Delft University of Technology
Faculty Eletrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

Multivariate generalisations of classical hypergeometric polynomials from Lie theory

A thesis submitted to the
Delft Institute of Applied Mathematics
in partial fulfillment of the requirements

for the degree

MASTER OF SCIENCE
in
APPLIED MATHEMATICS

by

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Delft, The Netherlands
August 2022

MSc THESIS APPLIED MATHEMATICS

"Multivariate generalisations of classical hypergeometric polynomials from Lie theory"

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August, 2022 Delft

MULTIVARIATE GENERALISATIONS OF CLASSICAL HYPERGEOMETRIC POLYNOMIALS FROM LIE THEORY

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Abstract

In this thesis, we will be studying Lie groups and their connection to certain orthogonal polynomials. We will look into the classical Krawtchouk, Meixner and Laguerre polynomials, and the multivariate Krawtchouk and Meixner polynomials as defined by Iliev [5, 6]. Using representations of the Lie groups $SU(2)$ and $SU(1,1)$, it will be shown that the three classical polynomials can be described as matrix coefficients of the representations. Using this connection of the polynomials to Lie groups, we derive various properties of the polynomials from the unitarity of the representation and the associated Lie algebra representation.

Next, the representations are generalised to higher dimensional spaces. Doing so, a new connection is shown between the Lie group $SU(d+1)$ and the multivariate Krawtchouk polynomials, extending the known theory for the univariate polynomials. Another new result that will be established is the connection between the multivariate Meixner polynomials and Lie theory. This will be done by defining a representation of $SU(1,d)$ in the Bergman space of the d-dimensional unit ball. Similar as for the univariate polynomials, we will derive the orthogonality, recurrence relations and difference equations from the associated Lie theory.

Keywords. Lie groups, Lie algebras, representation theory, orthogonal polynomials, Krawtchouk polynomials, Meixner polynomials, Laguerre polynomials.

Date: August 21, 2022.

Preface

This thesis was written as a final assignment in obtaining my degree Master of Science in Applied Mathematics at the Technical University of Delft. After five years, it is time for me to conclude my studies. Whilst I am happy to have achieved this feat, it is also with regret that I end this fun and intriguing part of my life.

Firstly, I want to thank my daily supervisor Wolter. Our weekly meetings were always enjoyable and I would like to thank you for all the valuable feedback and guidance during the project. I also want to thank Jan and Cor for willing to take part in my thesis committee and taking the time to read this thesis. Furthermore, I would like to extend my sincere thanks to my friends and family who motivated and helped me in my studies. Special thanks to my roommate Björn for proofreading my thesis and helping me stay motivated by studying together. Lastly, I want to thank Thao for proof reading my thesis and fixing numerous spelling mistakes. She judged me for every spelling mistake and forgotten comma or dot, and so helped tremendously in making this report readable (I hope).

Joop Vermeulen

Delft, August 2022.

Contents

Introduction

In this thesis, we will be studying Lie groups and their connection to certain orthogonal polynomials. We start by looking into the well studied connections between the Lie group $SU(2)$ with Krawtchouk polynomials and the Lie group $SU(1,1)$ and Meixner polynomials. We will also look into the connection of the Laguerre polynomials with the Lie group $SU(1,1)$. Thereafter, we will establish connections between the Lie group $SU(d+1)$ with d-variate Krawtchouk polynomials and between $SU(1,d)$ and d-variate Meixner polynomials, where d is some positive integer. Although Iliev used a character algebra to establish a connection between Lie theory and the multivariate Krawtchouk polynomials, we will be associating the Krawtchouk polynomials with a Lie group directly. For the multivariate Meixner polynomials, the relation with Lie theory which we will show is a new result.

In Section 1, the basic theory and language of representation theory and special functions will be described. To help develop a basic understanding, examples will be given to illustrate the definitions, theorems and other important tools. While we aim to develop a good understanding of this theory, the use of Lie theory and its connections to orthogonal polynomials ultimately come to live when given a proper examination. In order to achieve this, in Section 2 we work out the simple theory for Krawtchouk polynomials using the Lie group $SU(2)$. A representation of $SU(2)$ will be given on the space of homogeneous polynomials in two variables of fixed degree. Using this representation, the Krawtchouk polynomials will be recovered in terms of the matrix elements of the representation. Next, proving unitarity of the representation, the orthogonality of the Krawtchouk polynomials will be found from the orthogonality of the matrix elements. Lastly, the three-term recurrence relation will be established through the action of the Lie algebra $\mathfrak{su}(2)$. Having gone through the theory for the univariate Krawtchouk polynomials, we will move on to Meixner polynomials in Section 3. Whilst the Meixner polynomials are very similar to the Krawtchouk polynomials, the theory necessary to relate $SU(1,1)$ to the Meixner polynomials will prove to be more involved. Namely, we will have to work with an infinite dimensional representation for which we have to be more careful with the boundedness and smoothness of the representation. We will define a representation of $SU(1,1)$ on a Hilbert space of holomorphic functions defined on the unit ball, the Bergman space. We will give an extensive motivation on why this space is suitable as representation space. Having build the representation space, we proceed as for the Krawtchouk polynomials by relating the Meixner polynomials to matrix elements of the representation, and using properties of the representation to derive orthogonality and recurrence relations.

With the theory for the univariate polynomials explained, we extend the theory to the multivariate case. First, in Section 4, we create a representation of $SU(d+1)$ on homogeneous polynomials in $d+1$ variables of fixed degree. Then, proceeding as in the univariate case, we will relate the resulting matrix elements to the multivariate Krawtchouk polynomials. The orthogonality then follows by the unitarity of the representation. Lastly, we recover recurrence relations from the action of the Lie algebra $\mathfrak{su}(d+1)$. We will also highlight similarities and differences with Iliev's work. Next, in Section 5, we will look into the Lie group $SU(1,d)$ to define a representation on the Bergman space of holomorphic functions of the d-dimensional unit ball. We first use properties of the Lie group to show well definedness of the representation of choice. Thereafter, using Iliev's work as a stepping stone, we proceed as is custom. Lastly, we will briefly look into the (univariate) Laguerre polynomials in Section 6. Here we will broadly follow the work of Vilenkin and Klimyk [7] in order to establish a generating function and look into possibilities for multivariate extensions of the polynomials.

Notation. In this thesis the non-negative integers will be denoted by $\mathbb{N}_0$ or $\mathbb{N}$. All matrices that will be considered are square and the transpose and complex conjugate of a matrix A are denoted by A^t and $A^\dagger$, respectively. The standard basis vectors for the vector space $\mathbb{C}^n$ will be denoted by $\mathbf{v}_k$, for $k = 1, \ldots, n$, where $\mathbf{v}_k = [0, \ldots, 0, 1, 0, \ldots, 0]^t$ with a one on the k-th position. The matrix units in $\mathbb{C}^{n\times n}$ will be denoted by $e_{i,j}$ for $i, j = 1, \ldots, n$ where $e_{i,j}$ is the matrix with (i,j)'th component 1 and all others equal to 0. Vectors will be written in bold, while their elements are written normally, e.g. $\mathbf{z} \in \mathbb{C}^n$ will be written as $\mathbf{z} = \left[z_1, \ldots, z_n\right]^t$.

By a representation, we will mean a smooth representation of a Lie group or a Lie algebra, which of the two is meant should be clear from the context. Lie groups are denoted in capital letters, such as G, H and $SU(2)$. Lie algebras are denoted by small letters using the mathfrak typeset in LaTeX, such as $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{su}(2)$.

In Sections 4 and 5, we will use multi-index notation. This mainly concerns operations with vectors consisting of integers. Let $\mathbf{n} = (n_1, \dots n_k) \in \mathbb{N}_0^k$ be a vector of length k, consisting of non-negative integers. We will write $|\mathbf{n}| = n_1 + \cdots + n_k$ for the sum of the integers and $\mathbf{n}! = n_1! \cdots n_k!$ for the product of the factorials. Furthermore, if $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{C}^k$, we write $\mathbf{x}^\mathbf{n} = x_1^{n_1} \cdots x_k^{n_k}$. Lastly, $\delta_{m,n}$ will denote the Kronecker delta, defined by

$$\delta_{m,n} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Here m and n will be taken as integers, as well as multi-indices.

1. Basics of representation theory

In this section some basics of representation theory and Lie theory will be given. This will serve as an elementary introduction if the reader is not yet familiar with the subjects. We will start by defining Lie groups and representations of these, focusing on matrix Lie groups. We strive to give many examples to help the reader grasp the basics of the theory. Next, we define Lie algebras and their representations. We will clarify the connection between matrix Lie algebras and matrix Lie groups, concluding our theory on these. Lastly we will go over some special functions that will be used in the thesis, and the theory of orthogonal polynomials.

1.1. **Lie groups and representation theory.**

Although we will mostly encounter so-called matrix Lie groups, we will start by giving the general definition.

Definition 1.1. A Lie group is a smooth manifold equipped with a group structure such that multiplication and inversion are smooth.

In the example below, we give the most important Lie groups that will surface in this thesis.

Example.

(1) For any positive integer n, we denote by $GL(n,F)$ the set of invertible $n \times n$ matrices over the field $F = \mathbb{R}, \mathbb{C}$. This forms a group under the usual matrix multiplication and inversion. If we give $GL(n,F)$ the topology induced by the Hilbert-Schmidt norm, we can see it as a Lie group.

(2) If V is a vector space, the set of all bijective linear transformations of V forms a group under functional composition. This group is called the general linear group and will be denoted by $GL(V)$.

A very useful result to construct new Lie groups from old ones is given by the theorem below.

Theorem 1.2. *Let H be a subgroup of a Lie group G. If H is closed in the sense of topology (so that limits can be taken), then H itself is a Lie group.*

Now some other important examples of (matrix) Lie groups will be given.

Example.

(1) The special linear group, $SL(n,F)$, of matrices of determinant 1; this follows as the determinant is a smooth function.
(2) The group of unitary matrices, $U(n) := \{g \in GL(n,\mathbb{C}) \,|\, g^\dagger g = I\}$ where $g^\dagger$ is the conjugate transpose of the matrix g and I is the identity. This forms a closed subgroup of $GL(n,\mathbb{C})$ as the map $g \mapsto g^\dagger$ is continuous and matrix multiplication is smooth.
(3) The group of special unitary matrices, $SU(n) = SL(n,\mathbb{C}) \cap U(n)$.
(4) Let $J_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, where I_p and I_q are the diagonal matrices with diagonal 1 of lengths p and q respectively. The group $SU(p,q)$ is the group of matrices g of determinant one preserving the (standard) hermitian inner product of signature (p,q) given by the matrix $J_{p,q}$, that is it holds that $g^\dagger J_{p,q} g = J_{p,q}$. Analogous as for $SU(n)$, $SU(p,q)$ forms a closed subgroup of $SL(p+q,\mathbb{C})$ as the map $g \mapsto g^\dagger J_{p,q} g$ is continuous.
(5) The orthogonal group $O(n) := \{g \in GL(n,\mathbb{R}) \,|\, g^t g = I\}$ where g^t is the transpose of the matrix g. This group is clearly closed as the transpose and matrix multiplication are smooth.

Recall that a homomorphism between two groups G, H is a map $\phi : G \to H$ satisfying

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2).$$

Thus a homomorphism is a map preserving the group structure. For Lie groups, we want to retain the smoothness of maps as well.

Definition 1.3. Let G, H be Lie groups. A *Lie group homomorphism* is a smooth group homomorphism $\phi : G \to H$. This means that $\pi(gh) = \pi(g)\pi(h)$ for any $g, h \in G$ and the map $g \mapsto \phi(g)$ is smooth. If ϕ is also bijective with smooth inverse, it is called a *Lie group isomorphism*. Lastly, two Lie groups G, H are called *isomorphic* if there exists a Lie group isomorphism between them.

For finite dimensional H, the smoothness of ϕ can be relaxed to ϕ being continuous, as continuity of the homomorphism then implies smoothness (see f.e. [4, 10]).

We are mostly interested in (closed) subgroups of $GL(n,F)$. These subgroups will be called *matrix Lie groups*. The main purpose for Lie groups for us is by means of so called representations.

Definition 1.4. Let G be a (finite dimensional) Lie group and V a complex vector space. A *representation* of a Lie group G on a complex vector space V is a smooth map $\pi : G \longrightarrow GL(V)$ satisfying the following:

- For every $g \in G$, the map $\pi(g) : V \longrightarrow V$, $v \mapsto \pi(g)v$ is in $GL(V)$, i.e. it is linear and invertible.
- The map $g \mapsto \pi(g)$ is a Lie group homomorphism between G and $GL(V)$.

If there is a representation π of G on V, we will also say that G acts on V via π.

We can also define representations on infinite dimensional complex vector spaces. This however brings more difficulties regarding boundedness, so that also smoothness need not be attained everywhere. As definition of a representation on an infinite dimensional vector space, we retain the definition above, except that we require the map $g \mapsto \pi(g)$ to only be smooth on a dense subspace of V.

Let G be a Lie group and V a vector space. Possibly the most simple example of a representation is the *trivial representation* given by $\pi(g)v = v$ for all $g \in G$ and $v \in V$. Another elementary example is a representation of a matrix Lie group by standard matrix-vector multiplication. That is $G \subseteq GL(n, F)$ acts on $V = F^n$ by $\pi(g)v = gv$. This map is called the *standard representation* of G. A representation π of a matrix Lie group G on V induces a representation $\tilde{\pi}$ of G on $V^* = \{f | f : V \longrightarrow F, f \text{ linear}\}$ given by the action

$$[\tilde{\pi}(g)f](z) = f(\pi(g^t)z),$$

where $g \in G, f \in V^*$ and $z \in V$. Lastly, we will look at the *adjoint representation.* As this representation is particularly important, we put it inside a definition.

Definition 1.5. Let G be a Lie group. The *adjoint representation* $\mathrm{Ad} : G \longrightarrow GL(G)$ is defined by

$$\mathrm{Ad}_g(h) = ghg^{-1}, \qquad g, h \in G.$$

For finite dimensional V, a representation is just a way of expressing the group G as a group of matrices acting on V. Let π be a representation of the Lie group G on an N-dimensional vector space V. Evidently, if we fix an ordered basis $\{e_1, \dots, e_N\}$ of V and let $g \in G$ be arbitrary, the *matrix elements* of $\pi(g)$ with respect to this basis are determined by

$$\pi(g)e_n = \sum_{m=1}^{N} \pi_{m,n} e_m. \tag{1.1}$$

This definition will also be used in more general settings, mainly for countably infinite bases. To make this precise, let $\mathcal{N}$ be some countable index set and $\{e_n\}_{n\in\mathcal{N}}$ be a basis of a vector space V. Then the *matrix elements* of $\pi(g)$ with respect to this basis are determined by

$$\pi(g)e_n = \sum_{m\in\mathcal{N}} \pi_{m,n} e_m.$$

If moreover V is a Hilbert space and $\{e_n\}_{n\in\mathcal{N}}$ is an orthonormal basis, it follows that the matrix elements are given by

$$\pi_{m,n}(g) = \langle \pi(g)e_n, e_m \rangle.$$

Definition 1.6. Let π be a representation of a Lie group G on a Hilbert space V. We say that π is an *unitary representation* if $\pi(g)$ is a unitary operator on V for each $g \in G$.

According to this definition a representation is unitary if and only if

$$\pi_{m,n}(g) = \langle \pi(g)e_n, e_m \rangle = \langle e_n, \pi(g^{-1})e_m \rangle = \overline{\pi_{n,m}(g^{-1})}, \qquad \text{for all } g \in G, \text{ and } n, m \in \mathcal{N}.$$

As way of example, we compute the matrix elements of the trivial and the standard representations.

Example.
(1) Let G be some Lie group, V an Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathcal{N}}$ for some countable index set $\mathcal{N}$. Suppose π is the trivial representation, i.e. $\pi(g)v = v$ for all $g \in G$ and $v \in V$. Clearly for any $m, n \in \mathcal{N}$ it holds

$$\langle \pi(g)e_n, e_m \rangle = \langle e_n, e_m \rangle = \delta_{m,n}.$$

This means $\pi(g)_{m,n} = \delta_{m,n}$, hence the matrix of $\pi(g)$ is the identity matrix for all $g \in G$. Clearly π defines a unitary representation.

(2) Let N be some natural number and suppose $G \subseteq GL(N; F)$, $F = \mathbb{R}, \mathbb{C}$. Furthermore let $V = F^N$ with standard basis vectors $e_n = (0, \dots, 1, \dots 0)^t$ with a 1 on the n'th position. This basis is orthonormal with respect to the standard vector dot product. We take the standard representation of G on F^d by matrix-vector multiplication, that is $\pi(g)v = gv$. Evidently, the matrix elements of $\pi(g)$ are precisely those of g itself; $\pi_{m,n}(g) = \langle ge_n, e_m \rangle = g_{m,n}$. This

representation is unitary if and only if G consists solely of unitary matrices, meaning $G \subseteq U(N)$ if $F = \mathbb{C}$ or $G \subseteq O(N)$ if $F = \mathbb{R}$.

Instead of how we defined the 'standard representation' of $GL(N, F)$ on F^N by multiplying a column vector from the left by g, we can also define a similar representation of $GL(N, F)$ by multiplying a row vector from the right by g^t. If we take the standard row-vectors e_n as basis, we even get that the two matrices of the representations are identical. We introduce an equivalence notion for representations.

Definition 1.7. Let $\pi^a : G \longrightarrow V_a$ and $\pi^b : G \longrightarrow V_b$ be two representations of G. A continuous linear map $A : V_a \longrightarrow V_b$ is called an *intertwining map* if

$$A\pi^a(g) = \pi^b(g)A, \qquad \text{for all } g \in G.$$

If A is invertible with continuous inverse, the representations π^a, π^b are said to be *equivalent* and we write $\pi^a \cong \pi^b$.

Note that in case the representations are finite dimensional one does not have to require the map A to be continuous as this holds for any linear map $V^a \longrightarrow V^b$.

For the example above we find that the map A acting on F^N by transposition, $Av = v^t$ is an intertwining map. If V^a, V^b are Hilbert spaces, we can sometimes retain the matrix coefficients.

Lemma 1.8. *Let π^a, π^b be representations of a Lie group G on Hilbert spaces V^a, V^b. Suppose there is a unitary continuously invertible map $A : V^a \longrightarrow V^b$ intertwining the representations π^a and π^b. Then there exist bases of V^a and V^b such that the matrix elements of $\pi^a(g)$ and $\pi^b(g)$ are identical.*

Proof. Let A be as given and denote the inner products of V^x by $\langle \cdot, \cdot \rangle_x$, $x = a, b$. Fix some ordered orthonormal basis $\{e_n\}_{n\in\mathcal{N}}$ of V^a. As A is assumed to be unitary, an orthonormal basis of V^b is given by $\{Ae_n\}_{n\in\mathcal{N}}$. By equivalence of π^a and π^b we then find

$$\langle \pi^1(g)e_n, e_m\rangle_1 = \langle A\pi^1(g)e_n, Ae_m\rangle_2 = \langle \pi^2(g)Ae_n, Ae_m\rangle_2,$$

from which the lemma follows. □

1.2. **Lie algebras and representation theory.**

Definition 1.9. A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that for all $X, Y, Z \in \mathfrak{g}$ we have

- $[X, Y] = -[Y, X]$ (anti-symetry),
- $[X[Y, Z]] + [Y, [Z, X]] + [Z[X, Y]] = 0$ *(Jacobi-identity)*

The map $[\cdot, \cdot]$ is called the *Lie bracket or commutator.*

Example. (1) For any positive integer n, the Lie algebra $\mathfrak{gl}(n; F)$ is the vector space of $n\times n$ matrices with Lie bracket given by

$$[X, Y] = XY - YX.$$

This map is clearly anti-symmetric. Furthermore it holds that

$$\begin{aligned}[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] =& X(YZ - ZY) - (YZ - ZY)X \\ &+ Y(ZX - XZ) - (ZX - XZ)Y \\ &+ Z(XY - XY) - (XY - XY)Z \\ =&0,\end{aligned}$$

so that the Jacobi identity holds.

(2) If V is a vector space, we denote by $\mathfrak{gl}(V)$ the space of linear maps $V \longrightarrow V$. This becomes a Lie algebra if we take the Lie bracket defined by $[X, Y] = XY - YX$ (here, XY is the composition of X with Y).

We will mainly be focusing on matrix Lie algebras; subalgebras of $\mathfrak{gl}(n; F)$. For this, we first define what a subalgebra is.

Definition 1.10. Let $\mathfrak{g}$ be a Lie algebra. A (Lie) subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace of the vector space $\mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$.

Clearly a subalgebra of a Lie algebra is a Lie algebra itself. The special linear algebra, $\mathfrak{sl}(n; F)$, is the subalgebra of $\mathfrak{gl}(n; F)$ of matrices with trace 0. That it is closed with respect to the bracket follows as for $X, Y \in \mathfrak{sl}(n; F)$

$$\mathrm{Tr}([X, Y]) = \mathrm{Tr}(XY) - \mathrm{Tr}(YX) = 0,$$

where the last equality follows by the cyclic property of the trace.

Next, we define homomorphisms for Lie algebras.

Definition 1.11. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras with Lie brackets $[\cdot,\cdot]_{\mathfrak{g}}$, and $[\cdot,\cdot]_{\mathfrak{h}}$ respectively. A linear map $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is called a *Lie algebra* homomorphism if it preserves the bracket:

$$\phi([X,Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}},$$

for all $X, Y \in \mathfrak{g}$. If ϕ is bijective, then ϕ is called a *Lie algebra isomorphism.*

As for Lie groups, we will mainly use Lie algebras by means of representations.

Definition 1.12. Let $\mathfrak{g}$ be a Lie algebra and V a vector space. A *representation* of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

One very basic example of a Lie algebra representation is that of the algebra $\mathfrak{gl}(n;F)$ on the vectors $V = F^n$ by the map $\phi(g)v = gv$, where gv is the standard matrix-vector multiplication.

An important property for (matrix) Lie algebras is their correspondence with Lie groups. We roughly sketch this correspondence for matrix Lie groups and give the main results. Let G be a matrix Lie group. We call $g : \mathbb{R} \longrightarrow G$ a *one-parameter subgroup* if it is continuous, $g(0) = I$ and $g(s+t) = g(s)g(t)$ for all $s, t \in \mathbb{R}$. Using the smoothness of the Lie group G, it can be shown that every such one-parameter subgroup is differentiable and satisfies the initial value problem

$$g'(t) = Xg(t), \qquad g(0) = I,$$

for some $n \times n$ matrix X. That is, g is of the form

$$g(t) = \exp(tX),$$

where $\exp(tX)$ is the matrix exponential defined by the series

$$\exp(Y) = \sum_{k=0}^{\infty} \frac{Y^k}{k!} = I + Y + \frac{Y}{2} + \dots. \tag{1.2}$$

The collection of such X is then the associated Lie algebra.

Theorem 1.13. *Let G be a matrix Lie group. Its Lie algebra $\mathfrak{g}$ is given by*

$$\mathfrak{g} = \{X \mid \exp(tX) \in G, \text{ for all } t \in \mathbb{R}\} = \{g'(0) \mid g(t) \text{ is a one-parameter subgroup of } G\}.$$

Using the above theorem, it becomes clear that $\mathfrak{gl}(n;F)$ is the Lie algebra of $GL(n;F)$. We give some further examples of basic matrix Lie algebras.

Example.

(1) The special linear algebra $\mathfrak{sl}(n;F)$ is the subalgebra of $\mathfrak{gl}(n;F)$ of matrices with trace 0; this follows as for $X, Y \in \mathfrak{sl}(n;F)$

$$\mathrm{Tr}([X,Y]) = \mathrm{Tr}(XY) - \mathrm{Tr}(YX) = 0,$$

where the last equality follows by the cyclic property of the trace. From the identity $\det e^X = e^{\mathrm{Tr}\, X}$ it can be shown that it is the Lie algebra of $SL(n;F)$.

(2) The unitary algebra $\mathfrak{u}(n)$ is the Lie algebra of $U(n)$. It follows that $\mathfrak{u}(n)$ is the set of anti-hermitian matrices, $U^\dagger = -U$. This follows by considering a one-parameter subgroup $U(t)$ of $U(n)$ and considering the equation $(U(t))^\dagger U(t) = I$. Differentiating both sides with respect to t and setting t equal to zero then gives the above equality.

(3) The special unitary algebra $\mathfrak{su}(n)$ is the subalgebra of $\mathfrak{u}(n)$ of trace zero. Evidently, it is also the Lie algebra of $SU(n)$.

(4) The Lie group $SU(p,q)$ induces the Lie algebra $\mathfrak{su}(p,q)$ of matrices of trace zero satisfying

$$X^\dagger J = -JX.$$

This follows just as for $\mathfrak{u}(n)$ using the defining equation $g^\dagger J_{p,q} g = J_{p,q}$ of $SU(p,q)$, where $J_{p,q}$ is the $(p+q) \times (p+q)$ diagonal matrix with first p entries 1 and last q entries -1.

(5) Like for $\mathfrak{u}(n)$, the Lie algebra of $O(n)$ is the set of anti-symmetric matrices,

$$X^t = -X.$$

This algebra is denoted by $\mathfrak{o}(n)$.

Recall the adjoint representation Ad for a Lie group G on itself. We can also define an adjoint representation from G onto $\mathfrak{g}$:

$$\mathrm{Ad}_g(X) = gXg^{-1}, \qquad \text{for } g \in G, X \in \mathfrak{g}. \tag{1.3}$$

It is clear that Ad acts linearly on $\mathfrak{g}$ and as a homomorphism on G. To see that Ad maps $\mathfrak{g}$ to $\mathfrak{g}$, we use Theorem 1.13. That is, it suffices to show that $\exp(t\,\mathrm{Ad}_g(X)) \in G$ for all $t \in \mathbb{R}$. But this follows directly as $\exp(tgXg^{-1}) = g\exp(tX)g^{-1} = \mathrm{Ad}_g(\exp(tX)) \in G$, where the last Ad is the adjoint representation acting on G instead.

The correspondence between a Lie group and its Lie algebra also induces that a Lie group homomorphism gives rise to a Lie algebra homomorphism.

Theorem 1.14. *Let G be a Lie group with corresponding Lie algebra $\mathfrak{g}$. Suppose G acts on a finite-dimensional vector space V by the representation π. Then there is a corresponding Lie algebra representation $d\pi$ of $\mathfrak{g}$ on V such that for $X \in \mathfrak{g}$*

$$\pi(\exp(X)) = \exp(d\pi(X)), \tag{1.4a}$$

and

$$d\pi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi(\exp(tX)). \tag{1.4b}$$

As way of example, we calculate the Lie algebra representations corresponding to some of the Lie group representations we have given.

Example. Let $G \subseteq GL(n;F)$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ and V a vector space.

(1) Consider the trivial Lie group representation $\pi(g)v = v$ for all $g \in G, v \in V$. Using Equation (1.4b), we can define a Lie algebra representation for $\mathfrak{g}$. Let $X \in \mathfrak{g}$, $v \in V$ arbitrary, then

$$\mathrm{d}\pi(X)v = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi(\exp(tX)v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} v = 0.$$

Thus the corresponding Lie algebra representation is given by multiplication by 0.

(2) Suppose $V = F^n$ and let π be the standard representation, i.e., $\pi(g)v = gv$. Then the corresponding Lie algebra representation is also given by

$$\mathrm{d}\pi(X)v = Xv, \qquad X \in \mathfrak{g}, v \in V.$$

Indeed, let $X \in \mathfrak{g}$ and $v \in V$ be arbitrary, then $\mathrm{d}\pi(X)v = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{tX}v = Xv$.

(3) Consider the adjoint representation $\mathrm{Ad} : G \longrightarrow \mathfrak{g}$ as defined in Equation (1.3). Take some $X, Y \in \mathfrak{g}$ and consider the differential

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Ad}_{\exp(tY)}(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\exp(tY)X\exp(-tY) = YX - XY = [Y, X],$$

where $[\cdot,\cdot]$ is the Lie bracket. Define this map by ad_Y, that is

$$\mathrm{ad}_Y X = [Y, X],$$

by the above it then holds that $\mathrm{ad} = \mathrm{dAd}$.

Lastly, we will look unto the *complexification* of a Lie algebra.

Definition 1.15. Let $\mathfrak{g}$ be a real Lie algebra; this means it is closed under scalar multiplication of the real field. Then the *complexification* of $\mathfrak{g}$ is the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ defined by extending the field to the complex numbers as follows.

(1) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ as sum of vector spaces where scalar multiplication is defined as usual:

$$(a + ib)(X + iY) = (aX - bY) + i(bX + aY), \qquad a, b \in \mathbb{R},\ X, Y \in \mathfrak{g}.$$

(2) The Lie bracket of $\mathfrak{g}_{\mathbb{C}}$ is defined by

$$[X_1 + iY_1, X_2 + iY_2] := [X_1, X_2] - [Y_1, Y_2] + i([X_1, Y_2] + [Y_1, X_2]), \qquad X_1, X_2, Y_1, Y_2 \in \mathfrak{g}.$$

If ϕ is a representation of the real Lie algebra $\mathfrak{g}$ on a complex vector space, then it can be extended naturally to a representation $\hat{\phi}$ on the complexification $\mathfrak{g}_{\mathbb{C}}$; for $X, Y \in \mathfrak{g}$, we define $\hat{\phi}(X + iY) := \phi(X) + i\phi(Y)$. This allows us to use a bigger Lie algebra. The extended representation $\hat{\phi}$ will usually be again denoted by ϕ.

1.3. **Special functions.** We start by defining the gamma and beta functions and the pochhammer symbol.

Definition 1.16. For $\mathrm{Re}(x) > 0$, the gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t.$$

Using integration by parts, one can find the following :

Theorem 1.17. *For* $\mathrm{Re}(x) > 0$, $\Gamma(x+1) = x\Gamma(x)$.

Using that $\Gamma(1) = 1$, we also see that $\Gamma(1+n) = n!$ for natural numbers n. By the above theorem one finds another interesting result. Namely, for any natural number n and complex number x with positive real part it holds $\Gamma(x+n) = (x+n-1)(x+n-2)\cdots x\Gamma(x)$. This resulting product will be denoted by $(x)_n$, and is called the shifted factorial, or Pochhammer symbol. For general $x \in \mathbb{C}$, $n \in \mathbb{N}_0$ we define

$$(x)_0 = 1, \qquad (x)_n = x(x+1)\cdots(x+n-1).$$

Note that for $x = 1$, we get back the standard factorial, $(1)_n = n!$. By definition it also holds that $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. We give a limit concerning the gamma function that we will use later.

Lemma 1.18. *Let $a, b \in \mathbb{C}$ with positive real part. The following limit holds:*

$$\lim_{k\to\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} k^{b-a} = 1. \tag{1.5}$$

Next, we consider the beta function, which we will see is closely related to the gamma function.

Definition 1.19. For $\mathrm{Re}(x), \mathrm{Re}(y) > 0$, the *beta function* is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t.$$

The beta function can be written in terms of the gamma function as follows:

Theorem 1.20. *For* $\mathrm{Re}(x), \mathrm{Re}(y) > 0$,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Definition 1.21. For non-negative integers p, q, the *hypergeometric series* is the series given by

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right) = \sum_{k=0}^\infty \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!},$$

where all parameters are taken from the complex field.

In general, we omit the negative integers as values of the b_i. If for some i, we have that a_i is a negative integer, then this series terminates. More precise, suppose $a_i = -n$, then

$$(a_i)_k = (-n)(-n+1)\cdots(-n+k-1) \begin{cases} \neq 0, & \text{for } k \leq n, \\ = 0, & \text{for } k > n. \end{cases}$$

In particular, the hypergeometric series terminates after the term $k = n$ and so the series converges. All the hypergeometric functions we will consider are of this type. However, for the sake of completeness, we look at the convergence of a general hypergeometric series.

Lemma 1.22. *The hypergeometric series* ${}_pF_q\left(\begin{smallmatrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{smallmatrix}; x\right)$ *converges absolutely for all x if $p < q+1$ and for $|x| < 1$ if $p = q+1$. It diverges for all x if $p > q+1$ (if no a_i is a negative integer).*

Proof. This follows directly by using the ratio test. □

As for $p \leq q+1$ the series converges absolutely, it defines an analytic function on the open disk with center 0 and radius r ($r = \infty$ if $p < q+1$ and $r = 1$ if $p = q+1$). This will be called the *hypergeometric function.* As an elementary example of an hypergeometric function, we have the exponential function,

$$e^x = \sum_{k=0}^\infty \frac{x^n}{n!} = {}_0F_0\left(\begin{matrix} - \\ - \end{matrix}; x\right).$$

We can also generalize the binomial series,

$$(1-x)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} x^k = {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; x\right), \qquad |x| < 1, \tag{1.6}$$

where $a \in \mathbb{C}$.

Lastly, we will develop some theory of orthogonal polynomials. Let μ be positive measure on $\mathbb{R}$ and assume all moments of μ are finite, i.e.

$$\int_{\mathbb{R}} x^n \, \mathrm{d}\mu(x) < \infty, \qquad \text{for all } n \in \mathbb{N}.$$

The Hilbert space $L^2(\mu)$ is the space of functions $f : \mathbb{C} \to \mathbb{C}$ of finite 2-norm, with 2-norm induced by the inner product

$$\langle f, g \rangle_\mu = \int_{\mathbb{R}} f(x)\overline{g(x)} \, \mathrm{d}\mu(x).$$

A sequence $(p_n)_{n \in \mathbb{N}}$ of polynomials is called a *sequence of orthogonal polynomials* in $L^2(\mu)$ if

(1) $\deg(p_n) = n$, and
(2) $\langle p_m, p_n \rangle_\mu = C_m \delta_{m,n}$, for some $C_m > 0$.

One can show that orthogonal polynomials are unique up to a multiplicative constant. That is, for two sequences $(p_n)_{n \in \mathbb{N}}$, $(q_n)_{n \in \mathbb{N}}$ of orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_\mu$, there exists a sequence $(C_n)_{n \in \mathbb{N}}$ of nonzero numbers such that $p_n = C_n q_n$.

It can be shown that orthogonal polynomials satisfy a three-term recurrence relation.

Lemma 1.23. *Let $(p_n)_{n \in \mathbb{N}}$ be a sequence of orthogonal polynomials. Then, they satisfy a three-term recurrence relation*

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b(n) p_n(x) + a_n p_{n-1}(x), \qquad n \geq 0,$$

where $p_{-1} \equiv 0$ by convention and the constants a_n, b_n are given by

$$a_n = \langle x p_n, p_{n-1} \rangle_\mu, \qquad b_n = \langle x p_n, p_n \rangle_\mu.$$

By construction, it also holds that $a_n \neq 0$.

Evidently, if the constants a_n, b_n are known, as well as the initial value p_0, the sequence of polynomials can be reconstructed. Another way of encoding polynomials is by means of a generating function. Let $(p_n)_{n \in \mathbb{N}}$ be a sequence of polynomials. A generating function for $(p_n)_{n \in \mathbb{N}}$ is a formal power series in a 'variable' t written as

$$G(x, t) = \sum_{m=0}^{\infty} C(m, n) p_m(x) t^m,$$

where the $C(m, n)$ are some constants depending on m and n. As an example, we can use the binomial series from Equation (1.6).

Example. Let x be some nonzero complex number and consider the generating function

$$G(x, t) := (1-t)^{-x} = \sum_{m=0}^{\infty} p_m(x) t^m.$$

Then p_n is a polynomial in x given by $p_n(x) = \frac{(x)_n}{n!}$.

In this thesis, we will look at a couple of classes of orthogonal polynomials. This will be done by establishing a connection between Lie theory and orthogonal polynomials.

2. Krawtchouk polynomials in one variable

This section serves as an introduction to some of the known properties for the classical Krawtchouk polynomials. In particular, we will start by taking a finite dimensional representation of the special unitary group, $SU(2)$, and showing that the Krawtchouk polynomials can be seen as overlap coefficients between two orthonormal bases. We also show that the representation is unitary, so that the orthogonality relations of the Krawtchouk polynomials can be derived from the orthogonality of the matrix elements of the representation. Lastly, we also derive the recurrence relations from the action of the Lie algebra $\mathfrak{su}(2)$.

In section 4, we will use the methods developed here to derive analogous statements of the orthogonality and recurrence relations for multivariate Krawtchouk polynomials.

2.1. The Lie group $SU(2)$ and Krawtchouk polynomials.

We proceed analogous to the lecture notes in [4]. Let $N \in \mathbb{N}$. Write $\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$ and consider the space $\mathbb{C}^N[\mathbf{z}]$ of homogeneous polynomials in the variables z_0 and z_1 of degree N. A basis of $\mathbb{C}^N[\mathbf{z}]$ can be given by the homogeneous polynomials

$$e_n^N(\mathbf{z}) = \binom{N}{n}^{1/2} z_0^{N-n} z_1^n, \qquad \text{for } n = 0, 1, \ldots, N.$$

Throughout this section, we will consider the representation of $GL(2;\mathbb{C})$ on $\mathbb{C}^N[\mathbf{z}]$ given by

$$\left[\pi^N(g)p\right](\mathbf{z}) = p(g^t\mathbf{z}). \tag{2.1}$$

Recall from Equation (1.1) that the matrix elements are determined by the equation

$$\pi^N(g)e_n^N(\mathbf{z}) = \sum_{m=0}^{N} \pi_{m,n}^N(g)e_m^N(\mathbf{z}).$$

The main goal of this subsection is to express the matrix elements in terms of the classical Krawtchouk polynomials. From this, the orthogonality relations follow, see Theorem 2.8. We state the main result here:

Theorem 2.1. *For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2;\mathbb{C})$ with $b, c \neq 0$,*

$$\pi_{m,n}^N \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{N}{n}^{1/2} \binom{N}{m}^{1/2} a^{N-m-n} b^n c^m K_m\left(n; 1 - \frac{ad}{bc}, N\right). \tag{2.2}$$

Here, the classical Krawtchouk polynomials are given by the hypergeometric function

$$K_m(n;p,N) = {}_2F_1\left(\begin{matrix} -m, -n \\ -N \end{matrix}; \frac{1}{p}\right) = \sum_{k=0}^{\infty} \frac{(-m)_k(-n)_k}{(-N)_k} \frac{p^{-k}}{k!}, \tag{2.3}$$

where $N \in \mathbb{N}, 0 < p < 1$ are the parameters and $m, n \in \mathbb{N}, 0 \leq m, n \leq N$ are respectively the degree and the variable index. From the definition, it is clear that the Krawtchouk polynomials are self-dual, that is we have that

$$K_m(n;p,N) = K_n(m;p,N). \tag{2.4}$$

This theorem will be proved further down in this subsection. We start by writing out the basic properties of the representation with respect to the above basis of $\mathbb{C}^N[\mathbf{z}]$.

Proposition 2.2. *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2;\mathbb{C})$. The matrix elements of π^N are determined by*

$$\binom{N}{n}^{1/2} (az_0 + cz_1)^{N-n}(bz_0 + dz_1)^n = \sum_{m=0}^{N} \binom{N}{m}^{1/2} \pi_{m,n}^N(g) z_0^{N-m} z_1^m. \tag{2.5}$$

Furthermore, $\pi_{m,n}^N(g)$ is a homogeneous polynomial of degree N in a, b, c, d with real coefficients and the following symmetry relations hold:

$$\pi_{m,n}^N(g) = \pi_{n,m}^N(g^t) \qquad \textit{and} \qquad \pi_{m,n}^N \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pi_{N-m,N-n}^N \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{2.6}$$

In order to prove Theorem 2.1, we first derive an explicit formula for the matrix elements.

Lemma 2.3. *The matrix elements are given by*

$$\pi^N_{m,n}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \binom{N}{n}^{1/2}\binom{N}{m}^{-1/2}\sum_{i=\max\{0,m+n-N\}}^{\min\{m,n\}}\binom{N-n}{m-i}\binom{n}{i}a^{N-m-n+i}b^{n-i}c^{m-i}d^i \tag{2.7}$$

Proof. By the binomial formula we have

$$(az_0+cz_1)^{N-n} = \sum_{j=0}^{N-n}\binom{N-n}{j}(az_0)^{N-n-j}(cz_1)^j,$$
$$(bz_0+dz_1)^n = \sum_{i=0}^{n}\binom{n}{i}(bz_0)^{n-i}(dz_1)^i.$$

Using this in the left-hand side of Equation (2.5) yields

$$\begin{aligned}\binom{N}{n}^{1/2}(az_0+cz_1)^{N-n}(bz_0+dz_1)^n &= \binom{N}{n}^{1/2}\sum_{i=0}^{n}\sum_{j=0}^{N-n}\binom{n}{i}\binom{N-n}{j}a^{N-n-j}b^{n-i}c^jd^iz_0^{N-(i+j)}z_1^{i+j}\\ &= \binom{N}{n}^{1/2}\sum_{i,m\in I}\binom{N-n}{m-i}\binom{n}{i}a^{N-n-m+i}b^{n-i}c^{m-i}d^iz_0^{N-m}z_1^m,\end{aligned}$$

where we have set $m=i+j$ and the set I is given by

$$I := \{(i,m)\,|\,0\le i\le n,\ i\le m\le i+(N-n)\}.$$

Rewriting this set for fixed bounds for m yields

$$I = \{(i,m)\,|\,0\le m\le N,\ \max\{0,m+n-N\}\le i\le \min\{m,n\}\}$$

Lastly, writing out the last summation with respect to this parameterization of I and comparing it to the right-hand side of Equation (2.5) the claim follows □

As can be seen from the summation, there are four cases to be considered:

(1) $m+n-N\le 0$ and $m\le n$,
(2) $m+n-N\le 0$ and $m\ge n$,
(3) $m+n-N\ge 0$ and $m\le n$,
(4) $m+n-N\ge 0$ and $m\ge n$.

However, from the symmetry relations in Proposition 2.2, it follows that if we compute one case the others will follow easily. Before we prove the main theorem, we state some transformation formulas for ${}_2F_1$ hypergeometric functions.

Lemma 2.4. *We have the following transformation formulas [8]*

$$\begin{aligned}{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right) &= (1-z)^{-a}{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix};\frac{z}{z-1}\right) \qquad \textit{(Pfaff)},\\ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right) &= (1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}c-a,c-b\\c\end{matrix};z\right) \qquad \textit{(Euler)},\\ {}_2F_1\left(\begin{matrix}-n,b\\c\end{matrix};z\right) &= \frac{(b)_n}{(c)_n}(-z)^n{}_2F_1\left(\begin{matrix}-n,\ -c-n+1\\-b-n+1\end{matrix};\frac{1}{z}\right).\end{aligned}$$

In the last equation it is assumed that, if $b=-m$ or $c=-N$ is a negative integer, that then $m,N\ge n$.

A proof of the first two formulas can be found in, e.g., [1, 4]. The third equation is found by reversing the summation, using that $(a)_k=(a)_{k-i}(-1)^i(-a-k+1)_i$. From these transformation rules, one can find the following transformation:

Corollary 2.5.

$${}_2F_1\left(\begin{matrix}-n,b\\c\end{matrix};z\right) = \frac{(c-b)_n}{(c)_n}{}_2F_1\left(\begin{matrix}-n,b\\-c+b-n+1\end{matrix};1-z\right), \tag{2.8}$$

where it is assumed that if $b=-m$ or $c=-N$ is a negative integer, we have that $N-m\ge n$ and $N\ge n$.

Proof. By first reversing the summation, then applying Pfaff's transformation and then again reversing the summation, we find the wanted formula. □

We are now in a position to prove the main theorem, Theorem 2.1. Below we will restate the theorem for the less general case $g \in SU(2)$, to show that we obtain the classical Krawtchouk polynomials with $p \in [0, 1]$. However, the proof of the general case follows in the same way.

Theorem 2.6. *For $g \in SU(2)$,*

$$\pi^N_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{N}{n}^{1/2}\binom{N}{m}^{1/2} a^{N-m-n}b^n c^m K_m(n; -bc, N). \tag{2.9}$$

Moreover, we also have $-bc \in [0, 1]$

Proof. We restrict the proof to case 2) from above. By Lemma 2.3, we then have

$$\begin{aligned}\pi^N_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \binom{N}{n}^{1/2}\binom{N}{m}^{-1/2}\sum_{i=0}^{n}\binom{N-n}{m-i}\binom{n}{i}a^{N-m-n+i}b^{n-i}c^{m-i}d^i \\ &= \binom{N}{n}^{1/2}\binom{N}{m}^{1/2} a^{N-m-n}b^n c^m \sum_{i=0}^{n}\binom{N}{m}^{-1}\binom{N-n}{m-i}\binom{n}{i}\left(\frac{ad}{bc}\right)^i.\end{aligned}$$

Thus we have to prove that the remaining sum is precisely the given Krawtchouk polynomial. Writing out the binomial coefficients, and using the identities

$$(k-i)! = \frac{(-1)^i k!}{(-k)_i}, \qquad (k+i)! = k!(k+1)_i,$$

on the terms containing an i, we find

$$\begin{aligned}\sum_{i=0}^{n}\binom{N}{m}^{-1}\binom{N-n}{m-i}\binom{n}{i}\left(\frac{ad}{bc}\right)^i &= \sum_{i=0}^{n}\frac{(N-m)!(N-n)!}{N!(N-(m+n))!}\frac{(-n)_i(-m)_i}{(N-(m+n)+1)_i i!}\left(\frac{ad}{bc}\right)^i \\ &= \frac{(N-m)!(N-n)!}{N!(N-(m+n))!}\,{}_2F_1\left(\begin{matrix}-n,\ -m \\ N-m-n+1\end{matrix}; \frac{ad}{bc}\right).\end{aligned}$$

Rewriting the remaining fraction as $\frac{(-N+m)_n}{(-N)_n}$ and applying Corollary 2.5 (here we use that we are in case 2)), we find

$$\pi^N_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{N}{n}^{1/2}\binom{N}{m}^{1/2} a^{N-m-n}b^n c^m\,{}_2F_1\left(\begin{matrix}-n,\ -m \\ -N\end{matrix}; 1-\frac{ad}{bc}\right). \tag{2.10}$$

As $g \in SU(2)$, $ad - bc = 1$ and the claims follow for m, n in case 2). Case 1) now follows from the first symmetry relation of Proposition 2.2 and the self-duality of the Krawtchouk polynomials. The remaining cases follow from cases 1) and 2) by using the second transformation formula in Proposition 2.2 and Euler's transformation formula from Lemma 2.4. □

Next we will prove the orthogonality relations for the Krawtchouk polynomials. To this end, define an inner product, $\langle\cdot,\cdot\rangle$, on $\mathbb{C}^N[\mathbf{z}]$ by requiring $\langle e^N_n, e^N_m\rangle = \delta_{n,m}$ for $n, m = 0, \dots, N$.

Theorem 2.7. *Let L be a subgroup of $GL(2;\mathbb{C})$. The representation $\pi^N\big|_L$ of L on $\mathbb{C}^N[\mathbf{z}]$ is unitary if and only if $L \subseteq U(2)$. That is, $U(2)$ is the largest subgroup of $GL(2;\mathbb{C})$ on which π^N is unitary.*

Proof. We need to verify that $\pi^N_{m,n}(g) = \overline{\pi^N_{n,m}(g^{-1})}$ for all $m, n = 0, \dots, N$ if $g \in L$. To this end, note that by Proposition 2.2, we know that $\pi^N_{m,n}(g) = \pi^N_{n,m}(g^t) = \overline{\pi^N_{n,m}(g^\dagger)}$. Thus, we have $\pi^N_{m,n}(g) = \overline{\pi^N_{n,m}(g^{-1})}$ if and only if $\overline{\pi^N_{n,m}(g^\dagger)} = \overline{\pi^N_{n,m}(g^{-1})}$ for all $n, m = 0, \dots, N$. This is nothing more than needing $g^\dagger g \in \mathrm{Ker}(\pi^N)$. Some calculation shows that $\mathrm{Ker}(\pi^N) = \{aI \,|\, a \in \mathbb{C}, a^N = 1\}$, so that we require $g^\dagger g = aI$. Writing out $g^\dagger g$ it surely holds that $(g^\dagger g)_{0,0} \in \mathbb{R}_{\geq 0}$. Therefore it becomes clear that $a = 1$ is the only possibility, but this means $g^\dagger g = I$. This can only be if $g \in U(2)$. □

As we proved that the representation is unitary, we can derive the orthogonality of the Krawtchouk polynomials from the orthogonality of the matrix elements.

Theorem 2.8. *For $p \in (0, 1)$, we have*

$$\sum_{k=0}^{N}\binom{N}{k}(1-p)^{N-k}p^k K_m(k; p, N)K_n(k; p, N) = \binom{N}{m}^{-1}\left(\frac{1-p}{p}\right)^m\delta_{m,n}, \tag{2.11a}$$

$$\sum_{k=0}^{N}\binom{N}{k}(1-p)^{N-k}p^k K_k(m; p, N)K_k(n; p, N) = \binom{N}{m}^{-1}\left(\frac{1-p}{p}\right)^m\delta_{m,n}. \tag{2.11b}$$

Proof. As $\pi^N(g)$ is unitary we have that for $0 \le m, n \le N$

$$\langle \pi^N(g)e_n^N, \pi^N(g)e_m^N \rangle = \delta_{m,n}.$$

On the other hand, by writing out the left-hand side using Proposition 2.2, we get

$$\begin{aligned}\langle \pi^N(g)e_n^N, \pi^N(g)e_m^N \rangle &= \sum_{k=0}^{N} \pi_{k,n}^N(g)\langle e_k^N, \pi^N(g)e_m^N \rangle \\ &= \sum_{k=0}^{N} \pi_{k,n}^N(g)\overline{\pi_{k,m}^N(g)}.\end{aligned}$$

Combining the two equations and expressing the matrix elements in terms of Krawtchouk polynomials, we find

$$\binom{N}{n}^{1/2}\binom{N}{m}^{1/2}\left(\frac{b}{a}\right)^n\left(\frac{\overline{b}}{\overline{a}}\right)^m\sum_{k=0}^{N}\binom{N}{k}(|a|^2)^{N-k}(|b|^2)^k K_k(n;-bc,N)K_k(m;-bc,N) = \delta_{m,n},$$

where we have used that the Krawtchouk polynomials are real polynomials for $g \in SU(2)$. Moving all constants to the right-hand side, writing $p = -bc = |b|^2$ and using the self-duality of the Krawtchouk polynomials both assertions follow. □

2.2. **The Lie algebra $\mathfrak{su}(2)$ and Krawtchouk polynomials.** The complexification of the Lie algebra $\mathfrak{su}(2)$, $\mathfrak{su}_{\mathbb{C}}(2) = \mathfrak{su}(2) + i\mathfrak{su}(2)$ is (isomorphic to) $\mathfrak{sl}(2;\mathbb{C})$. A basis of $\mathfrak{sl}(2;\mathbb{C})$ is given by

$$H = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

together with the matrix units $e_{0,1}$ and $e_{1,0}$. We can compute the representation of $\mathfrak{sl}(2;\mathbb{C})$ corresponding to the representation given in Equation (2.1) on this basis, yielding the following:

Lemma 2.9. *The representation of $\mathfrak{sl}(2;\mathbb{C})$ on $\mathbb{C}^N[\mathbf{z}]$ is defined by*

$$\begin{aligned}\mathrm{d}\pi^N(H) &= \frac{1}{2}\left(z_0\partial_{z_0} - z_1\partial_{z_1}\right),\\ \mathrm{d}\pi^N(e_{0,1}) &= z_0\partial_{z_1},\\ \mathrm{d}\pi^N(e_{1,0}) &= z_1\partial_{z_0},\end{aligned}$$

where $\partial_{z_i} = \frac{\partial}{\partial z_i}$ is the partial derivative with respect to the variable z_i. For the basis $\{e_n\}_{n=0}^N$ of $\mathbb{C}^N[\mathbf{z}]$, we get the explicit actions

$$\begin{aligned}\mathrm{d}\pi^N(H)e_n &= \left(\frac{N}{2} - n\right)e_n,\\ \mathrm{d}\pi^N(e_{0,1})e_n &= \sqrt{n(N-n+1)}e_{n-1},\\ \mathrm{d}\pi^N(e_{1,0})e_n &= \sqrt{(n+1)(N-n)}e_{n+1}.\end{aligned} \tag{2.12}$$

Here we have set $e_i = 0$ for $i = -1, N+1$.

Proof. We have

$$\exp(tH) = \begin{pmatrix}e^{\frac{t}{2}} & 0\\ 0 & e^{-\frac{t}{2}}\end{pmatrix}, \qquad \exp(te_{0,1}) = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, \qquad \exp(te_{1,0}) = \begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}.$$

So that for $p \in \mathbb{C}^N[\mathbf{z}]$, we get

$$\mathrm{d}\pi^N(H)p(\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi^N(\exp tH)p(\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}p\left(\begin{bmatrix}e^{t/2}z_0\\ e^{-t/2}z_1\end{bmatrix}\right) = \frac{1}{2}z_0\frac{\partial p}{\partial z_0} - \frac{1}{2}z_1\frac{\partial p}{\partial z_1}.$$

Likewise, for $e_{0,1}$, we find

$$\mathrm{d}\pi^N(e_{0,1})p(\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}p\left(\begin{bmatrix}z_0\\ tz_0 + z_1\end{bmatrix}\right) = z_0\frac{\partial p}{\partial z_1},$$

and for $e_{1,0}$, we get

$$\mathrm{d}\pi^N(e_{1,0})p(\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}p\left(\begin{bmatrix}z_0 + tz_1\\ z_1\end{bmatrix}\right) = z_1\frac{\partial p}{\partial z_0},$$

which yield the first claim. To prove Equation (2.12), apply the above equations to the basis vector $e_n^N(z) = \binom{N}{n}^{1/2}z_0^{N-n}z_1^n$ and use that, for $n \neq 0$, $\binom{N}{n-1} = \frac{n}{N-n+1}\binom{N}{n}$ and, for $n \neq N$, $\binom{N}{n+1} = \frac{N-n}{n+1}\binom{N}{n}$. □

Since the representation π^N is unitary on $\mathbb{C}^N[\mathbf{z}]$, for each $g \in SU(2)$, $\pi(g)$ sends the orthonormal basis $\{e_n\}_{n=0}^N$ to another orthonormal basis. In what follows, we fix g and define the new basis by

$$\tilde{e}_n := \pi^N(g)e_n.$$

We will now construct a new basis of $\mathfrak{sl}(2;\mathbb{C})$ which acts tridiagonally on the basis $\{\tilde{e}_n\}_{n=0}^N$. Consider the automorphism Ad_g on $\mathfrak{sl}(2;\mathbb{C})$ defined by $\mathrm{Ad}_g(X) = gXg^{-1}$ (the adjoint representation). Using that $(gXg^{-1})^k = gX^kg^{-1}$ in the summation expansion of $\exp(t\,\mathrm{Ad}_g(X))$, it is easy to check that

$$\mathrm{d}\pi^N \circ \mathrm{Ad}_g = \mathrm{Ad}_{\pi(g)} \circ \mathrm{d}\pi^N\,. \tag{2.13}$$

Here $\mathrm{Ad}_{\pi(g)}$ is the automorphism of $\mathfrak{gl}(\mathbb{C}^N[\mathbf{z}])$ defined by $\mathrm{Ad}_{\pi(g)}\,\phi = \pi(g)\phi(\pi(g))^{-1}$, for $\phi \in \mathfrak{gl}(\mathbb{C}^N[\mathbf{z}])$. Hence, denoting by $\tilde{H}$ and $\tilde{e}_{i,j}$ the images of H and $e_{i,j}$ respectively under Ad_g, we get the following action on the basis $\{\tilde{e}_n\}_{n=0}^N$:

$$\begin{aligned} \mathrm{d}\pi^N(\tilde{H})\tilde{e}_n &= \left(\frac{N}{2} - n\right)\tilde{e}_n, \\ \mathrm{d}\pi^N(\tilde{e}_{0,1})\tilde{e}_n &= \sqrt{n(N-n+1)}\tilde{e}_{n-1}, \\ \mathrm{d}\pi^N(\tilde{e}_{1,0})\tilde{e}_n &= \sqrt{(n+1)(N-n)}\tilde{e}_{n+1}. \end{aligned} \tag{2.14}$$

Here it should again be understood that $\tilde{e}^N_{-1} = \tilde{e}^N_{N+1} = 0$.

As the actions of the $e_{i,j}, \tilde{e}_{i,j}$, $i \neq j$ act as raising and lowering operators on their corresponding bases, we can use them to determine a recurrence relation for the Krawtchouk polynomials. Before we determine this, we first derive the following lemma.

Lemma 2.10. *Let ρ be a unitary representation of a Lie group G on a complex Hilbert space V and $\mathrm{d}\rho$ be the corresponding Lie algebra representation of $\mathfrak{g}$. Then for any $X \in \mathfrak{g}$, it holds*

$$\langle \mathrm{d}\rho(X)u, v\rangle = -\langle u, \mathrm{d}\rho(X)v\rangle,$$

for any $u, v \in V$.

Proof. Let $u, v \in V$ and $X \in \mathfrak{g}$ be arbitrary. As the representation ρ is unitary, for $h \in G$ it holds

$$\langle \rho(h)u, v\rangle = \langle u, \rho(h^{-1})v\rangle.$$

Using the definition of the Lie algebra representation, we have

$$\langle \mathrm{d}\rho(X)u, v\rangle = \langle \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\rho(\exp(tX))u, v\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle u, \rho(\exp(-tX)v\rangle = \langle u, \mathrm{d}\rho(-X)v\rangle,$$

where in the second equality the derivative can be taken out of the inner product by smoothness of the maps ρ and $t \mapsto \exp(tX)$. The lemma now follows by linearity of $\mathrm{d}\rho$. $\square$

Using the linearity of the representation $\mathrm{d}\pi^{\mathrm{N}}$ on $\mathfrak{su}_{\mathbb{C}}(2) \cong \mathfrak{sl}(2;\mathbb{C})$ and that $X^\dagger = -X$ for $X \in \mathfrak{su}(2)$, we obtain the following relation for $\mathrm{d}\pi^{\mathrm{N}}$.

Corollary 2.11. *For $X \in \mathfrak{sl}(2;\mathbb{C})$,*

$$\langle \mathrm{d}\pi^N(X)u, v\rangle = \langle u, \mathrm{d}\pi^N(X^\dagger)v\rangle, \qquad u, v \in \mathbb{C}^N[\mathbf{z}].$$

We are now in the position to derive the recurrence relation and difference equation for the Krawtchouk polynomials.

Theorem 2.12. *Let $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$ and write $p = |b|^2$, we have*

$$\begin{aligned} -nK_m(n;p,N) =& p(N-m)K_{m+1}(n;p,N) \\ &- [p(N-m) + (1-p)m]K_m(n;p,N) \\ &+ (1-p)mK_{m-1}(n;p,N) \end{aligned} \tag{2.15a}$$

and

$$\begin{aligned} -mK_m(n;p,N) =& p(N-n)K_m(n+1;p,N) \\ &- [p(N-n) + (1-p)n]K_m(n;p,N) \\ &+ (1-p)nK_m(n-1;p,N). \end{aligned} \tag{2.15b}$$

Proof. Let g be as given and write $\tilde{H} = \mathrm{Ad}_g(H)$, $\tilde{e}_n = \pi^N(g)e_n$. From Corollary 2.11 and as $\tilde{H}^\dagger = \tilde{H}$, we have the equality

$$\langle \mathrm{d}\pi^N(\tilde{H})\tilde{e}_n, e_m\rangle = \langle \tilde{e}_n, \mathrm{d}\pi^N(\tilde{H})e_m\rangle. \tag{*}$$

Now recall the matrix elements from Equation (2.9); $\pi^N_{m,n}(g) = \langle \pi^N(g)e_n, e_m\rangle = B(m,n)K_m(n)$, where

$$B(m,n) = \binom{N}{n}^{1/2}\binom{N}{m}^{1/2} a^{N-m-n}b^n c^m.$$

From Equation (2.14), the left-hand side becomes $(\frac{N}{2} - n)B(m,n)K_m(n)$.

By a direct computation, we can express $\tilde{H} = \mathrm{Ad}_g(H)$ in terms of the basis $\{H, e_{0,1}, e_{1,0}\}$ of $\mathfrak{sl}(2;\mathbb{C})$ as follows:

$$\tilde{H} = (|a|^2 - |b|^2)H - abe_{0,1} - \overline{ab}e_{1,0}.$$

Using this in addition to Lemma 2.9 (and the conjugate linearity of the inner product), we can evaluate the right-hand side of Equation (*) as

$$\begin{aligned}\langle \tilde{e}_n, \mathrm{d}\pi^N(\tilde{H})e_m\rangle =&(|a|^2 - |b|^2)\left(\frac{N}{2} - m\right)B(m,n)K_{m,n}\\ &- \overline{ab}\sqrt{m(N-m+1)}B(m-1,n)K_{m-1}(n)\\ &- ab\sqrt{(m+1)(N-m)}B(m+1,n)K_{m+1}(n).\end{aligned}$$

With some calculation, one finds

$$\begin{aligned}B(m-1,n) &= -\frac{a}{\overline{b}}\sqrt{\frac{m}{N-m+1}}B(m,n)\\ B(m+1,n) &= -\frac{\overline{b}}{a}\sqrt{\frac{N-m}{m+1}}B(m,n).\end{aligned}$$

Equating the found expressions for the left- and right-hand side of Equation (*) and dividing by $B(m,n)$ using the above relations, we arrive at

$$\begin{aligned}\left(\frac{N}{2} - n\right)K_m(n) =&(|a|^2 - |b|^2)\left(\frac{N}{2} - m\right)K_{m,n}\\ &+ |a|^2 mK_{m-1}(n)\\ &+ |b|^2(N-m)K_{m+1}(n).\end{aligned}$$

So that if we move terms around, set $p := |b|^2$ (and thus $1 - p = |a|^2$), we get the recurrence relation (2.15a). The proof of the difference equation, Equation (2.15b), can be found similarly expressing H in terms of $\tilde{H}, \tilde{e}_{0,1}, \tilde{e}_{1,0}$, or simply by using the self-duality of the Krawtchouk polynomials (2.4). $\square$

3. Meixner polynomials in one variable

In this section, we will look into the connection between the Lie group $SU(1,1)$ and Meixner polynomials.

The classical Meixner polynomials (see e.g. [1, 8]) of degree m are defined on $\mathbb{N}$ by the generating function

$$(1-t)^{-\beta-n}\left(1-\frac{t}{c}\right)^{n}=\sum_{m\in\mathbb{N}_0}\frac{(\beta)_m}{m!}M_m(n;c,\beta)t^n, \tag{3.1}$$

where $\beta, c\in\mathbb{R}$, c nonzero, are the parameters. As hypergeometric function, they are given as

$$M_m(n;c,\beta)={}_2F_1\left(\begin{matrix}-m,-n\\ \beta\end{matrix};1-\frac{1}{c}\right)=\sum_{k=0}^{\infty}\frac{(-m)_k(-n)_k}{(\beta)_k}\frac{(1-{}^1/_c)^k}{k!}. \tag{3.2}$$

Comparing this to hypergeometric definition of the Krawtchouk polynomials, Equation (2.3), it is clear that the Meixner polynomials are closely related to the Krawtchouk polynomials via

$$K_m(n;p,N)=M_m\left(n;\frac{p}{p-1},-N\right).$$

However, as opposed to the Krawtchouk polynomials, for the Meixner polynomials the parameter β is often taken as a positive real number (as opposed to a negative integer for the Krawtchouk) causing the Meixner polynomials to be defined on the whole of $\mathbb{N}$. This will in particular have implications for the choice of representation, as we have to work with an infinite dimensional representation instead.

First, we will determine a suitable (infinite dimensional) Hilbert space and define a unitary representation of $SU(1,1)$ here. Next, we calculate the matrix elements of this representation and show that they can be written in terms of the Meixner polynomials. Using the unitarity of the representation, we derive the orthogonality of the Meixner polynomials. Lastly, using the Lie algebra of $SU(1,1)$, we derive recurrence and difference equations, similar as for the Krawtchouk polynomials.

In section 5, we will use the methods developed here to acquire similar results for the Lie group $SU(1,d)$ and multivariate Meixner polynomials as defined by Iliev in [5].

3.1. **$SU(1,1)$ and the Bergman space on the unit ball.** Recall that the Lie group $SU(1,1)$ is defined as the group of matrices of unit determinant preserving the hermitian form given by the matrix $J=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, i.e.

$$\langle g\mathbf{z},g\mathbf{w}\rangle_J=\langle\mathbf{z},\mathbf{w}\rangle_J,\qquad g\in GL(2;\mathbb{C}),$$

where $\langle\mathbf{z},\mathbf{w}\rangle_J=z_0\overline{w_0}-z_1\overline{w_1}$.

We will consider a representation of $SU(1,1)$ on the the space $H(\mathbb{B})$ of holomorphic functions on the unit ball, where $\mathbb{B}:=\{\mathbf{z}\in\mathbb{C}\,|\,|z|^2<1\}$. This representation will be given by the map

$$\left[\pi^\beta\begin{pmatrix}a&b\\c&d\end{pmatrix}p\right](z)=(a+cz)^{-\beta}p\left(\frac{a+cz}{b+dz}\right). \tag{3.3}$$

In order to construct a representation from this mapping, we have to make a choice for β. The motivation on why we take this map and its well-definedness is topic of 3.1.1. Next, in 3.1.2, we define a measure on $H(\mathbb{B})$ so that we can work on a Hilbert space. We will also make a choice of β and prove the unitarity of π^β. We note that although the map is a homomorphism on the whole of $H(\mathbb{B})$, it does not need to act smoothly everywhere. Lastly, in 3.1.3, we will compute the matrix elements and find an orthogonality relation for the Meixner polynomials.

3.1.1. *Detour: The space of holomorphic functions on the unit ball.* As discussed above, we want to have an infinite dimensional space on which we will act. As the Meixner polynomials are very much alike to the Krawtchouk polynomials, a natural choice is to also act on polynomials. We will restrict to one variable polynomials instead of two variables to circumvent problems around zero. The choice will be to work with the space of holomorphic functions on the unit ball $\mathbb{B}$, which will be denoted by $H(\mathbb{B})$. In particular, $f:\mathbb{B}\longrightarrow\mathbb{C}$ is holomorphic if and only if $f(z)=\sum_{k=0}^{\infty}a_kz^k$, where convergence is pointwise.

Let $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL(2;\mathbb{C})$. As for a representation, we like to have that

$$\pi(g)z^n=(a+cz)^{-\beta-n}(b+dz)^n=(a+cz)^{-\beta}\left(\frac{b+dz}{a+cz}\right)^n,$$

as to imitate the representation π^N (see Equation (2.1)). The following lemma asserts that this map is well defined for $g\in SU(1,1)$.

Lemma 3.1. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$*, then the map* $z \mapsto \frac{b+dz}{a+cz}$ *maps* $\mathbb{B}$ *onto itself.*

Proof. We look at the set $\tilde{\mathbb{B}} := \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 - |z_1|^2 > 0\}$. Note that the condition $0 < |z_0|^2 - |z_1|^2$ enforces z_0 to be nonzero, hence dividing by $|z_0|^2$ we get $0 < 1 - |\frac{z_1}{z_0}|^2$. The above tells us that we can map $\tilde{\mathbb{B}}$ surjectively onto $\mathbb{B}$ by the map

$$(z_0, z_1) \mapsto \frac{z_1}{z_0}. \tag{*}$$

On the space $\tilde{\mathbb{B}}$ we can define the indefinite hermitian form

$$\langle (z_0, z_1), (w_0, w_1) \rangle = z_0 \overline{w_0} - z_1 \overline{w_1},$$

then $\tilde{\mathbb{B}} = \{\mathbf{z} = (z_0, z_1) \mid \langle \mathbf{z}, \mathbf{z} \rangle > 0\}$.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$ and $(z_0, z_1) \in \tilde{\mathbb{B}}$. As $U(1,1)$ is the group preserving the above hermitian form, using Equation (*) on $g^t(z_0, z_1)$ we find the map $(z_0, z_1) \mapsto \frac{bz_0 + dz_1}{az_0 + cz_1}$. Writing $z = \frac{z_1}{z_0}$ and using that the map (*) is surjective, we conclude that the map $\mathbf{z} \mapsto \frac{b+dz}{a+cz}$ is well defined and maps $\mathbb{B}$ onto itself. ☐

As by the above $a + cz \neq 0$, for $p \in H(\mathbb{B})$ the function $z \mapsto (a+cz)^{-\beta} p\left(\frac{b+dz}{a+cz}\right)$ is also holomorphic as composition and product of holomorphic functions. This motivates us to define the map $\pi^\beta : SU(1,1) \longrightarrow GL(H(\mathbb{B}))$ by

$$[\pi^\beta(g)p](z) = (a+cz)^{-\beta} p\left(\frac{b+dz}{a+cz}\right). \tag{3.3}$$

That the above map is also a homomorphism can be seen by the correspondence between $\mathbb{B}$ and $\tilde{\mathbb{B}}$ as in the proof of Lemma 3.1, or by a direct calculation of $\pi^\beta(gh)$. For smoothness, we first need to define an inner product on $H(\mathbb{B})$. This will be done in the next subsection.

3.1.2. *The Bergman space on the unit ball.* For $\alpha > -1$, define the (probability) measure $\mathrm{d}v_\alpha$ on $\mathbb{B}$ by

$$\mathrm{d}v_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha \,\mathrm{d}v(z), \tag{3.4}$$

where

$$c_\alpha = \frac{1}{B(1, \alpha+1)} = \alpha + 1, \tag{3.5}$$

and $\mathrm{d}v$ is the standard volume measure on $\mathbb{B}$. The weighted Bergman space $\mathcal{A}_\alpha$ is the collection of holomorphic functions f in $L^2(\mathbb{B}, \mathrm{d}v_\alpha)$. In terms of sets we have $\mathcal{A}_\alpha = L^2(\mathbb{B}, \mathrm{d}v_\alpha) \cap H(\mathbb{B})$. As $\mathcal{A}_\alpha$ is closed in $L^2(\mathbb{B}, \mathrm{d}v_\alpha)$ (see Corollary 2.5 in [11]), it becomes a Hilbert space if we take the inner product

$$(f, g)_\alpha = \int_{\mathbb{B}} f(z) \overline{g(z)} \,\mathrm{d}v_\alpha. \tag{3.6}$$

The condition $\alpha > -1$ is enforced in order to make the norm of the constant function 1 finite. Using polar coordinates, it is easy to show that $||z^n||_\alpha = 2c_\alpha \int_0^1 r^{2n+1}(1-r^2)^\alpha \,\mathrm{d}r$. Changing variables using $u = r^2$, it follows that $||z^n||_\alpha = c_\alpha B(n+1, \alpha+1) = \frac{n!\Gamma(\alpha+2)}{\Gamma(\alpha+2+n)} = \frac{n!}{(\alpha+2)_n}$. Note furthermore that the measure is unitarily invariant (see [11]), so that for $f \in L^1(\mathbb{B}, \mathrm{d}v_\alpha)$ and $U \in U(1)$,

$$\int_{\mathbb{B}} f(Uz) \,\mathrm{d}v_\alpha(z) = \int_{\mathbb{B}} f(z) \,\mathrm{d}v_\alpha(z).$$

In particular, as the integral is invariant under the rotation $Uz = ze^{i\phi}$ the monomials $\{z^n\}_{n \in \mathbb{N}_0}$ form an orthogonal system. As the monomials are also dense in $\mathcal{A}_\alpha$ (see Proposition 2.6 in [11]), we may conclude that the functions

$$e_n^\alpha(z) := \sqrt{\frac{(\alpha+2)_n}{n!}} z^n, \qquad n \in \mathbb{N}_0 \tag{3.7}$$

form an orthonormal basis of $\mathcal{A}_\alpha$.

Now that we have treated all technicalities in forming a representation, we only need to verify whether the restriction of the homomorphism π^β indeed defines a representation on $\mathcal{A}_\alpha$ for certain β, α. To do so, we must check two things:

(1) π^β maps $\mathcal{A}_\alpha$ into $\mathcal{A}_\alpha$,
(2) π^β acts smooth on $\mathcal{A}_\alpha$.

For the first, we will prove in the next section that the representation is unitary for $\beta = \alpha + 2$. For the second we refer to Lemma 2.10 in [2] for the proof that π^β acts smoothly on (finite) polynomials. Note that this does not imply that π^β acts smooth everywhere on $\mathcal{A}_\alpha$. Denote by $\mathcal{A}_\alpha^\infty$ the subspace of $\mathcal{A}_\alpha$ on which π^β acts smoothly. An element of $\mathcal{A}_\alpha^\infty$ will be called a smooth vector. As the polynomials are dense in $\mathcal{A}_\alpha$ and π^β acts smooth on the polynomials, $\mathcal{A}_\alpha^\infty$ is dense in $\mathcal{A}_\alpha$. Lastly we note that the space $\mathcal{A}_\alpha^\infty$ is invariant under $SU(1,1)$ (see section 2.1.5 of [7]), which will allow us to consider the Lie algebra representation in section 3.2.

3.1.3. *Meixner polynomials in one variable.* In the remainder of this section α is assumed to be a non-negative integer and we set $\beta = \alpha + 2$. We have argued that we can represent $SU(1,1)$ on the Bergman space $\mathcal{A}_\alpha$ via the map (3.3), short for that it maps $\mathcal{A}_\alpha$ back into $\mathcal{A}_\alpha$. In this subsection, we will first compute the image of the orthonormal basis $\{e_n^\alpha(z)\}_{n\in\mathbb{N}_0}$ under π^β as elements in $H(\mathbb{B})$. Next we use this to show that $\pi^\beta(g)$ is unitary for all $g \in SU(1,1)$, so that it indeed defines a representation as wanted. Lastly we use the unitarity of the representation to derive the orthogonality relation of the Meixner polynomials.

Fix some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$ with $b, c \neq 0$. By definition of $SU(1,1)$ it holds that $ad = |a|^2$, $bc = |b|^2$ and $|a|^2 - |b|^2 = 1$. In particular, it follows that $a, d \neq 0$ and $0 < \frac{bc}{ad} < 1$. Let $n \in \mathbb{N}_0$ be arbitrary. Per definition, we have that $\pi^\beta(g)z^n = (a+cz)^{-\beta-n}(b+dz)^n$. Using that β is an integer and that $a, b \neq 0$, we can take these out of the brackets resulting in:

$$(a+cz)^{-\beta-n}(b+dz)^n = a^{-\beta-n}b^n(1+\frac{c}{a}z)^{-\beta-n}(1+\frac{d}{b}z)^n.$$

Writing $t = -\frac{c}{a}z$, $\hat{c} = \frac{bc}{ad}$, we find the equality

$$\pi^\beta(g)z^n = a^{-\beta-n}b^n(1-t)^{-\beta-n}\left(1-\frac{t}{c}\right)^n = a^{-\beta-n}b^n \sum_{m\in\mathbb{N}_0} \frac{(\beta)_m}{m!} M_m(n;\hat{c},\beta)t^m,$$

where the second equality follows by the definition of the Meixner polynomials (3.1). Undoing the substitutions just made, we get the following theorem.

Theorem 3.2. *For $n \in \mathbb{N}_0$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$, with $b, c \neq 0$, it holds*

$$\pi^\beta(g)z^n = \sum_{m\in\mathbb{N}_0} \frac{(\beta)_m}{m!} a^{-\beta-m-n}b^n(-c)^m M_m\left(n;\frac{bc}{ad},\beta\right)z^m.$$

Hence, the matrix elements of the map π^β with respect to the basis $\{e_n^\alpha(z)\}_{n\in\mathbb{N}_0}$ are given by

$$\pi^\beta_{m,n}(g) = \sqrt{\frac{(\beta)_m}{m!}}\sqrt{\frac{(\beta)_n}{n!}} a^{-\beta-m-n}b^n(-c)^m M_m\left(n;\frac{bc}{ad},\beta\right). \tag{3.8}$$

Moreover, we also have $0 < \frac{bc}{ad} < 1$.

Remark. Note that in all of the above, we only used the unit determinant property of $SU(1,1)$ to verify that the representation is smooth and that $0 < \frac{bc}{ad} < 1$ in the above theorem. In particular, one can define the homomorphism π^β on $U(1,1)$ for which the matrix elements will still be given as in the above theorem, provided that $b, c \neq 0$.

Next, we want to prove that π^β maps $SU(1,1)$ to unitary maps on $\mathcal{A}_\alpha$. Let $g \in GL(2;\mathbb{C})$ be written as usual and $J = \operatorname{diag}(1,-1)$ (as in the definition of $SU(1,1)$). Suppose furthermore that $\pi^\beta(g)$ is well defined on $H(\mathbb{B})$, so that in particular the matrix elements are given by Equation (3.8). As $Jg^\dagger J = \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & \overline{d} \end{pmatrix}$, from Equation (3.8) and the hypergeometric representation of the Meixner polynomial (3.2), it is easy to see that $\pi^\beta_{m,n}(g) = \overline{\pi^\beta_{n,m}(Jg^\dagger J)}$.

Suppose now that $\pi^\beta(g)$ acts unitarily on $\mathcal{A}_\alpha$, we must have that $\pi^\beta_{m,n}(g) = \overline{\pi^\beta_{n,m}(g^{-1})}$. Suppose this equality holds, then by the above we can deduce that it must hold that $\pi^\beta(Jg^\dagger Jg) = I$, or equivalently, $Jg^\dagger Jg \in \operatorname{Ker}(\pi^\beta)$. Looking at the action of π^β on e_0 and e_1, it becomes clear that $\operatorname{Ker}(\pi^\beta) = \{\gamma I \mid \gamma^\beta = 1\}$. As the diagonal of $Jg^\dagger Jg$ is certainly real, the only remaining possibilities are $\gamma^\beta = 1$ and $\gamma^\beta = -1$ (the latter only in case of even numbers β). By explicitly looking at the case $Jg^\dagger Jg = -I$, one finds that this is impossible for $g \in GL(2;\mathbb{C})$, hence it must hold that $Jg^\dagger Jg = I$. By definition, it follows $g \in U(1,1)$.

By the above, it can be concluded that $U(1,1)$ is the maximal subgroup of $GL(2;\mathbb{C})$ on which π^β can be unitary. Now restrict to matrices of unit determinant. As the span of $\{e_n^\alpha(z)\}_{n\in\mathbb{N}_0}$ is dense in $\mathcal{A}_\alpha$ and $\pi^\beta(g)$ is smooth on this basis, by taking limits one can show the following theorem.

Theorem 3.3. *For $\beta = \alpha+2$, $\pi^\beta(g)$ is unitary on $\mathcal{A}_\alpha$ for each $g \in SU(1,1)$.*

As discussed in section 3.1.2, π^β acts smoothly on the subspace $\mathcal{A}_\alpha^\infty$. As a corollary to the above discussion, using that $\pi^\beta_{m,n}(\overline{g}) = \overline{\pi^\beta_{m,n}(g)}$, we also find $\pi^\beta_{m,n}(g) = \pi^\beta_{n,m}(Jg^tJ)$. This corresponds to the self-duality of the Meixner polynomials in the sense that

$$M_m(n;c,\beta) = M_n(m;c,\beta). \tag{3.9}$$

Using the unitarity of the representation, we can also derive that the Meixner polynomials are orthogonal with respect to the negative binomial distribution.

Theorem 3.4. *For an integer $\beta \geq 2$ and any $0<c<1$, we have*

$$\sum_{k=0}^\infty \frac{(\beta)_k}{k!}c^k M_m(k;c,\beta)M_n(k;c,\beta) = \delta_{m,n}\frac{m!c^{-m}}{(\beta)_m(1-c)^\beta}, \tag{3.10a}$$

$$\sum_{k=0}^\infty \frac{(\beta)_k}{k!}c^k M_k(m;c,\beta)M_k(n;c,\beta) = \delta_{m,n}\frac{m!c^{-m}}{(\beta)_m(1-c)^\beta}. \tag{3.10b}$$

Proof. This proof follows just as for the Krawtchouk polynomials. Firstly, as $\pi^\beta(g)$ is unitary, for any $m,n\in\mathbb{N}_0$, it holds that

$$\begin{aligned}\delta_{m,n} &= \langle \pi^\beta(g)e_m^\beta, \pi^\beta(g)e_n^\beta\rangle \\ &= \sum_{k=0}^\infty \pi^\beta_{k,m}(g)\langle e_k^\beta, \pi^\beta(g)e_n\rangle \\ &= \sum_{k=0}^\infty \pi^\beta_{k,m}(g)\overline{\pi^\beta_{k,n}(g)}.\end{aligned}$$

Expressing the matrix elements in terms of Meixner polynomials using Theorem 3.2 and moving terms independent of k to the other side, we find

$$\sum_{k=0}^\infty \frac{(\beta)_k}{k!}\left|\frac{c}{a}\right|^{2k} M_k(m;\hat{c},\beta)\overline{M_k(n;\hat{c},\beta)} = \frac{(\beta)_m}{m!}\left|\frac{a}{b}\right|^{2m}|a|^{2\beta}\delta_{m,n},$$

where $\hat{c} = \frac{bc}{ad}$. Now use that $\hat{c} = \left|\frac{b}{a}\right|^2 = \left|\frac{c}{a}\right|^2$ and $1-\hat{c} = |a|^{-2}$ for $g\in SU(1,1)$ to evaluate the above equation, which gives us Equation (3.10b). The other orthogonality relation, Equation (3.10a), easily follows by using the self-duality of the Meixner polynomials (see (3.9)).

Lastly, note that the map $SU(1,1) \ni \begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto \frac{bc}{ad}$ exhausts the interval $(0,1)$, f.e. by taking the matrices $\begin{pmatrix} \sqrt{x} & \sqrt{x-1} \\ \sqrt{x-1} & \sqrt{x}\end{pmatrix}$ where $x\in(1,\infty)$. Thus the orthogonality can be stated for any $0<c<1$. □

Classically, the Meixner polynomials satisfy the orthogonality relations (3.10) for all $\beta>0$. To construct a mapping on the Bergman spaces, we needed to have $\beta=\alpha+2>1$. Furthermore, to prove this map was a homomorphism, we restricted to integer values of β, thus yielding the smaller range for our parameter β.

In the next subsection, we will derive the Lie algebra representation and derive recurrence relations and difference equations for the Meixner polynomials.

3.2. **The Lie algebra $\mathfrak{su}(1,1)$ and Meixner polynomials.** The Lie algebra of $SU(1,1)$ is given by matrices g with trace zero such that $g^\dagger J = -Jg$, where $J=\mathrm{diag}(1,-1)$. Working out this definition shows that $\mathfrak{su}(1,1) = \left\{\begin{pmatrix} ia & b \\ \overline{b} & -ia\end{pmatrix} \,\middle|\, a\in\mathbb{R}, b\in\mathbb{C}\right\}$. It follows that the complexification of $\mathfrak{su}(1,1)$ is isomorphic to $\mathfrak{sl}(2;\mathbb{C})$. In what follows, we will denote the representation of the complexification again by π^β.

We can, as in the case of $\mathfrak{su}(2)$, consider the representation of the complexification $\mathfrak{su}_{\mathbb{C}}(1,1)\cong\mathfrak{sl}(2;\mathbb{C})$ instead (wherever the original Lie algebra representation is defined). As already mentioned above, the

Lie group $SU(1,1)$ acts smoothly on polynomials, so that the Lie algebra representation can certainly be defined here. The set of elements of $\mathcal{A}_\alpha$ on which π^β acts smoothly will be denoted by $\mathcal{A}_\alpha^\infty$, which is a dense $SU(1,1)$-invariant subspace of $\mathcal{A}_\alpha$ as discussed in 3.1.2.

As before, for a basis of $\mathfrak{sl}(2;\mathbb{C})$, we take

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

together with $e_{0,1}$ and $e_{1,0}$. Next, we calculate the action of $\mathfrak{sl}(2;\mathbb{C})$ on $\mathcal{A}_\alpha^\infty$.

Lemma 3.5. *The representation of $\mathfrak{sl}(2;\mathbb{C})$ on $\mathcal{A}_\alpha^\infty$ is defined by*

$$\begin{aligned} \mathrm{d}\pi^\beta(H) &= -\frac{1}{2}\beta - z\frac{\mathrm{d}}{\mathrm{d}z}, \\ \mathrm{d}\pi^\beta(e_{0,1}) &= \frac{\mathrm{d}}{\mathrm{d}z}, \\ \mathrm{d}\pi^\beta(e_{1,0}) &= -\beta z - z^2\frac{\mathrm{d}}{\mathrm{d}z}. \end{aligned}$$

So that for the basis $\{e_n^\alpha\}_{n\in\mathbb{N}_0}$ of $\mathcal{A}_\alpha^\infty$ we get

$$\begin{aligned} \mathrm{d}\pi^\beta(H)e_n^\alpha &= (-\frac{1}{2}\beta - n)e_n^\alpha, \\ \mathrm{d}\pi^\beta(e_{0,1})e_n^\alpha &= \sqrt{n(\beta+n-1)}e_{n-1}^\alpha, \\ \mathrm{d}\pi^\beta(e_{1,0})e_n^\alpha &= -\sqrt{(n+1)(\beta+n)}e_{n+1}^\alpha. \end{aligned} \tag{3.11}$$

Proof. We will naively calculate the action of $\mathfrak{sl}(2;\mathbb{C})$ directly, noting that this is not a 'proper' proof (see the remark below). Recall that we have that

$$\exp(tH) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}, \qquad \exp(te_{0,1}) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \exp(te_{1,0}) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \tag{*}$$

Let $p \in \mathcal{A}_\alpha^\infty$ be arbitrary. Calculating the action of H on p, we see

$$\mathrm{d}\pi^\beta(H)p(z) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi^\beta(\exp(tH))p(z) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-\frac{1}{2}t\beta}p\left(e^{-t}z\right) = -\frac{1}{2}\beta p - z\frac{\mathrm{d}p}{\mathrm{d}z}$$

Likewise for $e_{0,1}$ and $e_{1,0}$, we find

$$\begin{aligned} \mathrm{d}\pi^\beta(e_{0,1})p(z) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}p(t+z) = \frac{\mathrm{d}p}{\mathrm{d}z}, \\ \mathrm{d}\pi^\beta(e_{1,0})p(z) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(1+tz)^{-\beta}p\left(\frac{z}{1+tz}\right) = -\beta zp - z^2\frac{\mathrm{d}p}{\mathrm{d}z}. \end{aligned}$$

The first claim is clear from the above equalities. The second claim follows trivially by applying the above to the basis vectors $e_n^\alpha(z) := \sqrt{\frac{(\beta)_n}{n!}}z^{\mathbf{n}}$. ☐

Remark. In the above proof, we have used elements of the complexification of $\mathfrak{su}(1,1)$ directly in the computation of the Lie algebra representation, even though we have not shown the map π^β to be smooth here. By definition of the complexification of the Lie algebra representation, one should express $H, e_{0,1}$ and $e_{1,0}$ in the form $X+iY$, with $X, Y \in \mathfrak{su}(1,1)$ and compute the action by computing $\mathrm{d}\pi^\beta(X) + i\,\mathrm{d}\pi^\beta(Y)$. The action resulting by properly using the definition through complexification to calculate the action as depicted here, must coincide with the action 'calculated' as in the proof above. Hence, the action as presented in the above lemma still holds.

For what follows, let $g \in SU(1,1)$ be fixed. As for each $g \in SU(1,1)$ the representation $\pi^\beta(g)$ is unitary on $\mathcal{A}_\alpha^\infty$, and as $SU(1,1)$ leaves $\mathcal{A}_\alpha^\infty$ invariant, the vectors $\tilde{e}_n^\beta := \pi^\beta(g)e_n^\beta$ form a new orthonormal basis of $\mathcal{A}_\alpha^\infty$. If we adjoin the basis of $\mathfrak{sl}(2;\mathbb{C})$ by g, we get a new basis of $\mathfrak{sl}(2;\mathbb{C})$ which acts traditionally on the basis vectors $\tilde{e}_n^\alpha$. Denote this new basis as in section 2, i.e., write $\tilde{\phi} = g\phi g^{-1}$ for ϕ in the basis. Using the identity 2.13, we have the following action on the new basis:

$$\begin{aligned} \mathrm{d}\pi^\beta(\tilde{H})\tilde{e}_n^\alpha &= \left(-\frac{1}{2}\beta - n\right)\tilde{e}_n^\alpha, \\ \mathrm{d}\pi^\beta(\tilde{e}_{0,1})\tilde{e}_n^\alpha &= \sqrt{n(\beta+n-1)}\tilde{e}_{n-1}^\alpha, \\ \mathrm{d}\pi^\beta(\tilde{e}_{1,0})\tilde{e}_n^\alpha &= -\sqrt{(n+1)(\beta+n)}\tilde{e}_{n+1}^\alpha, \end{aligned} \tag{3.12}$$

where we have set $\tilde{e}^{\alpha}_{-1} = 0$. Lastly, using Lemma 2.10 and as $-X = JX^{\dagger}J$ for $X \in \mathfrak{su}(1,1)$, the following corollary can be shown.

Corollary 3.6. *For $X \in \mathfrak{sl}(2;\mathbb{C})$,*

$$\langle \mathrm{d}\pi^{\beta}(X)u, v\rangle = \langle u, \mathrm{d}\pi^{\beta}(JX^{\dagger}J)v\rangle, \qquad u, v \in \mathcal{A}^{\infty}_{\alpha}.$$

Proof. Let $X = Y + iZ \in \mathfrak{su}_{\mathbb{C}}(1,1)$. By conjugate linearity of the inner product and Lemma 2.10, it follows that

$$\langle \mathrm{d}\pi^{\beta}(Y+iZ)u, v\rangle = \langle u, \mathrm{d}\pi^{\beta}(-Y+iZ)v\rangle.$$

The claim now follows by using that $-\phi = J\phi^{\dagger}J$ for $\phi \in \mathfrak{su}(1,1)$. □

With this, we can show the recurrence relation and difference equation of the Meixner polynomials.

Theorem 3.7. *Let $\beta \geq 2$ be an integer and $0 < c < 1$. The Meixner polynomials satisfy the following recurrence relation*

$$\begin{aligned}(c-1)nM_m(n;c,\beta) =& c(\beta+m)M_{m+1}(n;c,\beta) \\ &- [m + c(\beta+m)]\, M_m(n;c,\beta) \\ &+ mM_{m-1}(n;c,\beta).\end{aligned} \tag{3.13a}$$

Furthermore, they also satisfy the difference equation

$$\begin{aligned}(c-1)mM_m(n;c,\beta) =& c(\beta+n)M_m(n+1;c,\beta) \\ &- [n + c(\beta+n)]\, M_m(n;c,\beta) \\ &+ nM_m(n-1;c,\beta)\end{aligned} \tag{3.13b}$$

Proof. Let $g \in SU(1,1)$ be arbitrary with the usual conditions. It can be shown that g has the form $g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a}\end{pmatrix}$ where $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$ (with $b \neq 0$ by our conditions). Let $\tilde{H} = \mathrm{Ad}_g(H)$ and $\tilde{e}_n = \pi^{\beta}(g)e_n$ be as above. By Corollary 3.6 above, and as $J\tilde{H}J = \tilde{H}$, it follows that

$$\langle \mathrm{d}\pi^{\beta}(\tilde{H})\tilde{e}_m, e_n\rangle = \langle \tilde{e}_m, \mathrm{d}\pi^{\beta}(\tilde{H})e_n\rangle. \tag{*}$$

Write $\hat{c} = \frac{|b|^2}{|a|^2}$ and abbreviate $M_m(n;\hat{c},\beta)$ by $M_m(n)$. Lastly, write $\pi^{\beta}_{m,n} = \langle \tilde{e}_n, e_m\rangle = B(m,n)M_m(n)$, where

$$B(m,n) = \sqrt{\frac{(\beta)_m}{m!}}\sqrt{\frac{(\beta)_n}{n!}}a^{-\beta-m-n}b^n(-c)^m$$

as in Theorem 3.2. The left-hand side of (*) can be easily computed using Equation (3.12) to be $\left(-\frac{1}{2}\beta - n\right) B(m,n)M_m(n)$.

Calculating $\tilde{H}$ and expressing it in terms of $H, e_{0,1}$ and $e_{1,0}$, one finds

$$\tilde{H} = \frac{1}{2}(|a|^2 + |b|^2)H - abe_{0,1} + \overline{ab}e_{1,0}.$$

By simple calculations, we find

$$\begin{aligned} B(m-1,n) &= -\frac{a}{\overline{b}}\sqrt{\frac{m}{\beta+m-1}}B(m,n), \\ B(m+1,n) &= -\frac{\overline{b}}{a}\sqrt{\frac{\beta+n}{n+1}}B(m,n).\end{aligned}$$

Using these rules, the expansion of $\tilde{H}$ as above and the action of the basis $\{H, e_{0,1}, e_{1,0}\}$ from Equation (3.11), we can express the right-hand side of (*) as

$$\begin{aligned}\langle \tilde{e}_m, \mathrm{d}\pi^{\beta}(\tilde{H})e_n\rangle =& -(|a|^2+|b|^2)\left(\frac{1}{2}\beta + m\right)B(m,n)M_m(n) \\ &+ m|a|^2B(m,n)M_{m-1}(n) \\ &+ (\beta+m)|b|^2B(m,n)M_{m+1}(n).\end{aligned}$$

Combining the expressions for the left- and right-hand side of (*) and dividing by $B(m,n)$, we gather

$$\begin{aligned}-\left(\frac{1}{2}\beta+n\right)M_m(n) =& -\left(|a|^2+|b|^2\right)\left(\frac{1}{2}\beta+m\right)M_m(n) \\ &+ |a|^2mM_{m-1}(n) \\ &+ |b|^2(\beta+m)M_{m+1}(n).\end{aligned}$$

Moving the terms with β to the right-hand side and doing some reordering, we get

$$\begin{aligned} -nM_m(n) = & |b|^2(\beta+m)M_{m+1}(n) \\ & - \left[|a|^2 m + |b|^2(\beta+m)\right] M_m(n) \\ & + |a|^2 m M_{m-1}(n). \end{aligned}$$

Using $\hat{c} = \frac{|b|^2}{|a|^2}$ and $|a|^2 - |b|^2 = 1$, we can write $|a|^2 = \frac{1}{1-\hat{c}}$ and $|b|^2 = \frac{\hat{c}}{1-\hat{c}}$. Equation (3.13a) now follows by substituting this into the above equation and multiplying both sides by $1-\hat{c}$.

The difference equation, Equation (3.13b), can be proven similarly by acting with H instead of $\tilde{H}$ and expressing H in terms of the basis $\{\tilde{H}, \tilde{e}_{0,1}, \tilde{e}_{1,0}\}$, or more simply by using the self-duality of the Meixner polynomials (3.9). $\square$

4. Krawtchouk polynomials in multiple variables

In Section 2, we have seen how to construct the classical (univariate) Krawtchouk polynomials, orthogonal with respect to the binomial distribution. We have also determined a connection of these polynomials to the Lie group $SU(2)$ and used this connection to extract the orthogonality, recurrence and difference relations from properties of $SU(2)$ and its Lie algebra.

Griffiths [3] used a generating function to define multivariate polynomials orthogonal to the multivariate distribution. In [6], Iliev used the Lie algebra $\mathfrak{sl}(d+1;\mathbb{C}), d > 1$ an integer, together with the space $\mathbb{C}^N[\mathbf{z}]$ of homogeneous polynomials in $d+1$ variables of total degree N, to describe the the d-variate orthogonal polynomials. His methodology consisted of defining two subalgebras $H, \tilde{H}$ of $\mathfrak{sl}(d+1;\mathbb{C})$ and two orthogonal bases of $\mathbb{C}^N[\mathbf{z}]$, each diagonalizing one of the aforementioned subalgebras, and using the transition matrix between the two bases of $\mathbb{C}^N[\mathbf{z}]$ to describe the Krawtchouk polynomials.

In this present thesis, we will determine similar results using the Lie group $SU(d+1)$. The methods used to derive the properties here are highly similar to those used for the univariate case in Section 2.

4.1. **The Lie group $SU(d+1)$ and multivariate Krawtchouk polynomials.** Let $N \in \mathbb{N}$. We will consider the space $\mathbb{C}^N[\mathbf{z}]$ of homogeneous polynomials in the variables $z_0, \ldots, z_d$ of degree N. Throughout the remainder of this thesis, we use multi-index notation. To make this precise, we write the conventions down below.

Notation. Let $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}_0^k$. Write $|\mathbf{n}| = n_1 + \cdots + n_k$ and $\mathbf{n}! = n_1! \cdots n_k!$. Furthermore, if $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{C}^k$, we write $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_k^{n_k}$ and $\mathbf{x}' = (x_2, \ldots, x_k)$.

For this section only, we will also define some more notation. Let $\mathbb{I}_N = \{\mathbf{n} \in \mathbb{N}_0^d \,|\, |\mathbf{n}| \leq N\}$. For $\mathbf{n} \in \mathbb{I}_N$, we define the multinomial coefficient by

$$\binom{N}{\mathbf{n}} = \frac{N!}{\mathbf{n}!(N-|\mathbf{n}|)!}.$$

With the above notation, $\mathbb{C}^N[\mathbf{z}]$ is the vector space spanned by the monomials $\mathbf{z}^{\mathbf{n}}$, where $\mathbf{z} = (z_0, \ldots, z_d)^t \in \mathbb{C}^{d+1}$ and $\mathbf{n} = (n_0, \ldots, n_d) \in \mathbb{N}_0^{d+1}, |\mathbf{n}| = N$. Evidently, a basis of $\mathbb{C}^N[\mathbf{z}]$ is given by the elements

$$e_{\mathbf{n}}^N(\mathbf{z}) := \binom{N}{\mathbf{n}}^{1/2} z_0^{N-|\mathbf{n}|}\mathbf{z}'^{\mathbf{n}}, \qquad \mathbf{n} \in \mathbb{I}_N.$$

If we define an inner product on $\mathbb{C}^N[\mathbf{z}]$ by $\langle e_{\mathbf{m}}^N, e_{\mathbf{n}}^N \rangle = \delta_{\mathbf{m},\mathbf{n}}$, $\mathbb{C}^N[\mathbf{z}]$ becomes a Hilbert space.

Let now $g \in GL(d+1;\mathbb{C})$ be arbitrary. We define the representation of $GL(d+1;\mathbb{C})$ on $\mathbb{C}^N[\mathbf{z}]$ again by

$$[\pi^N(g)p](\mathbf{z}) = p(g^t\mathbf{z}).$$

We will write $g \in GL(d+1;\mathbb{C})$ in the form $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix}$ where $a \in \mathbb{C}$, $\mathbf{b}, \mathbf{c} \in \mathbb{C}^d$ and D is a $d \times d$ complex matrix. The matrix elements of the representation with respect to the above basis are then determined by the equations

$$\binom{N}{\mathbf{n}}^{1/2} (az_0 + \mathbf{c}\cdot\mathbf{z}')^{N-|\mathbf{n}|} \prod_{i=1}^d \left(b_i z_0 + \sum_{j=1}^d D_{j,i} z_j \right)^{n_i} = \sum_{\mathbf{m}\in\mathbb{I}_N} \binom{N}{\mathbf{n}}^{1/2} \pi_{\mathbf{m},\mathbf{n}}^N(g) z_0^{N-|\mathbf{m}|}\mathbf{z}'^{\mathbf{m}}. \tag{4.1}$$

Suppose now that the elements in the first row and column of g are all nonzero (that is $a \neq 0$ as well as $\mathbf{b}$ and $\mathbf{c}$ having all entries nonzero). Define the matrix $U(g)$ by

$$U_{i,j} = \frac{aD_{j,i}}{b_i c_j}, \qquad i,j \in \{1, \ldots, d\},$$

and U has exclusively 1's in its first row and column (indexed by $i = 0$ and/or $j = 0$). Now pull out the first term in each bracket in the left-hand side of Equation (4.1) and set $\mathbf{x} = (x_1, \ldots, x_d)$ with $x_j = \frac{c_j z_j}{a z_0}$, we get

$$\binom{N}{\mathbf{n}}^{1/2} a^{N-|\mathbf{n}|}\mathbf{b}^{\mathbf{n}} z_0^N \left(1 + \sum_{j=1}^d x_j\right)^{N-|\mathbf{n}|} \prod_{i=1}^d \left(1 + \sum_{j=1}^d U_{i,j}x_j\right)^{n_i}.$$

Using the generating function for multivariate Krawtchouk polynomials from Griffits and Iliev [3, 6],

$$\left(1 + \sum_{j=1}^d x_j\right)^{N-|\mathbf{n}|} \prod_{i=1}^d \left(1 + \sum_{j=1}^d U_{i,j}x_j\right)^{n_i} = \sum_{\mathbf{n}\in\mathbb{I}_N} \binom{N}{\mathbf{n}} K_{\mathbf{m}}(\mathbf{n}; U, N)\mathbf{x}^{\mathbf{m}} \tag{4.2}$$

and expanding $\mathbf{x}^{\mathbf{m}}$, we arrive at

$$\pi^N(g)e^N_{\mathbf{n}}(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{I}_N} \binom{N}{\mathbf{n}}^{1/2}\binom{N}{\mathbf{m}}^{1/2} a^{N-|\mathbf{n}|-|\mathbf{m}|}\mathbf{b}^{\mathbf{n}}\mathbf{c}^{\mathbf{m}}K_{\mathbf{m}}(\mathbf{n};U,N)e^N_{\mathbf{m}}.$$

Comparing this with Equation (4.1), we see that we have shown the following theorem holds.

Theorem 4.1. *For $g \in SU(d+1)$ with the elements of the first row and column nonzero, we have*

$$\pi^N_{\mathbf{m},\mathbf{n}}(g) = \binom{N}{\mathbf{m}}^{1/2}\binom{N}{\mathbf{n}}^{1/2} a^{N-|\mathbf{n}|-|\mathbf{m}|}\mathbf{b}^{\mathbf{n}}\mathbf{c}^{\mathbf{m}}K_{\mathbf{m}}(\mathbf{n};U,N). \tag{4.3}$$

Mizukawa and Tanaka [9] have given the following formula for multivariate Krawtchouk polynomials in terms of the Gelfand hypergeometric series,

$$K_{\mathbf{m}}(\mathbf{n};U,N) = \sum_{A=(a_{i,j})\in M_{d,N}} \frac{\prod_{j=1}^d(-m_j)_{\sum_{i=1}^d a_{i,j}}\prod_{i=1}^d(-n_i)_{\sum_{j=1}^d a_{i,j}}}{(-N)_{\sum_{i,j=1}^d a_{i,j}}}\prod_{i,j=1}^d \frac{(1-U_{i,j})^{a_{i,j}}}{a_{i,j}!}, \tag{4.4}$$

where $m_1,\dots,m_d$ are the degree indices, $n_1,\dots,n_d$ are the variables and $M_{d,N}$ is the set of all $d\times d$ matrices with non-negative integer entries such that $\sum_{i,j=1}^d a_{i,j} \le N$. In particular, there is a clear symmetry in the Krawtchouk polynomials between the variables and the degree indices by

$$K_{\mathbf{m}}(\mathbf{n};U,N) = K_{\mathbf{n}}(\mathbf{m};U^t,N).$$

This corresponds to sending g to g^t.

Remark. hg

From the formula of the Krawtchouk polynomials in Equation (4.4) and the use of Theorem 4.1 above, we immediately see

Theorem 4.2. *For $g \in GL(d+1;\mathbb{C})$, $\pi^N_{\mathbf{m},\mathbf{n}}(g) = \overline{\pi^N_{\mathbf{n},\mathbf{m}}(g^\dagger)}$. In particular, $U(d+1)$ is the largest subgroup of $GL(d+1,\mathbb{C})$ on which the representation is unitary.*

Proof. The first claim follows by noting that from the above discussion and Theorem (4.1), it holds that $\pi^N_{\mathbf{m},\mathbf{n}}(g) = \pi^N_{\mathbf{n},\mathbf{m}}(g^t)$ and $\overline{\pi^N_{\mathbf{m},\mathbf{n}}(g)} = \pi^N_{\mathbf{m},\mathbf{n}}(\overline{g})$. The last claim follows similarly to the proof of Theorem 2.7. □

As the representation restricted to $SU(d+1)$ is unitary, we can again use the orthogonality of matrix elements to compute the orthogonality relations for the Krawtchouk polynomials. By unitarity of the representation, we have

$$\begin{aligned}\delta_{\mathbf{m},\mathbf{n}} &= \sum_{\mathbf{k}\in\mathbb{I}_N}\pi^N_{\mathbf{m},\mathbf{k}}(g)\pi^N_{\mathbf{k},\mathbf{n}}(g^\dagger)\\ &= \sum_{\mathbf{k}\in\mathbb{I}_N}\pi^N_{\mathbf{m},\mathbf{k}}(g)\overline{\pi^N_{\mathbf{n},\mathbf{k}}(g)}.\end{aligned}$$

Using the explicit expression for the matrix elements, Equation (4.3), moving the terms dependent on $\mathbf{m}$ and $\mathbf{n}$ to the left-hand side and by some additional rewriting. we find the following expression:

$$\sum_{\mathbf{k}\in\mathbb{I}_N}\binom{N}{\mathbf{k}}|a|^{2(N-|\mathbf{k}|)}\left(\prod_{i=1}^d|b_i|^{2k_i}\right)K_{\mathbf{m}}(\mathbf{k};U,N)\overline{K_{\mathbf{n}}(\mathbf{k};U,N)} = \binom{N}{\mathbf{n}}^{-1}\frac{\delta_{\mathbf{n},\mathbf{m}}}{|a|^{-2|\mathbf{n}|}\left(\prod_{i=1}^d|c_i|^{2n_i}\right)}. \tag{4.5}$$

Set $\mathbf{p} = [p_1,\dots,p_d]$, $\tilde{\mathbf{p}} = [\tilde{p}_1,\dots,\tilde{p}_d]$, where $p_i = |b_i|^2$, $\tilde{p}_i = |c_i|^2$. Then as g is unitary, we have $\sum_{i=1}^d p_i = \sum_{i=1}^d \tilde{p}_i = 1-|a|^2$. Using this notation and the above remark, we can rewrite Equation (4.5) more compactly. Using the duality property of the Krawtchouk polynomials (that is sending g to g^t), or writing out the other orthogonality formula for the representation, we also get an orthogonality relation in the degree indices. The above is summarised in the following theorem.

Theorem 4.3. *For $g \in SU(d+1)$ and $\mathbf{p},\tilde{\mathbf{p}}$ as above, we have*

$$\sum_{\mathbf{k}\in\mathbb{I}_N}\binom{N}{\mathbf{k}}\mathbf{p}^{\mathbf{k}}(1-|\mathbf{p}|)^{N-|\mathbf{k}|}K_{\mathbf{m}}(\mathbf{k};U,N)\overline{K_{\mathbf{n}}(\mathbf{k};U,N)} = \binom{N}{\mathbf{n}}^{-1}\prod_{i=1}^d\left(\frac{1-|\tilde{\mathbf{p}}|}{\tilde{p}_i^{n_i}}\right)^{n_i}\delta_{\mathbf{m},\mathbf{n}}, \tag{4.6a}$$

$$\sum_{\mathbf{k}\in\mathbb{I}_N}\binom{N}{\mathbf{k}}\tilde{\mathbf{p}}^{\mathbf{k}}(1-|\tilde{\mathbf{p}}|)^{N-|\mathbf{k}|}K_{\mathbf{k}}(\mathbf{m};U,N)\overline{K_{\mathbf{k}}(\mathbf{n};U,N)} = \binom{N}{\mathbf{n}}^{-1}\prod_{i=1}^d\left(\frac{1-|\mathbf{p}|}{p_i^{n_i}}\right)^{n_i}\delta_{\mathbf{m},\mathbf{n}}. \tag{4.6b}$$

Remark. In [6], Iliev used a matrix U such that the first row and column has all elements equal to 1, in combination with two diagonal matrices $P, \tilde{P}$ such that $p_0 = \tilde{p}_0$ and the following holds:

$$\frac{1}{p_0} P U \tilde{P} U^t = I_{d+1}.$$

For our setup, the matrices U, P and $\tilde{P}$ obtained as above from a matrix $g \in GL(d+1)$ have the same form. However, for $g \in SU(d+1)$, we have the different although similar equation

$$\frac{1}{p_0} P U \tilde{P} U^\dagger = I_{d+1}.$$

As the generating function (4.2) from Iliev is well defined for all U, we can still use it for our case.

This does have impact on the orthogonality relations, as Iliev found orthogonality relations without a complex conjugate as opposed to ours above. This can be explained by the fact that Iliev used a real-inner product space to derive the orthogonality relations.

The multivariate Krawtchouk polynomials we found use an integer N and a matrix g to create a parameter space $(\mathbf{p}(g), \tilde{\mathbf{p}}(g), U(g), N)$. The so called duality property of the Krawtchouk polynomials, $K_{\mathbf{m}}(\mathbf{n}; U, N) = K_{\mathbf{n}}(\mathbf{m}; U^t, N)$ correspond to the involution

$$(\mathbf{p}(g), \tilde{\mathbf{p}}(g), U(g), N) \longmapsto (\mathbf{p}(g^t), \tilde{\mathbf{p}}(g^t), U(g^t), N) = (\tilde{\mathbf{p}}(g), \mathbf{p}(g), U^t(g), N). \tag{4.7}$$

4.2. **The Lie algebra $\mathfrak{su}(d+1)$ and Krawtchouk polynomials.** For simplicity, we again consider the Lie algebra $\mathfrak{sl}(d+1;\mathbb{C})$ instead (this is isomorphic to the complexification of $\mathfrak{su}(d+1)$). We denote by H the (standard Cartan) subalgebra of $\mathfrak{sl}(d+1;\mathbb{C})$ consisting of the diagonal matrices. As basis for H, we take the matrices

$$H_i = e_{i,i} - \frac{1}{d+1} I_{d+1} \qquad i = 1, \dots, d.$$

For the remainder of this section, fix some $g \in SU(d+1)$ (with elements in the first row and column nonzero). We create another (Cartan) subalgebra by conjugating by g, that is we have the subalgebra $\tilde{H} = \operatorname{span}\{\tilde{H}_1, \dots, \tilde{H}_d\}$ where

$$\tilde{H}_i = \operatorname{Ad}_g(H_i).$$

These two subalgebras complement each other as is stated in the lemma below.

Lemma 4.4. *The subalgebras H and $\tilde{H}$ together generate $\mathfrak{sl}(d+1;\mathbb{C})$.*

Proof. Let $i, j \in \{0, \dots, d\}$, $i \neq j$ be arbitrary. We want to construct $e_{i,j}$. Define

$$H_0 := e_{0,0} - \frac{1}{d+1} I_{d+1}.$$

Note that $H_0 = -\sum_{j=1}^d H_j$, so that H_0 is an element of H. Let $X \in M_{d+1}(\mathbb{C})$ be arbitrary. We have that $[H_k, X] = [e_{k,k}, X]$ for any $k \in \{0, \dots, d\}$. Secondly, by basic calculations, $[e_{i,i}, [e_{j,j}, X]] = -X_{i,j} e_{i,j} - X_{j,i} e_{j,i}$. Using that $[e_{j,j}, e_{j,i}] = e_{j,i}$ and $[e_{j,j}, e_{i,j}] = -e_{i,j}$, we find

$$\begin{aligned} [e_{j,j}, [e_{i,i}, [e_{j,j}, X]]] - [e_{i,i}, [e_{j,j}, X]] &= X_{i,j} e_{i,j} - X_{j,i} e_{j,i} - (-X_{i,j} e_{i,j} - X_{j,i} e_{j,i}) \\ &= 2 X_{i,j} e_{i,j}. \end{aligned}$$

Let $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix} \in SU(d+1)$ be given as usual and set $\tilde{H}_0 = \operatorname{Ad}_g(H_0)$. As $\operatorname{Ad}_g(H_0) = g e_{0,0} g^{-1} - \frac{1}{d+1} I$, its off-diagonal matrix elements are easily computed to be $(\tilde{H}_0)_{k,l} = c_k \overline{c_l}$. As it is assumed that the c_i are nonzero, we know that all off-diagonal elements are nonzero. Hence, we can take $X = \tilde{H}_0$ from which we get

$$e_{i,j} = \frac{[e_{j,j}, [e_{i,i}, [e_{j,j}, X]]] - [e_{i,i}, [e_{j,j}, X]]}{2 c_i \overline{c_j}}.$$

□

Next, we calculate the Lie algebra representation $d\pi^N$ of $\mathfrak{sl}(d+1;\mathbb{C})$ on $\mathbb{C}^N[\mathbf{z}]$:

$$\mathrm{d}\pi^N(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^N(\exp(tX)). \tag{4.8}$$

By doing some basic calculations, we can rewrite this representation in terms of derivations.

Lemma 4.5. *The Lie algebra representation $d\pi^N : \mathfrak{sl}(d+1;\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^N[\mathbf{z}])$ is given by*

$$\begin{aligned} \mathrm{d}\pi^N(e_{i,j}) &= z_i \partial_{z_j}, \\ \mathrm{d}\pi^N(H_k) &= z_k \partial_{z_k} - \frac{1}{d+1}\sum_{j=0}^{d} z_j \partial_{z_j}, \end{aligned} \tag{4.9}$$

for any $0 \leq i \neq j \leq d$ and $1 \leq k \leq d$.

Proof. Let $i \neq j$ and $\mathbf{z} = (z_0, \ldots, z_d) \in \mathbb{C}^{d+1}$. Then we have

$$[\mathrm{d}\pi^N(e_{i,j})p](\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^N(\exp(te_{i,j}))p(\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p((I + te_{i,j})^t\,\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p\left(\mathbf{z} + t \cdot z_i \mathbf{v}_j\right).$$

Now, using the chain rule in accordance with $\mathbf{y}(t) = \mathbf{z} + t \cdot z_i \mathbf{v}_j$, we find

$$[\mathrm{d}\pi^N(e_{i,j})p](\mathbf{z}) = \left[\frac{\partial p}{\partial y_j}(\mathbf{y}(t)) \cdot z_i\right]_{t=0} = [z_i \partial_{z_j} p](x).$$

Likewise, setting $\mathbf{z}(t) = \mathbf{z} + (e^t - 1)z_i \mathbf{v}_i + (e^{-t} - 1)z_j \mathbf{v}_j$, we see

$$\begin{aligned} [\mathrm{d}\pi^N(e_{i,i} - e_{j,j})p](\mathbf{z}) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^N(\exp(t(e_{i,i} - e_{j,j}))p(\mathbf{z}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(\mathbf{z}(t)) \\ &= \left[\frac{\partial p}{\partial z_i}(\mathbf{z}(t)) \cdot e^t z_i - \frac{\partial p}{\partial z_j}(\mathbf{z}(t)) \cdot e^{-t} z_j\right]_{t=0}. \end{aligned}$$

Now setting $t = 0$, we can conclude

$$[\mathrm{d}\pi^N(e_{i,i} - e_{j,j})p](\mathbf{z}) = [(z_i \partial_{z_i} - z_j \partial_{z_j})p](\mathbf{z}).$$

Applying $d\pi^N$ to $H_i = \frac{1}{d+1}\sum_{\substack{j=0 \\ j \neq i}}^{d} e_{i,i} - e_{j,j}$ and using linearity gives the statement. □

From the lemma above, it is clear that $d\pi^N$ preserves the total degree of a polynomial. In particular, $d\pi^N$ sends $\mathbb{C}^N[\mathbf{z}]$ onto itself. Let $\mathbf{v}_i$ be the standard basis vector of $\mathbb{C}^N$ with a 1 as i'th component and all others 0. Using the form of $d\pi^N$ given in the above lemma, we can compute the action on the basis $\{e^N_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{I}_N}$.

Corollary 4.6. *Let $\mathbf{n} \in \mathbb{I}_N$ be arbitrary, we have*

$$\begin{aligned} \mathrm{d}\pi^N(H_k)e^N_{\mathbf{n}} &= \left(n_k - \frac{N}{d+1}\right) e^N_{\mathbf{n}}, \\ \mathrm{d}\pi^N(e_{i,j})e^N_{\mathbf{n}} &= \sqrt{(n_i+1)n_j}\, e^N_{\mathbf{n}+\mathbf{v}_i - vj}, \qquad i \neq j \\ \mathrm{d}\pi^N(e_{0,j})e^N_{\mathbf{n}} &= \sqrt{(N - |\mathbf{n}| + 1)n_j}\, e^N_{\mathbf{n}-\mathbf{v}_j}, \\ \mathrm{d}\pi^N(e_{i,0})e^N_{\mathbf{n}} &= \sqrt{(n_i+1)(N - |\mathbf{n}|)}\, e^N_{\mathbf{n}+\mathbf{v}_i}, \end{aligned}$$

where $k = 1, \ldots, d$ and $i, j = 1, \ldots, d$. Here we have set $e^N_{\mathbf{n}} = 0$ if some n_i equals -1 or $N + 1$.

Proof. Follows directly by applying Lemma 4.5 and using that $\binom{N}{\mathbf{n}+\mathbf{v}_i - \mathbf{v}_j} = \binom{N}{\mathbf{n}}\frac{n_j}{n_i+1}$, $\binom{N}{\mathbf{n}-\mathbf{v}_j} = \binom{N}{\mathbf{n}}\frac{n_j}{N-|\mathbf{n}|+1}$ and $\binom{N}{\mathbf{n}+\mathbf{v}_i} = \binom{N}{\mathbf{n}}\frac{N-|\mathbf{n}|}{n_i+1}$. □

Using the definition of our Lie algebra representation, we can easily construct a basis on which $\tilde{H}$ acts diagonally. Namely, similarly to what we did in Chapter 2, we take

$$\tilde{e}^N_{\mathbf{n}} = \pi^N(g^t)e^N_{\mathbf{n}}.$$

Then the following is evident.

Corollary 4.7. *Let $\mathbf{n} \in \mathbb{I}_N$ be arbitrary, we have*

$$\begin{aligned}
\mathrm{d}\pi^N(\tilde{H}_k)\tilde{e}^N_{\mathbf{n}} &= \left(n_k - \frac{N}{d+1}\right)\tilde{e}^N_{\mathbf{n}},\\
\mathrm{d}\pi^N(\tilde{e}_{i,j})\tilde{e}^N_{\mathbf{n}} &= \sqrt{(n_i+1)n_j}\,\tilde{e}^N_{\mathbf{n}+\mathbf{v}_i-\mathbf{v}_j},\\
\mathrm{d}\pi^N(\tilde{e}_{0,j})\tilde{e}^N_{\mathbf{n}} &= \sqrt{(N-|\mathbf{n}|+1)n_j}\,\tilde{e}^N_{\mathbf{n}-\mathbf{v}_j},\\
\mathrm{d}\pi^N(\tilde{e}_{i,0})\tilde{e}^N_{\mathbf{n}} &= \sqrt{(n_i+1)(N-|\mathbf{n}|)}\,\tilde{e}^N_{\mathbf{n}+\mathbf{v}_i},
\end{aligned}$$

where $k = 1, \dots, d$ and $i, j = 1, \dots, d$.

Proof. By Equation (2.13), we have $\mathrm{d}\pi^N(gXg^{-1}) = \pi^N(g) \circ \mathrm{d}\pi^N(X) \circ \pi^N(g^{-1})$ for any $X \in \mathfrak{sl}_{d+1}(\mathbb{C})$. By the definition of $\tilde{e}^N_{\mathbf{n}}$, the results follow from Corollary 4.6. □

We can use these actions to derive recurrence and difference equations for the multivariate Krawtchouk polynomials. To do so, note again that by Lemma 2.10, we have that

$$\langle \mathrm{d}\pi^N(X)u, v\rangle = \langle u, \mathrm{d}\pi^N(X^\dagger)v\rangle, \qquad u, v \in \mathbb{C}^N[\mathbf{z}], \tag{4.10}$$

for any $X \in \mathfrak{sl}(d+1;\mathbb{C})$

Theorem 4.8. *Let $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D\end{pmatrix} \in SU(d+1)$, where $a \neq 0$ as well as $\mathbf{b}$ and $\mathbf{c}$ having all entries nonzero. Write $p_i = |b_i|^2$, $\tilde{p}_i = |c_i|^2$ and $U_{i,j} = \frac{D_{j,i}a}{b_i c_j}$ for $i, j \in \{1, \dots, d\}$. Lastly, set $p_0 = |a|^2 = 1 - \sum_{i=1}^d p_i$ and write $K_{\mathbf{m}}(\mathbf{n}) := K_{\mathbf{m}}(\mathbf{n}; U, N)$. For each $k = 1, \dots, d$, we have the following recurrence relation:*

$$\begin{aligned}
-n_k K_{\mathbf{m}}(\mathbf{n}) =& \frac{p_k}{p_0}\sum_{i=1}^d \tilde{p}_i \overline{U_{k,i}}(N-|\mathbf{m}|)K_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n})\\
&- \left[p_k(N-|\mathbf{m}|) + \frac{p_k}{p_0}\sum_{l=1}^d \tilde{p}_l |U_{k,l}|^2 m_l\right]K_{\mathbf{m}}(\mathbf{n})\\
&+ p_k \sum_{j=1}^d U_{k,j} m_j K_{\mathbf{m}-\mathbf{v}_j}(\mathbf{n})\\
&+ \frac{p_k}{p_0}\sum_{\substack{i,j=0\\ i\neq j}}^d \tilde{p}_i \overline{U_{k,i}} U_{k,j} m_l K_{\mathbf{m}-\mathbf{v}_i+\mathbf{v}_j}(\mathbf{n}).
\end{aligned}\tag{4.11a}$$

Also for $k = 1, \dots, d$, we have the difference equations

$$\begin{aligned}
-m_k K_{\mathbf{m}}(\mathbf{n}) =& \frac{\tilde{p}_k}{p_0}\sum_{i=1}^d p_i U_{i,k}(N-|\mathbf{n}|)K_{\mathbf{m}}(\mathbf{n}+\mathbf{v}_i)\\
&- \left[\tilde{p}_k(N-|\mathbf{n}|) + \frac{\tilde{p}_k}{p_0}\sum_{l=1}^d p_l |U_{l,k}|^2 n_l\right]K_{\mathbf{m}}(\mathbf{n})\\
&+ \tilde{p}_k \sum_{j=1}^d \overline{U_{j,k}} n_j K_{\mathbf{m}}(\mathbf{n}-\mathbf{v}_j)\\
&+ \frac{\tilde{p}_k}{p_0}\sum_{\substack{i,j=0\\ i\neq j}}^d p_i U_{i,k}\overline{U_{j,k}} n_l K_{\mathbf{m}-\mathbf{v}_i+\mathbf{v}_j}(\mathbf{n}).
\end{aligned}\tag{4.11b}$$

Proof. Let $k \in \{1, \dots, d\}$ be arbitrary. By Equation (4.10) above applied to $H_k \in \mathfrak{sl}(d+1,\mathbb{C})$, we have

$$\langle \mathrm{d}\pi^N(\tilde{H}_k)\tilde{e}^N_{\mathbf{n}}, e^N_{\mathbf{m}}\rangle = \langle \tilde{e}^N_{\mathbf{n}}, \mathrm{d}\pi^N(\tilde{H}_k)e^N_{\mathbf{m}}\rangle. \tag{*}$$

Set

$$B(\mathbf{m},\mathbf{n}) = \binom{N}{\mathbf{m}}^{1/2}\binom{N}{\mathbf{n}}^{1/2} a^{N-|\mathbf{n}|-|\mathbf{m}|}\mathbf{b}^{\mathbf{n}}\mathbf{c}^{\mathbf{m}},$$

so that by Theorem 4.1 we have that $\langle \tilde{e}^N_{\mathbf{n}}, e^N_{\mathbf{m}}\rangle = B(\mathbf{m},\mathbf{n})K_{\mathbf{m}}(\mathbf{n})$. Using the action of $\tilde{H}_k$ on $\tilde{e}^N_{\mathbf{n}}$ as by Corollary 4.7 the left-hand side becomes $\left(n_k - \frac{N}{d+1}\right)B(\mathbf{m},\mathbf{n})K_{\mathbf{m}}(\mathbf{n})$.

For the right-hand side, we first express $\tilde{H}$ in terms of the basis matrices of $\mathfrak{sl}(d+1;\mathbb{C})$; $\{H_k, e_{i,j} \,|\, k \in \{1,\dots d\}, i,j \in \{0,\dots,d\}, i \neq j\}$. To do this, we first write out $\tilde{H}_k$ as follows:

$$\tilde{H}_k = \begin{bmatrix} b_k \\ D_{1,k} \\ \vdots \\ D_{d,k} \end{bmatrix} \otimes \begin{bmatrix} \overline{b_k} \\ \overline{D_{1,k}} \\ \vdots \\ \overline{D_{d,k}} \end{bmatrix} - \frac{1}{d+1} I_{d+1} = \begin{pmatrix} |b_k|^2 & b_k\overline{D_{1,k}} & \dots & b_k\overline{D_{d,k}} \\ \overline{b_k}D_{1,k} & |D_{1,k}|^2 & \dots & D_{1,k}\overline{D_{d,k}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b_k}D_{d,k} & D_{d,k}\overline{D_{1,k}} & \dots & |D_{d,k}|^2 \end{pmatrix} - \frac{1}{d+1} I_{d+1},$$

where $\otimes$ is the outer product so that $(\mathbf{u}\otimes\mathbf{w})_{i,j} = u_i w_j$ for two vectors $\mathbf{u},\mathbf{w} \in \mathbb{C}^d$. We want to express the diagonal of the above matrix in terms of H_l. Suppose the diagonal is equal to the sum $\sum_{l=1}^d x_l H_l, x_l \in \mathbb{C}$. We solve the system

$$\begin{aligned} -\frac{1}{d+1}\sum_{l=1}^d x_l &= |b_k|^2 - \frac{1}{d+1}, \\ x_1 - \frac{1}{d+1}\sum_{l=1}^d x_l &= |D_{1,i}|^2 - \frac{1}{d+1}, \\ &\vdots \\ x_d - \frac{1}{d+1}\sum_{l=1}^d x_l &= |D_{d,i}|^2 - \frac{1}{d+1}. \end{aligned}$$

Subtracting the first equation from the rest, we see that the solution must be given by $x_l = |D_{k,l}|^2 - |b_k|^2$ if the system is consistent. As for $g \in SU(d+1)$ we have $g^\dagger g = I$, we know that $\sum_{l=1}^d |D_{k,l}|^2 + |b_k|^2 = 1$ so that the first equation holds for the choice of the x_l and the system is consistent. We conclude that we can write $\tilde{H}_k$ as follows:

$$\begin{aligned} \tilde{H}_k = &\sum_{l=1}^d (|D_{l,k}|^2 - |b_k|^2) H_l + \sum_{\substack{i,j=1 \\ i\neq j}}^d D_{i,k}\overline{D_{j,k}} e_{i,j} \\ &+ b_k \sum_{j=1}^d \overline{D_{j,k}} e_{0,j} + \overline{b_k}\sum_{i=0}^d D_{i,k} e_{i,0}. \end{aligned}$$

Now the right-hand side of (*) can be easily expanded using Corollary 4.6:

$$\begin{aligned} &\sum_{l=1}^d (|D_{l,k}|^2 - |b_k|^2)\left(m_k - \frac{N}{d+1}\right) B(\mathbf{m},\mathbf{n}) K_{\mathbf{m}}(\mathbf{n}) \\ &+ \sum_{\substack{i,j=0 \\ i\neq j}}^d D_{i,k}\overline{D_{j,k}} \sqrt{(n_i+1)n_j}\, B(\mathbf{m}+\mathbf{v}_i-\mathbf{v}_j,\mathbf{n}) K_{\mathbf{m}+\mathbf{v}_i-\mathbf{v}_j}(\mathbf{n}) \\ &+ b_k \sum_{j=1}^d \overline{D_{j,k}} \sqrt{(N-|\mathbf{n}|+1)n_j}\, B(\mathbf{m}-\mathbf{v}_j,\mathbf{n}) K_{\mathbf{m}-\mathbf{v}_j}(\mathbf{n}) \\ &+ \overline{b_k} \sum_{i=0}^d D_{i,k} \sqrt{(n_i+1)(N-|\mathbf{n}|)}\, B(\mathbf{m}+\mathbf{v}_i,\mathbf{n}) K_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}). \end{aligned}$$

With a bit of calculation, one can express the constants B as follows:

$$\begin{aligned} B(\mathbf{m}-\mathbf{v}_j,\mathbf{n}) &= \frac{a}{c_j}\sqrt{\frac{m_j}{N-|\mathbf{m}|+1}} B(\mathbf{m},\mathbf{n}), \\ B(\mathbf{m}+\mathbf{v}_i,\mathbf{n}) &= \frac{c_i}{a}\sqrt{\frac{N-|\mathbf{m}|}{m_i+1}} B(\mathbf{m},\mathbf{n}), \\ B(\mathbf{m}+\mathbf{v}_i-\mathbf{v}_j,\mathbf{n}) &= \frac{c_i}{c_j}\sqrt{\frac{m_j}{m_i+1}}. \end{aligned}$$

Combining all of the above and dividing by $B(\mathbf{m},\mathbf{n})$, (*) becomes

$$\begin{aligned}\left(n_k - \frac{N}{d+1}\right) K_{\mathbf{m}}(\mathbf{n}) = &\sum_{l=1}^{d}(|D_{l,k}|^2 - |b_k|^2)\left(m_l - \frac{N}{d+1}\right) K_{\mathbf{m}}(\mathbf{n})\\ &+ \sum_{\substack{i,j=0\\ i\neq j}}^{d} \frac{c_i \overline{D_{i,k}} D_{j,k}}{c_j} m_j K_{\mathbf{m}+\mathbf{v}_i-\mathbf{v}_j}(\mathbf{n})\\ &+ \overline{b_k}\sum_{j=1}^{d} \frac{a D_{j,k}}{c_j} m_j K_{\mathbf{m}-\mathbf{v}_j}(\mathbf{n})\\ &+ b_k \sum_{i=0}^{d} \frac{c_i \overline{D_{i,k}}}{a}(N-|\mathbf{m}|)K_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}).\end{aligned}$$

Now, first add $\frac{N}{d+1}K_{\mathbf{m}}(\mathbf{n})$ to both sides and then multiply by -1. Using that for any unitary matrix each column is an orthonormal vector of $\mathbb{C}^{d+1}$, we must have that $|b_k|^2 + \sum_{l=1}^{d} |D_{l,k}|^2 = 1$. Using this to rewrite the equation, we find

$$\begin{aligned}-n_k K_{\mathbf{m}}(\mathbf{n}) = &-\left[|b_k|^2(N-|\mathbf{m}|) + \sum_{l=1}^{d} |D_{l,k}|^2 m_l\right] K_{\mathbf{m}}(\mathbf{n})\\ &+ \sum_{\substack{i,j=0\\ i\neq j}}^{d} \frac{c_i \overline{D_{i,k}} D_{j,k}}{c_j} m_j K_{\mathbf{m}+\mathbf{v}_i-\mathbf{v}_j}(\mathbf{n})\\ &+ \overline{b_k}\sum_{j=1}^{d} \frac{a D_{j,k}}{c_j} m_j K_{\mathbf{m}-\mathbf{v}_j}(\mathbf{n})\\ &+ b_k \sum_{i=0}^{d} \frac{c_i \overline{D_{i,k}}}{a}(N-|\mathbf{m}|)K_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}).\end{aligned}$$

Equation (4.11a) now follows by the definitions of the $p_i, \tilde{p}_i$ and $U_{i,j}$. The difference equations (4.11b) can be shown similarly by expressing H_k in terms of the basis $\left\{\tilde{H}_k, \tilde{e}_{i,j} \mid k \in \{1,\dots d\}, i,j \in \{0,\dots,d\}, i \neq j\right\}$ of $\mathfrak{sl}(d+1;\mathbb{C})$, or by simply using the duality of the multivariate Krawtchouk polynomials (4.7) (sending g to g^t). □

5. Meixner polynomials in multiple variables

In Section 3, we have seen the construction of the univariate Meixner polynomial form the Lie group $SU(1,1)$ and established the orthogonality, recurrence and difference relation. In [5], Iliev used a space of matrices as parameters to construct multivariate Meixner polynomials through the use of a generating function. However, establishing the orthogonality, recurrence relations and difference equations for the Meixner polynomials was done directly, without establishing a connection to Lie theory. The aim of this section is to acquire such a connection to the Lie group $SU(1,d)$, $d>1$.

First, we will look at properties of the matrix group $SU(1,d)$ and define an action on the Bergman space on the d-dimensional unit ball. Next, we use this connection to define multivariate Meixner polynomials through the resulting matrix coefficients of the representation. Proving that the representation is unitary, we will moreover show that the multivariate Meixner polynomials are orthogonal with respect to a multivariate negative binomial distribution. Lastly, we look into the Lie algebra representation and use it to find recurrence and difference equations. The methods used will be similar to those of Section 3, although the generalisation of these methods is more demanding.

5.1. **The Lie group $SU(1,d)$ and Bergman spaces.** $SU(1,d)$ is the group of matrices of determinant one preserving the hermitian form associated with the matrix $J=\operatorname{diag}(1,-1,-1,\dots,-1)$. That is, $g\in SU(1,d)$ if and only if the following equation holds:

$$g^\dagger J g = J. \tag{5.1}$$

Throughout this section, we will write $g\in SU(1,d)$ in the form $g=\begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D\end{pmatrix}$ where $a\in\mathbb{C}$, $\mathbf{b},\mathbf{c}\in\mathbb{C}^d$ and D is a $d\times d$ complex matrix. From the defining equation, Equation (5.1), the inverse of an $SU(1,d)$ matrix can be written as follows:

$$g^{-1}=Jg^\dagger J=\begin{pmatrix} \overline{a} & -\mathbf{c}^\dagger \\ -\overline{\mathbf{b}} & D^\dagger\end{pmatrix} \tag{5.2}$$

where g is written as above.

Just as for the univariate Meixner polynomials, we will act on holomorphic functions on the unit ball $\mathbb{B}_d:=\{\mathbf{z}\in\mathbb{C}^d\,|\,|\mathbf{z}|<1\}$. More precisely, we will act on the Bergman spaces $\mathcal{A}_\alpha$ $\alpha>-1$. The space $\mathcal{A}_\alpha$ consists of the holomorphic functions in $L^2(\mathbb{B}_d,\mathrm{d}v_\alpha)$, where the weighted Lebesgue measure is given by

$$\mathrm{d}v_\alpha=c_\alpha(1-|\mathbf{z}|^2)^\alpha\,\mathrm{d}v, \tag{5.3}$$

where $\mathrm{d}v$ is the standard volume measure on $\mathbb{B}_d$ and c_{alpha} is so that $v_\alpha(\mathbb{B}_d)=1$. A direct calculation shows that

$$c_\alpha=\frac{1}{B(\alpha+1,d)}=\frac{(\alpha+1)_d}{d!}. \tag{5.4}$$

Defining an inner product on $\mathcal{A}_\alpha$ by

$$(f,g)_\alpha=\int_{\mathbb{B}_d} f(\mathbf{z})\overline{g(\mathbf{z})}\,\mathrm{d}v_\alpha,\qquad f,g\in\mathcal{A}_\alpha, \tag{5.5}$$

we can turn $\mathcal{A}_\alpha$ into a Hilbert space (called the Bergman space).

Recall that a holomorphic function f is (locally) equal to its Taylor series. We will show that the monomials $\mathbf{z}^\mathbf{n}$ form an orthogonal basis of $\mathcal{A}_\alpha$. First, notice that the measure is invariant under unitary transformations (see also [11]), so that in particular for the rotations $U\mathbf{z}=\left[z_1e^{i\phi_1},\dots,z_de^{i\phi_d}\right]$ we find that

$$(\mathbf{z}^\mathbf{n},\mathbf{z}^\mathbf{m})_\alpha=((U\mathbf{z})^\mathbf{n},(U\mathbf{z})^\mathbf{m})_\alpha=\left(\prod_{i=1}^d e^{i\phi_i(\mathbf{n}_i-\mathbf{m}_i)}\right)(\mathbf{z}^\mathbf{n},\mathbf{z}^\mathbf{m})_\alpha.$$

Thus, surely if $\mathbf{m}\neq\mathbf{n}$ the inner product will be zero. When $\mathbf{m}=\mathbf{n}$, a simple calculation using polar coordinates shows that

$$||\mathbf{z}^\mathbf{m}||_\alpha^2=\frac{\mathbf{m}!}{(\alpha+d+1)_{|\mathbf{m}|}}.$$

From the fact that the polynomials are dense in $\mathcal{A}_\alpha$ (see Proposition 2.6 from [11]), we conclude that an orthonormal basis of $\mathcal{A}_\alpha$ is given by:

$$e^\alpha_\mathbf{m}(\mathbf{z})=\sqrt{\frac{(\alpha+n+1)_{|\mathbf{m}|}}{\mathbf{m}!}}\mathbf{z}^\mathbf{m}. \tag{5.6}$$

In what follows, we write $\beta=\alpha+d+2$.

Next, we want to construct a representation of $SU(1,d)$ on $\mathcal{A}_\alpha$. For this, we first show an analog of Lemma 3.1.

Lemma 5.1. *Let* $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix} \in SU(1,d)$, *then the fractional linear transformation* $\mathbf{z} \mapsto \frac{\mathbf{b}+D^t\mathbf{z}}{a+\mathbf{c}\cdot\mathbf{z}}$ *maps* $\mathbb{B}_d$ *onto itself.*

Proof. Define on $\mathbb{C}^{d+1}$ the indefinite hermitian form

$$\langle (z_0, \mathbf{z}), (w_0, \mathbf{w}) \rangle = z_0\overline{w_0} - z_1\overline{w_1} - \cdots - z_d\overline{w_d},$$

where $\mathbf{z} = (z_1, \ldots, z_d), \mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{C}^d$. We look at the set $\widetilde{\mathbb{B}_d} := \{(z_0, \mathbf{z}) \in \mathbb{C}^{d+1} \mid \langle (z_0, \mathbf{z}), (z_0, \mathbf{z}) \rangle > 0\}$. By definition of $U(1,d)$, it preserves the above hermitian form and hence also the space $\widetilde{\mathbb{B}_d}$.

As the condition $|z_0|^2 - |\mathbf{z}|^2 > 0$ implies z_0 is nonzero, we can divide by $|z_0|^2$ which tells us $|\frac{\mathbf{z}}{z_0}|^2 < 1$. But this means we can map $\widetilde{\mathbb{B}_d}$ surjectively onto $\mathbb{B}_d$ by the map

$$(z_0, \mathbf{z}) \mapsto \frac{\mathbf{z}}{z_0}. \tag{*}$$

Now let $g \in SU(1,d)$ be as given. Using (*) on $g^t(z_0, z_1)$, we find the map $(z_0, z_1) \mapsto \frac{bz_0+D^t\mathbf{z}}{az_0+\mathbf{c}\cdot\mathbf{z}}$ to be surjective as well. Writing $\hat{z} = \frac{\mathbf{z}}{z_0}$ and using that the map (*) is surjective, we conclude that the map $\mathbf{z} \mapsto \frac{\mathbf{b}+D^t\mathbf{z}}{a+\mathbf{c}\cdot\mathbf{z}}$ is well defined and maps $\mathbb{B}_d$ onto itself. ☐

In particular, it follows that the function $\mathbf{z} \mapsto \frac{1}{a+\mathbf{c}\cdot\mathbf{z}}$ is holomorphic everywhere on $\mathbb{B}_d$. As compositions and products of holomorphic functions are holomorphic, we can define the map $\pi^\beta : SU(1,d) \longrightarrow GL(H(\mathbb{B}_d))$ by [2],

$$\pi^\beta \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix} f(\mathbf{z}) = (a + \mathbf{c}\cdot\mathbf{z})^{-\beta} f\left(\frac{\mathbf{b}+D^t\mathbf{z}}{a+\mathbf{c}\cdot\mathbf{z}}\right), \tag{5.7}$$

where $H(\mathbb{B}_d)$ is the space of holomorphic functions on $\mathbb{B}_d$. As in Section 3, the values for β have to be restricted in order to produce a representation. It will follow that if β is integral and $\beta = \alpha + d + 1$, it indeed defines a representation.

Denote by $\mathcal{A}_\alpha^\infty$ the space of functions in $\mathcal{A}_\alpha$ on which $\pi^\beta(g)$ is smooth for all $g \in SU(1,d)$. In [2], Lemma 2.10, it is shown that the (finite) polynomials are contained in $\mathcal{A}_\alpha^\infty$. Furthermore, $\mathcal{A}_\alpha^\infty$ is invariant under $SU(1,d)$ (see section 2.1.5 of [7]). Next, we will show that this map defines a representation of $SU(1,d)$ on $\mathcal{A}_\alpha^\infty$ for integral α (hence, β is integral).

Lemma 5.2. *Let* $\alpha \in \mathbb{N}$ *and write* $\beta = \alpha + d + 1$. *The map* π^β *can be restricted to* $\mathcal{A}_\alpha$. *Moreover, it defines a representation of* $SU(1,d)$ *on* $\mathcal{A}_\alpha^\infty$.

Proof. Write $\beta = \alpha + d + 1$ as above. By the preceding arguments, we know that $\pi^\beta(g)$ maps the space of holomorphic functions on $\mathbb{B}_d$ into itself. As $\pi^\beta(g)$ is clearly linear, the well definedness of π^β on $\mathcal{A}_\alpha$ then follows from the fact that $\pi^\beta(g)$ defines a unitary transformation (see Theorem 5.4).

Next, we will prove that π^β is a homomorphism of $SU(1,d)$. Let $g, h \in SU(1,d)$, $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix}, h = \begin{pmatrix} u & \mathbf{v}^t \\ \mathbf{w} & X \end{pmatrix}$. Writing out both $[\pi^\alpha(gh)f](z)$ and $\big[\pi^\alpha(g)[\pi^\alpha(h)f]\big](z)$ for arbitrary $f \in \mathcal{A}_\alpha$ and comparing terms, one finds that we need to have $\tilde{a}^{-\beta}\tilde{u}^{-\beta} = (\tilde{a}\tilde{u})^{-\beta}$, where $\tilde{a} = a + \mathbf{c}\cdot\left(\frac{\mathbf{v}+\mathbf{X^t z}}{\tilde{u}}\right)$ and $\tilde{u} = u + \mathbf{w}\cdot\mathbf{z}$. As α (hence β) is assumed to be integral, this equality trivially holds. Hence, the map π^β is a homomorphism of $SU(1,d)$ sending $\mathcal{A}_\alpha$ into $\mathcal{A}_\alpha$. As $\pi^\beta(g)$ acts smooth on $\mathcal{A}_\alpha^\infty$ for each g and it maps $\mathcal{A}_\alpha^\infty$ into itself, it follows that it defines a representation here. ☐

In all that follows, α will be assumed to be a non-negative integer. In the next section, we will use this representation to define multivariate Meixner polynomials.

5.2. **Multivariate Meixner polynomials.** In the preceding subsection, we found a representation of $SU(1,d)$ in the space $\mathcal{A}_\alpha^\infty$. In particular for the basis vectors, $\pi^\beta(g)e_\mathbf{m}(\mathbf{z})$ is holomorphic and hence must equal its Taylor series. As the Taylor series is uniquely determined wherever it converges, we can use it to define the matrix elements of the representation:

$$\pi^\beta(g)e_\mathbf{n}(\mathbf{z}) = \sum_{|\mathbf{m}|=0}^{\infty} \pi^\beta_{\mathbf{m},\mathbf{n}}(g)e_\mathbf{m}(\mathbf{z}). \tag{5.8}$$

Now if $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix}$, with b_i, c_i nonzero $i = 1, \dots, d$, we can write out the left-hand side of the above equation as follows:

$$\begin{aligned}\pi^\beta(g)\mathbf{z}^\mathbf{n} &= \left(a + \sum_{j=0}^d c_j z_j\right)^{-\beta} \prod_{i=1}^d \left(\frac{b_i + \sum_{j=0}^d D_{j,i} z_j}{a + \sum_{j=0}^d c_j z_j}\right)^{n_i} \\ &= a^{-\beta-|\mathbf{n}|}\mathbf{b}^\mathbf{n} \left(1 - \sum_{i=1}^d \frac{-c_i}{a} z_i\right)^{-\beta-|\mathbf{n}|} \prod_{j=1}^d \left(1 - \sum_{i=0}^d \frac{a D_{i,j}}{b_j c_i}\left(\frac{-c_i}{a} z_i\right)\right)^{n_j}.\end{aligned}$$

Now if we set $x_i = \frac{-c_i}{a} z_i$ and define the matrix U by $U_{i,j} = \frac{aD_{j,i}}{b_i c_j}$ for $i, j > 0$ and $U_{0,j} = U_{i,0} = 1$, we can express it in terms of the generating function for Meixner polynomials found in [5];

$$\pi^\beta(g)\mathbf{z}^\mathbf{n} = a^{-\beta-|\mathbf{n}|}\mathbf{b}^\mathbf{n} \left(1 - \sum_{j=1}^d x_j\right)^{-\beta-|\mathbf{n}|} \prod_{i=1}^d \left(1 - \sum_{j=1}^d U_{i,j} x_j\right)^{n_i}.$$

Using the definition of the Meixner polynomials of Iliev (see [5]), and undoing the substitution for $\mathbf{z}$, one finds the expression

$$\pi^\beta(g)\mathbf{z}^\mathbf{n} = \sum_{|\mathbf{m}|=0}^\infty \frac{(\beta)_{|\mathbf{m}|}}{\mathbf{m}!} a^{-\beta-|\mathbf{m}|-|\mathbf{n}|}\mathbf{b}^\mathbf{n}(-\mathbf{c})^\mathbf{m} M_\mathbf{m}(\mathbf{n}; U, \beta)\mathbf{z}^\mathbf{n},$$

where $M_\mathbf{n}(\mathbf{m}; U, \beta)$ is the Meixner polynomial with parameter matrix U determined by g. Iliev [5] gave the following formula in terms of the Gelfand hypergeometric series,

$$M_\mathbf{m}(\mathbf{n}; U, \beta) = \sum_{A=(a_{i,j}) \in M_d} \frac{\prod_{j=1}^d (-m_j)_{\sum_{i=1}^d a_{i,j}} \prod_{i=1}^d (-n_i)_{\sum_{j=1}^d a_{i,j}}}{(\beta)_{\sum_{i,j=1}^d a_{i,j}}} \prod_{i,j=1}^d \frac{(1 - U_{i,j})^{a_{i,j}}}{a_{i,j}!}, \tag{5.9}$$

where $m_1, \dots, m_d$ are the degree indices, $n_1, \dots, n_d$ are the variables and M_d is the set of all $d \times d$ matrices with non-negative integer entries. Note that the sum is finite as for $\sum_{i=1}^d a_{i,j} > n_i$, $(-n_i)_{\sum_{j=1}^d a_{i,j}} = 0$. With the above discussion, we have shown the following theorem:

Theorem 5.3. *Let $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix} \in SU(1, d)$ with b_i, c_i nonzero, it holds*

$$\pi^\beta(g) e_\mathbf{n}^\alpha(\mathbf{z}) = \sum_{\mathbf{k} \in \mathbb{N}_0^d} \sqrt{\frac{(\beta)_{|\mathbf{m}|}}{\mathbf{m}!}} \sqrt{\frac{(\beta)_{|\mathbf{n}|}}{\mathbf{n}!}} a^{-\beta-|\mathbf{m}|-|\mathbf{n}|}\mathbf{b}^\mathbf{n}(-\mathbf{c})^\mathbf{m} M_\mathbf{m}(\mathbf{n}; U, \beta) e_\mathbf{m}^\alpha(\mathbf{z}), \tag{5.10}$$

where $U_{i,j} = \frac{aD_{j,i}}{b_i c_j}$ and $U_{0,j} = U_{i,0} = 1$ and the Meixner polynomials are given by Equation (5.9). Hence, the matrix elements of the representation $\pi^\beta(g)$ with respect to the basis $\{e_\mathbf{n}^\alpha\}_{\mathbf{n} \in \mathbb{N}_0^d}$ are given by

$$\pi_{\mathbf{m},\mathbf{n}}^\beta(g) = \sqrt{\frac{(\beta)_{|\mathbf{m}|}}{\mathbf{m}!}} \sqrt{\frac{(\beta)_{|\mathbf{n}|}}{\mathbf{n}!}} a^{-\beta-|\mathbf{m}|-|\mathbf{n}|}\mathbf{b}^\mathbf{n}(-\mathbf{c})^\mathbf{m} M_\mathbf{m}(\mathbf{n}; U, \beta). \tag{5.11}$$

Next, we prove the unitarity of the representation. In the process we also want to show that if the map π^β, as defined in Equation (5.7), can be defined as a unitary representation on a larger Lie group, then this group must at least be contained in $U(1, d)$.

Let $g \in GL(d+1; \mathbb{C})$ be written as usual and suppose $\pi^\beta(g)$ is well defined on $H(\mathbb{B}_d)$, so that in particular the matrix elements given in Theorem 5.3 hold. Write $J = \mathrm{diag}(1, -1, \dots, -1)$ as in the definition of $SU(1, d)$, then $Jg^\dagger J = \begin{pmatrix} \overline{a} & -\mathbf{c}^\dagger \\ -\overline{\mathbf{b}} & D^\dagger. \end{pmatrix}$. Comparing the matrix elements of $Jg^\dagger J$ to those of g using the hypergeometric representation of the Meixner polynomials (5.9), it becomes clear that $\pi_{\mathbf{m},\mathbf{n}}^\beta(g) = \overline{\pi_{\mathbf{n},\mathbf{m}}^\beta(Jg^\dagger J)}$.

Now suppose furthermore that $\pi^\beta(g)$ acts unitarily on $\mathcal{A}_\alpha$. then it must hold that $\pi_{\mathbf{m},\mathbf{n}}^\beta(g) = \overline{\pi_{\mathbf{n},\mathbf{m}}^\beta(g^{-1})}$. By the above, we find that it must hold that $\pi^\beta(Jg^\dagger Jg)$ is the identity on $\mathcal{A}_\alpha$, or equivalently $Jg^\dagger Jg \in \mathrm{Ker}\,\pi^\beta$. Looking at the action of π^β on $e_\mathbf{0}$ and $e_\mathbf{n}$ for $|n| = 1$, it easily follows that $\mathrm{Ker}(\pi^\beta) = \{\gamma I \mid \gamma^\beta = 1\}$. As the diagonal of $Jg^\dagger Jg$ is certainly real, by verifying that $Jg^\dagger Jg = -I$ can never be true, it follows that $Jg^\dagger Jg = I$. This is noting else than having $g \in U(1, d)$.

By the above, it can be concluded that $U(1,d)$ is the maximal subgroup of $GL(2;\mathbb{C})$ on which π^β can be unitary. Now restrict to matrices of unit determinant. As the span of $\{e^\alpha_{\mathbf{n}}(\mathbf{z})\}_{n\in\mathbb{N}_0}$ is dense in $\mathcal{A}^\infty_\alpha$ and $\pi^\beta(g)$ is smooth on this basis, by taking limits one can show the following theorem.

Theorem 5.4. *For an integer $\beta \geq d+1$, the representation π^β is unitary on $SU(1,d)$.*

By the above discussion and as $\pi^\beta_{\mathbf{m},\mathbf{n}}(\overline{g}) = \overline{\pi^\beta_{\mathbf{m},\mathbf{n}}(g)}$, we also find $\pi^\beta_{\mathbf{m},\mathbf{n}}(g) = \pi^\beta_{\mathbf{n},\mathbf{m}}(Jg^tJ)$. This corresponds to the a duality between the variables and the degree indices of the Meixner polynomials in the sense that

$$M_{\mathbf{m}}(\mathbf{n};U,\beta) = M_{\mathbf{n}}(\mathbf{m};U^t,\beta). \tag{5.12}$$

This will be further refined as the involution in Equation (5.14).

As a first application of the unitarity, we prove orthogonality relations for the multivariate Meixner polynomials. For this, define the vectors $C = [C_1,\dots,C_d]$, $\tilde{C} = [\tilde{C}_1,\dots,\tilde{C}_d]$ by $C_i = \frac{|b_i|^2}{|a|^2}$ and $\tilde{C}_i = \frac{|c_i|^2}{|a|^2}$. Remark that by this definition and by the equality's $|a|^2 - |\mathbf{b}|^2 = |a|^2 - |\mathbf{c}|^2 = 1$ (which hold as $g \in SU(1,d)$), it holds that

$$0 < \sum_{i=1}^d C_i = \sum_{i=1}^d \tilde{C}_i < 1.$$

Lastly, we will set $C_0 = 1 - \sum_{i=1}^d C_i$. Using this notation, we can show the following orthogonality relations.

Theorem 5.5. *For an integer $\beta \geq d+1$ and $C,\tilde{C} \in \mathbb{C}^d$ as above, we have*

$$\sum_{\mathbf{k}\in\mathbb{N}_0^d} \frac{(\beta)_{|\mathbf{k}|}}{\mathbf{k}!} C^{\mathbf{k}} M_{\mathbf{m}}(\mathbf{k};U,\beta)\overline{M_{\mathbf{n}}(\mathbf{k};U,\beta)} = \delta_{\mathbf{m},\mathbf{n}} \frac{\mathbf{m}!\tilde{C}^{-\mathbf{m}}}{(\beta)_{|\mathbf{m}|}C_0^\beta}, \tag{5.13a}$$

$$\sum_{\mathbf{k}\in\mathbb{N}_0^d} \frac{(\beta)_{|\mathbf{k}|}}{\mathbf{k}!} \tilde{C}^{\mathbf{k}} M_{\mathbf{k}}(\mathbf{m};U,\beta)\overline{M_{\mathbf{k}}(\mathbf{n};U,\beta)} = \delta_{\mathbf{m},\mathbf{n}} \frac{\mathbf{m}!C^{-\mathbf{m}}}{(\beta)_{|\mathbf{m}|}C_0^\beta}. \tag{5.13b}$$

Proof. Let $g \in SU(1,d)$ be arbitrary and written as usual. Also let $U,C,\tilde{C}$ be defined as usual. Using the unitarity of the representation, we have

$$\begin{aligned}\delta_{\mathbf{m},\mathbf{n}} &= \langle \pi^\beta(g)e^\beta_{\mathbf{m}}, \pi^\beta(g)e^\beta_{\mathbf{n}}\rangle \\ &= \sum_{\mathbf{k}\in\mathbb{N}_0^d} \pi^N_{\mathbf{k},\mathbf{m}}(g)\overline{\pi^N_{\mathbf{k},\mathbf{n}}(g)}.\end{aligned}$$

Use Theorem 5.3 to write the matrix elements in terms of multivariate Meixner polynomials. Moving all terms independent of $\mathbf{k}$ to the other side, we find

$$\sum_{\mathbf{k}\in\mathbb{N}_0^d} \frac{(\beta)_{|\mathbf{k}|}}{\mathbf{k}!} \left(\prod_{i=1}^d \left|\frac{c_i}{a}\right|^{2k_i}\right) M_{\mathbf{k}}(\mathbf{m};U,\beta)\overline{M_{\mathbf{k}}(\mathbf{n};U,\beta)} = \frac{\mathbf{m}!}{(\beta)_{|\mathbf{m}|}} \left(\prod_{i=1}^d \left|\frac{a}{b_i}\right|^{2m_i}\right) |a|^{2\beta}\delta_{\mathbf{m},\mathbf{n}}.$$

Using the definitions of C and $\tilde{C}$ and using that $|a|^2 = \frac{1}{C_0}$, the orthogonality of the Meixner polynomials in the variables, Equation (5.13b) follows. To prove the other orthogonality relation, one can use the other orthogonality formula of the representation or apply the duality of the Meixner polynomials, that is sending g to Jg^tJ (see also the discussion below). □

Remark. In [5], Iliev used a matrix U such that the first row and column has all elements equal to 1, in combination with two diagonal matrices $C,\tilde{C}$ such that $C_{0,0} = \tilde{C}_{0,0} = 1$ and the following holds:

$$U^tCU\tilde{C} = (1 - \sum_{i=1}^d C_i)I_{d+1}.$$

For our approach, the matrix U together with the matrices $C := \mathrm{diag}(1,-C)$ and $\tilde{C} := \mathrm{diag}(1,-\tilde{C})$ obtained from a matrix $g \in SU(1,d)$ as above have the same shapes. Although, for our matrices one can show the slightly differing equality

$$U^\dagger CU\tilde{C} = (1 - \sum_{i=1}^d C_i)I_{d+1}.$$

As the generating function from Iliev is well defined for all U, we can still use it under our conditions.

The multivariate polynomials as we defined above use an integer β and a matrix $g \in SU(1,d)$ to construct parameters $(C(g), \tilde{C}(g), U(g), \beta)$. We already saw a duality for the Meixner polynomials between the degree and variable indices in Equation (5.12) by sending g to Jg^tJ. This duality induces an involution in the parameter space by

$$(C(g), \tilde{C}(g), U(g), \beta) \longmapsto (C(Jg^tJ), \tilde{C}(Jg^tJ), U(Jg^tJ), \beta) = (\tilde{C}(g), C(g), U^t(g), \beta). \tag{5.14}$$

5.3. **Multivariate Meixner polynomials and the Lie algebra $\mathfrak{su}(1,d)$.** The Lie algebra of $SU(1,d)$ is the algebra of matrices g with zero trace such that $g^\dagger J = -Jg$, where $J = \mathrm{diag}(1,-1,\dots,-1)$. Further working out the definition shows that $\mathfrak{su}(1,d) = \left\{ \begin{pmatrix} -Tr(D) & b^t \\ \bar{b} & D \end{pmatrix} \,\middle|\, b \in \mathbb{C}^d, D \in \mathfrak{u}(d) \right\}$. Note that $D \in \mathfrak{u}(d)$ means $D^\dagger = -D$, and so the diagonal of D and a are purely imaginary. It can be shown that the complexification of $\mathfrak{su}(1,d)$ is isomorphic to $\mathfrak{sl}(d+1;\mathbb{C})$. For all purposes, the representation of the complexification is again denoted by π^β. As usual, we take as a basis of $\mathfrak{sl}(d+1;\mathbb{C})$ the standard matrix elements $e_{i,j}, i \neq j$ together with the matrices H_k, $k = 1,\dots,d$, where

$$H_k = e_{k,k} - \frac{1}{d+1} I_{d+1} \qquad k = 1,\dots,d.$$

As the representation π^β is smooth on $\mathcal{A}_\alpha^\infty$, we can define a Lie algebra representation on this space. Define the Lie algebra representation by

$$\mathrm{d}\pi^\beta(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^\beta(\exp(tX)),$$

for $X \in \mathfrak{su}(1,d)$ and extend it to $\mathfrak{sl}(d+1;\mathbb{C})$ via the complexification. Proceeding as usual, we find the action is defined as in the following lemma.

Lemma 5.6. *The representation of $\mathfrak{sl}(d+1;\mathbb{C})$ on $\mathcal{A}_\alpha^\infty$ is given by*

$$\begin{aligned}
\mathrm{d}\pi^\beta(H_k) &= \frac{1}{d+1}\beta + z_k \partial_{z_k}, \\
\mathrm{d}\pi^\beta(e_{0,j}) &= \partial_{z_j}, \\
\mathrm{d}\pi^\beta(e_{i,0}) &= z_i \Bigg(-\beta - \sum_{k=1}^d z_k \partial_{z_k} \Bigg), \\
\mathrm{d}\pi^\beta(e_{i,j}) &= z_i \partial_{z_j}.
\end{aligned}$$

So for the basis $\{e_{\mathbf{n}}^\alpha\}_{\mathbf{n}\in\mathbb{N}_0^d}$ of $\mathcal{A}_\alpha^\infty$, we find

$$\begin{aligned}
\mathrm{d}\pi^\beta(H_k)e_{\mathbf{n}}^\alpha &= \Big(\frac{1}{d+1}\beta + n_k\Big) e_{\mathbf{n}}^\alpha, \\
\mathrm{d}\pi^\beta(e_{0,j})e_{\mathbf{n}}^\alpha &= \sqrt{(\beta + |\mathbf{n}| - 1)n_j}\, e_{\mathbf{n}-\mathbf{v}_j}^\alpha, \\
\mathrm{d}\pi^\beta(e_{i,0})e_{\mathbf{n}}^\alpha &= -\sqrt{(n_i+1)(\beta + |\mathbf{n}|)}\, e_{\mathbf{n}+\mathbf{v}_i}^\alpha, \\
\mathrm{d}\pi^\beta(e_{i,j})e_{\mathbf{n}}^\alpha &= \sqrt{(n_i+1)n_j}\, e_{\mathbf{n}+\mathbf{v}_i-\mathbf{v}_j}^\alpha, \qquad i \neq j
\end{aligned} \tag{5.15}$$

for any $1 \le i,j \le d$ and $1 \le k \le d$. Here $e_{\mathbf{n}}^\alpha$ is set as 0 when $n_i = -1$ for some $1 \le i \le d$.

Proof. We proceed analogous to the univariate case in Lemma 3.5, noting that calculating the action directly is not 'proper' but it yields the correct formulas nonetheless. For a formal proof, one should compute the action of $\mathfrak{su}(1,d)$ and compute the action of $\mathfrak{sl}(d+1;\mathbb{C})$ via linear combinations.

Let $p \in \mathcal{A}_\alpha^\infty$ be arbitrary and $1 \le i,j \le d$. As $\exp(te_{i,j}) = I + te_{i,j}$, we find

$$[\mathrm{d}\pi^\beta(e_{i,j})p](\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^\beta(I + te_{i,j})p](\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(\mathbf{z} + tz_i\mathbf{v}_j) = [z_i\partial_{z_j}p](z).$$

Likewise, for i or j zero, we can calculate the action as

$$[\mathrm{d}\pi^\beta(e_{i,0})p](\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (1+tz_i)^{-\beta} p\left(\frac{\mathbf{z}}{1+tz_i}\right) = -\beta z_i p - z_i \sum_{k=1}^d z_k \frac{\partial p}{\partial z_k},$$

$$[\mathrm{d}\pi^\beta(e_{0,j})p](\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(t\mathbf{v}_j + \mathbf{z}) = \frac{\partial p}{\partial z_j}.$$

The actions as listed in the lemma for the $e_{i,j}$ become clear from this.

For the elements H_k, $k = 1, \ldots, d$, note that $\exp(tH_k)$ is the diagonal matrix with k-th diagonal entry $e^{t\frac{d}{d+1}}$ and others equal to $e^{-t\frac{1}{d+1}}$. Therefore, the action of H_k can be computed as follows

$$[\mathrm{d}\pi^\beta(H_k)p](\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{\frac{t}{d+1}\beta} p(\left[z_1, \ldots, e^t z_k, \ldots, z_d\right]) = \frac{1}{d+1}\beta + z_k \frac{\partial p}{\partial z_k}.$$

This gives us the stated formula for the action of H_k.

Lastly, write $Q(\mathbf{n}) := \frac{(\beta)_{|\mathbf{n}|}}{\mathbf{n}!}$, so that $e_{\mathbf{n}}(\mathbf{z}) = \sqrt{Q(\mathbf{n})} z^{\mathbf{n}}$. Using the identities

$$\begin{aligned} Q(\mathbf{n} - \mathbf{v}_j) &= Q(\mathbf{n}) \frac{n_j}{\beta + |\mathbf{n}| - 1}, \\ Q(\mathbf{n} + \mathbf{v}_i) &= Q(\mathbf{n}) \frac{\beta + |\mathbf{n}|}{n_i + 1}, \\ Q(\mathbf{n} - \mathbf{v}_j + \mathbf{v}_i) &= Q(\mathbf{n}) \frac{n_j}{n_i + 1}, \end{aligned} \tag{5.16}$$

the action of $\mathfrak{sl}(d+1;\mathbb{C})$ on the basis vectors $\{e^\alpha_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^d}$ as in Equation (5.15) becomes clear. ☐

Fix some $g \in SU(1,d)$ and set the basis elements $\tilde{e}^\alpha_{\mathbf{n}} = \pi^\beta(g)e^\alpha_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}_0^d$, of $\mathcal{A}^\infty_\alpha$. As usual, we define a basis of $\mathfrak{sl}(d+1;\mathbb{C})$ by adjoining the old one by g, that is $\tilde{\phi} = g\phi g^{-1}$ for ϕ in the old basis. It should be clear that the action of the tilde basis of $\mathfrak{sl}(d+1;\mathbb{C})$ on the tilde basis of $\mathcal{A}^\infty_\alpha$ is given akin to Equation (5.15) in the lemma above. Using Lemma 2.10, using again that $-X = JX^\dagger J$, we have the following:

Corollary 5.7. *For $X \in \mathfrak{sl}(d+1;\mathbb{C})$, it holds that*

$$\langle \mathrm{d}\pi^\beta(X)u, v\rangle = \langle u, \mathrm{d}\pi^\beta(JX^\dagger J)v\rangle, \qquad u, v \in \mathcal{A}^\infty_\alpha.$$

Proof. Let $X = Y + iZ \in \mathfrak{su}_{\mathbb{C}}(1,d)$. By conjugate linearity of the inner product and Lemma 2.10, it follows

$$\langle \mathrm{d}\pi^\beta(Y+iZ)u, v\rangle = \langle u, \mathrm{d}\pi^\beta(-Y+iZ)v\rangle.$$

Using that $-\phi = J\phi^\dagger J$ for $\phi \in \mathfrak{su}(1,d)$, the claim follows. ☐

We are now in the position to show the recurrence relations and difference equations for the Meixner polynomials.

Theorem 5.8. *Let $\beta \geq d+1$ be an integer and let $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix} \in SU(1,d)$, where $\mathbf{b}$ and $\mathbf{c}$ have all entries nonzero. Write $C_i = \frac{|b_i|^2}{|a|^2}$, $\tilde{C}_i = \frac{|c_i|^2}{|a|^2}$, $C_0 = 1 - \sum_{i=1}^d C_i$ and $U_{i,j} = \frac{D_{j,i}a}{b_i c_j}$ for $i,j \in \{1,\ldots,d\}$. Lastly, write $M_{\mathbf{m}}(\mathbf{n}) := M_{\mathbf{m}}(\mathbf{n}; U, \beta)$. For each $k = 1, \ldots, d$ we have the following recurrence relation:*

$$\begin{aligned} C_0 n_k M_{\mathbf{m}}(\mathbf{n}) =& C_k \sum_{i=1}^d \tilde{C}_i \overline{U_{k,i}} (\beta + |\mathbf{m}|) M_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}) \\ &- \Big[C_k(\beta + |\mathbf{m}|) + \sum_{l=1}^d C_k \tilde{C}_l |U_{k,l}|^2 m_l\Big] M_{\mathbf{m}}(\mathbf{n}) \\ &+ C_k \sum_{j=1}^d U_{k,j} m_j M_{\mathbf{m}-\mathbf{v}_j}(\mathbf{n}) \\ &- C_k \sum_{\substack{i,j=1 \\ i\neq j}}^d \tilde{C}_i \overline{U_{k,i}} U_{k,j} m_j M_{\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i}(\mathbf{n}). \end{aligned} \tag{5.17a}$$

Also for $k = 1, \dots, d$, we have the difference equations

$$\begin{aligned} C_0 m_k M_{\mathbf{m}}(\mathbf{n}) =& \tilde{C}_k \sum_{i=1}^{d} C_i \overline{U_{i,k}}(\beta + |\mathbf{n}|) M_{\mathbf{m}}(\mathbf{n} + \mathbf{v}_i) \\ &- \Big[\tilde{C}_k(\beta + |\mathbf{n}|) + \sum_{l=1}^{d} \tilde{C}_k C_l |U_{l,k}|^2 n_l\Big] M_{\mathbf{m}}(\mathbf{n}) \\ &+ \tilde{C}_k \sum_{j=1}^{d} U_{j,k} n_j M_{\mathbf{m}}(\mathbf{n} - \mathbf{v}_j) \\ &- \tilde{C}_k \sum_{\substack{i,j=1 \\ i \neq j}}^{d} C_i \overline{U_{i,k}} U_{j,k} n_j M_{\mathbf{m}}(\mathbf{n} - \mathbf{v}_j + \mathbf{v}_i). \end{aligned} \tag{5.17b}$$

Proof. We proceed as usual. Let $k \in \{1, \dots, d\}$ be arbitrary. We will look at the action of $\tilde{H}_k$. We first claim that $J(\tilde{H}_k)^\dagger J = \tilde{H}_k$. Using the definition of $\tilde{H}_k$ and that $I = J^2$, we get

$$J(\tilde{H}_k)^\dagger J = (Jg^{-1}J)^\dagger (JH_kJ)(Jg^\dagger J).$$

As for $g \in SU(1,d)$, $g^{-1} = Jg^\dagger J$ and as diagonal matrices commute, the claim follows. By Corollary 5.7, we then have the equality

$$\langle \mathrm{d}\pi^\beta(\tilde{H}_k)\tilde{e}_{\mathbf{n}}, e_{\mathbf{m}}\rangle = \langle \tilde{e}_{\mathbf{n}}, \mathrm{d}\pi^\beta(\tilde{H}_k)e_{\mathbf{m}}\rangle. \tag{*}$$

We will expand both sides in terms of Meixner polynomials to derive the recurrence relations (5.17a).

Write $\langle \tilde{e}_m, e_n\rangle = B(\mathbf{m}, \mathbf{n}) M_{\mathbf{m}}(\mathbf{n})$ where

$$B(\mathbf{m}, \mathbf{n}) = \sqrt{\frac{(\beta)_{|\mathbf{m}|}}{\mathbf{m}!}} \sqrt{\frac{(\beta)_{|\mathbf{n}|}}{\mathbf{n}!}} a^{-\beta - |\mathbf{m}| - |\mathbf{n}|} \mathbf{b}^{\mathbf{n}} (-\mathbf{c})^{\mathbf{m}},$$

in accordance to Theorem 5.3. Using the action of $\tilde{H}_k$ on $\tilde{e}^\alpha_{\mathbf{m}}$ in accordance to Equation (5.15), we can write out the left-hand side of (*) as

$$\langle \mathrm{d}\pi^\beta(\tilde{H}_k)\tilde{e}_{\mathbf{n}}, e_{\mathbf{m}}\rangle = \left(\frac{1}{d+1}\beta + n_k\right) B(\mathbf{m}, \mathbf{n}) M_{\mathbf{m}}(\mathbf{n}).$$

For the right-hand side, we first expand $\tilde{H}_k$ in terms of the basis elements of $\mathfrak{sl}(d+1; \mathbb{C})$ given by $\{H_l, e_{i,j} \,|\, l \in \{1, \dots d\}, i, j \in \{0, \dots, d\}, i \neq j\}$. By definitions of $\tilde{H}_k$ and H_k, we find

$$\tilde{H}_k = g e_{k,k} g^{-1} - \frac{1}{d+1} I_{d+1}.$$

If we write $g = \begin{pmatrix} a & \mathbf{b}^t \\ \mathbf{c} & D \end{pmatrix}$, then its inverse is given as $g^{-1} = Jg^\dagger J = \begin{pmatrix} \overline{a} & -c^\dagger \\ -\overline{\mathbf{b}} & D^\dagger \end{pmatrix}$. Using this, one can show that $\tilde{H}_k$ can be written as

$$\tilde{H}_k = \begin{bmatrix} b_k \\ D_{1,k} \\ \vdots \\ D_{d,k} \end{bmatrix} \otimes \begin{bmatrix} -\overline{b_k} \\ \overline{D_{1,k}} \\ \vdots \\ \overline{D_{d,k}} \end{bmatrix} - \frac{1}{d+1} I_{d+1} = \begin{pmatrix} -|b_k|^2 & b_k\overline{D_{1,k}} & \dots & b_k\overline{D_{d,k}} \\ -\overline{b_k}D_{1,k} & |D_{1,k}|^2 & \dots & D_{1,k}\overline{D_{d,k}} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{b_k}D_{d,k} & D_{d,k}\overline{D_{1,k}} & \dots & |D_{d,k}|^2 \end{pmatrix} - \frac{1}{d+1} I_{d+1},$$

where $\otimes$ is the outer product so that $(\mathbf{u} \otimes \mathbf{w})_{i,j} = u_i w_j$ for two vectors $\mathbf{u}, \mathbf{w} \in \mathbb{C}^d$. To express the diagonal of the above matrix in terms of the sum $\sum_{l=1}^d x_l H_l, x_l \in \mathbb{C}$, we solve the system

$$\begin{aligned} -\frac{1}{d+1}\sum_{l=1}^{d} x_l &= -|b_k|^2 - \frac{1}{d+1}, \\ x_1 - \frac{1}{d+1}\sum_{l=1}^{d} x_l &= |D_{1,i}|^2 - \frac{1}{d+1} \\ &\vdots \\ x_d - \frac{1}{d+1}\sum_{l=1}^{d} x_l &= |D_{d,i}|^2 - \frac{1}{d+1}. \end{aligned}$$

Subtracting the first equation from the rest, we get $x_l = |b_k|^2 + |D_{k,l}|^2$. From the defining equation of $SU(1,d)$, we know that $|b_k|^2 - \sum_{l=1}^d |D_{k,l}|^2 = -1$ so that the first equation also holds and the system is consistent. We conclude that we can write $\tilde{H}_k$ as follows:

$$\begin{aligned}\tilde{H}_k = &\sum_{l=1}^d (|D_{l,k}|^2 + |b_k|^2) H_l + \sum_{\substack{i,j=1\\ i\neq j}}^d D_{i,k}\overline{D_{j,k}} e_{i,j}\\ &+ b_k \sum_{j=1}^d \overline{D_{j,k}} e_{0,j} - \overline{b_k} \sum_{i=0}^d D_{i,k} e_{i,0}.\end{aligned}$$

Using this and the action of the Lie algebra on the basis of $\mathcal{A}_\alpha^\infty$ as given in Equation (5.15), we can expand the right-hand side of (*) as

$$\begin{aligned}&\sum_{l=1}^d (|D_{l,k}|^2 + |b_k|^2)\Big(\frac{1}{d+1}\beta + m_k\Big) B(\mathbf{m},\mathbf{n}) M_{\mathbf{m}}(\mathbf{n})\\ &+ \sum_{\substack{i,j=1\\ i\neq j}}^d \overline{D_{i,k}} D_{j,k} \sqrt{(m_i+1)m_j}\, B(\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i,\mathbf{n}) M_{\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i}(\mathbf{n})\\ &+\overline{b_k} \sum_{j=1}^d D_{j,k} \sqrt{(\beta+|\mathbf{m}|-1)m_j}\, B(\mathbf{m}-\mathbf{v}_j,\mathbf{n}) M_{\mathbf{m}-\mathbf{v}_j}(\mathbf{n})\\ &+b_k \sum_{i=0}^d \overline{D_{i,k}} \sqrt{(m_i+1)(\beta+|\mathbf{m}|)}\, B(\mathbf{m}+\mathbf{v}_i,\mathbf{n}) M_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}).\end{aligned}$$

Note that we have taken the complex conjugate of the constants in accordance with the conjugate-linearity of the inner product.

Next, using Equation (5.16) we can relate the constants B using

$$\begin{aligned}B(\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i,\mathbf{n}) &= B(\mathbf{m},\mathbf{n})\sqrt{\frac{m_j}{m_i+1}\frac{c_i}{c_j}},\\ B(\mathbf{m}-\mathbf{v}_j,\mathbf{n}) &= -B(\mathbf{m},\mathbf{n})\sqrt{\frac{m_j}{\beta+|\mathbf{m}|-1}\frac{a}{c_j}},\\ B(\mathbf{m}+\mathbf{v}_i,\mathbf{n}) &= -B(\mathbf{m},\mathbf{n})\sqrt{\frac{\beta+|\mathbf{m}|}{m_i+1}\frac{c_i}{a}}.\end{aligned}$$

Combining the formulas for the right- and left-hand side of (*) and dividing by $B(\mathbf{m},\mathbf{n})$, we get

$$\begin{aligned}\Big(\frac{1}{d+1}\beta + n_k\Big) M_{\mathbf{m}}(\mathbf{n}) = &\sum_{l=1}^d (|D_{l,k}|^2 + |b_k|^2)\Big(\frac{1}{d+1}\beta + m_l\Big) M_{\mathbf{m}}(\mathbf{n})\\ &+ \sum_{\substack{i,j=1\\ i\neq j}}^d \frac{c_i \overline{D_{i,k}} D_{j,k}}{c_j} m_j M_{\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i}(\mathbf{n})\\ &- \sum_{j=1}^d \frac{a\overline{b_k} D_{j,k}}{c_j} m_j M_{\mathbf{m}-\mathbf{v}_j}(\mathbf{n})\\ &- \sum_{i=0}^d \frac{b_k c_i \overline{D_{i,k}}}{a} (\beta+|\mathbf{m}|) M_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}).\end{aligned}$$

Now, subtract $\frac{1}{d+1}\beta M_{\mathbf{m}}(\mathbf{n})$ from both sides and multiply by -1. Using again that $|b_k|^2 - \sum_{l=1}^d |D_{k,l}|^2 = -1$ for $g \in SU(1,d)$, we gather

$$\begin{aligned}
-n_k M_{\mathbf{m}}(\mathbf{n}) = &- \left[|b_k|^2(\beta + |\mathbf{m}|) + \sum_{l=1}^d |D_{l,k}|^2 m_l \right] M_{\mathbf{m}}(\mathbf{n}) \\
&- \sum_{\substack{i,j=1 \\ i\neq j}}^d \frac{c_i \overline{D_{i,k}} D_{j,k}}{c_j} m_j M_{\mathbf{m}-\mathbf{v}_j+\mathbf{v}_i}(\mathbf{n}) \\
&+ \sum_{j=1}^d \frac{a\overline{b_k} D_{j,k}}{c_j} m_j M_{\mathbf{m}-\mathbf{v}_j}(\mathbf{n}) \\
&+ \sum_{i=0}^d \frac{b_k c_i \overline{D_{i,k}}}{a} (\beta + |\mathbf{m}|) M_{\mathbf{m}+\mathbf{v}_i}(\mathbf{n}).
\end{aligned}$$

Next, divide both sides by $|a|^2$ and use the definitions of $C_i = \frac{|b_i|^2}{|a|^2}$, $\tilde{C}_i = \frac{|c_i|^2}{|a|^2}$ and $U_{i,j} = \frac{D_{j,i}a}{b_i c_j}$ to rewrite the equation. Lastly, use that $\frac{1}{|a|^2} = (1 - \sum_{i=1}^d C_i)$ to conclude the recurrence relations (5.17a).

To prove the difference equations (5.17b), one can proceed similar as above by acting with H_k instead and expressing it in terms of the basis $\left\{ \tilde{H}_l, \tilde{e}_{i,j} \,|\, l \in \{1, \dots d\}, i,j \in \{0,\dots,d\}, i \neq j \right\}$ of $\mathfrak{sl}(d+1;\mathbb{C})$. Another, even simpler, method is by using the duality of the multivariate Meixner polynomials (5.14) to interchange C and $\tilde{C}$ and transpose the matrix U. $\square$

We conclude this section by relating the above found difference equations (5.17b) to the difference operators from Iliev [5]. Iliev defined shift operators E_{n_i} acting on functions of $\mathbf{n} = (n_1, \dots, n_d)$ as follows

$$E_{n_i} f(\mathbf{n}) = f(\mathbf{n} + \mathbf{v}_i).$$

As commented in the remark on page 33, Iliev used a matrix U, such that the first row and column has all elements equal to 1, and two diagonal matrices $C, \tilde{C}$, such that $C_{0,0} = \tilde{C}_{0,0} = 1$, so that the equality is valid:

$$U^t C U \tilde{C} = (1 - \sum_{i=1}^d C_i) I_{d+1},$$

(Whereas for our matrices $U, C, \tilde{C}$ from a matrix $g \in SU(1,d)$ the transpose is replace by a dagger). Write $\mathbf{C} = (C_{1,1}, \dots, C_{d,d})$ and $\tilde{\mathbf{C}} = (\tilde{C}_{1,1}, \dots, \tilde{C}_{d,d})$ as the diagonal entries of C respectively $\tilde{C}$. For these $U, \mathbf{C}, \tilde{\mathbf{C}}$ Iliev defined the difference operators

$$\begin{aligned}
\mathcal{L}_k^{\mathbf{n}} = &- \frac{\tilde{\mathbf{C}}_k}{1 - |\tilde{\mathbf{C}}|} \sum_{i=1}^d \mathbf{C}_i U_{i,k} (\beta + |\mathbf{n}|)(E_{n_i} - \mathrm{Id}\,) \\
&- \frac{\tilde{\mathbf{C}}_k}{1 - |\tilde{\mathbf{C}}|} \sum_{j=1}^d U_{j,k} n_j (E_{n_j}^{-1} - \mathrm{Id}\,) \\
&+ \frac{\tilde{\mathbf{C}}_k}{1 - |\tilde{\mathbf{C}}|} \sum_{\substack{i,j=1 \\ i\neq j}}^d \mathbf{C}_i U_{i,k} U_{j,k} n_j (E_{n_i} E_{n_j}^{-1} - \mathrm{Id}\,),
\end{aligned}$$

where Id denotes the identity operator. Iliev also showed that the Meixner polynomials, as defined using his matrix U, diagonalize the operators $\mathcal{L}_k^{\mathbf{n}}$ in the sense that

$$\mathcal{L}_k^{\mathbf{n}} M_{\mathbf{m}}(\mathbf{n}) = m_i M_{\mathbf{m}}(\mathbf{n}).$$

In comparison, in our approach by defining a matrix U and two vectors $C, \tilde{C}$ from a matrix $g \in SU(1,d)$ as usual, we get difference operators

$$\begin{aligned}\mathbb{L}_k^{\mathbf{n}} = &- \frac{\tilde{C}_k}{1-\sum_{i=1}^d \tilde{C}_i} \sum_{i=1}^d C_i \overline{U_{i,k}} (\beta + |\mathbf{n}|)(E_{n_i} - \mathrm{Id}\,) \\ &- \frac{\tilde{C}_k}{1-\sum_{i=1}^d \tilde{C}_i} \sum_{j=1}^d U_{j,k} n_j (E_{n_j}^{-1} - \mathrm{Id}\,) \\ &+ \frac{\tilde{C}_k}{1-\sum_{i=1}^d \tilde{C}_i} \sum_{\substack{i,j=1 \\ i\neq j}}^d C_i \overline{U_{i,k}} U_{j,k} n_j (E_{n_i} E_{n_j}^{-1} - \mathrm{Id}\,).\end{aligned}$$

The operators $\mathbb{L}_k^{\mathbf{n}}$ are diagonalized by our Meixner polynomials in the sense that

$$\mathbb{L}_k^{\mathbf{n}} M_{\mathbf{m}}(\mathbf{n}) = m_i M_{\mathbf{m}}(\mathbf{n}).$$

Indeed, this follows directly by rewriting the difference equations (5.17b), using that $\sum_{j=1}^d C_j U_{j,k} = 1$ (this is clear from the first column of $gJg^\dagger = J$).

6. Laguerre polynomials

In this section, we will look into the Laguerre polynomials. These polynomials are known to be related to the Lie group $SU(1,1)$ just as the Meixner polynomials. The Laguerre polynomials are also known to be given as a limit of the Meixner polynomials. Looking into this limit using the representation π^β for the Meixner polynomials, it becomes clear that this limit does not translate into the representation theory as is. Lastly, we briefly look into a new representation of $SL(2;\mathbb{C}) \cong SU(1,1)$ following Vilenkin and Klimyk [7], to give insight into the connection between $SU(1,1)$ and Laguerre polynomials.

6.1. **Laguerre polynomials.** The Laguerre polynomials (see e.g. [1]) are defined on the positive real line, $\mathbb{R}_+$, together with a parameter $\beta > 0$, by the generating function

$$(1-t)^{-\beta} \exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n^\beta(x) t^n. \tag{6.1}$$

As a hypergeometric function, the Laguerre polynomials are defined by

$$L_n^\beta(x) = \frac{(\beta)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \beta \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(-n)_k}{(\beta)_k} \frac{x^k}{k!}, \tag{6.2}$$

where the constant $\frac{(\beta)_n}{n!}$ ensures that the leading coefficient is ± 1. The Laguerre polynomials are known to be orthogonal with respect to the gamma distribution (with shape parameter β and rate or scale parameter 1):

$$\int_0^\infty x^{\beta-1} e^{-x} L_m^\beta(x) L_n^\beta(x)\, \mathrm{d}x = \frac{\Gamma(\beta+m)}{m!} \delta_{m,n}. \tag{6.3}$$

There is a known limit relation transforming the Meixner polynomials into the Laguerre polynomials. This is done by letting the parameter c for the Meixner polynomial go to 1, whilst dividing the variable n by $1-c$. This limit is given as follows:

$$\lim_{c\to 1} M_m\left(\frac{x}{1-c}; c, \beta\right) = \frac{m!}{(\beta)} L_m^{\beta-1}(x), \qquad x \in \mathbb{R}_+. \tag{6.4}$$

The term $\frac{x}{1-c}$ can be seen as to smoothen $\mathbb{N}$ to look like $\mathbb{R}_+$; suppose $\frac{x}{1-c} \in \mathbb{N}$, then surely $x \in (1-c)\mathbb{N} = \{(1-c)n \,|\, n \in \mathbb{N}\}$, taking the limit $c \to 1$ then makes this set dense in $\mathbb{R}_+$. Let us look into this limit more closely. Writing out the Meixner polynomial in the limit, we get

$$M_m\left(\frac{x}{1-c}; c, \beta\right) = \sum_{k=0}^{\infty} \frac{(-m)_k \left(-\frac{x}{1-c}\right)_k}{(\beta)_k} \frac{(1-1/c)^k}{k!}.$$

As the sum is finite (there are at most $m+1$ terms as $(-m)_{m+1} = 0$), the terms of interest for the limit are $\left(\frac{x}{c-1}\right)_k (1-1/c)^k$. Writing $\left(\frac{x}{c-1}\right)_k = \frac{x}{c-1}\left(\frac{x}{c-1}+1\right)\cdots\left(\frac{x}{c-1}+k-1\right)$, and $1-1/c = \frac{c-1}{c}$, the product can be rewritten to

$$\left(\frac{x}{c-1}\right)_k (1-1/c)^k = c^{-k} x(x+c-1)(x+2(c-1))\cdots(x+(k-1)(c-1)).$$

Taking the limit of c to 1, this term becomes x^k, which gives the Laguerre polynomial as

$$\lim_{c\to 1} M_m\left(\frac{x}{1-c}; c, \beta\right) = \sum_{k=0}^{\infty} \frac{(-m)_k}{(\beta)_k} \frac{x^k}{k!} = {}_1F_1\left(\begin{matrix} -m \\ \beta \end{matrix}; x\right).$$

The Limit (6.4) also transforms the generating function of the Meixner polynomials into the one for the Laguerre polynomials. Recall the generating function of the Meixner polynomials:

$$(1-t)^{-\beta-n}\left(1-\frac{t}{c}\right)^n = \sum_{m\in\mathbb{N}_0} \frac{(\beta)_m}{m!} M_m(n; c, \beta) t^n. \tag{3.1}$$

Looking at the left-hand side, we can rewrite it to $(1-t)^{-\beta}\left(1-\frac{ct}{t-1}(1-c)\right)^n$. Substituting n by $\frac{x}{1-c}$ and taking the limit c to 1, the left-hand side becomes exactly the left-hand side of the generating function of the Laguerre polynomials (6.1).

One can also use the orthogonality of the Meixner polynomials to derive those for the Laguerre polynomials. Recall the orthogonality relation for the Meixner polynomial

$$\sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} c^k (1-c)^{\beta-1} \left(\frac{(\beta)_m}{m!} M_m(k;c,\beta)\right) \left(\frac{(\beta)_n}{n!} M_n(k;c,\beta)\right)(1-c) = \delta_{m,n} \frac{(\beta)_m c^{-m}}{m!}, \tag{3.10a}$$

where we have already rearranged some terms for our benefit (note the $1-c$ at the end of the left-hand side). If we write $(\beta)_m = \frac{\Gamma(\beta+m)}{\Gamma(\beta)}$, it becomes clear that the right-hand side of the above equation will become $\frac{1}{\Gamma(\beta)}$ times the right-hand side of Equation (6.3). In particular, the limit of the sum on the left-hand side should also converge. Next, we want to show that the negative binomial measure converges to the measure of the gamma distribution $\frac{x^{\beta-1}e^{-x}}{\Gamma(\beta)}\,\mathrm{d}x$. We start with c^k. Substitute $k = \frac{x}{1-c}$ and write this as an exponential, that is $c^{\frac{x}{1-c}} = e^{\frac{x \ln c}{1-c}}$. Now using L'Hôpital's rule, the limit of the exponent is equal to $-x$, giving us the term e^{-x}. For the term $\frac{(\beta)_k}{k!}(1-c)^{\beta-1}$, we first rewrite the fraction as

$$\frac{(\beta)_k}{k!} = \frac{\Gamma(\beta+k)}{\Gamma(\beta)\Gamma(1+k)}.$$

Now setting $k = \frac{x}{1-c}$, we can write the whole as

$$\frac{(\beta)_k}{k!}(1-c)^{\beta-1} \longmapsto \frac{\Gamma(\beta+k)}{\Gamma(1+k)} \left(\frac{x}{1-c}\right)^{1-\beta} \frac{x^{\beta-1}}{\Gamma(\beta)}.$$

Using Equation (1.5), and as the limit $c \to 1$ translates to the limit $k \to \infty$, we see that in the limit the above becomes $\frac{x^{\beta-1}}{\Gamma(\beta)}$. Regarding the sum as a Riemann sum due to the left term (1-c), one can argue that the sum converges to $\frac{1}{\Gamma(\beta)}$ times the integral in 6.3.

As we have seen above, the limit relation between the Meixner and Laguerre polynomials also translates many of their properties. When seeing this, one has to wonder if this can also be seen from the lie group representation, by taking limits in the Lie group instead. Sadly this does not seem the case. Take for example the matrix elements of the representation π^β of $\mathcal{A}_\alpha$. If we let $n \in \mathbb{N}_0$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$, with $b, c \neq 0$ the matrix elements are given by:

$$\pi^\beta_{m,n}(g) = \sqrt{\frac{(\beta)_m}{m!}} \sqrt{\frac{(\beta)_n}{n!}} a^{-\beta-m-n} b^n (-c)^m M_m\left(n; \frac{bc}{ad}, \beta\right). \tag{3.8}$$

Letting $\frac{bc}{ac}$ go to 1 here would mean to either let the determinant $ad - bc$ go to zero, or else let elements go to ∞. Either way, using this approach the limiting matrix will not be an element of $GL(2;\mathbb{C})$. In order to create a relation between Lie theory and Laguerre polynomials, we first find a new family of representations in the next subsection.

6.2. **A representation on the upper half plane.** We want to act on a different space via the same representation, namely the space of holomorphic functions on the complex upper half plane $\mathbb{C}_+ := \{w \in \mathbb{C} \mid \mathrm{Im}(w) > 0\}$. It is known (see for instance [7]) that the upper half plane can be send to the unit disk by the fractional linear transformation

$$w \mapsto \frac{w-i}{w+i}.$$

As the unit ball is closed under rotations, we may multiply this map by i to obtain a bijection $\phi : \mathbb{C}_+ \longrightarrow \mathbb{B}$ defined by

$$\phi(w) = \frac{w-i}{1-iw}, \qquad w \in \mathbb{C}_+. \tag{6.5}$$

In Subsection 3.1.1, we encountered fractional linear transformations via a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2;\mathbb{C})$ by the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{b+dz}{a+cz}.$$

The matrix which represents the map ϕ as above is then connected to the matrix $T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \in SU(2)$ (the constant $\frac{1}{\sqrt{2}}$ does not change the associated map). The inverse of ϕ, is then given by

$$\phi^{-1}(z) = \frac{z+i}{1+iz}, \qquad z \in \mathbb{B}.$$

Let $\alpha \geq 0$ be an integer, $\beta = \alpha + 2$ and π^β be the map of $GL(2;\mathbb{C})$ on $GL(H(\mathbb{B}))$ as for the univariate Meixner polynomials in Section 3. Let the mapping $\rho : H(\mathbb{C}_+) \longrightarrow H(\mathbb{B})$ be defined as follows

$$\begin{aligned}[\rho F](z) =& 2^{\frac{\beta}{2}}(1+iz)^{-\beta}F(\phi^{-1}(z)) \\ =& 2^{\frac{\beta}{2}}(1+iz)^{-\beta}F\left(\frac{z+i}{1+iz}\right).\end{aligned}$$

Observe that ρ looks like $\pi^\beta(T^{-1})$. Clearly ρ is linear and invertible with its inverse given by

$$\begin{aligned}[\rho^{-1} f](w) =& 2^{\frac{\beta}{2}}(1-iw)^{-\beta}f(\phi(w)) \\ =& 2^{\frac{\beta}{2}}(1-iw)^{-\beta}f\left(\frac{w-i}{1-iw}\right).\end{aligned}$$

Using the above maps, we can construct a representation π_+^β of $SU(1,1)$ on $H(\mathbb{C}_+)$ as follows:

$$\pi_+^\beta(g) = \rho^{-1}\,\pi^\beta(g)\rho, \qquad \text{for all } g \in SU(1,1). \tag{6.6}$$

We define an inner product $\langle\cdot,\cdot\rangle_+$ on the space $H(\mathbb{C}_+)$ from the inner product on $\mathcal{A}_\alpha$ via ρ, that is we have

$$\langle F, G\rangle_+ := \langle \rho F, \rho G\rangle_\alpha .$$

Denote the Hilbert space of functions with finite norm in $H(\mathbb{C}_+)$ by $\mathcal{B}_\alpha^+$. Writing out the fraction $\frac{z+i}{1+iz}$, we get

$$\frac{z+i}{1+iz} = \frac{2\operatorname{Re}(z) + i(1-|z|^2)}{1+|z|^2-2\operatorname{Im}(z)}.$$

Using this identity, one can show the inner product on $\mathcal{B}_\alpha^+$ has the following form:

$$\langle F, G\rangle_+ = 2^{\alpha+2}c_\alpha \int_{\mathbb{C}_+} F(w)\overline{G(w)}\operatorname{Im}(w)^\alpha\,\mathrm{d}v(w), \tag{6.7}$$

where $c_\alpha = \alpha + 1$.

By construction, the representation π_+^β is equivalent to the representation π^β. Furthermore, the map $\rho : \mathcal{B}_\alpha^+ \longrightarrow \mathcal{A}_\alpha$ is a unitary map which intertwines the two. This implies that the new representation π_+^β is also unitary. We reuse the basis vectors $e_n^\alpha(z) := \sqrt{\frac{(\alpha+2)_n}{n!}}z^n$, $n \in \mathbb{N}_0$ of $\mathcal{A}_\alpha$ to define a basis of $\mathcal{B}_\alpha^+$:

$$e_n^{\alpha,+}(w) := [\rho^{-1}e_n^\alpha](w) = \sqrt{\frac{(\beta)_n}{n!}}2^{\frac{\beta}{2}}(1-iw)^{-\beta-n}(w-i)^n.$$

In particular with respect to this basis, the matrix elements of π_+^β and π^β will be identical (see also Lemma 1.8). As a last remark, we can write, by slight abuse of notation, that the representation π_+^β is equal to

$$\pi_+^\beta(g) = \pi^\beta(TgT^{-1}),$$

where TgT^{-1} is not a matrix of $SU(1,1)$ anymore, but rather a matrix in the special linear group of real matrices; $SL(2;\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \det g = 1,\ a,b,c,d \in \mathbb{R}\right\}$. This follows by the identity

$$T\begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix}T^{-1} = \begin{pmatrix} \operatorname{Re}(a) - \operatorname{Im}(b) & \operatorname{Re}(b) + \operatorname{Im}(a) \\ \operatorname{Re}(b) - \operatorname{Im}(a) & \operatorname{Re}(a) + \operatorname{Im}(b)\end{pmatrix}. \tag{6.8}$$

Thus the representation π_+^β can also be seen as a representation of $SL(2;\mathbb{R})$ instead. The action of $Sl(2;\mathbb{R})$ on $\mathcal{B}_\alpha^+$ will again be denoted by π^β and acts as 'usual':

$$\pi^\beta\begin{pmatrix} a & b \\ c & d\end{pmatrix}F(w) = (a+cz)^{-\beta}F\left(\frac{b+dz}{a+cz}\right), \qquad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL2;\mathbb{C}. \tag{6.9}$$

6.3. **Laguerre polynomials from $SL(2;\mathbb{C})$.** In this subsection we will broadly follow Vilenkin and Klimyk [7] to find the Laguerre polynomials from the representation π^β of $SL(2;\mathbb{C})$ on $\mathcal{B}_\alpha^+$ as in equation (6.9). From Vilenkin and Klimyk [7] we know that the functions in $\mathcal{B}_\alpha^+$ can be written as Fourier transforms of functions defined on the semi-axis $\lambda > 0$:

$$F(w) = \int_0^\infty \mathfrak{F}(\lambda)e^{i\lambda w}\,\mathrm{d}\lambda, \tag{6.10}$$

where the transform is taken over a strip $(-\infty + ia, \infty + ia)$, with a some positive real number.

$$\mathfrak{F}(\lambda) = \frac{1}{2\pi} \int_{-\infty+ia}^{\infty-ia} F(w) e^{-i\lambda w} \, \mathrm{d}w, \qquad w = x + ia. \tag{6.11}$$

Now following [7] the fourier transform as above can be shown bijective, so that the functions

$$\phi_\lambda^+(w) := \sqrt{\frac{(4\lambda)^{-\beta-1}}{\pi\Gamma(\beta)}} e^{i\lambda w}, \qquad \lambda > 0, \tag{6.12}$$

form a continuous basis of $\mathcal{B}_\alpha^+$ in the sense that

$$\left\langle \phi_\lambda^+, \phi_\mu^+ \right\rangle_+ = \delta(\lambda - \mu), \tag{6.13}$$

with $\delta(\lambda - \mu)$ the Dirac delta distribution. As these vectors form a basis of $\mathcal{B}_\alpha^+$, if we act with π^β on the basis vectors $e_n^{\alpha,+}$ we can write them as

$$\pi^\beta(g) e_n^{\alpha,+} = \int_0^\infty K(\lambda, n; g) e^{i\lambda w} \, \mathrm{d}\lambda, \tag{6.14}$$

where $K(\lambda, n; g)$ can be computed using the Fourier transform 6.11 on the function $\pi^\beta(g)e_n^{\alpha,+}$. We remark that solving for $K(\lambda, n; g)$ is not trivial, and will not be included in this report. An derivation of $K(\lambda, n; g)$ for certain one-parameter subgroup of $SL(2;\mathbb{C})$ is worked out in [7]. This turns out to be related to Laguerre polynomials.

Summary and concluding remarks

In this thesis, we studied Lie groups and their connection to certain orthogonal polynomials. We looked into the, in literature well-established, connection between $SU(2)$ and Krawtchouk polynomials, and the connections between $SU(1,1)$ and the Meixner and Laguerre polynomials. Furthermore, we showed a new connections between the d-variate Krawtchouk polynomials and $SU(d+1)$ and established a connection of the d-variate Meixner polynomials with the Lie group $SU(1,1)$. For all the polynomials, we have shown that they can be written in terms of matrix elements of a unitary representation. The unitarity then allowed us to show orthogonality relations. Passing to the Lie algebra representation, we also showed the recurrence relations for the polynomials.

To conclude this thesis, we will bring up some remaining questions and possibilities for further research. Throughout the thesis, we have restricted to certain subgroups of $Gl(n;\mathbb{C})$ in order to derive the polynomials and their properties. Here we have extensively used their property to of leaving an hermitian form invariant to prove the unitarity of the representation. This raises the question if the results can be extended to larger, or different groups, maybe leaving other hermitian or bilinear forms invariant. As Iliev discovered slightly different polynomials using a real inner product, we suspect that Lie groups leaving a real bilinear form invariant can be used to create a direct link to Iliev's work. As a second remark, we have seen that through our representation on the Bergman space $\mathcal{A}_\alpha$ we restricted to a far smaller field of possible values of the parameter β for the Meixner polynomials. Naturally, we wonder if this can be extended using other representations, and/or other Hilbert spaces.

We have briefly looked into the classical Laguerre polynomials, their relation to the Meixner polynomials via a limit, and their connection to the Lie group $SU(1,1)$. We have also seen that via the Lie group representation we cannot (yet) explain the limit from the Meixner to the Laguerre polynomials. Further research can be done as to how this relation can be seen through representation theory. Also, we have already established that the representation on Bergman space $\mathcal{A}_\alpha$ can be extended to higher dimensions using the Lie group $SU(1,d)$. A natural follow-up question is if we can use this to define multivariate Laguerre polynomials, and if a similar limit relation can be established with the multivariate Meixner polynomials. Another group of orthogonal polynomials, the Meixner-Pollaczek polynomials, are also highly similar to the Meixner polynomials and known to be connected to the Lie group $SU(1,1)$ as well (see for instance [7] Section 7.7.7). Therefore, another question that arises is if we can define multivariate extensions via the same methods as for the (regular) multivariate Meixner polynomials.

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