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Agresti, A.; Blessing, Alexandra ; Luongo, Eliseo

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Antonio Agresti[1,4] , **Alexandra Blessing (Neamţu)**[2]
and Eliseo Luongo[3,5,*]

[1] Delft University of Technology, Delft Institute of Applied Mathematics, P.O. Box 5031, 2600 GA Delft, The Netherlands
[2] University of Konstanz, Department of Mathematics and Statistics, Universitätsstrasse 10, 78464 Konstanz, Germany
[3] Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italia

E-mail: eliseo.luongo@sns.it, antonio.agresti92@gmail.com and alexandra.blessing@uni-konstanz.de

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Abstract

In this paper, we prove the global well-posedness and interior regularity for the 2D Navier–Stokes equations driven by a fractional noise acting as an inhomogeneous Dirichlet-type boundary condition. The model describes a vertical slice of the ocean with a relative motion between the two surfaces and can be thought of as a stochastic variant of the Couette flow. The relative motion of the surfaces is modeled by a Gaussian noise which is colored in space and fractional in time with Hurst parameter $\mathcal{H} > \frac{3}{4}$.

[4] Current Address: Department of Mathematics Guido Castelnuovo, Sapienza University of Rome, P.le Aldo Moro 5, 00185 Rome, Italy.
[5] Current Address: Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany.
[*] Author to whom any correspondence should be addressed.

Contents

1. Introduction

In many situations occurring in applied sciences, noise can affect the evolution of a system only through the boundary of a region where the system evolves. Such phenomena can be modeled via partial differential equations with boundary noise, as introduced by Da Prato and Zabczyck in the seminal paper [21]. Such a description presents several issues from a mathematical viewpoint. Indeed, nowadays it is well-known that in the one dimensional case, the solution of the heat equation with white noise Dirichlet or Neumann boundary conditions has low (space) regularity compared to the case of noise diffused inside the domain. This is due to the large amplitude of the fluctuations of the solutions close to the boundary. In particular, in the case of Dirichlet boundary conditions the solution is only a distribution. This allowed to treat only a restricted class of nonlinearities, exploiting specific properties of the heat semigroup and studying carefully the blow-up of the solution close to the boundary. For some results in this direction, the reader is referred to [2, 12, 25, 36]. On the contrary, partial differential equations with white noise Neumann boundary conditions with more severe nonlinearities have been considered in [14, 61], and in the last few years, maximal L^p regularity techniques provided new ideas to treat some of those arising in fluid dynamics. Indeed, some results on the global and local well-posedness of the 2D Navier–Stokes equations and the 3D primitive equations with boundary noise perturbations of Neumann type have been proven in [1] and

[10], respectively. Besides the physical interests in studying the Navier–Stokes equations with boundary noise in virtue of its connection with the Couette flow (see also below for further motivations), the present manuscript also aims at (partially) filling the gap in the literature between Dirichlet and Neumann type boundary conditions for fluid dynamical models.

Throughout the manuscript, we fix a finite time horizon $T > 0$ and consider a spatial domain $\mathcal{O} = \mathbb{T} \times (0,a)$ where $\mathbb{T}$ is the one-dimensional torus and $a > 0$. We further define the lower and upper parts of the boundary of $\mathcal{O}$ by

$$\Gamma_b = \mathbb{T} \times \{0\} \quad \text{and} \quad \Gamma_u = \mathbb{T} \times \{a\}. \tag{1.1}$$

In this work, we focus on the global well-posedness and interior regularity of the two-dimensional Navier–Stokes equations with fractional boundary noise. The unknowns are the velocity field $u(t,\omega,x,z) = (u_1,u_2) : (0,T) \times \Omega \times \mathcal{O} \to \mathbb{R}^2$ and the pressure $P : (0,T) \times \Omega \times \mathcal{O} \to \mathbb{R}$, which formally satisfy the system

$$\begin{cases} \partial_t u = \Delta u + \nabla P - (u \cdot \nabla) u & \text{on } (0,T) \times \mathcal{O}, \\ \operatorname{div} u = 0 & \text{on } (0,T) \times \mathcal{O}, \\ u_1 = g \, \dot{W}^{\mathcal{H}} & \text{on } (0,T) \times \Gamma_u, \\ u_2 = 0 & \text{on } (0,T) \times \Gamma_u, \\ u = 0 & \text{on } (0,T) \times \Gamma_b, \\ u(0) = u_{\text{in}} & \text{on } \mathcal{O}, \end{cases} \tag{1.2}$$

where (u_{in},g) are given data and $W^{\mathcal{H}}$ is a fractional Brownian motion with Hurst parameter $\mathcal{H} > \frac{3}{4}$, respectively. The assumptions on $(u_{\text{in}},g,W^{\mathcal{H}})$ are made precise below. Even if we consider a more regular noise in time than the one introduced in [21], the combination of the blow-up of the solution close to the boundary and the Navier–Stokes nonlinearity makes the global well-posedness and the interior regularity of (1.2) a non-trivial issue, which, indeed, cannot be treated simply by the techniques introduced in [1]. Indeed, to the best of our knowledge, this is the first instance of a *global* well-posedness result for a fluid dynamical system with non-homogeneous Dirichlet-type boundary conditions of a regularity class comparable with the time derivative of a fractional Brownian motion with Hurst parameter $\mathcal{H} > \frac{3}{4}$, see [15, 28, 37] and the references therein for some results in this direction. Moreover, the reader is referred to [26] for the analysis of some properties of (1.2) in the 3D case replacing $g \, \dot{W}^{\mathcal{H}}$ with an Ornstein Uhlenbeck process and to [2, 36] for some results on the existence and uniqueness of solutions for the heat equation with white noise Dirichlet type boundary conditions perturbed by some Lipschitz forcing. Finally, in [9, 56] the emphasis is on the non-penetration boundary conditions, namely it is studied the case $u_2 = g(x,t)$ on $\Gamma_u \cup \Gamma_b$, with g much more regular either in time and space than $g \, \dot{W}^{\mathcal{H}}$.

According to [35, 52, 53], see also the discussion in the introduction of [1], the geometry considered in (1.2) can be seen as an idealization of the ocean dynamics (more precisely, a vertical slice of the ocean). The model (1.2), describes a Couette flow, namely a viscous fluid in the space between two surfaces, one of which is moving tangentially relative to the other (see also remark 1.4). The relative motion of the surfaces imposes a shear stress on the fluid and induces the flow. Let us recall that the onset of turbulence is often related to the randomness of background movement [47]. Moreover, according to [54, chapter 3] in any turbulent flow there are unavoidably perturbations in boundary conditions and material properties. We model these features by the noise term $g \, \dot{W}^{\mathcal{H}}$. As introduced by Kolmogorov in [41], fractional

Brownian motion can be thought of as a model for turbulence. Moreover, to describe turbulence in 3D fluids, models of random vortex filaments have been introduced in [29]. These have been analyzed for fractional Brownian motion with $\mathcal{H} > 1/2$ in [51] and $\mathcal{H} < 1/2$ in [30].

1.1. Main result

We begin by introducing some basic notation. Throughout this manuscript, we work on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbf{P})$ and consider a separable Hilbert space U. A process Φ is said to be $\mathcal{F}$-progressive measurable if, for every $t > 0$, the restriction $\Phi|_{(0,t) \times \Omega}$ is measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}((0,t))$, where $\mathcal{B}$ denotes the Borel σ-algebra. Further notation concerning function spaces is deferred to section 1.3. On the noise $W^{\mathcal{H}}$ we enforce the following

Assumption 1.1. $W^{\mathcal{H}}$ is a U-cylindrical fractional Brownian motion with Hurst parameter $\mathcal{H} \in (\frac{3}{4}, 1)$ and $g \in \mathscr{L}_2(U, H^{-s}(\Gamma_u))$ with $s \in [0, \frac{1}{2})$ and $\mathcal{H} - \frac{s}{2} > \frac{3}{4}$.

Note that assumption 1.1 is consistent with the results obtained in [23] for the stochastic heat equation with Dirichlet fractional noise. The reader is referred to remark 1.4 for the case of a time-dependent g.

Following [22, chapter 15] and [20], we construct solutions to (1.2) by the splitting

$$u = w_g + v, \tag{1.3}$$

where w_g is a mild solution of the linear problem with non-homogeneous boundary conditions

$$\begin{cases} \partial_t w_g = \Delta w_g + \nabla P_g & \text{on } (0,T) \times \mathcal{O}, \\ \operatorname{div} w_g = 0 & \text{on } (0,T) \times \mathcal{O}, \\ w_{g,1} = g \dot{W}^{\mathcal{H}} & \text{on } (0,T) \times \Gamma_u, \\ w_{g,2} = 0 & \text{on } (0,T) \times \Gamma_u, \\ w_g = 0 & \text{on } (0,T) \times \Gamma_b, \\ w_g(0) = 0 & \text{on } \mathcal{O} \end{cases} \tag{1.4}$$

and v is a weak solution of

$$\begin{cases} \partial_t v = \Delta v + \nabla(P - P_g) \\ \qquad - \operatorname{div}((v + w_g) \otimes (v + w_g)) & \text{on } (0,T) \times \mathcal{O}, \\ \operatorname{div} v = 0 & \text{on } (0,T) \times \mathcal{O}, \\ v = 0 & \text{on } (0,T) \times (\Gamma_b \cup \Gamma_u), \\ v(0) = u_{\text{in}} & \text{on } \mathcal{O}. \end{cases} \tag{1.5}$$

In (1.5), due to the divergence-free of v and w_g, we rewrote the Navier–Stokes nonlinearity in the conservative form to accommodate the weak (PDE) setting.

As discussed in [22, chapter 13], if $g, u_{\text{in}}, W^{\mathcal{H}}(t)$ were sufficiently regular, then $u = v + w_g$ would be a classical solution of the Navier–Stokes equations with non-homogeneous boundary conditions (1.2).

Next, we introduce the class of solutions we are going to consider. To motivate them, let us first discuss the regularity of w_g. It is well-known that, in the case of Dirichlet-type boundary conditions, the solution of a linear problem with boundary noise and $\mathcal{H} = \frac{1}{2}$ is a distribution which blows-up close to the boundary, see [2, 21], and the same holds also in case of $\mathcal{H} \neq \frac{1}{2}$,

see [12]. Therefore, we cannot expect that the mild solution of (1.4) has arbitrarily good integrability properties as in [1, 10]. This has drastic consequences in our analysis. As we will show in proposition 3.1, we have, $\mathbf{P} - a.s.$,

$$w_g \in C\left([0,T]; L^{2q}\left(\mathcal{O}; \mathbb{R}^2\right)\right) \text{ for all } q \in (1, q_{\mathcal{H}}) \tag{1.6}$$

where

$$q_{\mathcal{H}} := \frac{2}{2s + 5 - 4\mathcal{H}} \in (1, 2). \tag{1.7}$$

Let us stress that $q_{\mathcal{H}} < 2$ and $\lim_{\mathcal{H} \downarrow 3/4} q_{\mathcal{H}} = 1$ even if $s = 0$. As we will see below, this fact creates major difficulties in our analysis of the auxiliary Navier–Stokes equations (1.5). In particular, $w_g \otimes w_g \in C([0, \infty); L^q(\mathcal{O}; \mathbb{R}^2)) \mathbf{P} - a.s.$ and, from parabolic regularity, the best regularity we can hope for is $v \in L^p([0, \infty); H^{1,q}(\mathcal{O}; \mathbb{R}^2)) \mathbf{P} - a.s.$ for all $p < \infty$. Thus, in general,

$$v \notin L^2\left(0, T; H^1\left(\mathcal{O}; \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s. \text{ for any } T < \infty.$$

Therefore, v is a solution of the Navier–Stokes equations with *infinite energy* and the argument used in [1] does not work. The case of infinite energy solutions of 2D Navier–Stokes equations already appeared in the literature [11, 33]. In [33] the unboundedness of the energy is due to a rough initial data $u_0 \notin L^2$ while in [11] to a rough forcing term $f \notin L^2(0, T; H^{-1})$ acting on the bulk. Our case does not fit in any of the above situations due to the presence of transport-type terms depending on the w_g in (1.5) and the fact that we are working on domains. For this reason, our proofs rely on different methods. For details, the reader is referred to the text before remark 1.4.

In light of the previous discussion, we are now ready to define solutions to (1.2). Below, we set $A : B = \sum_{i,j=1}^2 A^{i,j} B^{i,j}$ for two matrices A and B and $\mathbb{L}^q$ the image of $L^q(\mathcal{O}; \mathbb{R}^2)$ via the Leray projection $\mathbb{P}$ defined rigorously in section 2.1.

Definition 1.2. Let $T < \infty$, $u_{\text{in}} \in L^0_{\mathscr{F}_0}(\Omega; \mathbb{L}^2)$ and $q \in (1, q_{\mathcal{H}})$.

- (*q*-solution) A progressively measurable process u with $\mathbf{P} - a.s.$ paths in $L^{2q'}(0, T; \mathbb{L}^{2q})$, is a pathwise weak *q*-solution of (1.2) if for all divergence-free $\varphi = (\varphi_1, \varphi_2) \in C^\infty(\mathcal{O}; \mathbb{R}^2)$ such that $\varphi = 0$ on $\Gamma_b \cup \Gamma_u$ and a.e. $t \in (0, T)$,

$$\int_{\mathcal{O}} u(x, t) \varphi(x) \, dx - \int_{\mathcal{O}} u_{\text{in}}(x) \varphi(x) \, dx$$
$$= \int_0^t \int_{\mathcal{O}} (u \cdot \Delta \varphi + [u \otimes u] : \nabla \varphi) \, dx \, dr - \langle g, \partial_2 \varphi_1 \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} W_t^{\mathcal{H}}.$$

- (unique *q*-solution) A *q*-solution u to (1.2) is said to be a *unique solution* if for any other *q*-solution $\widetilde{u}$ we have $u = \widetilde{u}$ a.e. on $[0, T] \times \Omega$.
- (unique solution) A *q*-solution u is said to be a unique solution to (1.2) if it is also a $\widetilde{q}$-solution for all $\widetilde{q} \in (1, q_{\mathcal{H}})$.

Before stating our main result, let us first comment on the above definition. Due to the argument below (1.5), one cannot expect solutions to (1.2) with integrability in space larger or equal to $2q_{\mathcal{H}}$. Furthermore, even if solutions are constructed by (1.3), the uniqueness class is independent of this splitting. Moreover, the unique solution of (1.2) is independent of the choice of $q \in (1, q_{\mathcal{H}})$. Such independence is expected from solutions to (1.2) in light of (1.6). Finally,

let us discuss the regularity class chosen to define q-solutions. Since $\mathcal{O}$ is two-dimensional, the space $L^{2q'}(0,T;L^{2q}(\mathcal{O};\mathbb{R}^2))$ has Sobolev index given by (keeping in mind the parabolic scaling)

$$-\frac{2}{2q'}-\frac{2}{2q}=-1.$$

In particular, the regularity class chosen for q-solutions to (1.2) is *critical* for the Navier–Stokes equations in two dimensions and satisfies the classical Ladyzhenskaya–Prodi–Serrin condition. In light of the recent convex integration results [16, 17, 44] in absence of noise and with periodic boundary conditions in all directions, the regularity assumption in our definition is expected to be sharp for obtaining uniqueness and a-fortiori well-posedness.

The main result of the current work reads as follows.

Theorem 1.3. *Let assumption* 1.1 *be satisfied and* $u_{\text{in}} \in L^0_{\mathscr{F}_0}(\Omega;\mathbb{L}^2)$.

(1) *There exists a unique solution of (1.2) in the sense of definition* 1.2 *with paths in*

$$u \in C\left([0,T];\mathbb{L}^2\right) \quad \mathbf{P} - a.s.$$

(2) *The unique solution of (1.2) satisfies, for all* $t_0 \in (0,T)$ *and* $\mathcal{O}_0 \subset \mathcal{O}$ *such that* $\text{dist}(\mathcal{O}_0,\partial\mathcal{O}) > 0$,

$$u \in C\left([t_0,T];C^\infty\left(\mathcal{O}_0;\mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s.$$

The proof of theorem 1.3(1) and (2) are given in section 3.3 and section 4.2, respectively. Routine extensions of the above are commented in remark 1.4 below.

Next, let us discuss the main ideas behind the proof of theorem 1.3. As commented above, due to (1.6), we cannot deal with the techniques introduced in [1] to study (1.5). Indeed, contrary to [1, 20], the splitting introduced above is *not* enough to study the global well-posedness of (1.2) since (1.5) has no Leray solutions since $w_g \otimes w_g \notin L^2(0,T;L^2)$. Thus, we control the blow-up of the energy of v introducing further splittings depending on the regularity of w_g. As discussed above we will show that $w_g \in C([0,T];L^{2q})$ for some $q \in (1,q_{\mathcal{H}})$. Since the space $C([0,T];L^{2q})$, $q > 1$ is subcritical for 2D Navier–Stokes equations we have some hopes to exploit the strong time regularity of w_g to circumvent its rough behavior in space. The heuristic idea above is realized by writing

$$v = \sum_{i=0}^{N-1} v_i + \bar{v}, \tag{1.8}$$

where N depends only on q. The terms $\{v_i\}_{i\in\{0,...,N-1\}}$ are defined inductively solving homogeneous Stokes equations with forcing having mixed regularity in space and time, such that the regularity in space increases in i while the regularity in time decreases in i. On the contrary $\bar{v}$ is a Leray-type solution of the remainder equation. In particular, N is chosen large enough such that the equation for $\bar{v}$ has a forcing in $L^2(0,T;H^{-1})$ and therefore is regular enough to prove the existence and uniqueness of Leray solutions, see theorem 2.6 below. The reader is referred to section 3.2 for further discussions.

The interior regularity of u in theorem 1.3(2) is treated considering again the splitting $u = v + w_g$ introduced above. The interior regularity of w_g can be proved similarly to the linear part of [1]. On the contrary, the low regularity of v does not allow us to study directly its

interior regularity by Serrin's argument as in [1]. For this reason, we rely on the splitting (1.8) analyzed in section 3.2 to study the well-posedness of (1.5). Combining maximal L^p regularity techniques for studying the interior regularity of the v_i's, an induction argument and a Serrin argument for treating the interior regularity of $\overline{v}$, we obtain the required regularity of v. As shown in [60] (see also [43, section 13.1]) and similarly to [1], higher-order interior time regularity does not appear to be attainable for our Navier–Stokes problem with stochastic boundary conditions. This stands in contrast to the case of the heat equation with white noise boundary conditions, as studied in [12]. The underlying reason for this phenomenon is the presence of the unknown pressure P, whose non-local nature creates a link between interior and boundary regularity, where the noise acts.

To conclude, let us point out that, in contrast to [11, 33], we employ a different splitting scheme to prove existence due to the presence of the transport-type terms originated by w_g. Moreover, the number of splitting N depends on how much the Sobolev index of the space $C([0,T]; L^{2q}(\mathcal{O}; \mathbb{R}^2))$, i.e. $-\frac{1}{q}$, is far from the critical threshold -1. In particular, $N \to \infty$ as $q \downarrow 1$. As commented above, such a splitting is also convenient when proving the interior regularity for u which was not addressed in the above-mentioned works.

Remark 1.4 (Extensions). One can readily check that theorem 1.3 extends in the following cases:

- (Bounded domains) If $\mathcal{O}$ is replaced by a smooth C^2-bounded domain in $\mathbb{R}^2$. However, we prefer to keep the same geometry of [1] for two reasons. Firstly, and more importantly, as discussed in section 1, the model considered has a clear physical interpretation. Secondly, in this way, we can easily compare our results, techniques and assumptions with those of [1].
- (Fractional Volterra noise) If $W^{\mathcal{H}}$ is replaced by a α-regular Volterra process with $\alpha > \frac{1}{4}$. Let us recall that a fractional Brownian motion with Hurst parameter $\mathcal{H}$ is an example of a α-regular Volterra process with $\alpha = \mathcal{H} - \frac{1}{2}$. These are non-Markovian stochastic processes which can be represented as integrals of kernels with respect to the Brownian motion and include for example the fractional Liouville Brownian motion and the Rosenblatt process. Stochastic convolutions with respect to such processes were analyzed in [13, 18, 19].
- (Time-dependent g) The term g in the boundary noise $g\dot{W}^{\mathcal{H}}$ depends on time as long as it is progressively measurable and the corresponding process w_g satisfies (1.6).
- (Full stochastic Couette flow) More general boundary conditions like

$$\begin{cases} u_1 = U_{\text{up}} + g\dot{W}^{\mathcal{H}} & \text{on } (0,T) \times \Gamma_u, \\ u_1 = U_b & \text{on } (0,T) \times \Gamma_b. \end{cases}$$

can be considered for sufficiently smooth velocity fields U_{up}, U_b such that the corresponding process w_g satisfies (1.6). For example the case $U_{\text{up}}, U_b \in L^2((0,T) \times \mathbb{T})$ can be treated. The above can be seen as a Couette flow with uncertainty on the velocity of one of the two surfaces.

1.2. Overview

In section 2, we introduce the functional framework required to study problem (1.2). The proof of theorem 1.3 is developed in sections 3 and 4. Specifically, global well-posedness, i.e. item (1), is addressed in section 3, where we first analyze the linear problem (1.4) in section 3.1, followed by the nonlinear problem (1.5) in section 3.2. Interior regularity, i.e. item (2), is the focus of section 4. In particular, section 4.1 is devoted to the interior regularity of the solution to the linear problem (1.4), while section 4.2 deals with the nonlinear problem (1.5).

1.3. Notation

Here we collect some notation which will be used throughout the paper. Additional notation will be introduced where needed. We use C to denote a generic constant, which may vary from line to line. When it is important to emphasize the dependence of C on a parameter ξ, we write $C(\xi)$. Moreover, we sometimes write $a \lesssim b$ (resp. $a \lesssim_\xi b$) to mean that there exists a constant C (resp. $C(\xi)$) such that $a \leqslant Cb$ (resp. $a \leqslant C(\xi)b$).

Let $q \in (1, \infty)$ be fixed. For any integer $k \geqslant 1$, we denote by $W^{k,q}$ the standard Sobolev spaces. In the case of non-integer smoothness $s \in (0, \infty) \setminus \mathbb{N}$, we define $W^{s,q} = B^s_{q,q}$ where $B^s_{q,q}$ is the Besov space with smoothness s, and integrability q and microscopic integrability q. We also denote by $H^{s,q}$ the Bessel potential spaces. Both Besov and Bessel potential spaces can be defined via Littlewood–Paley theory (see, e.g. [58], [57, section 6]), or through interpolation methods based on the classical Sobolev spaces $W^{k,q}$ (see, e.g. [8, chapter 6]). For a domain $D \subset \mathbb{R}^n$, integer $d \geqslant 1$, and $\mathcal{A} \in \{W, H\}$, we define the vector-valued spaces by $\mathcal{A}^{s,q}(D; \mathbb{R}^d) = (\mathcal{A}^{s,q}(D))^d$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two separable Hilbert spaces. We denote by $\mathscr{L}_2(\mathcal{K}_1, \mathcal{K}_2)$ the set of Hilbert–Schmidt operators from $\mathcal{K}_1$ to $\mathcal{K}_2$. We will use the following Fubini-type identity:

$$H^s(D; \mathcal{K}_1) = \mathscr{L}_2(\mathcal{K}_1, H^s(D)) \quad \text{for all } s \in \mathbb{R},$$

which follows from [39, theorem 9.3.6] and interpolation theory.

2. Preliminaries

2.1. The Stokes operator and its spectral properties

In this section, we introduce the functional analytic setup to define all the objects necessary in the following. Throughout this subsection, we let $r \in (1, \infty)$. Recall that $\mathcal{O} = \mathbb{T} \times (0, a)$ where $a > 0$.

We begin by introducing the Helmholtz projection on $L^r(\mathcal{O}; \mathbb{R}^2)$, see e.g. [55, subsection 7.4]. As described also in [1, section 2.1], the projection can be defined via an elliptic problem. We recall its construction here for the sake of completeness. Let $f \in L^r(\mathcal{O}; \mathbb{R}^2)$ and let ψ_f be the unique solution of the following elliptic problem

$$\begin{cases} \Delta \psi_f = \operatorname{div} f & \text{on } \mathcal{O}, \\ \partial_{\widehat{n}} \psi_f = f \cdot \widehat{n} & \text{on } \Gamma_u \cup \Gamma_b. \end{cases} \tag{2.1}$$

Here $\widehat{n}$ denotes the exterior normal vector field on $\partial\mathcal{O}$. This problem is understood in its standard weak formulation:

$$\int_\mathcal{O} \nabla \psi_f \cdot \nabla \varphi \, \mathrm{d}x\mathrm{d}z = \int_\mathcal{O} f \cdot \nabla \varphi \, \mathrm{d}x\mathrm{d}z \quad \text{for all } \varphi \in C^\infty(\mathcal{O}). \tag{2.2}$$

By [55, Corollary 7.4.4], we have $\psi_f \in W^{1,r}(\mathcal{O})$ and $\|\nabla \psi_f\|_{L^r(\mathcal{O}; \mathbb{R}^2)} \lesssim \|f\|_{L^r(\mathcal{O}; \mathbb{R}^2)}$. The Helmholtz projection $\mathbb{P}_r : L^r(\mathcal{O}; \mathbb{R}^2) \to L^r(\mathcal{O}; \mathbb{R}^2)$ is then defined by

$$\mathbb{P}_r f = f - \nabla \psi_f, \quad f \in L^r(\mathcal{O}; \mathbb{R}^2).$$

Next, we define the Stokes operator on $L^r(\mathcal{O}; \mathbb{R}^2)$ corresponding to the boundary conditions considered in (1.2). For notational convenience, we define A_r as minus the Stokes operator so

that A_r is a positive operator for $r = 2$ (i.e. $\langle A_2 u, u \rangle \geqslant 0$ for all $u \in \mathsf{D}(A_2)$). Let

$$\mathbb{L}^r := \mathbb{P}\left(L^r\left(\mathcal{O};\mathbb{R}^2\right)\right), \quad \mathbb{H}^{s,r} := H^{s,r}\left(\mathcal{O};\mathbb{R}^2\right) \cap \mathbb{L}^r, \ s \in \mathbb{R}.$$

Then, we define the operator $A_r : \mathsf{D}(A_r) \subseteq \mathbb{L}^r \to \mathbb{L}^r$ where

$$\mathsf{D}\left(A_r\right) = \left\{f = (f_1, f_2) \in W^{2,r}\left(\mathcal{O};\mathbb{R}^2\right) \cap \mathbb{L}^r : \ f|_{\Gamma_b \cup \Gamma_u} = 0\right\},$$

and $A_r u = -\mathbb{P}\Delta u$ for $u \in \mathsf{D}(A_r)$.

In the main arguments, we need stochastic and deterministic maximal L^r-regularity estimates for convolutions. By [42, 48], it is enough to provide the boundedness of the H^∞-calculus for A_r. The reader is referred to [55, chapters 3 and 4] and [39, chapter 10] for the main notation and basic results on the H^∞-calculus.

Contrary to [1], the boundary conditions we are interested in here are much more classical. Indeed, the Stokes operator with no-slip boundary conditions is well-studied. The reader is referred e.g. to [38, section 2.8], [34, 49] and [40, section 9] for the proof of this nowadays classical statement.

Lemma 2.1. *For all $r \in (1,\infty)$, the operator A_r is invertible and has a bounded H^∞-calculus of angle 0. Moreover, the domain of the fractional powers of A_r is characterized as follows:*

$$\mathsf{D}\left(A_r^\alpha\right) = \begin{cases} \mathbb{H}^{2\alpha, r} & \text{if } \alpha < \dfrac{1}{2r}, \\[2mm] \left\{u \in \mathbb{H}^{2\alpha, r} : u|_{\partial\mathcal{O}} = 0\right\} & \text{if } \dfrac{1}{2r} < \alpha \leqslant 1. \end{cases}$$

The above implies that $-A_r$ generates an analytic semigroup on $\mathbb{L}^r$ which admits stochastic and deterministic maximal L^p-regularity for all $p \in (1, +\infty)$, see [55, chapter 3-4] and [48]. We denote such semigroup by $S_r(t)$. We continue introducing some known facts about the 'Sobolev tower' of spaces associated with the operator A_r. We denote by

$$\begin{aligned} X_{\alpha, A_r} &= \mathsf{D}\left(A_r^\alpha\right) & &\text{for } \alpha \geqslant 0, \\ X_{\alpha, A_r} &= \left(\mathbb{L}^r, \|A_r^\alpha \cdot \|_{\mathbb{L}^r}\right)^\sim & &\text{for } \alpha < 0, \end{aligned}$$

where $\sim$ denotes the completion. Indeed, since $0 \in \rho(A_r)$ by lemma 2.1, we have that $f \mapsto \|A_r^\alpha f\|_{\mathbb{L}^r}$ is a norm for all $\alpha < 0$. Since $(A_r)^* = A_{r'}$, it follows that (see e.g. [4, chapter 5, theorem 1.4.9])

$$\left(X_{\alpha, A_r}\right)^* = X_{-\alpha, A_{r'}}. \tag{2.3}$$

For notational convenience, we will write A, $S(t)$ instead of A_2 and $S(t)$. Moreover we define

$$H := \mathbb{L}^2, \quad V := \mathsf{D}\left(A^{1/2}\right).$$

We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and the norm in H, respectively. In the following, V^* denotes the dual of V and we identify H with its dual H^*. Whenever X is a reflexive Banach space such that the embedding $X \hookrightarrow H$ is continuous and dense, denoting by X^* the dual of X, the scalar product $\langle \cdot, \cdot \rangle$ in H extends to the dual pairing between X and X^*. We will simplify the notation accordingly.

For the convenience of the reader, we recall here the definition of the deterministic maximal L^p-regularity, since our techniques heavily rely on it.

Definition 2.2 ([55, Definition 3.5.1]) We let $\mathcal{X}$ stand for a Banach space and $\mathcal{A}$ a linear closed operator on $\mathcal{X}$ with domain $\mathsf{D}(\mathcal{A})$. We say that the inhomogeneous initial value problem on $L^p(0,T;\mathcal{X})$ given by

$$\dot{u}(t) + \mathcal{A}u(t) = f(t), \quad u(0) = u_0 \tag{2.4}$$

admits maximal L^p-regularity, if for each $f \in L^p(0,T;\mathcal{X})$ and $u_0 \in (\mathcal{X}, \mathsf{D}(\mathcal{A}))_{1-1/p,p}$ there exists a unique $u \in W^{1,p}(0,T;\mathcal{X}) \cap L^p(0,T;\mathsf{D}(\mathcal{A}))$ satisfying (2.4) a.e. in $(0,T)$.

2.2. The Dirichlet map

Now we are interested in L^2-estimates for the Dirichlet map, i.e. we are interested in studying the weak solutions of the elliptic problem

$$\begin{cases} -\Delta u + \nabla \pi = 0, & \text{on } \mathcal{O}, \\ \operatorname{div} u = 0 & \text{on } \mathcal{O}, \\ u(\cdot, 0) = 0, & \text{on } \Gamma_b, \\ u_1(\cdot, a) = g, & \text{on } \Gamma_u, \\ u_2 = 0, & \text{on } \Gamma_u. \end{cases} \tag{2.5}$$

To state the main result of this subsection, we formulate (2.5) in the very weak setting. To this end, we argue formally. Take $\varphi = (\varphi_1, \varphi_2) \in C^\infty(\mathcal{O}; \mathbb{R}^2)$ such that $\operatorname{div} \varphi = 0$,

$$\varphi = 0, \quad \text{on} \quad \Gamma_b \cup \Gamma_u.$$

A formal integration by parts shows that (2.5) implies

$$\int_{\mathcal{O}} u \cdot \Delta \varphi \, \mathrm{d}x \mathrm{d}z = \int_{\mathbb{T}} g(x) \partial_2 \varphi_1(x,a) \, \mathrm{d}x. \tag{2.6}$$

In particular, the RHS of (2.6) makes sense even in case g is a distribution if we interpret $\int_{\mathbb{T}} g(x) \partial_2 \varphi_1(x,a) \, \mathrm{d}x = \langle \partial_2 \varphi_1(\cdot, a), g \rangle$. The well-posedness of (2.5) is, as for the properties of the Stokes operator, a well-known fact. Indeed, theorem 2.3 below holds. The reader is referred to [6, 7, 27, 32, 59] for its proof and more general results on the Dirichlet boundary values problem above even in case of weighted L^r spaces of Muckenhoupt class and $u \cdot \widehat{n}|_{\Gamma_u \cup \Gamma_b} \neq 0$.

Theorem 2.3. *For all $g \in H^{-\frac{1}{2}}(\Gamma_u)$ there exists a unique very weak solution $(u, \pi) \in H \times H^{-1}(\mathcal{O})/\mathbb{R}$ of (2.5). Moreover (u, π) satisfy*

$$\|u\| + \|\pi\|_{H^{-1}(\mathcal{O})/\mathbb{R}} \leqslant C \|g\|_{H^{-\frac{1}{2}}(\Gamma_u)}. \tag{2.7}$$

Finally, if $g \in H^{\frac{3}{2}}(\Gamma_u)$, then $(u, \pi) \in \mathbb{H}^2 \times H^1(\mathcal{O})/\mathbb{R}$ and

$$\|u\|_{\mathbb{H}^2(\mathcal{O};\mathbb{R}^2)} + \|\pi\|_{H^1(\mathcal{O})/\mathbb{R}} \leqslant C \|g\|_{H^{3/2}(\Gamma_u)}. \tag{2.8}$$

Next, we denote by $\mathcal{D}$ the solution map defined by theorem 2.3 which associate to a boundary datum g the velocity u solution of (2.5), i.e. $\mathcal{D}g := u$. From the above result, we obtain

Corollary 2.4. *Let $\mathcal{D}$ and U be the Dirichlet map and a separable Hilbert space, respectively. Then*

$$\mathcal{D} \in \mathscr{L}(H^{-\alpha}(\Gamma_u; U), \mathscr{L}_2(U, \mathsf{D}(A^{-\frac{\alpha}{2} + \frac{1}{4}}))) \quad \text{for } \alpha \in \left[-\frac{1}{2}, 0\right).$$

Proof. To begin, recall that $H^s(\Gamma_u; U) = \mathscr{L}_2(U, H^s(\Gamma_u))$ for all $s \in \mathbb{R}$, see section 1.3. Hence, due to the ideal property of Hilbert–Schmidt operators, it is enough to consider the scalar case $U = \mathbb{R}$.

By complex interpolation, the estimates in theorem 2.3 yield

$$\mathcal{D} : H^{2\theta - \frac{1}{2}}(\Gamma_u) \to \mathbb{H}^{2\theta}(\mathcal{O}) \quad \text{for all } \theta \in (0, 1).$$

Hence, the claim now follows from the description of the fractional power of A in lemma 2.1. $\qquad\square$

2.3. Deterministic Navier–Stokes equations

Let us consider the deterministic Navier–Stokes equations with homogeneous boundary conditions

$$\begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{\pi} = \Delta \overline{u} + \overline{f}, & \text{on } (0, T) \times \mathcal{O}, \\ \operatorname{div} \overline{u} = 0, & \text{on } (0, T) \times \mathcal{O}, \\ \overline{u} = 0, & \text{on } (0, T) \times (\Gamma_b \cup \Gamma_u), \\ \overline{u}(0) = \overline{u}_0, & \text{on } \mathcal{O}. \end{cases} \tag{2.9}$$

Define the trilinear form

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \partial_i v_j w_j \, \mathrm{d}x\mathrm{d}z = \int_{\mathcal{O}} (u \cdot \nabla v) \cdot w \, \mathrm{d}x\mathrm{d}z \tag{2.10}$$

which is well–defined and continuous on $\mathbb{L}^p \times \mathbb{H}^{1,q} \times \mathbb{L}^r$ by Hölder's inequality, whenever

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

Finally, we introduce the operator

$$B : \mathbb{L}^p \times \mathbb{L}^r \to X_{-1/2, A_{q'}}$$

defined by the identity

$$\langle B(u, v), \phi \rangle_{X_{-1/2, A_{q'}}, X_{1/2, A_q}} = -b(u, \phi, v) = -\int_{\mathcal{O}} (u \cdot \nabla \phi) \cdot v \, \mathrm{d}x\mathrm{d}z$$

for all $\phi \in X_{1/2, A_q}$. Moreover, if $u \cdot \nabla v \in L^r(\mathcal{O}; \mathbb{R}^2)$ for some $r \in (1, \infty)$, it is explicitly given by

$$B(u, v) = \mathbb{P}(u \cdot \nabla v).$$

We have to define our notion of a weak solution for problem (2.9).

Definition 2.5. Given $\overline{u}_0 \in H$ and $\bar{f} \in L^2(0,T;V^*)$, we say that

$$\overline{u} \in C([0,T];H) \cap L^2(0,T;V)$$

is a weak solution of equation (2.9) if for all $\phi \in \mathsf{D}(A)$ and $t \in [0,T]$,

$$\langle \overline{u}(t), \phi \rangle - \int_0^t b(\overline{u}(s), \phi, \overline{u}(s)) \, \mathrm{d}s$$

$$= \langle \overline{u}_0, \phi \rangle - \int_0^t \langle \overline{u}(s), A\phi \rangle \, \mathrm{d}s + \int_0^t \langle \bar{f}(s), \phi \rangle_{V^*,V} \, \mathrm{d}s.$$

The well-posedness of (2.9) in the sense of definition 2.5 is a well-known fact. Indeed the following theorem holds, see for instance [45, 62, 63].

Theorem 2.6. *For every* $\overline{u}_0 \in H$ *and* $\bar{f} \in L^2(0,T;V^*)$ *there exists a unique weak solution of Equation (2.9). It satisfies*

$$\|\overline{u}(t)\|^2 + 2\int_0^t \|\nabla \overline{u}(s)\|_{L^2}^2 \, \mathrm{d}s = \|\overline{u}_0\|^2 + 2\int_0^t \langle \overline{u}(s), \bar{f}(s) \rangle_{V^*,V} \, \mathrm{d}s.$$

If $(\overline{u}_0^n)_{n \in \mathbb{N}}$ *is a sequence in* H *converging to* $\overline{u}_0 \in H$ *and* $\left(\bar{f}^n\right)_{n \in \mathbb{N}}$ *is a sequence in* $L^2(0,T;V^*)$ *converging to* $\bar{f} \in L^2(0,T;V^*)$, *then the corresponding unique solutions* $(\overline{u}^n)_{n \in \mathbb{N}}$ *converge to the corresponding solution* $\overline{u}$ *in* $C([0,T];H)$ *and in* $L^2(0,T;V)$.

We end this section with the following lemma, which generalizes [31, Lemma 1.14] to the $L_t^p L_x^q$ setting. In particular, [31, Lemma 1.14] corresponds to the specific case $p = q = 4$.

Lemma 2.7. *If* $q > 2$ *and* $p \geqslant \frac{2q}{q-2}$, $u \in C([0,T];H) \cap L^2(0,T;V)$, $v \in L^p(0,T;\mathbb{L}^q)$, *then*

$$B(u,v) \in L^2(0,T;V^*), \tag{2.11}$$
$$B(v,u) \in L^2(0,T;V^*). \tag{2.12}$$

In particular for each $t \in [0,T]$, ε, $\varepsilon' > 0$ *and* $\phi \in L^2(0,T;V)$ *it holds*

$$\int_0^t |\langle u(s) \cdot \nabla \phi(s), v(s) \rangle| \, \mathrm{d}s$$

$$\leqslant \varepsilon \|\phi\|_{L^2(0,t;V)}^2 + \varepsilon' \int_0^t \|u(s)\|_V^2 \mathrm{d}s + \frac{C}{\varepsilon^{\frac{q}{q-2}} \varepsilon'^{\frac{2}{q-2}}} \int_0^t \|u(s)\|^2 \|v(s)\|_{\mathbb{L}^q}^{\frac{2q}{q-2}} \, \mathrm{d}s, \quad (2.13)$$

$$\int_0^t |\langle v(s) \cdot \nabla \phi(s), u(s) \rangle| \, \mathrm{d}s$$

$$\leqslant \varepsilon \|\phi\|_{L^2(0,t;V)}^2 + \varepsilon' \int_0^t \|u(s)\|_V^2 \mathrm{d}s + \frac{C}{\varepsilon^{\frac{q}{q-2}} \varepsilon'^{\frac{2}{q-2}}} \int_0^t \|u(s)\|^2 \|v(s)\|_{\mathbb{L}^q}^{\frac{2q}{q-2}} \, \mathrm{d}s, \quad (2.14)$$

where C is a constant independent from ε, ε'.

Proof. By Hölder inequality, Sobolev embedding theorem and interpolation, for each $\phi \in V$ we have

$$
\begin{aligned}
|\langle B(u(s), v(s)), \phi \rangle| &= |\langle u(s) \cdot \nabla \phi, v(s) \rangle| \\
&\leqslant \|\phi\|_V \|v(s)\|_{\mathbb{L}^q} \|u(s)\|_{\mathbb{L}^{2q/(q-2)}} \\
&\lesssim_q \|\phi\|_V \|v(s)\|_{\mathbb{L}^q} \|u(s)\|_{\mathsf{D}(A^{1/q})} \\
&\leqslant \|\phi\|_V \|v(s)\|_{\mathbb{L}^q} \|u(s)\|^{1-\frac{2}{q}} \|u(s)\|_V^{\frac{2}{q}}.
\end{aligned}
$$

Therefore for each

$$
\begin{aligned}
\int_0^t &|\langle u(s) \cdot \nabla \phi(s), v(s) \rangle| \, ds \\
&\leqslant \varepsilon \|\phi\|_{L^2(0,t;V)}^2 + \frac{C}{\varepsilon} \int_0^t \|v(s)\|_{\mathbb{L}^q}^2 \|u(s)\|^{2\left(1-\frac{2}{q}\right)} \|u(s)\|_V^{\frac{4}{q}} \, ds \\
&\leqslant \varepsilon \|\phi\|_{L^2(0,t;V)}^2 + \frac{C}{\varepsilon} \left(\int_0^t \|u(s)\|_V^2 \, ds \right)^{2/q} \left(\int_0^t \|u(s)\|^2 \|v(s)\|_{\mathbb{L}^q}^{\frac{2q}{q-2}} \, ds \right)^{\frac{q-2}{q}} \\
&\leqslant \varepsilon \|\phi\|_{L^2(0,t;V)}^2 + \varepsilon' \|u\|_{L^2(0,t,V)}^2 + \frac{C}{\varepsilon^{\frac{q}{q-2}} \varepsilon'^{\frac{2}{q-2}}} \int_0^t \|u(s)\|^2 \|v(s)\|_{\mathbb{L}^q}^{\frac{2q}{q-2}} \, ds.
\end{aligned}
$$

The relation above implies (2.11) and (2.13). The proof of (2.12) and (2.14) is analogous and we omit the details. $\qquad \square$

2.4. Stochastic convolutions with fractional noise

Definition 2.8. Let U be a separable Hilbert space. A U-cylindrical fractional Brownian motion $(W^{\mathcal{H}}(t))_{t \geqslant 0}$ with Hurst index $\mathcal{H} \in (0,1)$ is defined by the formal series

$$
W^{\mathcal{H}}(t) = \sum_{n=1}^{\infty} b_n^{\mathcal{H}}(t) e_n,
$$

where $\{e_n\}$ is an orthonormal basis in U and $(b^{\mathcal{H}}(t))_{n \in \mathbb{N}}$ is a sequence of independent standard one-dimensional fractional Brownian motions, i.e. $\mathbf{E}[b_n^{\mathcal{H}}(t)] = 0$ and

$$
\mathbf{E}\left[b_n^{\mathcal{H}}(t) b_n^{\mathcal{H}}(s) \right] = \frac{1}{2} \left(t^{2\mathcal{H}} + s^{2\mathcal{H}} - |t-s|^{2\mathcal{H}} \right), \quad s, t \geqslant 0.
$$

For $\mathcal{H} = 1/2$ one obtains a cylindrical Brownian motion. However for $\mathcal{H} \neq 1/2$ the fbm exhibits a totally different behavior, in particular is neither Markov nor a semimartingale.

For our aims in proposition 3.1, we need the following results on the regularity of stochastic convolutions established in [23, corollary 3.1] and [24, proposition 11.6].

Lemma 2.9. *([23, corollary 3.1] and [24, proposition 11.6]) Let A be the generator of an analytic C_0-semigroup $(S(t))_{t \geqslant 0}$ on a separable Hilbert space U_1, $\Phi \in \mathscr{L}(U, U_1)$. Assume that*

$$
\|S(t) \Phi\|_{\mathscr{L}_2(U, U_1)} \leqslant t^{-\gamma} \text{for } \gamma < \mathcal{H}. \tag{2.15}
$$

Then the stochastic convolution $\int_0^t S(t-s) \Phi \, dW^{\mathcal{H}}(s)$ has P-a.s. γ_1-Hölder continuous trajectories in $\mathsf{D}(A^{\gamma_2})$, for $0 \leqslant \gamma_1 + \gamma_2 < \mathcal{H} - \gamma$. If $\Phi \in \mathscr{L}_2(U, U_1)$, then the assumption (2.15) is satisfied for $\gamma = 0$.

3. Global well-posedness

Here we prove theorem 1.3(1). This section is organized as follows. Firstly, in section 3.1 we prove that the solution w_g of the 2D Stokes equations with boundary noise (1.4) satisfies (1.6). Secondly, in section 3.2, we prove the existence of a q-solution to (1.2) by studying the auxiliary Navier–Stokes problem (1.5) for a given forcing term $w = w_g$ satisfying the regularity assumption as in (1.6) for a given q. Finally, in section 3.3, we prove the uniqueness of solutions to (1.2) therefore concluding the proof of theorem 1.3(1). Recall that (q-)solutions of (1.2) are defined in definition 1.2.

3.1. Stokes equations

As discussed in section 1.1, we start by considering the linear problem (1.4). According to [21] and [22, chapter 15], the mild solution w_g of the former problem is formally given by

$$w_g(t) = A \int_0^t S(t-s) \mathcal{D}[g] \, dW^{\mathcal{H}}(s). \tag{3.1}$$

Here A is (minus) the Stokes operator with homogeneous boundary conditions as defined in section 2.1.

Next, we prove that w_g is well-defined in sufficiently regular function spaces therefore allowing us to treat the nonlinearity in the Navier–Stokes equations.

Proposition 3.1. *Let assumption* 1.1 *be satisfied. Then the process w_g is well-defined, progressively measurable, and for all $T > 0$ and $\varepsilon > 0$,*

$$w_g \in L^p\left(\Omega; C\left([0,T]; \mathsf{D}\left(A^{\mathcal{H}-\frac{3}{4}-\frac{s}{2}-\varepsilon}\right)\right)\right) \text{ for all } p \in (1,\infty). \tag{3.2}$$

In particular, for all $r \in (2, 2q_{\mathcal{H}})$,

$$w_g \in C\left([0,T]; \mathbb{L}^r\right) \text{ a.s.} \tag{3.3}$$

Proof. Note that, thanks to corollary 2.4

$$\mathcal{D}g \in \mathscr{L}_2\left(U, \mathsf{D}\left(A^{\frac{1}{4}-\frac{s+\varepsilon}{2}}\right)\right).$$

Hence, by lemma 2.9, a.s.

$$w_g = A^{\frac{3}{4}+\frac{s+\varepsilon}{2}} \underbrace{\underbrace{\int_0^{\cdot} S(\cdot-s) A^{\frac{1}{4}-\frac{s+\varepsilon}{2}} \mathcal{D}g \, dW^{\mathcal{H}}(s)}_{\in C\left([0,T]; \mathsf{D}\left(A^{\mathcal{H}-\frac{\varepsilon}{2}}\right)\right)}}_{\in C\left([0,T]; \mathsf{D}\left(A^{\mathcal{H}-\frac{3}{4}-\frac{s}{2}-\varepsilon}\right)\right)}.$$

The arbitrariness of $\varepsilon > 0$ yields (3.2). To prove (3.3), note that, by lemma 2.1,

$$\mathsf{D}\left(A^{\mathcal{H}-\frac{3}{4}-\frac{s}{2}-\varepsilon}\right) \subset H^{2\mathcal{H}-\frac{3}{2}-s-2\varepsilon}\left(\mathcal{O}; \mathbb{R}^2\right).$$

The above space embeds into $L^r(\mathcal{O};\mathbb{R}^2)$ for some $r > 2$ provided

$$2\mathcal{H} - \frac{3}{2} - s - 2\varepsilon > 0.$$

The above is exactly our assumption due to the arbitrariness of $\varepsilon > 0$. In particular, by the arbitrariness of ε and Sobolev's embedding we can choose whatever $r < 2q_{\mathcal{H}}$.   □

Remark 3.2 (Necessity of the L^p-setting for v). In the setting of proposition 3.1, we have $2\mathcal{H} - \frac{3}{2} - s < \frac{1}{2}$. Therefore, for all choices of $\mathcal{H}$ and s in assumption 1.1, it follows that

$$H^{2\mathcal{H} - \frac{3}{2} - s - 2\varepsilon}\left(\mathcal{O};\mathbb{R}^2\right) \not\hookrightarrow L^4\left(\mathcal{O};\mathbb{R}^2\right).$$

Thus, (3.3) holds with $r < 4$ and therefore $B(w_g, w_g) \notin L^2(0, T; V^*)$. In particular, in the next subsection, we cannot avoid the use of L^p-setting in space, cf the comments below assumption 3.6.

Remark 3.3. Previous results with white noise boundary conditions [1, 10] exploited stochastical maximal L^p regularity techniques to study the linear part of the problem. Here is worth mentioning that we employed the more standard Hilbert value framework because it produces the sharpest result on the regularity of the stochastic convolution in terms of the Hurst parameter $\mathcal{H}$. Indeed, assuming just for simplicity the case $s = 0$ and $g \in L^p(\Gamma_u; U)$ for some $p \in [2, +\infty)$, then by corollary 2.4, [18, Proposition 4.5] and arguing as above we have

$$w_g \in C\left([0, T]; \mathsf{D}\left(A_p^{\mathcal{H} - 1 + \frac{1}{2p} - \varepsilon}\right)\right).$$

In particular $w_g \in C([0, T]; \mathbb{L}^r)$ for some $r > 2$ if $\mathcal{H} > 1 - \frac{1}{2p}$. Therefore the right-hand side is minimized and we can use the rougher noise for $p = 2$.

We end this subsection showing a lemma concerning the relation between the mild and the weak formulation of (1.4) as defined below.

Definition 3.4. Let assumption 1.1 be satisfied. A stochastic process w is a weak solution of (1.4) if it is $\mathcal{F}$-progressively measurable with $\mathbf{P} - a.s.$ paths in

$$w_g \in C\left(0, T; \mathbb{L}^r\right)$$

for some $r \geqslant 2$, and $\mathbf{P} - a.s.$ for all $\phi \in \mathsf{D}(A)$ and $t \in [0, T]$,

$$\langle w_g(t), \phi \rangle = -\int_0^t \langle w_g(s), A\phi \rangle \, \mathrm{d}s - \langle g, \widehat{n} \cdot \nabla\phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} W_t^{\mathcal{H}}. \tag{3.4}$$

As above, $\widehat{n}$ denotes the exterior normal vector field on $\partial\mathcal{O}$. Since g is time-independent, the last term in (3.4) can be rewritten as a stochastic integral as

$$\langle g, \widehat{n} \cdot \nabla\phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} W_t^{\mathcal{H}} = \int_0^t \langle g, \widehat{n} \cdot \nabla\phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} \, \mathrm{d}W_s^{\mathcal{H}}.$$

Lemma 3.5. *Let assumption 1.1 be satisfied. There exists a unique weak solution of (1.4) in the sense of definition 3.4 and it is given by the formula (3.1).*

Proof. We split the proof into two steps.

Step 1: There exists a unique weak solution of (1.4) and it is necessarily given by the mild formula (3.1). Let $\psi \in C^1([0,T]; \mathsf{D}(A))$. Arguing as in the first step of the proof of [31, theorem 1.7], see also [46, Lemma 3], one can readily check that w_g satisfies

$$
\begin{aligned}
\langle w_g(t), \psi(t) \rangle = & \int_0^t \langle w_g(s), \partial_s \psi(s) \rangle \, \mathrm{d}s - \int_0^t \langle w_g(s), A\psi(s) \rangle \, \mathrm{d}s \\
& - \int_0^t \langle g, \widehat{n} \cdot \nabla \psi(s) \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} \, \mathrm{d}W_s^{\mathcal{H}}
\end{aligned}
\tag{3.5}
$$

for each $t \in [0,T]$, $\mathbf{P} - a.s.$ The stochastic integral in the relation above is well-defined as a real-valued stochastic integral. Indeed, recalling that $(W_t^{\mathcal{H}})_{t \geqslant 0}$ is a U-cylindrical fractional Brownian motion we observe that $\langle g, \widehat{n} \cdot \nabla \psi(s) \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)}$ is given by the linear operator on U

$$
h' \mapsto \langle gh', \widehat{n} \cdot \nabla \psi(s) \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} = L_\psi(gh'),
$$

where $L_\psi := \langle \cdot, \widehat{n} \cdot \nabla \psi(\cdot) \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)}$. By the ideal property of the Hilbert–Schmidt operators we have $\mathscr{L}_2(U, H^{-s}(\Gamma_u)) = H^{-s}(\Gamma_u; U)$ and obtain that

$$
\|\langle g, \widehat{n} \cdot \nabla \psi(s) \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)}\|_{U^*} \lesssim \|g\|_{H^{-s}(\Gamma_u)} \|\nabla \psi(s)\|_{H^s(\Gamma_u)} \quad \text{a.e. on } \Omega \times (0,T).
$$

In conclusion, the stochastic integral in (3.5) is well-defined as a real-valued one (see [23, (2.16)]), since

$$
\begin{aligned}
& \mathbf{E} \left| \int_0^t \langle g, \widehat{n} \cdot \nabla \psi(s) \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} \, \mathrm{d}W_s^{\mathcal{H}} \right|^2 \\
& \leqslant \mathcal{H}(2\mathcal{H}-1) \int_0^t \int_0^t \|\langle g, \widehat{n} \cdot \nabla \psi(s) \rangle\|_{U^*} \|\langle g, \widehat{n} \cdot \nabla \psi(v) \rangle\|_{U^*} |s-v|^{2\mathcal{H}-2} \, \mathrm{d}s \, \mathrm{d}v \\
& \leqslant \mathcal{H}(2\mathcal{H}-1) \|g\|_{H^{-s}(\Gamma_u; U)}^2 \int_0^t \int_0^t \|\nabla \psi(s)\|_{H^s(\Gamma_u)} \|\nabla \psi(v)\|_{H^s(\Gamma_u)} |s-v|^{2\mathcal{H}-2} \, \mathrm{d}s \, \mathrm{d}v,
\end{aligned}
$$

which is finite since $\psi \in C^1([0,T]; \mathsf{D}(A))$ and $\mathcal{H} > 1/2$. Now consider $\phi \in \mathsf{D}(A^2)$ and use $\psi_t(s) = S(t-s)\phi$, $s \in [0,t]$ as test function in (3.5) obtaining

$$
\langle w_g(t), \phi \rangle = - \int_0^t \langle g, \widehat{n} \cdot \nabla S(t-s)\phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} \, \mathrm{d}W_s^{\mathcal{H}}.
\tag{3.6}
$$

Recalling the definition of the Dirichlet map $\mathcal{D}$, (3.6) can be rewritten as

$$
\langle w(t), \phi \rangle = \int_0^t \langle \mathcal{D}[g], AS(t-s)\phi \rangle \, \mathrm{d}W_s^{\mathcal{H}}.
\tag{3.7}
$$

Then, exploiting the self-adjointness property of S and A we have that weak solutions of (1.4) satisfy the mild formulation. Therefore they are unique.

Step 2: The mild formula (3.1) is a weak solution of (1.4) in the sense of definition 3.4. We begin by noticing that w_g has the required regularity due to proposition 3.1. Let us test our mild formulation (3.1) against functions $\phi \in \mathsf{D}(A^2)$. It holds, exploiting self-adjointness property of S and A

$$
\begin{aligned}
\langle w(t), \phi \rangle &= \int_0^t \langle \mathcal{D}[g], AS(t-s)\phi \rangle \, \mathrm{d}W_s^{\mathcal{H}} \\
&= \int_0^t \langle g, \widehat{n} \cdot \nabla S(t-s)\phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} \, \mathrm{d}W_s^{\mathcal{H}} \quad \mathbf{P} - a.s.,
\end{aligned}
$$

where in the last step we used the definition of Dirichlet map. To complete the proof of this step it is enough to show that

$$
\int_0^t \langle g, \widehat{n} \cdot \nabla S(t-s)\phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} \, \mathrm{d}W_s^{\mathcal{H}} = -\int_0^t \langle w_g(s), A\phi \rangle \, \mathrm{d}s \\
+ \langle g, \widehat{n} \cdot \nabla \phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} W_t^{\mathcal{H}} \quad \mathbf{P} - a.s. \quad (3.8)
$$

The relation (3.8) is true. Indeed,

$$
\int_0^t \langle w_g(s), A\phi \rangle \, \mathrm{d}s = \int_0^t \mathrm{d}s \int_0^s \langle \mathcal{D}[g], S(s-\tau)A^2\phi \rangle \, \mathrm{d}W^{\mathcal{H}}(\tau) \quad \mathbf{P} - a.s. \quad (3.9)
$$

The double integrals in (3.9) can be exchanged via stochastic Fubini's theorem, see [3, 50]. Therefore the double integral in the right-hand side of (3.9) can be rewritten as

$$
\begin{aligned}
\int_0^t \mathrm{d}s &\int_0^s \langle \mathcal{D}[g], S(s-\tau)A^2\phi \rangle \, \mathrm{d}W_\tau^{\mathcal{H}} \\
&= \int_0^t \mathrm{d}W^{\mathcal{H}}(\tau) \int_\tau^t \langle \mathcal{D}[g], S(s-\tau)A^2\phi \rangle \, \mathrm{d}s \\
&= \langle \mathcal{D}[g], A\phi \rangle W_t^{\mathcal{H}} - \int_0^t \langle \mathcal{D}[g], AS(t-\tau)\phi \rangle \, \mathrm{d}W_\tau^{\mathcal{H}} \\
&= \langle g, \widehat{n} \cdot \nabla \phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} W_t^{\mathcal{H}} \\
&\quad - \int_0^t \langle g, \widehat{n} \cdot \nabla S(t-\tau)\phi \rangle_{H^{-s}(\Gamma_u), H^s(\Gamma_u)} \, \mathrm{d}W_\tau^{\mathcal{H}} \quad \mathbf{P} - a.s.
\end{aligned}
$$

Inserting this expression in (3.9), (3.8) holds and the proof is complete. □

3.2. Auxiliary Navier–Stokes type equations

Motivated by the auxiliary problem (1.5) and by the results of the previous subsection, here we study the well-posedness of the following abstract PDE

$$
\begin{cases}
\partial_t v + A_q v + B(v+w, v+w) = 0, & t \in [0,T], \\
v(0) = u_{\text{in}},
\end{cases} \quad (3.10)
$$

with A_q is the Stokes operator on $\mathbb{L}^q$ and B is the bilinear nonlinearity as in section 2.3. Finally, $q = r/2$ and (w, r) satisfies the following

Assumption 3.6. $w \in C([0,T]; \mathbb{L}^r)$ for some $r \in (2,4)$.

Note that the above assumption is satisfied $\mathbf{P} - a.s.$ with $w = w_g$, as it follows from proposition 3.1. Moreover, the limitation $r < 4$ is motivated by remark 3.2. In particular, the arguments used in [1] do not apply to (3.10). Indeed, if assumption 3.6 holds, then

$$B(w,w) \notin L^2 \left(0,T; H^{-1} \left(\mathcal{O}; \mathbb{R}^2\right)\right).$$

Hence, the (potential) energy of solutions for (3.10), i.e.

$$\int_0^t \int_{\mathcal{O}} |\nabla v|^2 \, \mathrm{d}x \, \mathrm{d}s \ \text{ for } \ t > 0,$$

is *ill-defined* even in absence of the terms $B(v,v), B(w,v)$ and $B(v,w)$. In particular, one cannot expect energy (or Leray's) type solutions for (3.10) to be defined and the analysis carried on in [1] does not work in our framework. However, $B(v,v) \in L^{\infty}(0,T; H^{-1,q}(\mathcal{O}; \mathbb{R}^2))$ for some $q > 1$ as $r > 2$, and therefore L^q-theory for (3.10) can be built.

Next, let us describe the main idea behind our construction of a solution to (3.10). In what follows, the subcriticality of $L^{\infty}(0,T; \mathbb{L}^r)$ with $r > 1$ for the 2D Navier–Stokes equations (cf the discussion below definition 1.2) plays a central role. Indeed, by subcriticality, given $q_0 = \frac{r}{2}$, the solution v_0 to

$$\begin{cases} \partial_t v_0 + A_{q_0} v_0 + B(w,w) = 0, \\ v_0(0) = 0, \end{cases}$$

satisfies $v_0 \in L^p(0,T; L^{r_0}(\mathcal{O}; \mathbb{R}^2))$ for some $r_0 > r$ and each $p < +\infty$. Hence, we obtained a small gain of space regularity. In particular, $\bar{v}_1 = v - v_0$ solves

$$\begin{cases} \partial_t \bar{v}_1 + A_{q_1} \bar{v}_1 + B(\bar{v}_1, \bar{v}_1) + B(\bar{v}_1, w + v_0) + B(w + v_0, \bar{v}_1) \\ \qquad\qquad\qquad + B(v_0, w + v_0) + B(w, v_0) = 0, \\ \bar{v}_1(0) = u_{\mathrm{in}}, \end{cases}$$

In the above, we would like to take $q_1 > q_0$ due to the increased regularity of the forcing terms. Indeed, as $v_0 \in L^p(0,T; L^{r_0}(\mathcal{O}; \mathbb{R}^2))$ for some $r_0 > r$ and each $p < +\infty$ one obtains that the terms $B(w + v_0, v_0)$ and $B(w, v_0)$ belong to $L^p(0,T; H^{-1,q_1}(\mathcal{O}; \mathbb{R}^2))$ where $\frac{1}{q_1} = \frac{1}{r_0} + \frac{1}{r}$ satisfies $q_1 > q_0$. In particular, the terms appearing in the problem above are more regular in space than $B(w,w)$. This opens the door to a further iteration. In particular, by considering the solution v_1 to

$$\begin{cases} \partial_t v_1 + A_{q_1} v_1 + B(v_0, w + v_0) + B(w, v_0) = 0, \\ v_1(0) = 0, \end{cases}$$

and studying the problem for $\bar{v}_2 = \bar{v}_1 - v_1$, one can check that the above procedure leads to a further improvement. The idea is to stop the iteration whenever the forcing terms appearing in the procedure are regular enough to build Leray-type solutions to the corresponding PDE. Before going further, let us stress that the above procedure is reminiscent of the so-called 'DaPrato–Debussche trick' introduced in [20] and now is widely used in the context of stochastic PDEs.

Let us now turn to the construction of a solution to (3.10). The above argument motivates the following splitting. Let N be a positive integer such that

$$r \in \left[\frac{2(N+2)}{N+1}, \frac{2(N+1)}{N} \right) \tag{3.11}$$

then we search of a solution v to (3.10) given by a sum of $N+1$ terms

$$v = \sum_{i=0}^{N-1} v_i + \overline{v} \tag{3.12}$$

where v_i and $\overline{v}$ solve the following system of PDEs on $[0,T]$:

$$\begin{cases} \partial_t v_0 + A_{q_0} v_0 + B(w,w) = 0, \\[2mm] \partial_t v_i + A_{q_i} v_i + B\left(v_{i-1}, w + \sum_{j=0}^{i-1} v_j \right) + B\left(w + \sum_{j=0}^{i-2} v_j, v_{i-1} \right) = 0, \\[4mm] \partial_t \overline{v} + A\overline{v} + B(\overline{v},\overline{v}) + B\left(\overline{v}, w + \sum_{j=0}^{N-1} v_j \right) + B\left(w + \sum_{j=0}^{N-1} v_j, \overline{v} \right) \\[4mm] \qquad + B\left(v_{N-1}, w + \sum_{j=0}^{N-1} v_j \right) + B\left(w + \sum_{j=0}^{N-2} v_j, v_{N-1} \right) = 0, \\[4mm] v_i(0) = 0, \\[1mm] \overline{v}(0) = u_{\mathrm{in}}, \end{cases} \tag{3.13}$$

where, $\sum_{j=0}^{-1} := 0$, $i \in \{1,\ldots,N-1\}$ and

$$q_i = \frac{2r}{r + 2 + (i+1)(2-r)}. \tag{3.14}$$

Note that v_i for $i \in \{1,\ldots,N-1\}$ solves a (linear) Stokes problem, while the problem for $\overline{v}$ is a modified version of the Navier–Stokes equations.

At least formally, it is clear that v solves (3.10). The latter fact is a straightforward consequence of the following identity involving $B(v+w,v+w)$ (letting $v_{-1} = w$, $v_N := \overline{v}$ for simplicity)

$$\begin{aligned} B(v+w,v+w) &= \sum_{i,j=-1}^{N} B(v_i,v_j) \\ &= \sum_{i=-1}^{N} B(v_i,v_i) + \sum_{i=-1}^{N} \sum_{j=-1}^{i-1} B(v_i,v_j) + \sum_{j=-1}^{N} \sum_{i=-1}^{j-1} B(v_i,v_j) \\ &= B(\overline{v},\overline{v}) + \sum_{i=-1}^{N-1} B(v_i,v_i) + \sum_{i=-1}^{N-1} B(\overline{v},v_i) + \sum_{i=-1}^{N-1} \sum_{j=-1}^{i-1} B(v_i,v_j) \\ &\quad + \sum_{i=-1}^{N-1} B(v_i,\overline{v}) + \sum_{j=-1}^{N-1} \sum_{i=-1}^{j-1} B(v_i,v_j) \end{aligned}$$

$$
= B(\bar{v}, \bar{v}) + \sum_{i=-1}^{N-1} B(v_i, \bar{v}) + \sum_{i=-1}^{N-1} B(\bar{v}, v_i)
$$

$$
+ \sum_{i=-1}^{N-1} \sum_{j=-1}^{i} B(v_i, v_j) + \sum_{j=-1}^{N-1} \sum_{i=-1}^{j-1} B(v_i, v_j)
$$

$$
= \left[B(\bar{v}, \bar{v}) + B\left(w + \sum_{j=0}^{N-1} v_j, \bar{v} \right) + B\left(\bar{v}, w + \sum_{j=0}^{N-1} v_j \right) \right.
$$

$$
+ B\left(v_{N-1}, w + \sum_{j=0}^{N-1} v_j \right) + B\left(w + \sum_{j=0}^{N-2} v_j, v_{N-1} \right) \Bigg]
$$

$$
+ \sum_{i=-1}^{N-2} \left[B\left(v_i, w + \sum_{j=0}^{i} v_j \right) + B\left(w + \sum_{j=0}^{i-1} v_j, v_i \right) \right].
$$

Looking at the last line, we can formally identify the first bracketed term as the one appearing in the equation for $\bar{v}$ and the *i*th summand as the one appearing in the equation for v_{i+1}, $i+1 \in \{0, 1, \dots, N-1\}$. To show rigorously that v given in (3.12) with $(v_0, \dots, v_{N-1}, \bar{v})$ solving (3.13) is a solution to (3.10) we need to check that $v_0, \dots, v_{N-1}$ and $\bar{v}$ are sufficiently regular. The appropriate regularity class for v in (3.12) to obtain a solution is given in the following definition, see also remark 3.8 below.

Definition 3.7. Given $r \in (2, 4)$, N given by (3.11), $p \geqslant 2^N \frac{r}{r-2}$, we say that

$$
(v_0, \dots, v_{N-1}, \bar{v})
$$

is a (p, r)-solution of (3.13) if

$$
\begin{aligned}
&v_i \in W^{1, p/2^i}\left(0, T; X_{-1/2, A_{q_i}} \right) \cap L^{p/2^i}\left(0, T; X_{1/2, A_{q_i}} \right), \\
&\bar{v} \in C\left([0, T]; H \right) \cap L^2\left(0, T; V \right),
\end{aligned}
\tag{3.15}
$$

where q_i is as in (3.14), and for each

$$
(\phi_0, \dots, \phi_{N-1}, \overline{\phi}) \quad \text{s.t. } \phi_i \in \mathsf{D}\left(A_{q_i'} \right), \ \overline{\phi} \in \mathsf{D}(A)
$$

we have, for all $t \in [0, T]$,

$$
\langle v_0(t), \phi_0 \rangle = -\int_0^t \langle v_0(s), A_{q_0'} \phi_0 \rangle \, \mathrm{d}s + \int_0^t \langle w(s) \otimes w(s), \nabla \phi_0 \rangle \, \mathrm{d}s, \tag{3.16}
$$

$$
\langle v_i(t), \phi_i \rangle = -\int_0^t \langle v_i(s), A_{q_i'} \phi_i \rangle \, \mathrm{d}s + \int_0^t \langle v_{i-1}(s) \otimes v_{i-1}(s), \nabla \phi_i \rangle \, \mathrm{d}s \tag{3.17}
$$

$$
+ \int_0^t \left\langle v_{i-1}(s) \otimes \left(w(s) + \sum_{j=0}^{i-2} v_j(s) \right), \nabla \phi_i \right\rangle \, \mathrm{d}s
$$

$$
+ \int_0^t \left\langle \left(w(s) + \sum_{j=0}^{i-2} v_j(s) \right) \otimes v_{i-1}(s), \nabla \phi_i \right\rangle \, \mathrm{d}s,
$$

$$\langle \overline{v}(t), \overline{\phi} \rangle = \langle u_{\text{in}}, \overline{\phi} \rangle - \int_0^t \langle \overline{v}(s), A\overline{\phi} \rangle \, ds + \int_0^t \langle \overline{v}(s) \otimes \overline{v}(s), \nabla \overline{\phi} \rangle \, ds$$

$$+ \int_0^t \left\langle \overline{v}(s) \otimes \left(w(s) + \sum_{j=0}^{N-1} v_j(s) \right), \nabla \overline{\phi} \right\rangle \, ds$$

$$+ \int_0^t \left\langle \left(w(s) + \sum_{j=0}^{N-1} v_j(s) \right) \otimes \overline{v}(s), \nabla \overline{\phi} \right\rangle \, ds$$

$$+ \int_0^t \left\langle v_{N-1}(s) \otimes v_{N-1}(s), \nabla \overline{\phi} \right\rangle \, ds$$

$$+ \int_0^t \left\langle v_{N-1}(s) \otimes \left(w(s) + \sum_{j=0}^{N-2} v_j(s) \right), \nabla \overline{\phi} \right\rangle \, ds$$

$$+ \int_0^t \left\langle \left(w(s) + \sum_{j=0}^{N-2} v_j(s) \right) \otimes v_{N-1}(s), \nabla \overline{\phi} \right\rangle \, ds. \qquad (3.18)$$

Remark 3.8. We observe that $q_i > 1$ for all $i \in \{0, \ldots, N-1\}$ and is increasing in i. As an immediate consequence of definition 3.7, Sobolev embedding theorem and interpolation we have that

$$v_i \in L^{p/2^i}(0, T; \mathbb{L}^{r_i}) \cap C([0, T]; H), \quad r_i = \frac{2r}{(i+1)(2-r)+2}.$$

In particular, $r_i > 2$ for all $i \in \{0, \ldots, N-1\}$ and is increasing in i. Therefore one can easily check that all the duality pairings in definition 3.7 are well defined. Moreover, for all $i \in \{1, \ldots, N-1\}$,

$$v_i, \overline{v} \in L^{\frac{2r}{r-2}}(0, T; \mathbb{L}^r).$$

Indeed, the above assertion for v_i follows from $p \geqslant 2^N \frac{r}{r-2}$. While for $\overline{v}$ we can use the standard interpolation inequality $L^2(0, T; H^1) \cap L^\infty(0, T; L^2) \subseteq L^{2/\theta}(0, T; H^\theta)$ with $\theta = \frac{r-2}{r} \in (0, 1)$ and the Sobolev embedding $H^\theta(\mathcal{O}) \hookrightarrow L^r(\mathcal{O})$.

In particular, if $(v_0, \ldots, v_{N-1}, \overline{v})$ is a (p, r) solution of (3.13), then, given $v := \sum_{i=0}^{N-1} v_i + \overline{v}$, $u = v + w$ is a $r/2$-solution of (1.2) in the sense of definition 1.2.

The following yields the well-posedness of (3.10) in the sense of definition 3.7.

Theorem 3.9. *Let assumption 3.6 be satisfied. For each $u_{\text{in}} \in H$, $p \geqslant 2^N \frac{r}{r-2}$ there exists a unique (p, r)-solution $(v_0, \ldots, v_{N-1}, \overline{v})$ of (3.13) in the sense of definition 3.7. Moreover $\overline{v}$ satisfies the energy relation*

$$\|\bar{v}(t)\|^2 + 2\int_0^T \|\nabla\bar{v}(s)\|_{L^2}^2 \, \mathrm{d}s = \|u_{\mathrm{in}}\|^2 + 2\int_0^t \left\langle \bar{v}(s)\cdot\nabla\bar{v}(s), w(s) + \sum_{j=0}^{N-1} v_j(s) \right\rangle \mathrm{d}s$$

$$+ 2\int_0^t \left\langle v_{N-1}(s)\cdot\nabla\bar{v}(s), w(s) + \sum_{j=0}^{N-1} v_j(s) \right\rangle \mathrm{d}s$$

$$+ 2\int_0^t \left\langle \left(w(s) + \sum_{j=0}^{N-2} v_j(s) \right)\cdot\nabla\bar{v}(s), v_{N-1}(s) \right\rangle \mathrm{d}s. \quad (3.19)$$

If $(u_{\mathrm{in}}^n)_{n\in\mathbb{N}}$ is a sequence in H converging to $u_{\mathrm{in}} \in H$ and $(w^n)_{n\in\mathbb{N}}$ is a sequence in $C([0,T];\mathbb{L}^r)$ converging to $w \in C([0,T];\mathbb{L}^r)$, then the corresponding unique solutions $((v_0^n,\ldots,v_{N-1}^n,\bar{v}^n))_{n\in\mathbb{N}}$ converge to the corresponding solution $(v_0,\ldots,v_{N-1},\bar{v})$, each one in the topologies of definition 3.7.

Proof. We exploit strongly the triangle structure of (3.13) and split the proof in several steps.

Step 1: Linear part of (3.13). We argue by induction and exploit maximal L^p regularity techniques, see [55, chapter 3]. The existence and uniqueness of v_0 satisfying the corresponding PDE in the sense of definition 3.7 and the continuous dependence from data, i.e. w in the topology of $C([0,T];\mathbb{L}^r)$, follows if

$$B(w,w) \in L^p\left(0,T;X_{-1/2,A_{q_0}}\right).$$

The claim is true, indeed $q_0 = \frac{r}{2}$ and by Hölder's inequality we have

$$\int_0^T \|B(w(s),w(s))\|_{X_{-1/2,A_{r/2}}}^p \, \mathrm{d}s \leqslant \int_0^T \|w(s)\|_{\mathbb{L}^r}^{2p} \, \mathrm{d}s \leqslant T\|w\|_{C([0,T];\mathbb{L}^r)}^{2p}.$$

Now assume we have already proved the existence and uniqueness of $(v_i)_{i\in\{0,\ldots,l-1\}}, l \leqslant N-1$ solving (3.13) in the sense of definition 3.7 and depending continuously from the data, i.e. w in the topology of $C([0,T];\mathbb{L}^r)$. Let us check that there exists a unique v_l solving the corresponding PDE in (3.13) in the sense of definition 3.7 and depending continuously from w in the topology of $C([0,T];\mathbb{L}^r)$. Again, due to maximal L^p regularity techniques, it is enough to show that

$$B(v_{l-1},v_{l-1}) + B\left(v_{l-1}, w + \sum_{j=0}^{l-2} v_j\right) - B\left(w + \sum_{j=0}^{l-2} v_j, v_{l-1}\right) \in L^{p/2^l}\left(0,T;X_{-1/2,A_{q_l}}\right).$$

The claim is true, indeed due to remark 3.8

$$w, \, v_i \in L^{p/2^{l-1}}(0,T;\mathbb{L}^r) \quad \text{if } i \in \{0,\ldots,l-2\}$$

and by induction hypothesis

$$v_{l-1} \in L^{p/2^{l-1}}(0,T;\mathbb{L}^{r_{l-1}}).$$

Moreover all v_i, $i \in 1, \ldots, l-1$ depends continuously from $w \in C([0,T]; \mathbb{L}^r)$ in the corresponding topologies. Therefore by Hölder's inequality we have

$$
\int_0^T \left\| B\left(v_{l-1}(s), w(s) + \sum_{j=0}^{l-2} v_j(s) \right) \right\|_{X_{-1/2, A_{q_l}}}^{p/2^l} \, ds
$$

$$
\leqslant \int_0^T \left\| v_{l-1}(s) \otimes \left(w(s) + \sum_{j=0}^{l-2} v_j(s) \right) \right\|_{\mathbb{L}^{q_l}}^{p/2^l} \, ds
$$

$$
\lesssim_{p,l} \int_0^T \| v_{l-1}(s) \|_{\mathbb{L}^{r_{l-1}}}^{p/2^l} \left(\| w(s) \|_{\mathbb{L}^{r_{l-1}}}^{p/2^l} + \sum_{j=0}^{l-2} \| v_j(s) \|_{\mathbb{L}^{r_{l-1}}}^{p/2^l} \right) \, ds
$$

$$
\lesssim \| v_{l-1} \|_{L^{p/2^{l-1}}(0,T; \mathbb{L}^{r_{l-1}})}^{p/2^{l-1}} + \| w \|_{L^{p/2^{l-1}}(0,T; \mathbb{L}^q)}^{p/2^{l-1}} + \sum_{j=0}^{l-2} \| v_j \|_{L^{p/2^{l-1}}(0,T; \mathbb{L}^r)}^{p/2^{l-1}}.
$$

Step 2: Introduction to the nonlinear part of (3.13). First, we observe that due to *Step 1* we have that

$$
\bar{f} = -B\left(v_{N-1}, w + \sum_{j=0}^{N-1} v_j \right) - B\left(w + \sum_{j=0}^{N-2} v_j, v_{N-1} \right) \in L^2(0,T; V^*), \quad (3.20)
$$

$$
\tilde{v} = w + \sum_{j=0}^{N-1} v_j \in L^{p/2^{N-1}}(0,T; \mathbb{L}^r). \tag{3.21}
$$

Therefore we are left to study the well-posedness in the weak setting of the following PDE

$$
\begin{cases}
\partial_t \bar{v} + A\bar{v} + B(\bar{v}, \bar{v}) + B(\bar{v}, \tilde{v}) + B(\tilde{v}, \bar{v}) = \bar{f}, \\
\bar{v}(0) = u_{\text{in}}.
\end{cases} \tag{3.22}
$$

This can be treated similarly to [1, section 3.2] and is the object of the remaining steps.

Step 3: Uniqueness. Let $\bar{v}^{(i)}$ be two solutions. The function $z = \bar{v}^{(1)} - \bar{v}^{(2)}$ satisfies hence

$$
\langle z(t), \overline{\phi} \rangle + \int_0^t \langle z(s), A\overline{\phi} \rangle \, ds - \int_0^t \langle z(s) \cdot \nabla \overline{\phi}, z(s) \rangle \, ds = \int_0^t \langle \widetilde{f}(s), \overline{\phi} \rangle \, ds
$$

where

$$
\widetilde{f} = -B\left(\bar{v}^{(2)} + \tilde{v}, z \right) - B\left(z, \bar{v}^{(2)} + \tilde{v} \right).
$$

By lemma 2.7, $\widetilde{f} \in L^2(0,T; V^*)$. Then, by theorem 2.6 ,

$$
\| z(t) \|^2 + 2 \int_0^t \| \nabla z(s) \|_{L^2}^2 \, ds = 2 \int_0^t \langle z(s) \cdot \nabla z(s), \bar{v}^{(2)}(s) + \tilde{v}(s) \rangle \, ds.
$$

Again by lemma 2.7, we have

$$
\int_0^t \langle z(s) \cdot \nabla z(s), \overline{v}^{(2)}(s) + \tilde{v}(s) \rangle \, ds
$$

$$
\leqslant \left| \int_0^t \langle z(s) \cdot \nabla z(s), \overline{v}^{(2)}(s) \rangle \, ds \right| + \left| \int_0^t \langle z(s) \cdot \nabla z(s), \tilde{v}(s) \rangle \, ds \right|
$$

$$
\leqslant 2\varepsilon \|z\|_{L^2(0,t;V)}^2 + \frac{C}{\varepsilon^3} \int_0^t \|z(s)\|^2 \|\overline{v}^{(2)}(s)\|_{\mathbb{L}^4}^4 \, ds
$$

$$
+ 2\varepsilon \|z\|_{L^2(0,t;V)}^2 + \frac{C}{\varepsilon^{\frac{r+2}{r-2}}} \int_0^t \|z(s)\|^2 \|\tilde{v}(s)\|_{\mathbb{L}^r}^{\frac{2r}{r-2}} \, ds
$$

$$
= 4\varepsilon \int_0^t \|\nabla z(s)\|_{L^2}^2 \, ds + \frac{C}{\varepsilon^{\frac{r+2}{r-2}}} \int_0^t \|z(s)\|^2 \left(\|\overline{v}^{(2)}(s)\|_{\mathbb{L}^4}^4 + \|\tilde{v}(s)\|_{\mathbb{L}^r}^{\frac{2r}{r-2}} \right) \, ds.
$$

Applying the above with $4\varepsilon = \frac{1}{2}$ and renaming the constant C, it follows that

$$
\|z(t)\|^2 + \int_0^t \|\nabla z(s)\|_{L^2}^2 \, ds \leqslant C \int_0^t \|z(s)\|^2 \left(\|\overline{v}^{(2)}(s)\|_{\mathbb{L}^4}^4 + \|\tilde{v}(s)\|_{\mathbb{L}^r}^{\frac{2r}{r-2}} \right) \, ds.
$$

We conclude $z = 0$ by the Grönwall lemma, using (3.21) and the integrability properties of $\overline{v}^{(2)}$.

Step 4: Existence. Define the sequence $(\overline{v}^n)$ by setting $\overline{v}^0 = 0$ and for every $n \geqslant 0$, given $\overline{v}^n \in C([0,T];H) \cap L^2(0,T;V)$, let $\overline{v}^{n+1}$ be the solution of equation (2.9) with initial condition u_{in} and with

$$
f = -B(\overline{v}^n, \tilde{v}) - B(\tilde{v}, \overline{v}^n) + \overline{f}.
$$

In particular

$$
\langle \overline{v}^{n+1}(t), \overline{\phi} \rangle + \int_0^t \langle \overline{v}^{n+1}(s), A\overline{\phi} \rangle \, ds - \int_0^t \langle \overline{v}^{n+1}(s) \cdot \nabla \overline{\phi}, \overline{v}^{n+1}(s) \rangle \, ds
$$

$$
= \langle u_{\mathrm{in}}, \overline{\phi} \rangle + \int_0^t \langle f(s), \overline{\phi} \rangle \, ds
$$

for every $\overline{\phi} \in \mathsf{D}(A)$. The above is well-defined as

$$
B(\overline{v}^n, \tilde{v}), B(\tilde{v}, \overline{v}^n), \overline{f} \in L^2(0,T;V^*)
$$

by lemma 2.7 and (3.20).

Then let us investigate the convergence of $(\overline{v}^n)$. First, let us prove a bound. From the previous identity and theorem 2.6 we get

$$\|\overline{v}^{n+1}(t)\|^2 + 2\int_0^t \|\nabla \overline{v}^{n+1}(s)\|_{L^2}^2 \, \mathrm{d}s$$
$$= \|u_{\mathrm{in}}\|^2 + 2\int_0^t \left(b\left(\overline{v}^n, \overline{v}^{n+1}, \tilde{v}\right) + b\left(\tilde{v}, \overline{v}^{n+1}, \overline{v}^n\right) + \langle \bar{f}, \overline{v}^{n+1}\rangle \right)(s) \, \mathrm{d}s.$$

It gives us using lemma 2.7 and (3.20)

$$\|\overline{v}^{n+1}(t)\|^2 + \int_0^t \|\nabla \overline{v}^{n+1}(s)\|_{L^2}^2 \, \mathrm{d}s = \|u_{\mathrm{in}}\|^2 + \varepsilon \int_0^t \|\overline{v}^n(s)\|_V^2 \mathrm{d}s$$
$$+ C_\varepsilon \int_0^t \|\overline{v}^n(s)\|^2 \|\tilde{v}(s)\|_{\mathbb{L}^r}^{\frac{2r}{r-2}} \, \mathrm{d}s + C_\varepsilon \int_0^t \|\bar{f}(s)\|_{V^*}^2 \, \mathrm{d}s.$$

Choosing a small constant ε, one can find $R > \|u_{\mathrm{in}}\|^2$ and $\overline{T}$ small enough, depending only from $\|u_{\mathrm{in}}\|$ and $\|\tilde{v}\|_{L^{\frac{2r}{r-2}}(0,T;\mathbb{L}^r)}$, such that if

$$\sup_{t\in[0,\overline{T}]} \|\overline{v}^n(t)\|^2 \leqslant R, \qquad \int_0^{\overline{T}} \|\overline{v}^n(s)\|_V^2 \mathrm{d}s \leqslant R \qquad (3.23)$$

then the same inequalities hold for $\overline{v}^{n+1}$.

Set $z_n = \overline{v}^n - \overline{v}^{n-1}$, for $n \geqslant 1$. From the identity above,

$$\langle z_{n+1}(t), \overline{\phi}\rangle - \int_0^t \left(b\left(\overline{v}^{n+1}, \overline{\phi}, \overline{v}^{n+1}\right) - b\left(\overline{v}^n, \overline{\phi}, \overline{v}^n\right) \right)(s) \, \mathrm{d}s$$
$$= -\int_0^t \langle z_{n+1}(s), A\overline{\phi}\rangle \, \mathrm{d}s - \int_0^t \langle \left(B\left(\overline{v}^n, \tilde{v}\right) - B\left(\overline{v}^{n-1}, \tilde{v}\right)\right)(s), \overline{\phi}\rangle \, \mathrm{d}s$$
$$- \int_0^t \langle \left(B\left(\tilde{v}, \overline{v}^n\right) - B\left(\tilde{v}, \overline{v}^{n-1}\right)\right)(s), \overline{\phi}\rangle \, \mathrm{d}s.$$

Since

$$b\left(\overline{v}^{n+1}, \overline{\phi}, \overline{v}^{n+1}\right) - b\left(\overline{v}^n, \overline{\phi}, \overline{v}^n\right) - b\left(z_{n+1}, \overline{\phi}, z_{n+1}\right)$$
$$= b\left(\overline{v}^n, \overline{\phi}, z_{n+1}\right) + b\left(z_{n+1}, \overline{\phi}, \overline{v}^n\right)$$

we may rewrite it as

$$\langle z_{n+1}(t), \overline{\phi}\rangle - \int_0^t b\left(z_{n+1}(s), \overline{\phi}, z_{n+1}(s)\right) \, \mathrm{d}s$$
$$= -\int_0^t \langle z_{n+1}(s), A\overline{\phi}\rangle \, \mathrm{d}s - \int_0^t \langle \left(B\left(z_n, \tilde{v}\right) + B\left(\tilde{v}, z_n\right)\right)(s), \overline{\phi}\rangle \, \mathrm{d}s$$
$$+ \int_0^t \left(b\left(\overline{v}^n, \overline{\phi}, z_{n+1}\right) + b\left(z_{n+1}, \overline{\phi}, \overline{v}^n\right) \right)(s) \, \mathrm{d}s.$$

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One can check as above the applicability of theorem 2.6 and get

$$\|z_{n+1}(t)\|^2 + 2\int_0^t \|\nabla z_{n+1}(s)\|_{L^2}^2 \, \mathrm{d}s$$
$$= 2\int_0^t \left(b(z_n, z_{n+1}, \tilde{v}) + b(\tilde{v}, z_{n+1}, z_n)\right)(s)\, \mathrm{d}s$$
$$+ 2\int_0^t b(z_{n+1}, z_{n+1}, \overline{v}^n)(s)\, \mathrm{d}s.$$

As above, thanks to lemma 2.7 we deduce that

$$\int_0^t |b(z_{n+1}, z_{n+1}, \overline{v}^n)(s)|\, \mathrm{d}s \leqslant \frac{1}{4}\int_0^t \|z_{n+1}(s)\|_V^2 \, \mathrm{d}s + C\int_0^t \|z_{n+1}(s)\|^2 \|\overline{v}^n(s)\|_{\mathbb{L}^4}^4 \, \mathrm{d}s.$$

But

$$\int_0^t |b(z_n, z_{n+1}, \tilde{v})(s) + b(\tilde{v}, z_{n+1}, z_n)(s)|\, \mathrm{d}s$$
$$\leqslant \frac{1}{4}\int_0^t \|z_{n+1}(s)\|_V^2 \, \mathrm{d}s + \frac{1}{8}\int_0^t \|z_n(s)\|_V^2 \, \mathrm{d}s + C\int_0^t \|z_n(s)\|^2 \|\tilde{v}(s)\|_{\mathbb{L}^r}^{\frac{2r}{r-2}} \, \mathrm{d}s.$$

Hence

$$\|z_{n+1}(t)\|^2 + \int_0^t \|\nabla z_{n+1}(s)\|_{L^2}^2 \, \mathrm{d}s \leqslant C\int_0^t \|z_{n+1}(s)\|^2 \|\overline{v}^n(s)\|_{\mathbb{L}^4}^4 \, \mathrm{d}s$$
$$+ \frac{1}{4}\int_0^t \|z_n(s)\|_V^2 \, \mathrm{d}s + C\int_0^t \|z_n(s)\|^2 \|\tilde{v}(s)\|_{\mathbb{L}^r}^{\frac{2r}{r-2}} \, \mathrm{d}s.$$

Now we work under the bounds (3.23) and deduce, using the Grönwall lemma, for $\overline{T}$, depending only from $\|u_{\mathrm{in}}\|$ and $\|\tilde{v}\|_{L^{\frac{2r}{r-2}}(0,T;\mathbb{L}^r)}$, possibly smaller than the previous one,

$$\sup_{t\in[0,\overline{T}]} \|z_{n+1}(t)\|^2 + \int_0^{\overline{T}} \|z_{n+1}(s)\|_V^2 \, \mathrm{d}s \leqslant \frac{1}{2}\left(\sup_{t\in[0,\overline{T}]} \|z_n(t)\|^2 + \int_0^{\overline{T}} \|z_n(s)\|_V^2 \, \mathrm{d}s\right).$$

Now we can proceed as in the second step of the proof of [1, theorem 3.3], showing that the sequence $(\overline{v}^n)$ is Cauchy in $C([0,\overline{T}];H) \cap L^2(0,\overline{T};V)$. Its limit $\overline{v}$ is a weak solution of (3.22) on $[0,\overline{T}]$ and , hence, by the previous step, it is the unique solution. We refer the reader to [1, theorem 3.3] for further details. After proving existence and uniqueness in $[0,\overline{T}]$ we can reiterate the existence procedure and in a finite number of steps cover the interval $[0,T]$.

Step 5: Continuity dependence on the data Let $\overline{v}^n$ (resp. $\overline{v}$) the unique solution of (3.22) with data u_{in}^n, $\overline{f}^n$, $\tilde{v}^n$ (resp. u_{in}, $\overline{f}$, $\tilde{v}$). Since $u_0^n \to u_0$ in H (resp. $\overline{f}^n \to \overline{f}$ in $L^2(0,T;V^*)$, $\tilde{v}^n \to \tilde{v}$ in $L^{p/2^{N-1}}(0,T;\mathbb{L}^r)$) the family $(u_{\mathrm{in}}^n)_{n\in\mathbb{N}}$ is bounded in H (resp. the family $(\overline{f}^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;V^*)$, the family $(\widetilde{v}^n)_{n\in\mathbb{N}}$ is bounded in $L^{p/2^{N-1}}(0,T;\mathbb{L}^r)$), by (3.19) one can show easily

that the family $(\overline{v}^n)_{n\in\mathbb{N}}$ is bounded in $C([0,T];H) \cap L^2(0,T;V)$. Moreover for each $t \in [0,T]$, $z^n = \overline{v}^n - \overline{v}$ satisfies the energy relation

$$
\begin{aligned}
\frac{1}{2}\|z^n(t)\|^2 + \int_0^t \|\nabla z^n(s)\|_{L^2}^2 \, ds &= \frac{1}{2}\|u_{\text{in}}^n - u_{\text{in}}\|^2 \\
&\quad + \int_0^t b\left(z^n(s), z^n(s), \overline{v}(s)\right) ds \\
&\quad + \int_0^t b\left(\overline{v}^n(s), z^n(s), \widetilde{v}^n(s) - \widetilde{v}(s)\right) ds \\
&\quad + \int_0^t b\left(z^n(s), z^n(s), \widetilde{v}(s)\right) ds \\
&\quad + \int_0^t b\left(\widetilde{v}^n(s) - \widetilde{v}(s), z^n(s), \overline{v}(s)\right) ds \\
&\quad + \int_0^t \langle \overline{f}^n(s) - \overline{f}(s), z^n(s) \rangle \, ds.
\end{aligned}
\tag{3.24}
$$

Thanks to lemma 2.7, we can easily bound the right-hand side of relation (3.24) by Young's inequality and Hölder's inequality obtaining

$$
\begin{aligned}
\frac{1}{2}\|z^n(t)\|^2 + \frac{1}{2}\int_0^t \|\nabla z^n(s)\|_{L^2}^2 \, ds &\leqslant \frac{1}{2}\|u_{\text{in}}^n - u_{\text{in}}\|^2 + \int_0^t \|\overline{f}^n(s) - \overline{f}(s)\|_{V^*}^2 \, ds \\
&\quad + C\int_0^t \|z^n(s)\|^2 \left(\|\overline{v}(s)\|_{\mathbb{L}^4}^4 + \|\widetilde{v}(s)\|_{\mathbb{L}^r}^{\frac{2r}{r-2}} \right) ds \\
&\quad + C\|\widetilde{v}^n - \widetilde{v}\|_{L^{\frac{2r}{r-2}}(0,T;\mathbb{L}^r)}^2 \|\overline{v}^n\|_{C([0,T];H)}^{\frac{2(r-2)}{r}} \|\overline{v}^n\|_{L^2(0,T;V)}^{\frac{4}{r}} \\
&\quad + C\|\widetilde{v}^n - \widetilde{v}\|_{L^{\frac{2r}{r-2}}(0,T;\mathbb{L}^r)}^2 \|\overline{v}\|_{C([0,T];H)}^{\frac{2(r-2)}{r}} \|\overline{v}\|_{L^2(0,T;V)}^{\frac{4}{r}}.
\end{aligned}
\tag{3.25}
$$

Applying Grönwall's inequality to relation (3.25) the claimed continuity follows. $\qquad\square$

Remark 3.10. Freezing the variable $\omega \in \Omega$ and solving (3.10) for each ω does not allow us to obtain information about the measurability properties of v. However, the measurability of v with respect to the progressive σ-algebra follows from the continuity of the solution map with respect to u_{in} and w. Therefore we have the required measurability properties for v with w being the mild solution of (1.4). In particular v has **P**-a.s. paths in $C(0,T;H) \cap L^{\frac{2q}{q-1}}(0,T;\mathbb{L}^{2q})$ for each $r \in (1, q_{\mathcal{H}})$, it is progressively measurable with respect to these topologies.

Combining proposition 3.1, lemma 3.5, theorem 3.9 and remark 3.10 we get immediately the existence of a q-solution of equation (1.2) in the sense of definition 1.2 for each $q \in (1, q_{\mathcal{H}})$.

3.3. Proof of theorem 1.3(1)

As discussed above, the results of section 3.1, section 3.2 provide the existence of a q-solution of equation (1.2) in the sense of definition 1.2 for each $q \in (1, q_{\mathcal{H}})$, moreover such a solution is adapted with paths in $C([0,T];H)$ due to remark 3.8. Here we are left to discuss the problem of uniqueness. In order to reach our goal, we start providing a lemma which shows equivalence of q-solutions in the sense of definition 1.2 and those of the form $u = w_g + v$ as described by lemma 3.11 below. Then we conclude the proof by providing uniqueness for solutions of the form $u = w_g + v$.

Lemma 3.11. *Let $T < \infty$, $u_{\mathrm{in}} \in L^0_{\mathscr{F}_0}(\Omega; \mathbb{L}^2)$, $q \in (1, q_{\mathcal{H}})$ and assuming assumption 1.1. Then u is a q-solution to (1.2) in the sense of definition 1.2 with paths in $C([0,T]; \mathbb{L}^2)\, \mathbf{P} - a.s.$ if and only if $v := u - w_g$ is progressively measurable with paths in $L^{2q'}(0,T; \mathbb{L}^{2q}) \cap C([0,T]; \mathbb{L}^2)\, \mathbf{P} - a.s.$ and it solves, for all divergence-free $\varphi \in C^\infty(\mathcal{O}; \mathbb{R}^2)$ such that $\varphi = 0$ on $\Gamma_b \cup \Gamma_u$ and a.e. $t \in (0, T)$,*

$$\int_{\mathcal{O}} v(x,t)\,\varphi(x)\,\mathrm{d}x - \int_{\mathcal{O}} u_{\mathrm{in}}(x)\,\varphi(x)\,\mathrm{d}x \tag{3.26}$$
$$= \int_0^t \int_{\mathcal{O}} (v \cdot \Delta\varphi + [(v + w_g) \otimes (v + w_g)] : \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}s.$$

Proof. The proof is a trivial consequence of the notion of q-solution in definition 1.2, the regularity of w_g in case of assumption 1.1, i.e. proposition 3.1, and the equivalence between weak and mild solutions for the linear stochastic problem, see lemma 3.5. $\qquad\square$

We call a v progressively measurable with paths in $L^{2q'}(0,T; \mathbb{L}^{2q}) \cap C([0,T]; \mathbb{L}^2)\, \mathbf{P} - a.s.$ satisfying (3.26) in the sense of lemma 3.11 an *auxiliary q-solution* of (3.10). The uniqueness of auxiliary q-solutions and their independence on q is the content of the following result, which, combined with lemma 3.11, concludes the proof of the first item in theorem 1.3.

Proposition 3.12 (Uniqueness). *Let v_1 be an auxiliary q_1-solution of (3.10) and v_2 be an auxiliary q_2-solution of (3.10). Then $v_1 \equiv v_2$.*

The uniqueness result in the Ladyzhenskaya–Prodi–Serrin class of proposition 3.12 might be known to experts. Here, for completeness, we provide a relatively short proof relying on maximal L^p-regularity techniques which seem not standard even in the absence of noise.

Proof of proposition 3.12. We split the proof into two cases.

Case $q_1 = q_2 = q$. Letting $\delta := v_1 - v_2$, for all divergence free vector field $\varphi \in C^\infty(\mathcal{O}; \mathbb{R}^2)$ such that $\varphi = 0$ on $\Gamma_b \cup \Gamma_u$ and a.a. $t \in (0, T)$, we have

$$\langle \delta(t), \varphi \rangle - \int_0^t \langle \delta(s), \Delta\varphi \rangle\,\mathrm{d}s$$
$$= \int_0^t b(\delta(s), \varphi, v_1(s) + w_g(s))\,\mathrm{d}s + \int_0^t b(v_2(s) + w_g(s), \varphi, \delta(s))\,\mathrm{d}s.$$

As $v_i \in L^{2q'}(0,T; L^{2q}(\mathcal{O}; \mathbb{R}^2))$ for $i \in \{1, 2\}$, we obtain

$$B(\delta, v_1 + w_g), B(v_2 + w_g, \delta) \in L^{q'}\left(0,T; X_{-1/2, A_q}\right)\ \mathbf{P} - a.s.$$

Hence, by the density of divergence-free $\varphi \in C^\infty(\mathcal{O}; \mathbb{R}^2)$ such that $\varphi = 0$ on $\Gamma_b \cup \Gamma_u$ in the domain of the Stokes operator A_q and from the maximal L^q-regularity of A_q, it follows that

$$\delta \in W^{1,q'}\left(0,T; X_{-1/2, A_q}\right) \cap L^{q'}\left(0,T; X_{1/2, A_q}\right)$$
$$\subset C\left([0,T]; B_{q,q'}^{1-2/q'}(\mathcal{O}; \mathbb{R}^2)\right)\ \mathbf{P} - a.s.$$

where in the last step we used the trace embedding [55, theorem 3.4.8] applied with $A = A_q$.

By real interpolation (see e.g. [8, chapter 6]), we obtain

$$\left(B_{q,q'}^{1-2/q'}(\mathcal{O}), H^{1,q}(\mathcal{O})\right)_{1/2,1} \hookrightarrow B_{q,1}^{1-1/q'}(\mathcal{O}) \hookrightarrow L^{2q}(\mathcal{O})$$

where in the last step we applied the Sobolev embedding and $1 - \frac{1}{q'} - \frac{2}{q} = -\frac{1}{q}$. In particular,

$$\|f\|_{L^{2q}(\mathcal{O})} \lesssim \|f\|_{B_{q,q'}^{1-2/q'}(\mathcal{O})}^{1/2} \|f\|_{H^{1,q}(\mathcal{O})}^{1/2} \tag{3.27}$$

for all f for which the right-hand side is finite.

Hence, by maximal L^q-regularity of A_q, again the trace embedding [55, theorem 3.4.8] as well as the Hölder inequality, there exists a constant $C_0 > 0$ independent of v_1, v_2 and δ such that, for all $t \in [0, T]$ and $\mathbf{P} - a.s.$,

$$
\sup_{r \in [0,t]} \|\delta(r)\|_{B_{q,q'}^{1-2/q'}}^{q'} + \int_0^t \|\delta(r)\|_{H^{1,q}}^{q'} \, dr
$$
$$
\leqslant C_0 \int_0^t \left(\max_i \|v_i(r)\|_{L^{2q}}^{q'} + \|w_g(r)\|_{L^{2q}}^{q'}\right) \|\delta(r)\|_{L^{2q}}^{q'} \, dr
$$
$$
\leqslant C_1 \int_0^t \left(\max_i \|v_i(r)\|_{L^{2q}}^{2q'} + \|w_g(r)\|_{L^{2q}}^{2q'}\right) \|\delta(r)\|_{B_{q,q'}^{1-2/q'}}^{q'} \, dr + \frac{1}{2} \int_0^t \|\delta(r)\|_{H^{1,q}}^{q'} \, dr
$$

where in the last step we used the Young inequality and (3.27).

Now the conclusion follows from the Grönwall lemma and the integrability conditions on v_1, v_2 and $w_g \in C([0, \infty); L^{2q})$ for all $q < q_{\mathcal{H}}$ by proposition 3.1.

Case $q_1 \neq q_2$. In the case of $q_1 \neq q_2$ we start by observing that by previous case and the results of section 3.2, for $k \in \{1, 2\}$, we have that $v_k = \sum_{i=1}^{N_k-1} v_{k,i} + \bar{v}_k$ where $(v_{k,0}, \ldots, v_{k,N_k-1}, \bar{v}_k)$ is the (p_k, r_k)-solution of (3.13) in the sense of definition 3.7 with $p_k = 2^{N_k} \frac{q_k}{q_k-1}$ and $r_k = 2q_k$. The claim is then a particular case of lemma 3.13 below on the compatibility of the (p, r) solutions of (3.10) in the sense of definition 3.7. $\qquad\square$

Lemma 3.13 (Compatibility). *Let $w \in C([0, T]; \mathbb{L}^r)$ for some $r \in (2, 4)$ and $2 < \tilde{r} \leqslant r$. If $(v_0, \ldots, v_{N-1}, \bar{v})$ is a solution of (3.13) in the sense of definition 3.7 with*

$$p \geqslant 2^N \frac{r}{r-2}, \qquad q_i = \frac{2r}{r+2+(i+1)(2-r)}$$

and $(\tilde{v}_0, \ldots, \tilde{v}_{\tilde{N}-1}, \tilde{v})$ is a solution of (3.10) in the sense of definition 3.7 with

$$\tilde{p} \geqslant 2^{\tilde{N}} \frac{\tilde{r}}{\tilde{r}-2}, \qquad \tilde{q}_i = \frac{2\tilde{r}}{\tilde{r}+2+(i+1)(2-\tilde{r})}$$

then $v = \tilde{v}$.

Proof. The case of $r = \tilde{r}$ is obvious since in such a case $N = \tilde{N}$, $q_i = \tilde{q}_i$ and our construction does not rely on the choice of p so far that $p \geqslant 2^N \frac{r}{r-2}$. In the general case we have two sequences $(v_0, \ldots, v_{N-1}, \bar{v})$ and $(\tilde{v}_0, \ldots, \tilde{v}_{\tilde{N}-1}, \tilde{v})$. If $N = \tilde{N}$ the claim is still trivial since our construction does not rely on the choice of p so far that $p \geqslant 2^N \frac{r}{r-2}$ and of the precise choice of the q_i since $S_{\tilde{q}_i}(t)|_{\mathbb{L}^{q_i}} = S_{q_i}(t)$. If $\tilde{N} > N$ arguing as above we have

$$v_i = \tilde{v}_i \quad \forall i \in 0, \ldots, N-1$$

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and we are left to show that $\bar{v} = \sum_{i=N}^{\tilde{N}-1} \tilde{v}_i + \tilde{\bar{v}} =: \hat{v}$. Due to previous steps we can assume that v is $(\tilde{p}, r)$ solution since $\tilde{p} \geqslant 2^{\tilde{N}} \frac{\tilde{r}}{\tilde{r}-2} > 2^N \frac{r}{r-2}$. We observe that due to definition 3.7 and remark 3.8,

$$\bar{v}, \hat{v} \in C\left([0,T];H\right) \cap L^{\frac{2\tilde{r}}{\tilde{r}-2}}\left(0,T;\mathbb{L}^{\tilde{r}}\right), \quad f := w + \sum_{i=0}^{N-1} v_i \in L^{\frac{2\tilde{r}}{\tilde{r}-2}}\left(0,T;\mathbb{L}^{\tilde{r}}\right). \quad (3.28)$$

Therefore either $\bar{v}$ and $\hat{v}$ satisfy for all divergence free vector field $\varphi \in C^\infty(\mathcal{O};\mathbb{R}^2)$ such that $\varphi = 0$ on $\Gamma_b \cup \Gamma_u$ equation (3.18). Therefore, denoting by $\delta(t) = \bar{v} - \hat{v}$ we have that δ satisfies

$$\langle \delta(t), \varphi \rangle - \int_0^t \langle z(s), \Delta\varphi \rangle \, \mathrm{d}s$$
$$= \int_0^t b\left(\delta(s), \varphi, \bar{v}(s) + f(s)\right) \mathrm{d}s + \int_0^t b\left(\hat{v}(s) + f(s), \varphi, \delta(s)\right) \mathrm{d}s.$$

Denoting by $\tilde{q} = \frac{\tilde{r}}{2} \in (1,2)$, due to relation (3.28)

$$B\left(\delta, \bar{v} + f\right), \, B\left(\hat{v} + f, \delta\right) \in L^{\tilde{q}'}\left(0,T;X_{-1/2,A_{\tilde{q}}}\right).$$

Now the proof proceeds as in the first case of proposition 3.12 and we omit the details. $\qquad\square$

4. Interior regularity

As announced at the end of section 1.1, we prove theorem 1.3(2). To this end, we first prove the interior regularity of w_g and afterwards the one of v by exploiting the decomposition introduced in section 3.2.

4.1. Stokes equations

Let $(v_0, v_1, \ldots, v_{N-1}, \bar{v})$ be the (p,r)-solution to (3.13) as defined in definition 3.7 given by theorem 3.9. Let N_0 be the **P** null measure set where at least one between

$$w_g \notin C\left([0,T];\mathbb{L}^r\right), \quad v_i \notin W^{1,p/2^i}\left(0,T;X_{-1/2,A_{q_i}}\right) \cap L^{p/2^i}\left(0,T;X_{1/2,A_{q_i}}\right),$$
$$\bar{v} \notin C\left([0,T];H\right) \cap L^2\left(0,T;V\right),$$

(3.16)–(3.18), (3.4) is not satisfied. In the following, we will work pathwise in $\Omega \setminus N_0$ even if not specified. Thanks to the weak formulation guaranteed by lemma 3.5 we can easily obtain the interior regularity of the linear stochastic problem (1.4). Indeed, we are exactly in the same position of [1, corollary 4.4] and the following holds. We omit the proof as it follows verbatim the one of [1, corollary 4.4].

Lemma 4.1. *Let assumption* 1.1 *be satisfied. Let w_g be the unique weak solution of* (1.4) *in the sense of definiion* 3.4. *Then, for all* $0 < t_1 \leqslant t_2 < T, x_0 \in \mathcal{O}, \ \rho > 0$ *such that* $\mathrm{dist}(B(x_0,\rho), \partial\mathcal{O}) > 0$,

$$w_g \in C\left([t_1, t_2], C^\infty\left(B(x_0,\rho);\mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s.$$

4.2. Auxiliary Navier–Stokes equations and proof of theorem 1.3(2)

To deal with the interior regularity of (3.10) we perform a Serrin type argument, see [43, 60]. In contrast to [1], as $w_g \notin C([0,T]; \mathbb{L}^4)$, we cannot work directly on v. However, recalling that the solution v to (1.2) proven in section 3.3 satisfies

$$v = \sum_{i=0}^{N-1} v_i + \overline{v} \tag{4.1}$$

where, again, $(v_0, \ldots, v_{N-1}, \overline{v})$ is the (p, r)-solution to (3.13), cf section 3.2. The advantage of having the splitting (4.1) at our disposal is that v_i satisfies a linear problem where the forcing terms only depend on $v_0, \ldots, v_{i-1}$. Thus, by lemma 4.1 and an induction argument, we can prove that v_i is smooth inside $(0,T) \times \mathcal{O}$. While to prove the corresponding statement for $\overline{v}$, we can exploit that $\overline{v}$ is a Leray solution (i.e. it has finite energy) and therefore the Serrin regularization can be adjusted to our situation.

We begin with analyzing the interior regularity of v_i for $i \in \{0, \ldots, N-1\}$.

Lemma 4.2. *Let assumption* 1.1, $r \in (2,4)$ *and* $p \geqslant 2^N \frac{r}{r-2}$. *Let* $(v_0, \ldots, v_{N-1}, \overline{v})$ *be the* (p, r)-*solution of* (3.13) *in the sense of definition* 3.7. *Then for all* $i \in \{0, \ldots, N-1\}$, $0 < t_1 \leqslant t_2 < T$, $x_0 \in \mathcal{O}$, $\rho > 0$ *such that* $\mathrm{dist}(B(x_0, \rho), \partial \mathcal{O}) > 0$,

$$v_i \in C\left([t_1, t_2], C^\infty\left(B(x_0, \rho); \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s.$$

Proof. As in the first step of the proof of theorem 3.9 we argue by induction exploiting strongly the linear and triangle structure of (3.13). Before starting we observe that, by [55, theorem 3.4.8], it follows that

$$v_i \in C\left([0,T]; \mathbb{L}^{q_i}\right) \cap L^{p/2^i}\left(0, T; X_{1/2, A_{q_i}}\right). \tag{4.2}$$

Step 1: Interior regularity of v_0. First let us observe that, since $\mathrm{dist}(B(x_0, \rho), \partial \mathcal{O}) > 0$, $0 < t_1 \leqslant t_2 < T$, we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_1 \leqslant t_2 < t_2 + 2\varepsilon < T$, $\mathrm{dist}(B(x_0, \rho + 2\varepsilon), \partial \mathcal{O}) > 0$. As described in lemma 3.5, arguing as in the proof of [31, theorem 7], we can extend the weak formulation satisfied by v_0 to time dependent test functions $\phi \in C^1([0,T]; \mathbb{L}^{q_0'}) \cap C([0,T]; \mathsf{D}(A_{q_0'}))$ obtaining that for each $t \in [0,T]$

$$\langle v_0(t), \phi(t) \rangle = \int_0^t \langle v_0(s), \partial_s \phi(s) \rangle \, ds - \int_0^t \langle v_0(s), A_{q_0'} \phi(s) \rangle \, ds$$
$$+ \int_0^t b(w_g(s), \phi(s), w_g(s)) \, ds \quad \mathbf{P} - a.s.$$

Choosing $\phi = -\nabla^\perp \chi$, $\chi \in C_c^\infty((0,T) \times \mathcal{O})$ in the weak formulation above and denoting by

$$\omega_0 = \mathrm{curl}\, v_0 \in C\left([0,T]; H^{-1,q_0}(\mathcal{O})\right) \cap L^p\left(0, T; L^{q_0}(\mathcal{O})\right),$$
$$\omega_w = \mathrm{curl}\, w_g \in C\left([t_1 - 2\varepsilon, t_2 + 2\varepsilon], C^\infty(B(x_0, \rho + 2\varepsilon))\right) \quad \mathbf{P} - a.s.$$

it follows that

$$-\int_0^t \langle \omega_0(s), \partial_s \chi(s) \rangle + \langle \omega(s), \Delta \chi(s) \rangle \, ds = \int_0^t \langle \mathrm{curl}(w_g(s) \otimes w_g(s)), \nabla \chi(s) \rangle \, ds.$$

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This means that ω_0 is a distributional solution in $(0,T) \times \mathcal{O}$ of the partial differential equation

$$\partial_t \omega_0 = \Delta \omega_0 - \operatorname{div} \operatorname{curl} (w_g(s) \otimes w_g(s)).$$

Let us consider $\psi_0 \in C_c^\infty((0,T) \times \mathcal{O})$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, \rho + \varepsilon)$ such that it is equal to one in $[t_1 - \varepsilon/2, t_2 + \varepsilon/2] \times B(x_0, \rho + \varepsilon/2)$. Let us denote by $\omega_0^* = \omega_0 \psi_0 \in L^p(0,T;L^{q_0}(\mathbb{R}^2))$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, \rho + \varepsilon)$, then ω_0^* is a distributional solution in $(0,T) \times \mathbb{R}^2$ of

$$\partial_t \omega_0^* = \Delta \omega_0^* + h_0 \tag{4.3}$$

with

$$h_0 = \partial_t \psi_0 \omega_0 - 2\nabla \psi_0 \cdot \nabla \omega_0 - \Delta \psi_0 \omega_0 - \psi_0 w_g \cdot \nabla \omega_w.$$

Due to lemma 4.1

$$h_0 \in L^p\left(0,T;H^{-1,q_0}\left(\mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s.$$

Then, again by maximal L^p-regularity techniques for the heat equation (see e.g. [55, theorem 4.4.4] [39, theorems 10.2.25 and 10.3.4]) and the trace embedding of [55, theorem 3.4.8],

$$\omega_0^* \in C\left([0,T];L^{q_0}\left(\mathbb{R}^2\right)\right) \cap L^p\left(0,T;H^{1,q_0}\left(\mathbb{R}^2\right)\right).$$

Therefore,

$$\begin{aligned} \omega_0 \in &C\left([t_1 - \varepsilon/4, t_2 + \varepsilon/4], L^{q_0}\left(B\left(x_0, \rho + \varepsilon/4\right)\right)\right) \\ &\cap L^p\left(t_1 - \varepsilon/4, t_2 + \varepsilon/4, H^{1,q_0}\left(B\left(x_0, \rho + \varepsilon/4\right)\right)\right) \quad \mathbf{P} - a.s. \end{aligned}$$

Introducing $\phi_0 \in C_c^\infty(B(x_0, \rho + \varepsilon/4))$ equal to one in $B(x_0, \rho + \varepsilon/8)$, since $\omega_0 = \operatorname{curl} v_0$, then $\phi_0 v_0$ satisfies

$$\Delta(\phi_0 v_0) = \nabla^\perp \omega_0 \phi_0 + \Delta \phi_0 v_0 + 2\nabla \phi_0 \cdot \nabla v_0, \quad (\phi_0 v)|_{\partial B(x_0, \rho + \varepsilon/4)} = 0. \tag{4.4}$$

From the regularity of ω_0, by standard elliptic regularity theory (see for example [64, chapter 4]), it follows that $\phi_0 v_0 \in C([t_1 - \varepsilon/4, t_2 + \varepsilon/4]; H^{1,q_0}(B(x_0, \rho + \varepsilon/4); \mathbb{R}^2)) \cap L^p(t_1 - \varepsilon/4, t_2 + \varepsilon/4; H^{2,q_0}(B(x_0, \rho + \varepsilon/4); \mathbb{R}^2)) \, \mathbf{P} - a.s.$ Therefore, since $\phi_0 \equiv 1$ on $B(x_0, \rho + \varepsilon/8)$

$$\begin{aligned} v_0 \in &C\left([t_1 - \varepsilon/16, t_2 + \varepsilon/16]; H^{1,q_0}\left(B\left(x_0, \rho + \varepsilon/16\right); \mathbb{R}^2\right)\right) \\ &\cap L^p\left(t_1 - \varepsilon/16, t_2 + \varepsilon/16; H^{2,q_0}\left(B\left(x_0, \rho + \varepsilon/16\right); \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s. \tag{4.5} \end{aligned}$$

Reiterating the argument, i.e. considering for each $j \in \mathbb{N}$, $j \geqslant 0$, first $\psi_j \in C_c^\infty((0,T) \times \mathcal{O})$ supported in $[t_1 - \varepsilon/2^{4j}, t_2 + \varepsilon/2^{4j}] \times B(x_0, \rho + \varepsilon/2^{4j})$ identically equal to one in $[t_1 - \varepsilon/2^{4j+1}, t_2 + \varepsilon/2^{4j+1}] \times B(x_0, \rho + \varepsilon/2^{4j+1})$ and $\phi_j \in C_c^\infty(B(x_0, \rho + \varepsilon/2^{4j+2}))$ identically equal to one in $B(x_0, \rho + \varepsilon/2^{4j+3})$ we get iteratively that $\mathbf{P} - a.s.$

$$\omega_0 \in C\left(\left[t_1 - \varepsilon/2^{4j+2}, t_2 + \varepsilon/2^{4j+2}\right], H^{j,q_0}\left(B\left(x_0, \rho + \varepsilon/2^{4j+2}\right)\right)\right)$$
$$\cap L^p\left(t_1 - \varepsilon/2^{4j+2}, t_2 + \varepsilon/2^{4j+2}, H^{j+1,q_0}\left(B\left(x_0, \rho + \varepsilon/2^{4j+2}\right)\right)\right)$$
$$v_0 \in C\left(\left[t_1 - \varepsilon/2^{4(j+1)}, t_2 + \varepsilon/2^{4(j+1)}\right], H^{j+1,q_0}\left(B\left(x_0, \rho + \varepsilon/2^{4(j+1)}\right); \mathbb{R}^2\right)\right)$$
$$\cap L^p\left(t_1 - \varepsilon/2^{4(j+1)}, t_2 + \varepsilon/2^{4(j+1)}, H^{j+2,q_0}\left(B\left(x_0, \rho + \varepsilon/2^{4(j+1)}\right); \mathbb{R}^2\right)\right).$$

and the claimed interior regularity for v_0 follows.

Step 2: Inductive step. Assume that we have already shown that the claim holds for v_j, $j \in \{0, l-1\}$, and $l \leqslant N-1$. Now let us prove that it holds also for v_l. Since $\mathrm{dist}(B(x_0, \rho), \partial\mathcal{O}) > 0$, $0 < t_1 \leqslant t_2 < T$, we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_1 \leqslant t_2 < t_2 + 2\varepsilon < T$, $\mathrm{dist}(B(x_0, \rho + 2\varepsilon), \partial\mathcal{O}) > 0$. As described in lemma 3.5, arguing as in the proof of [31, theorem 7], we can extend the weak formulation satisfied by v_l to time dependent test functions $\phi \in C^1([0,T]; \mathbb{L}^{q_l'}) \cap C([0,T]; \mathsf{D}(A_{q_{l'}}))$ obtaining that for each $t \in [0,T]$

$$\langle v_l(t), \phi(t) \rangle = \int_0^t \langle v_l(s), \partial_s \phi(s) \rangle \, ds - \int_0^t \left\langle v_0(s), A_{q_{l'}} \phi(s) \right\rangle \, ds$$
$$+ \int_0^t b\left(v_{l-1}(s), \phi(s), w(s) + \sum_{j=0}^{l-1} v_j(s)\right) ds$$
$$+ \int_0^t b\left(w(s) + \sum_{j=0}^{i-2} v_j(s), \phi(s), v_{i-1}(s)\right) ds \quad \mathbb{P}-a.s.$$

Choosing $\phi = -\nabla^\perp \chi$, $\chi \in C_c^\infty((0,T) \times \mathcal{O})$ in the weak formulation above and, for $i \in \{0, \ldots, l-1\}$, denoting by

$$\omega_l = \mathrm{curl}\, v_l \in C\left([0,T]; H^{-1,q_l}(\mathcal{O})\right) \cap L^{p/2^l}\left(0, T; L^{q_l}(\mathcal{O})\right),$$
$$\omega_i = \mathrm{curl}\, v_i \in C\left([t_1 - 2\varepsilon, t_2 + 2\varepsilon], C^\infty\left(B(x_0, \rho + 2\varepsilon)\right)\right),$$
$$\omega_w = \mathrm{curl}\, w_g \in C\left([t_1 - 2\varepsilon, t_2 + 2\varepsilon], C^\infty\left(B(x_0, \rho + 2\varepsilon)\right)\right) \quad \mathbb{P}-a.s.$$

arguing as in *Step 1* it follows that ω_l is a distributional solution in $(0,T) \times \mathcal{O}$ of the partial differential equation

$$\partial_t \omega_l = \Delta \omega_l - \mathrm{div}\,\mathrm{curl}\,(v_{l-1}(s) \otimes v_{l-1}(s))$$
$$- \mathrm{div}\,\mathrm{curl}\left(v_{l-1}(s) \otimes \left(w_g(s) + \sum_{j=0}^{l-2} v_j(s)\right)\right)$$
$$- \mathrm{div}\,\mathrm{curl}\left(\left(w_g(s) + \sum_{j=0}^{l-2} v_j(s)\right) \otimes v_{l-1}(s)\right).$$

Let us consider $\psi_0 \in C_c^\infty((0,T) \times \mathcal{O})$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, \rho + \varepsilon)$ such that it is equal to one in $[t_1 - \varepsilon/2, t_2 + \varepsilon/2] \times B(x_0, \rho + \varepsilon/2)$. Let us denote by $\omega_l^* = \omega_l \psi_0 \in L^p(0,T; L^{q_l}(\mathbb{R}^2))$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, \rho + \varepsilon)$, then ω_l^* is a distributional solution in $(0,T) \times \mathbb{R}^2$ of

$$\partial_t \omega_l^* = \Delta \omega_l^* + h_l \tag{4.6}$$

with

$$h_l = \partial_t \psi_0 \omega_l - 2\nabla\psi_0 \cdot \nabla\omega_l - \Delta\psi_0\omega_l - \psi_0 w_{l-1} \cdot \nabla\omega_{l-1}$$
$$- \psi_0 w_{l-1} \cdot \nabla\left(\omega_w + \sum_{j=0}^{l-2}\omega_j\right) - \psi_0\left(w_j + \sum_{j=0}^{l-2} v_j\right) \cdot \nabla\omega_{l-1}.$$

Due to lemma 4.1 and the inductive hypothesis

$$h_l \in L^p\left(0,T;H^{-1,q_l}\left(\mathbb{R}^2\right)\right) \quad \mathbf{P}-a.s.$$

Now we can argue as in *Step 1* obtaining the claim. We omit the easy details. ☐

Now we are in the position to apply similar ideas of [1, section 4.2] for the equation satisfied by $\bar{v}$. For the sake of completeness, we provide some details.

Lemma 4.3. *Let assumption* 1.1, $r \in (2,4)$ *and* $p \geqslant 2^N \frac{r}{r-2}$. *Let* $(v_0,\dots,v_{N-1},\bar{v})$ *be the (p, r)-solution of* (3.13) *in the sense of definition* 3.7. *Then, for all* $0 < t_1 \leqslant t_2 < T$, $x_0 \in \mathcal{O}$, $\rho > 0$ *such that* $\mathrm{dist}(B(x_0,\rho),\partial\mathcal{O}) > 0$,

$$\bar{v} \in C\left([t_1,t_2],H^{3/2}\left(B\left(x_0,\rho\right);\mathbb{R}^2\right)\right) \quad \mathbf{P}-a.s.$$

Proof. First let us observe that, since $\mathrm{dist}(B(x_0,\rho),\partial\mathcal{O}) > 0$, $0 < t_1 \leqslant t_2 < T$, we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_1 \leqslant t_2 < t_2 + 2\varepsilon < T$, $\mathrm{dist}(B(x_0,\rho+2\varepsilon),\partial\mathcal{O}) > 0$. To simplify the notation let us call

$$\tilde{v} = w + \sum_{j=0}^{N-1} v_j, \ \widetilde{\omega} = \mathrm{curl}\,\widetilde{v}.$$

As described in lemma 3.5, arguing as in the proof of [31, theorem 7], we can extend the weak formulation satisfied by $\bar{v}$ to time-dependent test functions $\phi \in C^1([0,T];H) \cap C([0,T];\mathsf{D}(A))$ obtaining that for each $t \in [0,T]$

$$\langle \bar{v}(t), \phi(t)\rangle - \langle u_{\mathrm{in}}, \phi(0)\rangle = \int_0^t \langle \bar{v}(s), \partial_s\phi(s)\rangle \,\mathrm{d}s - \int_0^t \langle \bar{v}(s), A\phi(s)\rangle \,\mathrm{d}s$$
$$+ \int_0^t b\left(\bar{v}(s) + \widetilde{v}(s), \phi(s), \bar{v}(s)\right) \mathrm{d}s$$
$$+ \int_0^t b\left(\bar{v}(s), \phi(s), \widetilde{v}(s)\right) \mathrm{d}s$$
$$+ \int_0^t b\left(v_{N-1}(s), \phi(s), \widetilde{v}(s)\right) \mathrm{d}s$$
$$+ \int_0^t b\left(\widetilde{v}(s) - v_{N-1}(s), \phi(s), v_{N-1}(s)\right) \mathrm{d}s \quad \mathbf{P}-a.s.$$

Choosing $\phi = -\nabla^{\perp}\chi$, $\chi \in C_c^{\infty}((0,T) \times \mathcal{O})$ in the weak formulation above and, for $i \in \{0,\dots,N-1\}$, denoting by

$$\omega = \operatorname{curl} v \in C\left([0,T];H^{-1}\right) \cap L^2\left((0,T) \times \mathcal{O}\right),$$
$$\omega_i = \operatorname{curl} v_i \in C\left([t_1 - 2\varepsilon, t_2 + 2\varepsilon], C^{\infty}\left(B(x_0, \rho + 2\varepsilon)\right)\right),$$
$$\omega_w = \operatorname{curl} w \in C\left([t_1 - 2\varepsilon, t_2 + 2\varepsilon], C^{\infty}\left(B(x_0, \rho + 2\varepsilon)\right)\right) \quad \mathbf{P} - a.s.$$

it follows that

$$-\int_0^t \langle \omega(s), \partial_s \chi(s) \rangle + \langle \omega(s), \Delta \chi(s) \rangle \, ds$$
$$= \int_0^t \langle \operatorname{curl}(v_{N-1}(s) \otimes \widetilde{v}(s)), \nabla \chi(s) \rangle \, ds$$
$$+ \int_0^t \langle \operatorname{curl}((\widetilde{v}(s) - v_{N-1}(s)) \otimes v_{N-1}(s)), \nabla \chi(s) \rangle \, ds$$
$$+ \int_0^t \langle \operatorname{curl}(\overline{v}(s) \otimes \widetilde{v}(s)), \nabla \chi(s) \rangle \, ds$$
$$+ \int_0^t \langle \operatorname{curl}(\widetilde{v}(s) \otimes \overline{v}(s)), \nabla \chi(s) \rangle \, ds$$
$$+ \int_0^t \langle \omega(s), \overline{v}(s) \cdot \nabla \chi(s) \rangle \, ds.$$

This means that ω is a distributional solution in $(0,T) \times \mathcal{O}$ of the partial differential equation

$$\partial_t \omega + \overline{v} \cdot \nabla \omega = \Delta \omega - \operatorname{div}\left(\operatorname{curl}(v_{N-1}(s) \otimes \widetilde{v}(s))\right.$$
$$+ \operatorname{curl}(\widetilde{v}(s) - v_{N-1}(s) \otimes v_{N-1}(s))$$
$$\left. + \operatorname{curl}(\widetilde{v}(s) \otimes \overline{v}(s)) + \operatorname{curl}(\overline{v}(s) \otimes \widetilde{v}(s))\right).$$

Let us consider $\psi \in C_c^{\infty}((0,T) \times \mathcal{O})$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, \rho + \varepsilon)$ such that it is equal to one in $[t_1 - \varepsilon/2, t_2 + \varepsilon/2] \times B(x_0, \rho + \varepsilon/2)$. Let us denote by $\omega^* = \omega\psi \in L^2((0,T) \times \mathbb{R}^2)$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, \rho + \varepsilon)$, then ω^* is a distributional solution in $(0,T) \times \mathbb{R}^2$ of

$$\partial_t \omega^* = \Delta \omega^* - \overline{v} \cdot \nabla \omega^* - \widetilde{v} \cdot \nabla \omega^* + h \tag{4.7}$$

with

$$h = \partial_t \psi\, \omega - 2\nabla\psi \cdot \nabla\omega - \Delta\psi\, \omega + \overline{v} \cdot \nabla\psi\, \omega + \widetilde{v} \cdot \nabla\psi\, \omega - \psi(\widetilde{v} - v_{N-1}) \cdot \nabla\omega_{N-1}$$
$$- \psi\overline{v} \cdot \nabla\widetilde{\omega} - \psi v_{N-1} \cdot \nabla\widetilde{\omega}.$$

Due to lemma 4.1 and lemma 4.2 the terms

$$\widetilde{v} \cdot \nabla\psi\, \omega - \psi(\widetilde{v} - v_{N-1}) \cdot \nabla\omega_{N-1} - \psi\overline{v} \cdot \nabla\widetilde{\omega}$$
$$- \psi v_{N-1} \cdot \nabla\widetilde{\omega} \in L^2\left((0,T) \times \mathbb{R}^2\right) \quad \mathbf{P} - a.s.$$

Therefore $h \in L^2(0,T;H^{-1}(\mathbb{R}^2)) + L^1(0,T;L^2(\mathbb{R}^2))$ $\mathbf{P} - a.s.$ Then, arguing as in the first step of the proof of [43, theorem 13.2], the fact that ω^* is a distributional solution of (4.7) implies that $\omega^* \in C([0,T];L^2(\mathbb{R}^2)) \cap L^2(0,T;H^1(\mathbb{R}^2))$. Therefore

$$\omega \in C\left([t_1 - \varepsilon/4, t_2 + \varepsilon/4], L^2\left(B\left(x_0, \rho + \varepsilon/4\right)\right)\right)$$
$$\cap L^2\left(t_1 - \varepsilon/4, t_2 + \varepsilon/4, H^1\left(B\left(x_0, \rho + \varepsilon/4\right)\right)\right) \quad \mathbf{P} - a.s.$$

Introducing $\phi \in C_c^\infty(B(x_0, \rho + \varepsilon/4))$ equal to one in $B(x_0, \rho + \varepsilon/8)$, since $\omega = \operatorname{curl} \bar{v}$, then $\phi \bar{v}$ satisfies

$$\Delta\left(\phi \bar{v}\right) = \nabla^\perp \omega \phi + \Delta \phi \bar{v} + 2\nabla \phi \cdot \nabla \bar{v}, \quad \left(\phi \bar{v}\right)|_{\partial B(x_0, \rho + \varepsilon/4)} = 0. \tag{4.8}$$

From the regularity of ω, by standard elliptic regularity theory (see for example [5]), it follows that $\phi \bar{v} \in C([t_1 - \varepsilon/4, t_2 + \varepsilon/4]; H^1(B(x_0, \rho + \varepsilon/4); \mathbb{R}^2)) \cap L^2(t_1 - \varepsilon/4, t_2 + \varepsilon/4; H^2(B(x_0, \rho + \varepsilon/4); \mathbb{R}^2))$ $\mathbf{P} - a.s.$ Therefore, since $\phi \equiv 1$ on $B(x_0, \rho + \varepsilon/8)$

$$\bar{v} \in C\left([t_1 - \varepsilon/16, t_2 + \varepsilon/16]; H^1\left(B\left(x_0, \rho + \varepsilon/16\right); \mathbb{R}^2\right)\right)$$
$$\cap L^2\left(t_1 - \varepsilon/16, t_2 + \varepsilon/16; H^2\left(B\left(x_0, \rho + \varepsilon/16\right); \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s. \tag{4.9}$$

Let us now consider $\widehat{\psi} \in C_c^\infty((t_1 - \varepsilon/16, t_2 + \varepsilon/16) \times B(x_0, \rho + \varepsilon/16))$ such that it is equal to one in $[t_1 - \varepsilon/32, t_2 + \varepsilon/32] \times B(x_0, \rho + \varepsilon/32)$. Let us denote by $\widehat{\omega} = \omega \widehat{\psi} \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2))$ supported in $(t_1 - \varepsilon/16, t_2 + \varepsilon/16) \times B(x_0, \rho + \varepsilon/16)$, then $\widehat{\omega}$ is a distributional solution in $(0, T) \times \mathbb{R}^2$ of

$$\partial_t \widehat{\omega} = \Delta \widehat{\omega} + \widehat{h} \tag{4.10}$$

with

$$\widehat{h} = -\bar{v} \cdot \nabla \widehat{\omega} - \widetilde{v} \cdot \nabla \widehat{\omega} + \partial_t \widehat{\psi} \omega - 2\nabla \widehat{\psi} \cdot \nabla \omega - \Delta \widehat{\psi} \omega + \bar{v} \cdot \nabla \widehat{\psi} \omega + \widetilde{v} \cdot \nabla \widehat{\psi} \omega$$
$$- \widehat{\psi}\left(\widetilde{v} - v_{N-1}\right) \cdot \nabla \omega_{N-1} - \widehat{\psi} \bar{v} \cdot \nabla \widetilde{\omega} - \widehat{\psi} v_{N-1} \cdot \nabla \widetilde{\omega}.$$

By lemma 4.1, lemma 4.2 and relation (4.9) it follows that

$$\widehat{h} \in L^2\left(0, T; H^{-1/2}\left(\mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s.$$

Therefore $\widehat{\omega} \in C([0, T]; H^{1/2}(\mathbb{R}^2)) \cap L^2(0, T; H^{3/2}(\mathbb{R}^2))$ $\mathbf{P} - a.s.$ and arguing as above

$$\bar{v} \in C\left([t_1 - \varepsilon/64, t_2 + \varepsilon/64], H^{3/2}\left(B\left(x_0, r + \varepsilon/64\right); \mathbb{R}^2\right)\right)$$
$$\cap L^2\left(t_1 - \varepsilon/64, t_2 + \varepsilon/64, H^{5/2}\left(B\left(x_0, \rho + \varepsilon/64\right); \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s.$$

This concludes the proof of lemma 4.3.

$\square$

Corollary 4.4. *Let assumption* 1.1, $r \in (2, 4)$ *and* $p \geqslant 2^N \frac{r}{r-2}$. *Let* $(v_0, \ldots, v_{N-1}, \bar{v})$ *be the (p, r)-solution of* (3.13) *in the sense of definition* 3.7. *Then, for all* $0 < t_1 \leqslant t_2 < T$, $x_0 \in \mathcal{O}$, $\rho > 0$ *such that* $\operatorname{dist}(B(x_0, \rho), \partial \mathcal{O}) > 0$,

$$\bar{v} \in C\left([t_1, t_2]; C^\infty\left(B\left(x_0, \rho\right); \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s.$$

Proof. Since $\mathrm{dist}(B(x_0,\rho),\partial\mathcal{O}) > 0$, $0 < t_1 \leqslant t_2 < T$ we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_1 \leqslant t_2 < t_2 + 2\varepsilon < T$, $\mathrm{dist}(B(x_0,\rho + 2\varepsilon),\partial\mathcal{O}) > 0$ and $\psi \in C_c^\infty((0,T) \times \mathcal{O})$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0,\rho + \varepsilon)$ such that it is equal to one in $[t_1 + \varepsilon/2, t_2 + \varepsilon/2] \times B(x_0,\rho + \varepsilon/2)$. From lemma 4.3 and Sobolev embedding theorem we know that $\bar{v} \in C([t_1 - \varepsilon, t_2 + \varepsilon]; L^\infty(B(x_0,\rho + \varepsilon); \mathbb{R}^2))$ $\mathbf{P} - a.s.$ Denoting, as in lemma 4.3 by

$$\tilde{v} = w + \sum_{j=0}^{N-1} v_j, \ \widetilde{\omega} = \mathrm{curl}\,\widetilde{v},$$

$$\omega = \mathrm{curl}\,v \in C\left([0,T];H^{-1}\right) \cap L^2\left((0,T) \times \mathcal{O}\right),$$

$$\omega_i = \mathrm{curl}\,v_i \in C\left([t_1 - 2\varepsilon, t_2 + 2\varepsilon], C^\infty\left(B(x_0,\rho + 2\varepsilon)\right)\right),$$

$$\omega_w = \mathrm{curl}\,w \in C\left([t_1 - 2\varepsilon, t_2 + 2\varepsilon], C^\infty\left(B(x_0,\rho + 2\varepsilon)\right)\right) \quad \mathbf{P} - a.s.$$

and $\omega^* = \omega\psi \in L^2((0,T) \times \mathbb{R}^2)$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0,\rho + \varepsilon)$, then, arguing as in the proof of lemma 4.3, it follows that ω^* is a distributional solution in $(0,T) \times B(x_0,\rho + \varepsilon)$ of

$$\partial_t \omega^* = \Delta\omega^* + \widetilde{h} \tag{4.11}$$

with

$$\widetilde{h} = -\bar{v} \cdot \nabla\omega^* - \widetilde{v} \cdot \nabla\omega^* + \partial_t\psi\omega - 2\nabla\psi \cdot \nabla\omega - \Delta\psi\omega + \bar{v} \cdot \nabla\psi\omega + \widetilde{v} \cdot \nabla\psi\omega$$
$$- \psi\left(\widetilde{v} - v_{N-1}\right) \cdot \nabla\omega_{N-1} - \psi\bar{v} \cdot \nabla\widetilde{\omega} - \psi v_{N-1} \cdot \nabla\widetilde{\omega}.$$

From the regularity of ω, $\bar{v}$, $\widetilde{\omega}$, $\widetilde{v}$, ω_{N-1}, v_{N-1}, then $\widetilde{h} \in L^2(t_1 - \varepsilon, t_2 + \varepsilon; H^{-1}(B(x_0,\rho + \varepsilon)))$ $\mathbf{P} - a.s.$ By standard regularity theory for the heat equation, see for example Step 2 in [43, theorem 13.1], a solution of (4.11) with $\widetilde{h} \in L^2(t_1 - \varepsilon, t_2 + \varepsilon; H^{k-1}(B(x_0,\rho + \varepsilon)))$, $k \in \mathbb{N}$, belongs to $C([t_1 - \varepsilon/2, t_2 + \varepsilon/2]; H^k(B(x_0,\rho + \varepsilon/2))) \cap L^2(t_1 - \varepsilon/2, t_2 + \varepsilon/2; H^{k+1}(B(x_0,\rho + \varepsilon/2)))$. Therefore

$$\omega^* \in C\left([t_1 - \varepsilon/2, t_2 + \varepsilon/2]; L^2\left(B(x_0,\rho + \varepsilon/2)\right)\right)$$
$$\cap L^2\left(t_1 - \varepsilon/2, t_2 + \varepsilon/2; H^1\left(B(x_0,\rho + \varepsilon/2)\right)\right) \quad \mathbf{P} - a.s.$$

which implies

$$\omega \in C\left([t_1 + \varepsilon/4, t_2 - \varepsilon/4; L^2\left(B(x_0,\rho + \varepsilon/4)\right)\right)$$
$$\cap L^2\left(t_1 - \varepsilon/4, t_2 + \varepsilon/4; H^1\left(B(x_0,\rho + \varepsilon/4)\right)\right) \quad \mathbf{P} - a.s.$$

since $\psi \equiv 1$ on $(t_1 - \varepsilon/2, t_2 + \varepsilon/2) \times B(x_0,\rho + \varepsilon/2)$. Considering now $\phi \in C_c^\infty(\mathcal{O})$ supported on $B(x_0,\rho + \varepsilon/4)$ such that $\phi \equiv 1$ on $B(x_0,\rho + \varepsilon/8)$, since $\mathrm{curl}\,\bar{v} = \omega$ then $\phi\bar{v}$ satisfies

$$\Delta\left(\phi\bar{v}\right) = \nabla^\perp\omega\phi + \Delta\phi\bar{v} + 2\nabla\phi \cdot \nabla\bar{v}, \quad (\phi\bar{v})|_{\partial B(x_0,\rho + \varepsilon/4)} = 0. \tag{4.12}$$

Since

$$\nabla^\perp\omega\phi + \Delta\phi\bar{v} + 2\nabla\phi \cdot \nabla\bar{v} \in C\left([t_1 + \varepsilon/4, t_2 - \varepsilon/4; H^{-1}\left(B(x_0,\rho + \varepsilon/4); \mathbb{R}^2\right)\right)$$
$$\cap L^2\left(t_1 - \varepsilon/4, t_2 + \varepsilon/4; L^2\left(B(x_0,\rho + \varepsilon/4); \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s.,$$

by standard elliptic regularity theory (see for example [5]),

$$\phi\bar{v} \in C\left([t_1 + \varepsilon/4, t_2 - \varepsilon/4; H^1\left(B\left(x_0, \rho + \varepsilon/4\right); \mathbb{R}^2\right)\right)$$
$$\cap L^2\left(t_1 - \varepsilon/4, t_2 + \varepsilon/4; H^2\left(B\left(x_0, \rho + \varepsilon/4\right); \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s..$$

Since $\phi \equiv 1$ on $B(x_0, \rho + \varepsilon/8)$ then

$$\bar{v} \in C\left([t_1 + \varepsilon/16, t_2 - \varepsilon/16; H^1\left(B\left(x_0, \rho + \varepsilon/16\right)\right)\right)$$
$$\cap L^2\left(t_1 - \varepsilon/16, t_2 + \varepsilon/16; H^2\left(B\left(x_0, \rho + \varepsilon/16\right)\right)\right) \quad \mathbf{P} - a.s.$$

Reiterating the argument as in Step 3 in [43, theorem 13.1] the claim follows. $\qquad\square$

Proof of theorem 1.3(2). The claim follows by lemma 4.1, lemma 4.2 and corollary 4.4 and a localization argument. To begin, recall from the proof of theorem 1.3(1) in section 3.3 that there exists a solution (1.2) on the time interval $[0, T+1]$ and it is given by $\widetilde{u} = w_g + \sum_{i=0}^{N-1} v_i + \bar{v}$ where $(v_0, \ldots, v_{N-1}, \bar{v})$ is the (p, r)-solution to (3.13) on $[0, T+1]$ for $r < 2q_{\mathcal{H}}$, N as in (3.11) and $p \geqslant 2^N \frac{r}{r-2}$. Then, by lemma 4.1, lemma 4.2, corollary 4.4 and a standard covering argument, for all $t_0 \in (0, T)$, $\mathcal{O}_0 \subset \mathcal{O}$ such that $\text{dist}(\mathcal{O}_0, \partial\mathcal{O}) > 0$,

$$\widetilde{u} \in C\left([t_0, T]; C^\infty\left(\mathcal{O}_0; \mathbb{R}^2\right)\right) \quad \mathbf{P} - a.s. \tag{4.13}$$

Now, let u be the unique solution (1.2) provided by theorem 1.3(1) on $[0, T]$. By uniqueness, we have $u = \widetilde{u}|_{[0,T]}$ and the conclusion follows from (4.13). $\qquad\square$

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No new data were created or analysed in this study.

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ORCID iDs

Antonio Agresti https://orcid.org/0000-0002-9573-2962
Eliseo Luongo https://orcid.org/0000-0003-1981-340X

References

[1] Agresti A and Luongo E 2024 Global well-posedness and interior regularity of 2D Navier–Stokes equations with stochastic boundary conditions *Math. Ann.* **390** 2727–66

[2] Alòs E and Bonaccorsi S 2002 Stochastic partial differential equations with Dirichlet white-noise boundary conditions *Ann. Inst. Henri Poincaré Probab. Stat.* **38** 125–54

[3] Álos E and Nualart D 2003 Stochastic integration with respect to the fractional Brownian motion *Stoch. Stoch. Rep.* **75** 129–52

[4] Amann H 1995 *Linear and Quasilinear Parabolic Problems. Vol. I, Abstract Linear Theory* (*Monographs in Mathematics.*) vol 89 (Birkhäuser Boston Inc.)

[5] Ambrosio L, Carlotto A and Massaccesi A 2019 *Lectures on Elliptic Partial Differential Equations* vol 18 (Springer)

[6] Amrouche C and Girault V 1994 Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension *Czech. Math. J.* **44** 109–40

[7] Amrouche C and Rodríguez-Bellido M 2010 Very weak solutions for the stationary Stokes equations *C. R. Math.* **348** 223–8

[8] Bergh J and Löfström J 1976 *Interpolation Spaces. An Introduction* (*Grundlehren der Mathematischen Wissenschaften*) vol 223 (Springer)

[9] Berselli L C and Romito M 2006 On the existence and uniqueness of weak solutions for a vorticity seeding model *SIAM J. Math. Anal.* **37** 1780–99

[10] Binz T, Hieber M, Hussein A and Saal M 2024 The primitive equations with stochastic wind driven boundary conditions *J. Math. Pures Appl.* **183** 76–101

[11] Brzeźniak Z and Ferrario B 2009 2D Navier–Stokes equation in Besov spaces of negative order *Nonlinear Anal. Theory Methods Appl.* **70** 3902–16

[12] Brzeźniak Z, Goldys B, Peszat S and Russo F 2015 Second order PDEs with Dirichlet white noise boundary conditions *J. Evol. Equ.* **15** 1–26

[13] Brzeźniak Z, van Neerven J and Salopek D 2012 Stochastic evolution equations driven by Liouville fractional Brownian motion *Czech. Math. J.* **62** 1–27

[14] Cerrai S and Freidlin M 2011 Fast transport asymptotics for stochastic RDEs with boundary noise *Ann. Probab.* **39** 369–405

[15] Chang T and Jin B J 2015 Solvability of the initial–boundary value problem of the Navier–Stokes equations with rough data *Nonlinear Anal.* **125** 498–517

[16] Cheskidov A and Luo X 2022 Sharp nonuniqueness for the Navier-Stokes equations *Invent. Math.* **229** 987–1054

[17] Cheskidov A and Luo X 2023 L^2-critical nonuniqueness for the 2D Navier-Stokes equations *Ann. PDE* **9** 56

[18] Coupek P and Maslowski B 2017 Stochastic evolution equations with Volterra noise *Stoch. Process. Appl.* **127** 877–900

[19] Coupek P, Maslowski B and Ondrejat M 2018 L^p-valued stochastic convolution integral driven by Volterra noise *Stoch. Dyn.* **18** 1850048

[20] Da Prato G and Debussche A 2002 Two-dimensional Navier–Stokes equations driven by a space–time white noise *J. Funct. Anal.* **196** 180–210

[21] Da Prato G and Zabczyk J 1993 Evolution equations with white–noise boundary conditions *Stoch. Int. J. Probab. Stoch. Process.* **42** 167–82

[22] Da Prato G and Zabczyk J 1996 *Ergodicity for Infinite Dimensional Systems* vol 229 (Cambridge University Press)

[23] Duncan T E, Pasik-Duncan B and Maslowski B 2002 Fractional Brownian motion and stochastic equations in Hilbert spaces *Stoch. Dyn.* **2** 225–50

[24] Duncan T E, Pasik-Duncan B and Maslowski B 2006 Linear stochastic equations in a Hilbert space with a fractional brownian motion *Stochastic Processes, Optimization, and Control Theory: Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems* pp 201–21

[25] Fabbri G and Goldys B 2009 An LQ problem for the heat equation on the halfline with Dirichlet boundary control and noise *SIAM J. Control Optim.* **48** 1473–88

[26] Fan W T L, Jolly M and Pakzad A 2021 Three-dimensional shear driven turbulence with noise at the boundary *Nonlinearity* **34** 4764–86

[27] Farwig R, Galdi G P and Sohr H 2005 Very weak solutions of stationary and instationary Navier-Stokes equations with nonhomogeneous data *Nonlinear Elliptic and Parabolic Problems: A Special Tribute to the Work of Herbert Amann* (Springer) pp 113–36

[28] Farwig R, Kozono H and Sohr H 2011 Global weak solutions of the Navier-Stokes equations with nonhomogeneous boundary data and divergence *Rend. Semin. Mat. della Univ. Padova* **125** 51–70

[29] Flandoli F and Gubinelli M 2004 *Random Currents and Probabilistic Models of Vortex Filaments* (Birkhäuser)

[30] Flandoli F, Gubinelli M and Russo F 2009 On the regularity of stochastic currents, fractional brownian motion and applications to a turbulence model *Ann. Inst. Henri Poincaré B* **45** 545–75

[31] Flandoli F and Luongo E 2023 *Stochastic partial differential equations in fluid mechanics* (*Lecture Notes in Mathematics*) vol 2330 (Springer)

[32] Fröhlich A 2007 The Stokes operator in weighted L^q-spaces. II. Weighted resolvent estimates and maximal L^p-regularity *Math. Ann.* **339** 287–316

[33] Gallagher I and Planchon F 2002 On global infinite energy solutions to the Navier-Stokes equations in two dimensions *Arch. Ration. Mech. Anal.* **161** 307–37

[34] Giga Y 1985 Domains of fractional powers of the Stokes operator in L_r spaces *Arch. Rational Mech. Anal.* **89** 251–65

[35] Gill A E 1982 *Atmosphere–Ocean Dynamics* vol 30 (Academic)

[36] Goldys B and Peszat S 2023 Linear parabolic equation with Dirichlet white noise boundary conditions *J. Differ. Equ.* **362** 382–437

[37] Grubb G 2001 Nonhomogeneous Dirichlet Navier-Stokes problems in low regularity L_p Sobolev spaces *J. Math. Fluid Mech.* **3** 57–81

[38] Hieber M and Saal J 2018 The Stokes equation in the L^p-setting: Well-posedness and regularity properties *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids* pp 117–206

[39] Hytönen T P, van Neerven J M A M, Veraar M C and Weis L 2017 *Analysis in Banach Spaces. Vol. II. Probabilistic Methods and Operator Theory* (*Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge.*) vol 67 (Springer)

[40] Kalton N J, Kunstmann P C and Weis L W 2006 Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators *Math. Ann.* **336** 747–801

[41] Kolmogorov A N 1991 The local structure of turbulence in an incompressible fluid at very high Reynolds numbers *Proc. R. Soc. A* **434** 9–13

[42] Kunstmann P C and Weis L W 2004 Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H_∞-functional calculus *Functional Analytic Methods for Evolution Equations* (*Lecture Notes in Math*) vol 1855 (Springer) pp 65–311

[43] Lemarié-Rieusset P G 2018 *The Navier-Stokes Problem in the 21st Century* (CRC Press)

[44] Li X and Tan Z 2024 Non-uniqueness of weak solutions to 2D generalized Navier-Stokes equations (arXiv:2405.20754)

[45] Lions P L 1996 *Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models* vol 1 (Clarendon Oxford)

[46] Luongo E 2024 Inviscid limit for stochastic Navier-Stokes equations under general initial conditions *J. Differ. Equ.* **389** 114–49

[47] Mikulevicius R and Rozovskii B L 2004 Stochastic Navier–Stokes equations for turbulent flows *SIAM J. Math. Anal.* **35** 1250–310

[48] Neerven J M A M, Veraar M C and Weis L W 2012 Stochastic maximal L^p-regularity *Ann. Probab.* **40** 788–812

[49] Noll A and Saal J 2003 H^∞-calculus for the Stokes operator on L_q-spaces *Math. Z.* **244** 651–88

[50] Nualart D 1995 *The Malliavin calculus and related topics (Probability and its Applications)* vol 21 (Springer)

[51] Nualart D, Rovira C and Tind S 2003 Probabilistic models for vortex filaments based on fractional Brownian motion *Ann. Probab* **31** 1862–99

[52] Pedlosky J 1996 *Ocean Circulation Theory* (Springer)

[53] Pedlosky J 2013 *Geophysical Fluid Dynamics* (Springer)

[54] Pope S B 2001 Turbulent flows *Meas. Sci. Technol.* **12** 2020–1

[55] Prüss J and Simonett G 2016 *Moving Interfaces and Quasilinear Parabolic Evolution Equations* (*Monographs in Mathematics*) vol 105 (Birkhäuser/Springer)

[56] Raymond J P 2007 Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **24** 921–51

[57] Sawano Y 2018 *Theory of Besov Spaces* (*Developments in Mathematics*) vol 56 (Springer)

[58] Schmeisser H J and Triebel H 1987 *Topics in Fourier Analysis and Function Spaces* (Wiley)

[59] Schumacher K 2008 Very weak solutions to the stationary Stokes and Stokes resolvent problem in weighted function spaces *Ann. Univ. Ferrara* **54** 123–44

[60] Serrin J 1961 *On the Interior Regularity of Weak Solutions of the Navier-Stokes Equations* (Air Force Office of Scientific Research) Mathematics Division

[61] Sowers R B 1994 Multidimensional reaction-diffusion equations with white noise boundary perturbations *Ann. Probab.* **22** 2071–121

[62] Temam R 1995 *Navier–Stokes Equations and Nonlinear Functional Analysis* (SIAM)

[63] Temam R 2001 *Navier–Stokes Equations: Theory and Numerical Analysis* vol 343 (American Mathematical Soc)

[64] Triebel H 1983 *Theory of Function Spaces* (*Monographs in Mathematics*) vol 78 (Birkhäuser)