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A NEW APPROACH TO SPACE-TIME BOUNDARY INTEGRAL EQUATIONS FOR THE WAVE EQUATION*

OLAF STEINBACH[†] AND CAROLINA URZÚA-TORRES[‡]

Abstract. We present a new approach for boundary integral equations for the wave equation with zero initial conditions. Unlike previous attempts, our mathematical formulation allows us to prove that the associated boundary integral operators are continuous and satisfy inf-sup conditions in trace spaces of the same regularity, which are closely related to standard energy spaces with the expected regularity in space and time. This feature is crucial from a numerical perspective, as it provides the foundations to derive sharper error estimates and paves the way to devise efficient adaptive space-time boundary element methods, which will be tackled in future work. On the other hand, the proposed approach is compatible with current time dependent boundary element method's implementations and we predict that it explains many of the behaviours observed in practice but that were not understood with the existing theory.

Key words. Transient wave equation, retarded potentials, integral equations, coercivity

AMS subject classifications. 35L05, 45P05, 47G10

1. Introduction. Different strategies have been used to derive variational methods for time domain boundary integral equations for the wave equation. The more established and successful ones include weak formulations derived via the Laplace transform, and also space-time energetic variational formulations, often referred as *energetic BEM* in the literature. These approaches started with the groundbreaking works of Bamberger and Ha Duong [5], and Aimi et al. [4], respectively. In spite of their extensive use [2, 3, 6, 13, 14, 15, 16, 17, 18, 19, 24, 25, 26] at the time of writing this article, the numerical analysis corresponding to these formulations was still incomplete and presents difficulties that are hard to overcome, if possible at all.

One of these difficulties is the fact that current approaches provide continuity and coercivity estimates which are not in the same space-time (Sobolev) norms. Indeed, there is a so-called *norm gap* arising from a loss of regularity in time of the related boundary integral operators. Yet, recent work by Joly and Rodríguez shows that these norm gaps are not present in 1D [19]. Moreover, to the best of the authors' knowledge, there is no proof nor numerical evidence that such loss of time regularity should hold for higher dimensions either. These two observations encouraged us to believe that one may be able to prove sharper results using different mathematical tools. Another disadvantage of current strategies is that they do not provide the foundations for space-time boundary element methods, which are basically boundary element discretizations where the time variable is treated simply as another space variable, in contrast to techniques such as time-stepping methods and convolution quadrature methods [26]. Therefore, we need to establish mapping properties of the related boundary integral operators in Sobolev trace spaces in the space-time domain.

Space-time discretization methods offer an increasingly popular alternative, since they allow the treatment of moving boundaries, adaptivity in space and time simultaneously, and space-time parallelization [12, 27, 28, 30]. However, in order to exploit these advantages, one needs to have a complete stability and error analysis of the

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43 corresponding space-time Galerkin boundary element methods.

44 We construct a new approach to boundary integral equations for the wave equa-
 45 tion by working directly in the time domain. Furthermore, we develop a mathematical
 46 framework that not only overcomes the aforementioned difficulties, but also paves the
 47 way to stable space-time FEM/BEM coupling, using either the symmetric approach
 48 or Johnson-Nédélec coupling [1, 15, 25]. We present these new results following the
 49 standard pieces and arguments from classical boundary integral equations. We hope
 50 this highlights some mathematical intuitions behind the obtained results and makes
 51 the article easier to read for those familiarized with the boundary integral equation
 52 literature. In addition to a new boundary integral equation formulation, we provide
 53 novel existence and uniqueness results for the Dirichlet and Neumann wave equation
 54 initial boundary value problems, when initial conditions are zero.

55 In order to understand some of the ideas and motivations we present in this work,
 56 it is worth mentioning that classical analysis for the wave equation considers the right
 57 hand side f in $L^2(Q)$. We refer the reader to [20, 21, 22] for further details. Indeed,
 58 it is the work by Lions and Magenes the one that paves the way to classical boundary
 59 integral equation analysis for the wave equation [5], which is based on Fourier analysis
 60 techniques. In contrast, the approach we pursue in this paper follows the cue from
 61 [8]. For this, we consider f to be in a functional space that is bigger than $L^2(Q)$,
 62 which naturally makes us enlarge the ansatz space in the same fashion as in [32].

63 The structure of this article is as follows. Section 2 introduces notation and sum-
 64 marizes results from the literature that will be needed later in the paper. We begin
 65 by using some key ideas of recent work on the wave equation in $H^1(Q)$ [31, 35]. Then,
 66 in Section 3, we introduce trace spaces, trace operators and their corresponding prop-
 67 erties for three different families of spaces. With this we aim, on the one hand, to
 68 emphasize the link between the existing space-time (volumetric) variational formula-
 69 tions and our new results. On the other hand, we prove that the related trace spaces
 70 are indeed connected, which provides a new and deeper understanding of the different
 71 existing boundary integral formulations and their relation. Section 4 presents some
 72 required results on initial boundary value problems for the wave equation, while all
 73 the remaining building blocks of the new boundary integral equation formulation are
 74 presented in Section 5. This final section concludes with the existence and uniqueness
 75 results for solutions of related boundary integral equations.

76 2. Preliminaries.

77 **2.1. Model problem.** Let $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, with boundary $\Gamma := \partial\Omega$. We
 78 assume Ω to be an interval for $n = 1$, or a bounded Lipschitz domain for $n = 2, 3$.
 79 Let $0 < T < \infty$. For a finite time interval $(0, T)$, we define the *space-time cylinder*
 80 $Q := \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, and its lateral boundary $\Sigma := \Gamma \times [0, T]$. We also introduce
 81 the initial boundary $\Sigma_0 := \Omega \times \{0\}$, and the final boundary $\Sigma_T := \Omega \times \{T\}$. We
 82 denote the D'Alembert operator by $\square := \partial_{tt} - \Delta_x$, and write the *interior Dirichlet*
 83 *initial boundary value problem for the wave equation* as

$$\begin{aligned}
 \square u(x, t) &= f(x, t) && \text{for } (x, t) \in Q, \\
 u(x, t) &= g(x, t) && \text{for } (x, t) \in \Sigma, \\
 u(x, 0) = \partial_t u(x, t)|_{t=0} &= 0 && \text{for } x \in \Omega.
 \end{aligned}$$

84 (2.1)

85 **2.2. Notation and mathematical framework.** Let $\mathcal{O} \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$. We
 86 stick to the usual notation for the space $C^\infty(\mathcal{O})$ of functions which are bounded
 87 and infinitely often continuously differentiable; the subspace $C_0^\infty(\mathcal{O})$ of compactly

88 supported smooth functions; the spaces $L^p(\mathcal{O})$ of Lebesgue integrable functions; and
 89 the Sobolev spaces $H^s(\mathcal{O})$. Moreover, inner products of Hilbert spaces X are denoted
 90 by standard brackets $(\cdot, \cdot)_X$, while angular brackets $\langle \cdot, \cdot \rangle_{\mathcal{O}}$ are used for the duality
 91 pairing induced by the extension of the inner product $(\cdot, \cdot)_{L^2(\mathcal{O})}$. For a Hilbert space
 92 X we denote by X' its dual with the norm

$$93 \quad \|f\|_{X'} = \sup_{0 \neq v \in X} \frac{|\langle f, v \rangle_{\mathcal{O}}|}{\|v\|_X} \quad \text{for } f \in X'.$$

94 In particular, we will use

$$95 \quad H^1(\mathcal{O}) := \overline{C^\infty(\mathcal{O})}^{\|\cdot\|_{H^1(\mathcal{O})}}, \quad H_0^1(\mathcal{O}) := \overline{C_0^\infty(\mathcal{O})}^{\|\cdot\|_{H^1(\mathcal{O})}},$$

96 where

$$97 \quad \|\phi\|_{H^1(\mathcal{O})} := \left(\|\phi\|_{L^2(\mathcal{O})}^2 + \sum_{i=1}^m \|\partial_{x_i} \phi\|_{L^2(\mathcal{O})}^2 \right)^{1/2}.$$

98 In the specific case $\mathcal{O} = Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$ we identify $H^1(Q)$ with the Sobolev
 99 space

$$100 \quad H^{1,1}(Q) := L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$$

101 using Bochner spaces, see, e.g., [21, Sect. 1.3, Chapt. 1] and [22, Sect. 2, Chapt. 4].
 102 Furthermore, let

$$103 \quad H_{0,0}^1(0, T; L^2(\Omega)) := \left\{ v \in L^2(Q) : \partial_t v \in L^2(Q), v(x, 0) = 0 \text{ for } x \in \Omega \right\},$$

$$104 \quad H_{,0}^1(0, T; L^2(\Omega)) := \left\{ v \in L^2(Q) : \partial_t v \in L^2(Q), v(x, T) = 0 \text{ for } x \in \Omega \right\}.$$

105 With this we introduce

$$106 \quad H_{,0}^{1,1}(Q) := L^2(0, T; H^1(\Omega)) \cap H_{,0}^1(0, T; L^2(\Omega)),$$

$$107 \quad H_{0,0}^{1,1}(Q) := L^2(0, T; H^1(\Omega)) \cap H_{0,0}^1(0, T; L^2(\Omega)),$$

108 with norms

$$109 \quad \|u\|_{H_{,0}^{1,1}(Q)} := \sqrt{\|\partial_t u\|_{L^2(Q)}^2 + \|\nabla_x u\|_{L^2(Q)}^2},$$

$$110 \quad \|v\|_{H_{0,0}^{1,1}(Q)} := \sqrt{\|\partial_t v\|_{L^2(Q)}^2 + \|\nabla_x v\|_{L^2(Q)}^2}.$$

111 Note that the space $H_{,0}^{1,1}(Q)$ corresponds to ${}_0H^1(Q)$ as used in [17, 21, 22]. In the case
 112 of zero Dirichlet boundary data along the lateral boundary Σ we define the subspaces

$$113 \quad H_{0,0}^{1,1}(Q) := L^2(0, T; H_0^1(\Omega)) \cap H_0^1(0, T; L^2(\Omega)),$$

$$114 \quad H_{,0}^{1,1}(Q) := L^2(0, T; H_0^1(\Omega)) \cap H_{,0}^1(0, T; L^2(\Omega)).$$

115 We remark that $H_{,0}^{1,1}(Q)$ and $H_{0,0}^{1,1}(Q)$ prescribe zero initial values at $t = 0$, while
 116 $H_{,0}^{1,1}(Q)$ and $H_{0,0}^{1,1}(Q)$ have zero final values at $t = T$.

117 In this paper we will consider, as in [32], a generalized variational formulation
 118 to describe solutions of the wave equation (2.1) also for $f \in [H_{0,0}^{1,1}(Q)]'$, instead of
 119 $f \in L^2(Q)$, as usually considered, e.g., [20]. Therefore we introduce the *extended*

120 space-time cylinder $Q_- := \Omega \times (-T, T)$. For $u \in L^2(Q)$, we define $\tilde{u} \in L^2(Q_-)$ as
 121 zero extension,

$$122 \quad \tilde{u}(x, t) := \begin{cases} u(x, t) & \text{for } (x, t) \in Q, \\ 0 & \text{for } (x, t) \in Q_- \setminus Q. \end{cases}$$

123 The application of the wave operator \square to $\tilde{u} \in L^2(Q_-)$ is defined as a distribution on
 124 Q_- , i.e., for all test functions $\varphi \in C_0^\infty(Q_-)$, we define

$$125 \quad \langle \square \tilde{u}, \varphi \rangle_{Q_-} := \int_{-T}^T \int_{\Omega} \tilde{u}(x, t) \square \varphi(x, t) dx dt = \int_0^T \int_{\Omega} u(x, t) \square \varphi(x, t) dx dt.$$

126 This motivates to consider the Sobolev space $H_0^1(Q_-)$ with the norm

$$127 \quad \|\phi\|_{H_0^1(Q_-)} = \sqrt{\|\partial_t \phi\|_{L^2(Q_-)}^2 + \|\nabla_x \phi\|_{L^2(Q_-)}^2} \quad \text{for } \phi \in H_0^1(Q_-),$$

128 the dual space $[H_0^1(Q_-)]'$, and the duality pairing $\langle \cdot, \cdot \rangle_{Q_-}$ as extension of the in-
 129 ner product in $L^2(Q_-)$. We also introduce the restriction operator $\mathcal{R} : H_0^1(Q_-) \rightarrow$
 130 $H_{0;0}^{1,1}(Q)$, i.e., $\mathcal{R}\phi := \phi|_Q$, and its adjoint $\mathcal{R}' : [H_{0;0}^{1,1}(Q)]' \rightarrow [H_0^1(Q_-)]'$. Moreover,
 131 let $\mathcal{E} : H_{0;0}^{1,1}(Q) \rightarrow H_0^1(Q_-)$ be any continuous and injective extension operator with
 132 norm

$$133 \quad \|\mathcal{E}\|_{H_{0;0}^{1,1}(Q), H_0^1(Q_-)} := \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{\|\mathcal{E}v\|_{H_0^1(Q_-)}}{\|v\|_{H_{0;0}^{1,1}(Q)}},$$

134 and its adjoint $\mathcal{E}' : [H_0^1(Q_-)]' \rightarrow [H_{0;0}^{1,1}(Q)]'$, satisfying $\mathcal{R}\mathcal{E}\phi = \phi$ for all $\phi \in H_{0;0}^{1,1}(Q)$.

135 In order to consider $f \in [H_0^1(Q_-)]'$, we need a solution space that is bigger than
 136 $H_{0;0}^{1,1}(Q)$. For this reason, we introduce the Banach space [32]

$$137 \quad \mathcal{H}(Q) := \left\{ u = \tilde{u}|_Q : \tilde{u} \in L^2(Q_-), \tilde{u}|_{\Omega \times (-T, 0)} = 0, \square \tilde{u} \in [H_0^1(Q_-)]' \right\},$$

138 with the norm $\|u\|_{\mathcal{H}(Q)} := \sqrt{\|u\|_{L^2(Q)}^2 + \|\square \tilde{u}\|_{[H_0^1(Q_-)]'}^2}$, where

$$139 \quad \|\square \tilde{u}\|_{[H_0^1(Q_-)]'} = \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|\langle \square \tilde{u}, \mathcal{E}v \rangle_{Q_-}|}{\|v\|_{H_{0;0}^{1,1}(Q)}}.$$

140 By completion, we finally define the Hilbert spaces

$$141 \quad \mathcal{H}_{0;0}(Q) := \overline{H_{0;0}^{1,1}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}} \subset \mathcal{H}_{;0}(Q) := \overline{H_{;0}^{1,1}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}} \subset \mathcal{H}(Q),$$

142 e.g.,

$$143 \quad \mathcal{H}_{;0}(Q) = \left\{ u \in \mathcal{H}(Q) : \exists (u_n)_{n \in \mathbb{N}} \subset H_{;0}^{1,1}(Q) \text{ with } \lim_{n \rightarrow \infty} \|u - u_n\|_{\mathcal{H}(Q)} = 0 \right\}.$$

144 Note that $H_{0;0}^{1,1}(Q) \subset \mathcal{H}_{0;0}(Q)$ and $H_{;0}^{1,1}(Q) \subset \mathcal{H}_{;0}(Q)$, see [32, Lemma 3.5] for the
 145 first inclusion.

146 **2.3. Transformation operator \mathcal{H}_T .** For $u \in L^2(0, T)$ we consider the Fourier
147 series

$$148 \quad u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right), \quad u_k = \frac{2}{T} \int_0^T u(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) dt.$$

149
150 As in [31] we introduce the transformation operator \mathcal{H}_T as

$$151 \quad \mathcal{H}_T u(t) := \sum_{k=0}^{\infty} u_k \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right),$$

152
153 and it's inverse, i.e., for $v \in L^2(0, T)$,

$$154 \quad \mathcal{H}_T^{-1} v(t) := \sum_{k=0}^{\infty} \bar{v}_k \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right), \quad \bar{v}_k = \frac{2}{T} \int_0^T v(t) \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) dt.$$

155
156 By construction we have $\mathcal{H}_T : H_{0,0}^1(0, T) \rightarrow H_{0,0}^1(0, T)$, and $\mathcal{H}_T^{-1} : H_{0,0}^1(0, T) \rightarrow$
157 $H_{0,0}^1(0, T)$. In the following, we summarize some additional properties fulfilled by
158 the operators \mathcal{H}_T and \mathcal{H}_T^{-1} , see [31, 35].

159 PROPOSITION 2.1.

160 1. For any $u, v \in L^2(0, T)$

$$161 \quad \langle \mathcal{H}_T u, v \rangle_{L^2(0, T)} = \langle u, \mathcal{H}_T^{-1} v \rangle_{L^2(0, T)}.$$

162
163 2. For all $u \in H_{0,0}^1(0, T)$

$$164 \quad \partial_t \mathcal{H}_T u = -\mathcal{H}_T^{-1} \partial_t u.$$

165
166 3. \mathcal{H}_T and \mathcal{H}_T^{-1} are norm preserving, i.e.,

$$167 \quad \|\mathcal{H}_T w\|_{L^2(0, T)} = \|w\|_{L^2(0, T)}, \quad \|\mathcal{H}_T^{-1} w\|_{L^2(0, T)} = \|w\|_{L^2(0, T)} \quad \forall w \in L^2(0, T).$$

168
169 4. For all $w \in L^2(0, T)$

$$170 \quad \langle w, \mathcal{H}_T w \rangle_{L^2(0, T)} \geq 0.$$

171 We conclude this subsection by extending the modified Hilbert transformation \mathcal{H}_T to
172 our functional spaces. For $u \in L^2(Q)$ we first have the decomposition

$$173 \quad u(x, t) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} u_{i,k} \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) \varphi_i(x),$$

$$174 \quad u_{i,k} = \frac{2}{T} \int_Q u(x, t) \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) dt \varphi_i(x) dx,$$

175
176 where φ_i are the Neumann eigenfunctions of the Laplacian, i.e.,

$$177 \quad -\Delta \varphi_i = \lambda_i \varphi_i \text{ in } \Omega, \quad \partial_{n_x} \varphi_i = 0 \text{ on } \Gamma, \quad \|\varphi_i\|_{L^2(\Omega)} = 1, \quad 0 = \lambda_0 < \lambda_i \quad \forall i \in \mathbb{N}.$$

178
179 They are an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H^1(\Omega)$, e.g., [20,
180 Chapt. 2]. With this we define

$$181 \quad \mathcal{H}_T u(x, t) := \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} u_{i,k} \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) \varphi_i(x), \quad (x, t) \in Q,$$

182
183 with $\mathcal{H}_T : H_{:,0}^{1,1}(Q) \rightarrow H_{:,0}^{1,1}(Q)$. Analogously, $\mathcal{H}_T^{-1} : H_{:,0}^{1,1}(Q) \rightarrow H_{:,0}^{1,1}(Q)$.

185 *Remark 2.2.* The time-reversal map κ_T , defined as [9, Eq. (2.36)]

$$186 \quad (2.2) \quad \kappa_T w(x, t) := w(x, T - t) \quad \text{for } (x, t) \in Q, w \in H^1(Q),$$

187 is often used instead of the transformation operator $\mathcal{H}_T : H_{;0}^{1,1}(Q) \rightarrow H_{;0}^{1,1}(Q)$. How-
188 ever, as we will see in Section 5, the composition \mathcal{H}_T with the standard boundary
189 integral operators turns out to be coercive and bounded in the same Sobolev trace
190 spaces.

191 **2.4. Fundamental solution and retarded potentials.** Let us briefly present
192 the *boundary layer potentials* for the wave equation, often called *retarded potentials*.
193 We refer the reader to [10] and [17] for further details. First, we introduce the funda-
194 mental solution of the wave equation,

$$195 \quad (2.3) \quad G(x, t) = \begin{cases} \frac{1}{2} \mathbf{H}(t - |x|), & n = 1, \\ \frac{1}{2\pi} \frac{\mathbf{H}(t - |x|)}{\sqrt{t^2 - |x|^2}}, & n = 2, \\ \frac{1}{4\pi} \frac{\delta(t - |x|)}{|x|}, & n = 3, \end{cases}$$

196 with δ the Dirac distribution, and \mathbf{H} the Heaviside step function. Let \mathcal{S} be the *single*
197 *layer potential* and \mathcal{D} the *double layer potential*, i.e., for $(x, t) \in Q$ and regular enough
198 densities w and z , respectively,
199

$$200 \quad (2.4) \quad (\mathcal{S}w)(x, t) := \int_0^t \int_{\Gamma} G(x - y, t - \tau) w(y, \tau) ds_y d\tau,$$

$$201 \quad (2.5) \quad (\mathcal{D}z)(x, t) := \int_0^t \int_{\Gamma} \partial_{n_y} G(x - y, t - \tau) z(y, \tau) ds_y d\tau.$$

203 In the following, we will make use of the fact that \mathcal{S} and \mathcal{D} can also be interpreted
204 as distributional pairings. Concretely, for $n = 3$, these layer potentials are

$$205 \quad (2.6) \quad (\mathcal{S}w)(x, t) := \frac{1}{4\pi} \int_{\Gamma} \frac{w(y, t - |x - y|)}{|x - y|} ds_y,$$

$$206 \quad (2.7) \quad (\mathcal{D}z)(x, t) := \frac{1}{4\pi} \int_{\Gamma} \left[\partial_{n_y} \frac{z(y, t - |x - y|)}{|x - y|} - \frac{\partial_{n_y} |x - y|}{|x - y|} \partial_t z(y, t - |x - y|) \right] ds_y.$$

208 The fact that the time argument is the retarded time $\tau = t - |x - y|$ motivates that
209 \mathcal{S} and \mathcal{D} are usually called retarded potentials.

210 **3. Green's Formula, Trace Spaces and Trace Operators.** We introduce
211 the lateral interior trace operator $\gamma_{\Sigma}^i : u \mapsto u|_{\Sigma}$ as continuous extension of the trace
212 map defined in the pointwise sense for smooth functions. As in [23, Lemma 4.1] we
213 can write a space-time Green's formula for $\varphi \in C^2(Q)$ and $\psi \in C^1(Q)$ as

$$214 \quad \Phi(\varphi, \psi) = \int_0^T \int_{\Omega} \square \varphi \psi dx dt + \int_0^T \int_{\Gamma} \partial_{n_x} \varphi \gamma_{\Sigma}^i \psi ds_x dt - \int_{\Omega} \left[\partial_t \varphi \psi \right]_{t=0}^T dx,$$

215 where

$$216 \quad (3.1) \quad \Phi(\varphi, \psi) := - \int_0^T \int_{\Omega} \partial_t \varphi \partial_t \psi \, dx \, dt + \int_0^T \int_{\Omega} \nabla_x \varphi \cdot \nabla_x \psi \, dx \, dt.$$

217 In particular, for $\varphi \in C^2(Q)$ with $\partial_t \varphi(x, t)|_{t=0} = 0$ for $x \in \Omega$ and for $\psi \in C^1(Q)$ with
218 $\psi(x, T) = 0$ for $x \in \Omega$, this gives Green's first formula

$$219 \quad (3.2) \quad \Phi(\varphi, \psi) = \int_0^T \int_{\Omega} \square \varphi \psi \, dx \, dt + \int_0^T \int_{\Gamma} \partial_{n_x} \varphi \gamma_{\Sigma}^i \psi \, ds_x \, dt.$$

220 **3.1. Traces on $H_{;0}^{1,1}(Q)$, $H_{;0}^{1,1}(Q)$, and $\mathcal{H}_{;0}(Q)$.** Following [22, Theorem 2.1,
221 Chapt. 4 and p. 19] we get that the interior trace map γ_{Σ}^i is continuous and surjective
222 from $H^1(Q)$ to $H^{1/2}(\Sigma)$. In addition, let $\mathcal{E}_{\Sigma} : H^{1/2}(\Sigma) \rightarrow H^1(Q)$ be a continuous
223 right inverse.

224 Let us introduce the spaces

$$225 \quad H_{0,}^{1/2}(\Sigma) := L^2(0, T; H^{1/2}(\Gamma)) \cap H_{0,}^{1/2}(0, T; L^2(\Gamma)),$$

$$226 \quad H_{0,}^{1/2}(\Sigma) := L^2(0, T; H^{1/2}(\Gamma)) \cap H_{0,}^{1/2}(0, T; L^2(\Gamma)),$$

228 with $H_{0,}^{1/2}(0, T; L^2(\Gamma))$ and $H_{;0}^{1/2}(0, T; L^2(\Gamma))$ defined by complex interpolation as

$$229 \quad H_{0,}^{1/2}(0, T; L^2(\Gamma)) := [H_{0,}^1(0, T; L^2(\Gamma)), L^2(0, T; L^2(\Gamma))]_{1/2},$$

$$230 \quad H_{;0}^{1/2}(0, T; L^2(\Gamma)) := [H_{;0}^1(0, T; L^2(\Gamma)), L^2(0, T; L^2(\Gamma))]_{1/2}.$$

232 Then, we have the following result, which is stated in [17] without a proof. Here we
233 provide one for completeness.

234 **LEMMA 3.1.** *The interior trace map γ_{Σ}^i is continuous and surjective from $H_{;0}^{1,1}(Q)$
235 to $H_{0,}^{1/2}(\Sigma)$.*

236 *Proof.* We adapt the proof of [22, Theorem 2.1, Chapt. 4] to $H_{;0}^{1,1}(Q)$ (instead of
237 $H^1(Q)$). Recall that

$$238 \quad u \in H_{;0}^{1,1}(Q) = L^2(0, T; H^1(\Omega)) \cap H_{0,}^1(0, T; L^2(\Omega)).$$

239 Without loss of generality, we can take $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$ and $\Gamma = \{x \in \mathbb{R}^n : x_n = 0\}$. Then, by using the notation $x = \{x', x_n\}$, with $x' = \{x_1, \dots, x_{n-1}\}$, we can
240 write:
241

$$242 \quad u \in H_{;0}^{1,1}(Q) \Leftrightarrow u \in L^2(\mathbb{R}_{+,x_n}; L^2(0, T; H^1(\mathbb{R}_{x'}^{n-1}))) \cap L^2(0, T; L^2(\mathbb{R}_{x'}^{n-1})),$$

$$243 \quad u \in L^2(\mathbb{R}_{+,x_n}; L^2(0, T; L^2(\mathbb{R}_{x'}^{n-1}))) \cap H_{0,}^1(0, T; L^2(\mathbb{R}_{x'}^{n-1})),$$

245 where \mathbb{R}_{+,x_n} indicates that the variable x_n is considered in the positive real line.

246 Then, we can apply Theorem 4.2 from [21, Chapt. 1] with

$$247 \quad X = L^2(0, T; H^1(\mathbb{R}_{x'}^{n-1})) \cap H_{0,}^1(0, T; L^2(\mathbb{R}_{x'}^{n-1})), \quad Y = L^2(0, T; L^2(\mathbb{R}_{x'}^{n-1})),$$

249 to get that $u(x', 0, t) \in [X, Y]_{1/2}$. Now, let us point out that Theorem 13.1 in [21,
250 Chapt. 1] gives

$$251 \quad [X, Y]_{1/2} = [L^2(0, T; H^1(\mathbb{R}_{x'}^{n-1})) \cap H_{0,}^1(0, T; L^2(\mathbb{R}_{x'}^{n-1})), Y]_{1/2}$$

$$252 \quad = [L^2(0, T; H^1(\mathbb{R}_{x'}^{n-1})), Y]_{1/2} \cap [H_{0,}^1(0, T; L^2(\mathbb{R}_{x'}^{n-1})), Y]_{1/2}.$$

254 Consequently, by interpolation we get

$$255 \quad [X, Y]_{1/2} = L^2(0, T; H^{1/2}(\mathbb{R}_{x'}^{n-1})) \cap H_0^{1/2}(0, T; L^2(\mathbb{R}_{x'}^{n-1})),$$

257 which corresponds to $H_0^{1/2}(\mathbb{R}_{x'}^{n-1} \times [0, T])$. Hence, we conclude that $\gamma_\Sigma^i u \in H_0^{1/2}(\Sigma)$.
 258 Surjectivity also follows from Theorem 4.2 in [21, Chapt. 1]. \square

259 By similar arguments, one can also prove:

260 **LEMMA 3.2.** *The interior trace map γ_Σ^i is continuous and surjective from $H_{;0}^{1,1}(Q)$*
 261 *to $H_0^{1/2}(\Sigma)$.*

262 Finally, we define the lateral trace space

$$263 \quad \mathcal{H}_0(\Sigma) := \left\{ v = \gamma_\Sigma^i V \quad \text{for all } V \in \mathcal{H}_{;0}(Q) \right\}$$

264 with the norm

$$265 \quad \|v\|_{\mathcal{H}_0(\Sigma)} := \inf_{V \in \mathcal{H}_{;0}(Q): \gamma_\Sigma^i V = v} \|V\|_{\mathcal{H}(Q)}.$$

266 *Remark 3.3.* By the definition of $\mathcal{H}_0(\Sigma)$ and using the linearity of γ_Σ^i , we have
 267 that for any $v \in \mathcal{H}_0(\Sigma)$ there exists a sequence $(v_n)_{n \in \mathbb{N}} \subset H_0^{1/2}(\Sigma)$ such that
 268 $\lim_{n \rightarrow \infty} \|v - v_n\|_{\mathcal{H}_0(\Sigma)} = 0$.

269 *Remark 3.4.* The trace spaces investigated in this paper are closely related to
 270 the spaces used in the classical time dependent BEM approach for the wave equation,
 271 introduced by Bamberger and Ha-Duong [5]. Indeed, as pointed out in [17, Remark 2],
 272 $H_0^{1/2}(\Sigma)$ agrees with ¹

$$273 \quad H_{\sigma, \Gamma}^{1/2, 1/2} := \left\{ u \in LT(\sigma, H^{1/2}(\Gamma)); \int_{\mathbb{R}+i\sigma} |\hat{u}|_{1/2, \omega, \Gamma} d\omega < \infty \right\}$$

275 when $\sigma = 0$. Additionally, $\left(H_0^{1/2}(\Sigma) \right)'$ corresponds to

$$276 \quad H_{\sigma, \Gamma}^{-1/2, -1/2} := \left\{ u \in LT(\sigma, H^{-1/2}(\Gamma)); \int_{\mathbb{R}+i\sigma} |\hat{u}|_{-1/2, \omega, \Gamma} d\omega < \infty \right\}$$

278 when $\sigma = 0$. Remarkably, σ is taken to be zero for practical computations and
 279 numerical experiments, yet the analysis of classical time dependent BEM does not
 280 cover this case. We refer to [17] for the detailed definitions and a more comprehensive
 281 discussion.

282 4. Initial boundary value problems.

283 **4.1. Homogeneous Dirichlet data.** Instead of (2.1), let us first consider the
 284 Dirichlet initial boundary value problem with zero boundary conditions,

$$285 \quad (4.1) \quad \begin{aligned} \square u(x, t) &= f(x, t) && \text{for } (x, t) \in Q, \\ u(x, t) &= 0 && \text{for } (x, t) \in \Sigma, \\ u(x, 0) = \partial_t u(x, t)|_{t=0} &= 0 && \text{for } x \in \Omega. \end{aligned}$$

¹The norm $|\cdot|_{r, \omega, \Gamma}$ relates to Sobolev norms with a parameter ω .

286 A possible variational formulation of (4.1) is to find $u \in H_{0;0}^{1,1}(Q)$ such that

$$287 \quad (4.2) \quad - \int_0^T \int_{\Omega} \partial_t u \partial_t v \, dx \, dt + \int_0^T \int_{\Omega} \nabla_x u \cdot \nabla_x v \, dx \, dt = \int_0^T \int_{\Omega} f v \, dx \, dt$$

288 is satisfied for all $v \in H_{0;0}^{1,1}(Q)$. When assuming $f \in L^2(Q)$ we are able to construct
 289 a unique solution $u \in H_{0;0}^{1,1}(Q)$ of the variational formulation (4.1), satisfying the
 290 stability estimate [31, Theorem 5.1], see also [20, Chapt. IV, Theorem 3.1],

$$291 \quad \|u\|_{H_{0;0}^{1,1}(Q)} \leq \frac{1}{\sqrt{2}} T \|f\|_{L^2(Q)}.$$

292 While the variational formulation (4.2) is well posed also for $f \in [H_{0;0}^{1,1}(Q)]'$, it is
 293 not possible to prove a related inf-sup condition to ensure the existence of a unique
 294 solution $u \in H_{0;0}^{1,1}(Q)$, see [35, Theorem 4.2.24]. However, by definition we have the
 295 inf-sup condition

$$296 \quad (4.3) \quad \|\square\tilde{u}\|_{[H_0^1(Q_-)]'} = \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|\langle \square\tilde{u}, \mathcal{E}v \rangle_{Q_-}|}{\|v\|_{H_{0;0}^{1,1}(Q)}} \quad \text{for all } u \in \mathcal{H}_{0;0}(Q),$$

297 and therefore we conclude unique solvability of the variational formulation to find
 298 $u \in \mathcal{H}_{0;0}(Q)$ such that

$$299 \quad (4.4) \quad \langle \square\tilde{u}, \mathcal{E}v \rangle_{Q_-} = \langle f, v \rangle_Q$$

300 is satisfied for all $v \in H_{0;0}^{1,1}(Q)$, see [32, Theorem 3.9]. Moreover, for the solution u it
 301 holds

$$302 \quad \|\square\tilde{u}\|_{[H_0^1(Q_-)]'} = \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|\langle \square\tilde{u}, \mathcal{E}v \rangle_{Q_-}|}{\|v\|_{H_{0;0}^{1,1}(Q)}} = \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|\langle f, v \rangle_Q|}{\|v\|_{H_{0;0}^{1,1}(Q)}} \leq \|f\|_{[H_{0;0}^{1,1}(Q)]'}.$$

303 In fact, (4.4) is the variational formulation of the operator equation $\mathcal{E}'\square\tilde{u} = f$ in
 304 $[H_{0;0}^{1,1}(Q)]'$, i.e.,

$$305 \quad f_u(v) := \langle \mathcal{E}'\square\tilde{u}, v \rangle_Q = \langle \square\tilde{u}, \mathcal{E}v \rangle_{Q_-} \quad \text{for } v \in H_{0;0}^{1,1}(Q) \subset H_{0;0}^{1,1}(Q)$$

306 is a continuous linear functional with norm

$$307 \quad \|f_u\|_{[H_{0;0}^{1,1}(Q)]'} = \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|f_u(v)|}{\|v\|_{H_{0;0}^{1,1}(Q)}} = \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|\langle \square\tilde{u}, \mathcal{E}v \rangle_{Q_-}|}{\|v\|_{H_{0;0}^{1,1}(Q)}} = \|\square\tilde{u}\|_{[H_0^1(Q_-)]'}.$$

308 Recall that for $u \in H_{0;0}^{1,1}(Q) \subset \mathcal{H}_{0;0}(Q)$ we have

$$309 \quad f_u(v) = \langle \square\tilde{u}, \mathcal{E}v \rangle_{Q_-} = -\langle \partial_t u, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} \quad \text{for all } v \in H_{0;0}^{1,1}(Q).$$

310 Using the Hahn–Banach theorem, e.g., [34, Chapt. IV., Sect. 5], [7, Theorem 5.9-1],
 311 there exists a linear continuous functional $\tilde{f}_u : H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$ satisfying

$$312 \quad (4.5) \quad \tilde{f}_u(v) = f_u(v) \quad \text{for all } v \in H_{0;0}^{1,1}(Q),$$

$$313 \quad \|\tilde{f}_u\|_{[H_{0;0}^{1,1}(Q)]'} = \|f_u\|_{[H_{0;0}^{1,1}(Q)]'} = \|\square\tilde{u}\|_{[H_0^1(Q_-)]'}.$$

314

315 Indeed, for $u \in H_{0;0}^{1,1}(Q)$, we have the explicit representation

$$316 \quad (4.6) \quad \tilde{f}_u(v) := \langle \square \tilde{u}, \mathcal{E}v \rangle_{Q_-} = -\langle \partial_t u, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} \quad \forall v \in H_{;0}^{1,1}(Q).$$

317 In the following we assume $f \in [H_{;0}^{1,1}(Q)]'$, and we consider the variational formulation
318 to find $\lambda_i \in [H_{,0}^{1/2}(\Sigma)]'$ such that

$$319 \quad (4.7) \quad \langle (\gamma_\Sigma^i)' \lambda_i, v \rangle_Q = \langle \lambda_i, \gamma_\Sigma^i v \rangle_\Sigma = \tilde{f}_u(v) - \langle f, v \rangle_Q \quad \text{for all } v \in H_{;0}^{1,1}(Q).$$

320 For $v \in H_{0;0}^{1,1}(Q) \subset H_{;0}^{1,1}(Q)$, it holds

$$321 \quad \tilde{f}_u(v) - \langle f, v \rangle_Q = f_u(v) - \langle f, v \rangle_Q = \langle \square \tilde{u}, \mathcal{E}v \rangle_{Q_-} - \langle f, v \rangle_Q = 0,$$

322 i.e.,

$$323 \quad \tilde{f}_u - f \in (\ker \gamma_\Sigma^i)^0 = (H_{0;0}^{1,1}(Q))^0 := \left\{ g \in [H_{;0}^{1,1}(Q)]' : \langle g, v \rangle_Q = 0 \quad \forall v \in H_{0;0}^{1,1}(Q) \right\}.$$

324 By the closed range theorem, we obtain

$$325 \quad \tilde{f}_u - f \in \text{Im}_{[H_{,0}^{1/2}(\Sigma)]'}(\gamma_\Sigma^i)',$$

326 which ensures existence of a solution $\lambda_i \in [H_{,0}^{1/2}(\Sigma)]'$ of the variational formulation
327 (4.7). Since the norm in $[H_{,0}^{1/2}(\Sigma)]'$ is defined by duality, this immediately implies the
328 inf-sup condition

$$329 \quad \|\lambda\|_{[H_{,0}^{1/2}(\Sigma)]'} = \sup_{0 \neq v \in H_{;0}^{1,1}(Q)} \frac{|\langle \lambda, \gamma_\Sigma^i v \rangle_\Sigma|}{\|v\|_{H_{;0}^{1,1}(Q)}} \quad \text{for all } \lambda \in [H_{,0}^{1/2}(\Sigma)]',$$

330 and therefore uniqueness of $\lambda_i \in [H_{,0}^{1/2}(\Sigma)]'$. Moreover, this also gives

$$\begin{aligned} 331 \quad \|\lambda_i\|_{[H_{,0}^{1/2}(\Sigma)]'} &= \sup_{0 \neq v \in H_{;0}^{1,1}(Q)} \frac{|\langle \lambda_i, \gamma_\Sigma^i v \rangle_\Sigma|}{\|v\|_{H_{;0}^{1,1}(Q)}} \\ 332 \quad &= \sup_{0 \neq v \in H_{;0}^{1,1}(Q)} \frac{|\tilde{f}_u(v) - \langle f, v \rangle_Q|}{\|v\|_{H_{;0}^{1,1}(Q)}} \leq 2 \|f\|_{[H_{;0}^{1,1}(Q)]'}, \\ 333 \end{aligned}$$

334 where we used

$$\begin{aligned} 335 \quad \sup_{0 \neq v \in H_{;0}^{1,1}(Q)} \frac{|\tilde{f}_u(v)|}{\|v\|_{H_{;0}^{1,1}(Q)}} &= \|\tilde{f}_u\|_{[H_{;0}^{1,1}(Q)]'} \\ 336 \quad &= \|f_u\|_{[H_{;0}^{1,1}(Q)]'} = \|\square \tilde{u}\|_{[H_0^1(Q_-)]'} \leq \|f\|_{[H_{;0}^{1,1}(Q)]'}. \\ 337 \end{aligned}$$

338 We now rewrite the variational formulation (4.7) as

$$339 \quad \tilde{f}_u(v) = \langle f, v \rangle_Q + \langle \lambda_i, \gamma_\Sigma^i v \rangle_\Sigma, \quad v \in H_{;0}^{1,1}(Q).$$

340 In particular, for $u \in H_{0;0}^{1,1}(Q)$, and using (4.6), this gives

$$341 \quad -\langle \partial_t u, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} = \langle f, v \rangle_Q + \langle \lambda_i, \gamma_\Sigma^i v \rangle_\Sigma, \quad v \in H_{;0}^{1,1}(Q).$$

342 i.e.,

$$343 \quad \Phi(u, v) = \langle f, v \rangle_Q + \langle \lambda_i, \gamma_\Sigma^i v \rangle_\Sigma.$$

344 When comparing this with Green's first formula (3.2) for suitable chosen functions,
 345 we observe that λ_i corresponds to the spatial normal derivative of u . Hence, also in
 346 the general case we shall write $\gamma_N^i u := \partial_{n_x} u = \lambda_i$ and call this distribution the interior
 347 spatial normal derivative of $u \in \mathcal{H}_{0;0}(Q)$, i.e.,

$$348 \quad \gamma_N^i : \mathcal{H}_{0;0}(Q) \rightarrow [H_{0,0}^{1/2}(\Sigma)]'.$$

349 In a similar way, we also define

$$350 \quad \gamma_N^i : \mathcal{H}_{0;0}(Q) \rightarrow [H_{0,0}^{1/2}(\Sigma)]'.$$

351 For a related approach in the case of an elliptic equation, see also [23, pp. 116–117].

352 **4.2. Inhomogeneous Dirichlet data.** Next we consider the Dirichlet bound-
 353 ary value problem (2.1). For $g \in \mathcal{H}_0(\Sigma)$ there exists, by definition, an extension
 354 $u_g = \mathcal{E}_\Sigma g \in \mathcal{H}_{0;0}(Q)$, and the zero extension $\tilde{u}_g \in L^2(Q_-)$. Thus, it remains to find
 355 $u_0 := u - u_g \in \mathcal{H}_{0;0}(Q)$ satisfying

$$356 \quad \langle \square \tilde{u}_0, \mathcal{E}v \rangle_{Q_-} = \langle f, v \rangle_Q - \langle \square \tilde{u}_g, \mathcal{E}v \rangle_{Q_-} \quad \text{for all } v \in H_{0;0}^{1,1}(Q).$$

357 Note that $u_g \in \mathcal{H}_{0;0}(Q) \subset \mathcal{H}(Q)$ involves $\square \tilde{u}_g \in [H_0^1(Q_-)]'$. For the solution u_0 we
 358 obtain

$$359 \quad \|\square \tilde{u}_0\|_{[H_0^1(Q_-)]'} = \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|\langle \square \tilde{u}_0, \mathcal{E}v \rangle_{Q_-}|}{\|v\|_{H_{0;0}^{1,1}(Q)}} = \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|\langle f, v \rangle_Q - \langle \square \tilde{u}_g, \mathcal{E}v \rangle_{Q_-}|}{\|v\|_{H_{0;0}^{1,1}(Q)}} \\ 360 \quad \leq \|f\|_{[H_{0;0}^{1,1}(Q)]'} + \|\mathcal{E}\|_{H_{0;0}^{1,1}(Q), H_0^1(Q_-)} \|g\|_{\mathcal{H}_0(\Sigma)},$$

361 where we have used

$$362 \quad \|\square \tilde{u}_g\|_{[H_0^1(Q_-)]'} \leq \sqrt{\|u_g\|_{L^2(Q)}^2 + \|\square \tilde{u}_g\|_{[H_0^1(Q_-)]'}^2} = \|u_g\|_{\mathcal{H}(Q)} = \|g\|_{\mathcal{H}_0(\Sigma)}.$$

363 As before, we can determine $\lambda_i \in [H_{0,0}^{1/2}(\Sigma)]'$ as unique solution of the variational
 364 formulation (4.7), and where $\gamma_N^i u := \lambda_i$ is again the spatial normal derivative of the
 365 solution u of the Dirichlet boundary value problem (2.1), satisfying

$$366 \quad (4.8) \quad \|\lambda_i\|_{[H_{0,0}^{1/2}(\Sigma)]'} \leq 2 \|f\|_{[H_{0;0}^{1,1}(Q)]'} + \|\mathcal{E}\|_{H_{0;0}^{1,1}(Q), H_0^1(Q_-)} \|g\|_{\mathcal{H}_0(\Sigma)}.$$

367 Specially, for $f \equiv 0$, this describes the interior Dirichlet to Neumann map $g \mapsto \lambda_i =$
 368 $\gamma_N^i u$, where u is the solution of the homogeneous wave equation with zero initial data.

369 This can be written as $\lambda_i = S_i g$, where $S_i : \mathcal{H}_0(\Sigma) \rightarrow [H_{0,0}^{1/2}(\Sigma)]'$ is the so-called
 370 Steklov-Poincaré operator, and from (4.8) we immediately conclude

$$371 \quad (4.9) \quad \|S_i g\|_{[H_{0,0}^{1/2}(\Sigma)]'} \leq c_2^{S_i} \|g\|_{\mathcal{H}_0(\Sigma)} \quad \text{for all } g \in \mathcal{H}_0(\Sigma),$$

372 with $c_2^{S_i} := \|\mathcal{E}\|_{H_{0;0}^{1,1}(Q), H_0^1(Q_-)}$.

373 As before, we can write the variational formulation (4.7) as

$$374 \quad \tilde{f}_u(v) = \langle f, v \rangle_Q + \langle \lambda_i, \gamma_\Sigma^i v \rangle_\Sigma, \quad v \in H_{0;0}^{1,1}(Q).$$

375 Now, for $u \in \mathcal{H}_{;0}(Q)$ there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset H_{;0}^{1,1}(Q)$ with

$$376 \quad \lim_{n \rightarrow \infty} \|u - u_n\|_{\mathcal{H}(Q)} = 0.$$

377 Hence we can write

$$378 \quad \tilde{f}_u(v) = \lim_{n \rightarrow \infty} \tilde{f}_{u_n}(v) = \lim_{n \rightarrow \infty} \left[-\langle \partial_t u_n, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x u_n, \nabla_x v \rangle_{L^2(Q)} \right].$$

379 In particular, for $v \in H_{;0}^{1,1}(Q) \cap H^2(Q)$, we can apply integration by parts to obtain

$$380 \quad \tilde{f}_u(v) = \lim_{n \rightarrow \infty} \left[\langle u_n, \square v \rangle_{L^2(Q)} + \langle \gamma_\Sigma^i u_n, \gamma_N^i v \rangle_\Sigma - \langle u_n(T), \partial_t v(T) \rangle_{L^2(\Omega)} \right]$$

$$381 \quad = \langle u, \square v \rangle_{L^2(Q)} + \langle \gamma_\Sigma^i u, \gamma_N^i v \rangle_\Sigma - \langle u(T), \partial_t v(T) \rangle_{L^2(\Omega)}.$$

382 With this we finally obtain Green's second formula for the solution $u \in \mathcal{H}_{;0}(Q)$ of
383 (2.1) and $v \in H_{;0}^{1,1}(Q) \cap H^2(Q)$,

$$384 \quad (4.10) \quad \langle u, \square v \rangle_{L^2(Q)} + \langle \gamma_\Sigma^i u, \gamma_N^i v \rangle_\Sigma - \langle u(T), \partial_t v(T) \rangle_{L^2(\Omega)} = \langle f, v \rangle_Q + \langle \gamma_N^i u, \gamma_\Sigma^i v \rangle_\Sigma.$$

385 **4.3. The Neumann boundary value problem.** We now consider (as in (4.7))
386 the variational problem to find $u \in \mathcal{H}_{;0}(Q)$ such that

$$387 \quad (4.11) \quad \tilde{f}_u(v) = \langle f, v \rangle_Q + \langle \lambda, \gamma_\Sigma^i v \rangle_\Sigma \quad \text{for all } v \in H_{;0}^{1,1}(Q),$$

388 when $\lambda \in [H_0^{1/2}(\Sigma)]'$ is given. This is the generalized variational formulation of the
389 Neumann boundary value problem

$$390 \quad (4.12) \quad \square u = f \quad \text{in } Q, \quad \gamma_N^i u = \lambda \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0.$$

391 **LEMMA 4.1.** *For all $u \in \mathcal{H}_{;0}(Q)$ there holds the inf-sup stability condition*

$$392 \quad (4.13) \quad \frac{\sqrt{2}}{\sqrt{2+T^2}} \|u\|_{\mathcal{H}(Q)} \leq \sup_{0 \neq v \in H_{;0}^{1,1}(Q)} \frac{|\tilde{f}_u(v)|}{\|v\|_{H_{;0}^{1,1}(Q)}}.$$

393 *Proof.* Using (4.5) and the norm definition by duality, we first have

$$394 \quad \|\square \tilde{u}\|_{[H_0^1(Q_-)]'} = \|\tilde{f}_u\|_{[H_{;0}^{1,1}(Q)]'} = \sup_{0 \neq v \in H_{;0}^{1,1}(Q)} \frac{|\tilde{f}_u(v)|}{\|v\|_{H_{;0}^{1,1}(Q)}}.$$

395 Now, for $0 \neq u \in \mathcal{H}_{;0}(Q)$, there exists a non-trivial sequence $(u_n)_{n \in \mathbb{N}} \subset H_{;0}^{1,1}(Q)$,
396 $u_n \neq 0$, with

$$397 \quad \lim_{n \rightarrow \infty} \|u - u_n\|_{\mathcal{H}(Q)} = 0.$$

398 For each $u_n \in H_{;0}^{1,1}(Q)$ we can write, as in (4.6),

$$399 \quad \tilde{f}_{u_n}(v) = -\langle \partial_t u_n, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x u_n, \nabla_x v \rangle_{L^2(Q)} \quad \text{for all } v \in H_{;0}^{1,1}(Q),$$

400 and we define $w_n \in H_{;0}^{1,1}(Q)$ as the unique solution of the variational formulation

$$401 \quad -\langle \partial_t v, \partial_t w_n \rangle_{L^2(Q)} + \langle \nabla_x v, \nabla_x w_n \rangle_{L^2(Q)} = \langle u_n, v \rangle_{L^2(Q)} \quad \text{for all } v \in H_{;0}^{1,1}(Q).$$

402 This variational formulation corresponds to a Neumann boundary value problem for
 403 the wave equation with a volume source $u_n \in L^2(Q)$, and zero conditions at the
 404 terminal time $t = T$. As for the related Dirichlet problem we conclude the bound

$$405 \quad \|w_n\|_{H_{:,0}^{1,1}(Q)} \leq \frac{1}{\sqrt{2}} T \|u_n\|_{L^2(Q)}.$$

406 In particular, for the test function $v = u_n$, the variational formulation gives

$$407 \quad -\langle \partial_t u_n, \partial_t w_n \rangle_{L^2(Q)} + \langle \nabla_x u_n, \nabla_x w_n \rangle_{L^2(Q)} = \|u_n\|_{L^2(Q)}^2.$$

408 With this, we now conclude

$$409 \quad \begin{aligned} \|\tilde{f}_{u_n}\|_{[H_{:,0}^{1,1}(Q)]'} &= \sup_{0 \neq v \in H_{:,0}^{1,1}(Q)} \frac{|\tilde{f}_{u_n}(v)|}{\|v\|_{H_{:,0}^{1,1}(Q)}} \geq \frac{|\tilde{f}_{u_n}(w_n)|}{\|w_n\|_{H_{:,0}^{1,1}(Q)}} \\ 410 \quad &= \frac{|-\langle \partial_t u_n, \partial_t w_n \rangle_{L^2(Q)} + \langle \nabla_x u_n, \nabla_x w_n \rangle_{L^2(Q)}|}{\|w_n\|_{H_{:,0}^{1,1}(Q)}} = \frac{\|u_n\|_{L^2(Q)}^2}{\|w_n\|_{H_{:,0}^{1,1}(Q)}} \geq \frac{\sqrt{2}}{T} \|u_n\|_{L^2(Q)}. \end{aligned}$$

411 Completion for $n \rightarrow \infty$ now gives

$$412 \quad \|\tilde{f}_u\|_{[H_{:,0}^{1,1}(Q)]'} \geq \frac{\sqrt{2}}{T} \|u\|_{L^2(Q)}.$$

413 Hence, we can write, for some $\alpha \in (0, 1)$,

$$414 \quad \begin{aligned} \|\tilde{f}_u\|_{[H_{:,0}^{1,1}(Q)]'}^2 &= \alpha \|\tilde{f}_u\|_{[H_{:,0}^{1,1}(Q)]'}^2 + (1 - \alpha) \|\tilde{f}_u\|_{[H_{:,0}^{1,1}(Q)]'}^2 \\ 415 \quad &\geq \alpha \frac{2}{T^2} \|u\|_{L^2(Q)}^2 + (1 - \alpha) \|\square \tilde{u}\|_{[H_0^1(Q_-)]'}^2 \\ 416 \quad &= (1 - \alpha) \left(\|u\|_{L^2(Q)}^2 + \|\square \tilde{u}\|_{[H_0^1(Q_-)]'}^2 \right) = (1 - \alpha) \|u\|_{\mathcal{H}(Q)}^2, \end{aligned}$$

417 when

$$418 \quad 1 - \alpha = \alpha \frac{2}{T^2}$$

419 is satisfied, i.e.,

$$420 \quad \alpha = \frac{T^2}{2 + T^2}, \quad 1 - \alpha = \frac{2}{2 + T^2}.$$

421 This concludes the proof. \square

422 LEMMA 4.2. For all $0 \neq v \in H_{:,0}^{1,1}(Q)$, there exists a function $u_v \in \mathcal{H}_{:,0}(Q)$ such
 423 that

$$424 \quad (4.14) \quad \tilde{f}_{u_v}(v) > 0.$$

425 *Proof.* For $0 \neq v \in H_{:,0}^{1,1}(Q)$, there exists a unique solution $u_v \in H_{:,0}^{1,1}(Q) \subset$
 426 $\mathcal{H}_{:,0}(Q)$, satisfying

$$427 \quad -\langle \partial_t u_v, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u_v, \nabla_x w \rangle_{L^2(Q)} = \langle v, w \rangle_{L^2(Q)} \quad \text{for all } w \in H_{:,0}^{1,1}(Q),$$

428 and, for $w = v$, we obtain

$$429 \quad \tilde{f}_{u_v}(v) = -\langle \partial_t u_v, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x u_v, \nabla_x v \rangle_{L^2(Q)} = \|v\|_{L^2(Q)}^2 > 0. \quad \square$$

430 The inf-sup condition (4.13) and the surjectivity condition (4.14) ensure unique solv-
 431 ability of the variational formulation (4.11). Moreover, for the unique solution
 432 $u \in \mathcal{H}_{;0}(Q)$ we obtain

$$433 \frac{\sqrt{2}}{\sqrt{2+T^2}} \|u\|_{\mathcal{H}(Q)} \leq \sup_{0 \neq v \in H_{;0}^{1,1}(Q)} \frac{|\tilde{f}_u(v)|}{\|v\|_{H_{;0}^{1,1}(Q)}}$$

$$434 = \sup_{0 \neq v \in H_{;0}^{1,1}(Q)} \frac{|\langle f, v \rangle_Q + \langle \lambda, \gamma_{\Sigma}^i v \rangle_{\Sigma}|}{\|v\|_{H_{;0}^{1,1}(Q)}} \leq \|f\|_{[H_{;0}^{1,1}(Q)]'} + \|\lambda\|_{[H_{;0}^{1/2}(\Sigma)]'},$$

435 and when taking the lateral trace this gives

$$436 (4.15) \quad \|\gamma_{\Sigma}^i u\|_{\mathcal{H}_0(\Sigma)} \leq \|u\|_{\mathcal{H}(Q)} \leq \frac{1}{\sqrt{2}} \sqrt{2+T^2} \left[\|f\|_{[H_{;0}^{1,1}(Q)]'} + \|\lambda\|_{[H_{;0}^{1/2}(\Sigma)]'} \right].$$

437 In particular, for $f \equiv 0$, this defines the interior Neumann to Dirichlet map $\lambda \mapsto \gamma_{\Sigma}^i u$
 438 which can be written as $\gamma_{\Sigma}^i u = S_i^{-1} \lambda$ when using the inverse of the Steklov–Poincaré
 439 operator S_i . From (4.15) we then conclude

$$440 (4.16) \quad \|S_i^{-1} \lambda\|_{\mathcal{H}_0(\Sigma)} \leq c_2^{S_i^{-1}} \|\lambda\|_{[H_{;0}^{1/2}(\Sigma)]'}, \quad \text{for all } \lambda \in [H_{;0}^{1/2}(\Sigma)]', \quad c_2^{S_i^{-1}} := \frac{1}{\sqrt{2}} \sqrt{2+T^2}.$$

441 Now, using (4.9) and duality this gives

$$442 \|\lambda\|_{[H_{;0}^{1/2}(\Sigma)]'} = \|S_i \gamma_{\Sigma}^i u\|_{[H_{;0}^{1/2}(\Sigma)]'} \leq c_2^{S_i} \|\gamma_{\Sigma}^i u\|_{\mathcal{H}_0(\Sigma)} = c_2^{S_i} \sup_{0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'} \frac{|\langle \gamma_{\Sigma}^i u, \mu \rangle_{\Sigma}|}{\|\mu\|_{[\mathcal{H}_0(\Sigma)]'}},$$

443 i.e., the inf-sup stability condition

$$444 (4.17) \quad \frac{1}{c_2^{S_i}} \|\lambda\|_{[H_{;0}^{1/2}(\Sigma)]'} \leq \sup_{0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'} \frac{|\langle S_i^{-1} \lambda, \mu \rangle_{\Sigma}|}{\|\mu\|_{[\mathcal{H}_0(\Sigma)]'}} \quad \text{for all } \lambda \in [H_{;0}^{1/2}(\Sigma)]'.$$

445 Furthermore, using (4.16) for $g_i := \gamma_{\Sigma}^i u$ and duality we obtain

$$446 \|g_i\|_{\mathcal{H}_0(\Sigma)} = \|\gamma_{\Sigma}^i u\|_{\mathcal{H}_0(\Sigma)} = \|S_i^{-1} \lambda\|_{\mathcal{H}_0(\Sigma)}$$

$$447 \leq c_2^{S_i^{-1}} \|\lambda\|_{[H_{;0}^{1/2}(\Sigma)]'} = c_2^{S_i^{-1}} \sup_{0 \neq v \in H_{;0}^{1/2}(\Sigma)} \frac{|\langle \lambda, v \rangle_{\Sigma}|}{\|v\|_{H_{;0}^{1/2}(\Sigma)}},$$

$$448$$

449 i.e., the inf-sup condition

$$450 (4.18) \quad \frac{1}{c_2^{S_i^{-1}}} \|g_i\|_{\mathcal{H}_0(\Sigma)} \leq \sup_{0 \neq v \in H_{;0}^{1/2}(\Sigma)} \frac{|\langle S_i g_i, v \rangle_{\Sigma}|}{\|v\|_{H_{;0}^{1/2}(\Sigma)}} \quad \text{for all } g_i \in \mathcal{H}_0(\Sigma).$$

451 **4.4. Adjoint problems.** Related to the variational problem (4.11) we now con-
 452 sider the adjoint problem to find $w \in H_{;0}^{1,1}(Q)$ such that

$$453 (4.19) \quad \tilde{f}_u(w) = \langle f, u \rangle_Q + \langle g, \gamma_{\Sigma}^i u \rangle_{\Sigma}$$

454 is satisfied for all $u \in \mathcal{H}_{;0}(Q)$. For $w \in H_{;0}^{1,1}(Q)$, let $u_w \in \mathcal{H}_{;0}(Q)$ be the unique
 455 solution of the variational problem

$$456 \tilde{f}_{u_w}(v) = \langle \partial_t w, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x w, \nabla_x v \rangle_{L^2(Q)} \quad \text{for all } v \in H_{;0}^{1,1}(Q).$$

457 For $v = w$, this gives

$$458 \quad \tilde{f}_{u_w}(w) = \|w\|_{H_{:,0}^{1,1}(Q)}^2.$$

459 Moreover, using the inf-sup stability condition (4.13), we obtain

$$460 \quad \frac{\sqrt{2}}{\sqrt{2+T^2}} \|u_w\|_{\mathcal{H}(Q)} \leq \sup_{0 \neq v \in H_{:,0}^{1,1}(Q)} \frac{|\tilde{f}_{u_w}(v)|}{\|v\|_{H_{:,0}^{1,1}(Q)}} \\ 461 \quad = \sup_{0 \neq v \in H_{:,0}^{1,1}(Q)} \frac{|\langle \partial_t w, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x w, \nabla_x v \rangle_{L^2(Q)}|}{\|v\|_{H_{:,0}^{1,1}(Q)}} \leq \|w\|_{H_{:,0}^{1,1}(Q)},$$

462 and thus it follows that

$$463 \quad \tilde{f}_{u_w}(w) = \|w\|_{H_{:,0}^{1,1}(Q)}^2 \geq \frac{\sqrt{2}}{\sqrt{2+T^2}} \|u_w\|_{\mathcal{H}(Q)} \|w\|_{H_{:,0}^{1,1}(Q)}.$$

464 In other words, we have

$$465 \quad \frac{\sqrt{2}}{\sqrt{2+T^2}} \|w\|_{H_{:,0}^{1,1}(Q)} \leq \sup_{0 \neq u \in \mathcal{H}_{:,0}(Q)} \frac{|\tilde{f}_u(w)|}{\|u\|_{\mathcal{H}(Q)}} \quad \text{for all } w \in H_{:,0}^{1,1}(Q).$$

466 Since the inf-sup condition (4.13) also implies surjectivity, unique solvability of the
467 variational formulation (4.19) follows. In fact, for $f \in [\mathcal{H}_{:,0}(Q)]'$ and $g \in [\mathcal{H}_0(\Sigma)]'$ we
468 have $w \in H_{:,0}^{1,1}(Q)$ as the weak solution of the adjoint Neumann problem for the wave
469 equation

$$470 \quad (4.20) \quad \square w = f \quad \text{in } Q, \quad \gamma_N^i w = g \quad \text{on } \Sigma, \quad w = \partial_t w = 0 \quad \text{on } \Sigma_T.$$

471 5. Boundary Integral Equations.

472 **5.1. Representation formula.** Let $u \in \mathcal{H}_{:,0}(Q)$ be a solution of the generalized
473 wave equation $\mathcal{E}'\square u = f$ in $[H_{:,0}^{1,1}(Q)]'$. For $(x, t) \in Q$ and $v(y, \tau) = \kappa_t G(x - y, \tau)$,
474 with $G(\cdot, \cdot)$ being the fundamental solution introduced in (2.3) and κ_t the time-reversal
475 map from (2.2), formula (4.10) becomes the representation formula

$$476 \quad u(x, t) = \int_0^t \int_{\Omega} f(y, \tau) G(x - y, t - \tau) dy d\tau + \langle \gamma_N^i u, \gamma_{\Sigma}^i v \rangle_{\Sigma} - \langle \gamma_{\Sigma}^i u, \gamma_N^i v \rangle_{\Sigma}. \\ 477$$

478 In particular, for $f \equiv 0$, we conclude the following representation formula

$$479 \quad (5.1) \quad u(x, t) = (\mathcal{S}\gamma_N^i u)(x, t) - (\mathcal{D}\gamma_{\Sigma}^i u)(x, t), \quad (x, t) \in Q,$$

481 with the single and double layer potentials \mathcal{S} and \mathcal{D} , defined as in (2.4) and (2.5),
482 respectively.

483 **5.2. Single layer potential.** We first recall the definition (2.4) of the single
484 layer potential

$$485 \quad u_w(x, t) = (\mathcal{S}w)(x, t) = \int_0^t \int_{\Gamma} G(x - y, t - \tau) w(y, \tau) ds_y d\tau, \quad (x, t) \in Q.$$

486 PROPOSITION 5.1. *For the single layer potential we have*

$$487 \quad \mathcal{S} : [H_{:,0}^{1/2}(\Sigma)]' \rightarrow \mathcal{H}_{:,0}(Q).$$

488 *Proof.* For $u_w = \mathcal{S}w$ and a suitable ψ , we can write the duality pairing as
 489 extension of the inner product in $L^2(Q)$ as

$$\begin{aligned}
 490 \quad \langle u_w, \psi \rangle_Q &= \int_0^T \int_{\Omega} u_w(x, t) \psi(x, t) dx dt \\
 491 &= \int_0^T \int_{\Omega} \int_0^t \int_{\Gamma} G(x - y, t - \tau) w(y, \tau) ds_y d\tau \psi(x, t) dx dt \\
 492 &= \int_0^T \int_{\Gamma} w(y, \tau) \int_{\tau}^T G(x - y, t - \tau) \psi(x, t) dx dt ds_y d\tau \\
 493 &= \int_0^T \int_{\Gamma} w(y, \tau) \varphi_{\psi}(y, \tau) ds_y d\tau \\
 494 &= \langle w, \gamma_{\Sigma}^i \varphi_{\psi} \rangle_{\Sigma},
 \end{aligned}$$

495 where

$$496 \quad \varphi_{\psi}(y, \tau) = \int_{\tau}^T G(x - y, t - \tau) \psi(x, t) dx dt, \quad (y, \tau) \in Q,$$

497 is a solution of the adjoint problem (4.20). Hence, for $\psi \in [\mathcal{H}_{;0}(Q)]'$, we obtain $\varphi_{\psi} \in$
 498 $H_{;0}^{1,1}(Q)$, and therefore $\gamma_{\Sigma}^i \varphi_{\psi} \in H_{,0}^{1/2}(\Sigma)$. From this, we conclude that $u_w \in \mathcal{H}_{;0}(Q)$
 499 when $w \in [H_{,0}^{1/2}(\Sigma)]'$ is given. \square

500 As a corollary of the previous result, we can define the single layer boundary integral
 501 operator

$$502 \quad (5.2) \quad \mathbb{V} := \gamma_{\Sigma}^i \mathcal{S} : [H_{,0}^{1/2}(\Sigma)]' \rightarrow \mathcal{H}_0(\Sigma),$$

503 and the normal derivative of the single layer potential,

$$504 \quad (5.3) \quad \gamma_N^i \mathcal{S} : [H_{,0}^{1/2}(\Sigma)]' \rightarrow [H_{,0}^{1/2}(\Sigma)]'.$$

505 **5.3. Double layer potential.** We first recall the definition (2.5) of the double
 506 layer potential

$$507 \quad u_z(x, t) = (\mathcal{D}z)(x, t) = \int_0^t \int_{\Gamma} \partial_{n_y} G(x - y, t - \tau) z(y, \tau) ds_y d\tau, \quad (x, t) \in Q.$$

508 PROPOSITION 5.2. *For the double layer potential we have*

$$509 \quad \mathcal{D} : \mathcal{H}_0(\Sigma) \rightarrow \mathcal{H}_{;0}(Q).$$

510 *Proof.* The proof is analogous to that of Proposition 5.1 choosing $u_z = \mathcal{D}z$. \square

511 With the previous result we are in a position to consider the lateral trace of the double
 512 layer potential

$$513 \quad (5.4) \quad \gamma_{\Sigma}^i \mathcal{D} : \mathcal{H}_0(\Sigma) \rightarrow \mathcal{H}_0(\Sigma),$$

514 and the so-called hypersingular boundary integral operator as normal derivative of
 515 the double layer potential,

$$516 \quad (5.5) \quad \mathbb{W} := -\gamma_N^i \mathcal{D} : \mathcal{H}_0(\Sigma) \rightarrow [H_{,0}^{1/2}(\Sigma)]'.$$

517 **5.4. Boundary integral operators and Calderón identities.** Without loss
518 of generality, let us consider the complementary domains

$$519 \quad \Omega^c := B_R \setminus \overline{\Omega} \quad \text{and} \quad Q^c := \Omega^c \times (0, T),$$

520 with $B_R := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < R\}$ is a sufficiently large ball containing Γ . With this,
521 we define the exterior traces γ_Σ^e and γ_N^e following the same ideas from Subsection 3.1,
522 but using Q^c instead of Q .

523 *Remark 5.3.* Clearly, the mappings

$$524 \quad \begin{aligned} \gamma_\Sigma^e &: H_{:,0}^{1,1}(Q^c) \rightarrow H_0^{1/2}(\Sigma), & \gamma_\Sigma^e &: H_{:,0}^{1,1}(Q^c) \rightarrow H_0^{1/2}(\Sigma), \\ 525 \quad \gamma_\Sigma^e &: \mathcal{H}_{:,0}(Q^c) \rightarrow \mathcal{H}_0(\Sigma), & \gamma_\Sigma^e &: \mathcal{H}_{:,0}(Q^c) \rightarrow \mathcal{H}_0(\Sigma), \end{aligned}$$

527 are continuous and surjective, while

$$528 \quad \begin{aligned} \gamma_N^e &: H_{:,0}^{1,1}(Q^c) \rightarrow [H_0^{1/2}(\Sigma)]', & \gamma_N^e &: H_{:,0}^{1,1}(Q^c) \rightarrow [H_0^{1/2}(\Sigma)]', \\ 529 \quad \gamma_N^e &: \mathcal{H}_{:,0}(Q^c) \rightarrow [H_0^{1/2}(\Sigma)]', & \gamma_N^e &: \mathcal{H}_{:,0}(Q^c) \rightarrow [H_0^{1/2}(\Sigma)]', \end{aligned}$$

531 are continuous. Moreover, Green's formulae and other properties of the interior trace
532 operators γ_Σ^i and γ_N^i also apply to these exterior traces in their corresponding spaces.
533 Indeed, following Propositions 5.1 and 5.2, we have the continuity of the mappings

$$534 \quad \mathcal{S} : [H_0^{1/2}(\Sigma)]' \rightarrow \mathcal{H}_{:,0}(Q^c), \quad \mathcal{D} : \mathcal{H}_0(\Sigma) \rightarrow \mathcal{H}_{:,0}(Q^c).$$

535 We define the jumps across Σ by

$$536 \quad [\gamma_\Sigma u] := \gamma_\Sigma^e u - \gamma_\Sigma^i u, \quad [\gamma_N u] := \gamma_N^e u - \gamma_N^i u,$$

537 which clearly do not depend on the choice of B_R . Now we can state the following
538 result:

539 **PROPOSITION 5.4.** *The following jump relations hold for all $w \in [H_0^{1/2}(\Sigma)]'$ and*
540 *$z \in \mathcal{H}_0(\Sigma)$,*

$$541 \quad [\gamma_\Sigma \mathcal{S} w] = 0, \quad [\gamma_N \mathcal{S} w] = -w, \quad [\gamma_\Sigma \mathcal{D} z] = z, \quad [\gamma_N \mathcal{D} z] = 0.$$

542 *Proof.* The jump relations are known to hold when w and z are smooth, e.g., [11,
543 Sect. 2.2.1], and [26, Sect. 1.3]. We extend them to $(w, z) \in [H_0^{1/2}(\Sigma)]' \times \mathcal{H}_0(\Sigma)$ by
544 using that the combined trace map $(\gamma_\Sigma, \gamma_N) : u \mapsto (\gamma_\Sigma u, \gamma_N u)$ maps $C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)|_{\overline{Q}}$
545 onto a dense subspace of $[H_0^{1/2}(\Sigma)]' \times H_0^{1/2}(\Sigma)$ (cf. [8, Lemma 3.5]), and that $H_0^{1/2}(\Sigma)$
546 is dense in $\mathcal{H}_0(\Sigma)$. \square

547 We can now define the boundary integral operators as follows:

DEFINITION 5.5.

$$548 \quad \begin{aligned} \mathbb{V} w &:= \gamma_\Sigma^i \mathcal{S} w = \gamma_\Sigma^e \mathcal{S} w, \\ 549 \quad \mathbb{K} z &:= \frac{1}{2} (\gamma_\Sigma^i \mathcal{D} z + \gamma_\Sigma^e \mathcal{D} z), \\ 550 \quad \mathbb{K}' w &:= \frac{1}{2} (\gamma_N^i \mathcal{S} w + \gamma_N^e \mathcal{S} w), \\ 551 \quad \mathbb{W} z &:= -\gamma_N^i \mathcal{D} z = -\gamma_N^e \mathcal{D} z. \end{aligned}$$

553 From this definition and (5.2), (5.3), (5.4), (5.5), we obtain:

554 **THEOREM 5.6.** *The boundary integral operators introduced in Definition 5.5 are*
 555 *continuous in the following spaces:*

$$\begin{aligned}
 556 \quad & \mathbf{V} : [H_{,0}^{1/2}(\Sigma)]' \rightarrow \mathcal{H}_0(\Sigma), \\
 557 \quad & \mathbf{K} : \mathcal{H}_0(\Sigma) \rightarrow \mathcal{H}_0(\Sigma), \\
 558 \quad & \mathbf{K}' : [H_{,0}^{1/2}(\Sigma)]' \rightarrow [H_{,0}^{1/2}(\Sigma)]', \\
 559 \quad & \mathbf{W} : \mathcal{H}_0(\Sigma) \rightarrow [H_{,0}^{1/2}(\Sigma)]'.
 \end{aligned}$$

561 Next, we take traces on the representation formula (5.1) and get

$$\begin{aligned}
 562 \quad & \gamma_{\Sigma}^i u = \left(\frac{1}{2} \mathbf{I} - \mathbf{K}\right) \gamma_{\Sigma}^i u + \mathbf{V} \gamma_N^i u, \\
 563 \quad & \gamma_N^i u = \mathbf{W} \gamma_{\Sigma}^i u + \left(\frac{1}{2} \mathbf{I} + \mathbf{K}'\right) \gamma_N^i u.
 \end{aligned}$$

565 As usual, we can rewrite this as

$$566 \quad (5.6) \quad \begin{pmatrix} \gamma_{\Sigma}^i u \\ \gamma_N^i u \end{pmatrix} = \underbrace{\begin{pmatrix} \left(\frac{1}{2} \mathbf{I} - \mathbf{K}\right) & \mathbf{V} \\ \mathbf{W} & \left(\frac{1}{2} \mathbf{I} + \mathbf{K}'\right) \end{pmatrix}}_{=: \mathbf{C}_Q^i} \begin{pmatrix} \gamma_{\Sigma}^i u \\ \gamma_N^i u \end{pmatrix}$$

567 with the interior Calderon projection \mathbf{C}_Q^i .

569 Using standard arguments (see for example [26, Sect. 1.4]), we can now prove

$$570 \quad \begin{pmatrix} z \\ w \end{pmatrix} = \mathbf{C}_Q^i \begin{pmatrix} z \\ w \end{pmatrix}, \quad \forall w \in [H_{,0}^{1/2}(\Sigma)]', z \in \mathcal{H}_0(\Sigma).$$

572 Furthermore, this gives $(\mathbf{C}_Q^i)^2 = \mathbf{C}_Q^i$, from which we get

$$573 \quad \mathbf{V} \mathbf{W} = \left(\frac{1}{2} \mathbf{I} - \mathbf{K}\right) \left(\frac{1}{2} \mathbf{I} + \mathbf{K}\right), \quad \mathbf{W} \mathbf{V} = \left(\frac{1}{2} \mathbf{I} - \mathbf{K}'\right) \left(\frac{1}{2} \mathbf{I} + \mathbf{K}'\right), \quad \mathbf{V} \mathbf{K}' = \mathbf{K} \mathbf{V}, \quad \mathbf{K}' \mathbf{W} = \mathbf{W} \mathbf{K}.$$

574 **5.5. Coercivity of boundary integral operators.** In this subsection, we are
 575 going to prove coercivity properties of boundary integral operators, i.e., of the single
 576 layer boundary integral operator \mathbf{V} and the hypersingular boundary integral operator
 577 \mathbf{W} , which ensure unique solvability of related boundary integral equations.

578 **THEOREM 5.7.** *The single layer boundary integral operator $\mathbf{V} : [H_{,0}^{1/2}(\Sigma)]' \rightarrow$*
 579 *$\mathcal{H}_0(\Sigma)$ satisfies the inf-sup stability condition*

$$580 \quad (5.7) \quad c_1^V \|w\|_{[H_{,0}^{1/2}(\Sigma)]'} \leq \sup_{0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'} \frac{|\langle \mathbf{V} w, \mu \rangle_{\Sigma}|}{\|\mu\|_{[\mathcal{H}_0(\Sigma)]'}} \quad \text{for all } w \in [H_{,0}^{1/2}(\Sigma)]'.$$

581 *Proof.* For $w \in [H_{,0}^{1/2}(\Sigma)]'$ we consider the single layer potential $u = \mathcal{S}w$ which
 582 defines a solution $u \in \mathcal{H}_0(Q)$ of the homogeneous wave equation. When taking the
 583 lateral trace of u this gives $g = \gamma_{\Sigma}^i u = \mathbf{V} w \in \mathcal{H}_0(\Sigma)$. In fact, u is the unique solution
 584 of the Dirichlet boundary value problem

$$585 \quad \square u = 0 \quad \text{in } Q, \quad \gamma_{\Sigma}^i u = g \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0.$$

586 When using the interior Steklov–Poincaré operator S_i we can determine the related
587 interior Neumann trace

$$588 \quad \lambda_i = \gamma_N^i u = S_i g \in [H_0^{1/2}(\Sigma)]'.$$

589 Since the Steklov–Poincaré operator S_i is invertible, this gives $g = S_i^{-1} \lambda_i$, i.e., $g = \gamma_\Sigma^i u$
590 is the lateral trace of the solution of the Neumann boundary value problem

$$591 \quad \square u = 0 \quad \text{in } Q, \quad \gamma_N^i u = \lambda_i \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0.$$

592 From the inf-sup stability condition (4.17) of the inverse interior Steklov–Poincaré
593 operator S_i^{-1} we now conclude

$$594 \quad \frac{1}{c_2^{S_i}} \|\lambda_i\|_{[H_0^{1/2}(\Sigma)]'} \leq \sup_{0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'} \frac{|\langle S_i^{-1} \lambda_i, \mu \rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_0(\Sigma)]'}} = \sup_{0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'} \frac{|\langle \mathbf{V} w, \mu \rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_0(\Sigma)]'}}$$

595 For the exterior problem we can derive a related estimate, i.e.,

$$596 \quad \frac{1}{c_2^{S_e}} \|\lambda_e\|_{[H_0^{1/2}(\Sigma)]'} \leq \sup_{0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'} \frac{|\langle \mathbf{V} w, \mu \rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_0(\Sigma)]'}},$$

597 where λ_e is the exterior Neumann trace of the single layer potential $u = \mathcal{S}w$. Now,
598 and using the jump relation of the adjoint double layer potential, this gives

$$599 \quad \begin{aligned} \|w\|_{[H_0^{1/2}(\Sigma)]'} &= \|\lambda_i - \lambda_e\|_{[H_0^{1/2}(\Sigma)]'} \\ 600 \quad &\leq \|\lambda_i\|_{[H_0^{1/2}(\Sigma)]'} + \|\lambda_e\|_{[H_0^{1/2}(\Sigma)]'} \leq (c_2^{S_i} + c_2^{S_e}) \sup_{0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'} \frac{|\langle \mathbf{V} w, \mu \rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_0(\Sigma)]'}}, \end{aligned}$$

601 which implies the desired inf-sup condition. \square

602 While the inf-sup stability condition (5.7) ensures uniqueness of a solution of a related
603 boundary integral equation, the following result will provide solvability.

604 **LEMMA 5.8.** *For any $0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'$ there exists a $w_\mu \in [H_0^{1/2}(\Sigma)]'$ such that*

$$605 \quad \langle \mathbf{V} w_\mu, \mu \rangle_\Sigma > 0$$

606 *is satisfied.*

607 *Proof.* For given $0 \neq \mu \in [\mathcal{H}_0(\Sigma)]'$ we define the adjoint single layer potential
608 $u_\mu \in H_{;0}^{1,1}(Q)$ by

$$609 \quad u_\mu(y, \tau) = \int_\tau^T \int_\Gamma G(x - y, t - \tau) \mu(x, t) ds_x dt \quad \text{for } (y, \tau) \in Q.$$

610 For the lateral trace $\gamma_\Sigma^i u_\mu \in H_0^{1/2}(\Sigma)$ and arbitrary $w \in [H_0^{1/2}(\Sigma)]'$ we then have

$$611 \quad \begin{aligned} \langle w, \gamma_\Sigma^i u_\mu \rangle_\Sigma &= \int_0^T \int_\Gamma w(y, \tau) \int_\tau^T \int_\Gamma G(x - y, t - \tau) \mu(x, t) ds_x dt ds_y d\tau \\ 612 \quad &= \int_0^T \int_\Gamma \int_0^t \int_\Gamma G(x - y, t - \tau) w(y, \tau) ds_y d\tau \mu(x, t) ds_x dt = \langle \mathbf{V} w, \mu \rangle_\Sigma. \end{aligned}$$

613 Moreover, we compute

$$614 \quad U_\mu(x, t) := \int_0^t u_\mu(x, s) ds \quad \text{for } (x, t) \in Q,$$

615 with the lateral trace $g_\mu := \gamma_\Sigma^i U_\mu \in H_{0,0}^{1/2}(\Sigma) \subset \mathcal{H}_0(\Sigma)$. Hence, there exists a unique
616 solution $v_\mu \in \mathcal{H}_{0,0}(Q)$ of the Dirichlet problem for the wave equation,

$$617 \quad \square v_\mu = 0 \quad \text{in } Q, \quad v_\mu = g_\mu \quad \text{on } \Sigma, \quad v_\mu = \partial_t v_\mu = 0 \quad \text{on } \Sigma_0.$$

618 We then conclude

$$\begin{aligned} 619 \quad & \int_0^t \int_\Gamma \frac{\partial}{\partial n_x} v_\mu(x, s) \partial_s v_\mu(x, s) ds_x ds \\ 620 \quad &= \int_0^t \int_\Omega \left[\partial_{ss} v_\mu(x, s) \partial_s v_\mu(x, s) + \nabla_x v_\mu(x, s) \cdot \nabla_x \partial_s v_\mu(x, s) \right] dx ds \\ 621 \quad &= \frac{1}{2} \int_0^t \frac{d}{ds} \int_\Omega \left[[\partial_s v_\mu(x, s)]^2 + [\nabla_x v_\mu(x, s)]^2 \right] dx ds \\ 622 \quad &= \frac{1}{2} \|\partial_t v_\mu(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla_x v_\mu(t)\|_{L^2(\Omega)}^2 \geq 0 \quad \text{for all } t \in (0, T]. \end{aligned}$$

623 In the case

$$624 \quad \frac{1}{2} \|\partial_t v_\mu(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla_x v_\mu(t)\|_{L^2(\Omega)}^2 = 0 \quad \text{for all } t \in (0, T],$$

625 and together with the zero initial conditions, we would conclude $v_\mu \equiv 0$ in Q , which
626 then implies $g_\mu \equiv 0$ on Σ , and thus $u_\mu \equiv 0$. But this contradicts $\mu \neq 0$. Therefore we
627 have

$$628 \quad \int_0^T \int_\Gamma \frac{\partial}{\partial n_x} v_\mu(x, t) \partial_t v_\mu(x, t) ds_x dt = \frac{1}{2} \|\partial_t v_\mu(T)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla_x v_\mu(T)\|_{L^2(\Omega)}^2 > 0,$$

629 and with

$$630 \quad \partial_t U_\mu = u_\mu \quad \text{in } Q, \quad \partial_t v_\mu = \partial_t g_\mu \quad \text{on } \Sigma, \quad g_\mu = \gamma_\Sigma^i U_\mu, \quad w_\mu := \gamma_N^i v_\mu \in [H_{0,0}^{1/2}(\Sigma)]'$$

631 we finally conclude

$$632 \quad \langle \mathbb{V} w_\mu, \mu \rangle_\Sigma = \langle w_\mu, \gamma_\Sigma^i u_\mu \rangle_\Sigma = \int_0^T \int_\Gamma \frac{\partial}{\partial n_x} v_\mu(x, t) \partial_t v_\mu(x, t) ds_x dt > 0. \quad \square$$

633 The solution of the Dirichlet boundary value problem

$$634 \quad \square u = 0 \quad \text{in } Q, \quad u = g \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0$$

635 is given by the representation formula

$$636 \quad u(x, t) = (\mathcal{S} \gamma_N^i u)(x, t) - (\mathcal{D}g)(x, t) \quad \text{for } (x, t) \in Q,$$

637 where we can determine the yet unknown Neumann datum $w = \gamma_N^i u \in [H_{0,0}^{1/2}(\Sigma)]'$ as
638 the unique solution of the first kind boundary integral equation

$$639 \quad (5.8) \quad \mathbb{V} w = \left(\frac{1}{2} I + K\right)g \quad \text{on } \Sigma,$$

640 i.e., of the variational formulation

$$641 \quad (5.9) \quad \langle \mathbb{V} w, \mu \rangle_\Sigma = \langle (\frac{1}{2} \mathbb{I} + \mathbb{K})g, \mu \rangle_\Sigma \quad \text{for all } \mu \in [\mathcal{H}_0(\Sigma)]'.$$

642 Solvability of the variational formulation (5.9) follows from Lemma 5.8, while unique-
643 ness of the solution is a consequence of Theorem 5.7. Instead of the variational
644 formulation (5.9), we may use the modified Hilbert transformation \mathcal{H}_T as defined in
645 subsection 2.3 to end up with an equivalent variational problem to find $w \in [H_{,0}^{1/2}(\Sigma)]'$
646 such that

$$647 \quad (5.10) \quad \langle \mathcal{H}_T \mathbb{V} w, \mu \rangle_\Sigma = \langle \mathcal{H}_T (\frac{1}{2} \mathbb{I} + \mathbb{K})g, \mu \rangle_\Sigma \quad \text{for all } \mu \in [\mathcal{H}_{,0}(\Sigma)]'.$$

648 Due to the inclusion $H_{,0}^{1/2}(\Sigma) \subset \mathcal{H}_{,0}(\Sigma)$, we obviously have $[\mathcal{H}_{,0}(\Sigma)]' \subset [H_{,0}^{1/2}(\Sigma)]'$
649 which will allow for a Galerkin–Bubnov space-time boundary element discretization
650 of (5.10).

651 *Remark 5.9.* For a solution u of the homogeneous wave equation with zero initial
652 data but inhomogeneous Dirichlet boundary conditions and a suitable test function v
653 we can write Green’s first formula as

$$654 \quad \int_0^T \int_\Omega \partial_{n_x} u v \, dx \, dt = \int_0^T \int_\Omega \left[\partial_{tt} u v + \nabla_x u \cdot \nabla_x v \right] dx \, dt.$$

655 In particular, for $v = \partial_t u$, this results in the energy representation

$$656 \quad E(u) := \int_0^T \int_\Omega \left[\partial_{tt} u \partial_t u + \nabla_x u \cdot \nabla_x \partial_t u \right] dx \, dt$$

$$657 \quad = \frac{1}{2} \|\partial_t u(T)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla_x u(T)\|_{L^2(\Omega)}^2 > 0.$$

659 Note that this representation is the basis of the energetic BEM, see, e.g., [4]. Instead,
660 when using the particular test function $v = \mathcal{H}_T u$ and Proposition 2.1 this gives

$$661 \quad \int_0^T \int_\Gamma \frac{\partial}{\partial n_x} u \mathcal{H}_T u \, ds_x \, dt = \int_0^T \int_\Omega \left[\mathcal{H}_T \partial_t u \partial_t u + \nabla_x u \cdot \mathcal{H}_T \nabla_x u \right] dx \, dt \geq 0.$$

662 Specifically, for the single layer potential $u = \mathcal{S}w$ in $\mathbb{R}^{n+1} \setminus \Sigma$ we then conclude

$$663 \quad \langle w, \mathcal{H}_T \mathbb{V} w \rangle_\Sigma = \int_0^T \int_\Omega \left[\mathcal{H}_T \partial_t u \partial_t u + \nabla_x u \cdot \mathcal{H}_T \nabla_x u \right] dx \, dt \geq 0.$$

664 In fact, when considering the spatially one-dimensional case $n = 1$ we can prove the
665 following ellipticity estimate [29, 33]

$$666 \quad \langle w, \mathcal{H}_T \mathbb{V} w \rangle_\Sigma \geq c_1^V \|w\|_{[H_{,0}^{1/2}(\Sigma)]'}^2 \quad \text{for all } w \in [H_{,0}^{1/2}(\Sigma)]'.$$

667 Since the single layer boundary integral operator $\mathbb{V} : [H_{,0}^{1/2}(\Sigma)]' \rightarrow \mathcal{H}_0(\Sigma)$ is invertible,
668 we can write the solution of the boundary integral equation (5.8) as

$$669 \quad w = \gamma_N^i u = \mathbb{V}^{-1} (\frac{1}{2} \mathbb{I} + \mathbb{K})g = \mathbb{S}_i g,$$

670 representing the Dirichlet to Neumann map with the interior Steklov–Poincaré oper-
671 ator

$$672 \quad S_i = V^{-1}(\frac{1}{2}I + K) : \mathcal{H}_0(\Sigma) \rightarrow [H_{,0}^{1/2}(\Sigma)]'.$$

673 Hence we find that

$$674 \quad VS_i = \frac{1}{2}I + K : \mathcal{H}_0(\Sigma) \rightarrow \mathcal{H}_0(\Sigma)$$

675 is invertible. As we can formulate a related boundary integral equation also for the
676 exterior Dirichlet boundary value problem,

$$677 \quad V\gamma_N^e u = (-\frac{1}{2}I + K)g \quad \text{on } \Sigma,$$

678 this gives that the exterior Steklov–Poincaré operator

$$679 \quad S_e = -V^{-1}(\frac{1}{2}I - K) : \mathcal{H}_0(\Sigma) \rightarrow [H_{,0}^{1/2}(\Sigma)]'$$

680 is invertible, and so is

$$681 \quad \frac{1}{2}I - K = -VS_e : \mathcal{H}_0(\Sigma) \rightarrow \mathcal{H}_0(\Sigma).$$

682 Consequently

$$683 \quad VW = (\frac{1}{2}I - K)(\frac{1}{2}I + K) : \mathcal{H}_0(\Sigma) \rightarrow \mathcal{H}_0(\Sigma),$$

684 and thus

$$685 \quad W = V^{-1}(\frac{1}{2}I - K)(\frac{1}{2}I + K) : \mathcal{H}_0(\Sigma) \rightarrow [H_{,0}^{1/2}(\Sigma)]'.$$

686 This finally implies that the hypersingular boundary integral operator $W : \mathcal{H}_0(\Sigma) \rightarrow$
687 $[H_{,0}^{1/2}(\Sigma)]'$ satisfies the inf-sup stability condition

$$688 \quad (5.11) \quad c_1^W \|v\|_{\mathcal{H}_0(\Sigma)} \leq \sup_{0 \neq \eta \in H_{,0}^{1/2}(\Sigma)} \frac{|(Wv, \eta)_\Sigma|}{\|\eta\|_{H_{,0}^{1/2}(\Sigma)}} \quad \text{for all } v \in \mathcal{H}_0(\Sigma).$$

689 The solution of the Neumann boundary value problem

$$690 \quad \square u = 0 \quad \text{in } Q, \quad \partial_{n_x} u = \lambda \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0$$

691 is given by the representation formula

$$692 \quad u(x, t) = (\mathcal{S}\lambda)(x, t) - (\mathcal{D}z)(x, t) \quad \text{for } (x, t) \in Q,$$

693 where we can determine the yet unknown Dirichlet datum $z = \gamma_\Sigma^i u \in \mathcal{H}_0(\Sigma)$ as the
694 unique solution of the first kind boundary integral equation

$$695 \quad Wz = (\frac{1}{2}I - K)\lambda \quad \text{on } \Sigma.$$

696 Unique solvability follows as described above.

697 **6. Conclusions.** In this paper, we presented a new framework to describe the
 698 mapping properties of boundary integral operators for the wave equation. The results
 699 are similar as known for the boundary integral operators for elliptic partial differential
 700 equations, i.e., providing ellipticity and boundedness with respect to function spaces of
 701 the same Sobolev spaces. This will be the starting point to derive quasi-optimal error
 702 estimates for related boundary element methods which are not available so far, and
 703 which will be reported in forthcoming work. Other topics of interest include efficient
 704 implementations of the proposed scheme using the modified Hilbert transformation,
 705 a posteriori error estimates and adaptivity, an efficient solution of the resulting linear
 706 systems of algebraic equations, and the coupling with space-time finite element meth-
 707 ods. Ongoing work also includes the numerical properties of the modified Hilbert
 708 transformation and exploring other operators, like the usual Hilbert transform, in
 709 order to regularize the boundary integral equations for the wave equation.

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