

3D FULLY NONLINEAR SLOSHING FLUID IN TANKS

Bang-Fuh Chen *, Chih-Hua Wu * and Roger Nokes[†]

* Department of Marine Environment and Engineering,
National Sun Yat-sen University,
Kaohsiung 804 Taiwan
e-mail: chenbf@mail.nsysu.edu.tw

[†] Department of Civil Engineering,
University of Canterbury, Christchurch, New Zealand 8004
e-mail: roger.nokes@canterbury.ac.nz

Key words: Nonlinear sloshing fluid, Three-dimensional Tank, Resonance Waves

Abstract. *Three-dimensional tank in smooth tank is of concern in various engineering applications. Sloshing waves in moving tanks have been studied numerically, theoretically and experimentally in past several decades and many significant phenomena have been considered in those studies, especially the linear and nonlinear effects of sloshing wave with inviscid or viscid liquid in a tank. The reported studies were the tanks excited by limited exciting directions and with a fixed excitation frequency throughout the excitation. In reality, for earthquake induced tank sloshing, the excited directions are actually multi-degree of freedoms(coupled surge/sway/heave/pitch/roll/yaw). The sloshing waves may vary with various excited directions and oscillating frequencies. In present study, a 3-D tank with various fluid depths, excited amplitudes, excitation frequencies and multiple degrees of freedoms of excitation are considered. The developed time-independent finite difference method¹ is extended to incorporate the incompressible and inviscid Navier-Stokes equations, fully nonlinear kinematic and dynamic free surface conditions in the analysis of the seismic response of sloshing fluid in a 3-D rectangular tank with square basin. The boundary of the tank is mapped on to a fixed square domain through proper mapping functions. The benchmark study shows the acceptance and accuracy of the proposed numerical scheme. The sloshing waves of different water depths and various frequencies of excitation are simulated. The coupling effect of surge, sway and heave motions are also included and discussed. The present results show four types of waves, which is 'diagonal wave', 'square-like wave', 'swirling wave' and 'irregular wave', and discover that 'square-like wave' and 'irregular wave' not only occur at the tank under near the resonant frequency but also at the excitation frequencies far away from the first natural frequency of the tank. The swirling wave will occur when the tank is under near resonant excitation frequency, while the phenomenon is insignificant as the excited angle is close to 45°. The swirling waves will change swirling directions (counterclockwise to clockwise) according to the phase lags between swirling periods and the fundamental period of the tank system. For the effect of coupling Surge, sway and heave motions, the results shows there is an unstable influence when the excited frequency of heave motion is twice as large as the fundamental natural frequency*

1 INTRODUCTION

Predition of free surface sloshing in a moving container is associated with various engineering problems, such as liquid oscillations in a large storage tanks caused by earthquakes, tank trucks on highways, the motion of liquid fuel in aircraft and spacecraft, primatic membrane tanks for transportation of liquefied natural gas (LNG) and sloshing of liquid cargo in ocean-going vessel. It is known that partially filled tanks are prone to violent sloshing under certain motions, especially when the near-resonant excitation occurs. The large liquid movement would create highly localized impact pressure on tank walls which in turn cause structural damage and may even create large rolling moments to affect the stability of the vehicle which carries the container.

If the interior of a tank is smooth, the fluid viscosity plays a minor role and the potential flow solution is suitable for the sloshing in a rigid tank. In mid 1970s, Abramson² provides a rather comprehensive review and discussion of analytic and experimental studies of liquid sloshing which apply in aerospace industry. Theoretical studies of nonlinear sloshing in axial-symmetric tanks are presented by Narimanov, Dokuchaev & Lukovsky³, Miles⁴, Lukovsky⁵ and many others. These studies made significant progress in the theoretical treatment of 'swirling' (rotary) waves due to sway/pitch resonant forcing, which can cause significant longitudinal and transverse horizontal forces. The potential formulation of the problem is often used in studying sloshing by many others such as Nakayama and Washizu⁶, Flipse et al.⁷, Waterhouse⁸ and Ockendon et al.⁹ among many others. The most distinguished works might belong to Faltinsen's series of studies. Faltinsen¹⁰ used a perturbation technique to solve the 2D mathematical problem in a tank which assumed the fluid to be inviscid and incompressible. Faltinsen *et al.*¹¹ reported a Multidimensional modal analysis of nonlinear sloshing in a rectangular tank with finite water depth. Faltinsen & Timokha¹² further developed adaptive multimodal approach to nonlinear sloshing in a rectangular tank. Faltinsen et al.¹³ extended their asymptotic modal system to model nonlinear sloshing in a 3D rectangular tank. Faltinsen et al.¹⁴ considered the effect of higher modes of nonlinear sloshing in their adaptive systems.

Apart from the potential flow approaches, many numerical studies (computational fluid dynamics, CFD, simulation) of the problems with primitive variables were made, particularly for the fully nonlinear effects of the sloshing waves on free surface. Many papers give successful examples for two-dimensional sloshing (see, for instance, Chen and Chiang¹⁵, Celebi & Akyildiz¹⁶, Sames, Marcouly & Schellin¹⁷; Aliabadi, Johnson & Abedi¹⁸ and recent papers by Frandsen¹⁹, Chen and Nokes¹). Frandsen¹⁹ developed a fully nonlinear finite difference model on inviscid flow equations. He described the sloshing motion behavior in a 2-D wave tank based on potential theory according to a modified σ -transformation that stretches the grid from the bed to surface. The advantages of sigma-transformation can avoid re-meshing due to moving free surface and the mapping avoid the necessity to calculate the free surface velocity components explicitly. Moreover, the free surface smoothing by means of spatial filter is also not required. Kim²⁰ used the finite difference method to simulate two- and there- dimensional liquid containers with or without internal members. He used the SOLA scheme to give a solution of Navier–Stokes equation with free boundary. The free

surface profile is assumed to be single-valued function (SURF scheme) and the free-slip condition is applied in his computation. Most recently, Chen¹ developed a time-independent finite difference method to study viscous fluid sloshing in 2D rectangular tanks, the time varied moving boundary was mapped onto a time-independent domain through proper transformation functions and special finite difference approximation is made in order to overcome the difficulty of maintaining accuracy of finite difference expression of the terms with second derivative when the difference mesh is stretched near the boundary. Detailed transient sloshing phenomenon, streamline patterns and excitation frequency effects on hydrodynamic force coefficients were reported. For 3D tank sloshing, Feng²¹ used a three-dimensional marker and cell method to study fluid sloshing in a rectangular tank. This method takes large amount of computer memory and CPU time and the results reported indicate the present of instability. Wu et al.²² used an inviscid finite element method analyzed the fully nonlinear wave in a three dimensional tank.

Many previous works paid more attention to the effects of resonant waves due to near resonant excitation. Waterhouse⁸ worked on resonant sloshing near a critical depth. He found the tanks excited under a near-resonant frequency would produce a response which changes from a 'hard-spring'(increasing amplitude with increasing frequency) to 'soft-spring' (decreasing amplitude with increasing frequency) response as the depth passes through a critical value. Wu et al.²² focused on near resonant cases primarily based on tanks excited by both surge and sway motions and compared its results with 2D standing waves. For fixed acceleration amplitude, their results showed the displacement amplitude of the tank is reciprocal proportional to the excitation frequency. Hill²³ developed multiple-scales analysis to derive an evolution equation for the complex amplitude of resonant wave but is invalid in the vicinity of the "critical depth" and in shallow-water limit. He used shallow water equation to modify the results and compared with experimental data which showed a good agreement. Faltinsen et al.¹¹ developed multidimensional modal analysis of nonlinear sloshing in a rectangular tank. Their theory is validated by new experimental results. The asymptotic theory is not applicable to shallow water because of secondary parametric resonance and means that the primary mode is not dominating. Faltinsen and Timokha¹² created an adaptive multimodal approach to nonlinear sloshing in a rectangular tank. Adaptive procedures have been established for a wide range of excitation periods as long as the mean fluid depth h is larger than 0.24 times the tank breadth. Their theory is invalid when either water depth is small or water impacts heavily on the tank ceiling. As water depth is small, many modes have the same order and each mode have more than one main harmonic. Faltinsen et al.¹³ derived an asymptotic modal system for modeling nonlinear sloshing in a 3D rectangular tank. The theoretical part concentrated on periodic solutions of the model system (steady-state wave motions) for longitudinal and diagonal excitations and found out three types of solutions are established for each case: (1) 'planar/diagonal' resonant standing waves; (2) 'swirling' waves moving along tank walls and (3) 'square-like' resonant standing wave coupling in phase oscillations of both lowest modes. The results only for the tank under the longitudinal or diagonal excitations of a basin excited near the lowest natural frequency. Their theoretical results compared with experiment agreed very well for planar and diagonal motions. Faltinsen et al.²⁴ also used asymptotic modal system to solve sloshing waves and classify steady state

three-dimensional resonance waves in detail for a tank with various water depth. Most recently, Faltinsen *et al.*¹⁴ added higher mode effect in the asymptotic system and compared their calculated results with new experimental results which were more agreeable than that of by Faltinsen *et al.*¹³. However, their modal system cannot account for the random perturbations occurring due to the local phenomena.

The work for transient waves also have been a focus in previous research. The phenomena of modulated (beating) waves and ‘beating period’ is indeed important in an excited tank. The beating is a consequence of transients that does not disappear during a long period of time because of very small damping of the fluid motion inside a smooth tank with no internal members blocking the flow and no heavy water impact on the tank ceiling. As is well known, the frequency of the amplitude-modulated wave is $\Delta\omega = |\omega - \omega_0|$, and its time period is $2\pi / \Delta\omega$. If the excitation frequency closes to the fundamental natural frequency, the period of ‘beating’ phenomenon would be very huge in a transient sloshing problem. The transient sloshing is in general much larger than that at the steady-state sloshing and it could cause significant effect of the stability of the ships or moving cargo with carrying liquid. During the earthquakes, the transient sloshing waves and its cause large pressure and moment on the structure is more important than steady-state sloshing waves. Hill²³ compared the transient waves with steady-state waves and showed that the maximum response of a basin set into motion from rest can far exceed the steady-state response of the basin. In a recent measurement by Faltinsen *et al.*¹³, the experiments lasted for about 120 forcing periods and clear steady-state regimes were not achieved even for this long series. The transient behaviour during the first 80 seconds changed between being dominated by ‘planar/diagonal’, ‘square’ and ‘swirling’ wave modes. The main intensity of a major earthquake occurs within 60 seconds, the results of the transient sloshing can, therefore, offer enough knowledge in engineering practice.

As we know, a liquid container or reservoir undergoes an earthquake or a ship’s oil container is subjected to the forcing waves which not only has horizontal but also vertical oscillation. Faraday²⁵ studied the sloshing instability in vertical oscillating container, known as faraday wave, and found it has a parametric instability. Benjamin and Ursell²⁶ investigated the vertical motion theoretically and concluded that the solutions solved from Mathieu equation are always unstable for an external forcing frequency equal to twice the sloshing frequency. Wu *et al.*²² found under the vertical oscillation, the motion of the free surface is not at the excitation frequency but at the first natural frequency of the tank and the wave evolution during the transient period may be determined only by the acceleration amplitude of the excitation. The heave motion coupling with surge or pitch motion was discussed in recent years. Frandsen¹⁹ coupled surge and heave motion together and found that vertical excitation causes the instability associated with parametric resonance of the combined motion for a certain set of frequencies and amplitudes of vertical motion while the horizontal motion is related to classical resonances. Chen and Nokes¹ reported the initial disturbance of the fluid surface of the tank would result in huge difference of the surface elevation when vertical excitation is coupled with surge or pitch motion.

The 3D tank with different ratios of depth/excited amplitude, multiple degrees of freedom

of excitation, excitation frequencies of near or far from natural frequency and non-constant excitation frequencies are considered in present study. The developed time-independent finite difference method is extended to study the seismic response of sloshing fluid in a 3D rectangular tank with square basin. The inviscid fluid and incompressible flow are considered sloving the Navier-Stokes equations and fully nonlinear kinematic and dynamic free surface conditions are taken into account as well in present numerical work. The main study of this research simulates a 3D tank undergoing different combination of motions with varying vibrating directions. Most simulations made in the present study are lasting for 40 seconds to capture the leading sloshing wave groups and the maximum transient phenomenon can be recorded and studied. Section 2 introduces the equations of motion written in a moving coordinate system attached to the accelerating tank. The fully non-linear free surface boundary conditions are listed. The coordinate transformation functions that map the time-dependent domain into a fixed unit cubic and allow for mesh stretching at the boundaries are introduced in Section 3. The sensible non-dimensional governing equations are also derived. The proposed finite-difference method is developed in Section 4. Section 5 presents the detailed results and provides comprehensive discussion of all the phenomena found in present study. The discussion begins with the benchmark tests to validate the accuracy of numerical scheme of present study then comprehensive examples are made. Secondly, the sloshing waves include the diagonal, square-like, swirling (clockwise and counterclockwise) and irregular waves are discussed with various excited angle (means the moving direction of horizontal ground motion) and a wide range of excitation frequency of the tank. The effects of excited angle are presented as well. Finally, the influence of heave motion coupling with horizontal ground motion is shown in the end of Section 5 and finds out that an unstable influence when the excited frequency of heave motion is twice as large as the fundamental natural frequency. Eventually, a brief conclusion of present work is shown in section 6.

2 EQUATIONS OF MOTION

A fully non-linear model of inviscid 3-D waves in a numerical wave tank was developed. As shown in Fig 1, a rigid tank with breadth L , width B and still water depth d_0 . As the coordinate system is chosen to move with the tank motions (including surge, sway, heave, yaw, roll and pitch motions, see Fig. 1), the momentum equations can be derived and written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -g \cos \alpha - \frac{1}{\rho} \frac{\partial p}{\partial x} - \ddot{X}_c - (\ddot{\beta}z - \ddot{\gamma}y) - (2\dot{\alpha}\dot{\beta} - \dot{\beta}^2 - \dot{\gamma}^2)x - 2 \times (\dot{\beta}w - \dot{\gamma}v) \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -g \cos \beta - \frac{1}{\rho} \frac{\partial p}{\partial y} - \ddot{Y}_c - (\ddot{\gamma}x - \ddot{\alpha}z) - (2\dot{\beta}\dot{\gamma} - \dot{\alpha}^2 - \dot{\gamma}^2)y - 2 \times (\dot{\gamma}u - \dot{\alpha}w) \quad (2)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -g \cos \gamma - \frac{1}{\rho} \frac{\partial p}{\partial z} - \ddot{Z}_c - (\ddot{\alpha}y - \ddot{\beta}x) - (2\dot{\alpha}\dot{\gamma} - \dot{\alpha}^2 - \dot{\beta}^2)z - 2 \times (\dot{\alpha}v - \dot{\beta}u) \quad (3)$$

where u , v and w are velocity components in x , y and z directions, $\ddot{X}_c$, $\ddot{Y}_c$ and $\ddot{Z}_c$ are

the acceleration components of tank in x , y and z directions ; $(\dot{\alpha} \ \dot{\beta} \ \dot{\gamma})$ and $(\ddot{\alpha} \ \ddot{\beta} \ \ddot{\gamma})$ are the corresponding angular velocities and accelerations with respect to x , y and z -axes, p is the pressure, ρ is the fluid density and g the acceleration of gravity.

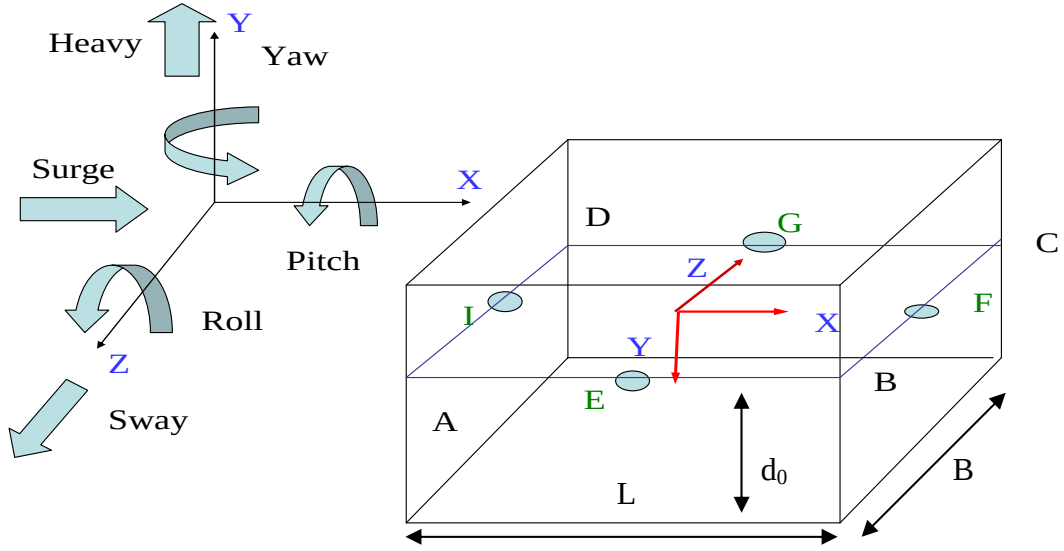


Figure 1: Definition sketch

The continuity equation for incompressible flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

The kinematic free surface condition is

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w \frac{\partial h}{\partial z} = 0 \quad (5)$$

The dynamic free surface condition is

$$p = 0 \quad (6)$$

The slip condition is applied at the boundaries between fluid and solid walls and the boundary condition at the solid walls must satisfied the momentum equations.

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = -g \cos \alpha - \ddot{X}_C - (\ddot{\beta}z - \ddot{\gamma}y) - (2\dot{\alpha}\dot{\beta} - \dot{\beta}^2 - \dot{\gamma}^2)x - 2(\dot{\beta}w - \dot{\gamma}v) \quad (7)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial y} = -g \cos \beta - \ddot{Y}_C - (\ddot{\gamma}x - \ddot{\alpha}z) - (2\dot{\beta}\dot{\gamma} - \dot{\alpha}^2 - \dot{\gamma}^2)y - 2(\dot{\gamma}u - \dot{\alpha}w) \quad (8)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = -g \cos \gamma - \ddot{Z}_C - (\ddot{\alpha}y - \ddot{\beta}x) - (2\dot{\alpha}\dot{\gamma} - \dot{\alpha}^2 - \dot{\beta}^2)z - 2(\dot{\alpha}v - \dot{\beta}u) \quad (9)$$

Taking partial differentiation of Eqs(1), (2) and (3) with respect to x , y and z respectively, and summing the results, one can obtain the following pressure wave equation which is used to solve for the pressure

$$\begin{aligned} \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = & -\rho \frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) - \rho \frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) - \rho \frac{\partial}{\partial z} \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) \\ & - 2\rho [\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\gamma} + \dot{\alpha}\dot{\gamma}] - (\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2) - 2\rho [\dot{\alpha}(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}) + \dot{\beta}(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}) + \dot{\gamma}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})] \end{aligned} \quad (10)$$

3 COORDINATE TRANSFORMATION AND DIMENSIONLESS EQUATIONS

In present study, a simple mapping functions is used to remove the time-dependence of the free surface of the fluid domain. The irregular boundary, such as time-varying fluid surface, non-vertical walls and non-horizontal bottom, can be mapped onto a square by the proper coordinate transformations as follow

$$x^* = \frac{x - b_1(y, z)}{b_2(y, z) - b_1(y, z)} \quad y^* = 1 - \frac{y + d(x, z)}{h(x, z, t)} \quad z^* = \frac{z - b_3(x, y)}{b_4(x, y) - b_3(x, y)} \quad (11)$$

where the instantaneous water surface, $h(x, z, t)$, is a single-valued function measured from tank bottom, $d(x, z)$ represents the vertical distance between still water surface and tank bottom, b_1 and b_2 are horizontal distance from the x -axis to the west and east walls respectively, and b_3 and b_4 are horizontal distance from the z -axis to the north and south walls respectively, (see Fig. 2). Through equations (11) to (13), one can map the west wall to $x^* = 0$ and east wall to $x^* = 1$, the north wall to $z^* = 0$ and south wall to $z^* = 1$, the free surface to $y^* = 0$ and the tank bottom to $y^* = 1$. In this way the computational domain is transformed to a fixed unit cubic domain. The main advantage of the transformations is to map a wavy and time-dependent fluid domain onto a time-independent unit cubic domain.

The coordinates (x^*, y^*, z^*) can be further transformed such that the layer near the boundary is stretched to capture the sharp local velocity gradients. The following exponential functions provide these stretching transformations

$$X = \lambda_1 + (x^* - \lambda_1)e^{k_1 x^* (x^* - 1)} \quad (12)$$

$$Y = \lambda_2 + (y^* - \lambda_2)e^{k_2 y^* (y^* - 1)} \quad (13)$$

$$Z = \lambda_3 + (z^* - \lambda_3)e^{k_3 z^* (z^* - 1)} \quad (14)$$

The constants k_2 and λ_2 control the mesh size near the free surface and tank bottom. The constants k_1, λ_1, k_3 and λ_3 have similar applications. Thus, the geometry of the flow field and the meshes in the computational domain (X-Y-Z system) become time-independent

throughout the computational analysis. Fig. 3 illustrates the coordinate transformations of X-Z plane in the present model.

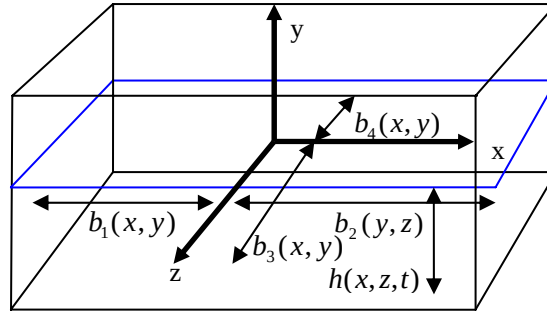


Figure 2: Definition sketch of coordinate transformation

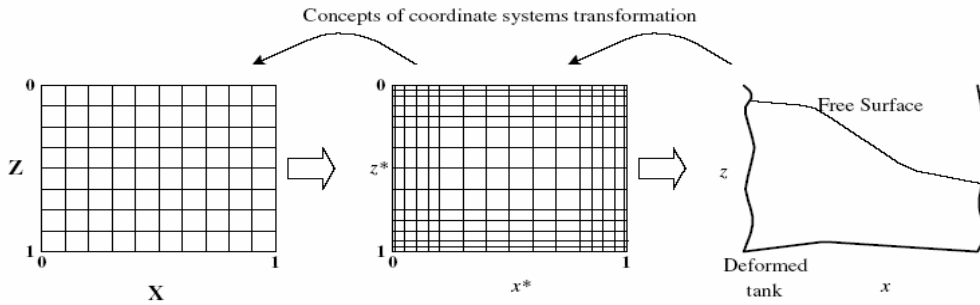


Figure 3: Concepts of coordinate transformation in x-z plane

Finally, the dimensional parameters are normalized as following definitions

$$\begin{aligned}
 U &= \frac{u}{\sqrt{gd_0}} \quad V = -\frac{v}{\sqrt{gd_0}} \quad W = \frac{w}{\sqrt{gd_0}} \quad P = \frac{p}{\rho gd_0} \quad T = t \sqrt{\frac{g}{d_0}} \quad H = \frac{h}{d_0} \\
 X_c &= \frac{x_c}{d_0} \quad Y_c = \frac{y_c}{d_0} \quad Z_c = \frac{z_c}{d_0} \quad \Theta_\alpha = \frac{\alpha}{2\pi} \quad \Theta_\beta = \frac{\beta}{2\pi} \quad \Theta_\gamma = \frac{\gamma}{2\pi} \quad \Theta_{\alpha T} = \frac{\dot{\alpha}}{2\pi} \sqrt{\frac{d_0}{g}} \\
 \Theta_{\beta T} &= \frac{\dot{\beta}}{2\pi} \sqrt{\frac{d_0}{g}} \quad \Theta_{\gamma T} = \frac{\dot{\gamma}}{2\pi} \sqrt{\frac{d_0}{g}} \quad \Theta_{\alpha TT} = \frac{\ddot{\alpha}}{2\pi} \frac{d_0}{g} \quad \Theta_{\beta TT} = \frac{\ddot{\beta}}{2\pi} \frac{d_0}{g} \quad \Theta_{\gamma TT} = \frac{\ddot{\gamma}}{2\pi} \frac{d_0}{g}
 \end{aligned} \tag{15}$$

Where u , v , and w are velocity components in x , y , and z directions; p is the pressure, g is the gravitational acceleration, d_0 is the undisturbed fluid depth, t is the real time of simulation, x_c , y_c and z_c are the ground displacement in the x , y and z directions, respectively; (α, β, γ) , $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ and $(\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma})$ are the responding angular displacements, velocities, and accelerations with respect to x , y and z axes, and the fluid depth h is normalized by d_0 .

With the aforementioned transformations and dimensionless variables, equations (1)-(10) can be written in the dimensionless forms. But all of them are tedious and lengthily, only the

dimensionless momentum equations are presented in this paper as follow

$$\begin{aligned}
 U_T + (C_{10}C_{13}U_X + C_{11}C_{14}U_Y + C_{12}C_{15}U_Z) + C_1C_{13}UU_X + C_2C_{14}UU_Y + C_3C_{15}UU_Z + C_4C_{13}VU_X \\
 + C_5C_{14}VU_Y + C_6C_{15}VU_Z + C_7C_{13}WU_X + C_8C_{14}WU_Y + C_9C_{15}WU_Z = -\cos(2\pi\Theta_\alpha) - (C_1C_{13}P_X \\
 + C_2C_{14}P_Y + C_3C_{15}P_Z) - X_{CTT} - \frac{2\pi}{d_0}(z\Theta_{\beta TT} - y\Theta_{\gamma TT}) - \frac{4\pi^2}{d_0}(2\Theta_{\alpha T}\Theta_{\beta T} - \Theta_{\beta T}^2 - \Theta_{\gamma T}^2)x - \\
 4\pi(\Theta_{\beta T}W - \Theta_{\gamma T}V)
 \end{aligned} \quad (16)$$

$$\begin{aligned}
 V_T + (C_{10}C_{13}V_X + C_{11}C_{14}V_Y + C_{12}C_{15}V_Z) + C_1C_{13}UV_X + C_2C_{14}UV_Y + C_3C_{15}UV_Z + C_4C_{13}VV_X \\
 + C_5C_{14}VV_Y + C_6C_{15}VV_Z + C_7C_{13}WV_X + C_8C_{14}WV_Y + C_9C_{15}WV_Z = \cos(2\pi\Theta_\beta) - (C_4C_{13}P_X \\
 + C_5C_{14}P_Y + C_6C_{15}P_Z) + Y_{CTT} + \frac{2\pi}{d_0}(x\Theta_{\gamma TT} - z\Theta_{\alpha TT}) + \frac{4\pi^2}{d_0}(2\Theta_{\beta T}\Theta_{\gamma T} - \Theta_{\alpha T}^2 - \Theta_{\gamma T}^2)y \\
 + 4\pi(\Theta_{\gamma T}U - \Theta_{\alpha T}W)
 \end{aligned} \quad (17)$$

$$\begin{aligned}
 W_T + (C_{10}C_{13}W_X + C_{11}C_{14}W_Y + C_{12}C_{15}W_Z) + C_1C_{13}UW_X + C_2C_{14}UW_Y + C_3C_{15}UW_Z + C_4C_{13}VW_X \\
 + C_5C_{14}VW_Y + C_6C_{15}VW_Z + C_7C_{13}WW_X + C_8C_{14}WW_Y + C_9C_{15}WW_Z = -\cos(2\pi\Theta_\gamma) - (C_7C_{13}P_X \\
 + C_8C_{14}P_Y + C_9C_{15}P_Z) - Z_{CTT} - \frac{2\pi}{d_0}(y\Theta_{\alpha TT} - z\Theta_{\beta TT}) - \frac{4\pi^2}{d_0}(2\Theta_{\alpha T}\Theta_{\gamma T} - \Theta_{\alpha T}^2 - \Theta_{\beta T}^2)z \\
 - 4\pi(\Theta_{\alpha T}V - \Theta_{\beta T}U)
 \end{aligned} \quad (18)$$

In the above equations, C_1 - C_{15} are the coefficients due to coordinate transformations and are listed in Table 2.1 P_x denotes a partial derivative of P with respect to X ; U_T expresses as a partial derivative U with respect to dimensionless time T ; X_{CTT} , Y_{CTT} and Z_{CTT} are dimensionless ground accelerations in x , y and z directions; the other terms have similar meanings. For a fully non-linear free surface condition, the kinematic free surface condition must be applied at the instantaneous free surface location i.e., at $y = h - d_0$. Thus, the coefficients C_1 - C_{15} that are related to the free surface position are updated during each iteration of the calculated process..

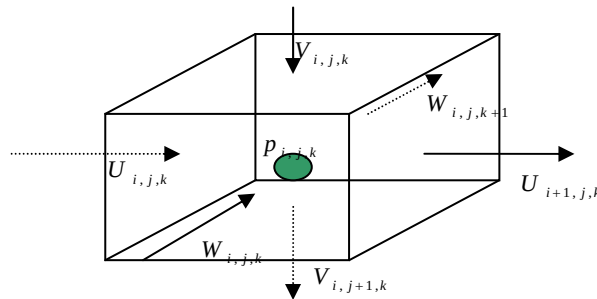
$C_1 = d_0 \frac{\partial x^*}{\partial x} = \frac{d_0}{b_2(y, z) - b_1(y, z)}$
$C_2 = d_0 \frac{\partial y^*}{\partial x} = \frac{1}{h^2}(y + d_0) \frac{\partial h}{\partial x} = \frac{1}{H^2}(y + d_0) \frac{\partial H}{\partial x}$
$C_3 = d_0 \frac{\partial z^*}{\partial x} = \frac{d_0}{[b_4(x, y) - b_3(x, y)]^2} \{ [(z - b_4(x, y))] \frac{\partial b_3}{\partial x} - [(z - b_3(x, y))] \frac{\partial b_4}{\partial x} \}$
$C_4 = -d_0 \frac{\partial x^*}{\partial y} = -\frac{d_0}{[b_2(y, z) - b_1(y, z)]^2} \{ [(x - b_2(y, z))] \frac{\partial b_1}{\partial y} - [(x - b_1(x, y))] \frac{\partial b_2}{\partial y} \}$
$C_5 = -d_0 \frac{\partial y^*}{\partial y} = \frac{d_0}{h} = \frac{1}{H}$
$C_6 = -d_0 \frac{\partial z^*}{\partial y} = \frac{d_0}{[b_4(x, y) - b_3(x, y)]^2} \{ [(z - b_4(x, y))] \frac{\partial b_3}{\partial y} - [(z - b_3(x, y))] \frac{\partial b_4}{\partial y} \}$
$C_7 = d_0 \frac{\partial x^*}{\partial x} = \frac{d_0}{[b_2(y, z) - b_1(y, z)]^2} \{ [(x - b_2(y, z))] \frac{\partial b_1}{\partial x} - [(x - b_1(x, y))] \frac{\partial b_2}{\partial x} \}$

$C_8 = d_0 \frac{\partial y^*}{\partial z} = \frac{1}{h^2} (y + d_0) \frac{\partial h}{\partial z} = \frac{1}{H^2} (y + d_0) \frac{\partial H}{\partial z}$
$C_9 = d_0 \frac{\partial z^*}{\partial z} = \frac{d_0}{b_4(x, y) - b_3(x, y)}$
$C_{10} = \sqrt{\frac{g}{d_0}} \frac{\partial x^*}{\partial t} = \sqrt{\frac{g}{d_0}} \frac{d_0}{[b_2(y, z) - b_1(y, z)]^2} \{ [(x - b_2(y, z)) \frac{\partial b_1}{\partial t} - (x - b_1(x, y)) \frac{\partial b_2}{\partial t}] \}$
$C_{11} = \sqrt{\frac{g}{d_0}} \frac{\partial y^*}{\partial t} = \sqrt{\frac{g}{d_0}} \frac{1}{h^2} (y + d_0) \frac{\partial h}{\partial t} = \sqrt{\frac{g}{d_0}} \frac{1}{H^2} (y + d_0) \frac{\partial H}{\partial t}$
$C_{12} = \sqrt{\frac{g}{d_0}} \frac{\partial z^*}{\partial t} = \sqrt{\frac{g}{d_0}} \frac{d_0}{[b_4(x, y) - b_3(x, y)]^2} \{ [(z - b_4(x, y)) \frac{\partial b_3}{\partial t} - (z - b_3(x, y)) \frac{\partial b_4}{\partial t}] \}$
$C_{13} = \frac{\partial X}{\partial x^*} = [1 + k_1(x^* - \beta_1)(2x^* - 1)] e^{k_1 x^* (x^* - 1)}$
$C_{14} = \frac{\partial Y}{\partial y^*} = [1 + k_2(y^* - \beta_2)(2y^* - 1)] e^{k_2 y^* (y^* - 1)}$
$C_{15} = \frac{\partial Z}{\partial z^*} = [1 + k_3(z^* - \beta_3)(2z^* - 1)] e^{k_3 z^* (z^* - 1)}$

 Table 1 : The coefficients C_1 - C_{15}

4 FINITE-DIFFERENCE METHOD

The numerical method used in this study is based on finite different method. In this three-dimensional analysis, the fluid flow is solved in a unit cubic mesh in the transformed flow domain. All the computations are using dimensionless equations in X-Y-Z coordinate systems. The difference equations for space derivatives are using central difference approximation, and become forward or backward difference approximation when they reach the boundaries. A staggered grid system is used in the analysis. That is, the pressure P is defined at the center of a cell, whereas the velocity components U , V and W are calculated $0.5\Delta X$, $0.5\Delta Y$ and $0.5\Delta Z$ behind, below or forward the cell center. The detail locations of them can be found in Fig. 4. The Crank-Nicholson second order finite difference scheme and the Gauss-Seidel Point successive over-relaxation iterative procedure are used to calculate the velocity and pressure, respectively. The numerical scheme and detailed iteration procedure are similar that reported in Chen and Roger¹ and is omitted in this paper.


 Figure 4: The location of U , V , W and P

5 RESULTS AND DISCUSSION

In present study, a rectangular tank with ratio breadth / width = $L / B = 1$, still water depth / breadth = $d_0 / L = 0.25$ are used in most of the simulations. Although the surge, sway, heave, pitch, roll, and yaw motions are considered in this study, the main focus of this paper is a tank under horizontal ground motion with various excited angle and the heave motion coupling with surge-sway motion is also discussed. The ground acceleration of surge, heave, and sway motion are given as $\ddot{X}_c = X_0 \omega_x^2 \sin \omega_x t$, $\ddot{Y}_c = Y_0 \omega_y^2 \sin \omega_y t$, and $\ddot{Z}_c = Z_0 \omega_z^2 \sin \omega_z t$, respectively, where X_0 , Y_0 and Z_0 are the maximum excited amplitudes and ω_x , ω_y , and ω_z are the corresponding excited frequency with respect to surge, heave, and sway motion.

5.1 Verification

In order to validate the accuracy of numerical simulation, the results obtained from the present numerical model are compared with those reported in the literature for the benchmark tests. Fig.5 shows the phenomenon of 'hard-spring' of present result comparing with Hill²³ and Lepelletier and Raichlen²⁷. It illustrates that the tank under only Surge motion with varying excited frequencies which are close to the fundamental natural frequency ω_1 of the tank and shows good agreements. The difference is due to neglecting viscous and damping effect in the present study. Fig.6 compares the present results with numerical results made by Frandsen¹⁹. The tank is under only heave motion with different excited frequencies. It also shows a great agreement. And the result of diagonal excitation made by Kim²⁰ is shown in Fig.7 as well and the agreement of the comparison is very well. Thus, the present numerical model can be demonstrated by the benchmark tests mentioned above.

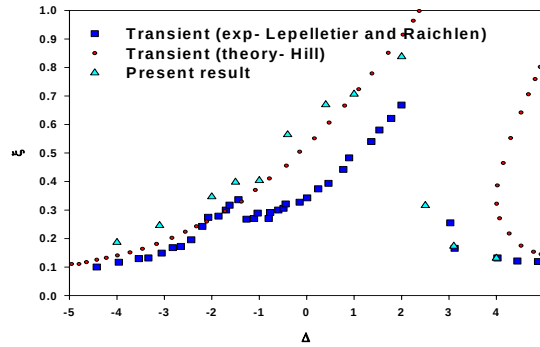


Figure 5: The 'hard-spring' phenomenon. The tank is under Surge motion. Tank's $L = 0.6095\text{m}$, $B = 0.23\text{m}$, water depth = 0.06 m and $a_0 = 0.00196\text{ m}$. ξ : the dimensionless maximum wave height during the transient period.

Δ : the 'detuning' frequency of the tank.

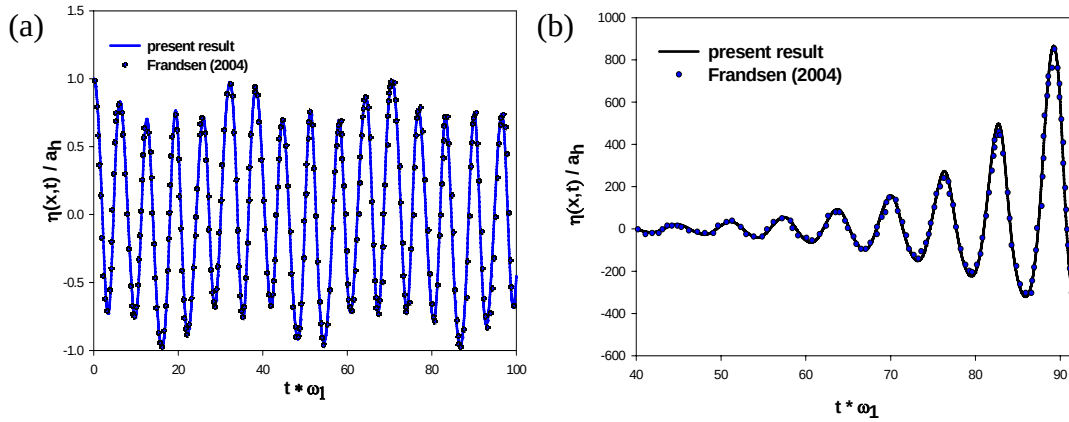


Figure 6: The wave history on tank's corner under Heave motion, the ratio d_0 (still water depth) / L and $d_0 / B = 0.5$. (a) Tank's displacement $a_0 / L = 0.272$, $\omega_y = 0.8\omega_1$; (b) Tank's displacement $a_0 / L = 0.272$, $\omega_y = 2\omega_1$

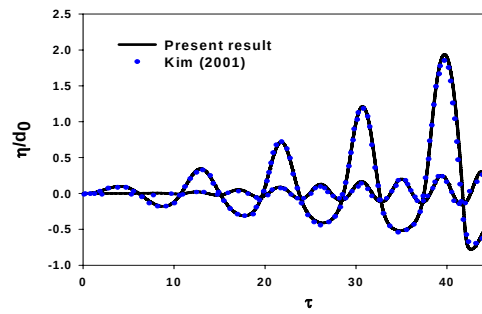


Figure 7: The wave history on tank's corner under Surge- Sway (diagonal) motion, the ratio d_0 / L and $d_0 / B = 0.25$, tank's displacement $a_0 / L = 0.0093$, $\omega_x = \omega_z = 0.99\omega_1$.

5.2 Surge-Sway motion

The results presented in this section are the cases of tank under horizontal ground motion only. The frequencies of excitation are in wide range, from $0.2\omega_1$ to $2.3\omega_1$, where ω_1 is the fundamental natural frequency. The excited angle θ (see Fig. 8) is chosen to be $5^\circ, 10^\circ, 15^\circ, 30^\circ$ and 45° . Thus, more than 100 simulation cases were made in this study, and only selected cases are reported in this paper. The depth/breadth ratio is fixed to 0.25, the maximum excited displacement is $0.005L$. Four different types of sloshing wave are found in those simulations.

The diagonal wave means the sloshing waves sloshing only in diagonal direction inside the tank, especially when the excited angle equals to 45° . Fig.9 illustrates the wave history of point A and I and the absolute peak sloshing displacement of fluid surface as a tank excited at $\theta = 45^\circ$. The traveling way of diagonal waves is obviously presented. However, when the

tank vibrates in other excited angles, the wave type would change. This type of sloshing waves does not sloshing in the absolute diagonal direction. On the other hand, the waves slosh in one direction that is almost close to the diagonal. We define this type of waves as “single-directional” waves. The single-directional waves were not found in previous studies because their studies only focused on a tank under longitudinal forcing (surge) or diagonal forcing (surge-sway). Fig.10 presents the free surface contour profiles of single-directional waves with various excited angles as the excitation frequency is fixed to $0.4\omega_1$. The traveling ways of single-directional waves are found to be the same as the excited angles of the tank.

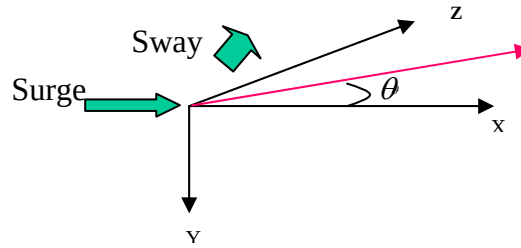
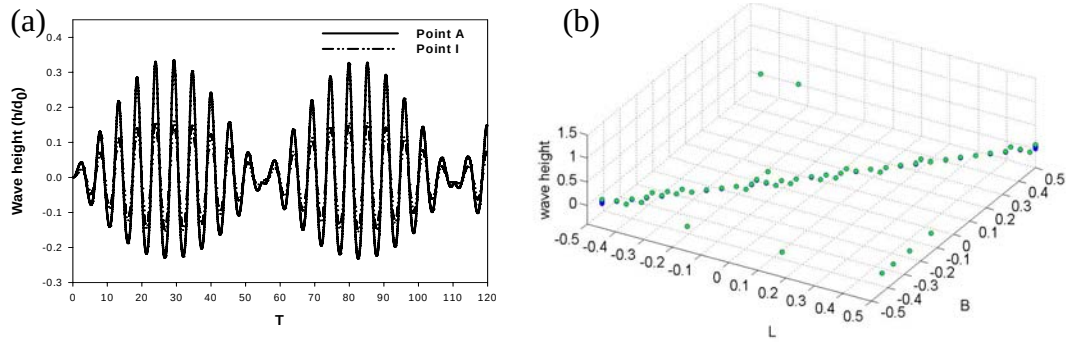
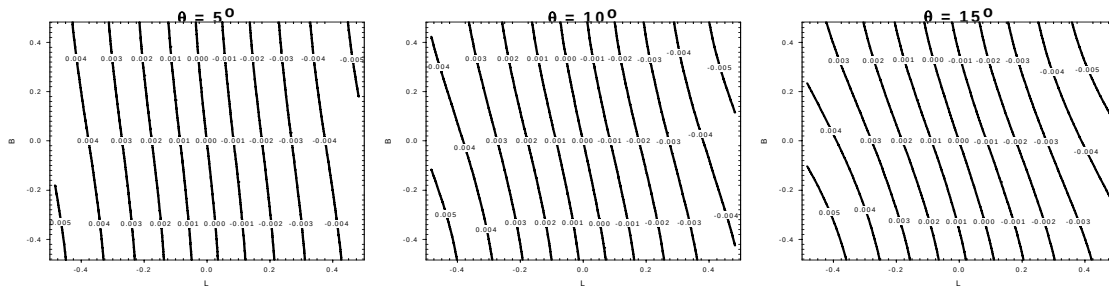


Figure 8: The definition of excited angle

Figure 9: The diagonal waves, $\omega = 1.1\omega_1, \theta = 45^\circ$. (a) the wave history ; (b) the distribution of peaks.

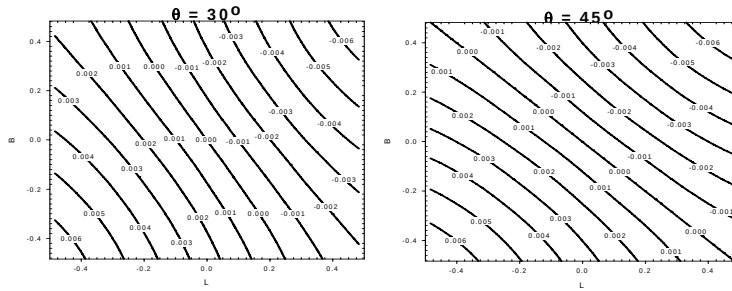


Figure 10: The free surface contour profiles of single-directional and diagonal waves with various excited angles. The excitation frequency is equal to $0.4\omega_1$

The meaning of “square-like” waves is the sloshing waves traveling mainly on the two opposite sides of the tank. In previous studies, this kind of waves could only be observed when the excitation frequencies of the tank was close to the first natural frequency during longitudinal forcing (Faltinsen¹³). However, the research also made by Faltinsen¹⁴, the square-like waves disappeared due to modifying their asymptotic modal system. In present study, the square-like waves are found when the tank’s excitation frequencies is far away from the first natural frequency. Fig.11 presents a tank excited at $\theta = 10^\circ$ and excitation frequency is $1.5\omega_1$. The absolute peaks of fluid surface shown in Fig11 (b) can be described the motion of square-like waves. Nevertheless, due to various excited angles, the square-like waves also be affected. Fig.12 represents the contour profiles of free surface with various excited angles and the same excitation frequency of the tank. We can see effortlessly the areas of the square-like waves vary with various excited angles and increase with increasing excited angle from 5° to 45° .

As the waves slosh irregularly inside the tank, it is named “irregular” waves and also called “chaotic” waves. In previous studies, it could be found as the tank’s excitation frequency being close to the first natural frequency within certain small range. By comparison, the irregular waves also can be found as the tank’s excitation frequencies being out of resonant frequency in present study. Fig.13 shows the result of the tank excited at $\theta = 15^\circ$ and excitation frequency is $2.3\omega_1$. From Fig 13 (b), we can find out that the sloshing wave peaks scattered irregularly inside the tank.

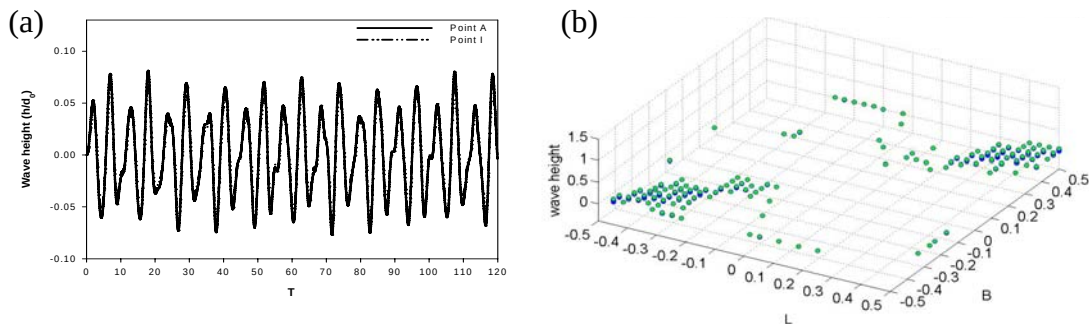


Figure 11: The square-like waves, $\omega = 1.5\omega_1$, $\theta = 10^\circ$. (a) the wave history ; (b) the distribution of peaks.

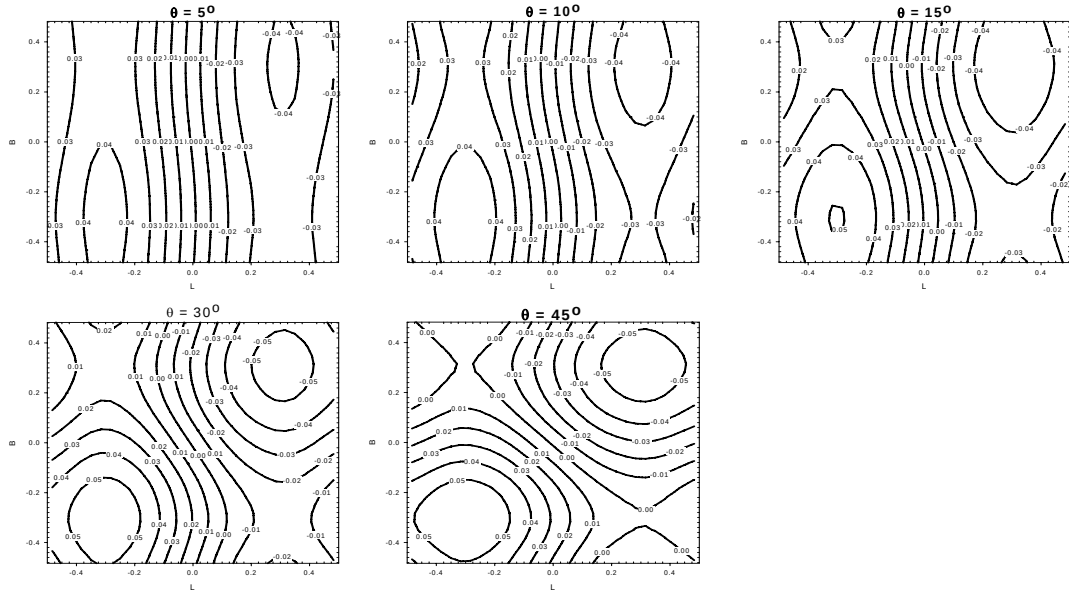


Figure 12: The free surface contour profiles of square-like waves with various excited angles. The excitation frequency is equal to $1.5\omega_1$.

The swirling waves describe the sloshing waves moving along the tank walls in a direction of clockwise or counterclockwise. In previous studies, the tank's excitation frequencies for swirling waves are limited within certain range in longitudinal forcing (surge) or diagonal forcing (surge-sway). However, the swirling waves can be discovered with a wider range under various excited angles. Fig.14 illustrates the results of swirling waves as the tank excited at $\theta = 10^\circ$ and the excitation frequency is $0.97\omega_1$. The “beating” phenomenon and a switching of rotation direction for swirling waves are found in Fig.14(a) and (b). This is a kind of “beating”, which was not observed for longitudinal forcing, but is well known from experimental studies on swirling motions in circular and spherical tanks. Fig. 14 (c) shows the absolute peaks of swirling waves scattered around the tank walls.

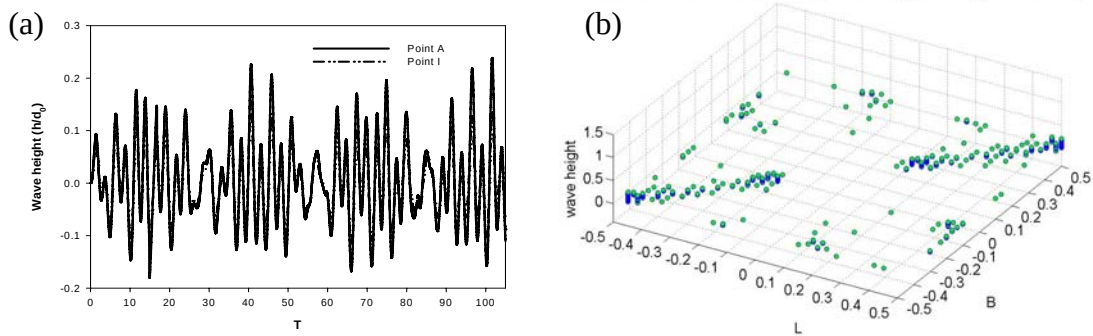


Figure 13: The square-like waves, $\omega = 2.3\omega_1$, $\theta = 15^\circ$. (a) the wave history ; (b) the distribution of peaks.

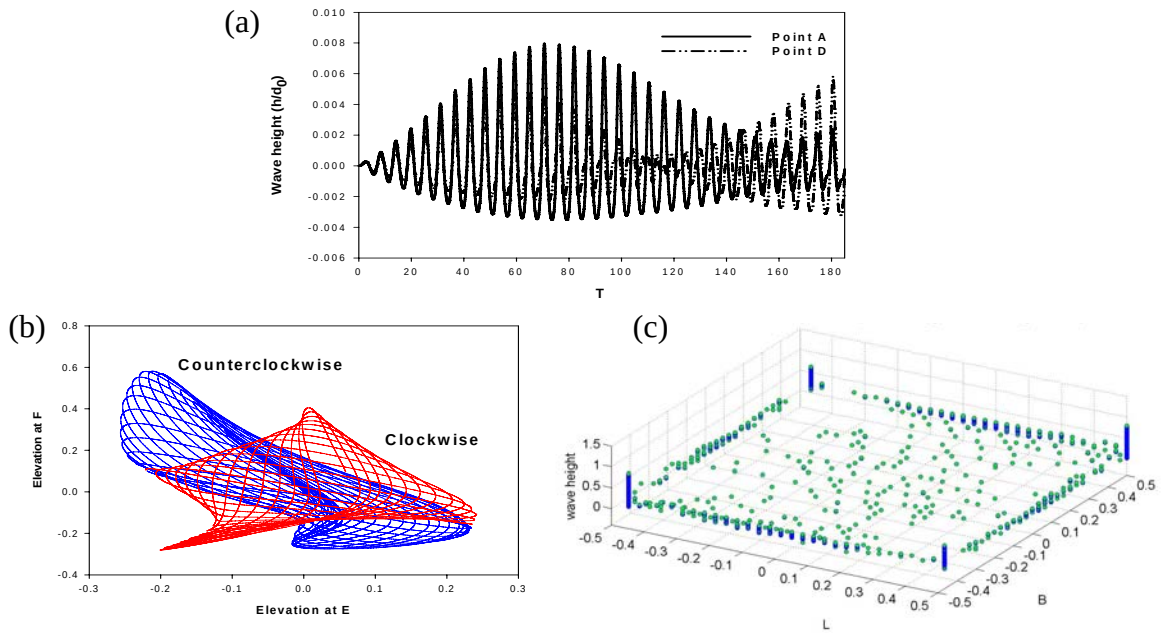


Figure 14: The swirling waves, $\omega = 0.97\omega_1$, $\theta = 10^\circ$. (a) the wave history ; (b) the pattern of counterclockwise and clockwise waves ; (c) the distribution of peaks.

The reason for the switching of rotation direction about swirling waves is made and discussed. Fig.15 presents the tank's displacement in X and Z direction compare with wave elevation of point A and D during 4 regions of dimensionless time. The beginning time regions of counterclockwise and clockwise swirling waves are at $T=30\sim50$ and $T=120\sim140$, respectively, and the switching time is at $T=130$. Fig.15 (a) and (c) indicate the beginning time region of counterclockwise and clockwise swirling waves, respectively. Fig.15 (b) illustrates the switching time region and (d) means the time region of clockwise swirling waves. In general, as the tank's displacement is positive or neglect, the water will be raised or lowered and this phenomenon can be found in Fig.15 (a) and (c). Fig. 15 (b) shows different behavior distinct from (a), obviously. As a result, we can discover that there must have a phase lag between the period of swirling waves and the fundamental period of the tank vibration system. This phase lag could be the reason for the switching of swirling waves. The other interesting phenomenon can be discovered in Fig. 14 (c) is that the larger absolute peaks of sloshing waves concentrate on the four corners of the tank. This means while the sloshing waves traveling in counterclockwise direction, the maximum sloshing displacement focus on the corner A and C of the tank. On the contrary, as the sloshing waves moving in clockwise direction, the maximum sloshing displacement become focusing on the corner B and D of the tank.

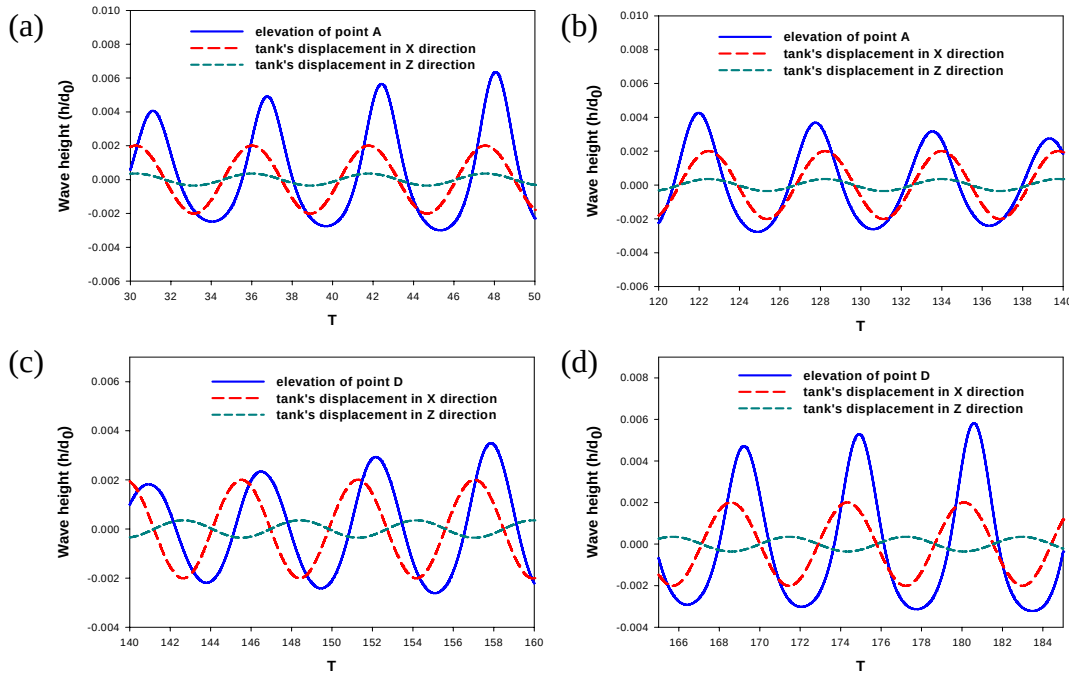


Figure 15: The link between wave elevation and tank's displacement. The tank excited at $\theta = 10^\circ$, $\omega = 0.97\omega_1$. (a) $T=30\sim50$ (The beginning of counterclockwise waves); (b) $T=110\sim130$ (The switching time region); (c) $T=140\sim160$ (The beginning of clockwise waves); (d) $T=165\sim185$ (The clockwise swirling waves).

5.3 Surge-Sway-Heave motion

A tank may be simultaneously excited by horizontal and vertical ground acceleration during earthquakes. In previous studies, the influence of heave motion for sloshing waves were made and found out its maximum effect for sloshing system was when the excited frequency of heave motion is twice as large as the first natural frequency. The same phenomenon has also been found in present study. Thus, coupled surge-sway-heave motion is made by a tank under various excitation frequencies but only chosen cases to present here as the excited frequency of heave motion is twice as large as the first natural frequency. The heave motion is just a change in gravitational acceleration, and its effect is expected to be small if the free surface of the tank is initially undisturbed. However, if the free surface is initially sloping or it is disturbed by other combination of ground motions, the vertical excitation would enlarge the free surface elevation during tank motion. The detailed relations between initial disturbed or plane free surface and heave motion are reported in Chen and Roger [2005]. Fig. 16 shows the two cases of sloshing waves affected by heave motion. The vertical displacement $y_0 = 0.005L$, excitation frequency $\omega_y = 2\omega_1$ and the excited angle $\theta = 5^\circ$. The heave motion starts at $T=40$ under the disturbed free surface. The effect of heave motion is shown in Fig 16(a) and found that the beating phenomenon, sloshing waves' period and the sloshing displacement are changed. The type of sloshing waves also differs from only surge-

sway motion as the tank's excitation frequency is equal to $0.9\omega_1$ and become swirling waves after coupling heave motion. Fig.16(b) depicts the distribution of absolute peaks and shows a swirling waves type. Fig.16 (c) and (d) present the square-like waves affected by heave motion finally changes its wave type to swirling waves. Thus, as the tank under heave motion coupling with other ground motions and vertical excitation frequency is equal to $2\omega_1$, the unstable influence of heave motion is obviously found in present study.

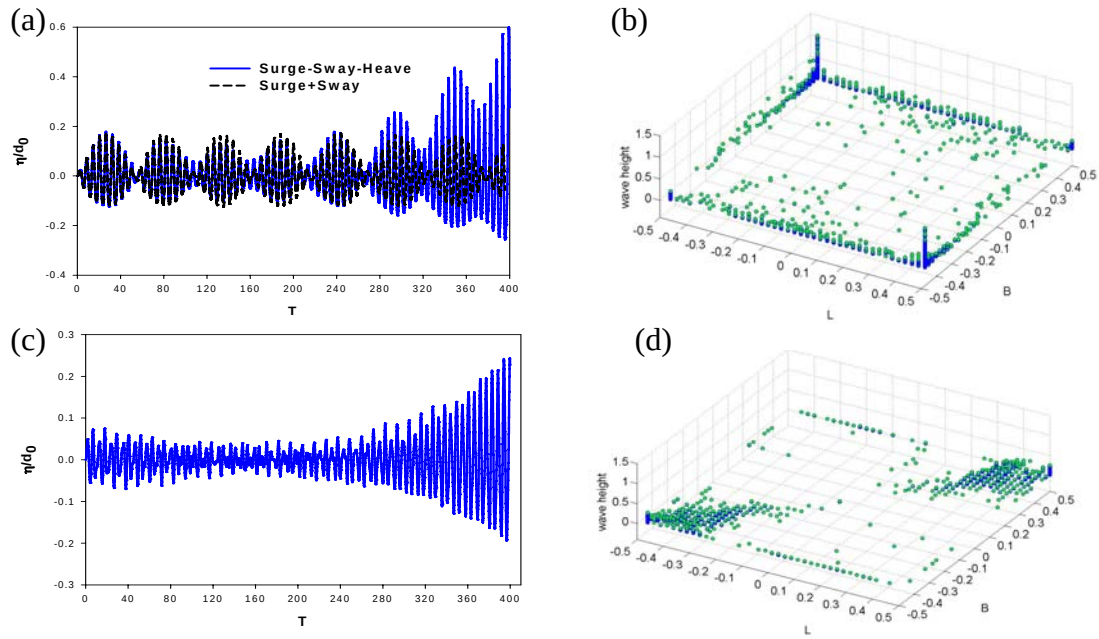


Figure 16: The results of surge-sway-heave motion, The tank excited at $\theta = 5^\circ$, $\omega_y = 2.0\omega_1$. (a)

$\omega_x = \omega_z = 0.9\omega_1$; (b) the distribution of peaks, $\omega_x = \omega_z = 0.9\omega_1$; (c) $\omega_x = \omega_z = 1.5\omega_1$; (d) the distribution of peaks, $\omega_x = \omega_z = 1.5\omega_1$.

6 CONCLUSIONS

- The developed time-independent finite difference method is extended to solve the incompressible and inviscid Navier-Stokes equations, fully nonlinear kinematic and dynamic free surface conditions in a rectangular tank with a square basin. The main study of this research simulates a 3-D tank undergoing different combination of motions with varying vibrating directions. In present study, the 3D tank with different ratios of depth/excited amplitude, multiple degrees of freedom of excitation, excitation frequencies of near or far from natural frequency are considered.
- The comparison of the results obtained by present simulation and those of reported data shows the acceptance and accuracy of the proposed numerical scheme.
- The results of Faltinsen *et al.*^{13, 14} indicated the square-like and irregular waves will be presented during the tank is under near resonant excited frequency. However, in

the present study, the tanks are under various excited angles of horizontal ground motion, the square-like and irregular waves not only occur at the tank under near resonant frequency excitation but also at the excitation frequency far away from the first natural frequency of the tank system.

- The effects of various excited angles for different sloshing wave are presented and demonstrate its influence can not be out of consideration.
- The swirling wave will occur when the tank is under near resonant excitation frequency, while the phenomenon is insignificant as the excited angle is close to 45° . However, the range of excitation frequency for swirling waves is wider than previous studies because of the effect of excited angle. The swirling waves will change swirling directions (counterclockwise to clockwise) according to the phase lags between swirling periods and the fundamental period of the tank system.
- The coupling effect of surge, sway and heave motions is also made in present study and the results shows an unstable influence when the excited frequency of heave motion is twice as large as the fundamental natural frequency.
- The developed numerical scheme can easily simulate the coupled excitations of six-degree of freedom and also the real fluid sloshing in the tanks

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