NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER
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THE CONVERGING FACTOR FOR THE MODIFIED BESSEL FUNCTION OF THE SECOND KIND

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by

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ABSTRACT

The converging factor for a specific mathematical function, such as the modified Bessel function of the second kind considered in this report, is that factor by which the last term of a truncated series (usually asymptotic) approximating the function must be multiplied to compensate for the omitted terms. This converging factor for the aforementioned Bessel function is discussed herein in detail and is shown to be related to the corresponding factor for the probability integral. Tables of this factor and its reduced derivatives, correct to 30 decimal places, are included to expedite the application of this procedure to the evaluation of this Bessel function to high precision for arguments between 5 and 20, and specific examples of such applications are presented.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

A variety of methods have been proposed in recent years for the calculation of Bessel functions as alternatives to the well-known use of power series for small arguments and asymptotic series for large arguments. These alternative procedures include recurrence relations, described by Abramowitz and Stegun[1] and by Goldstein and Thaler;[2] phase amplitude methods, also discussed by Goldstein and Thaler;[3] quadrature methods, treated by Fettis,[4] Luke,[5] and Hunter;[6] and continued fractions, discussed by Gargantini and Henrici.[7]

In addition to these methods, the use of converging factors to extend the precision attainable by asymptotic series for Bessel functions, in particular, has been advocated by several investigators including Airey[8] and Dingle.[9] Murnaghan[10] and the writer[11] have investigated in detail the use of converging factors in the numerical evaluation to high precision of the probability integral and the exponential integral.

This report will show that the converging factor for the probability integral can be directly applied to the evaluation of the converging factor for the modified Bessel function of the second kind, $K_n(x)$.

As explicitly noted by Hunter,[6] the evaluation of $K_n(x)$ by power series for even moderately large positive values of x presents special difficulty because of the loss of significant figures arising from the subtraction of nearly equal numbers. This computational difficulty

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[1]References are listed on page 55.

can be avoided if an asymptotic series is used and the remainder resulting from truncating the series is closely estimated by use of a converging factor. Thus, for arguments x exceeding 5, such a procedure more than doubles the number of significant figures of $K_n(x)$ that can be attained by the conventional use of asymptotic series (that is, terminating the series at the least numerical term).

As emphasized by Gargantini and Henrici,[7] the function K_n occupies a central position in the theory of Bessel functions, inasmuch as in the complex plane all other Bessel functions are expressible in terms of it. Accordingly, our attention in this study will be principally focussed upon the determination of the converging factor for this particular Bessel function.

A table of the basic converging factor $C_n(n)$ (or $\Lambda_{n-\frac{1}{2}}(n)$ in the notation of Dingle) and its reduced derivatives rounded to 30 decimal places for n = 10(1)40 is included; this permits the calculation of $K_p(x)$ to a precision ranging from 15 decimal places when x = 5 to 42 decimal places when x = 20 and p is either 0 or 1.

THE ASYMPTOTIC SERIES FOR THE BESSEL FUNCTION $K_p(z)$ AND ITS CONVERGING FACTOR

The modified Bessel functions $I_p(z)$ and $K_p(z)$ satisfy the second-order linear differential equation

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + p^2)\, w = 0 \tag{1}$$

These functions may be distinguished according to their behavior when the argument z is large in absolute value; thus, the modified Bessel function of the first kind, $I_p(z)$, behaves asymptotically like $z^{-1/2}\, e^{z}$, whereas that of the second kind, $K_p(z)$, behaves asymptotically like $z^{-1/2}\, e^{-z}$.

If in Eq. (1) we make the change of variable

$$w(z) = x^p\, e^{-x/2}\, f(x)\ , \quad x = 2z \tag{2}$$

we obtain after some simplification the differential equation

$$x\frac{d^2f}{dx^2} + (2p + 1 - x)\frac{df}{dx} - (p + \frac{1}{2})\, f = 0 \tag{3}$$

Since the confluent hypergeometric functions $f(a, c, x)$ satisfy Kummer's equation

$$x\frac{d^2f}{dx^2} + (c - x)\frac{df}{dx} - af = 0 \tag{4}$$

we infer that

$$\begin{aligned} w(z) &= x^p\, e^{-x/2}\, f(p + \tfrac{1}{2},\ 2p+1,\ x) \\ &= (2z)^p\, e^{-z}\, f(p + \tfrac{1}{2},\ 2p+1,\ 2z) \end{aligned} \tag{5}$$

Now, if $\Psi(a,c,x)$ represents the solution of Eq. (4) which behaves like x^{-a} when x tends to infinity, we have the relation

$$K_p(z) = \pi^{1/2}(2z)^p e^{-z}\, \Psi(p+\tfrac{1}{2},\ 2p+1,\ 2z) \tag{6}$$

expressing the modified Bessel function of the second kind of order p in terms of the confluent hypergeometric function Ψ.

To find the asymptotic series expansion of $\Psi(a,c,x)$ we set

$$\Psi(a,c,x) = x^{-a}\,\psi(a,c,x) \tag{7}$$

Differentiating this twice with respect to x and substituting the results in Eq. (4), we obtain

$$\frac{d^2\psi}{dx^2} - \left(1+\frac{2a-c}{x}\right)\frac{d\psi}{dx} + \frac{a(a-c+1)}{x^2}\,\psi = 0 \tag{8}$$

Next, we assume a solution to Eq. (8) of the form

$$\psi = a_1 + \frac{a_2}{x} + \frac{a_3}{x^2} + \cdots + \frac{a_n}{x^{n-1}} + \cdots \tag{9}$$

If we differentiate this series twice with respect to x and substitute the results in Eq. (8), we find the coefficient of x^{-r} to be $(r-1)\,a_r + (r+a-2)(r+a-c-1)a_{r-1}$, which must vanish; hence, we infer that

$$\frac{a_r}{a_{r-1}} = -\frac{(r+a-2)(r+a-c-1)}{r-1} \tag{10}$$

Thus, the function $\psi(a,c,x)$ may be defined by the asymptotic series

$$\begin{aligned}\psi(a,c,x) &\sim 1 - \frac{a(a-c+1)}{1!x} + \frac{a(a+1)(a-c+1)(a-c+2)}{2!x^2} - \cdots \\ &\sim \frac{1}{(a-1)!(a-c)!}\sum_{r=1}\frac{(r+a-2)!(r+a-c-1)!}{(r-1)!(-x)^{r-1}}\end{aligned} \tag{11}$$

We observe that if either a or a-c+1 is zero or a negative integer, this series terminates; indeed, we have

$$\Psi(-k, c, x) = x^{-a} \psi(a, c, x)$$

$$= (-1)^k k! \, L_k^{c-1}(x) \tag{12}$$

where

$$L_k^n(x) = (k!)^{-1} x^{-n} e^x \left(\frac{d}{dx}\right)^k (x^{k+n} e^{-x}) \tag{13}$$

is the generalized Laguerre polynomial.

In order to derive an expression for the converging factor associated with the asymptotic series (11), we proceed as follows.

From the definition of the Beta function we have

$$B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} \, dt$$

$$= \int_1^\infty (u-1)^{z-1} u^{-z-w} \, du \tag{14}$$

where $u = (1-t)^{-1}$. Furthermore, the Beta function can be expressed in terms of the Gamma or the factorial function, as follows:

$$B(z, w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}$$

$$= \frac{(z-1)!\,(w-1)!}{(z+w-1)!} \tag{15}$$

Consequently, we have the relation

$$\frac{(w-1)!}{(z+w-1)!} = \frac{1}{(z-1)!} \int_1^\infty (u-1)^{z-1} u^{-z-w} \, du$$

or

$$\frac{(r+a-2)!}{(r-1)!} = \frac{1}{(-a)!} \int_1^\infty (u-1)^{-a} u^{-r} \, du \tag{16}$$

if we set $w = r+a-1$ and $z = -a+1$.

Therefore, the series in (11) can be written

$$\psi(a,c,x) \sim \frac{1}{(a-1)!(a-c)!(-a)!} \int_1^\infty (u-1)^{-a} u^{-1} du \sum_{r=1}^\infty \frac{(r+a-c-1)!}{(-ux)^{r-1}} \tag{17}$$

$$\sim \frac{1}{(a-1)!(a-c)!} \{ \sum_{r=1}^n \frac{(r+a-2)!(r+a-c-1)!}{(r-1)!(-x)^{r-1}}$$

$$+ \frac{(n+a-c)!}{(-x)^n} \frac{1}{(-a)!} \int_1^\infty (u-1)^{-a} u^{-n-1} \Lambda_{n+a-c}(ux) du \} \tag{18}$$

where

$$\Lambda_{n+a-c}(ux) \sim 1 - \frac{n+a-c+1}{ux} + \frac{(n+a-c+1)(n+a-c+2)}{(ux)^2} - \cdots \tag{19}$$

We observe that

$$\Lambda_{n+a-c}(ux) = \frac{1}{(n+a-c)!} \{ (n+a-c)! - \frac{(n+a-c+1)!}{ux} + \frac{(n+a-c+2)!}{(ux)^2} - \cdots \}$$

$$= \frac{1}{(n+a-c)!} \int_0^\infty e^{-t} t^{n+a-c} (1 - \frac{t}{ux} + \frac{t^2}{(ux)^2} - \cdots) dt$$

$$= \frac{1}{(n+a-c)!} \int_0^\infty \frac{e^{-t} t^{n+a-c}}{1 + \frac{t}{ux}} dt \tag{20}$$

which is identifiable with the first basic converging factor of Dingle, independently derived by Murnaghan and the present writer in a joint study of the exponential integral, wherein the notation $\Gamma_{n+a-c}(ux)$ was used for this converging factor.

If we write

$$\psi(a,c,x) = \frac{1}{(a-1)!(a-c)!}\left\{\sum_{r=1}^{n}\frac{(r+a-2)!(r+a-c-1)!}{(r-1)!(-x)^{r-1}}\right.$$

$$\left. + \frac{(n+a-1)!(n+a-c)!}{n!(-x)^{n}}\ \Sigma_{n+a-c}(x)\right\} \tag{21}$$

we infer from (18) that the converging factor for this series is given by

$$\Sigma_{n+a-c}(x) = \int_1^{\infty}(u-1)^{-a}u^{-n-1}\Lambda_{n+a-c}(ux)du \Big/ \int_1^{\infty}(u-1)^{-a}u^{-n-1}du \tag{22}$$

Since the factor u^{-n-1} in the integrand in (18) forms a rapidly decreasing sequence for increasing values of u, we expand the factor $\Lambda_{n+a-c}(ux)$ in ascending powers of $u-1$ and then integrate term by term. Thus, we write

$$\Lambda_{n+a-c}(ux) = \Lambda_{n+a-c}(x) + x\Lambda^{(1)}_{n+a-c}(x)\ (u-1) + \cdots$$

$$+ x^{t}\Lambda^{(t)}_{n+a-c}(x)\,(u-1)^{t} + \cdots \tag{24}$$

where

$${}^{(k)}_{n+a-c}(x) = \frac{1}{k!}\ \frac{d^{t}}{d(ux)^{k}}\ \Lambda_{n+a-c}\ (u\,x)\Big|_{u=1} \tag{25}$$

which is the k^{th} reduced derivative of $\Lambda_{n+a-c}(u\,x)$, evaluated at $u = 1$.

Since

$$\int_1^{\infty}(u-1)^{-a}\ u^{-n-1}\Lambda^{(t)}_{n+a-c}(x)\ (u-1)^{t}du = \Lambda^{(t)}_{n+a-c}(x)\int_1^{\infty}(u-1)^{t-a}u^{-n-1}du$$

$$= \Lambda^{(t)}_{n+a-c}(x)\ \frac{(t-a)!(n-t+a-1)!}{n!} \tag{26}$$

we can write

$$\psi(a,c,x) = \frac{1}{(a-1)!(a-c)!}\left\{\sum_{r=1}^{n} \frac{(r+a-2)!(r+a-c-1)!}{(r-1)!(-x)^{r-1}}\right.$$

$$\left. + \frac{(n+a-1)!(n+a-c)!}{n!(-x)^n} \sum_{t=0}^{\infty} \frac{(t-a)!(n+a-1-t)!}{(-a)!(n+a-1)!} x^t \Lambda^{(t)}_{n+a-c}(x)\right\} \qquad (27)$$

which exhibits the converging factor $\Sigma_{n+a-c}(x)$ in Eq. (21) as an infinite series involving the basic converging factor $\Lambda_{n+a-c}(x)$ and its reduced derivatives.

Let us now consider again the modified Bessel function of the second kind. From Eqs. (6) and (7) we obtain

$$K_p(z) = \pi^{1/2}(2z)^{-1/2} e^{-z} \psi(p + \tfrac{1}{2},\ 2p+1,\ 2z) \qquad (28)$$

where the principal square root is selected, in accordance with the stipulations $\arg w^{1/2} = \frac{1}{2} \arg w$ and $-\pi < \arg w \leq \pi$.

Thus, setting $a = p+\frac{1}{2}$, $c = 2p+1$, and $x = 2z$ in (11), we deduce the well-known asymptotic expansion

$$K_p(x) \sim \pi^{1/2}(2z)^{-1/2} e^{-z}\left\{1 + \frac{4p^2-1}{1!8z} + \frac{(4p^2-1)(4p^2-9)}{2!(8z)^2} + \cdots\right.$$

$$\left. + \frac{(4p^2-1)(4p^2-9)\cdots[4p^2-(2r-3)^2]}{(r-1)!(8z)^{r-1}} \cdots\right\} \qquad (29)$$

This can be written in the form of the following terminating series:

$$K_p(z) = \pi^{1/2}(2z)^{-1/2}e^{-z}\{ \sum_{r=1}^{[p+\frac{1}{2}]} \frac{(r+p-\frac{3}{2})!}{(p+\frac{1}{2}-r)!(r-1)!(2z)^{r-1}}$$

$$+ \frac{\cos \pi p}{\pi} [\sum_{r=[p+\frac{3}{2}]}^{n} \frac{(r+1-\frac{3}{2})!(r-p-\frac{3}{2})!}{(r-1)!(-2z)^{r-1}}$$

$$+ \frac{(n+p-\frac{1}{2})!(n-p-\frac{1}{2})!}{n!\,(-2z)^n} \Sigma_n]\} \qquad (30)$$

where $[p+\frac{1}{2}]$ and $[p+\frac{3}{2}]$, respectively, represent the greatest integers not exceeding $p+\frac{1}{2}$ and $p+\frac{3}{2}$, and Σ_n designates the converging factor for this truncated series.

The coefficient $\frac{\cos \pi p}{p}$ arises from the relation

$$(-p-\frac{1}{2})!(p-\frac{1}{2})! = \Gamma(-p+\frac{1}{2})\,\Gamma(p+\frac{1}{2})$$

$$= \frac{\pi}{\sin(p+\frac{1}{2})\pi} = \frac{\pi}{\cos \pi p} \qquad (31)$$

From Eq. (27) we infer that the converging Σ_n can be evaluated by means of the series

$$\Sigma_n = \sum_{t=0}^{\infty} \frac{(t-p-\frac{1}{2})!(n+p-\frac{1}{2}-t)!}{(-p-\frac{1}{2})!\,(n+p-\frac{1}{2})!} (2z)^t \Lambda^{(t)}_{n-p-\frac{1}{2}}(2z) \qquad (32)$$

Since

$$\frac{(t-p-\frac{1}{2})!}{(-p-\frac{1}{2})!} = (t-p-\frac{1}{2})(t-p-\frac{3}{2}) \cdots (-p+\frac{1}{2})$$

$$= (-1)^t (p-\frac{1}{2})(p-\frac{3}{2}) \cdots (p-t+\frac{3}{2})(p-t+\frac{1}{2})$$

$$= (-1)^t \frac{(p-\frac{1}{2})!}{(p-\frac{1}{2}-t)!} \tag{33}$$

we can write alternatively

$$\Sigma_n = \sum_{t=0}^{\infty} \frac{(p-\frac{1}{2})!(n+p-\frac{1}{2}-t)!}{(p-\frac{1}{2}-t)!(n+p-\frac{1}{2})!} (-2z)^t \Lambda^{(t)}_{n-p-\frac{1}{2}}(2z) \tag{34}$$

If in Eq. (20) we make the substitutions $a = p+\frac{1}{2}$ $c = 2p+1$, and $ux = 2z$, we obtain for the basic converging factor $\Lambda_{n-p-\frac{1}{2}}$ the expression

$$\Lambda_{n-p-\frac{1}{2}}(2z) = \frac{1}{(n-p-\frac{1}{2})!} \int_0^{\infty} \frac{e^{-t}\, t^{n-p-\frac{1}{2}}}{1+\frac{t}{2z}}\, dt = C_{n-p}(2z) \tag{35}$$

where $C_n(y)$ designates the converging factor for the probability integral, as developed by Murnaghan.

To derive Eq. (35) we proceed as follows: If erfc(x) denotes the complementary error function, we have by definition

$$\text{erfc}(x) = \int_x^{\infty} e^{-t^2}\, dt$$

$$= \frac{1}{2} \int_y^{\infty} e^{-u}\, u^{-1/2}\, du \,, \tag{36}$$

where $u = t^2$ and $y = x^2$.

Furthermore, if we set $u = v + y$, we obtain

$$\operatorname{erfc}(x) = \frac{e^{-y}}{2y^{1/2}} \int_0^\infty e^{-v} \left(1 + \frac{v}{y}\right)^{-1/2} dv$$

$$= \frac{e^{-y}}{2y^{1/2}} C(y) \text{ , say.} \tag{37}$$

Then repeated integration by parts yields

$$C(y) = 1 - \frac{1}{2y} + \frac{1 \cdot 3}{(2y)^2} - \cdots + (-1)^{n-1} \frac{1 \cdot 3 \cdots (2n-3)}{(2y)^{n-1}}$$

$$+ (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{(2y)^n} C_n(y) \tag{38}$$

where the converging factor $C_n(y)$ is given by

$$C_n(y) = \int_x^\infty e^{-v} \left(1 + \frac{v}{y}\right)^{-n-\frac{1}{2}} dv \tag{39}$$

Moreover, from the definition of the exponential integral we have

$$-\operatorname{Ei}(-x) = \int_x^\infty e^{-t} t^{-1} dt$$

$$= \frac{e^{-x}}{x} \int_0^\infty e^{-u} \left(1 + \frac{u}{x}\right)^{-1} du \tag{40}$$

where $x = t-u$.

Thus, if we set $-\operatorname{Ei}(-x) = \frac{e^{-x}}{x} \Lambda(x)$, then repeated integration by parts yields

$$\Lambda(x) = 1 - \frac{1}{x} + \frac{2!}{x^2} - \cdots + (-1)^{n-1} \frac{(n-1)!}{x^{n-1}}$$

$$+ (-1)^n \frac{n!}{x^n} \Lambda_n(x) \tag{41}$$

where the converging factor $\Lambda_n(x)$ is given by

$$\Lambda_n(x) = \int_0^\infty e^{-u} \left(1 + \frac{u}{x}\right)^{-n-1} du \tag{42}$$

Comparison of Eqs. (39) and (42) reveals that

$$\Lambda_{n-\frac{1}{2}}(x) = C_n(x) \tag{43}$$

On the other hand, if we write

$$\left(1+\frac{u}{x}\right)^{-1} = 1 - \frac{u}{x} + \cdots + (-1)^{n-1} \frac{u^{n-1}}{x^{n-1}} + (-1)^n \frac{u^n}{1+\frac{u}{x}} \tag{44}$$

introduce this finite series in the integrand of Eq. (40), and integrate term by term, we obtain the series in Eq. (41), where $\Lambda_n(x)$ now assumes the equivalent form

$$\Lambda_n(x) = \frac{1}{n!} \int_0^\infty \frac{e^{-u} u^n}{1 + \frac{u}{x}} du \tag{45}$$

and thus the validity of Eq. (35) is finally established.

THE DIFFERENCE AND DIFFERENTIAL EQUATIONS SATISFIED BY THE CONVERGING FACTOR $C_n(y)$

From Eq. (38) we infer that $C_n(y)$ satisfies the difference equation

$$C_n(y) = 1 - \frac{2n+1}{2y} C_{n+1}(y) \tag{46}$$

as noted by Murnaghan[10].

To derive the differential equation satisfied by $C_n(y)$ we differentiate both sides of Eq. (39) with respect to y. This yields

$$\frac{d}{dy} C_n(y) = \frac{m}{y^2} \int_0^\infty e^{-v} v(1 + \frac{v}{y})^{-n-\frac{3}{2}} dv$$

$$= \frac{m}{y} \{ C_n(y) - C_{n+1}(y) \}$$

$$= (1 + \frac{m}{y}) C_n(y) - 1 \tag{47}$$

by virtue of Eq. (46). Here we have replaced $(2n+1)/2$ by m, for typographical simplicity.

Then from Eq. (47) we obtain by differentiation the relation

$$y \frac{d^2}{dy^2} C_n(y) = (y+m) \frac{d}{dy} C_n(y) - \frac{m}{y} C_n(y)$$

$$= (y+m-1) \frac{d}{dy} C_n(y) + C_n(y) - 1 \ . \tag{48}$$

Continuing this process, we find

$$y \frac{d^3}{dy^3} C_n(y) = (y+m-2) \frac{d^2}{dy^2} C_n(y) + 2 \frac{d}{dy} C_n(y) \tag{49}$$

and in general

$$y \frac{d^k}{dy^k} C_n(y) = (y+m-k+1) \frac{d^{k-1}}{dy^{k-1}} C_n(y) + (k-1) \frac{d^{k-2}}{dy^{k-2}} C_n(y) \tag{50}$$

if $k \geq 3$.

These derivatives are required in the Taylor series expansion

$$C_n(y_0+h) = C_n(y_0) + d_1 h + d_2 h^2 + \cdots \tag{51}$$

where

$$d_j = \left[\frac{1}{j!} \frac{d^j}{dy^j} C_n(y) \right]_{y=y_0} \tag{52}$$

is the j^{th} reduced derivative of $C_n(y)$, evaluated at $y = y_0$.

From Eqs. (47)-(50) we then obtain the following equivalent recurrence relations among the reduced derivatives:

$$y_0 d_1 = (y_0+m)d_0 - y_0 \tag{53}$$

$$2y_0 d_2 = (y_0+m-1)d_1 + d_0 - 1 \tag{54}$$

$$3y_0 d_3 = (y_0+m-2)d_2 + d_1 \tag{55}$$

. . . .

$$ky_0 d_k = (y_0+m-k+1)d_{k-1} + d_{k-2} \quad (k \geq 3) \tag{56}$$

Thus we have derived a systematic procedure for finding the successive reduced derivatives of the converging factor $C_n(y)$ (or $\Lambda_{n-\frac{1}{2}}(y)$ in the notation of Dingle) in terms of that factor.

In using Eq. (30) to calculate $K_p(z)$, it is generally most convenient to select n so that the asymptotic series is truncated at its least term numerically. Hence, if we are given the order p of the Bessel function and the argument z, we determine n such that

$$(n+p-\frac{1}{2})(n-p-\frac{1}{2}) \leqq 2nz \qquad (57)$$

which implies

$$n \leqq z + \frac{1}{2} + (z^2 + z + p^2)^{1/2} \qquad (58)$$

When $p = \pm\frac{1}{2}$, this reduces to the simpler inequality $n \leq 2z + 1$; consequently, when $p = 0$ or 1, we can select n to be the greatest integer not exceeding $2z+1$ if we assume z to be real and positive.

In that case, the appropriate converging factor is $C_n(n)$, which has been tabulated by Murnaghan[10] to 63 decimal places for integer values of n from 2 to 64, inclusive.

These fundamental data have been used to calculate correct to 30 decimal places the reduced derivatives d_j for $y_0 = n = 10(1)40$; that is, for successive integer values of the argument from 10 to 40, inclusive. These values are tabulated in the Appendix.

THE ASYMPTOTIC SERIES FOR THE CONVERGING FACTOR $C_n(n)$

In his study of the probability integral, Murnaghan[10] derived the Airey asymptotic series for the converging factor $C_n(n + \frac{1}{2} + h)$, and then by setting $h = -\frac{1}{2}$, he deduced therefrom the corresponding asymptotic series for the converging factor $C_n(n)$.

This result can be written

$$2C_n(n) \sim 1 + \sum_{i=1} \frac{c_i}{(4n+2)^i} \tag{59}$$

where for convenient reference we list here in Table 1 the exact values of the first 30 coefficients c_i, which have been taken from the more extended Table 2 on page 38 of Murnaghan's report[10]

It should be pointed out here that the first 25 terms suffice to yield an approximation to $C_n(n)$ that is correct to more than 30 decimal places when $n = 40$. Thus by means of series (59) and the relations (53)-(56), one can readily extend the range of the tables in the Appendix, in order to accommodate values of n exceeding 40, if such are required.

TABLE 1

Coefficients in the Asymptotic Series $2C_n(n) \sim 1 + \frac{c_1}{4m} + \frac{c_2}{(4m)^2} + \frac{c_3}{(4m)^3} + \cdots$

i	c_i						
1							0
2							−1
3							−1
4							4
5							−21
6							−23
7							916
8							−6619
9							−3099
10						6	40760
11						−72	98875
12						71	97679
13						10988	76024
14					−1	83598	69769
15					5	79797	07895
16					370	89637	19852
17					−8723	83728	95349
18					52107	67357	60217
19				21	41277	71661	78716
20				−696	02236	33844	34419
21				6549	63005	10513	05805
22			1	91213	38271	60645	86192
23			−85	96151	42501	57889	82715
24			1159	74216	37624	11668	68319
25			24305	68772	33650	41843	64656
26		−15	47926	19401	73625	04069	29169
27		282	97994	75909	84357	78487	50447
28		4037	22733	27480	09248	96541	72372
29	−3	88039	12428	77823	20544	49772	83413
30	92	71310	71991	43807	59763	38256	22729

APPLICATIONS

As the first illustration of the use of the converging factor in evaluating $K_n(x)$ we take $n = 0$ and $x = 2\pi$, which is a relatively small argument for the effective utilization of the conventionally truncated asymptotic series. The value of $K_0(2\pi)$ to nine decimal places has been included in a table published by Olver[12] for use in aerodynamic calculations of interference on lifting surfaces in rectangular wind tunnels.

We evaluate $K_0(2\pi)$ by means of the series

$$K_0(2\pi) = \frac{e^{-2\pi}}{2}\{1 - \frac{1}{1!16\pi} + \frac{(3!!)^2}{2!(16\pi)^2} - \frac{(5!!)^2}{3!(16\pi)^3}$$

$$+ \cdots + \frac{(23!!)^2}{12!(16\pi)^{12}} - \frac{(25!!)^2}{13!(16\pi)^{13}}\Sigma_{13}\}$$

where the symbol $(2k-1)!!$ represents the product $1 \cdot 3 \cdots (2k-1)$, and the converging factor Σ_{13} can be calculated from the series

$$\Sigma_{13} = C_{13}(4\pi) + \frac{4\pi}{25} C_{13}^{(1)}(4\pi) + \frac{1 \cdot 3}{23 \cdot 25}(4\pi)^2 C_{13}^{(2)}(4\pi) + \cdots .$$

We find the value of $C_{13}(4\pi)$ from the series

$$C_{13}(4\pi) = C_{13}(13) + d_1 h + d_2 h^2 + \cdots$$

where

$$h = -13 + 4\pi = -0.43362\ 93856\ 40827\ 046 \ldots$$

and $C_{13}(13)$, d_1, d_2, ... (for $n = 13$) are tabulated in the Appendix.

We thus find the approximation

$$C_{13}(4\pi) = 0.49150\ 20002\ 93166\ 9\ldots ,$$

and then by means of formulas (47)-(50) we deduce the values

$$C_{13}^{(1)}(4\pi) = 0.01952\ 05672\ 75096$$

$$C_{13}^{(2)}(4\pi) = -.0_3 76\ 34752\ 45781$$

$$C_{13}^{(3)}(4\pi) = .0_4 3\ 04115\ 66724$$

$$C_{13}^{(4)}(4\pi) = -.0_5\ 12332\ 67319$$

$$C_{13}^{(5)}(4\pi) = .0_7 508\ 95093$$

$$C_{13}^{(6)}(4\pi) = .0_8 21\ 36555$$

$$C_{13}^{(7)}(4\pi) = .0_{10}\ 91198$$

To seven decimal places, we infer that $\Sigma_{13} = 0.50074\ 87$ and consequently

$$\frac{(25!!)^2}{13!(16\pi)^{13}}\ \Sigma_{13} = 0.0_6 3843\ 518\ldots.$$

Combining this with the earlier terms in the asymptotic series, we obtain the result

$$2\,e^{2\pi}\ K_0(2\pi) = 0.98164\ 65536\ 976\ldots$$

whence

$$K_0(2\pi) = 0.00091\ 65843\ 60904\ 39\ldots$$

Use of the power series for $K_0(x)$ leads to the series

$$K_0(2\pi) = -(\ln \pi + \gamma) I_0(2\pi) + \frac{\pi^2}{(1!)^2} + (1+\frac{1}{2})\frac{\pi^4}{(2!)^2} + (1+\frac{1}{2}+\frac{1}{3})\frac{\pi^6}{(3!)^2} + \cdots$$

where $\gamma = 0.57721\ 56649\ 01532\ 86060\ldots$ is Euler's constant and

$$I_0(2\pi) = 1 + \frac{\pi^2}{(1!)^2} + \frac{\pi^4}{(2!)^2} + \frac{\pi^6}{(3!)^2} + \cdots$$

If we evaluate 22 terms of each series to 21 decimal places, we obtain

$$I_0(2\pi) = 87.10851\ \ 06533\ \ 90810\ \ 99853 \cdots \text{ and}$$

$$K_0(2\pi) = 0.00091\ \ 65843\ \ 60904\ \ 37031 \cdots$$

correct to 20 decimal places.

Consequently, the value of $K_0(2\pi)$ found by means of the converging factor is too large by less than 2×10^{-17}. This accuracy represents a gain of seven decimal places beyond that obtainable by the standard use of the asymptotic series in this case.

As a second example of the effectiveness of the converging-factor method in the calculation of $K_n(x)$ to high precision, we evaluate $K_1(10)$ to more than 25 decimal places by that procedure.

If we truncate the appropriate asymptotic series at the least numerical term and introduce the corresponding converging factor, we obtain

$$K_1(10) = (\frac{\pi}{20})^{1/2} e^{-10} \{1 + \frac{3!!}{80} - \frac{5!!}{2!}\frac{1}{80^2} + \frac{3!!7!!}{3!}\frac{1}{80^3} - \frac{5!!9!!}{4!}\frac{1}{80^4} + \cdots + \frac{39!!43!!}{21!}\frac{1}{80^{21}}\Sigma_{20}\}$$

where the converging factor Σ_{20} is computed from the series

$$\Sigma_{20} = C_{20}(20) - \frac{20}{43} C_{20}^{(1)}(20) - \frac{20^2}{41 \cdot 43} C_{20}^{(2)}(20)$$

$$- \frac{3 \cdot 20^3}{39 \cdot 41 \cdot 43} C_{20}^{(3)}(20) - \frac{3 \cdot 5 \cdot 20^4}{37 \cdot 39 \cdot 41 \cdot 43} C_{20}^{(4)}(20) - \cdots$$

Using the values of $C_{20}(20)$ and the reduced derivatives $C_{20}^{(k)}(20)$ $(= d_k)$ tabulated in the Appendix, we calculate from the last series the approximation

$$\Sigma_{20} = 0.49424\ 91283\ 39$$

Then we find

$$\frac{39!!43!!}{21!} \frac{1}{80^{21}} \Sigma_{20} = 0.0_9189149\ 702220 \ldots$$

and combining this with the sum of the first 21 terms of the series within the braces, we obtain

$$\left(\frac{20}{\pi}\right)^{1/2} e^{10} K_1(10) = 1.03641\ 84932\ 28924\ 58809\ 9\ldots$$

whence

$$K_1(10) = 0.00001\ 86487\ 73453\ 82558\ 45968\ 10\ldots$$

This approximation to the value of $K_1(10)$ was checked by means of power series, which entailed the calculation of $I_1(10)$ also.

Thus, evaluating 35 terms of the series

$$I_1(10) = 5 + \frac{5^3}{1!2!} + \frac{5^5}{2!3!} + \frac{5^7}{3!4!} + \cdots$$

we find

$$I_1(10) = 2670.98830\ 37012\ 54654\ 34103\ 19667\ 72152\ 5..$$

and then substituting this value in the series

$$K_1(10) = (\ln 5 + \gamma)I_1(10) + \frac{1}{10} - \frac{1}{2}\left\{\frac{5}{0!1!} + \frac{5^3}{1!2!}\left(1+1+\frac{1}{2}\right)\right.$$

$$\left. + \frac{5^5}{2!3!}\left(1+\frac{1}{2}+1+\frac{1}{2}+\frac{1}{3}\right) + \cdots \right\}$$

we obtain

$$K_1(10) = 0.00001\ 86487\ 73453\ 82558\ 45968\ 168581 \cdots$$

provided we evaluate 36 terms of the series within the braces, and approximate each term to at least 32 decimal places.

Comparison of this latter value with the earlier one reveals that the use of the converging factor here suffices to yield the value of $K_1(10)$ to within 7×10^{-27}. This is a gain in accuracy of 12 decimal places over that obtainable by use of the asymptotic series without the converging factor.

APPENDIX

TABLE OF THE CONVERGING FACTOR $C_n(n)$ $(=d_0)$ AND ITS REDUCED DERIVATIVES, d_i, TO 30 DECIMAL PLACES FOR $n = 10(1)40$

$n = 10$

d_0	0.49971	03665	11039	76983	07766	52551
d_1	.02440	62513	47631	52815	30921	37729
d_2	$-.0_2 121$	83866	10507	27155	91963	33087
d_3	$.0_4 6$	22033	01774	90014	35986	65854
d_4	$-.0_5$	32457	20736	16297	61554	92016
d_5		$.0_6 1729$	78192	56422	07406	60952
d_6		$-.0_8 94$	09312	52362	59112	54121
d_7		$.0_9 5$	22045	15673	77861	06803
d_8		$-.0_{10}$	29521	28634	58224	85154
d_9			$.0_{11} 1700$	32308	23889	44915
d_{10}			$-.0_{13} 99$	67570	89834	96186
d_{11}			$.0_{14} 5$	94298	30732	93045
d_{12}			$-.0_{15}$	36014	47482	26769
d_{13}				$.0_{16} 2216$	73285	64350
d_{14}				$-.0_{17} 138$	49270	28530
d_{15}				$.0_{19} 8$	77686	85859
d_{16}				$-.0_{20}$	56387	45352
d_{17}					$.0_{21} 3670$	25481
d_{18}					$-.0_{22} 241$	89756
d_{19}					$.0_{23} 16$	13427
d_{20}					$-.0_{24} 1$	08848
d_{21}						$.0_{26} 7424$
d_{22}						$-.0_{27} 512$
d_{23}						$.0_{28} 36$
d_{24}						$-.0_{29} 3$

$n = 11$

d_0	0.49975	89636	36729	31597	99366	68663
d_1	.02223	42438	02400	87359	53250	04083
d_2	$-.0_2100$	93088	45984	17735	09352	61071
d_3	$.0_44$	67700	74537	12842	12470	34913
d_4	$-.0_5$	22111	90738	86757	12867	74552
d_5		$.0_61066$	00833	96506	09534	85558
d_6		$-.0_852$	37517	34059	09787	99656
d_7		$.0_92$	62101	27071	83091	33652
d_8		$-.0_{10}$	13351	67777	79225	82137
d_9			$.0_{12}691$	93881	75599	16088
d_{10}			$-.0_{13}36$	45912	49169	42863
d_{11}			$.0_{14}1$	95206	40999	84548
d_{12}			$-.0_{15}$	10613	93012	66065
d_{13}				$.0_{17}585$	73527	04132
d_{14}				$-.0_{18}32$	78860	42707
d_{15}				$.0_{19}1$	86080	08128
d_{16}				$-.0_{20}$	10700	33987
d_{17}					$.0_{22}623$	14370
d_{18}					$-.0_{23}36$	73257
d_{19}					$.0_{24}2$	19066
d_{20}					$-.0_{25}$	13211
d_{21}						$.0_{27}805$
d_{22}						$-.0_{28}50$
d_{23}						$.0_{29}3$

$n = 12$

d_0	0.49979	62861	67303	14262	69706	94482
d_1	.02041	74175	91577	24953	00651	67902
d_2	$-.0_{3}84$	97666	84609	64555	90207	44159
d_3	$.0_{4}3$	60463	10773	89512	36694	00676
d_4	$-.0_{5}$	15577	29228	56042	50026	79784
d_5		$.0_{7}685$	47693	14010	68519	07752
d_6		$-.0_{8}30$	70127	94900	47415	34425
d_7		$.0_{9}1$	39884	83432	76087	32392
d_8			$-.0_{11}6480$	65987	78290	49141
d_9			$.0_{12}305$	12913	28095	31681
d_{10}			$-.0_{13}14$	59298	59940	10901
d_{11}			$.0_{15}$	70856	69386	08891
d_{12}				$-.0_{16}3491$	20300	19381
d_{13}				$.0_{17}174$	46574	57478
d_{14}				$-.0_{19}8$	83837	45586
d_{15}				$.0_{20}$	45368	22938
d_{16}					$-.0_{21}2358$	53790
d_{17}					$.0_{22}124$	12087
d_{18}					$-.0_{24}6$	60940
d_{19}					$.0_{25}$	35596
d_{20}						$-.0_{26}1938$
d_{21}						$.0_{27}107$
d_{22}						$-.0_{29}6$

$n = 13$

d_0	0.49982	55690	82240	27535	66822	29194
d_1	.01887	51985	13797	48438	09291	59510
d_2	$-.0_{3}72$	52641	85227	84126	65240	07819
d_3	$.0_{4}2$	83647	68608	08649	61818	19691
d_4	$-.0_{5}$	11286	94671	88093	47356	00867
d_5		$.0_{7}456$	79053	70408	40712	43080
d_6		$-.0_{8}18$	79423	29798	88103	06085
d_7		$.0_{10}$	78581	05599	24556	04048
d_8			$-.0_{11}3337$	42986	66915	96415
d_9			$.0_{12}143$	91968	76808	63849
d_{10}			$-.0_{14}6$	29873	33252	03685
d_{11}			$.0_{15}$	27965	44602	44777
d_{12}				$-.0_{16}1259$	03153	29549
d_{13}				$.0_{18}57$	45259	64298
d_{14}				$-.0_{19}2$	65616	19844
d_{15}				$.0_{20}$	12436	19058
d_{16}					$-.0_{22}589$	42311
d_{17}					$.0_{23}28$	26809
d_{18}					$-.0_{24}1$	37127
d_{19}						$.0_{26}6726$
d_{20}						$-.0_{27}333$
d_{21}						$.0_{28}17$
d_{22}						$-.0_{29}1$

$n = 14$

d_0	0.49984	89654	04695	36092	61517	24574
d_1	.01754	96795	73844	12759	96660	10739
d_2	$-.0_{3}62$	62445	11235	39750	29654	63575
d_3	$.0_{4}2$	27190	48240	62128	02638	38714
d_4		$-.0_{6}8376$	56805	34919	38613	85292
d_5		$.0_{7}313$	77950	13665	75808	55701
d_6		$-.0_{8}11$	93749	72783	02465	62813
d_7		$.0_{10}$	46107	97204	56840	12167
d_8			$-.0_{11}1807$	39579	32860	74118
d_9			$.0_{13}71$	87585	93914	24546
d_{10}			$-.0_{14}2$	89868	95368	09253
d_{11}			$.0_{15}$	11850	71620	80866
d_{12}				$-.0_{17}490$	96083	35679
d_{13}				$.0_{18}20$	60363	98583
d_{14}				$-.0_{20}$	87553	27335
d_{15}					$.0_{21}3765$	91201
d_{16}					$-.0_{22}163$	89938
d_{17}					$.0_{24}7$	21500
d_{18}					$-.0_{25}$	32114
d_{19}						$.0_{26}1445$
d_{20}						$-.0_{28}66$
d_{21}						$.0_{29}3$

$n = 15$

d_0	0.49986	79530	13128	16639	23752	27736
d_1	.01639	81711	26693	93833	11629	63063
d_2	$-.0_{3}54$	61999	58313	35509	46105	78730
d_3	$.0_{4}1$	84771	62550	29595	85502	54872
d_4		$-.0_{6}6346$	33136	33693	72413	09496
d_5		$.0_{7}221$	25125	83156	16220	74043
d_6		$-.0_{9}7$	82693	64035	68430	93571
d_7		$.0_{10}$	28086	96802	68472	97920
d_8			$-.0_{11}1022$	08243	10494	29937
d_9			$.0_{13}37$	70454	31721	12032
d_{10}			$-.0_{14}1$	40956	50189	93475
d_{11}				$.0_{16}5338$	46077	74217
d_{12}				$-.0_{17}204$	75842	63312
d_{13}				$.0_{19}7$	95092	25143
d_{14}				$-.0_{20}$	31246	32492
d_{15}					$.0_{21}1242$	34618
d_{16}					$-.0_{23}49$	95816
d_{17}					$.0_{24}2$	03119
d_{18}						$-.0_{26}8347$
d_{19}						$.0_{27}347$
d_{20}						$-.0_{28}15$
d_{21}						$.0_{29}1$

$n = 16$

d_0	0.49988	35735	94367	86390	45222	58666
d_1	.01538	85088	63559	72355	60608	37916
d_2	$-.0_{3}48$	05749	12609	40137	74862	92093
d_3	$.0_{4}1$	52286	25603	60378	21401	86022
d_4		$-.0_{6}4895$	38395	36077	81929	81319
d_5		$.0_{7}159$	59766	69777	00455	02730
d_6		$-.0_{9}5$	27550	11637	60618	92253
d_7		$.0_{10}$	17675	79119	46821	90697
d_8			$-.0_{12}600$	13625	71223	90855
d_9			$.0_{13}20$	64203	39943	30658
d_{10}			$-.0_{15}$	71905	28640	97627
d_{11}				$.0_{16}2535$	99122	28034
d_{12}				$-.0_{18}90$	52851	62473
d_{13}				$.0_{19}3$	26998	38449
d_{14}				$-.0_{20}$	11948	13896
d_{15}					$.0_{22}441$	49089
d_{16}					$-.0_{23}16$	49238
d_{17}					$.0_{25}$	62267
d_{18}						$-.0_{26}2375$
d_{19}						$.0_{28}92$
d_{20}						$-.0_{29}4$

$n = 17$

d_0	0.49989	65784	61135	97460	24139	91377
d_1	.01449	59974	65246	53669	31342	76619
d_2	$-.0_3 42$	61031	30973	67782	87525	80644
d_3	$.0_4 1$	26989	35462	78445	60524	58935
d_4		$-.0_6 3836$	27410	23481	56338	25356
d_5		$.0_7 117$	44699	41908	91673	03360
d_6		$-.0_9 3$	64301	73893	80901	80159
d_7		$.0_{10}$	11446	21730	54924	13183
d_8			$-.0_{12} 364$	19678	70371	23659
d_9			$.0_{13} 11$	73204	21503	83243
d_{10}			$-.0_{15}$	38252	77188	37359
d_{11}				$.0_{16} 1262$	09253	41540
d_{12}				$-.0_{18} 42$	12547	71133
d_{13}				$.0_{19} 1$	42203	30276
d_{14}					$-.0_{21} 4853$	68362
d_{15}					$.0_{22} 167$	46192
d_{16}					$-.0_{24} 5$	83888
d_{17}					$.0_{25}$	20568
d_{18}						$-.0_{27} 732$
d_{19}						$.0_{28} 26$
d_{20}						$-.0_{29} 1$

$n = 18$

d_0	0.49990	75205	19477	48462	38400	06143
d_1	.01370	13610	53384	89937	61200	12457
d_2	$-.0_338$	03933	91259	96076	45527	65324
d_3	$.0_41$	06997	97313	26394	44268	44607
d_4		$-.0_63048$	63628	68914	75868	53764
d_5		$.0_887$	96993	12074	05317	12192
d_6		$-.0_92$	57021	71653	53781	28886
d_7			$.0_{11}7601$	83147	94563	39533
d_8			$-.0_{12}227$	55338	81348	34116
d_9			$.0_{14}6$	89234	51704	54119
d_{10}			$-.0_{15}$	21118	83108	18588
d_{11}				$.0_{17}654$	47218	87685
d_{12}				$-.0_{18}20$	50828	82790
d_{13}				$.0_{20}$	64965	43843
d_{14}					$-.0_{21}2079$	92470
d_{15}					$.0_{23}67$	28568
d_{16}					$-.0_{24}2$	19890
d_{17}						$.0_{26}7258$
d_{18}						$-.0_{27}242$
d_{19}						$.0_{29}8$

$n = 19$

d_0	0.49991	68139	68100	10871	35781	20729
d_1	.01298	93335	66939	69397	22504	02531
d_2	$-.0_3 34$	16625	59780	56229	80797	83799
d_3	$.0_5$	90991	25174	54684	37252	33226
d_4		$-.0_6 2453$	10737	94933	34978	15846
d_5		$.0_8 66$	93733	84520	88215	85121
d_6		$-.0_9 1$	84830	29942	84032	86967
d_7			$.0_{11} 5163$	52717	13211	63599
d_8			$-.0_{12} 145$	91574	69196	48905
d_9			$.0_{14} 4$	17015	72530	51883
d_{10}			$-.0_{15}$	12050	58313	40097
d_{11}				$.0_{17} 352$	02921	52436
d_{12}				$-.0_{18} 10$	39377	06790
d_{13}				$.0_{20}$	31009	83492
d_{14}					$-.0_{22} 934$	68525
d_{15}					$.0_{23} 28$	45630
d_{16}					$-.0_{25}$	87488
d_{17}						$.0_{26} 2716$
d_{18}						$-.0_{28} 85$
d_{19}						$.0_{29} 3$

$n = 20$

d_0	0.49992	47740	47568	15571	94026	63730
d_1	.01234	76674	46325	51533	17903	94054
d_2	$-.0_3 30$	85590	45564	34971	68719	19278
d_3	$.0_5$	78024	03201	63418	72036	91697
d_4		$-.0_6 1996$	11568	78834	62091	68508
d_5		$.0_8 51$	65809	40859	55056	90412
d_6		$-.0_9 1$	35211	12319	33813	09657
d_7			$.0_{11} 3578$	75470	30275	03623
d_8			$-.0_{13} 95$	76775	40122	49614
d_9			$.0_{14} 2$	59057	05423	85506
d_{10}				$-.0_{16} 7082$	39096	35531
d_{11}				$.0_{17} 195$	65513	56826
d_{12}				$-.0_{19} 5$	46068	52538
d_{13}				$.0_{20}$	15394	46383
d_{14}					$-.0_{22} 438$	28846
d_{15}					$.0_{23} 12$	59940
d_{16}					$-.0_{25}$	36564
d_{17}						$.0_{26} 1071$
d_{18}						$-.0_{28} 32$
d_{19}						$.0_{29} 1$

$n = 21$

d_0	0.49993	16441	09513	89342	89211	09944
d_1	.01176	64226	02587	64146	32927	22505
d_2	$-.0_{3}28$	00432	82930	92823	43912	12050
d_3	$.0_{5}$	67407	87998	17536	46102	95785
d_4		$-.0_{6}1640$	73297	65477	77652	92006
d_5		$.0_{8}40$	37771	79491	82918	71939
d_6		$-.0_{9}1$	00451	23289	95382	54717
d_7			$.0_{11}2525$	86254	47969	08672
d_8			$-.0_{13}64$	18519	38050	47600
d_9			$.0_{14}1$	64800	71879	51145
d_{10}				$-.0_{16}4274$	73952	79449
d_{11}				$.0_{17}111$	99863	26273
d_{12}				$-.0_{19}2$	96342	30166
d_{13}					$.0_{21}7917$	30059
d_{14}					$-.0_{22}213$	54399
d_{15}					$.0_{24}5$	81364
d_{16}					$-.0_{25}$	15973
d_{17}						$.0_{27}443$
d_{18}						$-.0_{28}12$

$n = 22$

d_0	0.49993	76144	26027	46952	38495	87970
d_1	.01123	74473	61737	38153	68775	75667
d_2	$-.0_{3}25$	53051	21327	19167	32267	24330
d_3	$.0_{5}$	58633	28868	66265	79506	33207
d_4		$-.0_{6}1361$	01969	06467	46394	93707
d_5		$.0_{8}31$	92719	28675	75913	73982
d_6		$-.0_{10}$	75678	46392	23468	19859
d_7			$.0_{11}1812$	32743	99171	35126
d_8			$-.0_{13}43$	84195	98037	17345
d_9			$.0_{14}1$	07119	14499	06323
d_{10}				$-.0_{16}2643$	02878	72922
d_{11}				$.0_{18}65$	84566	87151
d_{12}				$-.0_{19}1$	65605	63839
d_{13}					$.0_{21}4204$	13854
d_{14}					$-.0_{22}107$	71193
d_{15}					$.0_{24}2$	78462
d_{16}						$-.0_{26}7263$
d_{17}						$.0_{27}191$
d_{18}						$-.0_{29}5$

$n = 23$

d_0	0.49994	28354	98693	55310	83704	31386
d_1	.01075	39935	08228	27041	47489	15627
d_2	$-.0_{3}23$	37056	49498	26397	87076	93643
d_3	$.0_{5}$	51317	69645	73222	26413	99254
d_4		$-.0_{6}1138$	44238	14035	10196	39414
d_5		$.0_{8}25$	51213	25884	61244	06297
d_6		$-.0_{10}$	57745	56447	99772	23030
d_7			$.0_{11}1319$	98694	04164	40209
d_8			$-.0_{13}30$	46782	78996	07798
d_9			$.0_{15}$	71002	68902	48985
d_{10}				$-.0_{16}1670$	35631	09873
d_{11}				$.0_{18}39$	66278	13196
d_{12}				$-.0_{20}$	95046	22252
d_{13}					$.0_{21}2298$	27243
d_{14}					$-.0_{23}56$	06862
d_{15}					$.0_{24}1$	37983
d_{16}						$-.0_{26}3425$
d_{17}						$.0_{28}86$
d_{18}						$-.0_{29}2$

$n = 24$

d_0	0.49994	74276	31398	40823	28114	58076
d_1	.01031	04266	71784	28330	38064	88195
d_2	$-.0_{3}21$	47355	30392	66947	57579	24014
d_3	$.0_{5}$	45170	07062	84934	27925	41966
d_4		$-.0_{7}959$	55302	42733	72780	96506
d_5		$.0_{8}20$	58300	87360	69493	10395
d_6		$-.0_{10}$	44577	87795	44096	04822
d_7			$.0_{12}974$	64916	99079	82771
d_8			$-.0_{13}21$	51009	06418	14166
d_9			$.0_{15}$	47912	26801	59764
d_{10}				$-.0_{16}1076$	97698	97931
d_{11}				$.0_{18}24$	42671	93521
d_{12}				$-.0_{20}$	55894	10211
d_{13}					$.0_{21}1290$	18336
d_{14}					$-.0_{23}30$	03748
d_{15}					$.0_{25}$	70525
d_{16}						$-.0_{26}1670$
d_{17}						$.0_{28}40$
d_{18}						$-.0_{29}1$

$n = 25$

d_0	0.49995	14879	19156	07409	19444	17838
d_1	$.0_2990$	20055	96695	26966	57277	24032
d_2	$-.0_319$	79847	00888	56154	90906	64852
d_3	$.0_5$	39966	34714	66712	71310	73049
d_4		$-.0_7814$	45519	41873	01036	46950
d_5		$.0_816$	75344	49356	94184	91919
d_6		$-.0_{10}$	34782	29974	21437	48431
d_7			$.0_{12}728$	75517	16801	23924
d_8			$-.0_{13}15$	40724	88702	91789
d_9			$.0_{15}$	32865	37541	89880
d_{10}				$-.0_{17}707$	24722	85647
d_{11}				$.0_{18}15$	35222	78622
d_{12}				$-.0_{20}$	33611	40934
d_{13}					$.0_{22}742$	10316
d_{14}					$-.0_{23}16$	52155
d_{15}					$.0_{25}$	37084
d_{16}						$-.0_{27}839$
d_{17}						$.0_{28}19$

$n = 26$

d_0	0.49995	50954	20341	91191	10809	02262
d_1	$.0_{2}952$	47119	06459	62982	04518	21875
d_2	$-.0_{3}18$	31200	26865	13754	49163	51369
d_3	$.0_{5}$	35532	12176	54081	79766	16381
d_4		$-.0_{7}695$	77155	06122	16834	02313
d_5		$.0_{8}13$	74770	43131	97410	12340
d_6		$-.0_{10}$	27407	43316	36793	93052
d_7			$.0_{12}551$	23510	55420	28766
d_8			$-.0_{13}11$	18382	62572	93674
d_9			$.0_{15}$	22886	68251	81454
d_{10}				$-.0_{17}472$	35360	07309
d_{11}				$.0_{19}9$	83095	97450
d_{12}				$-.0_{20}$	20631	01645
d_{13}					$.0_{22}436$	50831
d_{14}					$-.0_{24}9$	31027
d_{15}					$.0_{25}$	20016
d_{16}						$-.0_{27}434$
d_{17}						$.0_{29}9$

$n = 27$

d_0	0.49995	83150	32556	55655	23951	89992
d_1	$.0_2917$	51173	80530	82711	50199	20540
d_2	$-.0_316$	98686	13130	44801	47044	27058
d_3	$.0_5$	31730	27051	63341	16424	38272
d_4		$-.0_7597$	93703	43821	58807	30149
d_5		$.0_811$	36629	83706	30345	59746
d_6		$-.0_{10}$	21793	37347	89856	17424
d_7			$.0_{12}421$	43504	40858	84205
d_8			$-.0_{14}8$	21855	96523	43137
d_9			$.0_{15}$	16161	32520	65549
d_{10}				$-.0_{17}320$	42840	12447
d_{11}				$.0_{19}6$	40492	03743
d_{12}				$-.0_{20}$	12905	66820
d_{13}					$.0_{22}262$	11151
d_{14}					$-.0_{24}5$	36519
d_{15}					$.0_{25}$	11067
d_{16}						$-.0_{27}230$
d_{17}						$.0_{29}5$

$n = 28$

d_0	0.49996	12004	30562	21298	60435	26779
d_1	$.0_{2}885$	02794	40241	60834	68378	30822
d_2	$-.0_{3}15$	80051	89929	08078	15081	58261
d_3	$.0_{5}$	28451	97489	36554	46981	33400
d_4		$-.0_{7}516$	70752	21450	12514	10905
d_5		$.0_{9}9$	46307	12931	59214	21863
d_6		$-.0_{10}$	17475	80389	72101	08244
d_7			$.0_{12}325$	40322	70755	66100
d_8			$-.0_{14}6$	10867	92721	85653
d_9			$.0_{15}$	11560	42951	37151
d_{10}				$-.0_{17}220$	52687	61325
d_{11}				$.0_{19}4$	24003	17323
d_{12}					$-.0_{21}8215$	90247
d_{13}					$.0_{22}160$	42723
d_{14}					$-.0_{24}3$	15642
d_{15}						$.0_{26}6257$
d_{16}						$-.0_{27}125$
d_{17}						$.0_{29}3$

$n = 29$

d_0	0.49996	37963	16978	38157	69576	10228
d_1	$.0_{2}854$	76580	87697	76973	28282	82701
d_2	$-.0_{3}14$	73424	76558	61825	49209	67835
d_3	$.0_{5}$	25610	13357	88319	91930	29885
d_4		$-.0_{7}448$	81337	89741	98250	67321
d_5		$.0_{9}7$	92968	56878	49567	36972
d_6		$-.0_{10}$	14123	67510	01416	07123
d_7			$.0_{12}253$	57451	24508	49079
d_8			$-.0_{14}4$	58874	01246	67584
d_9				$.0_{16}8369$	01768	39602
d_{10}				$-.0_{17}153$	81943	83129
d_{11}				$.0_{19}2$	84882	42188
d_{12}					$-.0_{21}5316$	17469
d_{13}					$.0_{23}99$	94774
d_{14}					$-.0_{24}1$	89299
d_{15}						$.0_{26}3611$
d_{16}						$-.0_{28}69$
d_{17}						$.0_{29}1$

$n = 30$

d_0	0.49996	61401	63493	40969	01799	09275
d_1	$.0_{2}826$	50493	29711	70954	18628	17038
d_2	$-.0_{3}13$	77237	45310	99787	61497	07949
d_3	$.0_{5}$	23134	46989	09259	76344	98911
d_4		$-.0_{7}391$	71195	31811	26013	83505
d_5		$.0_{9}6$	68496	35746	15710	42206
d_6		$-.0_{10}$	11498	04154	99744	91895
d_7			$.0_{12}199$	30044	27998	15400
d_8			$-.0_{14}3$	48111	60841	01533
d_9				$.0_{16}6126$	61051	27722
d_{10}				$-.0_{17}108$	63722	33413
d_{11}				$.0_{19}1$	94069	91941
d_{12}					$-.0_{21}3492$	39256
d_{13}					$.0_{23}63$	30482
d_{14}					$-.0_{24}1$	15575
d_{15}						$.0_{26}2125$
d_{16}						$-.0_{28}39$
d_{17}						$.0_{29}1$

$n = 31$

d_0	0.49996	82635	70250	98919	28166	19019
d_1	$.0_2 800$	05313	91635	05885	64851	18990
d_2	$-.0_3 12$	90170	29745	04663	11830	41340
d_3	$.0_5$	20967	85936	12621	15044	20623
d_4		$-.0_7 343$	40859	23820	19892	74309
d_5		$.0_9 5$	66746	26252	51298	83055
d_6			$-.0_{11} 9424$	45775	81022	63434
d_7			$.0_{12} 157$	90045	71076	12899
d_8			$-.0_{14} 2$	66525	15670	55434
d_9				$.0_{16} 4531$	98806	56243
d_{10}				$-.0_{18} 77$	62514	57892
d_{11}				$.0_{19} 1$	33920	20871
d_{12}					$-.0_{21} 2326$	94578
d_{13}					$.0_{23} 40$	71823
d_{14}					$-.0_{25}$	71750
d_{15}						$.0_{26} 1273$
d_{16}						$-.0_{28} 23$

$n = 32$

d_0	0.49997	01933	36182	88359	92436	35771
d_1	$.0_2775$	24209	43243	62475	47254	53350
d_2	$-.0_312$	11105	74497	60850	74326	57445
d_3	$.0_5$	19063	54553	57388	58352	53782
d_4		$-.0_7302$	32573	85011	35044	10546
d_5		$.0_94$	83023	97276	26176	15098
d_6			$-.0_{11}7774$	20557	67591	47459
d_7			$.0_{12}126$	03993	98312	83432
d_8			$-.0_{14}2$	05823	84236	73243
d_9				$.0_{16}3385$	23225	47726
d_{10}				$-.0_{18}56$	07328	82108
d_{11}				$.0_{20}$	93533	53616
d_{12}					$-.0_{21}1571$	05374
d_{13}					$.0_{23}26$	57023
d_{14}					$-.0_{25}$	45243
d_{15}						$.0_{27}776$
d_{16}						$-.0_{28}13$

$n = 33$

d_0	0.49997	19523	08451	71102	68177	84255
d_1	$.0_{2}751$	92372	27637	53888	73752	31908
d_2	$-.0_{3}11$	39092	31534	68927	04561	29178
d_3	$.0_{5}$	17383	00945	96041	35652	01009
d_4		$-.0_{7}267$	20617	16820	46141	35342
d_5		$.0_{9}4$	13711	35119	77283	74195
d_6			$-.0_{11}6451$	35893	70925	20820
d_7			$.0_{12}101$	31660	39118	21925
d_8			$-.0_{14}1$	60235	22891	63319
d_9				$.0_{16}2551$	85016	69252
d_{10}				$-.0_{18}40$	92074	03580
d_{11}				$.0_{20}$	66068	41231
d_{12}					$-.0_{21}1073$	93220
d_{13}					$.0_{23}17$	57368
d_{14}					$-.0_{25}$	28948
d_{15}						$.0_{27}480$
d_{16}						$-.0_{29}8$

$n = 34$

d_0	0.49997	35600	61604	03366	13365	71789
d_1	$.0_2729$	96724	77055	18546	47516	22575
d_2	$-.0_310$	73315	84370	16099	21754	25065
d_3	$.0_5$	15894	32514	11254	39714	29958
d_4		$-.0_7237$	04078	64551	00150	49727
d_5		$.0_93$	55996	71457	14588	27780
d_6			$-.0_{11}5383$	76112	85410	75910
d_7			$.0_{13}81$	98169	76371	49510
d_8			$-.0_{14}1$	25693	64847	66107
d_9				$.0_{16}1940$	20925	12418
d_{10}				$-.0_{18}30$	15058	24345
d_{11}				$.0_{20}$	47165	82321
d_{12}					$-.0_{22}742$	70443
d_{13}					$.0_{23}11$	77155
d_{14}					$-.0_{25}$	18778
a_{15}						$.0_{27}301$
d_{16}						$-.0_{29}5$

$n = 35$

d_0	0.49997	50334	44358	32502	31614	34201
d_1	$.0_{2}709$	25673	66493	19754	66537	46034
d_2	$-.0_{3}10$	13076	36919	49207	83486	17378
d_3	$.0_{5}$	14570	87976	26657	31406	99578
d_4		$-.0_{7}211$	01418	01070	27953	67113
d_5		$.0_{9}3$	07678	16317	04985	64495
d_6			$-.0_{11}4516$	65868	11197	11394
d_7			$.0_{13}66$	74970	70950	08896
d_8			$-.0_{15}$	99304	38592	38023
d_9				$.0_{16}1487$	13202	94071
d_{10}				$-.0_{18}22$	41647	46150
d_{11}				$.0_{20}$	34009	17278
d_{12}					$-.0_{22}519$	28972
d_{13}					$.0_{24}7$	97961
d_{14}					$-.0_{25}$	12339
d_{15}						$.0_{27}192$
d_{16}						$-.0_{29}3$

$n = 36$

d_0	0.49997	63870	22708	95104	40127	80620
d_1	$.0_2689$	68905	31844	41529	69701	83194
d_2	$-.0_49$	57769	43755	76882	25363	76681
d_3	$.0_5$	13390	37009	83993	80384	78029
d_4		$-.0_7188$	46330	36036	89504	31623
d_5		$.0_92$	67018	77863	70274	10621
d_6			$-.0_{11}3808$	16112	20814	82475
d_7			$.0_{13}54$	66692	07087	65183
d_8			$-.0_{15}$	78985	35265	88066
d_9				$.0_{16}1148$	57044	56281
d_{10}				$-.0_{18}16$	80869	26706
d_{11}				$.0_{20}$	24754	33175
d_{12}					$-.0_{22}366$	84691
d_{13}					$.0_{24}5$	47029
d_{14}						$-.0_{26}8207$
d_{15}						$.0_{27}124$
d_{16}						$-.0_{29}2$

$n = 37$

d_0	0.49997	76334	41220	86107	89144	22884
d_1	$.0_{2}671$	17213	88404	16622	64628	24455
d_2	$-.0_{4}9$	06870	87987	47190	91360	53779
d_3	$.0_{5}$	12334	00984	79687	21990	89419
d_4		$-.0_{7}168$	84578	20605	09925	75408
d_5		$.0_{9}2$	32639	03389	33903	92017
d_6			$-.0_{11}3225$	97004	71187	40226
d_7			$.0_{13}45$	01963	57751	22342
d_8			$-.0_{15}$	63224	53793	85075
d_9				$.0_{17}893$	48890	27072
d_{10}				$-.0_{18}12$	70544	54356
d_{11}				$.0_{20}$	18178	78922
d_{12}					$-.0_{22}261$	69241
d_{13}					$.0_{24}3$	79005
d_{14}						$-.0_{26}5522$
d_{15}						$.0_{28}81$
d_{16}						$-.0_{29}1$

$n = 38$

d_0	0.49997	87837	19331	70565	27416	90713
d_1	$.0_2653$	62356	45496	72322	19668	24724
d_2	$-.0_48$	59924	34811	39067	22205	66350
d_3	$.0_5$	11385	89930	24665	03643	38874
d_4		$-.0_7151$	71545	64645	96739	58284
d_5		$.0_92$	03436	16251	74947	49280
d_6			$-.0_{11}2745$	00011	60473	65723
d_7			$.0_{13}37$	26937	72035	91977
d_8			$-.0_{15}$	50913	94749	92511
d_9				$.0_{17}699$	80209	54985
d_{10}				$-.0_{19}9$	67712	11924
d_{11}				$.0_{20}$	13462	56847
d_{12}					$-.0_{22}188$	40764
d_{13}					$.0_{24}2$	65238
d_{14}						$-.0_{26}3756$
d_{15}						$.0_{28}53$
d_{16}						$-.0_{29}1$

$n = 39$

d_0	0.49997	98474	95686	93118	27964	25927
d_1	$.0_2636$	96930	36190	36148	33210	11161
d_2	$-.0_48$	16531	05122	56479	30746	82168
d_3	$.0_5$	10532	52088	15516	93428	01926
d_4		$-.0_7136$	70336	32595	83826	48313
d_5		$.0_91$	78523	24036	54863	35911
d_6			$-.0_{11}2345$	63315	85262	26320
d_7			$.0_{13}31$	00672	66385	89479
d_8			$-.0_{15}$	41234	68072	66277
d_9				$.0_{17}551$	64579	09735
d_{10}				$-.0_{19}7$	42384	16768
d_{11}				$.0_{20}$	10049	56553
d_{12}					$-.0_{22}136$	83439
d_{13}					$.0_{24}1$	87392
d_{14}						$-.0_{26}2581$
d_{15}						$.0_{28}36$

$n = 40$

d_0	0.49998	08332	30668	59330	22731	21769
d_1	$.0_2 621$	14268	76720	54402	08246	57561
d_2	$-.0_4 7$	76341	25875	60196	30270	82527
d_3		$.0_6 9762$	33295	71491	60266	55660
d_4		$-.0_7 123$	50284	11059	98185	07930
d_5		$.0_9 1$	57182	80627	01495	53995
d_6			$-.0_{11} 2012$	42598	83480	29922
d_7			$.0_{13} 25$	91810	76365	04731
d_8			$-.0_{15}$	33576	58658	27913
d_9				$.0_{17} 437$	52287	88836
d_{10}				$-.0_{19} 5$	73425	18565
d_{11}					$.0_{21} 7558$	66432
d_{12}					$-.0_{22} 100$	20420
d_{13}					$.0_{24} 1$	33592
d_{14}						$-.0_{26} 1791$
d_{15}						$.0_{28} 24$

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13. ABSTRACT

The converging factor for a specific mathematical function, such as the modified Bessel function of the second kind considered in this report, is that factor by which the last term of a truncated series (usually asymptotic) approximating the function must be multiplied to compensate for the omitted terms. This converging factor for the aforementioned Bessel function is discussed herein in detail and is shown to be related to the corresponding factor for the probability integral. Tables of this factor and its reduced derivatives, correct to 30 decimal places, are included to expedite the application of this procedure to the evaluation of this Bessel function to high precision for arguments between 5 and 20, and specific examples of such applications are presented.

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