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Applied mathematics

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The application of continued fractions in Christiaan Huygens' planetarium

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Preamble

In this project, we will show how Christiaan Huygens used continued fractions in the design of his planetarium. We will introduce continued fractions and discuss some basic properties. Most of the knowledge about continued fractions, is obtained from 'Ergodic theory numbers', written by K. Dajani and C. Kraaikamp, see [2]. By using an algorithm devised by Alexander Ostrowski, we will find that in a certain sense, the approximation using continued fractions is a best approximation possible. Furthermore, we will describe an algorithm which helps us to construct a gear train, which leads us to even better approximations in comparison with the approximations Huygens made. Most of the knowledge about the gear train, the Ostrowski algorithm and the proof on best approximations can be found in a condensed way in 'Continued fractions', written by A.M. Rocket and P. Szüsz, see [6]. The appendix contains the code from the algorithm which helps us to find values for the gear train.

Introduction

Christiaan Huygens, known for his contributions in physics, astronomy and mathematics, designed in 1680 a planetarium on behalf of l'Académie Royale des Sciences. A planetarium displays multiple planets and their movement around the sun. Huygens was requested to design the planetarium by the minister of Finance in France Jean-Baptiste Colbert, under the mandate of King Louis XIV. In the planetarium, the ratio between two gears determine the orbital periods of two planets. Huygens used continued fractions to approximate such ratios and made this approximation very precise.

These continued fractions have a long history, with their origin connected with Euclid's algorithm. Later on, several mathematicians showed that the approximation with continued fractions is very accurate. Alexander Ostrowski, a mathematician from Russia, is one of the mathematicians who proved that the approximation by convergents is a best approximation possible.

In six months, Huygens completed his design and minister Colbert gave him permission to construct the planetarium. Huygens went to his clockmaker Johannes van Ceulen who finished the planetarium in less than a year. Huygens tried to contact Colbert to inform him that the cost of this planetarium will be 620 pound: 520 for the material and the work of van Ceulen and 100 pound for the design. In 1682 Huygens got a response of Callois, the spokesman of Colbert, that the costs for the planetarium were acceptable. A few months later, Colbert died and Francois Michel Le Tellier becomes first minister. In the following years, Huygens tried to convince the new minister to let him continue his work for the academy, as he was first chairman to the French academy at that time. However, Le Tellier decided to dismiss Huygens from the academy. For this reason, Huygens never got the promised 620 pound and the relationship between the academy and Huygens ended badly; see also [1] for more information.

The design of his planetarium is an interesting example of an application of continued fractions and deserves proper attention. This project describes the way Christiaan Huygens designed the planetarium and eventually shows in which way the approximations can be improved. But first, continued fractions are introduced and the theorem of Ostrowski is presented to show that the approximation by convergents is a best approximation possible in a certain, well-defined sense.

Contents

1 Continued fractions

Huygens used continued fractions to approach numbers. The following example introduces this method.

Example 1. The rational number $\frac{19}{15} = 1.2\overline{6}$ can be written as:

$$\frac{19}{15} = 1 + \frac{1}{\frac{15}{4}} = 1 + \cfrac{1}{3 + \cfrac{1}{\frac{4}{3}}} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{3}}} \tag{1.1}$$

In (1.1) the Divisor Algorithm of Euclid is applied to convert $\frac{19}{15}$, $\frac{15}{4}$ and $\frac{4}{3}$ into mixed fractions. In order to approximate the number $\frac{19}{15}$ by rationals with smaller denominators, parts of the continued fraction are truncated. For example, the following rational numbers could be approximations of $\frac{19}{15}$:

$$1 \qquad 1 + \frac{1}{3} = 1.\overline{3} \qquad 1 + \cfrac{1}{3 + \cfrac{1}{1}} = 1.25 \qquad 1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{3}}} = 1.2\overline{6}$$

In Example 1 a finite continued fraction is obtained, while in the following example, an infinite continued fraction is given.

Example 2. The irrational number $\frac{1}{\sqrt{2} - 1} \approx 2.41421356$ can be written as:

$$\frac{1}{\sqrt{2} - 1} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} \tag{1.2}$$

1.1 The Gauss map

In order to get a better understanding of continued fractions such as (1.1) and (1.2), the Gauss map will be introduced. The Gauss map is also known as the continued fraction operator; see [2]. Define $T : [0, 1) \to [0, 1)$ such that

$$T(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & x \neq 0, \\ 0 & x = 0, \end{cases}$$

in which $\lfloor x \rfloor$ denotes the largest $m \in \mathbb{Z}$ such that $m \leq x$. If $x \in [0, 1)$ is a rational number then according to the Divisor Algorithm of Euclid, there exists a $n \in \mathbb{N}$ such that $T^n(x) = 0$, where $T^n(x)$ denotes the nth iteration of the Gauss map. If $x \in [0, 1)$ is an irrational number, $T(x)$ is an irrational number and eventually $T^n(x)$ is irrational for all positive integers n. For $x \in [0, 1)$, define the sequence $(a_i)_{i \geq 1} \in \mathbb{N}$ such that $a_1 = \left\lfloor \frac{1}{x} \right\rfloor, a_2 = \left\lfloor \frac{1}{T(x)} \right\rfloor, a_3 = \left\lfloor \frac{1}{T(T(x))} \right\rfloor = \left\lfloor \frac{1}{T^2(x)} \right\rfloor$, and so on. As long as $T^n(x) \neq 0$ we find that

$$T(x) = \frac{1}{x} - a_1 \text{ and } x = \frac{1}{a_1 + T(x)}$$

Similary,

$$T^2(x) = \frac{1}{T(x)} - a_2 \text{ and } T(x) = \frac{1}{a_2 + T^2(x)}$$

and it follows that

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T^2(x)}}$$

After $n \in \mathbb{N}$ steps in this process, we find that

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n(x)}}}}} \tag{1.3}$$

In (1.3), one can easily see that this continued fraction will be finite if there exists an $m \in \mathbb{N}$ in which $T^m(x) = 0$. The Gauss map can be used to find the continued fraction of any $x \in \mathbb{R}$. This process is applied in Example 1, where the continued fraction of $\frac{19}{15}$ is examined, in which $a_1 = 3, a_2 = 1$ and $a_3 = 3$. In the first step of this process, 1 is subtracted from $\frac{19}{15}$ to make sure that $x \in [0, 1)$.

$$\begin{aligned}
x &= \frac{19}{15} - 1 = \frac{4}{15} & & \\
a_1 &= \left\lfloor \frac{1}{x} \right\rfloor = \left\lfloor \frac{15}{4} \right\rfloor = 3 & T(x) &= \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \frac{15}{4} - 3 = \frac{3}{4} \\
a_2 &= \left\lfloor \frac{1}{T(x)} \right\rfloor = \left\lfloor \frac{4}{3} \right\rfloor = 1 & T^2(x) &= \frac{1}{T(x)} - \left\lfloor \frac{1}{T(x)} \right\rfloor = \frac{4}{3} - 1 = \frac{1}{3} \\
a_3 &= \left\lfloor \frac{1}{T^2(x)} \right\rfloor = \left\lfloor \frac{3}{1} \right\rfloor = 3 & T^3(x) &= \frac{1}{T^2(x)} - \left\lfloor \frac{1}{T^2(x)} \right\rfloor = \frac{3}{1} - 3 = 0
\end{aligned}$$

1.2 Definition and notation

In general, a continued fraction is a representation of a real number t of the form:

$$t = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots}}}},$$

in which $a_0 \in \mathbb{Z}$ such that $t - a_0 \in [0, 1)$, $a_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}$ for $i \geq 1$. In this project, the main focus lies on the *regular continued fractions*, in which $b_i = 1$ for all i.

Definition 1 (Regular continued fraction)**.** *The unique regular continued fraction of $r \in \mathbb{R} \setminus \mathbb{Q}$ is*

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where $a_0 \in \mathbb{Z}$ such that $r - a_0 \in [0, 1)$ and $a_i \in \mathbb{N}$ for $i \geq 1$. The term $r - a_0$ is also known as $\{r\}$. A shorter notation for the unique regular continued fraction of $r \in \mathbb{R} \setminus \mathbb{Q}$ is

$$r = [a_0; a_1, a_2, \dots]$$

Note that for rational numbers there are two different continued fractions possible. Let x be a rational number with the continued fraction $[a_0; a_1, a_2, \dots, a_k]$. If $a_k > 1$ we also have the continued fraction $x = [a_0; a_1, a_2, \dots, a_k - 1, 1]$. In addition, if $a_k = 1$, we have $x = [a_0; a_1, a_2, \dots, a_{k-1} + 1]$. Example 1 described in which way continued fractions can be used for approximation of numbers. The first approximation is just the number $[a_0]$, the second becomes $[a_0; a_1]$, the third is $[a_0; a_1, a_2]$ and so on. These approximations are called convergents.

Definition 2 (Convergents of a continued fraction)**.** *The convergents* $\frac{A_k}{B_k}$ *of the continued fraction* $[a_0; a_1, a_2, \dots]$ *of the number* $x \in \mathbb{R}$ *are defined by:*

$$[a_0; a_1, a_2, \dots, a_k] = \frac{A_k}{B_k}, \text{ where } k \in \mathbb{N}_{\geq 0}$$

where $A_k, B_k \in \mathbb{Z}$ *such that* $B_k \geq 1$ *and* $gcd(A_k, B_k) = 1$.

Obviously, there are only finitely many convergents whenever $x \in \mathbb{Q}$.

1.3 Properties of continued fractions

Moving on now to consider some properties of continued fractions, together with their proofs.

Theorem 1 (Basic properties of continued fractions)**.** *Let* $[a_0; a_1, a_2, \dots]$ *be the continued fraction of the number* $x \in \mathbb{R}$ *with convergents* $\frac{A_k}{B_k}$ *for all* $k \in \mathbb{N}_{\geq 0}$*, where* $B_k > 0$ *and* $gcd(A_k, B_k) = 1$*. Let* $t = x - a_0$*, then this continued fraction has the following properties*

1. $A_k = A_{k-2} + a_k A_{k-1}$
   $B_k = B_{k-2} + a_k B_{k-1}$,
   where $k \geq 1$ *and* $A_{-1} = 1, A_0 = a_0, B_{-1} = 0, B_0 = 1$.
2. $A_{k-1}B_k - A_k B_{k-1} = (-1)^k$.
3. $t = \dfrac{A_k + A_{k-1}T^k(t)}{B_k + B_{k-1}T^k(t)}$, *where* $k \geq 1$.

Note that for $x \in \mathbb{Q}$ *there are only finitely many* $k \geq 1$ *such that* $1, 2$ *and* 3 *hold.*

Proof. Let $k \in \mathbb{N}_{\geq 0}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ where $ad - bc \neq 0$. Define the Mobiüs transformation $M : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ such that

$$M(x) = \frac{ax + b}{cx + d}$$

Note that $(AB)(x) = A(B(x))$ for all matrices A and B with integer valued entries and determinant ± 1. Let $E_0 = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix}$ and $E_k = \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix}$. This results in

$$\det(E_k) = 0 \cdot a_k - 1 \cdot 1 = -1 \text{ for } k \geq 1 \text{ and } \det(E_0) = 1 \cdot 1 - 0 \cdot a_0 = 1 \tag{1.4}$$

so E_k is an integer matrix of determinant -1 when $k \geq 1$ and E_0 is an integer matrix of determinant 1. Now let $p_k, q_k, r_k, s_k \in \mathbb{Z}$ such that

$$E_0 E_1 \cdots E_{k-1} E_k = \begin{pmatrix} p_k & q_k \\ r_k & s_k \end{pmatrix} \tag{1.5}$$

Hence,

$$\det(E_0 E_1 \cdots E_{k-1} E_k) = \det \begin{pmatrix} p_k & q_k \\ r_k & s_k \end{pmatrix}$$

and therefore we find that

$$\det(E_0)\det(E_1) \cdots \det(E_{k-1})\det(E_k) = p_k s_k - r_k q_k$$

As a result of (1.4), we have that

$$(-1)^k = p_k s_k - r_k q_k \tag{1.6}$$

In other words, gcd(p_k, r_k)=1 and gcd(q_k, s_k)=1. The calculations that follow show the relationship between matrices, the Möbius transformation and continued fractions. Since

$$E_k(0) = \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix}(0) = \frac{0 \cdot 0 + 1}{1 \cdot 0 + a_k} = \frac{1}{a_k}$$

we have that

$$E_{k-1}E_k(0) = \begin{pmatrix}0 & 1\\ 1 & a_{k-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & a_k\end{pmatrix}(0) = \begin{pmatrix}0 & 1\\ 1 & a_{k-1}\end{pmatrix}\left(\frac{1}{a_k}\right) = \frac{0\cdot\dfrac{1}{a_k}+1}{1\cdot\dfrac{1}{a_k}+a_{k-1}} = \frac{1}{a_{k-1}+\dfrac{1}{a_k}}$$

So after finitely many steps

$$E_1\cdots E_{k-1}E_k(0) = \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots\,\cfrac{1}{a_k}}}}$$

and finally we have that

$$E_0E_1\cdots E_{k-1}E_k(0) = \begin{pmatrix}1 & a_0\\ 0 & 1\end{pmatrix}\left(\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots\,\cfrac{1}{a_k}}}}\right) = \frac{1\cdot\left(\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots\,\cfrac{1}{a_k}}}}\right)+a_0}{0\cdot\left(\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots\,\cfrac{1}{a_k}}}}\right)+1} = a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots\,\cfrac{1}{a_k}}}}$$

By Definition 2 this finite continued fraction above equals $\frac{A_k}{B_k}$. Consequently and due to (1.5), we have that

$$\frac{A_k}{B_k} = E_0E_1\cdots E_{k-1}E_k(0) = \begin{pmatrix}p_k & q_k\\ r_k & s_k\end{pmatrix}(0) = \frac{p_k\cdot 0+q_k}{r_k\cdot 0+s_k} = \frac{q_k}{s_k} \tag{1.7}$$

As a result of (1.6), (1.7) and the fact that gcd(A_k, B_k)=1 and both $s_k > 0$ and $B_k > 0$,

$$q_k = A_k \text{ and } s_k = B_k \tag{1.8}$$

Since $E_0E_1\cdots E_{k-1}E_k = \begin{pmatrix}p_k & A_k\\ r_k & B_k\end{pmatrix}$ and $E_0E_1\cdots E_{k-1} = \begin{pmatrix}p_{k-1} & A_{k-1}\\ r_{k-1} & B_{k-1}\end{pmatrix}$, we have that

$$\begin{pmatrix}p_k & A_k\\ r_k & B_k\end{pmatrix} = \begin{pmatrix}p_{k-1} & A_{k-1}\\ r_{k-1} & B_{k-1}\end{pmatrix}E_k = \begin{pmatrix}p_{k-1} & A_{k-1}\\ r_{k-1} & B_{k-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & a_k\end{pmatrix} = \begin{pmatrix}A_{k-1} & p_{k-1}+a_k\cdot A_{k-1}\\ B_{k-1} & r_{k-1}+a_k\cdot B_{k-1}\end{pmatrix} \tag{1.9}$$

The equation above results in the following values for p_k and r_k,

$$p_k = A_{k-1} \text{ and } r_k = B_{k-1} \tag{1.10}$$

According to (1.5), (1.8) and (1.10) we find that,

$$E_0E_1\cdots E_{k-1}E_k = \begin{pmatrix}A_{k-1} & A_k\\ B_{k-1} & B_k\end{pmatrix} \tag{1.11}$$

From these values and (1.9), we know that

$$\begin{pmatrix}A_{k-1} & A_{k-2}+a_k\cdot A_{k-1}\\ B_{k-1} & B_{k-2}+a_k\cdot B_{k-1}\end{pmatrix} = \begin{pmatrix}A_{k-1} & A_k\\ B_{k-1} & B_k\end{pmatrix}$$

Consequently, the result (Part 1) follows:

$$A_k = A_{k-2}+a_kA_{k-1} \tag{1.12}$$

$$B_k = B_{k-2}+a_kB_{k-1} \tag{1.13}$$

In addition, the result (Part 2) follows, according to (1.6), (1.8) and (1.10)

$$A_{k-1}B_k - A_kB_{k-1} = (-1)^k$$

It remains to prove Part 3 of the theorem. Let $E_k^* = \begin{pmatrix} 0 & 1 \\ 1 & a_k + T^k(t) \end{pmatrix}$, where $t = x - a_0$ and where $T^k(t)$ denotes the kth iteration of the Gauss map, described in Section 1.1. Since

$$E_k^*(0) = \begin{pmatrix} 0 & 1 \\ 1 & a_k + T^k(t) \end{pmatrix}(0) = \frac{0 \cdot 0 + 1}{1 \cdot 0 + a_k + T^k(t)} = \frac{1}{a_k + T^k(t)}$$

we have that

$$E_{k-1}E_k^*(0) = \begin{pmatrix} 0 & 1 \\ 1 & a_{k-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_k + T^k(t) \end{pmatrix}(0) = \begin{pmatrix} 0 & 1 \\ 1 & a_{k-1} \end{pmatrix}\left(\frac{1}{a_k + T^k(t)}\right)$$

$$= \frac{0 \cdot \dfrac{1}{a_k + T^k(t)} + 1}{1 \cdot \dfrac{1}{a_k + T^k(t)} + a_{k-1}} = \frac{1}{a_{k-1} + \dfrac{1}{a_k + T^k(t)}}$$

So after finitely many steps

$$E_0E_1 \cdots E_{k-1}E_k^*(0) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_k + T^k(t)}}}}$$

According to (1.3), we have that

$$E_0E_1 \cdots E_{k-1}E_k^*(0) = x \tag{1.14}$$

On the other hand, the term $E_0E_1 \cdots E_{k-1}E_k^*(0)$ can also be described in a different way. Because of (1.11), we have that

$$E_0E_1 \cdots E_{k-1}E_k^*(0) = \begin{pmatrix} A_{k-2} & A_{k-1} \\ B_{k-2} & B_{k-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_k + T^k(t) \end{pmatrix}(0)$$

and after matrix multiplication

$$E_0E_1 \cdots E_{k-1}E_k^*(0) = \begin{pmatrix} A_{k-1} & A_{k-2} + a_kA_{k-1} + T^k(t)A_{k-1} \\ B_{k-1} & B_{k-2} + a_kB_{k-1} + T^k(t)B_{k-1} \end{pmatrix}(0)$$

Next, (1.12) and (1.13) are applied

$$\begin{aligned} E_0E_1 \cdots E_{k-1}E_k^*(0) &= \begin{pmatrix} A_{k-1} & A_k + T^k(t)A_{k-1} \\ B_{k-1} & B_k + T^k(t)B_{k-1} \end{pmatrix}(0) \\ &= \frac{A_{k-1} \cdot 0 + A_k + T^k(t)A_{k-1}}{A_{k-1} \cdot 0 + B_k + T^k(t)B_{k-1}} \\ &= \frac{A_k + T^k(t)A_{k-1}}{B_k + T^k(t)B_{k-1}} \end{aligned} \tag{1.15}$$

At last, as a result of (1.14) and (1.15), we have that

$$x = \frac{A_k + T^k(t)A_{k-1}}{B_k + T^k(t)B_{k-1}}$$

so that the proof of Part 3 is complete. □

1.4 Ostrowski's algorithm

In 1921 Alexander Ostrowski wrote an article about problems in Diophantine approximation; see [4]. In this article he describes an algorithm which uses a continued fraction expansion to create a representation of any positive integer. Ostrowski's algorithm can be used to prove that the approximation by continued fractions convergents, is a best approximation possible. In addition, it is important to define the statement 'best approximation'. We are interested in the distance from an irrational number t to a rational number $\frac{A}{B}$, so we would like to find integers A and B such that $\left|t - \frac{A}{B}\right| < \left|t - \frac{p}{q}\right|$, for all $\frac{p}{q} \in \mathbb{Q}$ where $\frac{p}{q} \neq \frac{A}{B}$ and $0 < q < B$. However, in this project we will focus on a different formulation of a best approximation.

Definition 3 (Best approximation)**.** *Let $t \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{A}{B} \in \mathbb{Q}$, where $gcd(A, B) = 1$. Then $\frac{A}{B}$ is a best approximation to t if and only if $\left|Bt - A\right| < \left|qt - p\right|$ for all $\frac{p}{q} \in \mathbb{Q}$ such that $\frac{p}{q} \neq \frac{A}{B}$ and $0 < q < B$.*

As a consequence of this definition we find that

$$\left|t - \frac{A}{B}\right| = \frac{1}{B}|Bt - A| < \frac{1}{B}|qt - p| < \frac{1}{q}|qt - p| = \left|t - \frac{p}{q}\right|$$

In order to understand the Ostrowski algorithm, this chapter starts off with an visual explanation which eventually leads to a representation of any natural number. The starting point is the continued fraction expansion of some irrational number $x = [a_0; a_1, a_2, \ldots]$ with the convergents $\frac{A_k}{B_k}$ in which $k \in \mathbb{N}_{\geq 0}$. The goal in this algorithm is to express any natural number m in a summation of B_ks, multiplied by a non negative integer c_{k+1}. Theorem 1.1 gives that:

$$1 = B_0 \leq B_1 < B_2 < \ldots$$

As a result of this, we have the following. For all $m \in \mathbb{N}$ there exists an index $N_1 \in \mathbb{N}$ such that $B_{N_1} \leq m < B_{N_1+1}$. A few B_ks and the number m are shown in the figure below.

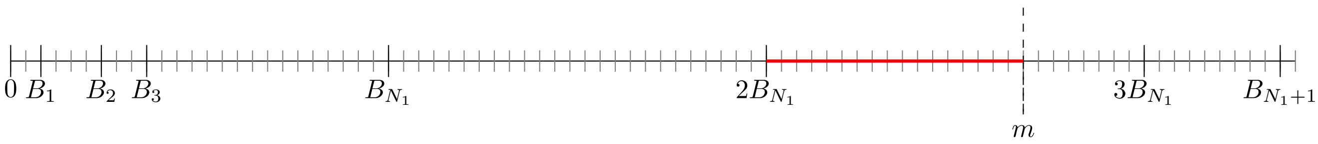


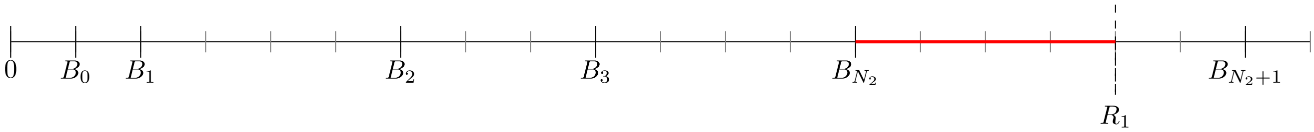
Figure 1: Position of natural number m with respect to the convergents of x

In the first step of the algorithm it is determined how many times B_{N_1} fits in m and what the so-called rest-term is, which is R_1. In this example B_{N_1} fits two times in m and the rest-term is the length of the red line, so

$$m = \left[\frac{m}{B_{N_1}}\right] \cdot B_{N_1} + R_1$$

Recall that $\left[\frac{m}{B_{N_1}}\right]$ denotes the largest $c \in \mathbb{Z}$ such that $c \leq \frac{m}{B_{N_1}}$. In addition, we define $c_{N_1+1} = \left[\frac{m}{B_{N_1}}\right] = 2$. Due to Theorem 1.1 we know that in the example above $B_{N_1+1} = 3B_{N_1} + B_{N_1-1}$. Therefore, we find that the difference between $3B_{N_1}$ and B_{N_1+1} is B_{N_1-1} and we also know that $a_{N_1+1} = 3$. In general we see that $c_{N_k+1} \leq a_{N_k+1}$, where $c_{N_k+1} = a_{N_k+1}$ if and only if $m \in [a_{N_k+1}B_{N_k}, B_{N_k+1})$. Since $B_{N_k+1} - a_{N_k+1}B_{N_k} = B_{N_k-1}$ we find that if $c_{N_k+1} = a_{N_k+1}$ then $m < B_{N_k-1}$ and consequently $c_{N_k} = 0$.

Next, the same idea is applied to the rest-term R_1, instead of m. There exists an index $N_2 \in \mathbb{N}$ such that $B_{N_2} \leq R_1 < B_{N_2+1}$ and $N_2 < N_1$.



The red line is the second rest-term, which we denote by R_2. Again, the following calculation is needed to find out how many times B_{N_2} fits in R_2 and what the second rest-term will be.

$$R_1 = \left[\frac{R_1}{B_{N_2}}\right] \cdot B_{N_2} + R_2$$

where $0 \leq R_2 < R_1$. In the example above B_{N_2} only fits one time in R_1, so define $c_{N_2+1} = \left[\frac{R_1}{B_{N_2}}\right] = 1$. This process, which is very similar to Euclid's Algorithm, continues until there is no rest-term left. This results in a summation of $\left[\frac{R_i}{B_{N_{i+1}}}\right]$ multiplied by B_i, where $0 \leq B_{N_{i+1}} \leq B_{N_1}$ for a non-negative integer i. In the following theorem, this representation is described.

Theorem 2 (Ostrowski representation of positive integers m)**.** *Let $[a_0; a_1, a_2, \dots]$ be the continued fraction of the irrational number t with convergents $\frac{A_k}{B_k}$ where $k \in \mathbb{N}_{\geq 0}$. For every $m \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $B_N \leq m < B_{N+1}$. Then there exists an unique finite sequence $\left(c_{k+1}\right)_{k \in \mathbb{N}_{\geq 0}}$ such that:*

$$m = \sum_{k=0}^{N} c_{k+1} B_k \tag{1.16}$$

where

$$\begin{cases} 0 \leq c_{k+1} \leq a_{k+1}, & \text{if } k > 0 \\ 0 \leq c_{k+1} < a_{k+1}, & \text{if } k = 0 \\ c_k = 0, & \text{if } c_{k+1} = a_{k+1} \end{cases} \tag{1.17}$$

Proof. Let $m \in \mathbb{N}$. We know that t is an irrational number so the continued fraction of t is infinite, see also Section 1.1 for more information. Therefore, t has an infinite number of convergents and there exists an N_1 such that $B_{N_1} \leq m < B_{N_1+1}$, where $N_1 = N$. In the process described above, the rest-term equals zero after finitely many steps, because of the fact that $B_k < B_{k+1}$ for all $k > 0$. Consequently, we find that there exists a finite sequence $\left(c_{k+1}\right)_{k \in \mathbb{N}_{\geq 0}}$ such that: $m = \sum_{k=0}^{N} c_{k+1} B_k$, where $c_{N_k+1} = \left[\frac{R_k}{B_{N_{k+1}}}\right]$ if $B_{N_{k+1}} \leq R_k < B_{N_{k+1}+1}$, otherwise $c_{k+1} = 0$. The sequence $\left(c_{k+1}\right)_k$ is unique because it is obtained by the biggest number of times in which $B_{N_{k+1}}$ fits in R_k. We also described in the process that $c_{k+1} \leq a_{k+1}$, but we find that $c_1 < a_1$ because of the fact that

$$c_1 = c_1 B_0 < B_1 = a_1 B_0 + B_{-1} = a_1$$

At last recall that if $c_{N_k+1} = a_{N_k+1}$ then $R_k < B_{N_k-1}$, because of the fact that the difference between $a_{N_k+1} B_{N_k}$ and B_{N_1+1} is B_{N_1-1}. Consequently, we know that $c_{N_k} = 0$. ☐

Example 3. In the Example 2, the continued fraction of the irrational number $\frac{1}{\sqrt{2}-1} = [2; \overline{2}]$ is discussed. Some convergents of this continued fraction are

$$\frac{A_0}{B_0} = \frac{2}{1}, \ \frac{A_1}{B_1} = \frac{5}{2}, \ \frac{A_2}{B_2} = \frac{12}{5}, \ \frac{A_3}{B_3} = \frac{29}{12}, \ \frac{A_4}{B_4} = \frac{70}{29}, \ \frac{A_5}{B_5} = \frac{169}{70}, \ \frac{A_6}{B_6} = \frac{408}{169}, \dots$$

Let $m_1 = 27$. It is known that that $27 = 2 \cdot 12 + 1 \cdot 2 + 1 \cdot 1$. Translated into the terms of the theorem:

$$m_1 = \sum_{k=0}^{N} c_{k+1} B_k = 2 \cdot B_3 + 1 \cdot B_1 + 1 \cdot B_0$$

Let $m_2 = 139$. It is known that that $139 = 1 \cdot 70 + 2 \cdot 29 + 2 \cdot 5 + 1 \cdot 1$. Translated into the terms of the theorem:

$$m_2 = \sum_{k=0}^{N} c_{k+1} B_k = 1 \cdot B_5 + 2 \cdot B_4 + 2 \cdot B_2 + 1 \cdot B_0$$

1.5 Error of the approximation

Before examining the precision of the approximation by convergents, it is important to take a look at the quantity $|Bt - A|$, where $t \in \mathbb{R} \setminus \mathbb{Q}$, $\frac{A}{B} \in \mathbb{Q}, B > 0$ and gcd(A, B)=1.

Definition 4. *Let $[a_0; a_1, a_2, \dots]$ be the continued fraction expansion of $t \in \mathbb{R} \setminus \mathbb{Q}$ with convergents $\frac{A_k}{B_k}$ for all $k \in \mathbb{N}_{\geq 0}$. Then*

$$D_k = B_k t - A_k$$

In addition to the previous definition and Theorem 1.1, D_{-1} and D_0 are defined by

$$D_{-1} = B_{-1} t - A_{-1} = -1 \text{ and } D_0 = B_0 t - A_0 = t - a_0 = \{t\} \tag{1.18}$$

The quantity D_k has a notable property which includes information about the sign of D_k.

Lemma 1. *Let* $[a_0; a_1, a_2, \dots]$ *be the continued fraction of* $t \in \mathbb{R}\setminus\mathbb{Q}$ *with convergents* $\frac{A_k}{B_k}$ *for all* $k \in \mathbb{N}_{\geq 1}$. *Then*

$$D_k = \frac{(-1)^k}{B_k[a_{k+1}; a_{k+2}, a_{k+3}, \dots] + B_{k-1}} \tag{1.19}$$

Consequently, we find that $D_k > 0$ *if and only if* k *is even.*

Proof. According to Definition 4 and Theorem 1.3 we have that

$$\begin{aligned}
D_k &= B_k t - A_k \\
&= B_k \cdot \frac{A_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + A_{k-1}}{B_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + B_{k-1}} - A_k \\
&= \frac{B_k A_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + B_k A_{k-1}}{B_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + B_{k-1}} - \frac{A_k B_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + A_k B_{k-1}}{B_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + B_{k-1}} \\
&= \frac{B_k A_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] - A_k B_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + B_k A_{k-1} - A_k B_{k-1}}{B_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + B_{k-1}} \\
&= \frac{B_k A_{k-1} - A_k B_{k-1}}{B_k \cdot [a_k; a_{k+1}, a_{k+2}, \dots] + B_{k-1}} \\
&= \frac{(-1)^k}{B_k[a_{k+1}; a_{k+2}, a_{k+3}, \dots] + B_{k-1}}
\end{aligned}$$

In the last step Theorem 1.2 is applied. Since the denominator is positive, we immediately find that $D_k > 0$ if and only if k is even. □

In addition to Lemma 1, we obtain the following Lemma.

Lemma 2. *Let* $[a_0; a_1, a_2, \dots]$ *be the continued fraction of* $t \in \mathbb{R}\setminus\mathbb{Q}$ *with convergents* $\frac{A_k}{B_k}$ *for all* $k \in \mathbb{N}_{\geq 1}$. *Then*

$$|D_k| > |D_{k+1}|$$

Proof. Note that

$$[a_k; a_{k+1}, a_{k+2}, \dots] = a_k + \frac{1}{[a_{k+1}; a_{k+2}, a_{k+3}, \dots]} \tag{1.20}$$

Next, we rewrite (1.19). Now,

$$\begin{aligned}
|D_k| &= \frac{1}{|B_k \cdot [a_{k+1}; a_{k+2}, a_{k+3}, \dots] + B_{k-1}|} \\
&= \frac{1}{\left| B_k \cdot \left(a_{k+1} + \frac{1}{[a_{k+2}; a_{k+3}, a_{k+4}, \dots]} \right) + B_{k-1} \right|} && \text{(since (1.20))} \\
&= \frac{1}{\left| B_{k+1} + \frac{B_k}{[a_{k+2}; a_{k+3}, a_{k+4}, \dots]} \right|} && \text{(due to Theorem 1.1)} \\
&= \frac{[a_{k+2}; a_{k+3}, a_{k+4,\dots}]}{|B_{k+1} \cdot [a_{k+2}; a_{k+3}, a_{k+4}, \dots] + B_k|} && \text{(since } [a_{k+2}; a_{k+3}, a_{k+4,\dots}] > 0) \\
&= [a_{k+2}; a_{k+3}, a_{k+4,\dots}] \cdot |D_{k+1}| && \text{(due to Lemma 1)} \\
&> |D_{k+1}| && \text{(since } [a_{k+2}; a_{k+3}, a_{k+4,\dots}] > 1 \text{ for } k \geq 0)
\end{aligned}$$

□

Furthermore, another result is worth mentioning.

Lemma 3. *Let* $[a_0; a_1, a_2, \dots]$ *be the continued fraction of* $t \in \mathbb{R}\setminus\mathbb{Q}$ *with convergents* $\frac{A_k}{B_k}$ *for all* $k \in \mathbb{N}_{\geq 1}$. *Then*

$$D_n = a_n D_{n-1} + D_{n-2} \tag{1.21}$$

Notice that the sequence of the D_n satisfies for $n \geq 1$ the same recursion relation as the A_n and B_n, with different starting values for the recursion.

Proof. We have that

$$\begin{aligned}
a_n D_{n-1} &= a_n(B_{n-1}t - A_{n-1}) \\
&= a_n B_{n-1} t - a_n A_{n-1} \\
&= (B_n - B_{n-2})t - (A_n - A_{n-2}) && \text{(due to Theorem 1.1)} \\
&= (B_n t - A_n) - (B_{n-2}t - A_{n-2}) \\
&= D_n - D_{n-2}
\end{aligned}$$

□

Lemma 4 shows another property of D_k.

Lemma 4. *Let $[a_0; a_1, a_2, \dots]$ be the continued fraction of $t \in \mathbb{R} \setminus \mathbb{Q}$ with convergents $\frac{A_k}{B_k}$ for all $k \in \mathbb{N}_{\geq 1}$. Then*

$$|D_k| < \frac{1}{B_k} \leq \tfrac{1}{2}$$

Proof.

$$\begin{aligned}
|D_k| &= \left| \frac{(-1)^k}{B_k[a_{k+1}; a_{k+2}, a_{k+3}, \dots] + B_{k-1}} \right| && \text{(due to Lemma 1)} \\
&= \frac{1}{B_k[a_{k+1}; a_{k+2}, a_{k+3}, \dots] + B_{k-1}} && \text{(due to Theorem 1.1)} \\
&< \frac{1}{B_k a_{k+1} + B_{k-1}} && \text{(since } a_{k+1} < [a_{k+1}; a_{k+2}, a_{k+3}, \dots]) \\
&= \frac{1}{B_{k+1}} && \text{(due to Theorem 1.1)} \\
&\leq \frac{1}{2} && \text{(due to Theorem 1.1 we know that } B_2 = a_2 B_1 + B_0 \geq 2)
\end{aligned}$$

□

At last, a property is shown, concerning the Ostrowski representation.

Lemma 5. *Let $[a_0; a_1, a_2, \dots]$ be the continued fraction of $t \in \mathbb{R} \setminus \mathbb{Q}$ with convergents $\frac{A_k}{B_k}$ for all $k \in \mathbb{N}_{\geq 0}$. Let $\left(c_{k+1}\right)_{k \in \mathbb{N}_{\geq 0}}$ be such that $m = \sum_{k=0}^{N} c_{k+1} B_k$ and such that (1.17) is satisfied. Then we have that*

$$||mt|| = \left|\left| \sum_{k=0}^{N} c_{k+1} D_k \right|\right|$$

Here $||x||$ denotes the distance from x to the nearest integer.

Proof. In the first step of this proof $\sum_{k=0}^{N} c_{k+1} A_k$ is being subtracted from mt. Since $\sum_{k=0}^{N} c_{k+1} A_k$ is an integer, this translation does not have an influence on the value of $||mt||$, because the distance to the nearest integer will not change. Now,

$$||mt|| = \left|\left| mt - \sum_{k=0}^{N} c_{k+1} A_k \right|\right| = \left|\left| \sum_{k=0}^{N} c_{k+1} B_k t - c_{k+1} A_k \right|\right| = \left|\left| \sum_{k=0}^{N} c_{k+1} (B_k t - A_k) \right|\right| = \left|\left| \sum_{k=0}^{N} c_{k+1} D_k \right|\right|$$

□

2 Best approximation

Christiaan Huygens used continued fractions to approach the ratio of the orbital periods of the planets. Thus far, this project has given an introduction to continued fractions and the Ostrowski representation. In this section we will discuss if the approximation by convergents of a continued fraction is a best approximation possible.

2.1 Lemma's prior to a best approximation

This section describes two lemmas prior to the theorem about a best approximation.

Lemma 6. *Let* $[a_0; a_1, a_2, \dots]$ *be the continued fraction of* $t \in \mathbb{R} \setminus \mathbb{Q}$ *with convergents* $\frac{A_k}{B_k}$ *for all* $k \in \mathbb{N}_{\geq 0}$ *and let* m *be a positive integer. The Ostrowski representation of* m *is given by (1.16), where* $c_{k+1} = 0$ *for* $0 \leq k < n \leq N$ *and* $c_{n+1} > 0$*. Then*

$$|(c_{n+1} - 1)D_n - D_{n+1}| < \left| \sum_{k=0}^{N} c_{k+1} D_k \right| < |c_{n+1} D_n - D_{n+1}| \tag{2.1}$$

Proof. Since c_{n+1} is the first non-zero term, we must have that $c_{n+2} \neq a_{n+2}$, according to Theorem 2. Consequently:

$$c_{n+2} < a_{n+2} \tag{2.2}$$

For this proof four possibilities for D_n and D_N are examined:

1. $D_n > 0$ and $D_N > 0$
2. $D_n > 0$ and $D_N < 0$
3. $D_n < 0$ and $D_N > 0$
4. $D_n < 0$ and $D_N < 0$

1. $D_n > 0$ and $D_N > 0$

Due to Lemma 1 and since we have that $D_n > 0$, the following holds

$$D_{n+2m} > 0 \text{ and } D_{n+2m+1} < 0 \text{ for all } m \in \mathbb{N}_{\geq 0}$$

Consequently, since $D_N > 0$ there exists an $m \in \mathbb{N}_{\geq 0}$ such that $N = n + 2m$. In the calculations that follow, it will be argued that $\sum_{k=0}^{N} c_{k+1} D_k > 0$. In the first step, some non-negative terms will be eliminated.

$$\begin{aligned}
& c_{n+1}D_n + c_{n+2}D_{n+1} + c_{n+3}D_{n+2} + c_{n+4}D_{n+3} + c_{n+5}D_{n+4} + c_{n+6}D_{n+5} + \cdots + c_{N+1}D_N \\
& \geq c_{n+1}D_n + c_{n+2}D_{n+1} + c_{n+4}D_{n+3} + c_{n+6}D_{n+5} + \cdots + c_N D_{N-1} \\
& \geq c_{n+1}D_n + c_{n+2}D_{n+1} + a_{n+4}D_{n+3} + a_{n+6}D_{n+5} + \cdots + a_N D_{N-1} && \text{(due to Theorem 2)} \\
& \geq c_{n+1}D_n + (a_{n+2} - 1)D_{n+1} + a_{n+4}D_{n+3} + a_{n+6}D_{n+5} + \cdots + a_N D_{N-1} && \text{(due to (2.2))} \\
& = c_{n+1}D_n - D_{n+1} + a_{n+2}D_{n+1} + a_{n+4}D_{n+3} + a_{n+6}D_{n+5} + \cdots + a_N D_{N-1} \\
& = c_{n+1}D_n - D_{n+1} + (D_{n+2} - D_n) + (D_{n+4} - D_{n+2}) + (D_{n+6} - D_{n+4}) + \cdots + (D_N - D_{N-2}) && \text{(due to Lemma 3)} \\
& = c_{n+1}D_n - D_n - D_{n+1} + D_N \\
& > c_{n+1}D_n - D_n - D_{n+1} && \text{(since } D_N > 0) \\
& = (c_{n+1} - 1)D_n - D_{n+1} && \text{(since } -D_{n+1} > 0) \\
& > 0
\end{aligned}$$

Note that this last step also holds when $c_{n+1} = 1$, since D_{n+1} is negative. These calculations demonstrate that in Case 1:

$$\sum_{k=0}^{N} c_{k+1} D_k > 0 \tag{2.3}$$

and also that

$$\sum_{k=0}^{N} c_{k+1} D_k > (c_{n+1} - 1)D_n - D_{n+1} \tag{2.4}$$

As a result of this

$$\begin{aligned}
\left|\sum_{k=0}^{N} c_{k+1}D_k\right| &= |c_{n+1}D_n + c_{n+2}D_{n+1} + \cdots + c_{N+1}D_N| && \text{(since } c_{n+1} \text{ is the first non-zero term)}\\
&= c_{n+1}D_n + c_{n+2}D_{n+1} + \cdots + c_{N+1}D_N && \text{(due to (2.3))}\\
&> (c_{n+1}-1)D_n - D_{n+1} && \text{(due to (2.4))}\\
&= |(c_{n+1}-1)D_n - D_{n+1}| && \text{(since } (c_{n+1}-1)D_n - D_{n+1} > 0)
\end{aligned}$$

In the following calculations it will be argued that $\left|\sum_{k=0}^{N} c_{k+1}D_k\right| < |c_{n+1}D_n - D_{n+1}|$. At the start of the calculations, some non-positive terms will be removed and some non-negative terms enlarged.

$$\begin{aligned}
\left|\sum_{k=0}^{N} c_{k+1}D_k\right| &= c_{n+1}D_n + c_{n+2}D_{n+1} + c_{n+3}D_{n+2} + c_{n+4}D_{n+3} + c_{n+5}D_{n+4} + \cdots + c_{N+1}D_N && \text{(due to (2.3))}\\
&\leq c_{n+1}D_n + c_{n+3}D_{n+2} + c_{n+5}D_{n+4} + \cdots + c_{N+1}D_N\\
&\leq c_{n+1}D_n + a_{n+3}D_{n+2} + a_{n+5}D_{n+4} + \cdots + a_{N+1}D_N && \text{(due to Theorem 2)}\\
&= c_{n+1}D_n + (D_{n+3} - D_{n+1}) + (D_{n+5} - D_{n+3}) + \cdots + (D_{N+1} - D_{N-1}) && \text{(due to Lemma 3)}\\
&= c_{n+1}D_n - D_{n+1} + D_{N+1}\\
&< c_{n+1}D_n - D_{n+1} && \text{(since } D_{N+1} < 0)\\
&= |c_{n+1}D_n - D_{n+1}|
\end{aligned}$$

In summary, we have proven (2.1) for $D_n > 0$ and $D_N > 0$.

2. $D_n > 0$ and $D_N < 0$

Note that there exists an $m \in \mathbb{N}_{\geq 0}$ such that $N = n + 2m + 1$. The rest of this part of the proof is similar to part 1, apart from the last term in the calculations.

3. $D_n < 0$ and $D_N > 0$

Due to Lemma 1 and $D_n < 0$, the following holds

$$D_{n+2m} < 0 \text{ and } D_{n+2m+1} > 0 \text{ for all } m \in \mathbb{N}_{\geq 0}$$

Consequently, we must have that there exists an $m \in \mathbb{N}_{\geq 0}$ such that $N = n + 2m + 1$. In the calculations that follow, it will be argued that $-\sum_{k=0}^{N} c_{k+1}D_k > 0$. Again, some non-negative terms will be eliminated. We have that

$$\begin{aligned}
&- c_{n+1}D_n - c_{n+2}D_{n+1} - c_{n+3}D_{n+2} - c_{n+4}D_{n+3} - c_{n+5}D_{n+4} - \cdots - c_{N+1}D_N\\
&\geq -c_{n+1}D_n - c_{n+2}D_{n+1} - c_{n+4}D_{n+3} - \cdots - c_{N+1}D_N\\
&\geq -c_{n+1}D_n - c_{n+2}D_{n+1} - a_{n+4}D_{n+3} - \cdots - a_{N+1}D_N && \text{(due to Theorem 2)}\\
&\geq -c_{n+1}D_n - (a_{n+2} - 1)D_{n+1} - a_{n+4}D_{n+3} - \cdots - a_{N+1}D_N && \text{(due to (2.2))}\\
&= -c_{n+1}D_n + D_{n+1} - a_{n+2}D_{n+1} - a_{n+4}D_{n+3} - \cdots - a_{N+1}D_N\\
&= -c_{n+1}D_n + D_{n+1} + (-D_{n+2} + D_n) + (-D_{n+4} + D_{n+2}) + \cdots + (-D_{N+1} + D_{N-1}) && \text{(due to Lemma 3)}\\
&= -c_{n+1}D_n + D_n + D_{n+1} - D_{N+1}\\
&> -c_{n+1}D_n + D_n + D_{n+1} && \text{(since } -D_{N+1} > 0)\\
&= (1 - c_{n+1})D_n + D_{n+1}\\
&> 0 && \text{(since } D_{n+1} > 0 \text{ and } D_n < 0)
\end{aligned}$$

These calculations yield:

$$\sum_{k=0}^{N} -c_{k+1}D_k > 0 \tag{2.5}$$

and we also found that

$$\sum_{k=0}^{N} -c_{k+1}D_k > (1 - c_{n+1})D_n + D_{n+1} \tag{2.6}$$

Consequently, we find that

$$\begin{aligned}
\left|\sum_{k=0}^{N} c_{k+1} D_k\right| &= \left|\sum_{k=n}^{N} -c_{k+1} D_k\right| && \text{(since } c_{n+1} \text{ is the first non-zero term)}\\
&= -c_{n+1}D_n - c_{n+2}D_{n+1} - \cdots - c_{N+1}D_N && \text{(due to (2.5))}\\
&> (1-c_{n+1})D_n + D_{n+1} && \text{(due to (2.6))}\\
&= |(1-c_{n+1})D_n + D_{n+1}| && \text{(since } D_n < 0 \text{ and } D_{n+1} > 0)\\
&= |(c_{n+1}-1)D_n - D_{n+1}|
\end{aligned}$$

In the following calculations it will be argued that $\left|\sum_{k=0}^{N} -c_{k+1}D_k\right| < |c_{n+1}D_n - D_{n+1}|$. Again, at the start of the calculations, some non-positive terms will be removed and some non-negative terms enlarged.

$$\begin{aligned}
\left|\sum_{k=0}^{N} c_{k+1} D_k\right| &= -c_{n+1}D_n - c_{n+2}D_{n+1} - c_{n+3}D_{n+2} - c_{n+4}D_{n+3} - c_{n+5}D_{n+4} - \cdots - c_{N+1}D_N && \text{(due to (2.5))}\\
&\leq -c_{n+1}D_n - c_{n+3}D_{n+2} - c_{n+5}D_{n+4} - \cdots - c_N D_{N-1}\\
&\leq -c_{n+1}D_n - a_{n+3}D_{n+2} - a_{n+5}D_{n+4} - \cdots - a_N D_{N-1} && \text{(due to Theorem 2)}\\
&= -c_{n+1}D_n + (-D_{n+3} + D_{n+1}) + (-D_{n+5} + D_{n+3}) + \cdots + (-D_N + D_{N-2}) && \text{(due to Lemma 3)}\\
&= -c_{n+1}D_n + D_{n+1} - D_N\\
&< -c_{n+1}D_n + D_{n+1} && \text{(since } -D_N < 0)\\
&= |-c_{n+1}D_n + D_{n+1}|\\
&= |c_{n+1}D_n - D_{n+1}|
\end{aligned}$$

In summary, we have proven (2.1) for $D_n < 0$ and $D_N > 0$.

4. $D_n < 0$ and $D_N < 0$

Note that there exists an $m \in \mathbb{N}_{\geq 0}$ such that $N = n + 2m$. The rest of this part of the proof is similar to part 3, apart from the last term in the calculations. ☐

Turning now to the last lemma previous to the theorem about the best approximation.

Lemma 7. *Let $[a_0; a_1, a_2, \ldots]$ be the continued fraction of $t \in \mathbb{R} \setminus \mathbb{Q}$ with convergents $\frac{A_k}{B_k}$ for all $k \in \mathbb{N}_{\geq 0}$ and let m be a positive integer. The Ostrowski representation of m is given by (1.16), where $c_{k+1} = 0$ for $0 \leq k < n \leq N$ and $c_{n+1} > 0$. Then*

1. *if $c_1 = c_2 = 0$ then $||mt|| = \left|\sum_{k=0}^{N} c_{k+1}D_k\right|$*
2. *if $c_1 = 0$ and $c_2 > 0$ then*
   (a) *if $\{t\} < \frac{1}{2}$ then $||mt|| = \left|\sum_{k=0}^{N} c_{k+1}D_k\right|$*
   (b) *if $\{t\} > \frac{1}{2}$ then*
      i. *if $c_2 > 1$ then $||mt|| > ||t||$*
      ii. *if $c_2 = 1$ then $||mt|| > D_2$*
3. *if $c_1 > 0$ then $||mt|| > |D_1|$*

Proof.

Case 1, where $c_1 = c_2 = 0$

First, we will prove that $\left|\sum_{k=0}^{N} c_{k+1}D_k\right| < \frac{1}{2}$ and therefore it is necessary to discuss two different cases:

(A) If $D_n > 0$, then $D_{n+1} < 0$, according to Lemma 1:

$$\begin{aligned}
\left|\sum_{k=0}^{N} c_{k+1} D_k\right| &< |c_{n+1}D_n - D_{n+1}| && \text{(due to Lemma 6)}\\
&= c_{n+1}D_n - D_{n+1} && \text{(since } D_n > 0, D_{n+1} < 0\text{)}\\
&\leq a_{n+1}D_n - D_{n+1} && \text{(due to Theorem 2)}\\
&= |a_{n+1}D_n - D_{n+1}| \\
&= |-D_{n-1}| && \text{(due to Lemma 3)}\\
&= |D_{n-1}| \\
&< \tfrac{1}{2} && \text{(due to Lemma 4 and since } c_1 = c_2 = 0 \text{ we have that } n \geq 2\text{)}
\end{aligned}$$

(B) If $D_n < 0$, then $D_{n+1} > 0$, according to Lemma 1:

$$\begin{aligned}
\left|\sum_{k=0}^{N} c_{k+1} D_k\right| &< |c_{n+1}D_n - D_{n+1}| && \text{(due to Lemma 6)}\\
&= |-c_{n+1}D_n + D_{n+1}| \\
&= -c_{n+1}D_n + D_{n+1} && \text{(since } D_n < 0, D_{n+1} > 0\text{)}\\
&\leq -a_{n+1}D_n + D_{n+1} && \text{(due to Theorem 2)}\\
&= |-a_{n+1}D_n + D_{n+1}| \\
&= |a_{n+1}D_n - D_{n+1}| \\
&= |-D_{n-1}| && \text{(due to Lemma 3)}\\
&= |D_{n-1}| \\
&< \tfrac{1}{2} && \text{(due to Lemma 4 and } n \geq 2\text{)}
\end{aligned}$$

Note that for all $x \in \mathbb{R}$ the following holds:

$$\text{if } |x| \leq \tfrac{1}{2}, \text{ then } ||x|| = |x| \tag{2.7}$$

In addition we also have that:

$$\text{if } \tfrac{1}{2} < |x| < 1, \text{ then } ||x|| = 1 - |x| \tag{2.8}$$

Due to (2.7) and Lemma 5 we have that

$$\left|\sum_{k=0}^{N} c_{k+1} D_k\right| = ||mt||$$

Case 2.a, where $c_1 = 0$ and $c_2 > 0$ and $\{t\} < \frac{1}{2}$.

In the same way like Case 1, in this case it is proven that $\left|\sum_{k=0}^{N} c_{k+1} D_k\right| < \frac{1}{2}$. Since $D_1 < 0$, we know that $D_2 > 0$, according to Lemma 1. Now,

$$\begin{aligned}
\left|\sum_{k=0}^{N} c_{k+1} D_k\right| &< |c_2 D_1 - D_2| && \text{(due to Lemma 6)}\\
&= |-c_2 D_1 + D_2| \\
&= -c_2 D_1 + D_2 && \text{(since } D_1 < 0 \text{ and } D_2 > 0\text{)}\\
&\leq -a_2 D_1 + D_2 && \text{(due to Theorem 2)}\\
&= |-a_2 D_1 + D_2| \\
&= |a_2 D_1 - D_2| \\
&= |-D_0| && \text{(due to Lemma 3)}\\
&= |-\{t\}| && \text{(by Definition 4)}\\
&= \{t\} < \tfrac{1}{2} && \text{(by the definition of } \{t\}\text{)}
\end{aligned}$$

In conclusion, $\left|\sum_{k=0}^{N} c_{k+1} D_k\right| < \frac{1}{2}$. Since (2.7) and Lemma 5 we find that $\left|\sum_{k=0}^{N} c_{k+1} D_k\right| = ||mt||$.

Case 2.b.i, where $c_1 = 0$ and $c_2 > 1$ and $\{t\} > \frac{1}{2}$.
In this case we have that $\{t\} > \frac{1}{2}$, so $\frac{1}{\{t\}} < 2$, which results in

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} < 2$$

Consequently, we know that $a_1 = 1$. Next, we will show that

$$1 - \{t\} < \left| \sum_{k=0}^{N} c_{k+1} D_k \right| < \{t\} \tag{2.9}$$

Again due to Lemma 1 we have that $D_1 < 0$ and $D_2 > 0$. Now,

$$\begin{aligned}
\left| \sum_{k=0}^{N} c_{k+1} D_k \right| &> |(c_2 - 1)D_1 - D_2| && \text{(due to Lemma 6)} \\
&= |D_2 - (c_2 - 1)D_1| \\
&= D_2 - (c_2 - 1)D_1 && \text{(since } c_2 > 1 \text{ and } D_1 < 0 \text{ and } D_2 > 0) \\
&> -(c_2 - 1)D_1 && \text{(since } D_2 > 0) \\
&> -D_1 \\
&= 1 - a_1\{t\} && \text{(due to Lemma 3 and by Definition 4)} \\
&= 1 - \{t\} && \text{(since } a_1 = 1)
\end{aligned}$$

and furthermore we have that

$$\begin{aligned}
\left| \sum_{k=0}^{N} c_{k+1} D_k \right| &< |c_2 D_1 - D_2| && \text{(due to Lemma 6)} \\
&= |D_2 - c_2 D_1| \\
&= D_2 - c_2 D_1 && \text{(since } D_1 < 0 \text{ and } D_2 > 0) \\
&\leq D_2 - a_2 D_1 && \text{(due to Theorem 2)} \\
&= D_0 && \text{(due to Lemma 3)} \\
&= \{t\} && \text{(by Definition 4)}
\end{aligned}$$

Note that since $\{t\} > \frac{1}{2}$ we have that $1 - \{t\} < \frac{1}{2}$. Next, we can observe two possibilities for $\left| \sum_{k=0}^{N} c_{k+1} D_k \right|$, which are $\left| \sum_{k=0}^{N} c_{k+1} D_k \right| \leq \frac{1}{2}$ or $\left| \sum_{k=0}^{N} c_{k+1} D_k \right| > \frac{1}{2}$. If $\left| \sum_{k=0}^{N} c_{k+1} D_k \right| \leq \frac{1}{2}$ we know from (2.7), (2.9) and since $1 - \{t\} < \frac{1}{2}$ that

$$\left|\left| \sum_{k=0}^{N} c_{k+1} D_k \right|\right| = \left| \sum_{k=0}^{N} c_{k+1} D_k \right| > 1 - \{t\} = ||\{t\}|| = ||\{t\} + a_0|| = ||t||$$

On the other hand, if $\left| \sum_{k=0}^{N} c_{k+1} D_k \right| > \frac{1}{2}$ we know since (2.8) and (2.9) that

$$\left|\left| \sum_{k=0}^{N} c_{k+1} D_k \right|\right| = 1 - \left| \sum_{k=0}^{N} c_{k+1} D_k \right| > 1 - \{t\} = ||t||$$

From Lemma 5 we know that in both situations the following holds

$$||mt|| = \left|\left| \sum_{k=0}^{N} c_{k+1} D_k \right|\right| > ||t||$$

Case 2.b.ii, where $c_1 = 0$ and $c_2 = 1$ and $\{t\} > \frac{1}{2}$.

For the same reasons described in Case 2.b.i we know that $a_1 = 1$. According to Lemma 1 we have that $D_1 < 0$ and $D_2 > 0$. Since $c_2 = 1$ and due to Lemma 6 we find that

$$|D_2| < \left|\sum_{k=0}^{N} c_{k+1}D_k\right| < |c_2D_1 - D_2| = D_2 - D_1 \tag{2.10}$$

Furthermore, due to Lemma 4 we find that

$$D_2 - D_1 = |D_2| + |D_1| < \tfrac{1}{2} + \tfrac{1}{2} = 1 \tag{2.11}$$

Next, we can differentiate two possibilities for $\left|\sum_{k=0}^{N} c_{k+1}D_k\right|$, which are $\left|\sum_{k=0}^{N} c_{k+1}D_k\right| \le \frac{1}{2}$ or $\left|\sum_{k=0}^{N} c_{k+1}D_k\right| > \frac{1}{2}$. If $\left|\sum_{k=0}^{N} c_{k+1}D_k\right| \le \frac{1}{2}$ we find that since (2.7) and (2.10)

$$\left\|\sum_{k=0}^{N} c_{k+1}D_k\right\| = \left|\sum_{k=0}^{N} c_{k+1}D_k\right| > D_2 \tag{2.12}$$

On the other hand, if $\left|\sum_{k=0}^{N} c_{k+1}D_k\right| > \frac{1}{2}$ we know since (2.8), (2.10) and (2.11) that

$$\left\|\sum_{k=0}^{N} c_{k+1}D_k\right\| = 1 - \left|\sum_{k=0}^{N} c_{k+1}D_k\right| > 1 - (D_2 - D_1) \tag{2.13}$$

It remains to be proven that $1 - D_2 + D_1 > D_2$, so that we obtain $\left\|\sum_{k=0}^{N} c_{k+1}D_k\right\| > D_2$. We know that $a_1 = 1$, so because of

$$a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}} < 2a_2$$

we have that

$$\frac{1}{\{t\}} = 1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} > 1 + \frac{1}{2a_2}$$

implying that

$$\{t\} < \frac{1}{1 + \frac{1}{2a_2}}$$

Therefore, we obtain

$$\begin{aligned}
& \{t\} < \frac{2a_2}{2a_2 + 1} \\
\iff\quad & \{t\}(2a_2 + 1) < 2a_2 \\
\iff\quad & 2\{t\}a_2 + \{t\} < 2a_2 \\
\iff\quad & -\{t\} + 2\{t\}a_2 < -2\{t\} + 2a_2 \\
\iff\quad & 1 - \{t\} + 2\{t\}a_2 - 2a_2 < 1 - 2\{t\} \\
\iff\quad & (2a_2 - 1)(\{t\} - 1) < 1 - 2\{t\} \\
\iff\quad & (2a_2 - 1)D_1 < 1 - 2D_0 \qquad \text{(due to Definition 4, Lemma 3 and } a_1 = 1) \\
\iff\quad & 2a_2D_1 - D_1 < 1 - 2D_0 \\
\iff\quad & 2a_2D_1 + 2D_0 < 1 + D_1 \\
\iff\quad & 2D_2 < 1 + D_1 \qquad \text{(since Lemma 6)} \\
\iff\quad & D_2 < 1 - D_2 + D_1
\end{aligned}$$

Owing to the calculations above, we know that $1 - D_2 + D_1 > D_2$. Consequently, we see that in both situations $\left\|\sum_{k=0}^{N} c_{k+1}D_k\right\| > D_2$. Due to Lemma 5, we find that $||mt|| > D_2$.

Case 3, where $c_1 > 0$

Note that since $c_1 \geq 1$, we know from Theorem 2 that $a_1 \geq 2$. Consequently, we find that

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} > 2$$

Which results in $\{t\} < \frac{1}{2}$. Next, we will prove that

$$1 - a_1\{t\} < \left| \sum_{k=0}^{N} c_{k+1} D_k \right| < 1 - \{t\} \tag{2.14}$$

Now,

$$\begin{aligned}
\left| \sum_{k=0}^{N} c_{k+1} D_k \right| &> |(c_1 - 1)D_0 - D_1| && \text{(due to Lemma 6)} \\
&= (c_1 - 1)D_0 - D_1 && \text{(since } D_0 > 0 \text{ and } D_1 < 0) \\
&\geq -D_1 && \text{(since } D_0 > 0) \\
&= -a_1 D_0 - D_{-1} && \text{(due to Lemma 3)} \\
&= 1 - a_1\{t\} && \text{(since (1.18))}
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
\left| \sum_{k=0}^{N} c_{k+1} D_k \right| &< |c_1 D_0 - D_1| && \text{(due to Lemma 6)} \\
&= c_1 D_0 - D_1 && \text{(since } D_0 > 0 \text{ and } D_1 < 0) \\
&\leq (a_1 - 1)D_0 - D_1 && \text{(due to Theorem 2)} \\
&= a_1 D_0 - D_0 - D_1 && \\
&= -D_{-1} - D_0 && \text{(due to Lemma 3)} \\
&= 1 - \{t\} && \text{(since (1.18))}
\end{aligned}$$

Next, we can observe two possibilities for $\left|\sum_{k=0}^{N} c_{k+1} D_k\right|$, which are $\left|\sum_{k=0}^{N} c_{k+1} D_k\right| \leq \frac{1}{2}$ or $\left|\sum_{k=0}^{N} c_{k+1} D_k\right| > \frac{1}{2}$. If $\left|\sum_{k=0}^{N} c_{k+1} D_k\right| \leq \frac{1}{2}$ we know from (2.7) and (2.14) that

$$\left\| \sum_{k=0}^{N} c_{k+1} D_k \right\| = \left| \sum_{k=0}^{N} c_{k+1} D_k \right| > 1 - a_1\{t\} = -D_1 \tag{2.15}$$

On the other hand, if $\left|\sum_{k=0}^{N} c_{k+1} D_k\right| > \frac{1}{2}$ we know from (2.14)

$$-\left| \sum_{k=0}^{N} c_{k+1} D_k \right| > \{t\} - 1$$

Therefore and due to (2.8) and $1 - \{t\} < 1$ we find that

$$\left\| \sum_{k=0}^{N} c_{k+1} D_k \right\| = 1 - \left| \sum_{k=0}^{N} c_{k+1} D_k \right| > \{t\} \tag{2.16}$$

It remains to prove that $\{t\} > 1 - a_1\{t\}$. Since

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} < a_1 + 1$$

we know that $\{t\} > \frac{1}{a_1+1}$. Obviously, we have that $\{t\}a_1 + \{t\} > 1$. Therefore we obtain $\{t\} > 1 - a_1\{t\}$ and we find from (2.15), (2.16) and Lemma 5 that in both situations the following holds

$$||mt|| = \left\| \sum_{k=0}^{N} c_{k+1} D_k \right\| > 1 - a_1\{t\} = -D_1 = |D_1|$$

□

2.2 Theorem on a best approximation

We have now enough tools to formulate and prove a theorem on a *best approximation.*

Theorem 3 (Best approximation)**.** *Let $t \in \mathbb{R} \setminus \mathbb{Q}$ and $A, B \in \mathbb{Z}$ with gcd$(A, B) = 1$ and $B > 0$. Then $\frac{A}{B}$ is a best approximation to t if and only if it is a convergent of t.*

Proof. Let the Ostrowski representation of B be given by (1.16). Define $n \in \mathbb{N}$ such that $c_{k+1} = 0$ for all $0 \leq k < n \leq N$ and $c_{n+1} > 0$. Note that N depends on B. In this proof we will check every case described in Lemma 7. In other words, we will check every possibility for c_1 and c_2.

Case 1, where $\mathbf{c_1 = c_2 = 0}$
Before we give the actual proof of the statement, we first will derive a handy inequality.

$$|Bt - A| > |B_N t - A_N| \text{ if } \tfrac{A}{B} \neq \tfrac{A_N}{B_N} \tag{2.17}$$

Notice that due to Lemma 7 Part 1, we find that

$$|Bt - A| = ||Bt|| = \left|\sum_{k=0}^{N} c_{k+1} D_k\right| \tag{2.18}$$

Next, we observe four cases in which Lemma 2 and parts of the proof of Lemma 7 are applied.

(A) $D_n > 0$ and $D_N > 0$

$$\left|\sum_{k=0}^{N} c_{k+1} D_k\right| \geq (c_{n+1} - 1)D_n - D_{n+1} + D_N > D_N = |D_N| = |B_N t - A_N|$$

(B) $D_n > 0$ and $D_N < 0$

$$\left|\sum_{k=0}^{N} c_{k+1} D_k\right| \geq (c_{n+1} - 1)D_n - D_{n+1} + D_{N+1} > -D_{n+1} = |D_{n+1}| \geq |D_N| = |B_N t - A_N|$$

Notice that $N \neq n$ because of the fact that D_n and D_N have different sign. Consequently, we have that $N \geq n + 1$.

(C) $D_n < 0$ and $D_N > 0$

$$\left|\sum_{k=0}^{N} c_{k+1} D_k\right| \geq (1 - c_{n+1})D_n + D_{n+1} - D_{N+1} > D_{n+1} = |D_{n+1}| > |D_N| = |B_N t - A_N|$$

Also in this case $N \geq n + 1$ holds.

(D) $D_n < 0$ and $D_N < 0$

$$\left|\sum_{k=0}^{N} c_{k+1} D_k\right| \geq (1 - c_{n+1})D_n + D_{n+1} - D_N > -D_N = |D_N| = |B_N t - A_N|$$

We see that $|B_N t - A_N|$ minimizes $|Bt - A|$ in all possible cases. Note that according to Theorem 2, we know that $0 < B_N \leq B$.

Now, we prove the theorem for this case, in which $c_1 = c_2 = 0$. Assume that $\frac{A}{B}$ is a best approximation. By Definition 3 we know that $|Bt - A| < |B_N t - A_N|$ for all $\frac{A}{B} \neq \frac{A_N}{B_N}$ such that $0 < B_N < B$. Due to (2.17) we know that $|Bt - A| > |B_N t - A_N|$ for all $\frac{A}{B} \neq \frac{A_N}{B_N}$. Consequently, we find that $A = A_N$ and $B = B_N$.

Assume that $\frac{A}{B}$ is a convergent. Then there exists an m such that $|Bt - A| = |B_m t - A_m| = |D_m|$. According to Theorem 2 we know that $B_N \leq B_m < B_{N+1}$. Consequently, we find that $B_N = B_m$, so $m = N$. Due to (2.17) and since we know from Theorem 2 that $B_N \leq B$, we conclude that $\frac{A_N}{B_N}$ is a best approximation.

Case 2a, where $\mathbf{c_1 = 0, c_2 > 0}$ and $\mathbf{\{t\} < \frac{1}{2}}$
The proof of this case is similar to Case 1.

Case 2bi, where $\mathbf{c_1 = 0, c_2 > 1}$ and $\mathbf{\{t\} > \frac{1}{2}}$
Before we start with the proof, first we will show that $|Bt - A| \geq ||Bt||$. Assume that $|Bt - A| \leq \frac{1}{2}$, then according to (2.7) we know that $|Bt - A| = ||Bt||$. Second, assume that $|Bt - A| > \frac{1}{2}$, then we find that $|Bt - A| > ||Bt||$.

From Lemma 7 and $\{t\} > \frac{1}{2}$ we know that $a_1 = 1$. Note that due to Theorem 1 we know that $B_1 = a_1 B_0 + B_{-1} = 1$. Furthermore we have that,

$$|Bt - A| \geq ||Bt|| > 1 - \{t\} = -D_1 = |D_1| = |B_1 t - A_1| \tag{2.19}$$

Assume that $B = B_0 = B_1 = 1$. Due to Lemma 4 we have that $|D_1| < \frac{1}{2}$, so $||B_1 t|| = ||B_1 t - A_1|| = |D_1|$. Consequently, we obtain $|B_1 t - A_1| > |B_1 t - A_1|$. Therefore $B > B_1$ and from (2.19) we find that $|Bt - A| > |B_1 t - A_1|$. By Definition 3 we know that $\frac{A}{B}$ is not a best approximation in this case.

For the contrary assume that $\frac{A}{B}$ is are convergent, then there exists an $m \in \mathbb{N}$, such that $|Bt - A| = |B_m t - A_m| = |D_m|$. From (2.19), we know that $|D_m| = |B_m t - A_m| = |Bt - A| > ||Bt|| > |D_1|$ and according to Lemma 2, we find that $|D_m| > |D_1|$ for all $k \in \mathbb{N}_{\geq 1}$. Since $B > B_1$ we find that $\frac{A}{B}$ is not a convergent.

Case 2bii, where $\mathbf{c_1 = 0, c_2 = 1}$ and $\mathbf{\{t\} > \frac{1}{2}}$
Note that in the proof of Lemma 7, it is found that $a_1 = 1$ in this case. We observe three possibilities for N, which are $N = 0$, $N = 1$ and $N > 1$.

(A) $N = 0$
Because of the fact that $c_1 = 0$, we know that $B = c_1 B_0 = 0$, which is not possible.

(B) $N = 1$
Next, we assume that $N = 1$. Due to Theorem 1.1 and because of the fact that $a_1 = 1$, we find that:

$$B = c_1 B_0 + c_2 B_1 = B_1 \text{ and } A = c_1 A_0 + c_2 A_1 = A_1 \tag{2.20}$$

So obviously, $\frac{A}{B}$ is a convergent. For the contrary assume that $\frac{A}{B}$ is a convergent. Because of the fact that $B = B_1 = 1$, there does not exist an $q \in \mathbb{N}$ where $0 < q < B$. So we find that $\frac{A}{B}$ is a best approximation.

(C) $N > 1$
At last, assume that $N > 1$. Since $c_1 = 0$, $c_2 = 1$ and $B_1 = a_1 B_0 + B_{-1} = 1$ we have that

$$B = c_1 B_0 + c_2 B_1 + c_3 B_2 + \cdots + c_{N+1} B_N = 1 + c_3 B_2 + \cdots + c_{N+1} B_N$$

Since $N > 1$, we find that there exists an $z \in [2, 3 \ldots, N]$ such that $c_{z+1} B_z \neq 0$. Consequently and due to Theorem 1.1 we see that $B > B_z \geq B_2$. Due to Lemma 7 we know that

$$|Bt - A| \geq ||Bt|| > D_2 = |D_2| \tag{2.21}$$

Since $B > B_2$ and $|Bt - A| > |B_2 t - A_2|$, we know by Definition 3 that $\frac{A}{B}$ is not a best approximation in this case.
For the contrary, assume that $\frac{A}{B}$ is a convergent, then there exists an $m \in \mathbb{N}$, such that $|Bt - A| = |B_m t - A_m| = |D_m|$. From (2.21), we know that $|D_m| = |Bt - A| > |D_2|$. According to Lemma 2, we find that $|D_m| > |D_k|$ for all $k \in \mathbb{N}_{\geq 2}$. We also know that in this case $B > 1$, so $B_m \neq B_1 = B_0 = 1$. So $\frac{A}{B}$ is not a convergent.

Case 3, where $\mathbf{c_1 > 0}$
Since $c_1 > 0$, we know that $a_1 \geq 2$. First we proof this case for $c_1 = 1$ and $N = 0$, afterwards we will proof this case for $c_1 \neq 1$ or $N \neq 0$.

(A) $c_1 = 1$ and $N = 0$
Since $c_1 = 1$ and $N = 0$, we find that

$$B = c_1 B_0 \text{ and } A = c_1 A_0 = A_0 \tag{2.22}$$

Obviously, we find that $\frac{A}{B}$ is an convergent. Because of the fact that $B = B_0 = 1$, there does not exist an $q \in \mathbb{N}$ where $0 < q < B$. So we find that $\frac{A}{B}$ is a best approximation.

(B) $N \neq 0$
We assume that $N \neq 0$, then $B > 1$. Assume that $\frac{A}{B}$ is a best approximation. Since $N > 0$ and since

$$B = c_1 B_0 + c_2 B_1 + \cdots + c_{N+1} B_N$$

we know that there exists an $z \in [1, 2, \ldots, N]$ such that $c_{z+1}B_z \neq 0$. Due to this and since $c_1 \geq 1$ we have that

$$B = c_1 B_0 + c_2 B_1 + \cdots + c_{N+1} B_N \geq c_1 + c_{z+1} B_z > B_z$$

Due to Theorem 1.1 we know that $B_z \geq B_1$, so $B > B_1$. From Lemma 7 we know that

$$|Bt - A| \geq ||Bt|| > |D_1| = |B_1 t - A_1| \tag{2.23}$$

Since $B > B_1$ and $|Bt - A| > |B_1 t - A_1|$, we know by Definition 3 that $\frac{A}{B}$ is not a best approximation in this case.

Conversely, assume that $\frac{A}{B}$ is a convergent, then there exists an $m \in \mathbb{N}$, such that $|Bt - A| = |B_m t - A_m| = |D_m|$. From (2.23), we know that $|D_m| = |Bt - A| > |D_1|$. According to Lemma 2, we find that $|D_m| > |D_k|$ for all $k \in \mathbb{N}_{\geq 1}$. We also know that in this case $B > 1$, so $B_m \neq B_0$. Therefore is $\frac{A}{B}$ not a convergent.

□

3 Planetarium

Having discussed what continued fractions are and why the approximation by convergents yields best approximations, this section describes the planetarium of Christiaan Huygens in greater detail. This section is primary based on the translation of *Opuscula postuma*; see [5], a detailed description of the planetarium, written by Huygens himself. As mentioned before, a planetarium shows multiple planets and their movement around the sun. In his design, Huygens made it possible to look into the past or future, where the planetarium shows the position of planets relative to each other. Huygens was not the first man to design a planetarium, even Archimedes, from the ancient Greece, designed a planetarium; see [3]. However, Huygens was the first mathematician who used continued fractions convergents to approach the ratio of the orbit of the planets. In 1682 Christiaan Huygens describes to his minister Colbert, that his planetarium is more precise than the planetarium of Ole Christensen Rømer, an astronomer from Denmark:

Het mijne geeft de beweging van alle planeten veel juister weer dan dat van Rømer, omdat ik een betere manier heb [met herhaalde breuken] om het aantal tanden van de raderen te vinden; see [1]

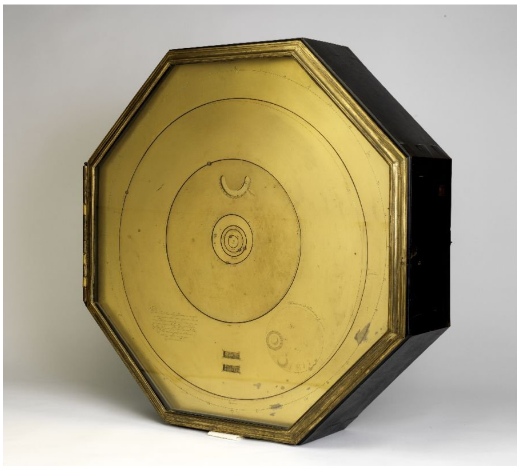


Figure 2: Front view planetarium

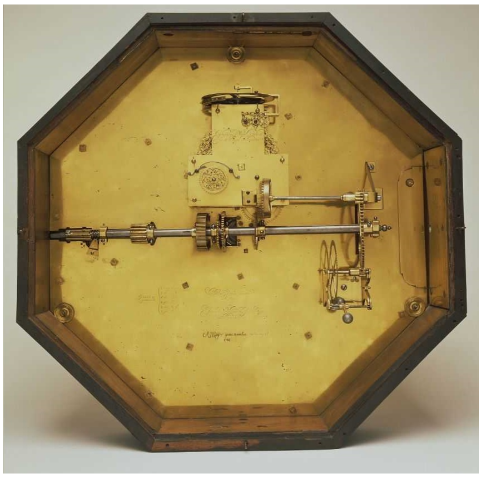


Figure 3: Back view planetarium

Figures 2 and 3 show the planetarium, currently situated in Museum Boerhaave in Leiden. On the front view, we see circles which represent the orbit of the planets around the sun, which are: Mercury, Venus, Earth, Mars, Jupiter and Saturn. On these circles, small balls move, which represent the planets itself. Not only the planets are visible on the planetarium, also moons from different planets rotate around their planet. Saturn shows five moons, Jupiter four and Earth obviously one. Also, between the orbits of Saturn and Jupiter, there are two lines on the bottom of the planetarium, in which the date is shown. The semicircle between the orbits of Jupiter and Mars, nearby the middle of the planetarium, shows the exact time in hours and minutes. On the right side of the planetarium there is an handle and if this handle is rotated ones to the right, one year has passed in the planetarium. Is the handle rotated the other way around, the planets are positioned one year back. Eventually, when the handle is removed from the planetarium, the planets rotate back into their position in the present.

3.1 Mechanism

Before explaining the use of continued fractions convergents, first the mechanism of the planetarium is briefly discussed. In order to rotate the planets around the sun, moving force is generated by a clock-mechanism, invented by Huygens himself. In figure 4 this clock-mechanism is visible right above an horizontal shaft and has a letter N on it. The horizontal shaft, which Huygens refers to as 'ijzeren as' ('iron bar'), is connected with six gears, as much as the number of planets in the planetarium. These gears have a specific number of teeth, which will be of great importance in the section that follows. The cover of the planetarium is a copper plate which consists concentric rings, which are shown in figure 5. The teeth of the gears on the horizontal shaft mesh with the teeth of the concentric rings when the planetarium is closed. On the right side of the horizontal shaft there is another gear, which is responsible for the movement of the planets, because it is connected to the clock-mechanism. A screw-thread is also connected with this gear, which causes the planets to rotate one extra time after 300 years as a slight correction to the leap day every four years.

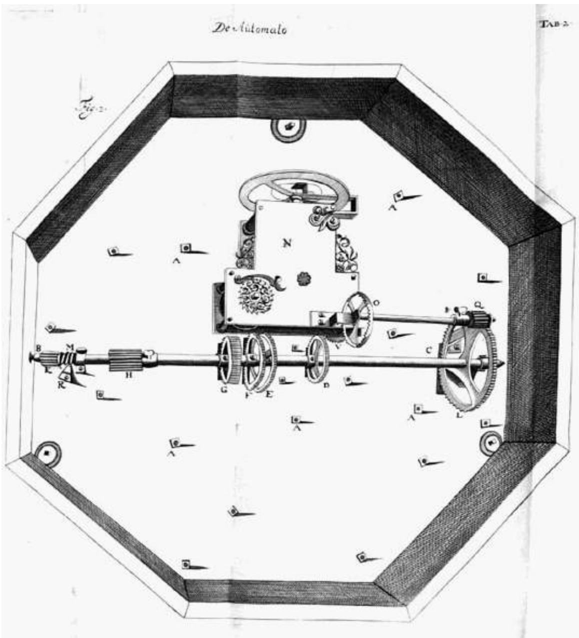


Figure 4: Inside view planetarium

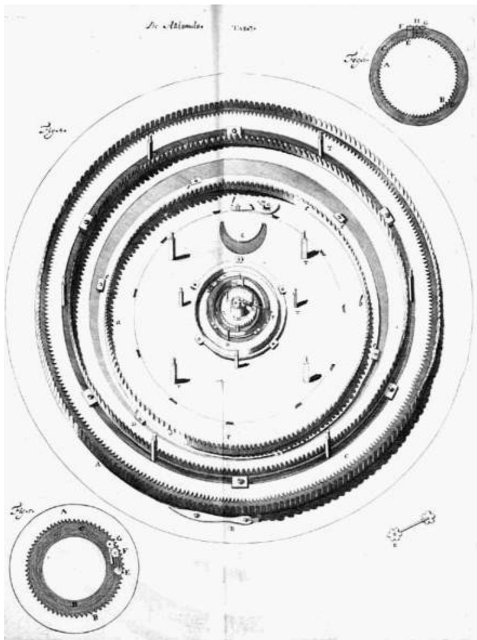


Figure 5: Inside view planetarium

3.2 Approximation by convergents

As indicated before, the ratios between the teeth on the gears on the horizontal shaft and the concentric rings appear to be important because they determine the orbital period of the planets in the planetarium. In the ideal situation, the ratio between the number of teeth of the horizontal shaft and the concentric ring equals the ratio between the orbital period of earth and any other planet.

First, Huygens approximated the orbital period of the planets by using the data derived by Johannes Kepler, a German mathematician and astronomer. Afterwards, he calculated the ratio between this orbital periods and the orbital period of the earth. Huygens approximated a year as $365\frac{35}{144}$ days. For example, the ratio determined for Mercury is $\frac{25335}{105190}$. In other words, it takes Mercury $\frac{25335}{105190}$ part of one year to finish one rotation around the sun. So if Mercury completes 105190 rotations around the sun, the earth would have completed 25335. Consequently, Mercury should have 25335 teeth on the gear connected to the concentric rings and 105190 teeth on the gear connected to the horizontal shaft, to make sure the orbital period is precise. However, it is impossible to make an planetarium with gears with this amount of teeth. It is clear that this ratio should be approximated and Huygens did so with continued fraction convergents. Hence, Huygens' problem was to construct a planetarium in which the number of teeth is workable, but the approximation of orbital period is still accurate. In the following example, the approximation of the orbital period of Mercury is further explained.

Example 4 (Mercury)**.** As mentioned before, the ratio which determines the number of teeth in the two gears is $\frac{25335}{105190}$. The continued fraction is

$$\frac{25335}{105190} = \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{2}}}}}}}}}}}} = [0; 4, 6, 1, 1, 2, 1, 1, 1, 1, 7, 1, 2]$$

with the following convergents

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{\frac{A_k}{B_k}}$	0	$\frac{1}{4}$	$\frac{6}{25}$	$\frac{7}{29}$	$\frac{13}{54}$	$\frac{33}{137}$	$\frac{46}{191}$	$\frac{79}{328}$	$\frac{125}{519}$	$\frac{204}{847}$	$\frac{1553}{6448}$	$\frac{1757}{7295}$	$\frac{25335}{21038}$

Initially, Huygens wanted to use the fifth convergent $\frac{33}{137}$, so that the amount of teeth is workable and the approximation accurate. In other words, the number of teeth on the gear connected to the horizontal shaft is 33 and the number of teeth on the gear connected to the concentric ring is 137. However, in section 4 we will see that Huygens invented another way to use the 8th convergent, so that the approximation is even more precise.

Similar to the example described above, the remaining planets are approximated by convergents so that the number of teeth is workable and the approximation precise. In some cases, the number of teeth is a multiple of the convergent. The following table shows these results [6]

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Ratio from data	$\frac{25335}{105190}$	$\frac{64725}{105190}$	1	$\frac{197836}{105190}$	$\frac{1247057}{105190}$	$\frac{3095277}{105190}$
Convergent	$\frac{A_5}{B_5} = \frac{33}{137}$	$\frac{A_5}{B_5} = \frac{8}{13}$	-	$\frac{A_5}{B_5} = \frac{79}{42}$	$\frac{A_3}{B_3} = \frac{83}{7}$	$\frac{A_1}{B_1} = \frac{59}{2}$
Gear teeth	33 : 127	32 : 52	60 : 60	158 : 84	166 : 14	118 : 4

Table 1: Approximation of orbital periods and the number of teeth on the gears for each planet

3.3 Error of the approximation

Huygens stated in his description of the planetarium that his way of approximation is very accurate. In this section the error of the approximation is determined. As explained earlier, Huygens obtained data concerning the ratio between the orbital period of any planet and the orbital period of the earth. Then he chose an accurate, workable convergent to approximate this ratio. The error of the approximation is, in this project, the difference between the ratio Huygens obtained from this data and the ratio Huygens used as an approximation. In the following example, the error of the approximation of the planet Mercury is determined.

Example 5 (Mercury)**.** As explained earlier, it Mercury takes $\frac{25335}{105190}$ part of a year to finish one rotation around the sun. Huygens approximated this ratio with $\frac{33}{137}$. The approximation error is

$$\left|\frac{25335}{105190} - \frac{33}{137}\right| \approx 0.00002602173$$

So in one rotation around the sun, Mercury is approximately 0.00002602173 ≈ 0.0026% behind because of the approximation with the 5th convergent.

For every planet it is determined how many percent of a rotation the planet is behind because of the approximation by convergents. This is shown in the following table

Planet	Mercury	Venus	Mars	Jupiter	Saturn
Error	0.0026%	0.0069%	0.020%	0.1862%	7.4418%

Table 2: Error of approximation

4 Improvements planetarium

4.1 Before construction

Before Johannes van Ceulen started building the planetarium, Huygens made some improvements. In his opinion the error of the approximation should be smaller for the planets Mercury and Saturn. For Saturn he recalculated the ratio for the orbital period, which will be $\frac{77708431}{2640858}$, see [5] for a detailed description of these calculations.

$$\frac{77708431}{2640858} = [29; 2, 2, 1, \dots]$$

Huygens chose to use the 3th convergent, which is $\frac{206}{7}$ and the error of approximation becomes

$$\left|\frac{77708431}{2640858} - \frac{206}{7}\right| \approx 0.003122956900$$

So the error of approximation of Saturn is reduced from 7.4418% to 0.3123%.

In addition, the error of approximation of Mercury was also improved. However, for the planet Mercury, Huygens used another method. For Mercury he was forced to use the 5th convergent in order to restrain the number of teeth on the gears, but Huygens found a way to use the 9th convergent $\frac{204}{847}$. He factorised the convergent into the following

$$\frac{204}{847} = \frac{12 \cdot 17}{7 \cdot 121} \tag{4.1}$$

and used this result to design four gears for the planet Mercury, in stead of two. The gear connected to the horizontal shaft has 121 teeth and is connected to a second, smaller gear, with 12 teeth. The gear connected to the concentric rings has 17 teeth and is connected to a second, smaller gear, with 7 teeth. The new error of approximation becomes

$$\left|\frac{25335}{105190} - \frac{204}{847}\right| \approx 1.683578588 \cdot 10^{-7}$$

So the error of approximation of Mercury is reduced from 0.0026% to $1.6836 \cdot 10^{-5}$%.

4.2 After the construction: more about gear trains

In the previous section, a method is discussed which Huygens applied on the approximation of the orbital period of Mercury. Equation (4.1) shows that the 9th convergent of the continued fraction can be broken into two fractions, such that the approximation of the ratio between the orbital periods is more precise. In the planetarium, a gear train is constructed for this planet, with four gears. In this subsection, an algorithm is described which results in four integers bigger than 20 and smaller than 120, which can be used in a new 'improved' planetarium. Rocket and Szüsz described this algorithm in 'Continued fractions', see [6] p.63. Note that these numbers (20 and 120) appear to be useful in the construction of the planetarium, but the algorithm also works with slightly different numbers, in the next subsection we will discuss this in greater detail.

In the end we will see that the approximation of the ratio between the orbital periods is better, when this method is used in comparison with the approximation in the original planetarium. As mentioned before, the goal in this algorithm is to find integers $20 \leq n_1, n_2, n_3, n_4 \leq 120$ such that

$$\left| r - \frac{n_1 \cdot n_2}{n_3 \cdot n_4} \right| \text{ is minimized} \tag{4.2}$$

were $r \in \mathbb{Q}$ represents the ratio between the orbital period of any planet and the orbital period of the Earth and $\gcd(n_1 \cdot n_2, n_3 \cdot n_4) = 1$.

4.2.1 An algorithm for the gear train

In the first step of the algorithm, we create an ordered list L which contains all possible values of $n \cdot m$ such that $20 \leq n, m \leq 120$. For all $b \in L$ we construct an integer a, which will be important for the algorithm. Let $\left(\frac{A_k}{B_k}\right)_k$ be the convergents of r. The denominators of the ratios for the orbital period of any planet are smaller than $120 \cdot 120$, see Table 1. Therefore we know that there exists an $N \in \mathbb{N}$ in which $b < B_{N+1}$ and Theorem 2 can be applied to $r \in \mathbb{Q}$. According to Theorem 2 we find that there exist an unique finite sequence $\left(c_{k+1}\right)_k$ such that

$$b = \sum_{k=0}^{N} c_{k+1} B_k \tag{4.3}$$

Next, we define A as

$$A = \sum_{k=0}^{N} c_{k+1} A_k \tag{4.4}$$

and a as

$$a = \begin{cases} A & \text{if } bt - A \geq 0 \text{ and } bt - A \leq \frac{1}{2} \text{ or } bt - A < 0 \text{ and } bt - A > -\frac{1}{2} \\ A + 1 & \text{if } bt - A \geq 0 \text{ and } bt - A > \frac{1}{2} \\ A - 1 & \text{if } bt - A < 0 \text{ and } bt - A \leq -\frac{1}{2} \end{cases} \tag{4.5}$$

In the following subsection, we find that $|bt - a| \leq \frac{1}{2}$ and therefore we know that $||bt|| = |bt - a|$. So in this way a is the nearest integer to bt. The next step defines $a' \in L$ and depends on the value of a:

- if $a \in L$ then $a' = a$
- if $a \notin L$ then
  - if $a > 120 \cdot 120$ then $a' = 120 \cdot 120$
  - if $a < 20 \cdot 20$ then $a' = 20 \cdot 20$
  - if $20 \cdot 20 < a < 120 \cdot 120$ then define a_{lower} as the greatest lower bound in L of a and define a_{upper} as the smallest upper bound in L of a, then
    - if $|br - a_{lower}| < |br - a_{upper}|$ then $a' = a_{lower}$
    - if $|br - a_{upper}| < |br - a_{lower}|$ then $a' = a_{upper}$

Thus in this way we find a list of combinations of a' and b for each $b \in L$. Afterwards, we create a list of the values of $|br - a'|$ and find the minimum of that list, so that we find the 'best' combination between a' and b. These a' and b are the $n_1 \cdot n_2$ and $n_3 \cdot n_4$ we were trying to find, according to (4.2).

4.2.2 Theory behind the algorithm

The following theorem shows that $|Br - A| < 1$ and $|br - a| \leq \frac{1}{2}$. Since bt is an irrational number and a non-negative integer, we find that $|bt - a| = ||bt||$.

Theorem 4. *Let $[a_0; a_1, a_2, \dots]$ be the continued fraction of the irrational number t with convergents $\frac{A_k}{B_k}$ for all $k \in \mathbb{N}_{\geq 0}$, where $B_k > 0$ and $gcd(A_k, B_k) = 1$. Let $b \in \mathbb{N}$ and let the Ostrowski representation of b given by (4.3). Let $A \in \mathbb{N}$ be given by (4.4) and define $a \in \mathbb{N}$ as in (4.5). Then*

1. $|bt - A| < 1$
2. $|bt - a| \leq \frac{1}{2}$

Proof. From Definition 4 and Lemma 6 we know that

$$|bt - A| = \left|\sum_{k=0}^{N} c_{k+1}B_k - \sum_{k=0}^{N} c_{k+1}A_k\right| = \left|\sum_{k=0}^{N} c_{k+1}B_k - A_k\right| = \left|\sum_{k=0}^{N} c_{k+1}D_k\right| < |c_{n+1}D_n - D_{n+1}| \tag{4.6}$$

In the first part of this proof two possibilities for n are examined, which are $n = 1$ and $n \geq 2$. First, let $n = 1$. From Lemma 1 we know that $D_1 < 0$ and $D_2 > 0$, so according to Lemma 3 and Theorem 2 it follows that

$$|c_2D_1 - D_2| = |-c_2D_1 + D_2| = -c_2D_1 + D_2 \leq -a_2D_1 + D_2 = D_0 = \{t\} < 1 \tag{4.7}$$

Let $n \geq 2$, then we have two different cases for D_n, which are $D_n > 0$ and $D_n < 0$. Let $D_n > 0$ then according to Lemma 1 we have that $D_{n+1} < 0$. By Lemma 3, Lemma 4 and Theorem 2 we find that

$$|c_{n+1}D_n - D_{n+1}| = c_{n+1}D_n - D_{n+1} \leq a_{n+1}D_n - D_{n+1} = -D_{n-1} = |D_{n-1}| \leq \tfrac{1}{2} \tag{4.8}$$

Let $D_n < 0$ then according to Lemma 1 we have that $D_{n+1} > 0$. By the same reasons as (4.8), we have that

$$|c_{n+1}D_n - D_{n+1}| = -c_{n+1}D_n + D_{n+1} \leq -a_{n+1}D_n + D_{n+1} = D_{n-1} = |D_{n-1}| \leq \tfrac{1}{2} \tag{4.9}$$

As a result of (4.6), (4.8) and (4.9), we have that for all $n \geq 1$

$$|bt - A| < 1 \tag{4.10}$$

In order to prove Part 2 of the theorem, the different cases of the definition of a are examined. First, assume that $bt - A \geq 0$ and $bt - A \leq \frac{1}{2}$. Since $a = A$ we have that

$$|bt - a| = |bt - A| = bt - A \leq \tfrac{1}{2}$$

Second, we assume that $bt - A < 0$ and $bt - A > -\frac{1}{2}$. Since $a = A$ we have that

$$|bt - a| = |bt - A| = -bt + a \leq \tfrac{1}{2}$$

In the second case of the definition of a we assume that $bt - A \geq 0$ and $bt - A > \frac{1}{2}$. According to (4.10) and $bt - A \geq 0$, we know that $bt - A < 1$. So $bt - A - 1 < 0$, which results in

$$bt - (A + 1) < 0 \tag{4.11}$$

Another property of this case is $bt - A > \frac{1}{2}$, which results in $bt - (A + 1) > -\frac{1}{2}$. So that $-bt + (A + 1) < \frac{1}{2}$. Since (4.11) and $a = A + 1$, we find that $|bt - a| < \frac{1}{2}$. In the last case of the definition of a, we assume that $bt - A < 0$ and $bt - A \leq -\frac{1}{2}$. According to (4.10) and $bt - A < 0$, we know that $-bt + A < 1$ so that $bt - A > -1$. Consequently, $bt - A + 1 > 0$, which results in

$$bt - (A - 1) > 0 \tag{4.12}$$

Another property of this case is $bt - A \leq -\frac{1}{2}$, which results in $bt - (A - 1) \leq \frac{1}{2}$. Since (4.12) and $a = A - 1$, we find that $|bt - a| \leq \frac{1}{2}$. So for every case in the definition of a it is proven that $|bt - a| \leq \frac{1}{2}$. □

4.2.3 Results

The algorithm described above is applied to the ratios between the orbital periods of any planet and the Earth, which is referred to as r in the algorithm. The algorithm is carried out by an program, which is shown in the Appendix. In Table 3 the comparison is shown between the approximation by convergents and the approximation after the algorithm is applied. Note that the ratios used in the gear train are slightly different from the ratios shown in 'Continued fractions', see [6].

Planet	Mercury	Venus	Mars	Jupiter	Saturn
Ratio from data (r)	$\frac{25335}{105190}$	$\frac{64725}{105190}$	$\frac{197836}{105190}$	$\frac{1247057}{105190}$	$\frac{3095277}{105190}$
Ratio used in planetarium	$\frac{12 \cdot 17}{7 \cdot 121}$	$\frac{A_5}{B_5} = \frac{8}{13}$	$\frac{A_5}{B_5} = \frac{79}{42}$	$\frac{A_3}{B_3} = \frac{83}{7}$	$\frac{A_3}{B_3} = \frac{206}{7}$
Ratio used in gear train	$\frac{37 \cdot 53}{69 \cdot 118}$	$\frac{38 \cdot 107}{56 \cdot 118}$	$\frac{34 \cdot 77}{12 \cdot 116}$	$\frac{85 \cdot 106}{8 \cdot 95}$	$\frac{107 \cdot 110}{4 \cdot 100}$

Table 3: Comparison between approximation by convergents and after the algorithm is applied

As mentoined before, the error of the approximation is more precise when using the algorithm described in the previous section. Table 4 shows the difference between the error of approximation by convergents like Huygens used them and the approximation as a result of the gear train.

Planet	Mercury	Venus	Mars	Jupiter	Saturn
Error planetarium	$1.6836 \cdot 10^{-5}\%$	0.0069%	0.020%	0.1862%	0.3123%
Error gear train	$2.3352 \cdot 10^{-6}\%$	$3.7405 \cdot 10^{-5}\%$	$1.0081 \cdot 10^{-4}\%$	$1.7762 \cdot 10^{-3}\%$	$5.8228 \cdot 10^{-2}\%$

Table 4: Error of approximation

5 References

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[2] Dajani K., Kraaikamp C., (2002) *Ergodic theory of numbers.* Carus Mathematical Monographs, 29.

[3] Wright M. (2017), *Archimedes, astronomy, and the planetarium.* Springer International Publishing AG

[4] Ostrowski, A. (1921), *Bemerkungen zur Theorie der Diophantischen Approximationen.* Abh. Math. Sem. Hamburg. Univ., 77-98.

[5] Huygens, C. (1703), *Opuscula postuma* (translated by dr. J.A. Vollgraff and dr. D.A.H. van Eck). Bayerische Staatsbibliothek.

[6] Andrew M. Rockett, Peter Szüsz (1992). *Continued fractions.* World Scientific, First edition.

Appendix

```
import math

############################continued fraction###################
x=int(input("a="))
y=int(input("b="))

cf=[]
a=x
b=y
if b>a:
cf.append(0)

while a>b:
v=a-b
n=1
while v>b:
v=v-b
n=n+1
cf.append(n)
a=b
b=v

while b>a:
v=b-a
n=1
while v>a:
v=v-a
n=n+1
cf.append(n)
b=a
a=v

cf[-1]=n+1
print("Continued fraction=",cf)

##########################convergents######################

denominators=[]
numerators=[]

numerators.append(cf[0])
numerators.append(cf[1]*cf[0]+1)
```

```
A=numerators[1]
for i in range(1,len(cf)-1):
A=cf[i+1]*numerators[i]+numerators[i-1]
numerators.append(A)
print("Numerators=",numerators)

denominators.append(1)
denominators.append(cf[1])

B=numerators[1]
for i in range(1,len(cf)-1):
B=cf[i+1]*denominators[i]+denominators[i-1]
denominators.append(B)
print("Denominators=",denominators)

###################creating list L##################
l=set()

for i in range(20,121):
for j in range(20,121):
n=i*j
l.add(n)

L=sorted(l)

############for every element in L create ostrowski representation##############
alphalist=[]
alphaaccentlist=[]
for i in L:
errorlist=[]
O1=[]
O2=[]
s=i
for j in range(0,len(denominators)-1):
if s>denominators[j] or s==denominators[j]:
N=j
while s>0:
n=0
while s>denominators[N] or s==denominators[N]:
s=s-denominators[N]
n=n+1
O1.append(n)
O2.append(N)
j=-1
while j<N-1 or j==N-1:
if s>denominators[j+1] or s==denominators[j+1]:
j=j+1
else:
N=j
Ostrowskilist=[]
for k in range(0,len(O1)-1):
Ostrowskilist.append(O1[k])
Ostrowskilist.append("*B"+str(O2[k])+ "+")
Ostrowskilist.append(str(O1[-1])+"*B"+str(O2[-1]))
###############create A####################################
A=0
for l in range(0,len(O1)):
A=A+O1[l]*numerators[O2[l]]
##################create alpha#####################
if abs(i*x/y-A)<0.5 or abs(i*x/y-A)==0.5:
alpha=A
```

```
elif b*x/y-A>0.5:
alpha=A+1
else:
alpha=A-1
alphalist.append(alpha)
######################create alpha accent###############
if alpha in L:
alphaaccent=alpha
else:
if alpha>120*120:
alphaaccent=120*120
elif alpha<20*20:
alphaaccent=20*20
else:
for q in range(0,len(L)-1):
if L[q]<alpha:
pass
else:
alphalower=L[q-1]
for q in range(0,len(L)-1):
if L[q]<alpha:
pass
else:
alphaupper=L[q+1]
if abs(i*x/y-alphalower)<abs(i*x/y-alphaupper):
alphaaccent=alphalower
else:
alphaaccent=alphaupper
alphaaccentlist.append(alphaaccent)

########################find best combination alpha' and b##################
error=[]

for i in range(0,len(L)-1):
error.append(abs(x/y-alphaaccentlist[i]/L[i]))

for i in range(0,len(error)-1):
if error[i]==min(error):
bestalpha=alphaaccentlist[i]
bestb=L[i]
else:
pass
print("best alpha=", bestalpha)
print("best b=", bestb)
print("Error=", min(error))


#############find n_1,n_2,n_3,n_4####################
for i in range(20,119):
if bestalpha%i==0:
firstn=bestalpha/i
secondn=bestalpha/firstn
else:
pass

for i in range(20,119):
if bestb%i==0:
thirdn=bestb/i
fourthn=bestb/thirdn
else:
pass
```

```
print("n1=",firstn)
print("n2=",secondn)
print("n3=",thirdn)
print("n4=",fourthn)
```