7	18	101	1010
8	22	110	1100
9	24	111	1110
10	28	120	1200

We first test if differences $a(n+1) - a(n)$ are equal to either 2 or 4. The test is carried out within the Dumont–Thomas system with most significant digit (msd) first representation:

```
eval test1 "?msd_dumthomabaaab An,s,t (n>0 & $shift0123(n,s)
&
$shift0123(n+1,t)) => t=s+2|t=s+4";
```

We then test if the difference is 4 if the index n is in the sequence:

```
eval test2 "?msd_dumthomabaaab An,s,t (n>0 &
  $shift0123(n,s) &
Ek ($shift0123(k,n+1)) & $shift0123(n+1,t)) => t=s+4";
```

Finally, we test that the difference is 2 if the index is not in the sequence:

```
eval test3 ″?msd_dumthomabaaab An,s,t (n>0 & $shift0123(n,s)
&
(~Ek ($shift0123(k,n+1))) & $shift0123(n+1,t)) => t=s+2″;
```

The sequence in `shift0123` passes the tests so that indeed A284753 is of the form $(0w, w0)$ in the Dumont–Thomas numeration system generated by $0 \mapsto 01,\ 1 \mapsto 0001$.

In the introduction, we observed that the two sequences defined in Eqs. 1 and 2 have many common entries. The following establishes the connection between these two hiccup sequences using simple `Walnut` checks.

Lemma 14 *We have A284753(n) =A086398$(n) + 1$, and*

$$\text{A086398}(n) = \begin{cases} \text{A080903}(n) + 2, & \text{if the DT representation of } n \text{ ends with a } 0 \\ \text{A080903}(n), & \text{otherwise.} \end{cases}$$

Proof With `Walnut`, we define a new synchronised sequence `a086398` that accepts one more than the term accepted in `shift0123` in parallel with n as follows:

```
def a086398 ″?msd_dumthomabaaab Ek ($shift0123(n,k) &
```

```
p+1=k)":
```

We can then verify that `a086398` indeed accepts n and the terms of the $(1, 1, 4, 2)$-hiccup sequence in parallel using the tests – `test1`, `test2`, and `test3` where `shift0123` replaced by `a086398`, and here we must modify our check – if one less than the index is in the sequence.

Now, for the conditional check, we define a DFA that accepts only the representations ending in a 0, followed by defining a DFA accepting n and A080903(n) in parallel:

```
reg endin0 msd_dumthomabaaab "(0|1|2|3)*0";
def a080903 "?msd_dumthomabaaab Es n>0 & $a086398(n,s) &
($endin0(n) => t+2=s) & (~$endin0(n) => t=s)";
```

Again, we will verify that `a086398` accepts all n and the terms the $(0, 1, 4, 2)$-hiccup sequence A080903 in parallel using the same tests – `test1`,`test2`, and `test3`, this time replacing `shift0123` with `a080903`. □

Additionally, there is a simple characterization of A080903.

Lemma 15 *The numbers in A080903 are those that end with a* 1 *in Dumont–Thomas numeration.*

Proof We define a DFA accepting all the representations ending with a 1 in `Walnut` with the following code:

```
reg endin1 msd_dumthomabaaab "(0|1|2|3)*1";
```

This is followed by a simple check described in the statement of the lemma, which returns `TRUE`.

```
eval test "?msd_dumthomabaaab As (En (n>0 & $a080903(n,s))
<=> $endin1(s))";
```

□

Now that we have a characterization of A284753 in terms of Dumont–Thomas numeration, we can address Kimberling's conjectured bounds on the numbers in this sequence. Let $\lambda = 1+\sqrt{3}$ and $\bar{\lambda} = 1-\sqrt{3}$. The following is a Binet type representation for the base B_n of our numeration system.

Lemma 16 *The basis B_n of the Dumont–Thomas numeration system for* $0 \mapsto 01$, $1 \mapsto 0001$ *satisfies*

$$B_n = \frac{\lambda^n - \bar{\lambda}^n}{2\sqrt{3}}.$$

Proof These numbers satisfy the recursion $B_{n+1} = 2B_n + 2B_{n-1}$ with initial condition $B_0 = 0$ and $B_1 = 1$. □

Lemma 17

$$B_{n+1} - \lambda B_n = \bar{\lambda}^n.$$

Proof This is a straightforward computation. □

In order to prove Kimberling's conjecture that $-3 < a(n) - \lambda n < 2$ we run into the problem of bounding

$$\sum_{i \geq 1} d_i \bar{\lambda}^i$$

over all admissible words $w = d_1 d_2 \cdots$ in least digit first format in our Dumont–Thomas numeration system. Since $\bar{\lambda}$ is negative and less than 1 in absolute value, a straightforward lower bound is $3\sum_{i=1}^{\infty} \bar{\lambda}^{2i-1}$ and an upper bound is $3\sum_{i=1}^{\infty} \bar{\lambda}^{2i}$, since digits are at most 3. These bounds are not sharp enough. We need to work a little harder.

Lemma 18 *If $w = d_1 d_2 d_3 d_4 d_5 d_6$ is any admissible word in lsd format in our Dumont–Thomas numeration system, then $\sum_{i=1}^{6} d_i \bar{\lambda}^i$ is minimized by* 312121. *For a word in seven digits it is maximized by* 0312121.

Proof This is a finite problem that we solved numerically. The minimum rounded to three decimals is -2.424. To understand why this is the minimizing word, without going into the computation, notice that the odd powers $\bar{\lambda}^i$ are negative. We would like to maximize the d_i for odd i. That is why $d_1 = 3$ is the logical candidate. If $d_i > 1$ then d_{i+1} is positive, which explains why 312121 is the minimizer. The maximizer does the same thing, but for the even indexed digits. Its value rounded to three decimals is 1.775. □

We write $m_6 = -2.424$ for the minimum, rounded to three decimals, and $M_7 = 1.775$ for the maximum.

Theorem 19 *The* $(0, 2, 4, 2)$*-hiccup sequence* $a(n)$ *satisfies*

$$-3 < a(n) - \lambda n < 2.$$

Proof Let $n = \sum d_i B_i$ be the Dumont–Thomas representation, for which we have $a(n) = \sum d_i B_{i+1}$. It follows that

$$a(n) - \lambda n = \sum d_i \bar{\lambda}_i^{i-1}.$$

The powers $\bar{\lambda}^i$ alternate in sign and the digits are bounded by 3. It follows that

$$m_6 + 3\sum_{i=4}^{\infty} \bar{\lambda}^{2i-1} < a(n) - \lambda n < M_7 + 3\sum_{i=4}^{\infty} \bar{\lambda}^{2i}.$$

These bounds rounded to three decimals are -2.667 for the minimum and 1.953 for the maximum. □

This solves Kimberling's conjecture that $-2 < n\left(1+\sqrt{3}\right) -$ A284753$(n) < 3$. Dekking, in his comments in the OEIS entry A284753, points out that Kimberling's conjecture implies $-1 < n(1+\sqrt{3})/2 -$ A026363$(n) < 3/2$, which is stronger than another of Kimberling's conjectures: $-1 < n(1+\sqrt{3})/2-$A026363$(n) < 2$. A natural question arising from this result is whether the results of Schaeffer, Shallit, and Zorcic on Beatty sequences can be extended to other hiccup sequences. More specifically, is it true that hiccup sequences are synchronized with respect to Dumont–Thomas numeration?

4 Iterated function sequences

The fourth characterization of the BDS theorem is the most intriguing. It connects Lambert's continued fraction expansion of $\frac{4}{\pi}$ to the (1, 1, 3, 2)-hiccup sequence. Lambert's expansion, which is of polynomial type [12], is given by

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{\ddots}}}},$$

It is a special case of his continued fraction expansion of the arctangent, see [5]. It is well known that the continued fraction converges, but that the rate of convergence is not as fast as for regular continued fractions, see [20, Table 2.3]. A shorthand notation for Lambert's continued fraction is

$$\frac{4}{\pi} = 1 + \mathop{\mathbf{K}}_{j=1}^{\infty} \frac{j^2}{2j+1}$$

The fractions $\frac{j^2}{2j+1}$ are called *partial fractions* and the truncated expansions $1 + \mathop{\mathbf{K}}_{j=1}^{n} \frac{j^2}{2j+1}$ are called *convergents*. The initial convergents are

$$\frac{1}{1}, \frac{4}{3}, \frac{24}{19}, \frac{204}{160}, \ldots.$$

The sequence of numerators is A012244, but the sequence of denominators is not in the OEIS. The convergents are defined by cutting subsequent tails from the continued fraction. These tails are given by

$$s_n = 2n - 1 + \mathop{\mathbf{K}}_{j=n}^{\infty} \frac{j^2}{2j+1},$$

starting from $s_1 = \frac{4}{\pi} = 1 + \frac{1}{s_2}$. Each next tail is connected to the previous tail by

$$s_n = 2n - 1 + \frac{n^2}{s_{n+1}}.$$

The fourth characterization of the (1, 1, 3, 2,)-hiccup sequence in the BDS theorem is that it is equal to $\lfloor s_n + \frac{1}{2} \rfloor$. It is suprising that an iterative sequence that starts from $\frac{4}{\pi}$ produces the same result as a Beatty sequence with slope $1+\sqrt{2}$. Bosma, Dekking, and Steiner write that they were only convinced after a numerical verification of the initial 130, 000 terms of the sequence.

In their proof of the fourth characterization, Bosma, Dekking, and Steiner consider the family of continued fractions of polynomial type given by

$$k - 1 + \mathop{\mathbf{K}}_{j=1}^{\infty} \frac{j^2}{(j+1)k - 1}, \tag{11}$$

for natural numbers k. For $k = 2$ we have the continued fraction of $\frac{4}{\pi}$. For $k = 1$, Beukers [7] found the following elegant expression in terms of the golden mean $\phi = \frac{1+\sqrt{5}}{2}$ and $\eta = \frac{1}{1+\phi^2}$

$$\frac{\phi}{\sum_{n=1}^{\infty} \frac{1}{n+\eta} \left(\frac{-1}{\phi^2}\right)^n} = \cfrac{1^2}{1 + \cfrac{2^2}{2 + \cfrac{3^2}{3 + \cfrac{4^2}{\ddots}}}}. \tag{12}$$

The sequence of numerators of its convergents is A288952, and appears in a study by Genitrini et al. on tree enumeration [18]. The tails of the continued fraction in Eq. 11 satisfy

$$s_n = kn - 1 + \frac{n^2}{s_{n+1}}. \tag{13}$$

Expressing s_{n+1} in terms of s_n gives

$$s_{n+1} = \frac{n^2}{s_n - kn + 1}.$$

In the final line of their paper, Bosma, Dekking, and Steiner conjecture a remarkable relation between this iterative sequence and a Beatty sequence that has slope α_k such that $\alpha_k^2 = k\alpha_k + 1$. Such a quadratic number is known as a *metallic mean* [13].

Conjecture 20 (Bosma, Dekking, and Steiner) Let α_k be the k-th metallic mean and let s_1 be the unique positive real number such that all $s_{n+1} = \frac{n^2}{s_n - kn + 1}$ are positive, where we suppress k in our notation of the sequence s_n. The rounded-down sequence $\lfloor s_n \rfloor$ is equal to the Beatty sequence

$$\left\lfloor \alpha_k . n - \frac{1+\alpha_k}{2\alpha_k - k} \right\rfloor.$$

These are not hiccup sequences, and none of these sequences appear in the OEIS. It is shown in [8] that $s_n - n\alpha - \beta$ is positive and converges to zero with order $O\left(\frac{1}{n}\right)$, where we write α for the k-th metallic mean, and $\beta = -\frac{1+\alpha}{2\alpha-k}$. These constants α and β can be retrieved from Eq. 13 if we expand it up to order $O\left(\frac{1}{n}\right)$ as

$$n\alpha + \beta + O\left(\frac{1}{n}\right) = kn - 1 + \frac{n}{\alpha} - \frac{\alpha+\beta}{\alpha^2} + O\left(\frac{1}{n}\right).$$

Equating the terms of order n to zero gives that α is the j-th metallic mean. By doing the same for the terms of order 1, we get that $\beta = -\frac{\alpha^2+\alpha}{\alpha^2+1}$, which gives the expression for β in the conjecture.

We extend the one-parameter family in Eq. 11 to the two-parameter family

$$u + v + \mathop{\mathbf{K}}_{j=1}^{\infty} \frac{j^2}{(j+1)u + v}, \tag{14}$$

for constants u and v. Its tails satisfy

$$s_n = nu + v + \frac{n^2}{s_{n+1}}.$$

If we assume that s_n converges to $\alpha n + \beta$ with order $O\left(\frac{1}{n}\right)$, and if we take u to be a natural number, then α is the u-th metallic mean and

$$\beta = \frac{v\alpha^2 - \alpha}{\alpha^2 + 1}.$$

The generalization of Conjecture 20 then says that the rounded down tails of the continued fraction give the Beatty sequence with slope α and offset β.

The continued fraction in Eq. 14 can be expressed in terms of hypergeometric functions. We are indebted to Frits Beukers for pointing that out to us [7]. The following equality follows from [22, Eq. 15.7.5].

$$\frac{c}{{}_2F_1(1,1;c+1;z)} = c - z + \mathop{\mathbf{K}}_{j=1}^{\infty} \frac{j^2 z(1-z)}{c - z + j(1-2z)},$$

which converges for $\mathrm{Re}(z) < \frac{1}{2}$, It is not hard to verify that

$$\frac{\lambda \cdot c}{{}_2F_1(1,1;c+1;z)} = \lambda(c - z) + \mathop{\mathbf{K}}_{j=1}^{\infty} \frac{(\lambda j)^2 z(1-z)}{\lambda(c - z + j(1-2z))}.$$

If we set

$$\lambda = \sqrt{4+u^2}\,, \quad z = \frac{1}{2} - \frac{u}{2\lambda}\,, \quad c = z + \frac{u+v}{\lambda},$$

then

$$\frac{\lambda \cdot c}{{}_2F_1(1,1;c+1;z)} = u + v + \mathop{\mathbf{K}}_{j=1}^{\infty} \frac{j^2}{(j+1)u+v}.$$

For the continued fraction in Eq. 12, the parameters are $\lambda = \sqrt{5},\ z = \frac{1}{2} - \frac{1}{2\sqrt{5}},\ c = z$. Beukers' formula in Eq. 12 follows from the transformation

$${}_2F_1(a,b;c;z) = \frac{1}{(1-z)^a} \cdot {}_2F_1(a, c-b; c; z/(z-1)).$$

The continued fraction is therefore equal to $\frac{\sqrt{5}\,(1-\eta)\,\eta}{{}_2F_1(1,\eta;\eta+1;\eta/(\eta-1))} = \frac{\phi}{\sum_{n=1}^{\infty} \frac{1}{n+\eta}\left(\frac{-1}{\phi^2}\right)^n}$.

We now restrict our attention to one particular case of the conjecture. The lower Wythoff sequence A000201, which is the (1, 1, 2, 1)-hiccup sequence, is the Beatty sequence $\lfloor n\phi \rfloor$ for the golden mean ϕ, i.e., the first metallic mean. According to the generalized conjecture, if we take $u = 1$ and $v = \frac{1}{\phi} = \phi - 1$, then the rounded down tails of the continued fraction are equal to the lower Wythoff sequence. We prove that the conjecture holds in this particular case. The backward recurrence for these values of u and v is given by $s_n = n + 1/\phi + \frac{n^2}{s_{n+1}}$, which in forward recurrence is equal to

$$s_{n+1} = \frac{n^2}{s_n - n - 1/\phi}.$$

It is convenient to scale the tails to $r_n = \frac{s_n}{n}$, which satisfy the recurrence

$$r_{n+1} = \frac{n}{n+1} \cdot \frac{1}{r_n - 1 - \frac{1}{n\phi}}. \tag{15}$$

Lemma 21 *There is a unique sequence of positive real numbers $(r_n)_{n\geq 1}$ that satisfies the recurrence 15. Moreover, this sequence is contained in $(1 + \frac{1}{n\phi}, 2)$.*

Proof We first prove that a positive sequence r_n must be contained in $(1 + \frac{1}{n\phi}, 2)$. The lower bound $r_n > 1 + \frac{1}{n\phi}$ is required to ensure that $r_{n+1} > 0$. If $r_n \geq 2$ and $n > 1$, then

$$r_{n+1} \leq \frac{n}{n+1} \cdot \frac{1}{1 - \frac{1}{n\phi}} = \frac{n^2\phi}{n^2\phi + n(\phi - 1) - 1} < 1, \tag{16}$$

which is impossible since we have already established that $r_{n+1} > 1$. If $r_1 \geq 2$, then $r_2 \leq \frac{1}{2} \cdot \frac{1}{1-\frac{1}{\phi}} = 1 + \frac{1}{2\phi}$, which is again impossible.

We denote the functions that produce the recurrence by

$$f_n(x) = \frac{n}{n+1} \cdot \frac{1}{x - 1 - \frac{1}{n\phi}},$$

and we denote $I_n = (1 + \frac{1}{n\phi}, 2)$. If $n > 1$, then by inequality 16 the image $f_n(I_n)$ contains $(1, \infty)$. Therefore, $f_n^{-1}(I_{n+1}) \subset I_n$ and $f_n^{-1}(I_{n+1})$ is an open subinterval in which the end points are elements of I_n. We have a descending chain

$$I_2 \supset f_2^{-1}(I_3) \supset f_2^{-1}(f_3^{-1}(I_4)) \supset \cdots \tag{17}$$

such that the end-points of each next interval are in the previous interval. Therefore, the intersection of the open intervals is equal to the intersection of their closures. It follows that the intersection of the chain is non-empty. We need to show that it is a singleton $\{r_2\}$.

On the interval I_n the derivate of f_n is bounded below by its value in 2:

$$|f_n'(x)| > \frac{n}{n+1} \cdot \frac{1}{\left(1 - \frac{1}{n\phi}\right)^2} = \frac{1}{(1+\frac{1}{n})(1 - \frac{2}{n\phi} + \frac{1}{n^2\phi^2})} = \frac{1}{1 - \frac{2-\phi}{n\phi} - \frac{2\phi-1}{n^2\phi^2} + \frac{1}{n^3\phi^2}},$$

which is greater than $1 + \frac{2-\phi}{n\phi}$. Each I_{n+1} has length less than 1 and the inverse f_n^{-1} shrinks its length by a factor that is greater than $1+\frac{2-\phi}{n\phi}$. The product of all these factors is infinite and therefore the intersection is indeed a singleton $\{r_2\}$. Now $f_1(I_1) \supset I_2$ since $f_1(2) = 1 + \frac{1}{2\phi}$ and there is a unique $r_1 \in I_1$ such that $f_1(r_1) = r_2$. □

To prove that $\lfloor s_n \rfloor = \lfloor n\phi \rfloor$ we need to establish that there is no integer m in between s_n and $n\phi$. This is equivalent to proving that there is no rational $\frac{m}{n}$ in between r_n and ϕ. To do this, we need to narrow down the intervals I_n that we found in the previous lemma, which is a straightforward but headache causing computation.

Lemma 22 *Let* $\psi = -\frac{1}{\phi}$. *For each* $n \geq 1$ *we have* $r_n \in \left(\phi + \frac{1}{n(n+1/2)\sqrt{5}}, \phi + \frac{1-1/(5n^2)}{n^2\sqrt{5}}\right)$.

Proof Let $J_n = \left(\phi + \frac{1}{n(n+1/2)\sqrt{5}}, \phi + \frac{1-\psi^{4n}}{n^2\sqrt{5}}\right)$. We prove that

$$f_n(J_n) \supset J_{n+1}.$$

Since f_n is decreasing, we need to verify the two inequalities

$$f_n\left(\phi + \frac{1}{n(n+1/2)\sqrt{5}}\right) > \phi + \frac{1-\psi^{4(n+1)}}{(n+1)^2\sqrt{5}}, \tag{18}$$

and

$$f_n\left(\phi + \frac{1-\psi^{4n}}{n^2\sqrt{5}}\right) < \phi + \frac{1}{(n+1)(n+3/2)\sqrt{5}}. \tag{19}$$

We have that

$$f_n(\phi + x) = \frac{n}{n+1} \cdot \frac{1}{\phi + x - 1 - \frac{1}{n\phi}} = \frac{n}{n+1} \cdot \frac{1}{x + \frac{n-1}{n\phi}}$$

$$= \frac{n^2\phi}{(n+1)n\phi x + n^2 - 1} = \frac{\phi}{1 + \frac{\phi x(n+1)}{n} - \frac{1}{n^2}}$$
$$= \frac{\phi}{1 - \frac{1-\phi x n(n+1)}{n^2}}.$$

For the first inequality 18 we have $x = \frac{1}{n(n+1/2)\sqrt{5}}$ and therefore we need to verify that

$$\frac{\phi}{1 - \frac{\sqrt{5}(n+1/2)-\phi(n+1)}{n^2(n+1/2)\sqrt{5}}} > \phi + \frac{1-1/(5(n+1)^2)}{(n+1)^2\sqrt{5}}.$$

We will prove the sharper inequality

$$\frac{\phi}{1 - \frac{\sqrt{5}(n+1/2)-\phi(n+1)}{n^2(n+1/2)\sqrt{5}}} > \phi + \frac{1}{(n+1)^2\sqrt{5}}.$$

This can be rewritten as

$$\frac{\phi(\sqrt{5}(n+1/2)-\phi(n+1))}{n^2(n+1/2)\sqrt{5}} > \frac{1}{(n+1)^2\sqrt{5}}\left(1 - \frac{\sqrt{5}(n+1/2)-\phi(n+1)}{n^2(n+1/2)\sqrt{5}}\right).$$

Since $\phi(\sqrt{5}-\phi) = 1$ this is

$$\frac{n+1-\phi/2}{n+1/2} > \frac{n^2}{(n+1)^2}\left(1 - \frac{\sqrt{5}(n+1/2)-\phi(n+1)}{n^2(n+1/2)\sqrt{5}}\right).$$

Both factors on the right-hand side are less than 1 and the inequality follows from

$$\frac{n+1-\phi/2}{n+1/2} > \frac{n^2}{(n+1)^2},$$

which reduces to $(n+1-\phi/2)(n+1)^2 > n^2(n+1/2)$. It is straightforward to check that this holds by comparing the coefficients of the polynomials.

For the second inequality in 19 we have that $x = \frac{1-1/(5n^2)}{n^2\sqrt{5}}$ and we need to verify that

$$\frac{\phi}{1 - \frac{n\sqrt{5}-\phi(n+1)(1-1/(5n^2))}{n^3\sqrt{5}}} < \phi + \frac{1}{(n+1)(n+3/2)\sqrt{5}}.$$

For $n = 1$ the left-hand side is 1.397... and the right-hand side is 1.707.... For $n = 2$ the left-hand side is 1.605... and the right-hand side is 1.660....

We may assume that $n > 2$.

The inequality can be rewritten as

$$\frac{\phi(n\sqrt{5}-\phi(n+1)(1-1/(5n^2)))}{n^3\sqrt{5}} < \frac{1}{(n+1)(n+3/2)\sqrt{5}}\left(1 - \frac{n\sqrt{5}-\phi(n+1)(1-1/(5n^2))}{n^3\sqrt{5}}\right),$$

which simplifies to

$$\frac{n - \phi^2 + \phi^2/(5n) + \phi^2/(5n^2)}{n} < \frac{n^2}{(n+1)(n+3/2)}\left(1 - \frac{n/\phi - \phi + \phi/(5n) + \phi/(5n^2)}{n^3\sqrt{5}}\right).$$

The left-hand side is

$$1 - \phi^2 \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

and the right-hand side is

$$1 - \frac{5}{2} \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

and therefore the inequality holds for sufficiently large n. We need to verify that it holds for all n. On the right-hand side of the equation, the factor

$$1 - \frac{n/\phi - \phi + \phi/(5n) + \phi/(5n^2)}{n^3\sqrt{5}}$$

is less than 1 if $n > 2$. If we ignore that factor, we get

$$\frac{n - \phi^2 + \phi^2/(5n) + \phi^2/(5n^2)}{n} < \frac{n^2}{(n+1)(n+3/2)}.$$

Since $\frac{n^2}{(n+1)(n+3/2)} > 1 - \frac{5}{2} \cdot \frac{1}{n}$ it suffices to show that

$$\frac{n - \phi^2 + \phi^2/(5n) + \phi^2/(5n^2)}{n} = 1 - \phi^2 \cdot \frac{1}{n} + \frac{\phi^2}{5n^2 + 5n^3} < 1 - \frac{5}{2} \cdot \frac{1}{n},$$

We arrive at

$$\frac{\phi^2}{5n + 5n^2} < \frac{1}{\phi} - \frac{1}{2} \text{ for } n > 2.$$

It suffices to check this at $n = 3$, when the left-hand side is 0.04... and the right-hand side is 0.11... The inequality holds. □

The intervals J_n are disjoint and descend on ϕ, which implies that r_n is a decreasing sequence with limit point ϕ. We need to show that $(\phi, r_n) \subset (\phi, \phi + \frac{1-1/(5n^2)}{\sqrt{5}n^2})$ does not contain a rational of denominator n.

Lemma 23 *No rational satisfies* $0 < \frac{m}{n} - \phi < \frac{1-1/(5n^2)}{\sqrt{5}n^2}$.

Proof The convergents of $\frac{1}{\phi}$ are ratios of consecutive Fibonacci numbers $\frac{F_k}{F_{k+1}}$. By Cassini's identity

$$F_k^2 - F_{k-1} \cdot F_{k+1} = (-1)^k,$$

which means that consecutive convergents are Farey neighbors. The convergents $\frac{F_{2k}}{F_{2k+1}}$ are smaller than $\frac{1}{\phi}$ and the convergents $\frac{F_{2k-1}}{F_{2k}}$ are greater. By Binet's formula $F_k =$

$\frac{\phi^k-\psi^k}{\sqrt{5}}$ with $\psi = -1/\phi$

$$\frac{F_{2k-1}}{F_{2k}} - \frac{1}{\phi} = \frac{(\phi-\psi)\psi^{2k}}{\phi^{2k}-\psi^{2k}} = \frac{\psi^{2k}}{F_{2k}}.$$

Now $\psi^{2k}F_{2k} = \frac{1-\psi^{4k}}{\sqrt{5}}$ and therefore

$$\frac{F_{2k-1}}{F_{2k}} - \frac{1}{\phi} = \frac{1-\psi^{4k}}{\sqrt{5}F_{2k}^2}.$$

Suppose that $0 < \frac{m}{n} - \phi < \frac{1-1/(5n^2)}{\sqrt{5}n^2}$, which is equivalent to $0 < \frac{m-n}{n} - \frac{1}{\phi} < \frac{1-1/(5n^2)}{\sqrt{5}n^2}$. Let F_{2k} be the largest even-index Fibonacci number $\leq n$. We will show that $\frac{m-n}{n}$ is closer to $\frac{1}{\phi}$ than $\frac{F_{2k-1}}{F_{2k}}$, which is impossible, since its first Farey neighbor that is larger than $\frac{1}{\phi}$ has denominator F_{2k+2}. We will show that

$$\frac{1-1/(5n^2)}{\sqrt{5}n^2} < \frac{1-\psi^{4k}}{\sqrt{5}F_{2k}^2}.$$

Now $\psi^{4k}F_{2k}^2 < \frac{1}{5}$ and therefore it suffices to show that

$$\frac{1-1/(5n^2)}{\sqrt{5}n^2} \leq \frac{1-1/(5F_{2k}^2)}{\sqrt{5}F_{2k}^2}.$$

This follows from the fact that $f(x) = \frac{1-1/(5x^2)}{x^2}$ is descending for $x > 1$. □

These lemmas prove our result.

Theorem 24 *The sequence of rounded down tails $\lfloor s_n \rfloor$ for the continued fraction in Eq. 14 with $z = 1$ and $v = 1/\phi$ is equal to the lower Wythoff sequence.*

This settles a particular case of the BDS conjecture. It depends on the computation in Lemma 22, which shows that the tails of the continued fraction of polynomial type with partial fractions $\frac{n^2}{n+1/\phi}$ converge to ϕ slightly faster than the convergents of its regular continued fraction. A proof of the BDS conjecture would require a conceptual explanation rather than a computation.

5 Conclusion and acknowledgements

We have extended the considerations by Bosma et al. [8] from the (1, 1, 3, 2)-hiccup sequence to other hiccup sequences. There is still much to be explored, most notably the conjectured relation between polynomial continued fractions and Beatty sequences.

Table 3 Table of (j, x, y, z)-hiccup sequences that are not yet recognized as such in the OEIS at the time of writing

OEIS	j	x	y	z
A000201	1	1	2	1
A001950	1	2	2	3
A003156	1	1	3	1
A026352	1	1	2	3
A026356	0	2	2	3
A284753	0	2	4	2

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Author contributions R. Fokkink and G. Joshi carried out the work in full cooperation and contributed equally to all aspects of the manuscript. All authors reviewed the manuscript.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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