CROSS - SWELL

A comparison of mathematical derivations with the

results of laboratory tests

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LIST OF SYMBOLS

a	Wave amplitude
$\vec{e}$	Vector of unity
E	Wave energy
g	Acceleration of gravity
h	Water depth
H	Wave height
$\bar{H}$	Mean wave height
k	Wave number $(2\pi/\lambda)$
K_D	Diffraction coefficient
L	Wave length within the cross-swell system
S	Radiation stress
T	Wave period
u	Velocity perpendicular to the coast
v	Velocity parallel to the coast
γ	Breakerindex
δ	Phase angle
η	Surface elevation
$\bar{\eta}$	Mean surface elevation, set-up
λ	Original wave length
ρ	Density of water
σ	Standard deviation
τ	Friction force
ϕ	Direction of the wave (angle between wave crest and beach)
ω	angular frequency $(2\pi/T)$

1. SUMMARY

The results of some initial laboratory tests regarding cross-swell are compared with theoretical considerations.
A first order linear wave model describes the surface of cross-swell quite accurately.
The breakerindex proved to be nearly equal to the breakerindex of standing waves.
Set-up is calculated with the theory of radiation stress.
A set-up formula for cross-swell is presented. Test results fit rather well with the data from this formula.
The tests were executed in the wave basin of the Laboratory of Fluid Mechanics, Delft University of Technology.

2. INTRODUCTION

The phenomenon of cross-swell occurs when two wave fields with different wave height, wave period and direction come together in the same area. When such a cross-swell reaches the coast it is not possible to calculate longshore current (and thus littoral sediment transport) with the usual formula. In previous reports (Ludikhuize & Verhagen 1978) some theoretical derivations regarding cross-swell were presented in order to calculate littoral sediment transport.
These derivations are summarized in this report.
Theoretical derivations and considerations about cross-swell were not found in literature. There was no proof at all that the derivation presented in Ludikhuize & Verhagen 1978 give an acceptable description of the phenomenon.
One of the basic assumptions is that both wave fields do not influence each other. Especially this assumption had to be proved experimentally. To prove this very extensive laboratory tests have to be made.

However, the authors got the opportunity to do some short tests in the Laboratory of Fluid Mechanics of the Delft University of Technology. Because of the limited time available (one and a half week) certainly not all the aspects of this phenomenon were examined. But these tests give some indications about cross-swell.

This report can be regarded as a starting point for further investigations, theoretical considerations and derivations on the subject of cross-swell.

The authors wish to thank ir. P. Visser of the Laboratory of Fluid Mechanics for his comments on this report.

3. THEORETICAL DERIVATIONS

3.a. General Theory

In this chapter the theory of cross-swell will be discussed. In the derivation of this theory several assumptions have been made, viz.:

- the two wave fields do not influence each other
- each wave field consists of stationary, long crested, regular waves
- each wave field can be described by the linear Airy-wave theory

The consequence of these assumptions is that at a certain point the surface elevation is the sum of the two original surface elevations (of the original wave fields).

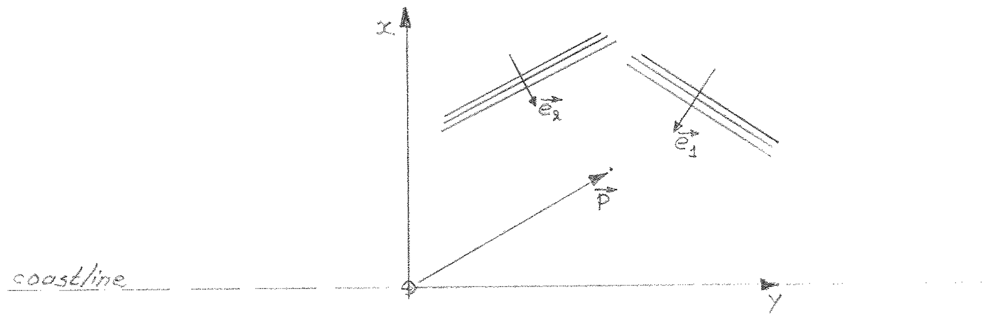


Fig. 3.1.

The surface can be described by:

$$\eta = \eta_1 + \eta_2 = a_1 \sin(\omega_1 t - \vec{k}_1 . \vec{p}) + a_2 \sin(\omega_2 t - \vec{k}_2 . \vec{p} + \delta) \qquad (3.1)$$

In which:

η - total surface elevation

a_1 a_2 - amplitude of the original waves

ω_1 ω_2 - angular frequency of the original waves

$\vec{k}_1$ $\vec{k}_2$ - wave number and wave direction of the original waves

$$\vec{k}_1 = \frac{2\pi}{\lambda_1} . \vec{e}_1 \quad ; \quad \vec{k}_2 = \frac{2\pi}{\lambda_2} . \vec{e}_2$$

in which λ is the local wave length of the original waves and $\vec{e}$ the vector of unity in the original direction of propagation.

δ - the phase angle, to fit the choosen coordinate system

The internal product $\vec{k}.\vec{p}$ can be written for a carthesian coordinate system as:

$$\vec{k}.\vec{p} = k_x x + k_y y \qquad (3.2)$$

The surface elevation is visualised in fig. 3.2. The maximum elevation (in C) is the sum of the original amplitudes:

$$\eta_{max} = a_1 + a_2 \qquad (3.3)$$

This maximum will only occur at points, and not along lines, like crest lines in long crested waves. Because of the special shape of the surface the term 'wave height' has to be defined somewhat more detailed than in case of long crested waves.
For cross-swell the wave height is defined as the difference between the maximum surface elevation in the wave field and the minimum surface elevation (i.e. a negative elevation) of the same field but not necessarily in the same point.

$$H_T = (a_1 + a_2) - (-a_1 - a_2) = 2a_1 + 2a_2 = H_1 + H_2 \quad (3.4)$$

Thus, the 'wave height' in cross-swell is the sum of the two original wave heights.

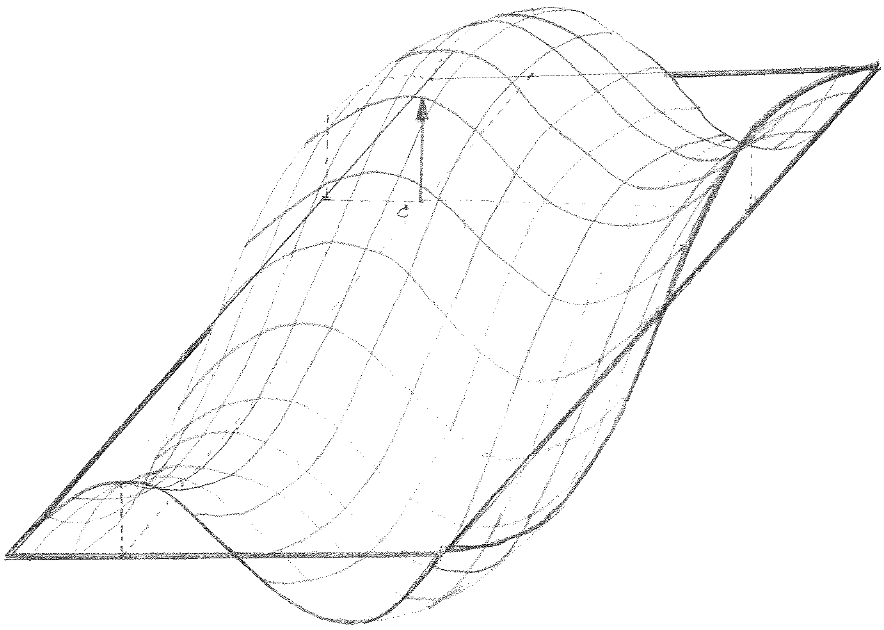


Fig. 3.2.

The distance between two wave crests can be measured in the direction of propagation (L_p) or transverse to this direction (L_s). This second distance is called L_s because in this direction the wave system resembles somewhat a standing wave.

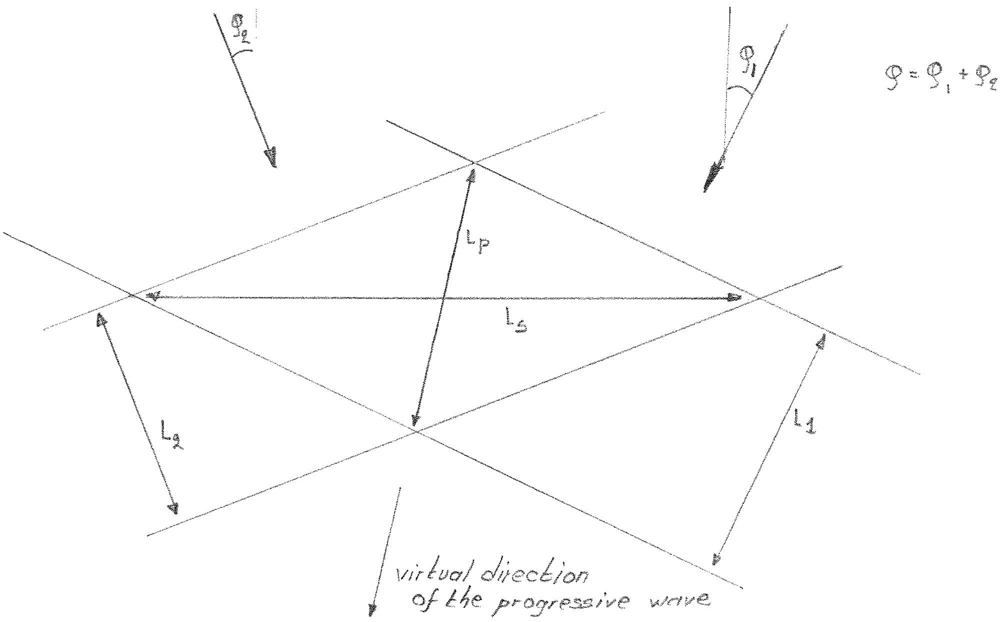


Fig. 3.3.

The distance L_p is:

$$L_p^{\,2} = \left(\frac{L_2}{\sin\phi}\right)^2 + \left(\frac{L_1}{\sin\phi}\right)^2 - \frac{2L_1L_2}{\sin^2\phi}\cos\phi$$

$$= \frac{1}{\sin^2\phi}\left(L_1^{\,2} + L_2^{\,2} - 2L_1L_2\cos\phi\right) \qquad (3.5)$$

The distance L_s is:

$$L_s^{\,2} = \left(\frac{L_2}{\sin\phi}\right)^2 + \left(\frac{L_1}{\sin\phi}\right)^2 + \frac{2L_1L_2}{\sin^2\phi}\cos\phi$$

$$= \frac{1}{\sin^2\phi}\left(L_1^{\,2} + L_2^{\,2} + 2L_1L_2\cos\phi\right) \qquad (3.6)$$

When the incoming waves are perpendicular ($\phi = 90^{o}$) then $L_p{}^2 = L_1{}^2 + L_2{}^2$ and also $L_s{}^2 = L_1{}^2 + L_2{}^2$.
L_p and L_s are called virtual wave lengths, because the direction of wave propagation does not necessarily match to the direction of L_p and L_s. They match only when $L_1 = L_2$. This will be discussed in the next chapter.
The energy of cross-swell is:

$$E = \frac{1}{8} \rho g \; (H_1{}^2 + H_2{}^2) \tag{3.7}$$

and not $E = \frac{1}{8} \rho g \; H_T{}^2$. The derivation of this formula is presented in Annex I to this report.
Another important aspect is the radiation stress in cross-swell.
In Annex II is shown that:

$$S_{xx} = (S_{xx})_1 + (S_{xx})_2 \tag{3.8}$$

$$S_{yy} = (S_{yy})_1 + (S_{yy})_2 \tag{3.9}$$

$$S_{xy} = (S_{xy})_1 + (S_{xy})_2 \tag{3.10}$$

These formulae are derived under the following restrictions:

- The periods of both wave fields are not identical
- The radiation stresses are an average over an interval T (see below).

The longshore current and the set-up can be calculated from the averaged radiation stresses. At a fixed point the radiation stress itself will fluctuate in time. This fluctuation has a certain return period T, which depends on the periods of the two wave fields.

This period T is the smallest whole multiple of T_1 and T_2

$$T = a\,T_1 = b\,T_2 \tag{3.11}$$

in which a and b are integers which are as small as possible.

When the waves reach the coast breaking will occur. Parallel to the coast the envelope of the surface resembles somewhat a standing wave, so it can be expected that the breakerindex (γ, relation between water depth and wave height; $H = \gamma h$) will be higher than in case of only one wave field.
The breaking criterion for standing waves is $\frac{H}{L} = 0.22 \tanh\left(\frac{2\pi}{L} h\right)$, which leads to $H = 1.38\ h$ in shallow water.
When the wave heights of both waves differ very much, it is possible that the waves of each field break independently of each other.
This is certainly not true when the wave heights of both fields are almost identical.

The position of the mean water level (set-up or set-down) and the currents in x- and y-direction can be calculated by solving the following equations:

$$\rho(h + \bar{\eta})\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial \bar{\eta}}{\partial y}\right) + \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} \tag{3.12}$$

$$\ldots . + \text{turbulence terms} + \bar{\tau}_{by} = 0$$

$$\rho(h + \bar{\eta})\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \bar{\eta}}{\partial x}\right) + \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \ldots . \tag{3.13}$$

$$\ldots . + \text{turbulence terms} + \bar{\tau}_{bx} = 0$$

in which:

$\bar{\eta}$	= position of the mean water level
u	= velocity perpendicular to the coast
v	= velocity parallel to the coast
$\bar{\tau}_{by}$, $\bar{\tau}_{bx}$	= mean friction forces

These equations can be simplified assuming that:

- the velocity u, perpendicular to the coast, is zero:

$$\frac{\partial u}{\partial t} = 0 \qquad \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$$

- the velocity v, parallel to the coast is constant in time and is constant along the y-axis:

$$\frac{\partial v}{\partial t} = 0 \qquad \frac{\partial v}{\partial y} = 0$$

- the turbulence terms are negligible.

The equations 3.12 and 3.13 become:

$$\rho g \; (h + \bar{\eta}) \; \frac{\partial \bar{\eta}}{\partial y} + \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} + \bar{\tau}_{by} = 0 \tag{3.14}$$

$$\rho g \; (h + \bar{\eta}) \; \frac{\partial \bar{\eta}}{\partial x} + \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \bar{\tau}_{bx} = 0 \tag{3.15}$$

Note: The second simplification is not correct. From the laboratory tests it appeared that the velocity in y-direction is certainly not constant. This simplification has to be a subject for further studies.

Near the coast ($^h/_{L_o}$ < 0.5) the functions of S_{xy}, S_{xx} and S_{yy} become very complex because of:

- shoaling (both wave height and wave length will change)
- refraction (the angle of incidence of both wave fields will change)

It is beyond the scope of this study to describe the effects of shoaling and refraction for cross-swell mathematically.
In this study is tried to show that for a cross-swell situation the generally used formula:

$$\bar{\eta}_{max} = \frac{5}{16} \gamma H_b \tag{3.16}$$

is not correct.

3.b. Special case theory

In this chapter is discussed the theory of a special case of cross-swell, viz. when the wave periods of the two wave fields are identical.
This special case is investigated in the laboratory tests.
In this case the surface extention (η) becomes a function of the location. This can be seen by means of formula 3.1 which becomes:

$$\eta = \eta_1 + \eta_2 = a_1 \sin(\omega t - \vec{k}_1 . \vec{p}) + a_2 \sin(\omega t - \vec{k}_2 . \vec{p} + \delta) \tag{3.17}$$

(wave number $|\vec{k}|$ and angular frequency (ω) of both wave fields are identical).
The phase-lag between η_1 and η_2 depends only on $\vec{k}_1 . \vec{p}$ and $\vec{k}_2 . \vec{p}$, which are functions of x and y. The surface extention η at one place becomes a sinusoidal function of the time; it will repeat itself after T ($T = \frac{2\pi}{\omega}$) seconds, thus after a period T the elevation of the surface will be the same as it was a period T before.

In this pattern lines with maximum surface elevations ($\eta_{max} = a_1 + a_2$) can be distinguished. At these lines the phase-lag between η_1 and η_2 is constantly zero. Also lines with minimum surface elevations can be distinguished ($\eta_{min} = |a_1 - a_2|$), for these lines the phase-lag is constantly π (see fig. 3.4).

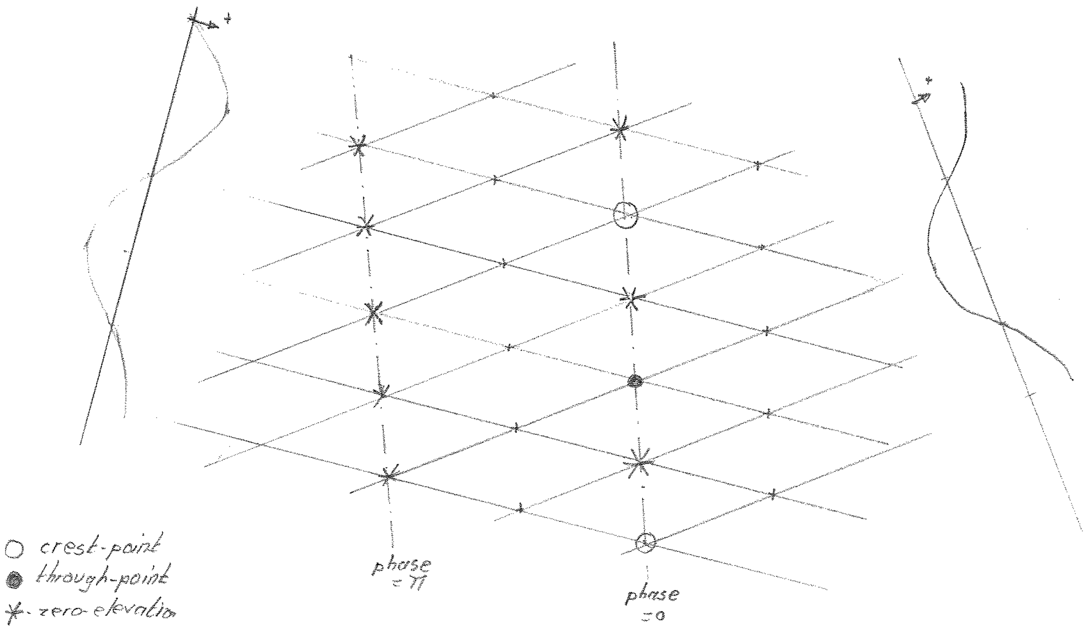


Fig. 3.4.

(If $a_1 = a_2$ then the maximum surface elevation at 'phase = 0'-lines is $2a_1$, at 'phase = π' the surface elevation is zero).
The crest and trough points propagate along lines which are the same as the 'phase = zero'-lines, because the celerities of both wave systems are identical. Also the other points of the surface propagate along the phase-lines.

The energy and radiation-stresses can also be calculated and it appears that they are place-dependent.

The radiation-stresses cannot be described with simple relations as in the general case. In Annex III these radiation-stress formulae are derivated (form 29-32 in that Annex).
In general the results can be summarized as:

$$S = f\ (H_1,\ H_2,\ \phi_1,\ \phi_2,\ (p - q)) \tag{3.18}$$

in which (p - q) varies from $-\pi$ to $+\pi$ and represents the location in respect to the phase-lines in fig. 3.4.
The radiation-stress formula is sinusoidal in (p - q) and therefore a sinusoidal set-up distribution along the coast and a sinusoidal longshore velocity distribution can be expected.

4. DIFFRACTION MODELS

The tests were executed in a basin in which the two-wave-fields were generated by a snake-type wave generator.
Each half of this wave generator produced a wave field. Because the width of the generated wave field was smaller than the width of the basin, diffraction occurred.
Two diffraction models are made for the analysis of the measurements. By means of these two diffraction models diffraction coefficients and phases can be computed.

The two diffraction models are based on the following assumptions:

- constant depth (which is true until the toe of the slope; fig. 5.1).
- both wave fields do not influence each other
- no reflection occurs against the sides or against the slope
- in the diffraction model of each wave field it is assumed that part of the wave board which produced the other wave field was an impermeable wall with a fixed position.

Diffraction model I

In this diffraction model is assumed that the width of the basin was much larger (theoretical infinitively times larger) than the width of the wave generator.
Because of the above-mentioned assumptions the diffraction of each wave field can be calculated separately. In order to get the total wave heights, the results of diffraction models of the two wave systems are added (fig. 4.1).

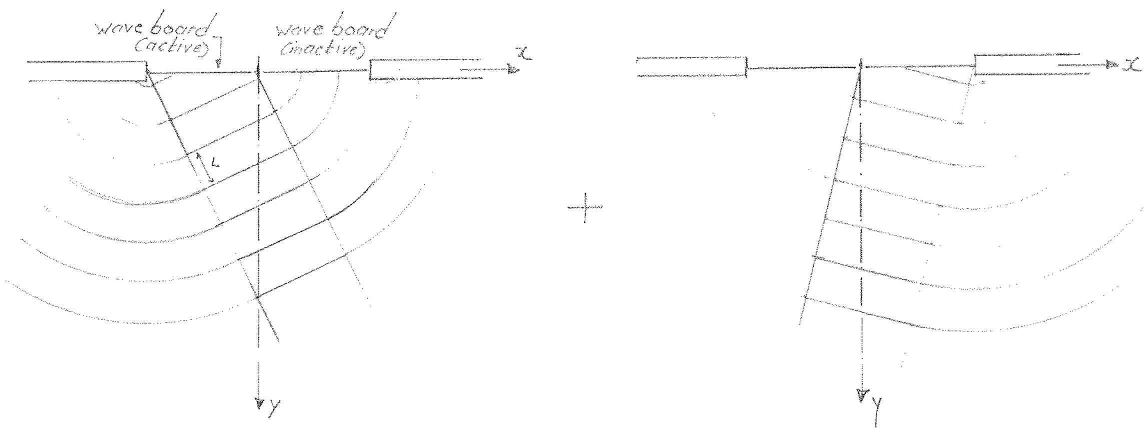


Fig. 4.1.

Diffraction model II

In this model also two diffracted wave fields are added (see fig. 4.2). In wave field A is assumed an impermeable wall at the left side of the waves, and perpendicular to the wave crest, while in wave field B an impermeable wall is assumed at the right side.
In both fields the other part of the wave board is assumed to be inactive and the impermeable wall which guides the other wave field is neglected.

Remark: In the model tests these two impermeable walls were placed in the wave basin, but they were not exactly perpendicular to the wave crests.

Reflection due to these walls is neglected.

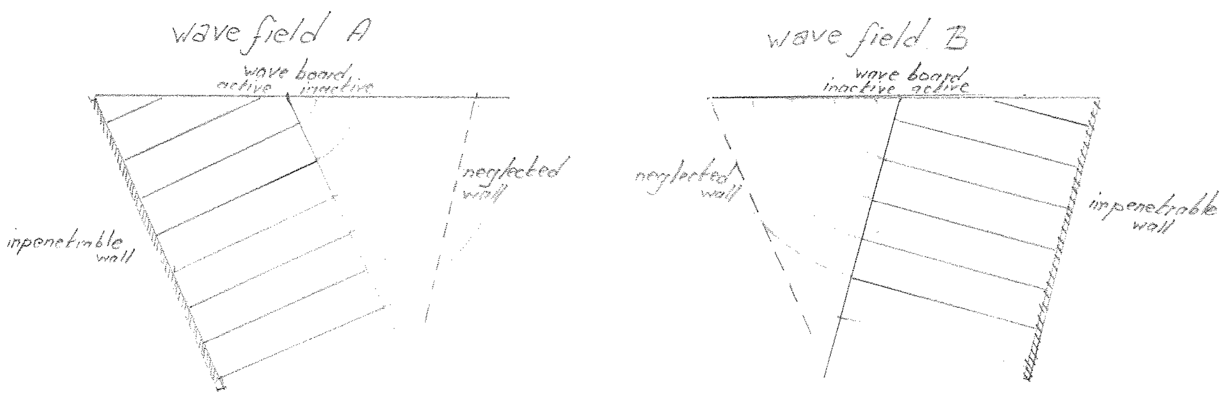


Fig. 4.2.

For both diffraction models (I and II) the diffraction coefficients and phases were calculated with a computer program based on the theory of Sommerfeld, using an approximation of the Fresnel-integrals. The maximum amplitude of the surface elevation at certain points is also calculated in this computer program in the following way:

$$a = K_{D1} \cdot a_1 \cdot \sin(\omega t - \text{Phase 1}) +$$

$$K_{D2} \cdot a_2 \cdot \sin(\omega t - \text{Phase 2})$$

in which:

K_{D1}, K_{D2} = diffraction coefficients (-)

Phase 1, Phase 2 = phase differences caused by diffraction (rad)

a_1, a_2 = amplitudes of incoming waves (m)

ω = angular frequency (rad/sec)

t = time (sec)

5. EXPERIMENTAL RESULTS

5.a. Description of the experiments

The experiments were made in the large wave basin of the Laboratory of Fluid Mechanics, Delft University of Technology. For the layout of this basin see fig. 5.1. The main dimensions of the experiment area were approx. 15 x 34 m. The water depth was 0.40 m. Along one long side a snake-type wave generator was available. This generator was adjusted in such a way that two wave fields were generated, both directed to the centre of the basin. Two series of testst were run:

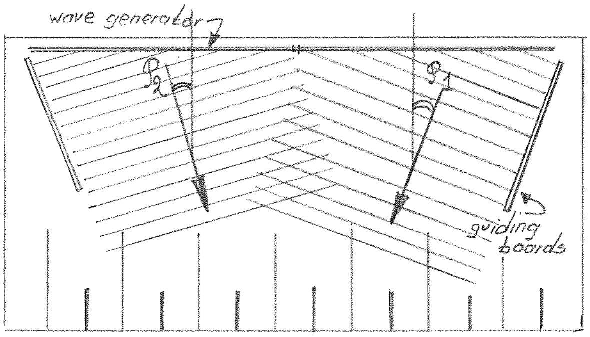


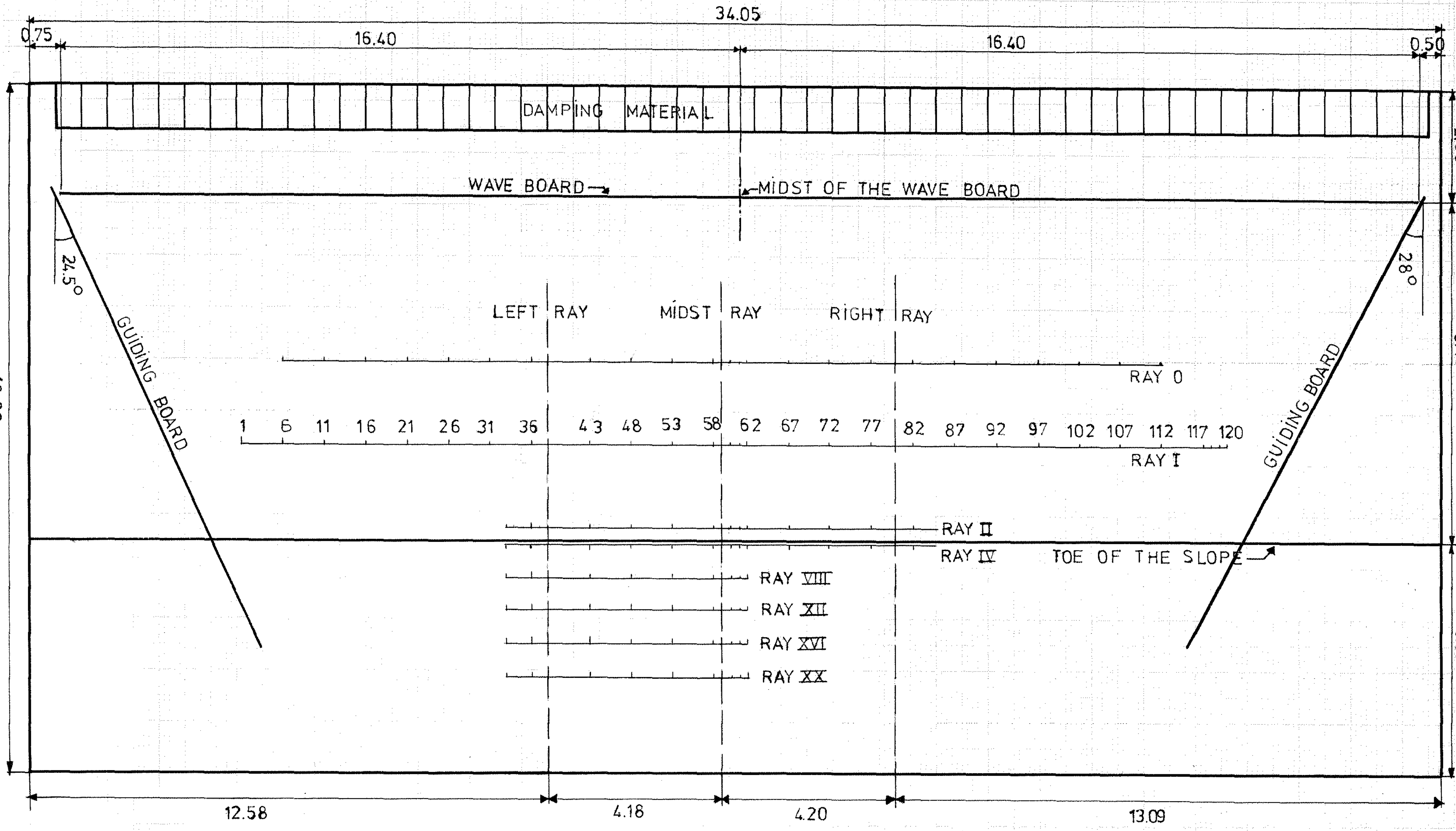
Fig. 5.2.

a. $\phi_1 = \phi_2 = 43^o$

b. $\phi_1 = 43^o$, $\phi_2 = 28^o$

On the other side of the basin was a 1:10 slope. The waves were guided by two guide boards on both sides of the basin.

The basin had a fixed cemented bottom.
The period of the waves was in all the tests constantly 1.62 sec.
During the test the wave height was measured in various places, and the set-up was measured in two rays perpendicular to the slope. Additional qualitative observations were made of the longshore velocities with dye.



SCHEMATIC PLAN OF THE WAVE BASIN (scale 1:125)

5.b. Measurement methods

The wave heights were measured with a wave height surveyor during a period of 30 seconds (for each point). This period is chosen, because during initial tests appeared that this measurement period gave reliable information with an acceptable standard deviation of $\pm$ 5%. In the basin certain measurement-rays were chosen, viz. (see fig. 5.1):

Ray 0 : 4.04 m from the wave board
Ray I : 6.04 m from the wave board

The Rays II, IV, VIII, XII, XVI, XVIII and XX are situated on the connection lines between the points number 2, 4, 8, 12, 16, 18 and 20 of the set-up measurement rays.
Between the left set-up measurement ray (L.R.) and the right set-up measurement ray (R.R.) the distance between the wave height measurement points was 20 cm. In the remaining part of the basin this distance was mostly 60 cm.

The set-up was measured through holes (no. 1 up to and including 25) in the set-up measurement rays. These holes are connected with small tubes, placed against one side of the basin. The water level in these tubes, represents the water level in the basin at that point. The water level is measured in a pot, because of the water level in the (small) tubes varied due to wave pressures. The surface of this pot is many times larger than the cross-section of a tube so damping will occur and the water level will stay almost constant.
In the right set-up measurement ray (R.R.) set-up could not be measured, because several holes were clogged.

With the mean depth and the wave height (at that point) the breakerindex γ can be calculated. Because it was not possible to measure the set-up in the Right Ray, it was also not very useful to measure the wave height in that Ray. The wave height measurements of Ray IV until XX are used for the determination of the breakerdepth the measurements of the other rays (0 up to and including II) for checking the envelope of the surface extensions with the computed one (which could not be done with Ray IV until XX because of refraction).

The set-up measurements are also important for the indication that, in this case, set-up does not depend on breakerheight and index ($\bar{\eta}=\frac{5}{16}\gamma H_b$), but probably on the radiation stress caused by the two wave systems.

5.c. Wave height measurements

The measured wave heights are compared with the computed wave heights. When the formula, which can be used to compute the wave height at a certain point (form 3.1. and 3.4.), is examined, it appears that the two incoming wave heights H_1 and H_2 are not known. Because of technical reasons, a part of the wave board had to be removed, these wave heights could not be measured. The computation is, therefore, reversed: the wave heights H_1 and H_2 are calculated by means of the measured wave height. The problem rises that two variables (H_1 and H_2) had to be calculated from the following equation:

$$\{H_1.KD_1.\sin(\omega t - \text{phase } 1) + H_2.KD_2.\sin(\omega t - \text{phase } 2)\}_{max} = H_{measured}$$

The maximum value of the left side of the equation depends on the ratio between H_1 and H_2. In the computations it is assumed that this ratio is known, viz.:

* When the angles of incidence of both systems are absolute the same ($FI_1 = - FI_2$) than $H_1 = H_2$
* When $FI_1 \neq - FI_2$ than:

  $$H_2 = H_1 . \sqrt{\cos (FI_1)/\cos (FI_2)}$$

  (FI_1 and FI_2 are angles of incidence of both systems).

This can be explained as follows:

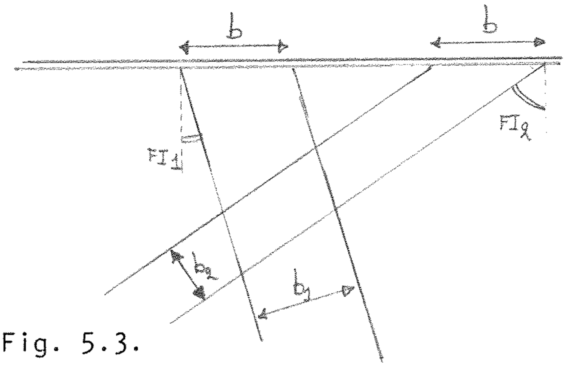


Fig. 5.3.

Assuming that the energy is equally distributed over the wave board the energy-flux through b_1 and b_2 is the same.

$$^1/_8 \rho g H_1{}^2 . b_1 = {}^1/_8 \rho g H_2{}^2 . b_2 \quad \} \rightarrow$$

$$b_1 = b.\cos (FI_1) \ ; \ b_2 = b.\cos (FI_2)$$

$$H_2 = H_1 . \sqrt{\cos (FI_1)/\cos (FI_2)}$$

Two types of tests were made:

- the wave systems had opposite angles of incidence, viz. $+43^o$, -43^o
- the wave systems had different angles of incidence, viz. $+28^o$, -43^o

(Both tests with the same wave period, T = 1.62 sec).

At first the incoming wave heights H_1 (respectively H_2) are computed; the results are presented in table 1.

Table 1: (+43°, -43°)

Ray	Points no.	Amount of points	Mean wave height ($\bar{H}$)	Standard deviation (σ)	$\sigma/\bar{H}$*100%	H_{max}	H_{min}
0	6- 33	10	7.31	2.84	38.8	12.03	3.46
0	112- 85	10	7.19	1.66	23.0	9.22	4.84
I	1- 37	13	8.66	5.71	65.9	21.95	1.97
I	81-120	14	8.22	4.91	59.8	22.94	3.32
II	38- 80	43	14.81	32.33	218.4	208.51	3.07
Tot.	All above	90	11.22	-	-	208.51	1.97

When the column of the maximum wave heights is examined it appears that very high wave heights (H_1 respectively H_2, in this case $H_1 = H_2$) are computed. This occurs mostly on 'phase = π'-lines. These are locations where the surface extension is theoretically almost zero, which does not happen in these tests (this is explained later). The very high wave heights are therefore excluded; than the following results are obtained:

Table 2: (+43°, -43°; excluding Ray I: no. 10, 111; Ray II: 45, 55, 67 and 77)

Ray	Points no.	Amount of points	Mean wave height ($\bar{H}$)	Standard deviation (σ)	$\sigma/\bar{H}$*100%	H_{max}	H_{min}
0	6- 33	10	7.31	2.84	38.8	12.03	3.46
0	112- 85	10	7.19	1.66	23.0	9.22	4.84
I	1- 37	12	7.55	4.27	56.5	17.12	1.97
I	81-120	13	7.09	2.59	36.5	12.44	3.32
II	38- 80	39	7.44	2.10	28.3	13.60	3.07
Tot.	All points	84	7.36	≈2.57	34.9	17.12	1.97

Tabel 3: (+28/-43°); using $H_2 = H_1 * \sqrt{\cos (FI_1)/\cos (FI_2)}$

Ray	Points no.	Amount of points	Mean wave height ($\bar{H}$)	Standard deviation (σ)	$\sigma/\bar{H} * 100\%$	H_{max}	H_{min}
0	33- 6	10	7.65	4.17	54.4	18.36	3.69
0	36- 83	48	9.04	7.50	82.9	54.00	3.03
0	85-112	10	7.11	0.89	12.5	8.46	6.37
0	85-118	12	7.30	0.98	13.4	9.36	6.03
I	1- 37	13	7.66	2.67	34.8	12.50	2.92
I	38- 80	44	9.08	6.54	72.0	32.05	3.56
I	84-120	13	7.96	2.51	31.5	11.90	4.69
II	38- 81	44	8.29	4.20	50.6	30.35	3.51
Tot.	All points	194	8.44	-	-	54.00	2.92

When the wave heights of Ray 0: no. 53; Ray I: no 39, 51, 62, 63 and Ray II: 61 are excluded the following result is obtained:

Table 4: (+28/-43); using $H_2 = H_1 * \sqrt{\cos (FI_1/\cos (FI_2)}$; excluded points

Ray	Points no.	Amount of points	Mean wave height ($\bar{H}$)	Standard deviation (σ)	$\sigma/\bar{H} * 100\%$	H_{max}	H_{min}
0	33- 6	10	7.65	4.17	54.4	18.36	3.69
0	36- 83	47	8.08	3.48	43.0	17.17	3.03
0	85-112	10	7.11	0.89	12.5	8.46	6.37
0	85-118	12	7.30	0.98	13.4	9.36	6.03
I	1- 37	13	7.66	2.67	34.8	12.50	2.92
I	38- 80	40	7.29	2.17	29.7	15.22	3.56
I	84-120	13	7.96	2.51	31.5	11.90	4.69
II	38- 81	43	7.78	2.49	32.0	16.24	3.51
Tot.	All points	188	7.68	≈2.67	34.8	18.36	2.92

In the case with different angles of incidence the incoming wave heights are also calculated assuming H_1 is H_2, the result becomes:

Table 5: (+28/-43; using $H_1 = H_2$)

Ray	Points no.	Amount of points	Mean wave height ($\bar{H}$)	Standard deviation (σ)	$\sigma/\bar{H}$*100%	H_{max}	H_{min}
0	33- 6	10	6.90	3.51	50.9	15.84	3.45
0	36- 83	48	8.30	4.67	56.3	25.84	2.86
0	85-112	10	7.09	0.93	13.2	8.56	5.37
0	85-118	12	7.29	1.03	14.1	9.44	5.95
I	1- 37	13	6.97	2.28	32.7	10.93	5.47
I	38- 81	44	8.44	5.31	62.9	31.62	3.43
I	84-120	13	8.00	2.65	33.1	12.38	4.64
II	38-81	44	7.95	3.55	44.6	25.41	3.33
Tot.	All points	194	7.95	≈3.98	50.1	31.62	2.86

When the wave heights of Ray 0: no. 53, 63, 64; Ray I: 51, 62, 63; and Ray II: 61 are excluded the following result is obtained:

Table 6: (+28/-43; using $H_1 = H_2$; excluded points)

Ray	Points no.	Amount of points	Mean wave height ($\bar{H}$)	Standard deviation (σ)	$\sigma/\bar{H}$*100%	H_{max}	H_{min}
0	33- 6	10	6.90	3.51	50.9	15.84	3.45
0	36- 83	45	7.43	3.15	42.3	15.59	2.86
0	85-112	10	7.09	0.93	13.2	8.56	5.37
0	85-118	12	7.29	1.03	14.1	9.44	5.95
I	1- 37	13	6.97	2.28	32.7	10.93	5.47
I	38- 81	41	7.21	2.40	33.4	14.52	3.43
I	84-120	13	8.00	2.65	33.1	12.38	4.64
II	38- 81	43	7.54	2.34	31.0	14.29	3.33
Tot.	All points	187	7.36	≈2.51	34.1	15.84	2.86

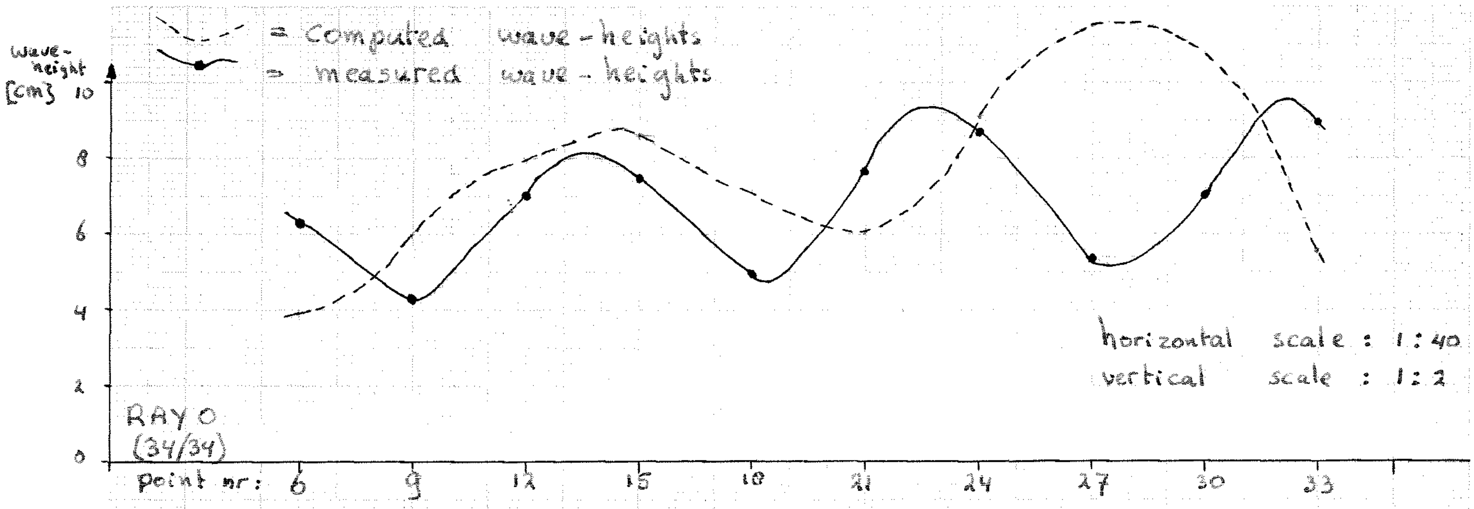


fig 5.4

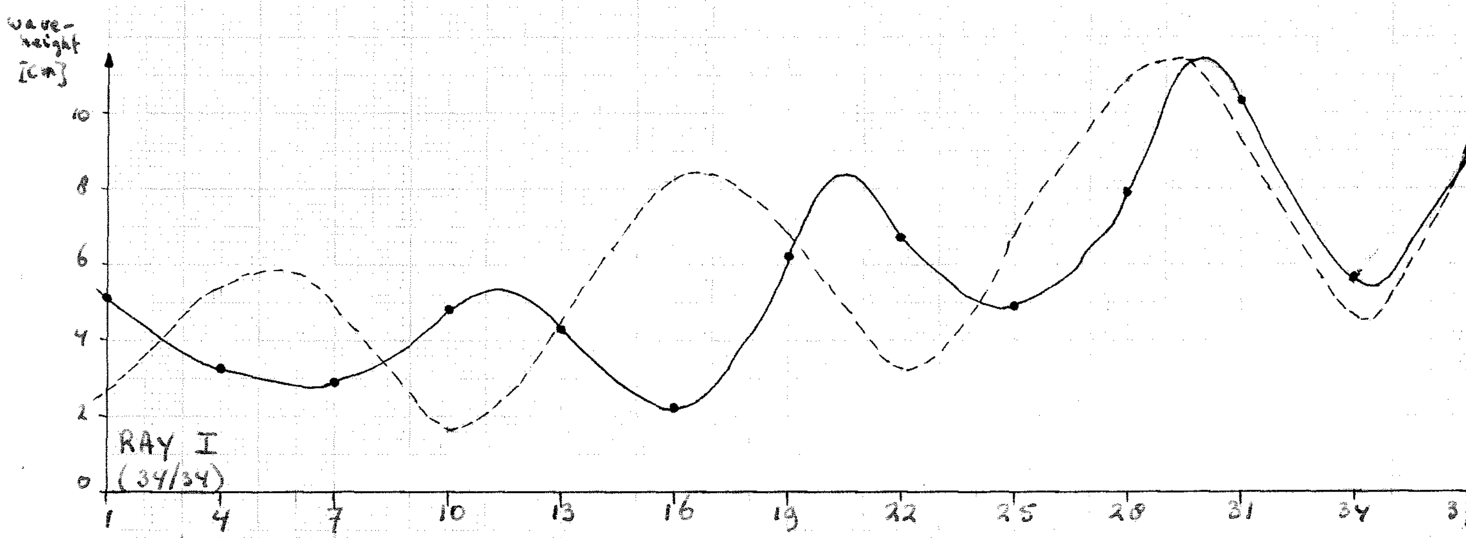


fig 5.5

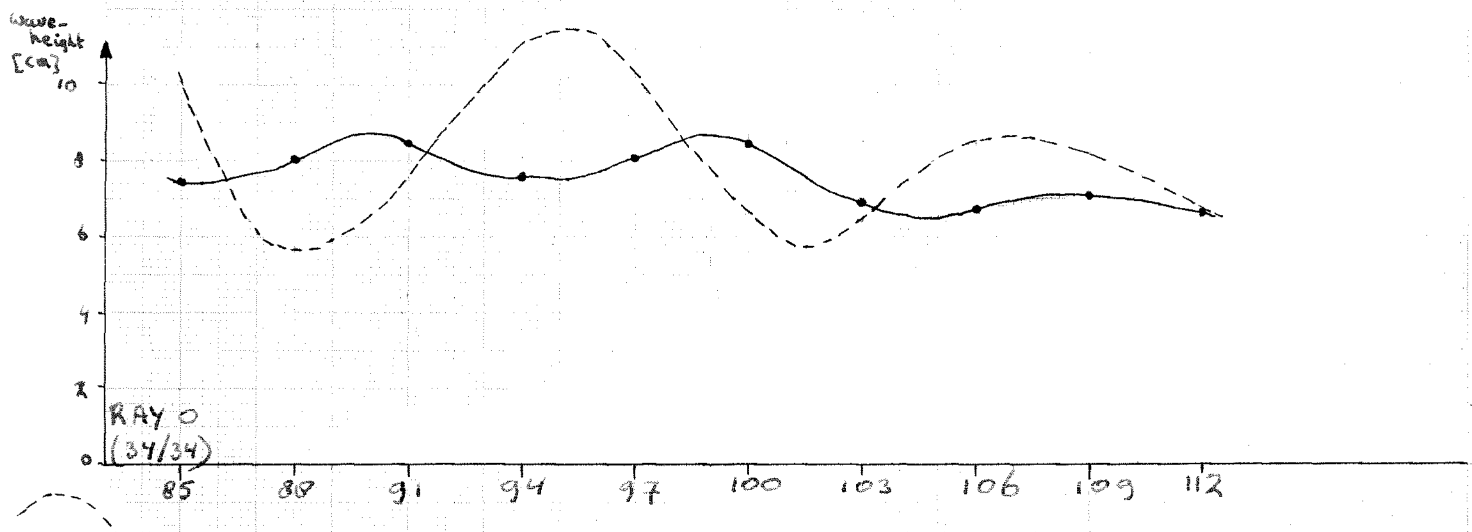


fig 5.6

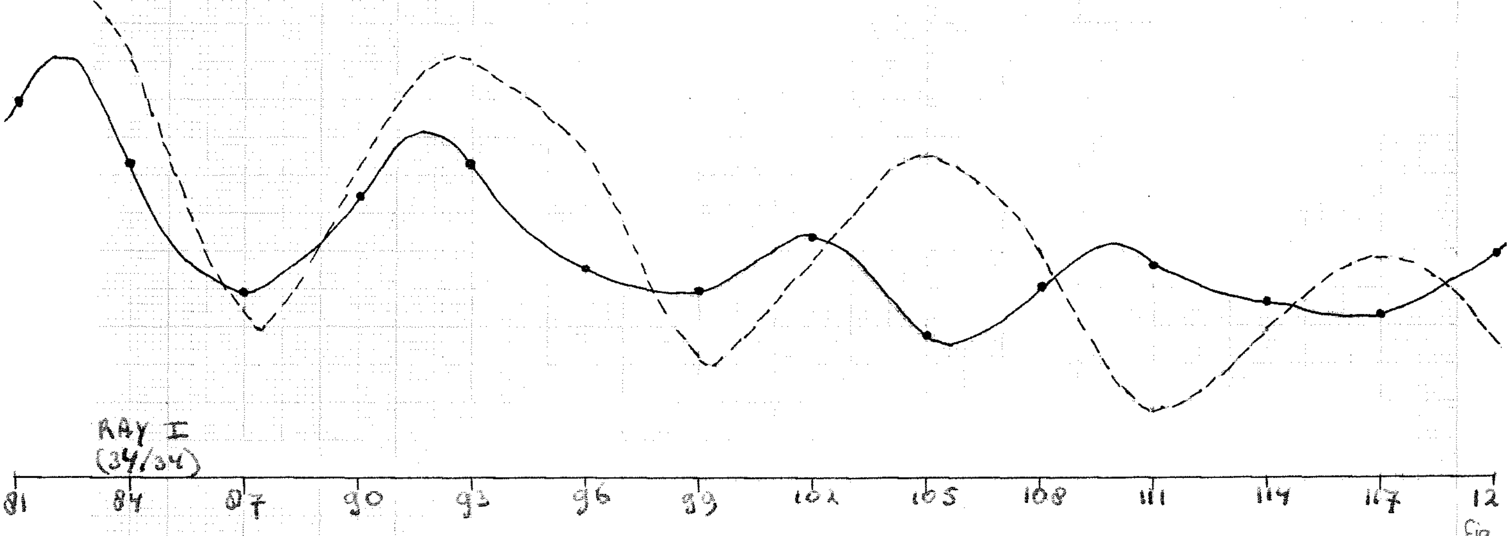


fig 5.7

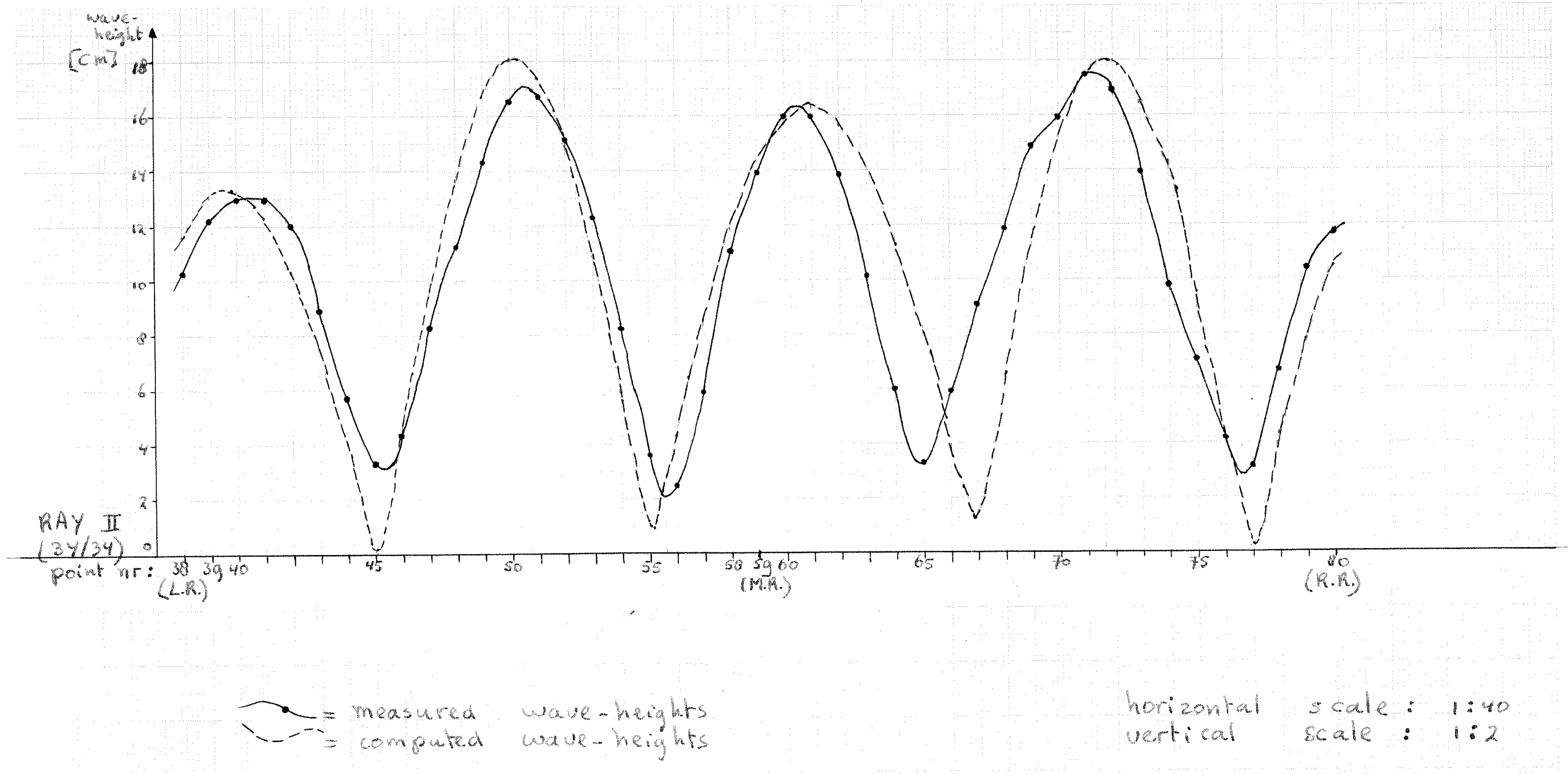

wave-height [cm]
0
2
4
6
8
10
12
14
16
18
RAY II
(34/34)
point nr:
38
(L.R.)
39
40
45
50
55
58
59
60
(M.R.)
65
70
75
80
(R.R.)
= measured wave-heights
= computed wave-heights
horizontal scale : 1:40
vertical scale : 1:2

fig 5.8

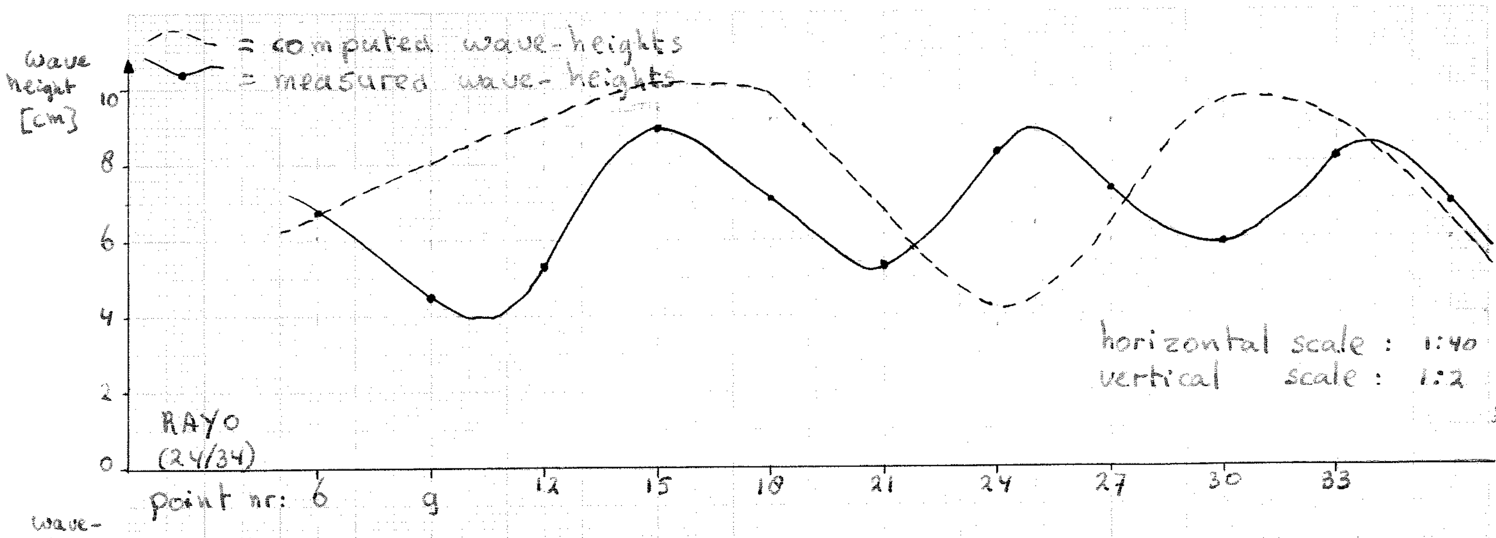


fig 5.9

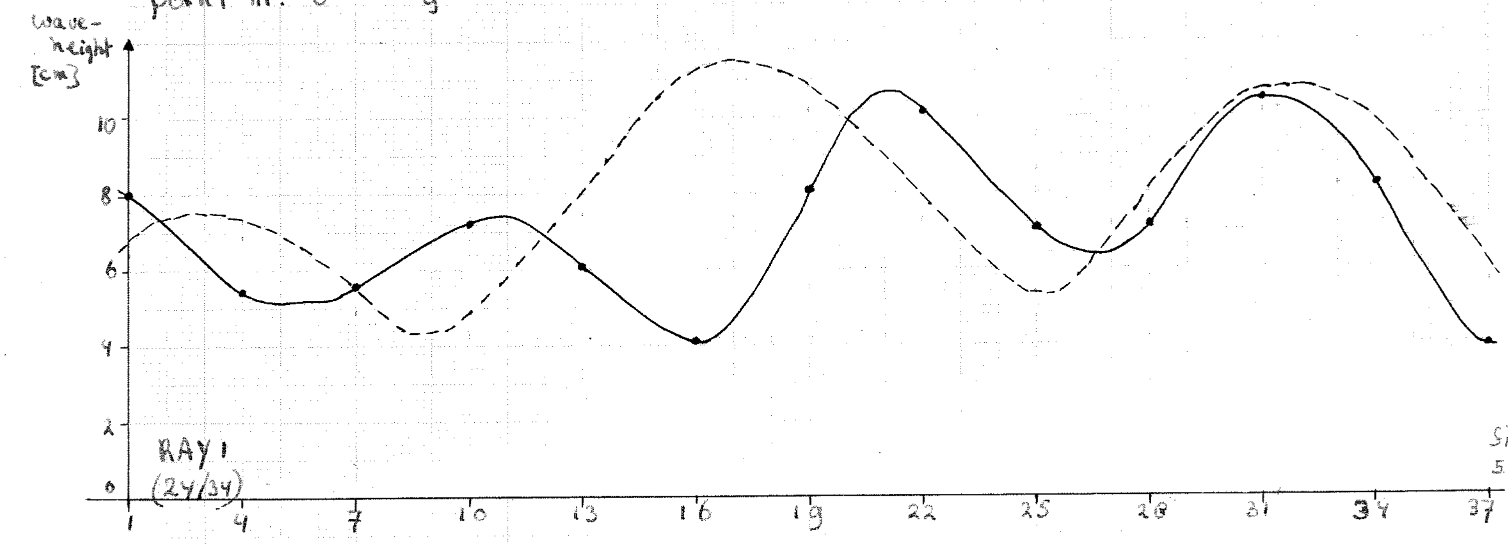


fig 5.10

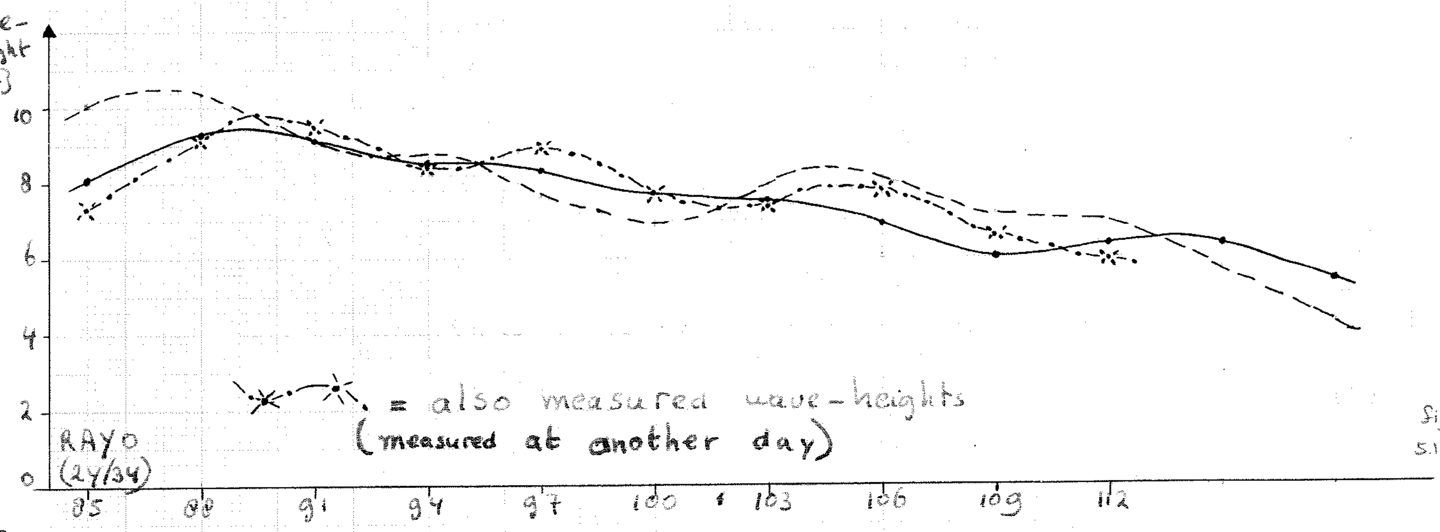


fig 5.11

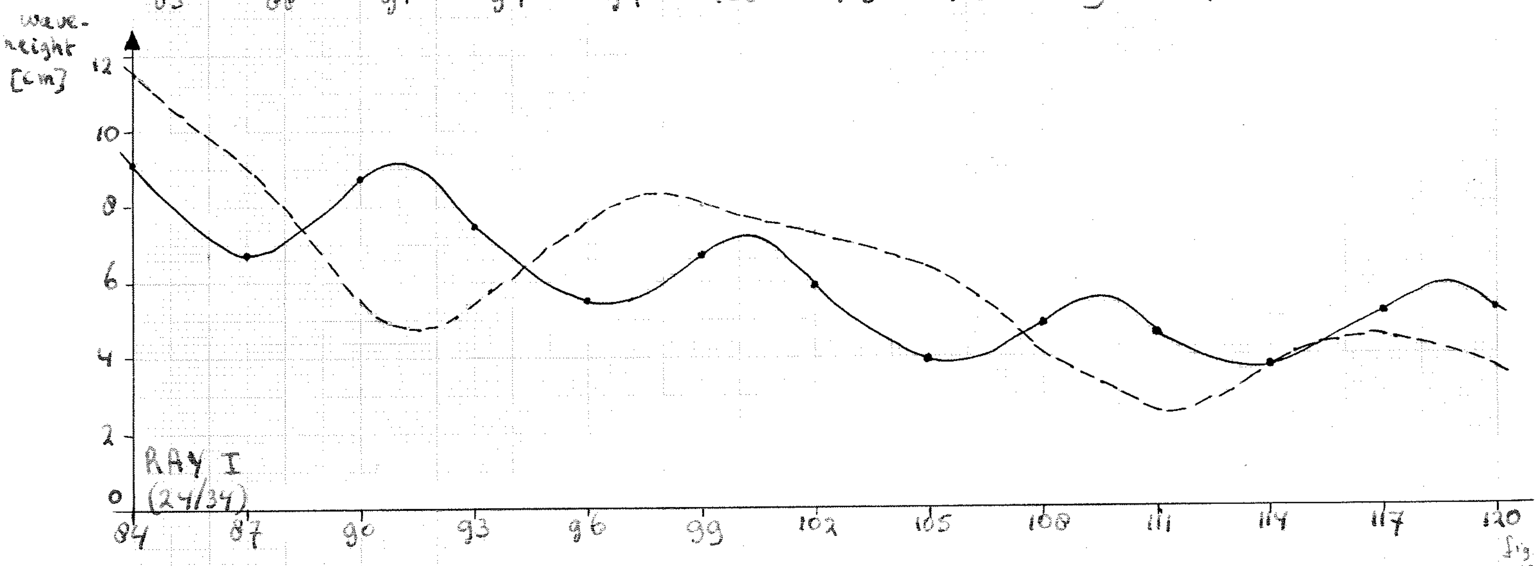


fig 5.12

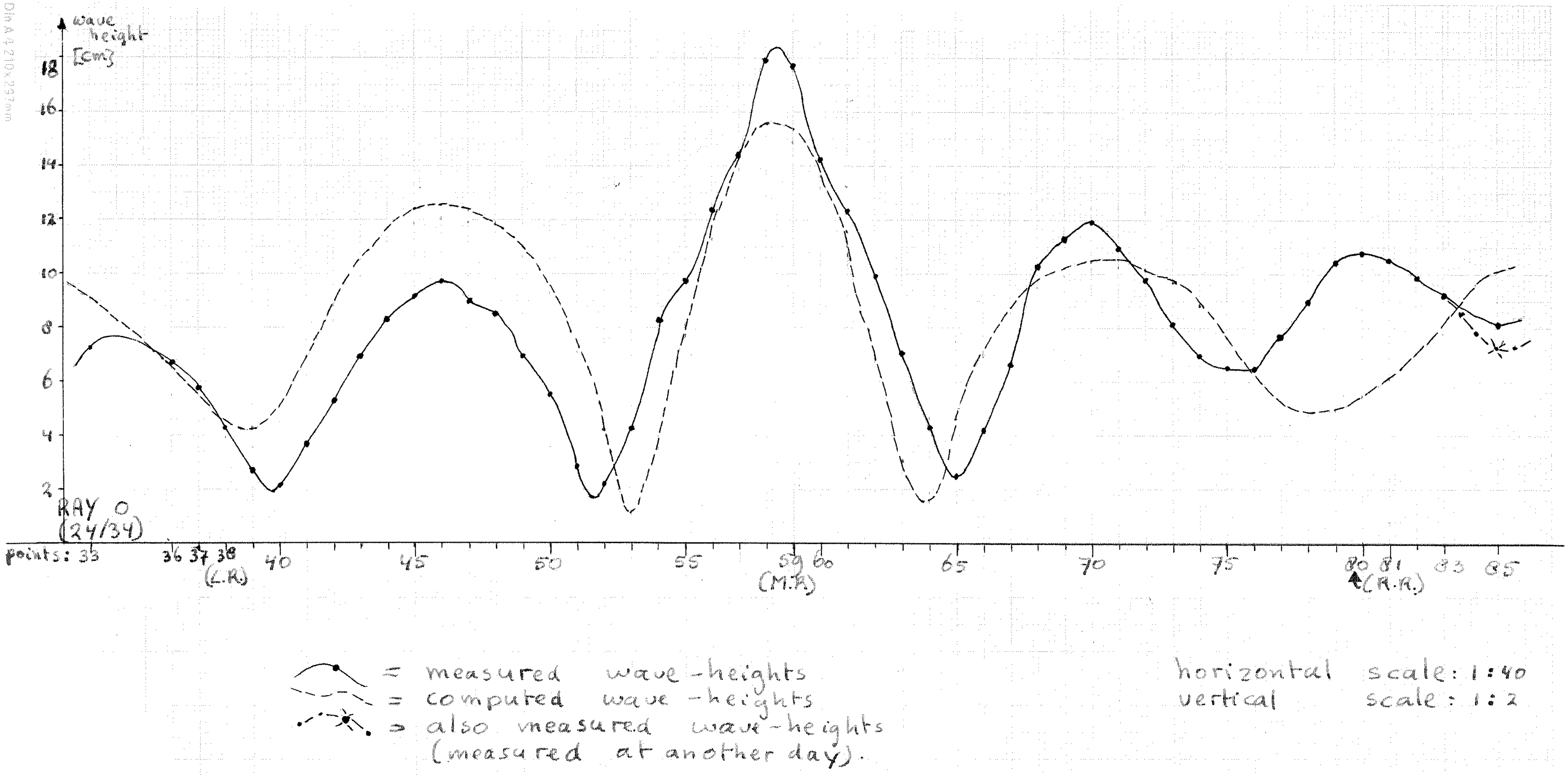

wave height [cm]
18
16
14
12
10
8
6
4
2
RAY 0
(24/34)
points: 33
36 37 38
(L.R.)
40
45
50
55
59 60
(M.R.)
65
70
75
80 81
(R.R.)
83
85
= measured wave-heights
= computed wave-heights
= also measured wave-heights
(measured at another day).
horizontal scale: 1:40
vertical scale: 1:2

fig. 5.13

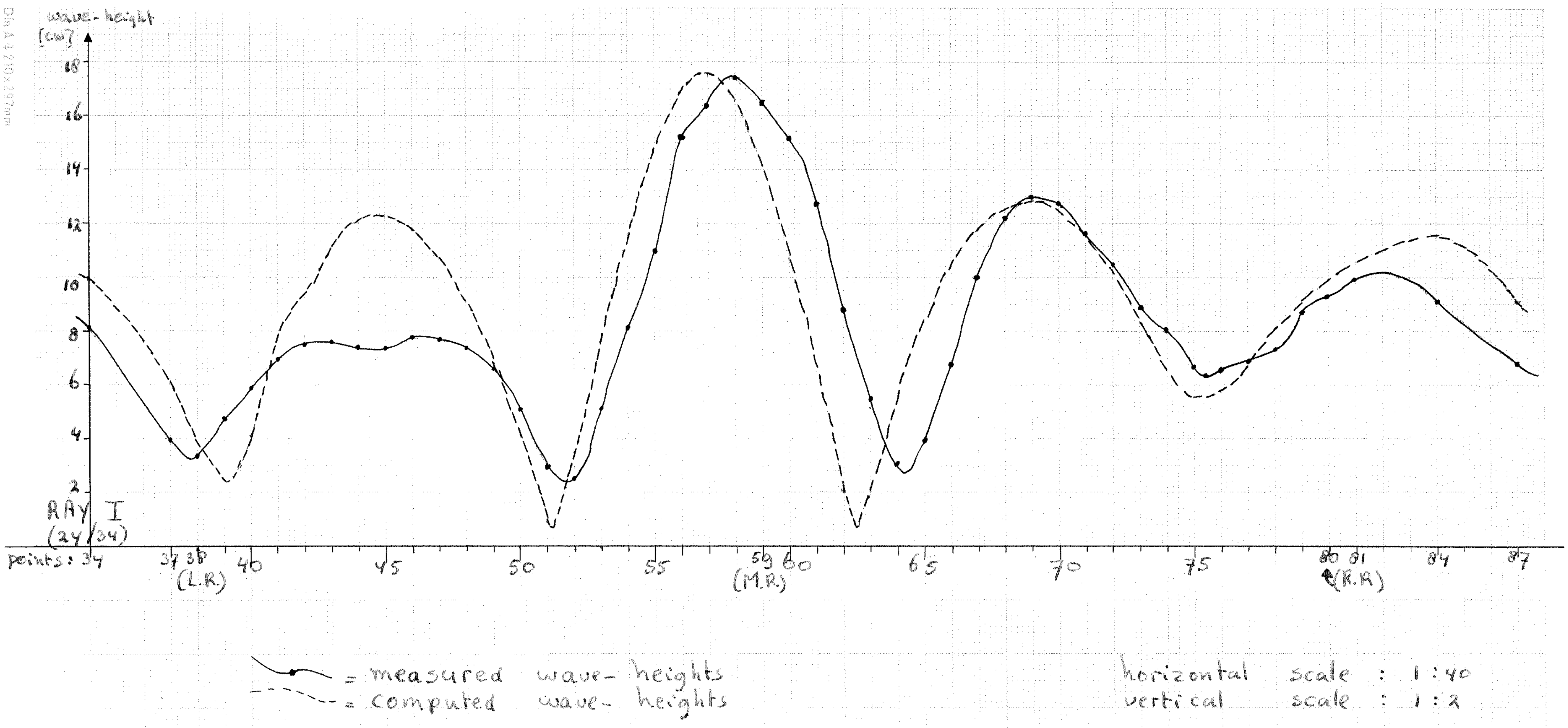

wave-height [cm]
18
16
14
12
10
8
6
4
2
RAY I
(24/34)
points: 34
37 38
(L.R.)
40
45
50
55
59 60
(M.R.)
65
70
75
80 81
(R.A)
84
87
= measured wave-heights
= computed wave-heights
horizontal scale : 1:40
vertical scale : 1:2

fig.5.14

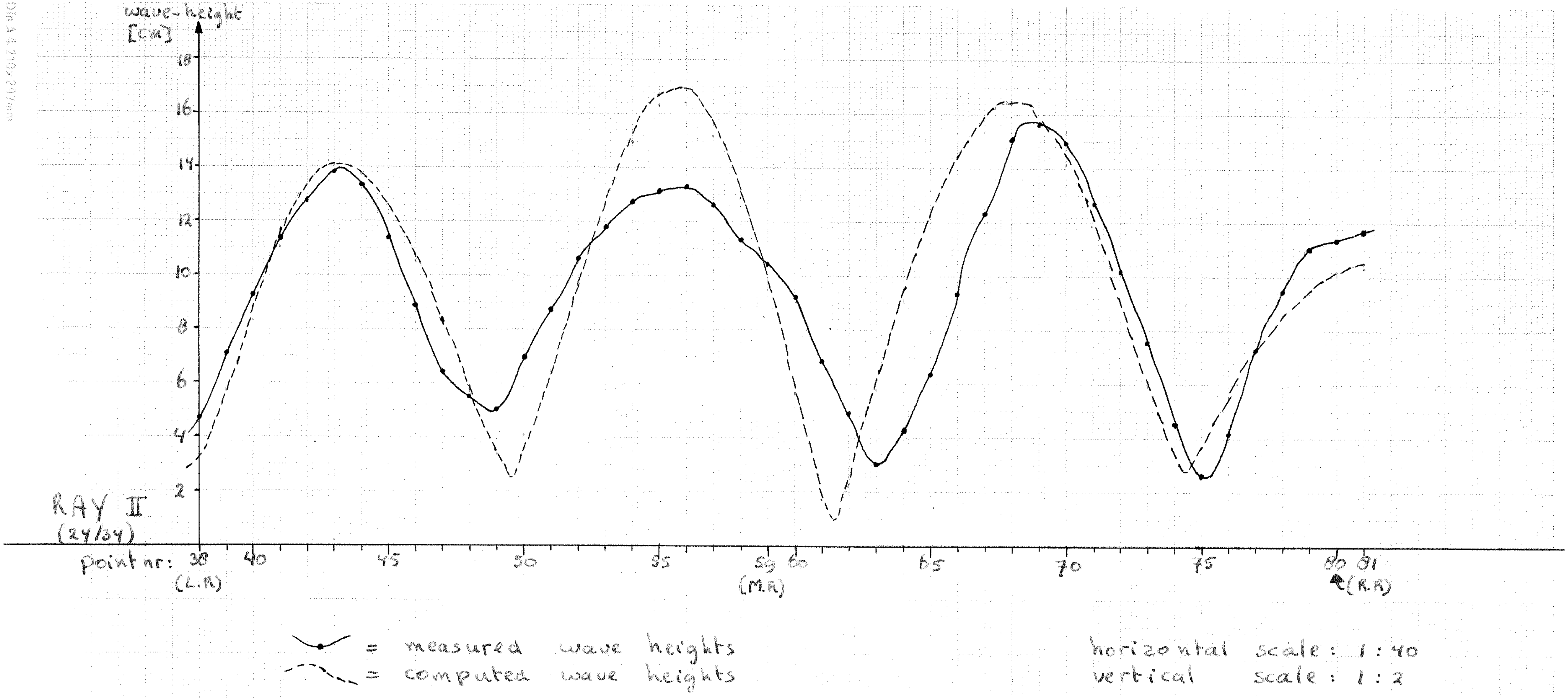

wave-height
[cm]
18
16
14
12
10
8
6
4
2
RAY II
(24/34)
point nr: 38
(L.R)
40
45
50
55
59 60
(M.R)
65
70
75
80 81
(R.R)
= measured wave heights
= computed wave heights
horizontal scale : 1 : 40
vertical scale : 1 : 2

fig 5.15

By means of the above tables the wave heights of both systems are determined. The wave height of the systems with an angle of incidence of plus or minus 43^o is valued at 7.5 cm.
With this mean wave height (of 7.5 cm) the wave heights are computed (with formula 3.1) in two ways:

* Assuming $H_1 = H_2$
* Assuming $H_2 = H_1 \cdot \sqrt{\cos(FI_1)/\cos(FI_2)}$

The results of the above calculation and the measured wave height are presented in figures 5.4. until 5.15; it appeared that the differences between both calculation methods was very small, so only one line of computed wave heights is drawn.

The distances between two points with minimum wave heights (or maximum wave heights) is the distance L_s (see chapter 3.a.). This distance can be calculated (theoretically) for both cases:

- L_s $(+43^o/-43^o)$ = 4.25 m
- L_s $(+28^o/-43^o)$ = 5.00 m

The measured values (in the middle area between L.R. and R.R.) are:

- L_s $(+43^o/-43^o)$ = 4.15 m
- L_s $(+28^o/-43^o)$ = 4.92 m

5.d. Set-up measurements

The set-up was measured only in case of the wave systems with different angles of incidence $(+28^o/-43^o)$. First the breakerindex (mean) will be valued, using the mean depths and the wave heights of Ray XVIII and XX. The wave heights of Ray IV until XX are presented in fig. 5.16 until 5.21. The mean depths are calculated by means of the measured (mean) set-up values.

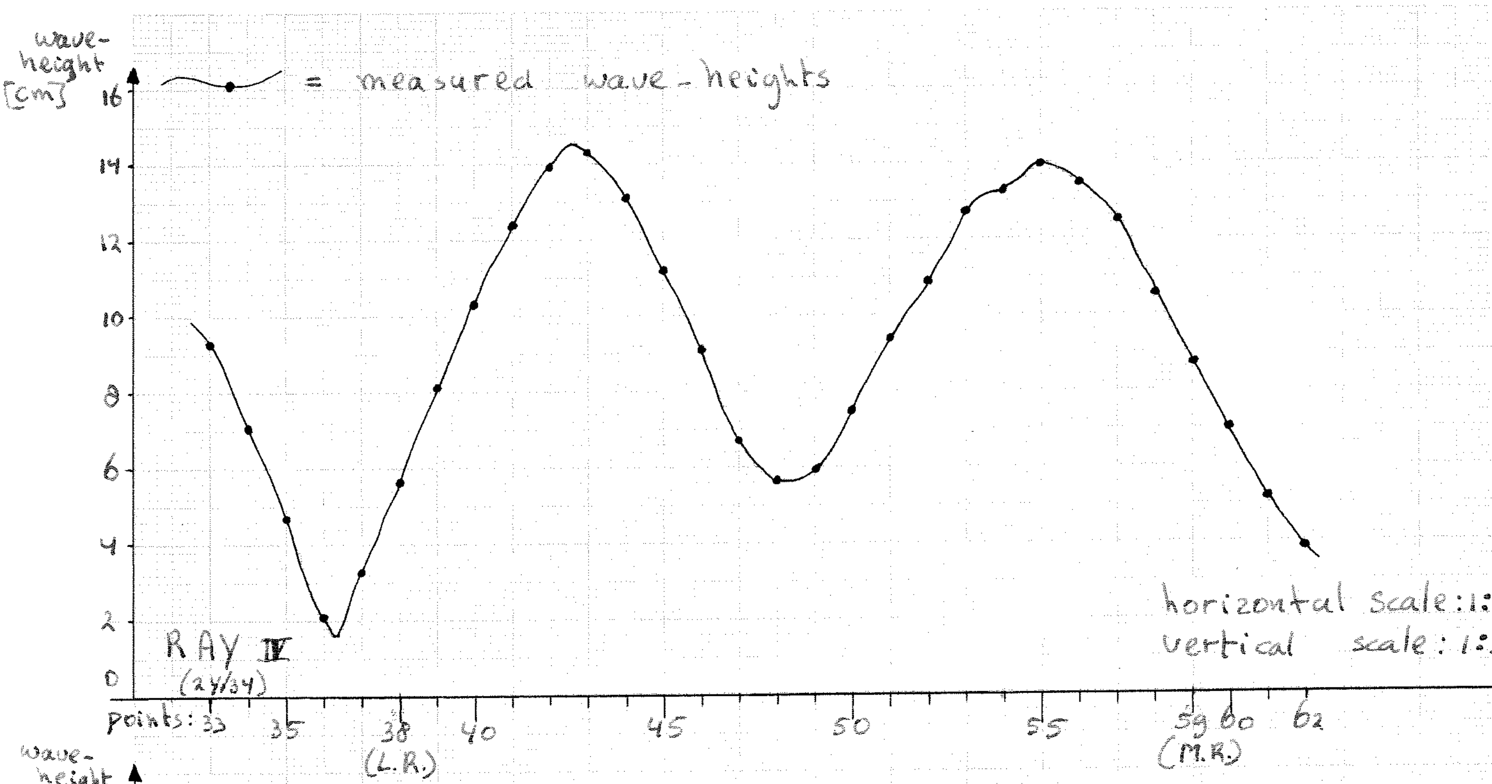


fig 5.16

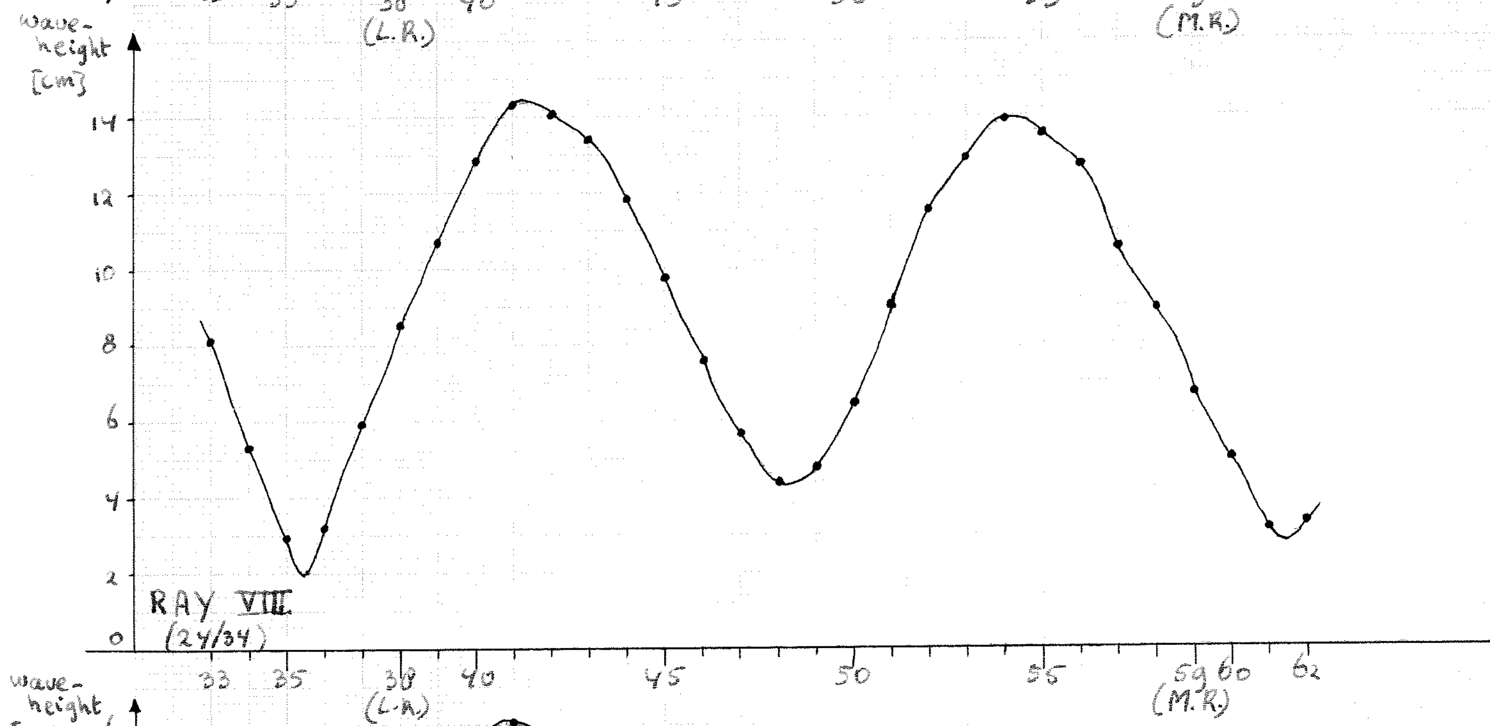


fig 5.17

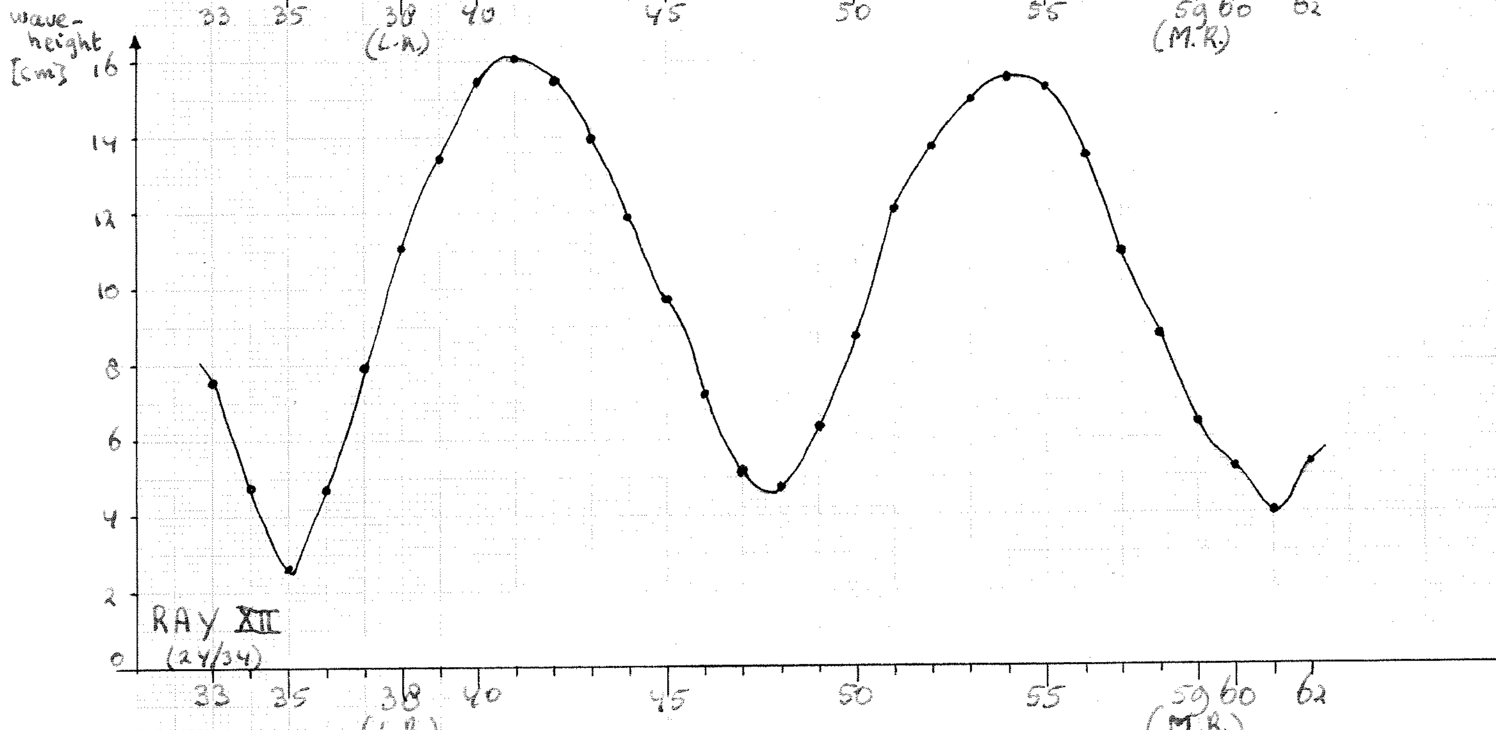
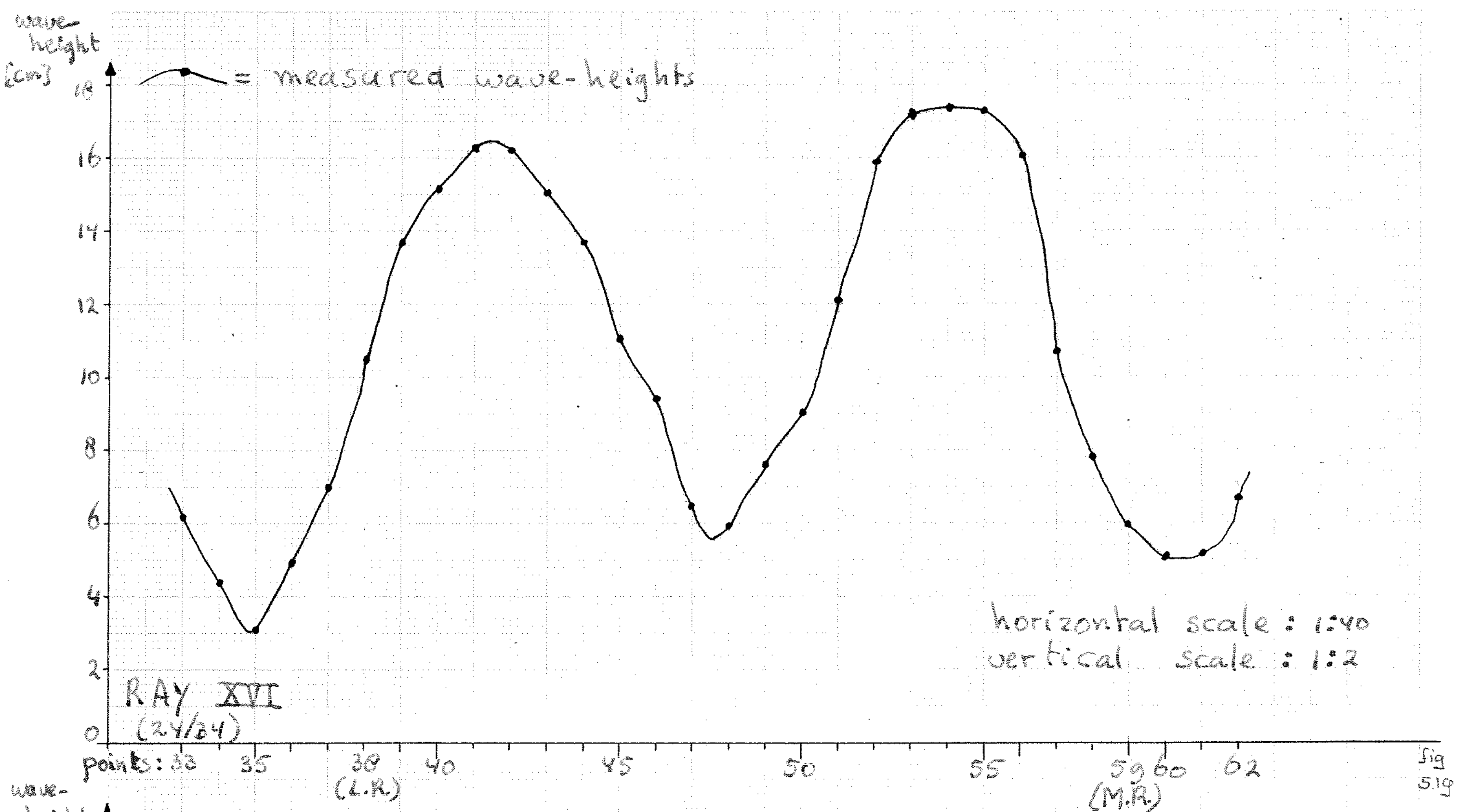
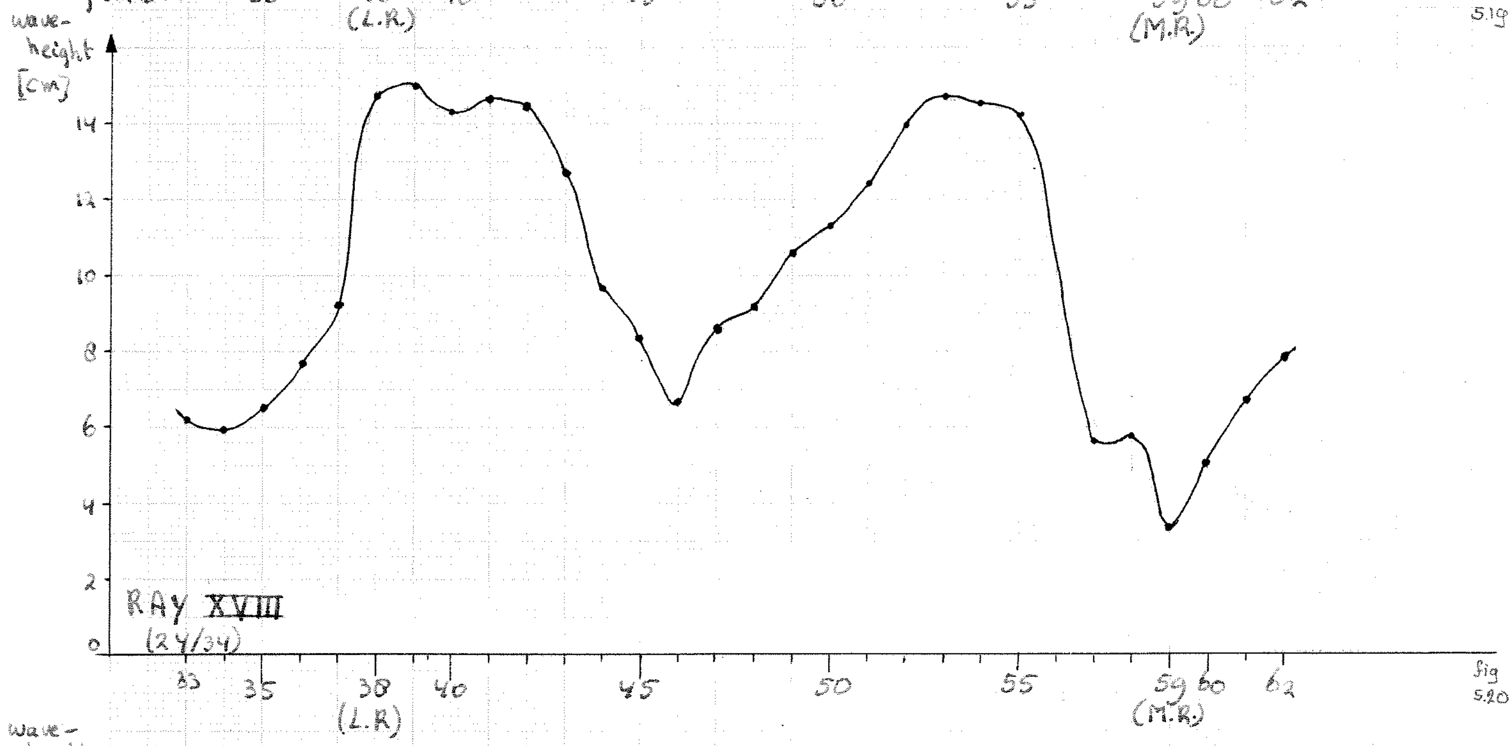
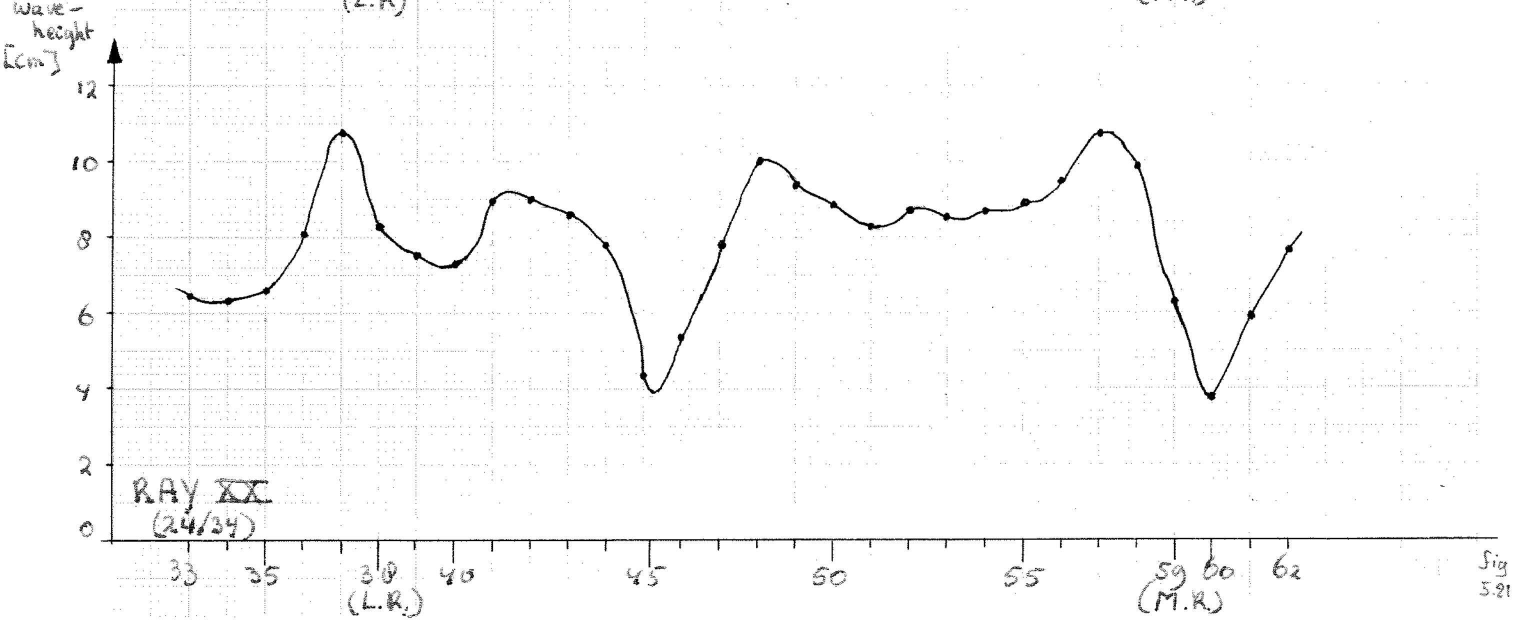


fig 5.18


wave-height [cm]
= measured wave-heights
horizontal scale : 1:40
vertical scale : 1:2
RAY XVI
(24/34)
points: 33 35 38 40 45 50 55 59 60 62
(L.R.)
(M.R.)
0 2 4 6 8 10 12 14 16 18
fig 5.19


wave-height [cm]
RAY XVIII
(24/34)
33 35 38 40 45 50 55 59 60 62
(L.R.)
(M.R.)
0 2 4 6 8 10 12 14
fig 5.20


wave-height [cm]
RAY XX
(24/34)
33 35 38 40 45 50 55 59 60 62
(L.R.)
(M.R.)
0 2 4 6 8 10 12
fig 5.21

Left set-up measurement ray (L.R.):

Measurement point	Amount of measurements	Set-up $\bar{\eta}$ (cm)	$\sigma_{\bar{\eta}}$ (cm)	$\frac{\sigma_{\bar{\eta}}}{\bar{\eta}}$*100%	Actual waterdepth h (cm)	$\sigma_{\bar{\eta}}/h$*100%
1	3	0	0.10	-	40.00	0.5%
4	1	+0.03	-	-	39.13	-
8	1	-0.03	-	-	31.07	-
12	1	-0.09	-	-	22.91	-
16	1	-0.25	-	-	14.75	-
18	1	-0.27	-	-	10.73	-
20	5	-0.20	0.02	6 %	6.70	0.3%
21	9	+0.85	0.08	9 %	5.85	1.3%
22	6	+1.66	0.07	4 %	4.66	1.5%
23	5	+2.57	0.09	3.6%	3.57	2.6%
24	5	+3.86	0.05	1.3%	2.86	1.7%
25	21	+4.64	0.04	0.9%	1.64	2.4%

Middle set-up measurement ray (M.R.):

Measurement point	Amount of measurements	Set-up $\bar{\eta}$ (cm)	$\sigma_{\bar{\eta}}$ (cm)	$\frac{\sigma_{\bar{\eta}}}{\eta}$*100%	Actual waterdepth h (cm)	$\sigma_{\bar{\eta}}/h$*100%
1	3	0	0.10	-	40.00	0.5%
4	1	0	-	-	39.05	-
8	1	0	-	-	31.00	-
12	1	-0.03	-	-	22.87	-
16	1	-0.03	-	-	15.07	-
18	1	-0.11	-	-	10.89	-
20	5	-0.21	0.06	27 %	6.69	0.8%
21	7	-0.11	0.08	69 %	4.84	1.6%
22	5	+0.34	0.03	10 %	3.34	1.0%
23	6	+1.22	0.20	16 %	2.22	9 %
24	8	+2.11	0.21	9 %	1.11	19 %
25	20	+3.30	0.20	6 %	0.30	67 %

Table 7: ratios between wave height and water depth in Ray XVIII

Ray XVIII (set-up point 18)			Ray XX (set-up point 20)		
Point	Measured wave height	H/h	Point	Measured wave height	H/h
33	6.19	(0.58)	33	6.42	(0.95)
34	5.95	(0.55)	34	6.35	(0.94)
35	6.52	(0.61)	35	6.60	(0.90)
36	7.65	(0.71)	36	8.04	'1.19'
37	9.22	(0.80)	37	10.74	'1.59'
38	14.74	'1.37'	38	8.22	'1.22'
39	15.00	'1.40'	39	7.59	'1.12'
40	14.38	'1.34'	40	7.23	'1.07'
41	14.67	'1.36'	41	8.89	'1.32'
42	14.49	'1.35'	42	8.98	'1.33'
43	12.73	'1.18'	43	8.57	'1.27'
44	9.66	(0.90)	44	7.80	'1.16'
45	8.39	(0.78)	45	4.34	(0.64)
46	6.61	(0.61)	46	5.35	(0.79)
47	8.64	(0.80)	47	7.89	'1.17'
48	9.13	(0.85)	48	9.98	'1.48'
49	10.59	(0.99)	49	9.37	'1.39'
50	11.27	(1.05)	50	8.83	'1.31'
51	12.42	'1.16'	51	8.23	'1.22'
52	13.99	'1.30'	52	8.69	'1.29'
53	14.69	'1.37'	53	8.53	'1.26'
54	14.54	'1.35'	54	8.82	'1.31'
55	14.20	'1.32'	55	9.42	'1.40'
56	10.11	(0.94)	56	10.74	'1.59'
57	5.73	(0.53)	57	9.90	'1.47'
58	5.85	(0.54)	58	6.29	(0.93)
59	3.36	(0.31)	59	3.77	(0.56)
60	5.07	(0.47)	60	5.84	(0.87)
61	6.65	(0.62)	61	7.10	(1.05)
62	7.84	(0.73)	62	7.60	'1.13'

By means of figures in above tables the water depth at Ray XVIII is valued at 10.8 cm (± 1.5%), at Ray XX 6.7 cm (± 2%). The breaker index, breaker height (H_b) devided by breaker depth (h_b) can be 'calculated'. Because the wave heights differ parallel to the coast not each value of $^H/_h$ (wave height devided by water depth) represents a value of $^{H_b}/_{h_b}$ (= γ). The values of $^H/_h$ are presented in Table 7; all values which (probably) represent a value of $^{H_b}/_{h_b}$ are in quotes.

The breaker index is valued at 1.30 (± 10%) by means of the figures in quotes from above table. Because the shape of wave envelope parallel to the coast resembles somewhat a standing wave, the breaker index is also valued by means of the breaker criterium for standing waves:

$$(^H/_L)_{breaking} \simeq 0.22 \tanh \frac{2\pi h}{L}$$

Using this equation it appears that the breaker index varies from 1.25 until 1.35 (depends on the chosen wave height).

The set-up measurements are also used for the determination of the maximum set-up ($\bar{\eta}_{max}$). This value is a measure for the amount of (wave) momentum perpendicular to the coast. The maximum set-up can be determined in several ways, viz.:

1. By extrapolation of the connection line between point 24 and 25 (last point at which the set-up is measured). The maximum set-up is determined in three ways, viz.:

   mean slope of the connection line: $(\bar{\eta}_{25} - \bar{\eta}_{24})/20$

   'max' slope of the connection line: $\{(\bar{\eta}_{25} + \sigma_{25})-(\bar{\eta}_{24} - \sigma_{24})\}/20$

   'min' slope of the connection line: $\{(\bar{\eta}_{25} - \sigma_{25})-(\bar{\eta}_{24} + \sigma_{24})\}/20$

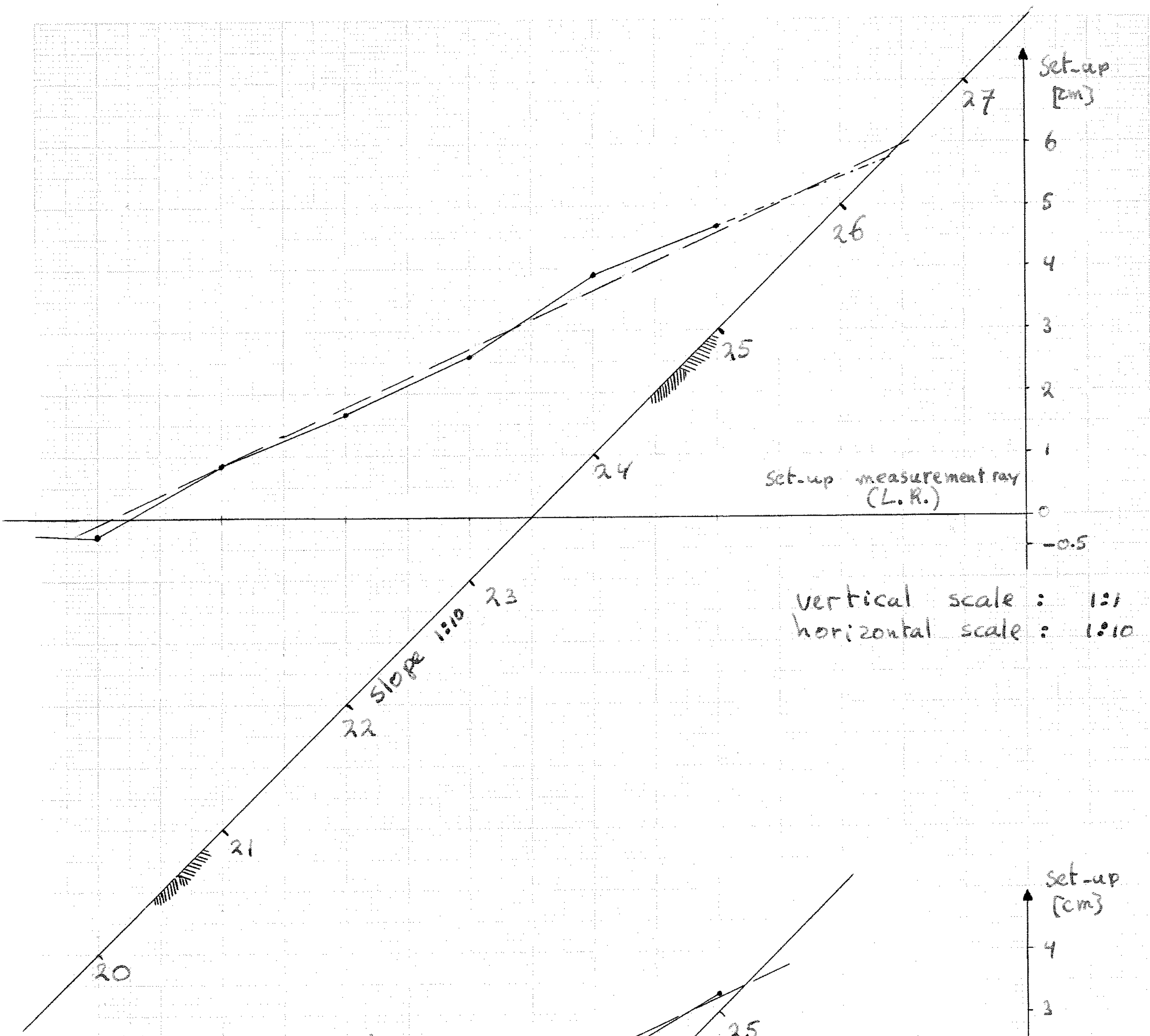


fig 5.22

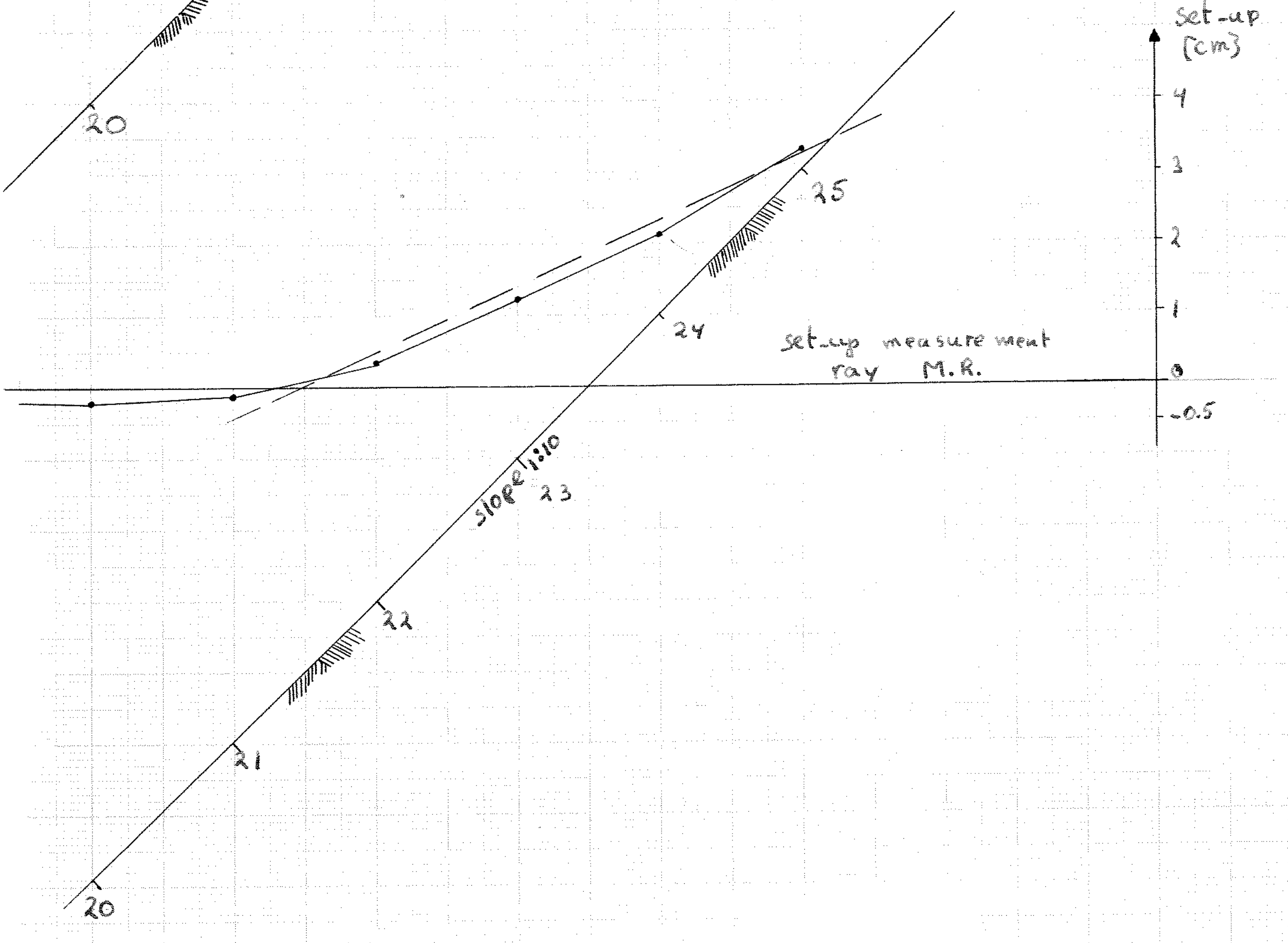


fig 5.23

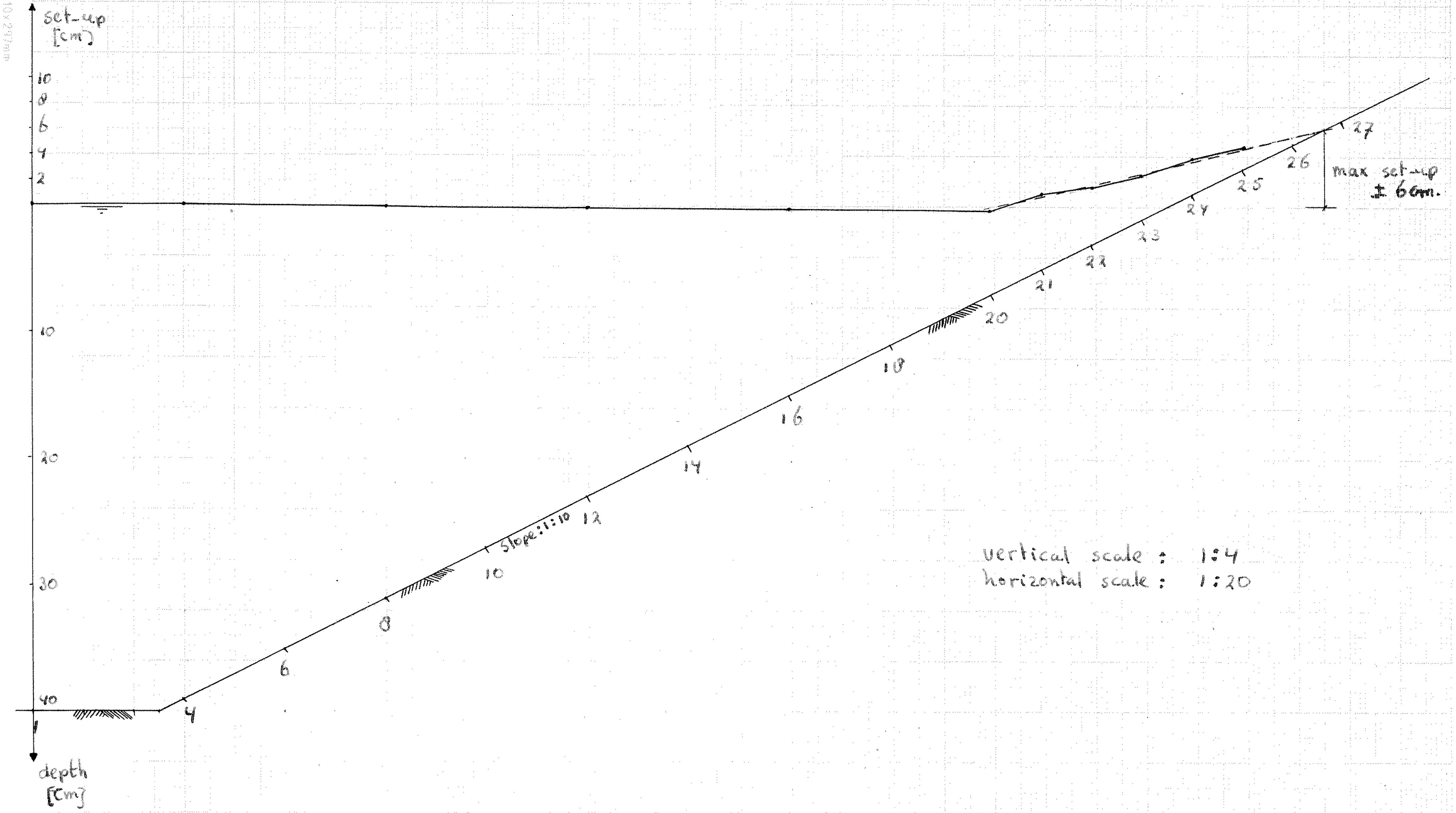

set-up [cm]
10
8
6
4
2
10
20
30
40
depth [cm]
4
6
8
10
slope: 1:10
12
14
16
18
20
21
22
23
24
25
26
27
max set-up ± 6cm.
vertical scale : 1:4
horizontal scale : 1:20

fig 5.24

2. First the mean slope (of the water level) is calculated in several points and than a line with that slope is drawn through the point with the largest set-down.

The results of the above calculations are:

Left ray:

1. Max. set-up :
   - mean value: 5.7 cm (slope: 0.78/20 through +1.64)
   - max. slope: 6.0 cm (slope: 0.87/20 through +1.68)
   - min. slope: 5.4 cm (slope: 0.69/20 through +1.60)

2. Mean slope : 0.99/20 max. set-up (through -0.30 cm): ≃ 6.3 cm

   The mean set-up is valued at 5.9 cm (σ= 0.35/6%)

Middle ray:

1. Max. set-up :
   - mean value: (slope 1.19/20; through 0.30) ≃ 3.75 cm
   - max. value: (slope 1.6/20; through 0.50) ≃ 5.5 cm
   - min. value: (slope 0.78/20; through 0.10) ≃ 3.15 cm

2. Mean slope : 0.85/20 → max. set-up (through -0.11) ≃ 3.5 cm

   The mean set-up is valued at:
   - (with max. value): 3.85 cm ($\sigma \simeq$ 0.95/25%)
   - (without max. value): 3.45 cm ($\sigma \simeq$ 0.25/7%)

   The mean set-up is valued at 3.6 cm (± 15%).

   The maximum set-up can be calculated assuming that the formula, derived by Battjes ($\bar{\eta}_{max} = {}^5/_{16}\ \gamma H_b$), is also valid in this case.

   Left ray :γ = 1.3 (± 10%); H_b = 14.8 cm (± 15%)

   $$\bar{\eta}_{max} = \frac{5}{16}\ \gamma H_b \simeq 6.0 \text{ cm } (\pm 12\%)$$

   Middle ray :γ = 1.3 (± 10%); H_b = 3.8 cm (± 5%)

   $$\bar{\eta}_{max} = \frac{5}{16}\ \gamma H_b \simeq 1.5 \text{ cm } (\pm 12\%)$$

Another method for the calculation of the maximum set-up is the addition of the set-up of each wave system, assuming that the radiation-stress, perpendicular to the coast, can be added.
In this method is the radiation-stress assumed not to depend on the phase difference between the two wave systems. With other words, S_{xx}, S_{yy} and S_{xy} do not depend on (p-q) in form 3.18.

Assuming each wave system breaks independently from the other (as if the other system does not exist), the following results are obtained:

Left ray : the diffraction factor K_D has to be valued for each system.
$K_D\ (-43^o) \simeq 1.25$, $K_D\ (-28^o) \simeq 1.20$;
assuming $\gamma = 0.8 \rightarrow \bar{\eta}_{max_{tot}} = \bar{\eta}_{max} + \bar{\eta}_{max} \simeq 2.5 + 2.5 \simeq 5.00$ cm

$\gamma = 0.9 \rightarrow \bar{\eta}_{max_{tot}} = \bar{\eta}_{max} + \bar{\eta}_{max} \simeq 2.9 + 2.9 \simeq 5.80$ cm

The mean value is $\bar{\eta}_{max} \simeq 5.4$ cm ($\pm$ 10%).

Middle ray : $K_D\ (-43^o) \rightarrow 0.95$, $K_D\ (-28^o) \simeq 1.05$
assuming $\gamma = 0.8 \rightarrow \bar{\eta}_{max} \simeq 2.0 + 2.2 = 4.2$ cm

$\gamma = 0.9 \rightarrow \bar{\eta}_{max} \simeq 2.3 + 2.6 = 4.9$ cm

The mean value is $\bar{\eta}_{max} \simeq 4.6$ cm ($\pm$ 10%)

A third method is using formula 50 of annex III

$$\bar{\eta} = \frac{1}{8} \left(\frac{\gamma}{1 + \alpha}\right)^2 Q\ h$$

in which h is the water depth of the breaking wave (11.4 cm).

α is the coefficient $\sqrt{\cos(\phi_1)/\cos(\phi_2)}$, which is 0.89, and γ is 1.3.
ϕ is the angle of wave approach on the breakerline.

For the waves in the experiment $\phi_{1br} = 16^o$ $\phi_{2br} = 11^o$

For a point on the phase = 0 line (p - q = 0) Q can be written as:

$$Q = 2\Phi + 2\alpha + (1 - \alpha^2) = 8.72$$

$$\Phi = (\cos\phi_1 + \alpha\cos\phi_2)^2 = 3.37$$

For a point on the phase = π line (p - q = $\frac{1}{4}$L) Q can be written as:

$$Q = \alpha\pi\cos\phi_1 \cos\phi_2 + 2\Phi + \frac{\pi}{L} + (1 + \alpha^2) = 4.37$$

$$\Phi = \cos^2\phi_1 + \alpha^2 \cos^2\phi_2 = 1.69$$

So, the maximum set-up is $\frac{1}{8} \left(\frac{1.3}{1.89}\right)^2 \; . \; 8.72 \; . \; 11.4 = 5.87$ cm

the minimum set-up is $\frac{1}{8} \left(\frac{1.3}{1.89}\right)^2 \; . \; 4.37 \; . \; 11.4 = 2.94$ cm

The results of the various calculation methods are summarized in the table below:

cm	measured	Battjes-formula	Addition	Annex III
maximum set-up	5.9	6.0	5.4	5.9
minimum set-up	3.6	1.5	4.6	2.9

5.e. Various observations

Because of the high turbulence it was not possible to measure longshore velocities with current meters. Also because of the short measurement section (only 8 m) and the high set-up (6 cm) large currents were caused by water level slopes. These set-up currents had velocities of 1.5 to 2 m/sec. (fig. 5.25)*). In the measurement section also a large influence of these currents has to be expected. Therefore only qualitative results regarding longshore currents are presented in this chapter.

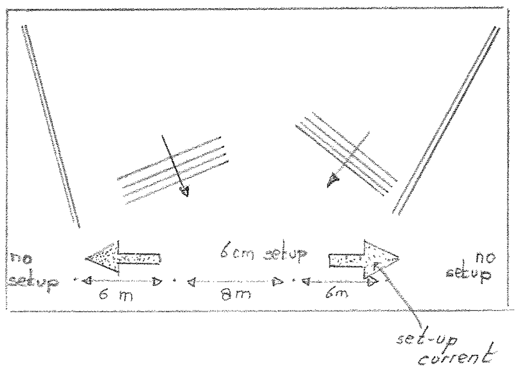


fig 5.25

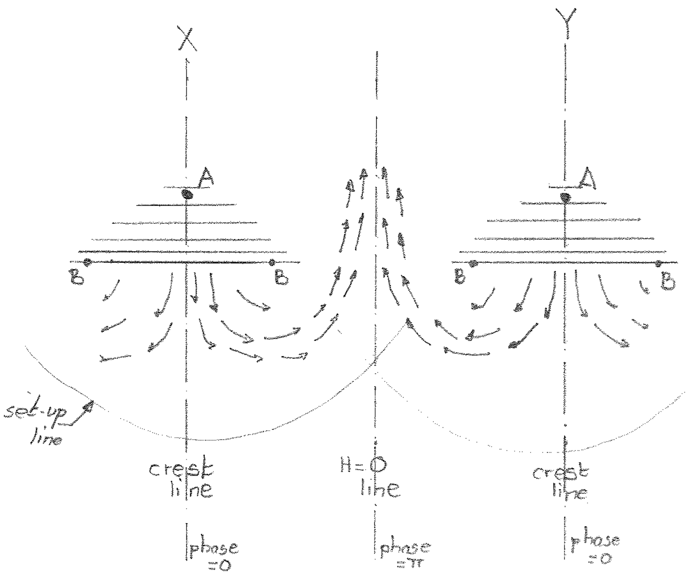


fig 5.26

By the breaking waves a large quantity of water is brought into the breakerzone. But because only on the crest lines waves have sufficient height to break. On the 'H = 0' lines (lines with phase = π) no wave breaking will occur. On these lines a fast rip current was observed.
The currents observed in the breaking area are indicated in fig. 5.26. The velocities were in the order of 0.7 - 1.5 m/sec.

The waves break on the indicated breaker line. Breaking starts at point A and the breaking front extends then towards B.

*) of the Chézy-formula, suppose C = 30 and the average water depth on the slope 0.2 m, then $v = C\sqrt{hi} = 30\sqrt{0.2 * 6/500} \simeq 1.5$ m/s.

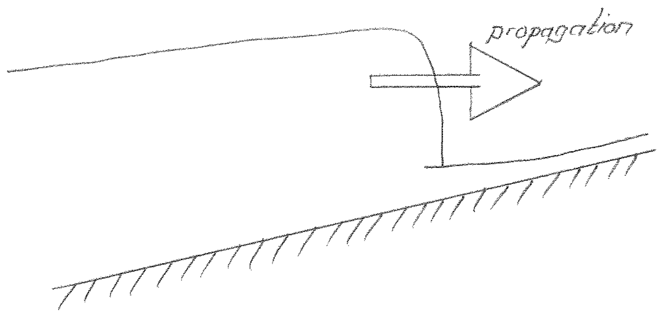


fig 5.27

One has to realize that the waves break alternating on crest line X and Y (when there is a crest at line X there is a trough at line Y). Due to this alternating breaking the rip current on the 'H = 0' line is also moving somewhat to-and-fro.

The breaking waves do not really plunge but they form some kind of a vertical water front which propagates towards the coast and than suddenly collapses (fig. 5.27)

6. CONCLUSIONS

6.a. Wave heights

The shape of the surface envelope was predicted almost correctly in the middle section of the wave basin. The distances between two minima or two maxima could be calculated with an accuracy of 98%. The variations in these distances were partly caused by diffraction.
It was not possible to calculate the wave height in any arbitrary point very accurate because of:

- diffraction; the applied diffraction is not totally correct.
- the waves became too high with regard to their (combined) length; therefore the maxima became somewhat lower, and the minima became higher.

6.b. Breakerindex

The breakerindex is quite high (1.3) but resembles very much the breakerindex of standing waves.
During the tests it appeared that the waves did not start breaking at one line parallel to the coast, but a certain breakerzone developed.
The ratio H/h was maximal at the points where the wave was just not yet broken.

6.c. Set-up

The assumption that the set-up of cross-swell can be predicted by addition of the set-ups of both individual systems appears to be not correct.
The Battjes-formula describes the set-up for cross-swell only in a realistic way for the maximum set-up.

An identical derivation as the derivation of the Battjes-formula leads to a very complicated set-up formula for cross-swell. It appeared that the set-up calculated with this formula agrees with the measurements for 80% and more.

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Calculation of the energy of cross-swell

The potential energy in a small area $dx\,dy$ is the product of the mass above (or below) S.W.L. and the distance from the mass-center to S.W.L.
This mass is $\rho g \eta\, dx\, dy$
The distance is $\frac{1}{2}\eta$
So $dE_p = \rho g \eta\, dx\, dy \cdot \frac{1}{2}\eta = \frac{1}{2}\eta^2 \rho g\, dx\, dy$

Hence [for perpendicular waves]

$$E_p = \int_0^{L_1}\int_0^{L_2} \frac{1}{2}\eta^2 \rho g\, dx\, dy \qquad L \text{ is the wave-length.}$$

$$= \frac{1}{2}\rho g \int_0^{L_1}\int_0^{L_2} \eta^2\, dx\, dy$$

$$\eta = a_1 \sin k_1 x + a_2 \sin k_2 y \qquad k = \frac{2\pi}{L}$$

$$\eta^2 = a_1^2 \sin^2 k_1 x + a_2^2 \sin^2 k_2 y + 2a_1 a_2 \sin k_1 x \sin k_2 y$$

$$\int_0^{L_1}\int_0^{L_2} \eta^2\, dx\, dy = \int_0^{L_1}\int_0^{L_2} (a_1^2 \sin^2 k_1 x + a_2^2 \sin^2 k_2 y + 2a_1 a_2 \sin k_1 x \sin k_2 y)\, dx\, dy$$

$$= a_1^2 \int_0^{L_1}\int_0^{L_2} \sin^2 k_1 x\, dx\, dy + a_2^2 \int_0^{L_1}\int_0^{L_2} \sin^2 k_2 y\, dx\, dy + 2a_1 a_2 \int_0^{L_1}\int_0^{L_2} \sin k_1 x \sin k_2 y\, dx\, dy$$

$$= a_1^2 \int_0^{L_2} \frac{1}{2}\left[x - \frac{1}{2k_1}\sin 2k_1 x\right]_0^{L_1} dy + a_2^2 \int_0^{L_1} \frac{1}{2}\left[y - \frac{1}{2k_2}\sin 2k_2 y\right]_0^{L_2} dx + 0$$

$$= \frac{1}{2}a_1^2 \int_0^{L_2} L_1\, dy + \frac{1}{2}a_2^2 \int_0^{L_1} L_2\, dx$$

$$= \frac{1}{2}a_1^2 L_1 [y]_0^{L_2} + \frac{1}{2}a_2^2 L_2 [x]_0^{L_1}$$

$$= \frac{1}{2}L_1 L_2 (a_1^2 + a_2^2)$$

So

$$E_p = \frac{1}{4}\rho g L_1 L_2 (a_1^2 + a_2^2)$$
$$= \frac{1}{4}\rho g L_1 L_2 \left(\frac{1}{4}H_1^2 + \frac{1}{4}H_2^2\right)$$
$$= \frac{1}{16}\rho g L_1 L_2 (H_1^2 + H_2^2)$$

The potential energy per unit surface is:

$$V_p = E_p / L_1 L_2 = \frac{1}{16}\rho g (H_1^2 + H_2^2)$$

And because the potential energy per unit surface is equal to the kinetic energy, the total energy becomes:

$$V_t = \frac{1}{8}\rho g (H_1^2 + H_2^2) = \frac{1}{8}\rho g H_1^2 + \frac{1}{8}\rho g H_2^2 = V_1 + V_2$$

and not $\frac{1}{8}\rho g H_T^2$

Note: When the waves are not perpendicular an identical derivation leads to the same result.

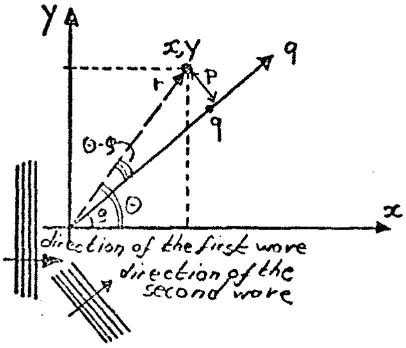
- η: deviation from S.W.L
- E: energy of a wave
- V: energy of a wave per unit surface
- L: wave length
- a: amplitude
- H: wave height
- ρ: density of water
- g: acceleration of gravity
- k: wave number $[2\pi/L]$

Radiation-stress caused by cross-swell

The formulae for radiation-stress were derived (by Longuet-Higgins and Stewart) for a one-wave system. It is not known whether the radiation-stress of a two-wave system can be found by addition, or that other relations have to be derived. The intention of this annex is to investigate the radiation-stress in a cross-swell situation.

In the following discussions the following conventions will be used.

- the origin of the coordinates will be placed at the still water level, the x-axis will have the direction of wave propagation of one of the waves. The positive z-axis extends upwards from the still water level [S.W.L.]
- the water density, ρ, is constant.
- the two waves do not influence each other.



wave 1: direction of propagation : x-axis
wave 2: direction of propagation : q-axis

[1] $\eta(x,y,t) = \eta_1 + \eta_2$

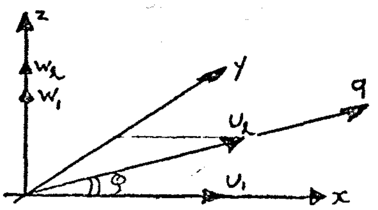
[2] $\eta_1 = a_1 \cos(k_1 x - \omega_1 t)$

[3] $\eta_2 = a_2 \cos(k_2 q - \omega_2 t + ph)$ $\qquad q = f(x,y)$

$r = \sqrt{x^2+y^2}$; $q = r\cos(\Theta - \varphi)$; $\Theta = \operatorname{arctg}\left(\frac{y}{x}\right)$

[4] $q = \sqrt{x^2+y^2} * \cos\left\{\operatorname{arctg}\left(\frac{y}{x}\right) - \varphi\right\}$

[5] $\eta = a_1\cos(k_1 x - \omega_1 t) + a_2\left(k_2\left[\sqrt{x^2+y^2} * \cos\left\{\operatorname{arctg}\left(\frac{y}{x}\right) - \varphi\right\}\right] - \omega_2 t + ph\right)$



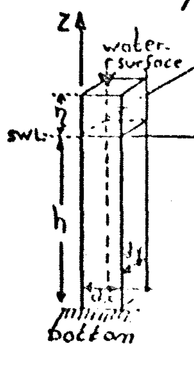
[6] $\vec{u_1} = \frac{a_1\omega_1}{\sinh(k_1 h)} \cosh k_1(z+h) \cos(k_1 x - \omega_1 t)\, \vec{e_x}$

[7] $\vec{w_1} = \frac{a_1\omega_1}{\sinh(k_1 h)} \sinh k_1(z+h) \sin(k_1 x - \omega_1 t)\, \vec{e_z}$

[8] $\vec{u_2} = \frac{a_2\omega_2}{\sinh(k_2 h)} \cosh k_2(z+h) \cos(k_2 q - \omega_2 t + ph)\, \vec{e_q}$

[9] $\vec{w_2} = \frac{a_2\omega_2}{\sinh(k_2 h)} \sinh k_2(z+h) \sin(k_2 q - \omega_2 t + ph)\, \vec{e_z}$

in which η = wave profile
a = amplitude of the original waves [maximum deviation of the surface]
$k = 2\pi/L$, wave number
$\omega = 2\pi/T$, frequency
t = time
x, y, z, position in the three dimensional carthesian frame of reference
h = waterdepth
u = instantaneous horizontal velocity of a particle
w = instantaneous vertical velocity of a particle
$\vec{e_q} = \cos\varphi\, \vec{e_x} + \sin\varphi\, \vec{e_y}$, vector of unity in q-direction



The total flux of horizontal momentum in the direction of the x-axis

[10] $P_1 = \left\{\int_{-h}^{\eta} (p + \rho u^2)\, dz\right\} dy \qquad u = u(x,y,t)$

The radiation-stress will be:

[11] $S_{xx} = (\overline{P_1} - \overline{P_0})\, dy$

Unlike in the one-wave system, the horizontal velocity in the x-direction will vary. Therefore dy is introduced, because of this the velocity (u) is only a function of x and t.

S_{xx} is separated into three parts:

[12] $S_{xx} = S_{xx}^{(1)} + S_{xx}^{(2)} + S_{xx}^{(3)}$

[13] $S_{xx}^{(1)} = \left(\overline{\int_{-h}^{\eta} \rho v^2 dz}\right) dy$

[14] $S_{xx}^{(2)} = \left(\overline{\int_{-h}^{0} (p-p_0) dz}\right) dy$

[15] $S_{xx}^{(3)} = \left(\overline{\int_{0}^{\eta} p\, dz}\right) dy$

$\overline{\int_a^b x\, dx}$ denotes the time average of the integral $\int_a^b x\, dx$

Equation [13] is split again:

[16] $S_{xx}^{(1)} = \left\{\overline{\int_{-h}^{0} \rho v^2 dz} + \overline{\int_{0}^{\eta} \rho v^2 dz}\right\} dy$

η is a function of the amplitude a too, so the last term will yield only a term of third order. Since only first and second order terms are considered this last term is neglected.

[17] $S_{xx}^{(1)} = \left\{\overline{\int_{-h}^{0} \rho v^2 dz}\right\} dy$

Both limits of the integral are constants, so $S_{xx}^{(1)}$ becomes

[18] $S_{xx}^{(1)} = \left\{\int_{-h}^{0} \rho\, \overline{v^2}\, dz\right\} dy$

For eq.[14] the same technique can be used.

[19] $S_{xx}^{(2)} = \left\{\overline{\int_{-h}^{0} (p-p_0) dz}\right\} dy = \left\{\int_{-h}^{0} (\bar{p} - \bar{p}_0)\, dz\right\} dy$

p_0 is excluded from the time average, because it is supposed to be a constant.

[20] $S_{xx}^{(2)} = \left\{\int_{-h}^{0} (\bar{p} - p_0) dz\right\} dy$

The mean flux of vertical momentum across a horizontal plane must be equal to the water above that plane. The average waterlevel is $z=0$, therefore:

[21] $\overline{p + \rho w^2} = -\rho g z = p_0$

[22] $\bar{p} - p_0 = -\rho \overline{w^2}$

Substitution into [20] yields:

[23] $S_{xx}^{(2)} = \left\{\int_{-h}^{0} -\rho \overline{w^2}\, dz\right\} dy$

Adding $S_{xx}^{(1)}$ and $S_{xx}^{(2)}$ yields using [18] and [23]:

[24] $S_{xx}^{(1)} + S_{xx}^{(2)} = \left\{\int_{-h}^{0} (\rho \overline{v^2} - \rho \overline{w^2}) dz\right\} dy$

The third term of S_{xx} was (eq 15)

$$S_{xx}^{(3)} = \left\{\overline{\int_{0}^{\eta} p\, dz}\right\} dy$$

Assumed is that p is nearly equal to the hydrostatic presure measured from the instantaneous surface η.

[25] $p = \rho g (\eta - z)$; so $S_{xx}^{(3)}$ becomes

[26] $S_{xx}^{(3)} = \left\{\overline{\int_{0}^{\eta} \rho g (\eta - z) dz}\right\} dy = \left\{\rho g \left(\overline{\int_0^{\eta} \eta\, dz - \int_0^{\eta} z\, dz}\right)\right\} dy$

[27] $S_{xx}^{(3)} = \left\{\frac{1}{2} \rho g\, \overline{\eta^2}\right\} dy$

The instantaneous watersurface is (eq.5)

$\eta = a_1 \cos A_1 + a_2 \cos A_2$ in which $A_1 = k_1 x - \omega_1 t$

$A_2 = k_2 \left[\sqrt{x^2+y^2} \cos\{\mathrm{arctg}(\tfrac{y}{x}) - \varphi\}\right] x - \omega_2 t + ph.$

$\overline{\eta^2} = \overline{(a_1 \cos A_1 + a_2 \cos A_2)^2} = \overline{a_1^2 \cos^2 A_1 + a_2^2 \cos^2 A_2 + 2 a_1 a_2 \cos A_1 \cos A_2}$

$= \overline{a_1^2 \cos^2 A_1} + \overline{a_2^2 \cos^2 A_2} + \overline{2 a_1 a_2 \cos A_1 \cos A_2} = a_1^2\, \overline{\cos^2 A_1} + a_2^2\, \overline{\cos^2 A_2} + 2 a_1 a_2\, \overline{\cos A_1 \cos A_2}$

Note $\overline{\cos^2 q} = \frac{1}{\pi} \int_0^{\pi} \cos^2 q\, dq = 1/2$ *)

$\overline{\cos q_1 x \cos q_2 x} = 0$

$\overline{\eta^2} = \frac{1}{2} a_1^2 + \frac{1}{2} a_2^2 + 0$

*) For a proof of these statements see page III-7

[28] $$S_{xx}^{(3)} = \left\{\tfrac{1}{2}\rho g\left(\tfrac{1}{2}a_1^2+\tfrac{1}{2}a_2^2\right)\right\}dy = \left\{\tfrac{1}{4}\rho g\left(a_1^2+a_2^2\right)\right\}dy$$

From the above equations it becomes clear that $S_{xx}^{(3)}$ is independent from y, so

[29] $$S_{xx}^{(3)} = \tfrac{1}{4}\rho g\left(a_1^2+a_2^2\right)$$

Calculation will be continued with the other parts of S_{xx}. (eq 24)

$$S_{xx}^{(1)}+S_{xx}^{(1)} = \left\{{}_{-h}\int^{0}\rho\left(\overline{U^2}-\overline{W^2}\right)dz\right\}dy$$

in which $U = \vec{U} = U\,\vec{e_x} = U_1\,\vec{e_x} + U_2\,\vec{e_x}$

using equation [6] and [8] yields:

[30] $$U = \frac{a_1\omega_1}{\sinh(k_1h)}\cosh k_1(z+h)\cos(k_1x-\omega_1t)\vec{e_x} + \frac{a_2\omega_2\cos\varphi}{\sinh(k_2h)}\cosh k_2(z+h)\cos(k_2q-\omega_2t)\vec{e_x}$$

using equation [7] and [9] yields:

[31] $$W = \vec{W_1}+\vec{W_2} = \frac{a_1\omega_1}{\sinh(k_1h)}\sinh k_1(z+h)\sin(k_1x-\omega_1t)\vec{e_z} + \frac{a_2\omega_2}{\sinh(k_2h)}\sinh k_2(z+h)\sin(k_2q-\omega_2t)\vec{e_z}$$

Equation [24] can be written in the following way:

$$S_{xx}^{(1)}+S_{xx}^{(1)} = \left\{{}_{-h}\int^{0}\rho\left(\overline{U^2}-\overline{W^2}\right)dz\right\}dy =$$

$$= \left\{\rho\,{}_{-h}\int^{0}\left\{\overline{(U_1+U_{2x})^2}-\overline{(W_1+W_2)^2}\right\}dz\right\}dy$$

[32] $$= \rho\,dy\left\{{}_{-h}\int^{0}\left\{\left(\overline{U_1^2}+\overline{2U_1U_{2x}}+\overline{U_{2x}^2}\right)-\left(\overline{W_1^2}+\overline{2W_1W_2}+\overline{W_2^2}\right)\right\}dz\right\}$$

It can be proved that $\overline{2U_1U_{2x}}=0$ and $\overline{2W_1W_2}=0$ [see page III-7]; so eq [32] becomes:

[33] $$S_{xx}^{(1)}+S_{xx}^{(2)} = \rho\,dy\left\{{}_{-h}\int^{0}\overline{U_1^2}\,dz + {}_{-h}\int^{0}\overline{U_{2x}^2}\,dz - {}_{-h}\int^{0}\overline{W_1^2}\,dz - {}_{-h}\int^{0}\overline{W_2^2}\,dz\right\}$$

Using equations [6], [7], [8] and [9], yields:

$$S_{xx}^{(1)}+S_{xx}^{(2)} = \rho\,dy\left\{\frac{a_1^2\omega_1^2}{\sinh^2(k_1h)}\overline{\cos^2(k_1x-\omega_1t)}\,{}_{-h}\int^{0}\cosh^2 k_1(z+h)\,dz\right.$$
$$+\frac{a_2^2\cos^2\varphi\,\omega_2^2}{\sinh^2(k_2h)}\overline{\cos^2(k_2q-\omega_2t)}\,{}_{-h}\int^{0}\cosh^2 k_2(z+h)\,dz$$
$$-\frac{a_1^2\omega_1^2}{\sinh^2(k_1h)}\overline{\sin^2(k_1x-\omega_1t)}\,{}_{-h}\int^{0}\sinh^2 k_1(z+h)\,dz$$
$$\left.-\frac{a_2^2\omega_2^2}{\sinh^2(k_2h)}\overline{\sin^2(k_1x-\omega t)}\,{}_{-h}\int^{0}\sinh^2 k_2(z+h)\,dz\right.$$

The time average of $\cos^2 q$ and $\sin^2 q$ is ½.; the integral ${}_{-h}\int^{0}\cosh^2 k_1(z+h)\,dz$ yields:

$$\frac{1}{k_1}\int_0^{k_1h}\cosh^2 q\,dq \qquad q = k_1(z+h)$$

$$= \frac{1}{k_1}\int_0^{k_1h}\left(\frac{\cosh 2q}{2}+\frac{1}{2}\right)dq$$

[34] $$= \frac{1}{k_1}\left[\frac{\sinh 2q}{4}+\frac{1}{2}q\right]_0^{k_1h} = \frac{\sinh 2k_1h}{4k_1}+\frac{1}{2}h$$

So:

[35] $${}_{-h}\int^{0}\cosh^2 k_2(z+h)\,dz = \frac{\sinh 2k_2h}{4k_2}+\frac{1}{2}h$$

Also can be proved that

[36] $${}_{-h}\int^{0}\sinh^2 k_1(z+h)\,dz = \frac{\sinh 2k_1h}{4k_1}-\frac{1}{2}h$$

[37] $${}_{-h}\int^{0}\sinh^2 k_2(z+h)\,dz = \frac{\sinh 2k_2h}{4k_2}-\frac{1}{2}h.$$

So, using [34] to [37], 33 yields:

[38] $$S_{xx}^{(1)}+S_{xx}^{2} = \rho\,dy\left\{\frac{a_1^2\omega_1^2}{\sinh^2(k_1h)}\cdot\frac{1}{2}\left[\frac{\sinh 2k_1h}{4k_1}+\frac{1}{2}h\right]\right.$$
$$+\frac{a_2^2\cos^2\varphi\,\omega_2^2}{\sinh^2 k_2h}\,\frac{1}{2}\left[\frac{\sinh 2k_2h}{4k_2}+\frac{1}{2}h\right]$$
$$-\frac{a_1^2\omega_1^2}{\sinh^2 k_1h}\,\frac{1}{2}\left[\frac{\sinh 2k_1h}{4k_1}-\frac{1}{2}h\right]$$
$$\left.-\frac{a_2^2\omega_2^2}{\sinh^2 k_2h}\,\frac{1}{2}\left[\frac{\sinh 2k_2h}{4k_2}-\frac{1}{2}h\right]\right.$$

From the above equation it becomes clear that $S_{xx}^{(1)}+S_{xx}^{(1)}$ is independent from y, so dy can be omitted

For a one-wave system the following equation is valid:

[39] $$\omega^2 = gk \tanh(kh)$$

Is this equation also valid for a two-wave system?

[5] $$\eta = a_1 \cos(k_1 x - \omega_1 t) + a_2 \cos(k_2 q - \omega_2 t + ph)$$

Suppose: the velocity potential $\Phi(x,y,z,t)$ is:

[40] $$\Phi(x,y,z,t) = \Phi_1(x,y,z,t) + \Phi_2(x,y,z,t) = \qquad q = f(x,y).$$

$$= \frac{\omega_1 a_1}{k_1}\frac{\cosh k_1(h+z)}{\sinh k_1 h}\sin(k_1 x - \omega_1 t) + \frac{\omega_2 a_2}{k_2}\frac{\cosh k_2(h+z)}{\sinh k_2 h}\sin(k_2 q - \omega_2 t + ph)$$

This equation of $\Phi(x,y,z,t)$ has to meet the following conditions:

[41] $$\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial x^2} = 0 \qquad \text{(Laplace equation)}$$

[42] $$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} \quad \text{on } z = 0$$

[43] $$\frac{\partial \Phi}{\partial z} = 0 \quad \text{on } z = -h$$

[44] $$\frac{\partial \eta}{\partial t} = a_1 \omega_1 \sin(k_1 x - \omega_1 t) + a_2 \omega_2 \sin(k_2 q - \omega_2 t + ph)$$

$$\frac{\partial \Phi}{\partial z} = \frac{\omega_1 a_1}{k_1} k_1 \frac{\sinh k_1(h+z)}{\sinh k_1 h}\sin(k_1 x - \omega_1 t) + \frac{\omega_2 a_2}{k_2} k_2 \frac{\sinh k_2(h+z)}{\sinh k_2 h}\sin(k_2 q - \omega_2 t + ph)$$

[45] $$\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = \omega_1 a_1 \sin(k_1 x - \omega_1 t) + \omega_2 a_2 \sin(k_2 q - \omega_2 t + ph)$$

With [44] and [45] the condition of [42] is proved.

[46] $$\left(\frac{\partial \Phi}{\partial z}\right)_{z=-h} = \frac{\omega_1 a_1}{k_1} k_1 \frac{\sinh 0}{\sinh k_1 h}\sin(k_1 x - \omega_1 t) + \frac{\omega_2 a_2}{k_2} k_2 \frac{\sinh 0}{\sinh k_2 h}\sin(k_2 q - \omega_2 t + ph) = 0 + 0 = 0$$

With [46] the condition of [43] is proved.

[47] $$\frac{\partial^2 \Phi}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z}\right) = \frac{\omega_1 a_1}{k_1} k_1^2 \frac{\cosh k_1(h+z)}{\sinh k_1 h}\sin(k_1 x - \omega_1 t) + \frac{\omega_2 a_2}{k_2} k_2^2 \frac{\cosh k_2(h+z)}{\sinh k_2 h}\sin(k_2 q - \omega_2 t + ph)$$

[48] $$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{\partial}{\partial x}\left\{\frac{\omega_1 a_1}{k_1}\frac{\cosh k_1(h+z)}{\sinh k_1 h} k_1 \cos(k_1 x - \omega_1 t) + \frac{\omega_2 a_2}{k_2}\frac{\cosh k_2(h+z)}{\sinh k_2 h} k_2 \cos(k_2 q - \omega_2 t + ph)\right\}$$

$$= \frac{\omega_1 a_1}{k_1}\frac{\cosh k_1(h+z)}{\sinh k_1 h}(-k_1^2)\sin(k_1 x - \omega_1 t) + \frac{\omega_2 a_2}{k_2}\frac{\cosh k_2(h+z)}{\sinh k_2 h}(-k_2^2)\sin(k_2 q - \omega_2 t + ph)\left(\frac{\partial q}{\partial x}\right)^2$$

$$+ \frac{\omega_2 a_2}{k_2}\frac{\cosh k_2(h+z)}{\sinh k_2 h} k_2 \cos(k_2 q - \omega_2 t + ph)\frac{\partial^2 q}{\partial x^2}$$

[49] $$\frac{\partial^2 \Phi}{\partial y^2} = 0 + \frac{\omega_2 a_2}{k_2}\frac{\cosh k_2(h+z)}{\sinh k_2 h}(-k_2^2)\sin(k_2 q - \omega_2 t + ph)\left(\frac{\partial q}{\partial y}\right)^2$$

$$+ \frac{\omega_2 a_2}{k_2}\frac{\cosh k_2(h+z)}{\sinh k_2 h} k_2 \cos(k_2 q - \omega_2 t + ph)\left(\frac{\partial^2 q}{\partial y^2}\right)$$

For the first wave can be proved that $\left(\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)_1 = 0$ with [47] [48] and [49]. Therefore [41] exists only out of terms of the second wave. So directly can be proved (by axis-transformation)

$$\left(\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)_2 = 0.$$

By means of eq. [40] can be proved $\left(\frac{\partial \Phi}{\partial t} + \frac{p}{\rho} + gz = 0 \text{ for } z = \eta \text{ and } p = 0\right)$ that

$$\omega_1^2 = g k_1 \tanh k_1 h$$
$$\omega_2^2 = g k_2 \tanh k_2 h$$

Equation [38] can be rewritten with equation [37] to:

[50] $$S_{xx}^{(1)} + S_{xx}^{(2)} = \frac{\rho g a_1^2 k_1}{\sinh 2k_1 h}\left[\frac{\sinh 2k_1 h}{4k_1} + \tfrac{1}{2}h\right] + \frac{\rho g a_2^2 k_2}{\sinh 2k_2 h}\left[1 - \sin^2\varphi\right]\left[\frac{\sinh 2k_2 h}{4k_2} + \tfrac{1}{2}h\right]$$

$$- \frac{\rho g a_1^2 k_1}{\sinh 2k_1 h}\left[\frac{\sinh 2k_1 h}{4k_1} - \tfrac{1}{2}h\right] - \frac{\rho g a_2^2 k_2}{\sinh 2k_2 h}\left[\frac{\sinh 2k_2 h}{4k_2} - \tfrac{1}{2}h\right]$$

$$= \frac{\rho g a_1^2 k_1 h}{\sinh 2k_1 h} + \frac{\rho g a_2^2 k_2 h}{\sinh 2k_2 h} - \rho g a_2^2 \sin^2\varphi\left[\frac{1}{4} + \frac{k_2 h}{2\sinh 2k_2 h}\right]$$

S_{xx} becomes, using eq [29], [12] and [50]:

[51] $$S_{xx} = \tfrac{1}{4}\rho g(a_1^2 + a_2^2) + \tfrac{1}{2}\rho g a_1^2 \frac{2k_1 h}{\sinh 2k_1 h} + \tfrac{1}{2}\rho g a_2^2 \frac{2k_2 h}{\sinh 2k_2 h} - \tfrac{1}{2}\rho g a_2^2\left[\frac{1}{2} + \frac{k_2 h}{\sinh 2k_2 h}\right]\sin^2\varphi$$

Using the energy-density of the waves: $E_1 = \frac{1}{2}\rho g a_1^2$ and $E_2 = \frac{1}{2}\rho g a_2^2$. [51] becomes:

[52] $$S_{XX} = E_1\left(\frac{2k_1h}{\sinh 2k_1h}+\frac{1}{2}\right) + E_2\left(\frac{2k_2h}{\sinh 2k_2h}+\frac{1}{2}\right) - E_2\left[\frac{1}{2}+\frac{k_2h}{\sinh 2k_2h}\right]\sin^2\varphi .$$

When S_{YY} has to be determined, it is necessary to examine the flow of momentum in the YZ-plane:

[53] $$S_{YY} = \overline{\left\{{}_{-h}\int^{\eta}(p+\rho v^2)\,dz - {}_{-h}\int^{0} p_0\,dz\right\}}dx$$

Like S_{XX}, S_{YY} is split into:

[54] $$S_{YY} = S_{YY}^{(1)} + S_{YY}^{(2)} + S_{YY}^{(3)}$$

[55] $$S_{YY}^{(1)} = \left\{\overline{{}_{-h}\int^{\eta}\rho v^2 dz}\right\}dx$$

[56] $$S_{YY}^{(1)} = \left\{\overline{{}_{-h}\int^{0}(p-p_0)\,dz}\right\}dx$$

[57] $$S_{YY}^{(3)} = \overline{{}_{0}\int^{\eta} p\,dz} = S_{XX}^{(3)} = \frac{1}{4}\rho g\,(a_1^2+a_2^2) \qquad [\text{cf. eq}[25] \text{ to } [29]]$$

Equation [55] can be rewritten as:

[58] $$S_{YY}^{(1)} = \left\{\overline{{}_{-h}\int^{0}\rho v^2 dz + \int_{0}^{\eta}\rho v^2 dz}\right\}dx$$

The second term is neglected, like in eq [16] and [17].

[59] $$S_{YY}^{(1)} = \left\{\overline{{}_{-h}\int^{0}\rho v^2 dz}\right\}dx$$

[60] $$S_{YY}^{(2)} = \left\{\overline{{}_{-h}\int^{0}\rho w^2 dz}\right\}dx \qquad (\text{cf eq.}[21] \text{ to } [23])$$

So

[61] $$S_{YY}^{(1)} + S_{YY}^{(1)} = \left\{{}_{-h}\int^{0}\left(\overline{\rho v^2} - \overline{\rho w^2}\right)dz\right\}dx$$

In the same way as $S_{XX}^{(1)} + S_{XX}^{(1)}$ this can be rewritten:

[62] $$v = v\vec{e}_y = \vec{U}_{2y} = \frac{a_2\omega_2\sin\varphi}{\sinh k_2h}\cosh k_2(z+h)\cos(k_2y-\omega_2t+ph)\,\vec{e}_y$$

[63] $$w = w\vec{e}_z = w_1 + w_2$$

Equation [61] yields, using [62] and [63]:

[64] $$S_{YY}^{(1)} + S_{YY}^{(1)} = \rho\,dx\left\{\frac{a_2^2\omega_2^2\sin^2\varphi}{\sinh^2 k_2h}\int_{-h}^{0}\cosh^2 k_2(z+h)\,dz\;\overline{\cos^2(k_2y-\omega_2t+ph)}\right.$$
$$-\frac{a_1^2\omega_1^2}{\sinh^2 k_1h}\,\overline{\sin^2(k_1x-\omega_1t)}\int_{-h}^{0}\sinh^2 k_1(z+h)\,dz$$
$$\left.+\frac{a_2^2\omega_2^2}{\sinh^2 k_2h}\,\overline{\sin^2(k_2y-\omega_2t+ph)}\;{}_{-h}\int^{0} k_2(z+h)\,dz\right\}$$

Using eq[34] to [37], equation [64] yields:

[65] $$S_{YY}^{(1)} + S_{YY}^{(1)} = \rho\,dx\left\{\frac{a_2^2\omega_2^2\sin^2\varphi}{\sinh^2 k_2h}\,\frac{1}{2}\left[\frac{\sinh 2k_2h}{4k_2}+\frac{1}{2}h\right]\right.$$
$$\left.-\frac{a_1^2\omega_1^2}{\sinh^2 k_1h}\,\frac{1}{2}\left[\frac{\sinh 2k_1h}{4k_1}-\frac{1}{2}h\right] - \frac{a_2^2\omega_2^2}{\sinh^2(k_2h)}\,\frac{1}{2}\left[\frac{\sinh 2k_2h}{4k_2}-\frac{1}{2}h\right]\right\}$$

From this equation it becomes clear that $S_{YY}^{(1)} + S_{YY}^{(1)}$ is independent from x, so dx can be omitted. The total radiation-stress in the Y-direction becomes, using eq. [54], [39], [57] and [65]:

$$S_{YY} = E_2\frac{2k_2}{\sinh 2k_2h}\left[\frac{\sinh 2k_2h}{4k_2}+\frac{1}{2}h\right](1-\cos^2\varphi) - E_1\frac{2k_1}{\sinh 2k_1h}\left[\frac{\sinh 2k_1h}{4k_1}-\frac{1}{2}h\right]$$
$$-E_2\frac{2k_2}{\sinh 2k_2h}\left[\frac{\sinh 2k_2h}{4k_2}-\frac{1}{2}h\right] + \frac{1}{2}(E_1+E_2)$$

[66] $$S_{YY} = E_1\frac{k_1h}{\sinh 2k_1h} + E_2\left(\frac{2k_2h}{\sinh 2k_2h}+\frac{1}{2}\right) - E_2\left(\frac{k_2h}{\sinh 2k_2h}+\frac{1}{2}\right)\cos^2\varphi .$$

Shear stresses

One has to investigate the possibility of the transfer of x-momentum across the plane y=constant. Since this momentum-transfer manifests itself as a shear stress, the pressure at the point does not contribute. This results in a more simple equation than [11] or [53], viz.:

[67] $$S_{XY} = \overline{\int_{-h}^{\eta}\rho uv\,dz}$$

This equation can be split

[68] $$S_{xy} = \int_{-h}^{0} \overline{\rho u v \, dz} + \int_{0}^{\eta} \overline{\rho u v \, dz}$$

$$= \int_{-h}^{0} \overline{\rho u v \, dz} + 0 \qquad \left(\text{because } \int_{0}^{\eta} \overline{\rho u v \, dz} \text{ is of the third order}\right)$$

in which [19] $u = u_1 + u_{2x}$ and [62] $v = v_{2y}$, so [68] yields:

[69] $$S_{xy} = \int_{-h}^{0} \overline{\rho (u_1 u_{2y} + u_{2x} u_{2y})} \, dz =$$

$$= \int_{-h}^{0} \rho \left(\overline{u_1 u_{2y}} + \overline{u_{2x} u_{2y}}\right) dz$$

It can be proved that $\overline{u_1 u_{2y}} = 0$, because it is assumed that $\omega_1 \neq \omega_2$ [$u_1 = f(1) \cos q_1 x$; $u_{2y} = f(2y) \cos q_2 z$ and $\overline{\cos q_1 x \cos q_2 x} = 0$]

Eq [69] becomes, using [8]:

[70] $$S_{xy} = \rho \int_{-h}^{0} \frac{a_2^2 \omega_2^2}{\sinh^2 k_2 h} \cosh^2 k_2 (z+h) \, \overline{\cos^2 (k_2 q - \omega_2 t + ph)} \sin\varphi \cos\varphi \, dz$$

$$= \frac{\rho a_2^2 \omega_2^2}{\sinh^2 k_2 h} \overline{\cos^2 (k_2 q - \omega_2 t + ph)} \sin\varphi \cos\varphi \int_{-h}^{0} \cosh^2 k_2 (z+h) \, dz$$

Using [34] [35] and [39], equation [70] yields:

[71] $$S_{xy} = \frac{\rho g a_2^2 k_2}{\sinh 2 k_2 h} \sin\varphi \cos\varphi \left[\frac{\sinh 2 k_2 h}{4 k_2} + \frac{1}{2} h\right]$$

$$= E_2 \left[\frac{k_2 h}{\sinh 2 k_2 h} + \frac{1}{2}\right] \sin\varphi \cos\varphi \qquad \left(E_2 = \tfrac{1}{2} \rho g a_2^2\right)$$

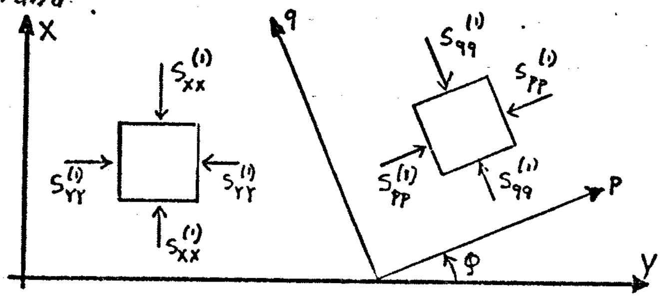
Now all radiation-stresses are known. The results have to be interpreted, because it is important to know whether radiation stresses of two different waves may be added. Summarising the results of the above derivation gives:

[58] $$S_{xx} = E_1 \left(\frac{2 k_1 h}{\sinh 2 k_1 h} + \frac{1}{2}\right) + E_2 \left(\frac{2 k_2 h}{\sinh 2 k_2 h} + \frac{1}{2}\right) - E_2 \left(\frac{k_2 h}{\sinh 2 k_2 h} + \frac{1}{2}\right) \sin^2 \varphi$$

[66] $$S_{yy} = E_1 \left(\frac{k_1 h}{\sinh 2 k_1 h}\right) + E_2 \left(\frac{2 k_2 h}{\sinh 2 k_2 h} + \frac{1}{2}\right) - E_2 \left(\frac{k_2 h}{\sinh 2 k_2 h} + \frac{1}{2}\right) \cos^2 \varphi$$

[71] $$S_{xy} = E_2 \left(\frac{k_2 h}{\sinh 2 k_2 h} + \frac{1}{2}\right) \sin\varphi \cos\varphi$$

When is supposed that radiation stresses may be added, the following formulae are valid:



$$S_{xx}^{(1)} = E_1 \left(\frac{2 k_1 h}{\sinh 2 k_1 h} + \tfrac{1}{2}\right)$$

$$S_{yy}^{(1)} = E_1 \left(\frac{k_1 h}{\sinh 2 k_1 h}\right)$$

$$S_{qq}^{(1)} = E_2 \left(\frac{2 k_2 h}{\sinh 2 k_2 h} + \tfrac{1}{2}\right)$$

$$S_{pp}^{(1)} = E_2 \left(\frac{k_2 h}{\sinh 2 k_2 h}\right)$$

When the radiation stresses in the xy-plane of reference are determined, the formulae [58], [66] and [71] are found.

So it is proved that radiation stresses can be added under the following assumptions

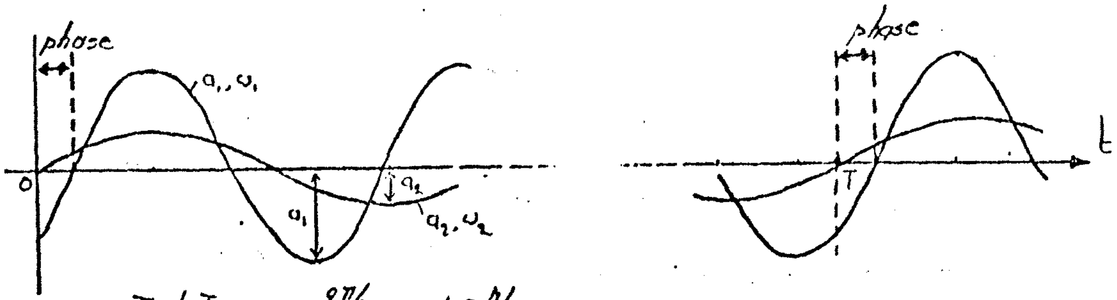
- the two incoming waves do not influence each other
- the periods of these two waves are not identical. In that case difficulties will arise with the determination of the time averages

Time average of a quadratic goniometric function

$$\overline{\{a_1 \cos \omega_1 t + a_2 \cos(\omega_2 t + ph)\}^2} = \frac{1}{T_0} \int^{T} \{a_1 \cos \omega_1 t + a_2 \cos(\omega_2 t + ph)\}^2 dt$$

$$= \frac{1}{T_0} \int^{T} \{a_1^2 \cos^2 \omega_1 t + 2 a_1 a_2 \cos \omega_1 t \cos(\omega_2 t + ph) + a_2^2 \cos^2(\omega_2 t + ph)\} \, dt$$

$$= \frac{1}{T_0} \int^{T} a_1^2 \cos^2 \omega_1 t \, dt + \frac{1}{T_0} \int 2 a_1 a_2 \cos \omega_1 t \cos(\omega_2 t + ph) \, dt + \frac{1}{T_0} \int^{T_1} a^2 \cos^2(\omega_2 t + ph) \, dt$$



$$T = k_1 T_1 \qquad \omega_1 = 2\pi / T_1 \qquad k_1 \in \mathbb{N}$$
$$T = k_2 T_2 \qquad \omega_2 = 2\pi / T_2 \qquad k_2 \in \mathbb{N}$$

$$\frac{1}{T}\int_0^T a_1^2 \cos^2 \omega_1 t \, dt = \frac{1}{k_1 T_1} \cdot k_1 \int_0^{T_1} a_1^2 \cos^2 \omega_1 t \, dt = \frac{k_1}{k_1 T_1} \cdot \frac{1}{2} a_1^2 T_1 = \frac{1}{2} a_1^2$$

$$\frac{1}{T}\int_0^T a_2^2 \cos^2 \omega_2 t \, dt = \frac{1}{k_2 T_2}\, k_2 \int_0^{T_2} a_2^2 \cos^2 \omega_2 t \, dt = \frac{k_2}{k_2 T_2} \cdot \frac{1}{2} a_2^2 T_2 = \frac{1}{2} a_2^2$$

$$\frac{1}{T}\int_0^T 2 a_1 a_2 \cos \omega_1 t \cos(\omega_2 t + ph)\, dt = \frac{a_1 a_2}{T}\int_0^T \{\cos 2(\omega_1 t + \omega_2 t + ph) + \cos 2(\omega_1 t - \omega_2 t - ph)\}\, dt$$

[Note: $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$= \frac{a_1 a_2}{T}\int_0^T \cos\{2(\omega_1+\omega_2)t + 2ph\}\, dt + \frac{a_1 a_2}{T}\int_0^T \cos\{2(\omega_1-\omega_2)t - 2ph\}\, dt$$

$$\omega_1 + \omega_2 = \frac{2\pi}{T_1} + \frac{2\pi}{T_2} = 2\pi\left(\frac{1}{T_1} + \frac{1}{T_2}\right) = 2\pi\left(\frac{k_1}{k_1 T_1} + \frac{k_2}{k_2 T_2}\right) = 2\pi\left(\frac{k_1}{T} + \frac{k_2}{T}\right) = \frac{2\pi}{T}(k_1 + k_2) = \frac{2\pi}{T_3} \qquad T_3 = \frac{T}{k_1 + k_2}$$

$$\omega_1 - \omega_2 = \frac{2\pi}{T_1} - \frac{2\pi}{T_2} = 2\pi\left(\frac{k_1}{T} - \frac{k_2}{T}\right) = \frac{2\pi}{T}(k_1 - k_2) = \frac{2\pi}{T_4} \qquad T_4 = \frac{T}{|k_1 - k_2|}$$

[Note: $\omega_1 \neq \omega_2$]

$$\frac{a_1 a_2}{T}\int_0^T \cos\{2(\omega_1+\omega_2)t + 2ph\}\, dt + \frac{a_1 a_2}{T}\int_0^T \cos\{2(\omega_1-\omega_2)t - 2ph\}\, dt$$

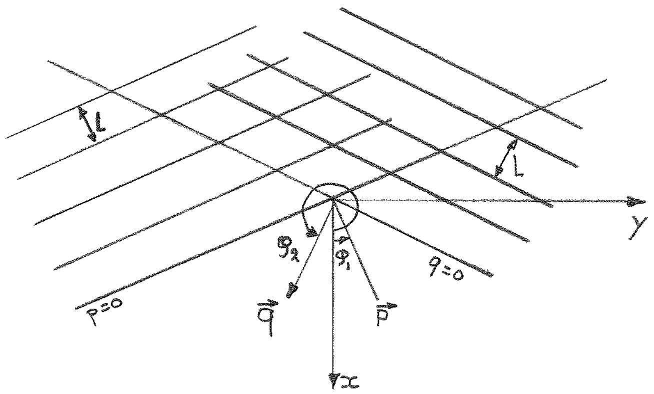
$$= \frac{a_1 a_2}{(k_1+k_2) T_3}(k_1+k_2)\int_0^{T_3} \cos(2\omega_3 + 2ph)\, dt + \frac{a_1 a_2}{|k_1-k_2|\, T_4}|k_1-k_2|\int_0^T \cos(2\omega_4 t - 2ph)\, dt$$

$$= \frac{a_1 a_2}{T_3} \cdot 0 + \frac{a_2 a_3}{T_4} \cdot 0$$

So $\overline{\{a_1 \cos\omega_1 t + a_2 \cos(\omega_2 t + ph)\}^2} = \frac{1}{2} a_1^2 + \frac{1}{2} a_2^2 = \frac{1}{2}(a_1^2 + a_2^2)$

Radiation stresses in cross-swell
a special case, $\omega_1 = \omega_2$

The radiation stresses for cross-swell are already calculated in Annex II. However the situation $\omega_1 = \omega_2$ was excluded in that annex. In this annex this special case will be calculated.
In the following discussions the same conventions will be used, as they were used in Annex II.



coastline

$\vec{p}$ and $\vec{q}$ are vectors of unity in the direction of wave propagation of the two wave-systems.
The x-coordinate is perpendicular to the coast, the y-coordinate is paralel to the coast.

[1] $\eta(x,y,t) = \eta_1 + \eta_2$ [η = surface elevation]

[2] $\eta_1 = a_1 \sin(\omega t - kp)$

[3] $\eta_2 = a_2 \sin(\omega t - kq)$

p and q are the distances to the origin of the p-q reference system [which is a non-carthesian frame-of-reference!!].
From [1], [2] and [3] follows

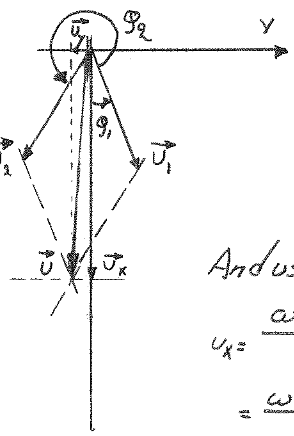
$$\eta(p,q,t) = a_1 \sin(\omega t - kp) + a_2 \sin(\omega t - kq)$$
$$= a_1\{\sin\omega t \cos kp - \cos\omega t \sin kp\} + a_2\{\sin\omega t \cos kq - \cos\omega t \sin kp\}$$
$$= \sin\omega t\{a_1 \cos kp + a_2 \cos kq\} - \cos\omega t\{a_1 \sin kp + a_2 \sin kq\}$$

[4] $$= A \sin\omega t - B\cos\omega t$$

in which [4A] $A = a_1 \cos kp + a_2 \cos kq$
[4B] $B = a_1 \sin kp + a_2 \sin kq$

- a - amplitude of the original waves
- k - wavenumber ($k = 2\pi/L$)
- ω - frequency ($\omega = 2\pi/T$)
- t - time
- h - waterdepth
- u - instantaneous horizontal velocity of a water particle
- v - instantaneous vertical velocity of a water particle

The horizontal velocities



[5] $\vec{u_1} = u_1 * \vec{e_p}$
[6] $\vec{u_2} = u_2 * \vec{e_q}$
[7] $\vec{U_X} = \vec{u_1}\cos\varphi_1 + \vec{u_2}\cos\varphi_2$

[8] $$u_1 = \frac{a_1\omega}{\sinh(kh)}\cosh\{k(z+h)\}\sin(\omega t - kp)$$

[9] $$u_2 = \frac{a_2\omega}{\sinh(kh)}\cosh\{k(z+h)\}\sin(\omega t - kq)$$

And using [8] and [9]

$$u_X = \frac{\omega\cosh\{k(z+h)\}}{\sinh(kh)} * \{a_1 \sin(\omega t - kp)\cos\varphi_1 + a_2 \sin(\omega t - kq)\cos\varphi_2$$

$$= \frac{\omega\cosh\{k(z+h)\}}{\sinh(kh)} * \{a_1(\sin\omega t\cos kp - \cos\omega t \sin kp)\cos\varphi_1 + a_2(\sin\omega t\cos kq - \cos\omega t\sin kq) * \cos\varphi_2\}$$

$$= \frac{\omega\cosh\{k(z+h)\}}{\sinh(kh)}\{\sin\omega t(a_1\cos kp\cos\varphi_1 + a_2\cos kq\cos\varphi_2) - \cos\omega t(a_1\sin kp\cos\varphi_1 + a_2\sin kq\cos\varphi_2)\}$$

[10] $$= \frac{\omega\cosh\{k(z+h)\}}{\sinh(kh)}\{C\sin\omega t - D\cos\omega t\}$$

in which $C = a_1 \cos kp \cos\varphi_1 + a_2 \cos kq \cos\varphi_2$ [10A]
$D = a_1 \sin kp \cos\varphi_1 + a_2 \sin kp \cos\varphi_2$ [10B]

The vertical velocities

[11] $\vec{W} = \vec{W_1} + \vec{W_2} = W_1 * \vec{e_z} + W_2 * \vec{e_z} = W * \vec{e_z}$

[12] $$W_1 = \frac{a_1 \omega}{\sinh(kh)} \sinh\{k(z+h)\} \cos(\omega t - kp)$$

[13] $$W_2 = \frac{a_2 \omega}{\sinh(kh)} \sinh\{k(z+h)\} \cos(\omega t - kq)$$

$$W = \frac{\omega \sinh\{k(z+h)\}}{\sinh(kh)} \{a_1(\cos\omega t \cos kp + \sin\omega t \sin k) + a_2(\cos\omega t \cos kq + \sin\omega t \sin kq)\}$$

$$= \frac{\omega \sinh\{k(z+h)\}}{\sinh(kh)} \{\cos\omega t(a_1 \cos kp + a_2 \cos kq) + \sin\omega t(a_1 \sin kp + a_2 \sin kq)\}$$

[14] $$= \frac{\omega \sinh\{k(z+h)\}}{\sinh(kh)} \{A \cos\omega t + B \sin\omega t\}$$ (using [4A] and [4B])

The total flux of horizontal momentum in the direction of the x-axis

[15] $$P_1 = \int_{-h}^{\eta} (p + \rho u^2)^2 \, dz$$

The radiation-stress in x-direction will be

[16] $$S_{xx} = \overline{(P_1 - P_0)}$$ (the bar indicates a time-average)

S_{xx} is seperated in three parts:

[17] $S_{xx} = S_{xx}^{(1)} + S_{xx}^{(2)} + S_{xx}^{(3)}$
[18] $S_{xx}^{(1)} = \int_{-h}^{\eta} \rho u_x^2 \, dz$
[19] $S_{xx}^{(2)} = \int_{-h}^{0} (P - P_0) \, dz$
[20] $S_{xx}^{(3)} = \int_{0}^{\eta} p \, dz$

After some mathematics (cf. eq 58-61 in annex) and neglecting 3rd order terms it appears that:

$$S_{xx}^{(1)} + S_{xx}^{(2)} = \int_{-h}^{0} \overline{(\rho u_x^2 - \rho w^2)} \, dz$$

$$= \rho \int_{-h}^{0} \frac{\omega^2 \cosh^2\{k(z+h)\}}{\sinh^2(kh)} \overline{\left[\{C^2 \sin^2\omega t + D^2 \cos^2\omega t - 2CD \sin\omega t \cos\omega t\} - \{A^2 \cos^2\omega t + B^2 \sin^2\omega t - 2AB \sin\omega t \cos\omega t\}\right]} dz$$

$$= \frac{\omega^2 \rho}{\sinh^2(kh)} \int_{-h}^{0} \left\{\tfrac{1}{2}(C^2+D^2)\cosh^2\{k(z+h)\} - \tfrac{1}{2}(A^2+B^2)\sinh^2\{k(z+h)\}\right\} dz$$

$$= \frac{\omega^2 \rho}{\sinh^2(kh)} \left\{\tfrac{1}{2}(C^2+D^2) \int_{-h}^{0} \cosh^2\{k(z+h)\} dz - \tfrac{1}{2}(A^2+B^2) \int_{-h}^{0} \sinh^2\{k(z+h)\} dz\right\}$$

[21] $$= \frac{\omega^2 \rho}{\sinh^2(kh)} \left\{\tfrac{1}{2}(C^2+D^2)\left(\frac{\sinh(2kh)}{4k} + \tfrac{1}{2}h\right) - \tfrac{1}{2}(A^2+B^2)\left(\frac{\sinh(2kh)}{4k} - \tfrac{1}{2}h\right)\right\}$$

[22] $\omega^2 = gk \tanh(kh)$

$$C^2 + D^2 = a_1^2 \cos^2 kp \cos^2\varphi_1 + a_2^2 \cos^2 kq \cos^2\varphi_2 + 2a_1 a_2 \cos kp \cos kq \cos\varphi_1 \cos\varphi_2 + a_1^2 \sin^2 kp \cos^2\varphi_1 + a_2^2 \sin^2 kq \cos^2\varphi_2 + 2a_1 a_2 \sin kp \sin kq \cos\varphi_1 \cos\varphi_2$$

[23] $$= a_1^2 \cos^2\varphi_1 + a_2^2 \cos^2\varphi_2 + 2a_1 a_2 \cos\{k(p-q)\} \cos\varphi_1 \cos\varphi_2$$

$$A^2 + B^2 = a_1^2 \cos^2 kp + a_2^2 \cos^2 kq + 2a_1 a_2 \cos kp \cos kq + a_1^2 \sin^2 kp + a_2^2 \sin^2 kq + 2a_1 a_2 \sin kp \sin kq$$

[24] $$= a_1^2 + a_2^2 + 2a_1 a_2 \cos\{k(p-q)\}$$

Using 22, 23 and 24 follows

$$S_{xx}^{(1)} + S_{xx}^{(1)} = \frac{\rho g k \tanh(kh)}{\sinh^2(kh)} \frac{1}{2}\left[\frac{\sinh(2kh)}{4k}\left\{(C^2+D^2)-(A^2+B^2)\right\} + \frac{1}{2}h\left\{(C^2+D^2)+(A^2+B^2)\right\}\right]$$

$$= \frac{\rho g k}{\sinh(2kh)}\left[\frac{\sinh(2kh)}{4k}\left\{-a_1^2\sin^2\varphi_1 - a_2^2\sin^2\varphi_2 + 2a_1a_2\cos\{k(p-q)\}(\cos\varphi_1\cos\varphi_2 - 1)\right\}\right.$$

[25] $$\left. + \frac{1}{2}h\left(a_1^2(1+\cos^2\varphi_1) + a_2^2(1+\cos^2\varphi_2) + 2a_1a_2\cos\{k(p-q)\}(\cos\varphi_1\cos\varphi_2+1)\right\}\right]$$

$$S_{xx}^{(3)} = \overline{\int_0^{\eta} p\,dz} \qquad p = \text{nearly equal to } \rho g(\eta - z) \qquad [10]$$

[26] $$= \rho g \int_0^{\eta} \overline{(\eta - z)}\,dz = \rho g \overline{\left(\eta z - \tfrac{1}{2}z^2\right)}\Big|_0^{\eta} = \tfrac{1}{2}\rho g \overline{\eta^2}$$

Using eq [4] yields:

$$S_{xx}^{(3)} = \tfrac{1}{2}\rho g \frac{1}{T}\int_0^{T}(A\sin\omega t - B\cos\omega t)\,dt$$

$$= \tfrac{1}{2}\rho g \frac{1}{T}\int_0^{T}(A^2\sin^2\omega t + B^2\cos\omega t - 2AB\sin\omega t\cos\omega t)\,dt$$

[27] $$= \tfrac{1}{2}\rho g \frac{1}{T}\left\{\tfrac{1}{2}A^2T + \tfrac{1}{2}B^2T\right\} = \tfrac{1}{4}\rho g(A^2+B^2)$$

[27] and [24] give:

[28] $$S_{xx}^{(3)} = \tfrac{1}{4}\rho g\left(a_1^2 + a_2^2 + 2a_1a_2\cos\{k(p-q)\}\right)$$

Using eq [17], eq [25] and eq [28] S_{xx} becomes

For set-up calculations it is necessary to determine $\frac{dS_{xx}}{dx}$. To simplify the equations a_2 is written as

[33] $$a_2 = \alpha a_1$$

Than eq. 29 becomes:

[34] $$S_{xx} = \frac{1}{4}\rho g a_1^2 \Big[\left(\cos^2\varphi_1 + \alpha^2 \cos\varphi_2^2 + 2\alpha\cos\{k(p-q)\}\cos\varphi_1\cos\varphi_2\right) + \frac{2kh}{\sinh(2kh)}\left((\cos^2\varphi_1 + 1) + \alpha^2(\cos^2\varphi_2 + 1) + 2\alpha\cos\{k(p-q)\}(\cos\varphi_1\cos\varphi_2 + 1)\right)\Big]$$

or

[35] $$S_{xx} = \frac{1}{4}\rho g a_1^2\left[\Phi + \Phi + 2\alpha\cos\{k(p-q)\} + 1 + \alpha^2\right]$$
$$= \frac{1}{2}\rho g a_1^2\left[\Phi + \alpha\cos\{k(p-q)\} + \frac{1}{2}(1+\alpha^2)\right]$$

in which $\Phi = \cos^2\varphi_1 + \alpha^2\cos^2\varphi_2 + 2\alpha\cos\{k(p-q)\}\cos\varphi_1\cos\varphi_2$

and $\frac{2kh}{\sinh(2kh)} \approx 1$ [shallow water assumption]

In cross-swell the wave-height is defined as

[36] $$H = 2(a_1 + a_2) \Rightarrow H = 2(a_1 + \alpha a_1) = 2a_1(1+\alpha)$$

thus:

[37] $$H^2 = 4a_1^2(1+\alpha)^2 \Rightarrow a_1^2 = \frac{1}{4(1+\alpha)^2}H^2$$

Combining 35 and 37 yields:

[38] $$S_{xx} = \frac{1}{8}\frac{\rho g}{(1+\alpha)^2}H^2\left[\Phi + \alpha\cos\{k(p-q)\} + \frac{1}{2}(1+\alpha^2)\right]$$

In the region in which the set-up has to be calculated, the wave is broken. In this region a linear relationship between H and the depth d is assumed

[39] $$H = \gamma d$$

Thus

[40] $$S_{xx} = \frac{1}{8}\frac{\rho g\gamma^2}{(1+\alpha)^2}\left[\Phi d^2 + \alpha\cos\{k(p-q)\}d^2 + \frac{1}{2}d^2(1+\alpha^2)\right]$$

[41] $$\frac{dS_{xx}}{dx} = \frac{1}{8}\frac{\rho g\gamma^2}{(1+\alpha)^2}\left[d^2\frac{d\Phi}{dx} + 2\Phi d\frac{dd}{dx} + 2\alpha d\cos\{k(p-q)\}\frac{dd}{dx} - d^2\sin\{k(p-q)\}\frac{-k}{2d}\frac{dd}{dx} + d(1+\alpha^2)\frac{dd}{dx}\right]$$

because $\frac{dk}{dd} = \frac{-k}{2d}\frac{dd}{dx}$

[42] $$\frac{dS_{xx}}{dx} = \frac{1}{8}\frac{\rho g\gamma^2}{(1+\alpha)^2}\Big[-d^2\alpha(p-q)\sin\{k(p-q)\}\cos\varphi_1\cos\varphi_2\frac{k}{2d}\frac{dd}{dx} + 2\Phi d\frac{dd}{dx} + 2\alpha d\cos\{k(p-q)\}\frac{dd}{dx} + d\sin\{k(p-q)\}\frac{k}{2}\frac{dd}{dx} + d(1+\alpha^2)\frac{dd}{dx}\Big]$$
$$= \frac{1}{8}\frac{\rho g\gamma^2}{(1+\alpha)^2}d\left[-k\alpha(p-q)\sin\{k(p-q)\}\cos\varphi_1\cos\varphi_2 + 2\Phi + 2\alpha\cos\{k(p-q)\} + \frac{k}{2}\sin\{k(p-q)\} + (1+\alpha^2)\right]\frac{dd}{dx}$$
$$= \frac{1}{8}\frac{\rho g\gamma^2}{(1+\alpha)^2}Qd\frac{dd}{dx}$$

in which was substituded

[43] $$\frac{d\Phi}{dx} = -\alpha(p-q)\sin\{k(p-q)\}\cos\varphi_1\cos\varphi_2\frac{k}{d}\frac{dd}{dx}$$

[44] $$k = \frac{2\pi}{L} = \frac{2\pi}{\sqrt{gd}} \qquad \frac{dk}{dx} = \frac{-k}{2d}\frac{dd}{dx}$$

[45] $$Q = Q[d, \alpha, (p-q), \varphi_1, \varphi_2]$$

The maximum value of Q_{Ξ} is reached for $(p-q)=0$, $\varphi_1=\varphi_2=0$ $(\alpha=1)$
Then

$$\Phi = 1+1+2\cos\{k(p-1\} = 4$$

and $Q = 0+8+2+0+2 = 12$

So $$\frac{dS_{xx}}{dx} = \frac{1}{8}\frac{\rho g\gamma^2}{2^2}\,12\,d\frac{dd}{dx} = \frac{3}{8}\rho g\gamma^2 d\frac{dd}{dx}$$

[46] $$= \left(\frac{3}{8}\rho g\gamma H_{br}\right)\frac{dd}{dx}$$

Formula [46] is equal to the normal used formula for $\frac{dS_{xx}}{dx}$ (cf Battjes 1977 formula 36)

The general set-up equation is

[47] $$\frac{dS_{xx}}{dx} + \rho g(h+\bar{\eta})\frac{d\bar{\eta}}{dx} = 0$$ [cf Battjes 1977, form 28]

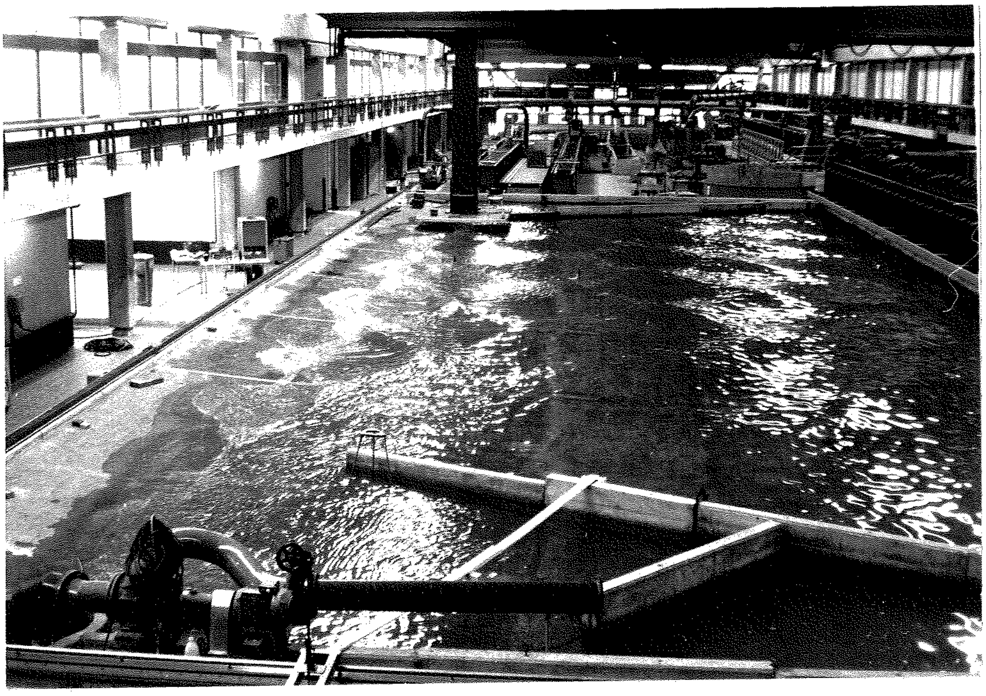
Substituating 42 in 47 yields:

$$\frac{1}{8}\frac{\rho g\gamma^2}{(1+\alpha)^2}Q\,d\frac{dd}{dx} + \rho g\,d\frac{d\bar{\eta}}{dx} = 0$$

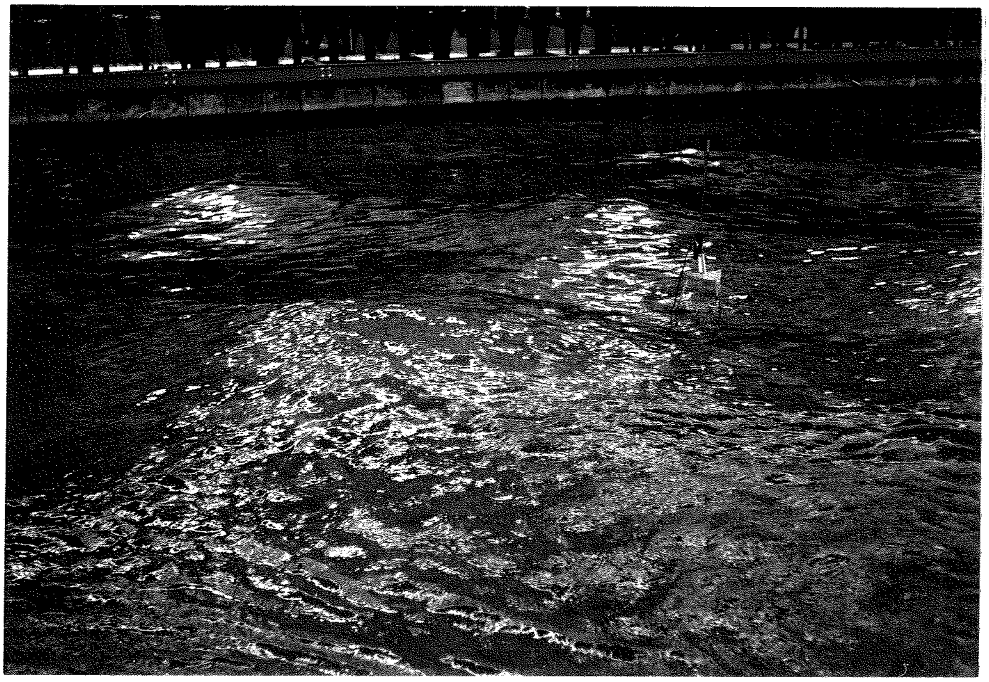
[48] $$\frac{1}{8}\left(\frac{\gamma}{1+\alpha}\right)^2 Q\frac{dd}{dx} + \frac{d\bar{\eta}}{dx} = 0$$

[49] $$\frac{d}{dx}\left(\frac{1}{8}\left(\frac{\gamma}{1+\alpha}\right)^2 Qd + \bar{\eta}\right) = 0$$

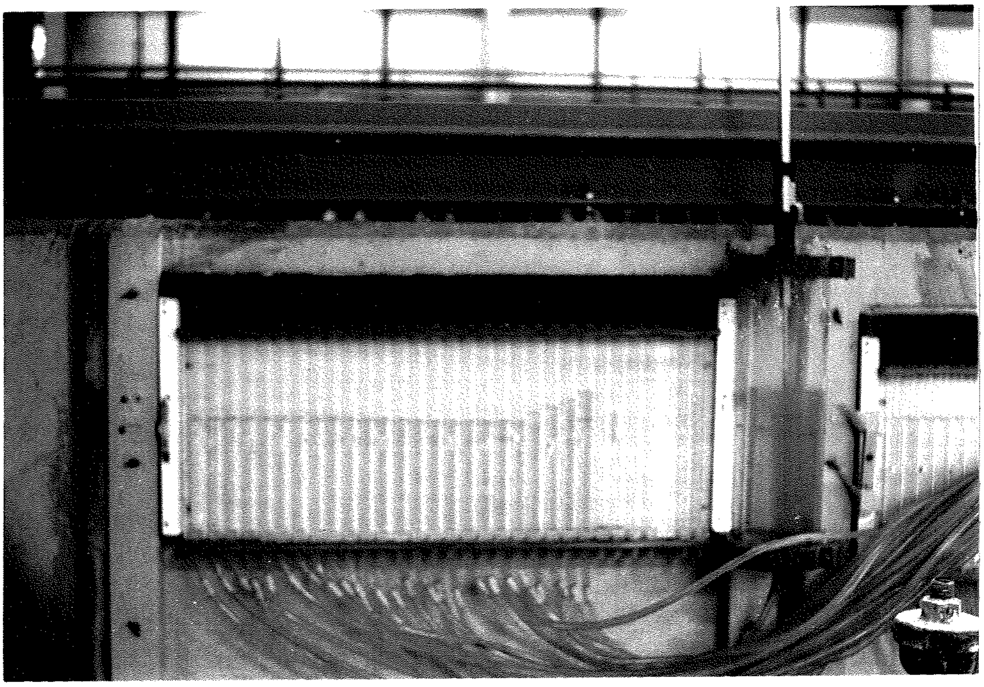
From eq 49 follows that the maximum elevation $\bar{\eta}_{max}$ is $\frac{1}{8}\left(\frac{\gamma}{1+\alpha}\right)^2 Qd$ [50]
above the set-down level



overall view of the wave-basin



cross-swell



set-up measurements