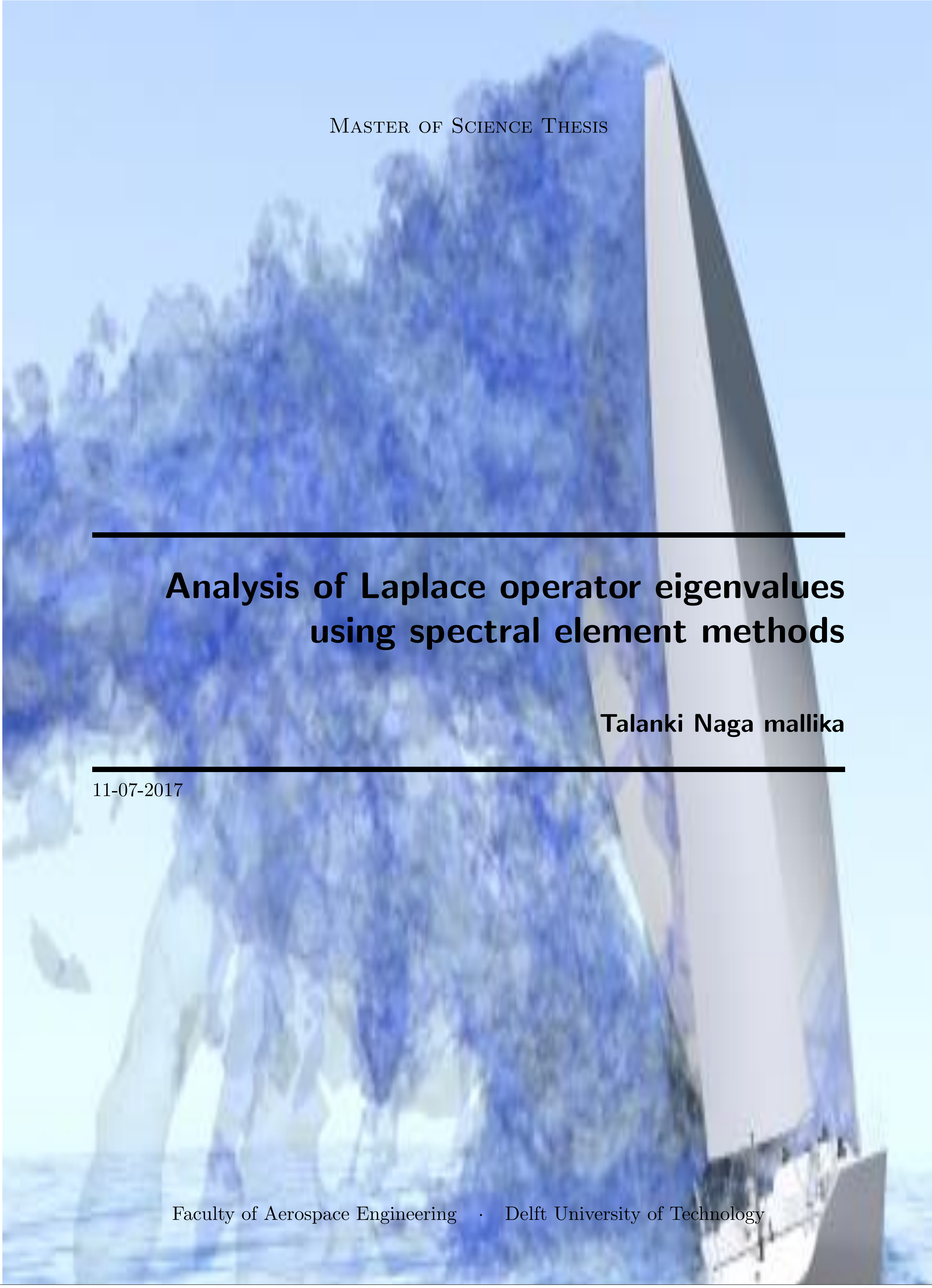

Master of Science Thesis

Analysis of Laplace operator eigenvalues using spectral element methods

Talanki Naga mallika

11-07-2017

Faculty of Aerospace Engineering · Delft University of Technology

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Master of Science Thesis

For obtaining the degree of Master of Science in Aerospace Engineering at Delft University of Technology

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11-07-2017

Faculty of Aerospace Engineering · Delft University of Technology

Delft University Of Technology
Department Of
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The undersigned hereby certify that they have read and recommend to the Faculty of Aerospace Engineering for acceptance a thesis entitled **"Analysis of Laplace operator eigenvalues using spectral element methods"** by **Talanki Naga mallika** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Summary

In order to solve a partial differential equation numerically it has to be replaced with a system of equations. The methods which can preserve these essential mathematical and physical structures are called mimetic methods.

The equation that we focus in this thesis is Laplace operator. Differential forms are used to describe the Laplace operator over spectral elements. In order to create a compatible discretization, we have to introduce a basis function that reconstructs a continuous field from integrated values. These functions are called edge functions.

In this thesis the main goal is to solve for the eigenvalues of Laplace operator in 1D, 2D domains of single element and multi element cases. Two types of formulations of Laplace operator are solved based on the type of differential form used, which are mixed and direct formulations. Mixed formulation is obtained when p-form is applied to Laplace operator. Direct formulation is obtained when 0-form is applied to Laplace operator.

The rate of convergence of eigenvalues for two different formulations of Laplace operator in single element case is exponential. In multi element case the rate of convergence based on h-refinement is around -6.84. There is alternative plus minus pattern is noticed in error values of eigenvalues.

Acknowledgements

I wish to thank the following persons Dr.Ir.M.I.Gerritsma and Ir.Varun jain for their interest in my work and the input that they had given which helped shape this tthesis to what it is. I also would like to thank my friends and my family to make everything possible for the thesis.

Delft, The Netherlands
11-07-2017

Talanki Naga mallika

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Chapter 1

Introduction

Nowadays science and engineering rely highly on numerical solutions because they do not have exact analytical solutions. For a numerical solution, partial differential equation must be replaced with a system of algebraic equations. It is really important that this system of equations give a physically and mathematically compatible system.[5].The methods which can preserve essential physical and mathematical structures in a discrete setting are called mimetic methods. Mimetic means retains analytical relations such as vector identities and integral theorems. The system of equations are discretized on spectral elements.

Spectral elements are used because these methods have exponential convergence which is the highest form of convergence available. Spectral element methods have been developed since 1971 and significant development has happened during 1980's. Spectral element methods has been a standard tool for linear variable seperable differential equation. Now spectral element methods are developed to the point where most of the problems in fluid dynamics are solved using this technique mostly.

In this thesis Laplace equation eigenvalues are analyzed because it is easier system to test. It is elliptic and it is used in many places like inverse square law theories of electromagnetism and gravitation , steady heat conduction, shape recognition e.t.c. Laplacian eigenfunctions appear as vibration modes in acoustics,as electron wave functions in quantum wave guides e.t.c. Differntial geometry is used to describe the Laplace operator and we discretize equations on spectral elements.

1.1 Research question

In this thesis the main goal is to do error analysis of Laplace operator eigenvalues using spectral element methods. Spectral element methods denotes fourier series methods, polynomial basis functions. In this thesis polynomial basis functions (Lagrangian polynomials) are used. There are two different methods to solve Laplace operator based on the p-form(differntial form). In this thesis mixed and direct formulations of Laplace equation are solved in 1D and 2D domains. The error analysis of eigenvalues is also done for multielement 1D mixed formulation. The goal is to see

1. How eigenvalues converge as compared to traditional methods like finite element methods.
2. Is there any pattern noticed in the results.
3. Are there any spurious modes in the results.
4. How does multielement case reduces error compared to single element case.
5. Spectral element methods are knonwn for exponential convergence. Is this convergence seen in the results obtained in this project.

1.2 Outline of thesis

In Literature chapter we discuss the necessary basics of differential geometry and algebraic topology, k-forms are explained by their analogues in vector calculus. We also discuss about Laplace equation and eigenvalues. In methodology chapter we discussed first how to use codifferential operator and how to create a mass matrix. Then detailed discussion of mixed and direct forms of laplace equation in 1D, 2D and multielement. The results obtained for mixed and direct forms of laplace equation in 1D, 2D and multielement cases are discussed in detail and compared with Boffis paper. In the last chapter the conclusions are made by analysisng the results and recommendations are made.

Chapter 2

Literature

2.1 State of the art

2.1.1 Algebraic topology

Differential geometry related topics are explained in detail by Flanders(1962)[20] and Bochev(2003)[24]. For Algebraic topology a detailed description is given by Tonti(1972)[15] and Mattiussi(2000)[19]. Brief description of Algebraic topology topics are discussed in further sections. Two types of basis functions are used in this thesis which are nodal basis functions and edge basis functions. Nodal basis functions are well known and explained extensively by Sherwin and Karniadakis(1999)[26]. The second type Edge basis functions are recently introduced by Gerritsma(2009)[2].

2.1.2 The approximation of Eigenvalues

A lot of research has been done on the approximation of eigenvalues in mixed form. For example Canuto(1978) to Mercier, Osborn, Rappaz and Raviart(1981)[27]. Eigenvalue Problems solved using Finite element methods by Babuska and Osborn(1991)[17].

Daniel Boffi in [23] discusses in detail about Mixed and direct formulations of Laplace eigenvalue problems in 1D and 2D. For mixed Laplace eigenvalue problem in 1D he uses P1-P0, P1-P1 and P2-P1 finite element methods. P1-P1 element approximation is used by Babuska and Narasimhan(1997)[16] and shown that it is not stable for the approximation of the one dimensionsal Laplace problem. Boffi has got better results by using P2-P0 element method. In 2D mixed Laplace eigen value problem well known mixed finite elements are Brezzi and fortin(1991)[25], Brezzi-Douglas-Marini(1985), Raviart and Thomas(1977)[27]. In Boffi paper, he uses Raviart and Thomas elements.

Partial Differential Equations (PDE's) solutions obtained by numerical methods have to be accurate and mathematically,physically consistent. Methods which can preserve these kind of qualities are called mimetic methods. Mimetic means retaining analytical relations such as vector identities and integral theorems. Mimetic discritizations give the relation between differential geometry and algebraic topology. Most of the structures are topological i.e independent of metric and involve integral relations. Integration is done using differential forms . Differential forms provide a framework from which to analye PDE's.[8] Differential forms can work at high dimensions and independent from the coordinate system. But scientists doesnot want higher order dimensions, since navier stokes equations and other kind of equations mostly require 3 dimensions which is not very high dimensionality. But the reason to use differential forms(differential geometry) is the clarity that they give in discritization process while describing the numerical solution of PDE's. Exterior calculus and differential geometry describes the relationship between different operators and operands. Numerical methods developed using exterior calculus are more mathematically and physically consistent.[8][9]

2.2 Differential geometry

2.2.1 Differential forms

Compatible spatial discritizations are those that inherit or mimic fundamental properties of the PDE such as topology, conservation, symmetries and positivity structures and maximum principles[1].

For example temperature and density are both scalars. But $T(x, y, z)$ is defined at each point and $\rho(x, y, z)$ is assosiated with the volume. Both variables are represented by a scalar field but one refers to point values(temperature) and the other to mass per volume(density). The same holds for vectors where it is not always clear to which geometric objects the quantities are associated. Hence traditional vector calculus is inadequate to give a full representation of the physical variable and its relation to the geometry. Differential geometry is used to rephrase differential equations in such a way that its geometric content is preserved.

Differential forms are the main building blocks for differential geometry. The generalization from differentials to differential forms and the assosciated vector calculus makes use of three operations the wedge product,the exterior derivative and the hodge dual.

Differential forms come in several types. Functions are called 0-forms,line elements 1-forms,surface elements 2-forms and volume forms are called 3-forms. These are all the types that exist in three dimensions. There are p-forms with p ranging from zero to the dimension d of the space.

In $\mathbb{R}^3$ a 0-form is given by

$$\phi^0 = f(x, y, z). \tag{2.1}$$

In $\mathbb{R}^3$ a one form is given by

$$\lambda^1 = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz. \tag{2.2}$$

Typical of the 1-form is associated with line segements more precisely with line integrals. The coefficients A,B,C form the vector proxy of one form. Classical vector calculus only works with these vector proxies and ignores the basis vectors dx, dy and dz ,which tells us that this vector is assosciated with one dimensional objects.

A 2-form is given by

$$\eta^2 = P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy. \tag{2.3}$$

dxdy is a formal antisymmetric product called the wedge product,symbolized by $'\wedge'$. Thus surface integrals will be written as integrals over the products $dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$. $'\wedge'$ is antisymmetric.

$$dx \wedge dy = -dy \wedge dx. \tag{2.4}$$

under the interchange of any two basis forms. This automatically gives the right orientation of the surface. 2-forms can be integrated over the surfaces.

The 3-form is given by

$$\omega^3 = g(x,y,z)dxdydz. \tag{2.5}$$

which changes sign if any pair of the basis elements are switched.

The 3-forms are assosciated with integration over volumes.

Both zero and three forms are described by scalar fields f and g respectively. But there is a different association with respect to geometry.

2.2.2 Exterior calculus

Wedge product

The product between two differential k-forms α and β of degrees p and q in an exterior algebra is called wedge product.

$$\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha. \tag{2.6}$$

The above equation (2.6) shows that wedge product is not commutative, which means it is an antisymmetric product.

Wedge products are used to calculate the determinants. It is shown in the following example of wedge product between two one forms in $\mathbb{R}^3$

$$a^{(1)} \wedge b^{(1)} = (a_x dx + a_y dy + a_z dz) \wedge (b_x dx + b_y dy + b_z dz) \tag{2.7}$$

$$a^{(1)} \wedge b^{(1)} = a_x b_x (dx \wedge dx) + a_x b_y (dx \wedge dy) + a_x b_z (dx \wedge dz) + ... \tag{2.8}$$

$$= a_y b_x dy \wedge dx + a_y b_y dy \wedge dy + a_y b_z dy \wedge dz + ... \tag{2.9}$$

$$= a_z b_x dz \wedge dx + a_z b_y dz \wedge dy + a_z b_z dz \wedge dz + ... \tag{2.10}$$

From above equations (2.8),(2.9) and(2.10) it is noticed that

$$dx \wedge dx = dy \wedge dy = dz \wedge dz = 0 \tag{2.11}$$

$$dx \wedge dz = -dz \wedge dx, dy \wedge dx = -dx \wedge dy, dz \wedge dy = -dy \wedge dz \tag{2.12}$$

by applying the changes as mentioned in (2.11) and (2.12) to (2.8),(2.9) and(2.10), the following equation is obtained

$$a^{(1)} \wedge b^{(1)} = (a_x b_y - a_y b_x) dx \wedge dy + (a_z b_x - a_x b_z) dz \wedge dx + (a_y b_z - a_z b_y) dy \wedge dz \tag{2.13}$$

The wedge product acts just like the exterior product of the proxies of the 1-forms.

Exterior derivative

The operator exterior derivative transforms p-form to p+1 form. It is denoted as $'d'$. The way exterior derivative acts on different p-forms can be related to the vector calculus operations.

When exterior derivative operating on the 0-form is similar as gradient of scalar field in $\mathbb{R}^3$.

$$d\phi^0 = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \tag{2.14}$$

whose vector proxy is,

$$\bigtriangledown \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \tag{2.15}$$

Similarly when exterior derivative is applied to one form for which vector proxy is curl. From vector calculus it is known that $div \times curl = 0$ and $curl \times grad = 0$.

$$dd\omega^{(k)} = 0^{(k+2)} \tag{2.16}$$

Hence in differential geometry also if exterior derivative is applied twice results in zero.

Hodge operator

The exterior derivative sequence like

$$\mathbb{R} \hookrightarrow \Lambda^0(\Omega) \overset{d}{\mapsto} \Lambda^1(\Omega) \overset{d}{\mapsto} \Lambda^2(\Omega) \overset{d}{\mapsto} \Lambda^3(\Omega) \overset{d}{\mapsto} 0,$$

The De-rham complex represents the fact that the exterior derivative maps k-forms into k+1-forms. We cannot apply div directly to a grad, because the divergence operates on surfaces,yet the grad maps onto lines.

By combining De-rham complex sequences in both ways(i.e normal and reverse) and define a map that connects them.

$$\begin{array}{ccccccccccc}
\mathbb{R} \longrightarrow & \Lambda^0(\Omega) & \xrightarrow{d} & \Lambda^1(\Omega) & \xrightarrow{d} & \Lambda^2(\Omega) & \xrightarrow{d} & \Lambda^3(\Omega) & \xrightarrow{d} & 0 \\
 & \star\updownarrow & & \star\updownarrow & & \star\updownarrow & & \star\updownarrow & & \\
0 \xleftarrow{d} & \tilde{\Lambda}^3\Omega) & \xleftarrow{d} & \Lambda^2(\Omega) & \xleftarrow{d} & \Lambda^1(\Omega) & \xleftarrow{d} & \Lambda^0(\Omega) & \longleftarrow & \mathbb{R}
\end{array}$$

From above sequence we can see that 0-forms map on to 3-forms, 1-forms map onto 2-forms and in reverse. That map is called the hodge operator($\star$). In terms of differential geometry the Laplace operator operating on a 0-form is then written as $d \star d\phi^0$.

When we apply the hodge to a k-form we essentially map information from one geometrical object to another.

The mappings of hodge operator are

$$\star 1 = dxdydz. \star dxdydz = 1. \star dx = dydz. \star dy = dzdx. \star dz = dxdy. \tag{2.17}$$

For example when hodge operator is applied to one form

$$u^1 = u_x dx + u_y dy + u_z dz. \star u^1 = u_x dydz + u_y dzdx + u_z dydx \tag{2.18}$$

Codifferential operator

The operator codifferential maps k-forms into (k-1) forms. It is denoted as d^*. The operation of codifferential operator is $d^* = (-1)^{(k+1+k(n-k))} * d*$ where d is the exterior derivative. The codifferential operator is the adjoint of the exterior derivative $(d^*\zeta, \eta) = (\zeta, d\eta)$. Codifferential operator is associated to the vector operators grad,curl and div. The major differences between exterior derivative and codifferential operator are,

1. The exterior derivative is purely topological, but codifferential operator depends on hodge operator which is metric dependent.
2. Codifferential operator does not satisfy generalized Stokes theorem where as exterior derivative satisfies it.
3. Codifferential operator does not commute with reduction, reconstruction and the projection, where as exterior derivative commutes with reduction, reconstruction and the projection.

Hence the above differences between codifferential operator and exterior derivative shows that, codifferential operator is not friendly as compared to exterior derivative. Whenever we come across a gradient in physical model ,it is required to determine whether the problem is the representation of exterior derivative or codifferential operator. If the gradient is representation of codifferential operator ,then it can be changed as exterior derivative with boundary integral by using integration by parts. This operation is done in the example problem .

2.2.3 Basis functions

The word spectral element method denotes Fourier series methods, but now a days spectral element method also includes polynomial basis functions. Spectral methods converge faster because these methods converge exponentially[2]. There are two types of basis functions used in this thesis

1. Nodal basis functions
2. Edge basis functions

Nodal basis functions

Nodal basis functions are a set of normalized unique Lagrangian polynomials. A function is expanded using nodal basis functions. For the basis of Lagrangian polynomials some form of the Legendre polynomial is used. One such basis function is Legendre-Gauss-Lobatto basis function. These basis functions ($h_i(\xi)$) are associated to the GL nodes (ξ_i) for which hold that

$$(1-\xi_i)^2 L'_N(\xi_i) = 0, for - 1 \leq \xi_i \leq 1, \tag{2.19}$$

where L'_N is the derivative of the Nth order Legendre polynomial. The basis functions which are constructed using

$$h_i(\xi) = \frac{(1-\xi^2)L'_N\xi}{N(N+1)L_n\xi_i(\xi_i-\xi)}, \tag{2.20}$$

$i = 0, ..., N$ The nodal basis functions orthogonal property is expressed by

$$h_i(\xi_k) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$$

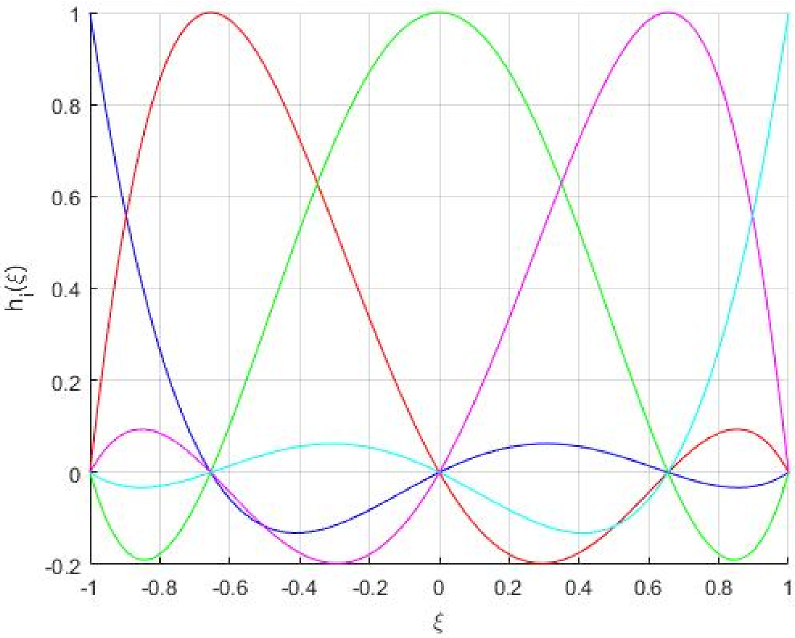


Figure 2.1: Nodal basis functions.

The above figure 2.1 shows nodal basis functions for polynomial degree N=5. Nodal basis functions are used to approximate a function $f(\xi)$ on the interval $-1 \leq \xi \leq 1$ by using

$$f(\xi) \approx f^N(\xi) = \sum_{i=1}^{N+1} f(\xi_i)h_i(\xi). \tag{2.21}$$

Edge functions

Lagrange polynomials are 0-form themselves. Lagrange polynomials are used to reconstruct a 0-form from a 0-cochain. Gerritsma(2009) and Robidoux(2009)[2] derived a similar projection of 1-form consisting of 1-cochain polynomials that is called edge polynomials. Hence edge polynomials are used to reconstruct a 1-form from a 1-cochain.

Consider the one dimensional equation

$$u(\xi) = \frac{d\phi}{d\xi}, \xi \in [a, b]. \tag{2.22}$$

where $a = \xi_0 < \xi_1 < ... < \xi_N = b$

$$\phi(\xi) = \sum_{i=0}^{N} \phi_i h_i(\xi). \tag{2.23}$$

where $h_i(\xi)$ are the lagrange basis functions.

where $'u'$ is a one form function and ϕ is a zero form function.

$$u^N(\xi) = \sum_{i=0}^{N} (\phi_i - \phi_k) e_i(\xi), \tag{2.24}$$

where $\bar{u_i} = (\phi_i - \phi_k)$

$$u^N(\xi) = -\sum_{j=1}^{k} (\sum_{i=0}^{j-1} dh_i(\xi)) \vec{u_j} + \sum_{j=k+1}^{N} (\sum_{i=j}^{N} dh_j(\xi)) \vec{u_j}. \tag{2.25}$$

hence

$$e_j(\xi) = -\sum_{i=0}^{j-1} dh_i(\xi). \tag{2.26}$$

where $e_j(\xi)$ are edge basis functions. The edge basis functions are constructed such that

$$\int_{\xi_{k-1}}^{\xi_k} e_i(\xi) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$$

$$u^N(\xi) = \sum_{i=1}^{N} u_i e_i(\xi). \tag{2.27}$$

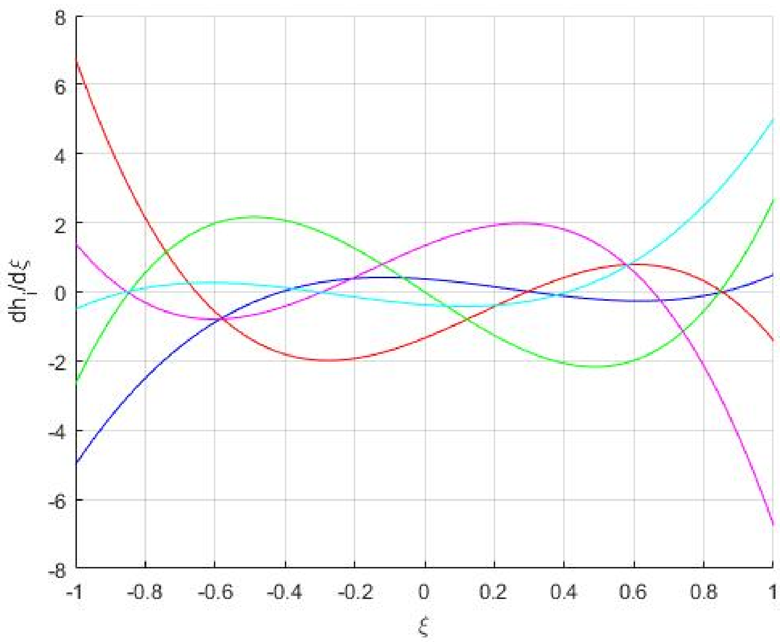


Figure 2.2: Edge functions.

Edge functions for polynomial degree N=5 are shown in the figure 2.2.

2.2.4 Algebraic topology

A discrete system of differential forms is required in order to find good approximate solution for PDE. A good discrete system setup involves right use of shape of grid on which the differential forms are discretized.

Chains and cells

The grid consists of different kind of cells in $\mathbb{R}^3$ each kind represents a node, an edge, a surface or a volume which are denoted as 0-cell, 1-cell, 2-cell or a 3-cell. k-cell is denoted with the symbol σ^k. A linear combination of oriented k-cells is called a k-chain which is expressed as

$$C_k = \sum_{i=1}^{r_k} m_i \sigma_i^k. \tag{2.28}$$

where r_k denotes the number of k-cells that make up the k-chain and the multiplicity m_i takes values -1,1 and 0 where 0 means that the cell is not part of the chain.

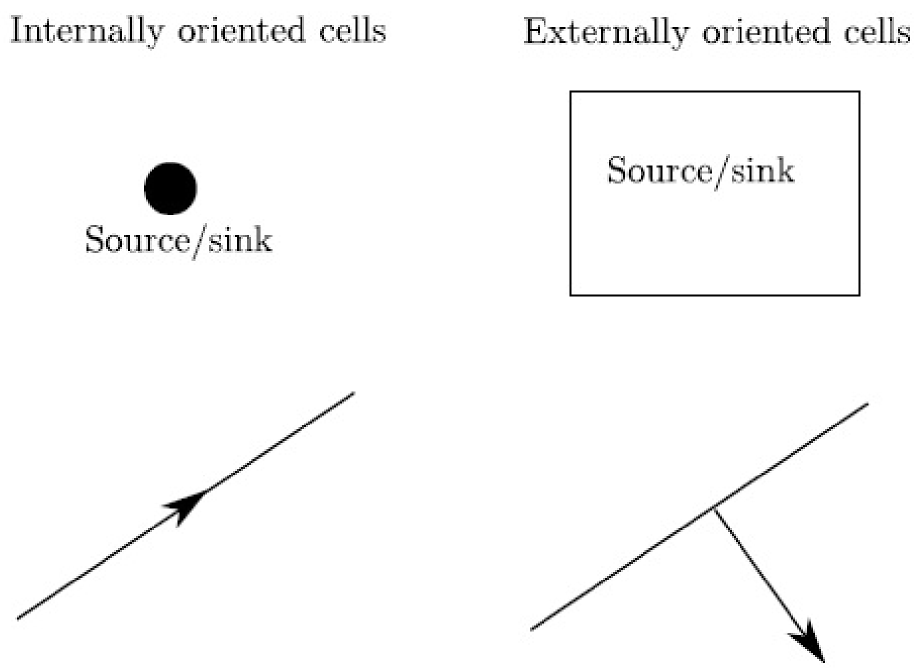


Figure 2.3: orientations of the Node,Edge and surface in $\mathbb{R}^2$

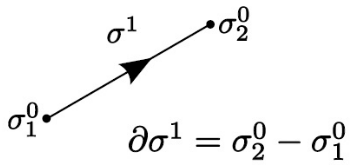
The orientation of a k-cell can be internal or external, the orientations important throughout this thesis are given in the figure 2.3. For example the surface can act like a source or a sink likewise the node. The internally oriented edge can be associated with gradients. The externally oriented edge with fluxes.

Now for $1 \le k \le n$ the boundary of a k-cell, σ^k consists of (k-1) chain.

$$\partial C_k = \sum_{i=1}^{r_k} m_i \partial \sigma_i^k. \tag{2.29}$$

The boundary of a k-cell is denoted by $\partial \sigma_i^k$, where the symbol ∂ represents the boundary operator. As an example, the boundary of an oriented volume (3-cell) is a collection of oriented surfaces (2-cells). The boundary of an oriented 2-cell is given by a collection of oriented lines (1-cells) and the boundary of an oriented edge is given by points (0-cells) as shown in the figure 2.4. The boundary of the boundary of any cell complex is empty

$$\partial \partial C_k \equiv 0. \tag{2.30}$$



empty,

Figure 2.4: Example of the boundary operator operating on 1-cell

One can think of the k-cells as the nodes, edges, surfaces and volumes (the geometrical objects) that make up a grid that is used in a numerical computation. Important is that in algebraic topology it is only the connectivity that matters. There is no notion of distance or angle. A grid can be coarse or dense, in algebraic topology this has no meaning. In the introduction of this section we talked about the shape of the grid, it is better to talk of connectivity. Since shape also has the notion of curves, distances and angles. But changing distances and angles in a grid does not influence the connectivity.

Cochains

Now that we have defined the k-chain we can associate values to these chains. Lets associate to a each k-cell in a chain (σ^k) the integral of k-form($\omega^{(k)}$). The assosciation

$$C_k \longrightarrow \int_{(\sigma_i^k)} \omega^k, \tag{2.31}$$

between a k-form and its representation on a k-chain is called cochain and is denoted by C_k^*. Next we define an operator δ, called the coboundary operator. Let C_{k+1} be a (k+1)-chain then its boundary ∂C_{k+1} is a k-chain (C_k) and we can formally write,

$$\langle \partial C_{k+1}, C_k^* \rangle = \langle C_{k+1}, \delta C_k^* \rangle, \tag{2.32}$$

where the coboundary operator δ is the formal adjoint of the boundary operator ∂, we can also interpret this as the discrete version of the Stokes theorem where δ assumes the role of the exterior derivative d. Note that,

$$\delta : C_k^* \longrightarrow C_{k+1}^*. \tag{2.33}$$

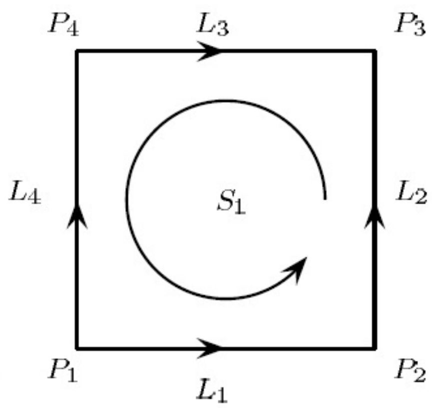
Similarly like exterior derivative,

$$\delta\delta C_k^* \equiv 0. \tag{2.34}$$

Remember that cell complex is a topological grid on which the cochains are defined that contain the information from k-forms.

2.2.5 Incidence Matrices

The action of coboundary operator on cochains can be seen as a matrix operator with the vector containing the cochains (reduced k-forms). These matrices are called incidence matrices, they are purely topological and remain unchanged under deformation of the cell-complex. As long as the topology (connection of points,lines and surfaces) remains the same.



empty,

Figure 2.5: Example of 2-cell

The incidence matrix $E^{1,0}$ relates 0 cochains to 1 cochains, i.e. this matrix is the discrete gradient operator. In this matrix, each row represents a 1-cochain. And such a 1-cochain consists of two 0-cochains, leaving a node results in -1 and arriving at a node in a+1. The value 0 indicates that the 0-cochain is not part of the boundary of the 1-cell. The matrix $E^{2,1}$ relates 1-cochains and 2-cochains, it can be seen as the discrete curl operator. Here a-1 corresponds to a line with an orientation in the opposite direction as the positive rotation, and a+1 to a line with an orientation in the same direction as the positive rotation as shown in the figure above.

$$E^{1,0}=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$E^{2,1} = [11 - 1 - 1]$$

Differential geometry and algebraic topology connected

When differential forms are expressed as k-cells representation then we are reducing contionous k-forms to discrete values using integration. The integration or reduction of a k-form ω onto a k-chain c as

$$\langle R, c\rangle = \int_c \omega \tag{2.35}$$

In this case c is a metric chain, not solely a topological chain. The mapping $\omega \longrightarrow R_\omega$ establishes discrete representation of k-forms in terms of global quantities assosciated with a chain complex. Thus, we encode discrete k-forms as k-cell quanties, k-cochains. this mapping has a property, which is known as the commuting diagram property, that is given by

$Rd = \delta R$

While solving a PDE, the continous k-forms are reduced to a discrete representation on a cell complex. The reconstruction process which is used to reconstruct continous k-forms from this discrete representation is denoted by symbol I. Reconstruction or interpolation is an approximate left inverse of reduction. Reduction is a straight forward process just integrating k-forms where as interpolation can be done in mulple ways and depends highly on the shape of the cell complex on which differential forms are discretized. This thesis however discusses the use of spectral elements for solving the poisson equation, this means that our interpolators are combinations of lagrangian interpolators based on the legendre polynomials. For mimetic reconstruction operators I we impose a condition that serves to coordinate the action of the exterior derivative and the coboundary operator, this condition is known as the second commuting diagram property,

$dI = I\delta$

such a map is called a conforming reconstruction operator.

2.3 Laplace operator and eigenvalues

2.3.1 Laplace operator

Scalar Laplacian is one of the important differential operators in vector calculus. Scalar Laplacian is denoted by $\bigtriangledown^2$. Scalar Laplacian is divergence of gradient. Laplacian is used in many fundamental concepts like any diffusion process and all forms of wave propagation including the Schrodinger equation. The spectrum of Laplacian gives all the details that is to be known about a given piece of geometry. Since Laplacian is a self adjoint operator it has a real spectrum. Hence understanding the spectrum of Laplacian is important. Laplacian eigen functions appear as vibration modes in acoustics, as electron wave functions in quantum wave guides, as natural basis for constructing heat kernels in the theory of diffusion e.t.c[5].

Inorder to analyze the various PDEs(Partial differential equation) that involve $\triangle$ (laplace) operator we need to understand the spectrum of laplacian. The spectrum of Laplacian operator changes as there is a change in region(Ω) and boundary conditions. For any function 'u' satisfying boundary conditions that make $'\Delta'$ self adjoint integration by parts gives

$$(u^\star, \Delta u) = \int_\Omega u^\star(\bigtriangledown^2 u)d\Omega = -\int_\Omega (\vec{\bigtriangledown} u^\star).(\vec{\bigtriangledown} u)d\Omega < 0 \tag{2.36}$$

Since scalar Laplacian is a self adjoint operator, surface integral term vanishes.

If the region Ω is bounded then $\triangle$ has a discrete spectrum. If Ω is unbounded then $\triangle$ has a continous spectrum and the infinite series is replaced by an integral over the continous spectrum.

The laplace beltrami operator is self adjoint and semi positive definite. It admits an ortho normal eigen system that is a basis of the space of square integrable function[18].

In general two types of boundary conditions are applied to laplace equation

1. Neumann boundary conditions
2. Dirchlet boundary conditions

General properties of eigen values of Laplace equation under neumann and dirchlet boundary conditions::

1. The boundary integral vanishes for both dirchlet and neumann boundary conditions.
2. For dirchlet boundary condition the minmax principle implies the property of domain monotonocity i.e eigenvalues monotonously decrease when the domain enlarges. This property doesnot hold for neumann boundary conditions.
3. The eigenvalues are invariant under translations and rotations of the domain. This is the key property for an efficient image recognition and analysis. When the domain is expanded by factor α, all the eigen values are rescaled by $1/\alpha^2$.
4. The first eigen value λ_1 is simple and strictly positive for dirchlet boundary conditions. For neumann boundary conditions $\lambda_1 = 0$.
5. For dirchlet boundary condition, the eigen values vary continously under a continous perturbation of the domain.

The general way to solve for eigenvalues of Laplace equation is method of seperation of variables.

2.3.2 Generalized Eigenvalue Problem

Consider a matrix $'A'$ which is of size $n \times n$.

$$Ax_i = \lambda_i x_i. \tag{2.37}$$

The above equation (2.37) is used to determine scalars(eigenvalues) and vectors x_i(eigenvectors). Eigenvalues should be computed in such a way that it satisfies the condition $det(A-\lambda_i I) = 0$. $det(A - \lambda_i I) = 0$ is a polynomial in λ of degree n, thus there are n eigenvalues of A. No need for all eigenvalues to be distinct.

If A consists of only real numbers, then the eigenvalues are real or complex conjugates. For each distinct eigenvalue there must exist atleast one eigenvector.

If eigenvalues are complex conjugate then eigenvectors are complex conjugates.

If $'A'$ is real and symmetric all n of the eigenvalues are real numbers(i.e no complex conjugates). For $i \neq j$ $x_i^T x_j = 0$ then all eigenvectors are orthogonal.

Consider two matrices $'A'$ and $'B'$ of size $n \times n$.

$$Ax_i = \lambda_i B x_i. \tag{2.38}$$

The above equation (2.38) is called generalized eigenvalue problem. The above equation (2.38) is used to determine scalars(eigenvalues) and vectors x_i(eigenvectors). Eigenvalues obtained in such a way that it should satisfy the condition $det(A - \lambda_i B) = 0$.

If B is non singular then the problem is equivalent to $B^{(-1)}Ax_i = \lambda_i x_i$ and all theory and properties of standard eigenvalue problem are applied to $B^{(-1)}A$.

If A is nonsingular then problem is equivalent to $A^{(-1)}Bx_i = \frac{1}{\lambda_i}x_i$ and all theory and properties of standard eigenvalue problem apply to $A^{(-1)}B$.

If B is singular, zero eigenvalues of $A^{(-1)}B$ correspond to infinite eigenvalues of the generalized eigenvalue problem.

If A and B are both singular, then there are two cases

1. $det(A - \lambda_i B)$ is a polynomial and there are n eigenvalues (some maybe zero or infinite).

2. $det(A - \lambda_i B)$ identically equals zero and every scalar in an eigenvalue e.g if z is in the nullspace of both A and B, then $Az = \lambda Bz$ for all scalar eigenvalues.

Chapter 3

Methodology

The following problem is solved first to understand codifferential operator indetail.

3.1 Usage of codifferential operator

$$d^{\star}u^{(1)} = f^{(0)}. \tag{3.1}$$

Codifferential operator is applied to function in 1-form which maps the function in 0-form. Dirichlet boundary conditions are applied to the equation. The domain range is $[-1,1](\Lambda^{(0)}(\Omega) = [-1,1])$.

(3.1) equation is multiplied with the trial function $v^{(0)}$ it can be written as

$$(d^{\star}u^{(1)}, v^{(0)}) = (f^{(0)}, v^{(0)}).\forall v^{(0)} \in \Lambda^{(0)}(\Omega) \tag{3.2}$$

As discussed previously in section 2.1.2 codifferential operator can be changed into exterior derivative which has much nicer properties by using integration by parts. After applying integration by parts to the equation ((3.2)) it can be written as

$$\int_{-1}^{1} u^{(1)}v^{(0)} - (u^{(1)}, dv^{(0)}) = (f^{(0)}, v^{(0)}).\forall v^{(0)} \in \Lambda^{(0)}(\Omega) \tag{3.3}$$

For Dirichlet boundary conditions the boundary integral $\int_{-1}^{1} u^{(1)}v^{(0)}$ vanishes and the above equation(3.3) can be written as

$$-(u^{(1)}, dv^{(0)}) = (f^{(0)}, v^{(0)}) \tag{3.4}$$

1-form can be written as follows using basis functions $\sum_{i=1}^{N} \bar{u}_i e_i(x)$. Since it is 1-form ,edge basis functions are used. 0-form can be written as follows using basis functions $\sum_{i=1}^{N} \bar{v}_i h_i(x)$. Since it is 0-form ,nodal basis functions are used. When the exterior derivative is applied to nodal basis functions, it changes to edge basis functions and incidence matrix.

$$-(u^{(1)}, dv^{(0)}) = -v^T E^{(1,0)T} (e_i)^T (e_j) u \forall v^{(0)} \in \Lambda^{(0)}(\Omega) \tag{3.5}$$

$$-v^T E^{(1,0)T} (e_i)^T (e_j) u = v^T (h_i)^T (h_j) f \forall v^{(0)} \in \Lambda^{(0)}(\Omega) \tag{3.6}$$

$(e_i), (e_j)$ are edge basis functions. $(h_i), (h_j)$ are nodal basis functions. The product of any two basis functions integrated over the domain produces an entry in the mass matrix. $\int_k (e_i)^T (e_j)$ gives mass matrix in one form and $\int_k (h_i)^T (h_j)$ gives mass matrix in zero form.

$$-v^T E^{(1,0)T} M^{(1)} u = v^T M^{(0)} f \forall v^{(0)} \in \Lambda^{(0)}(\Omega). \tag{3.7}$$

After removing trial functions on both sides

$$-E^{(1,0)T} M^{(1)} u = M^{(0)} f. \tag{3.8}$$

Inorder to make the equation symmetric on both sides

$$-M^{(1)} E^{(1,0)} M^{(0)^{-1}} E^{(1,0)T} M^{(1)} u = M^{(1)} E^{(1,0)} f. \tag{3.9}$$

3.1.1 Exact solution

Exact solution for codifferential problem is solved to compare it with the calculated solutions which are obtained from previous section. The function used for exact solution is given by $u^{(1)} = sin(2\pi x)dx$.

codifferential operator is given by

$$d^* = (-1)^{(n(k+1)+1)} \star d \star u^{(1)}, \tag{3.10}$$

where $n = 1$ and $k = 1$ (Hodge operations map from k-form to $(n-k)$form. To calculate $f^{(0)}$

$$u^{(1)} = \sin(2\pi x)dx, \tag{3.11}$$

$$\star u^{(1)} = \sin(2\pi x), \tag{3.12}$$

$$d \star u^{(1)} = 2\pi \cos(2\pi x)dx, \tag{3.13}$$

$$\star d \star u^{(1)} = 2\pi \cos(2\pi x), \tag{3.14}$$

$$(-1)^{(n(k+1)+1)} \star d \star u^{(1)} = -2\pi \cos(2\pi x), \tag{3.15}$$

Hence

$$d^* u^{(1)} = (-1)^{(n(k+1)+1)} \star d \star u^{(1)} = -2\pi \cos(2\pi x) = f^{(0)}. \tag{3.16}$$

3.2 Introduction to Mixed and Direct formulations

$$\triangle = d^*d + dd^* \tag{3.17}$$

In the above equation (3.17) Δ represents a Laplace operator. It is known that Laplace maps k-forms into k-forms. We know that $d\alpha^n = 0$ and $d^*\beta^0 = 0$.

When a Laplacian is applied to a 0-form

$$\triangle\phi^0 = dd^*\phi^0 + d^*d\phi^0 = d^*d\phi^0 = 0. \tag{3.18}$$

The above equation (3.18) is multiplied with a test function ψ^0. Then do the integration by parts

$$(d^*d\phi^0, \psi^0)_\Omega = -(d\phi^0, d\psi^0)_\Omega = 0. \tag{3.19}$$

For Dirchlet boundary conditions the boundary integral $\int_L^R \phi^0\psi^0$vanishes. This is called direct formulation.

When n-form is applied to laplacian

$$\triangle\phi^{(n)} = dd^*\phi^n + d^*d\phi^n = dd^*\phi^n = 0. \tag{3.20}$$

Multiply (3.20) with a test equation σ^n and do integration by parts. It would give two codifferentials. It is already known section 2.1.2 that codifferential operator does not have nicer properties compared to exterior derivative. Hence the equation (3.20) has to be solved by two different equations. Those two equations are given below,

$$p^{(n-1)} = d^*\phi^n, \tag{3.21}$$

$$p^{(n-1)} - d^*\phi^n = 0, \tag{3.22}$$

$$dp^{(n-1)} = 0. \tag{3.23}$$

Hence when n-form is applied to Laplacian,(two different equations need to be solved) the procedure is called mixed method.When 0-form is applied to Laplacian ,the procedure is called direct method. One has to be careful when applying n-form or 0-form to Laplacian and make sure which method to use.

3.3 Mixed Method

Poisson equation in 1-form is solved in this section. From the previous section it is known that n-form results in two codifferentials if it is tried to solve directly. Hence mixed method approach is used. The domain is $\Omega = [-1, 1]$. A detailed discussion of poisson equation in a mixed method approach is presented below

$$\frac{d^2\phi^{(1)}}{dx^2} = f^{(1)}. \tag{3.24}$$

$$dd^{*}\phi^{(1)} = f^{(1)}. \tag{3.25}$$

The above problem has two functions $'\phi'$ is known data and $'f'$ is unknown data.

The above equation (3.24) can be written as two different equations to reduce two codifferentials to one codifferential.

$$u^{(0)} - d^{\star}\phi^{(1)} = 0, \tag{3.26}$$

$$du^{(0)} = \lambda f^{(1)}. \tag{3.27}$$

$\Sigma = H\Lambda^{0}(\Omega)$ and $U = L^{2}\Lambda^{1}(\Omega)$ Mutiplying equations (3.26) and (3.27) with trial functions and integrating over the domain.

$$(u^{(0)} - d^{\star}\phi^{(1)})\tilde{u}^{(0)} = 0, \forall\tilde{u}^{(0)} \in \Sigma, \tag{3.28}$$

$$(du^{(0)})\tilde{u}^{(1)} = \lambda f^{(1)}\tilde{u}^{(1)}.\forall\tilde{u}^{(1)} \in U. \tag{3.29}$$

After integration by parts for (3.28) and (3.29) the remaining codifferential has changed into exterior derivative

$$(u^{(0)}, \tilde{u}^{(0)}) + (\phi^{(1)}, d\tilde{u}^{(0)}) = 0, \forall\tilde{u}^{(0)} \in \Sigma. \tag{3.30}$$

$$((du^{(0)}), \tilde{u}^{(1)}) = \lambda(f^{(1)}, \tilde{u}^{(1)}).\forall\tilde{u}^{(1)} \in U. \tag{3.31}$$

After writing in terms of basis functions to the above equations (3.30) and (3.31)

$$(\tilde{u}^{T}h_i^{T}h_j u) + (\tilde{u}^{T}E^{(1,0)T}e_i^{T}e_j\phi) = 0, \forall\tilde{u}^{(0)} \in \Sigma, \tag{3.32}$$

$$(\tilde{u}^{T}E^{(1,0)}e_i^{T}e_j u) = \lambda\tilde{u}^{T}e_i^{T}e_j f.\forall\tilde{u}^{(1)} \in U. \tag{3.33}$$

$$M^{(0)}\vec{u} + E^{(1,0)T}M^{(1)}\vec{\phi} = 0, \tag{3.34}$$

$$M^{(1)}E^{(1,0)}\vec{u} = \lambda M^{(1)}\vec{f}. \tag{3.35}$$

The equations (3.34) and (3.35) are in the form of mixed Laplace eigenvalue problem as mentioned in Boffi paper. The equations (3.34) and (3.35) are in the following format as mentioned in Boffi's PAPER

$$\begin{bmatrix} A & B^{T} \\ B & 0 \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix}$$

where $'A'$ is a mass matrix in zero form is M^{0} and $'B'$ is $M^{1}E^{(1,0)}$ and M is M^{1} mass matrix in one form. The homogeneous Dirichlet boundary conditions are enforced,so that the matrices A,B and M do not include boundary conditions.

1. The domain taken is [-1,1] in my program, in Boffi's paper the domain is $[0, \pi]$
2. The number of subdivisions is equal to the $polynomialdegree(N) + 1$.
3. It gives N eigenvalues.
4. The exact eigenvalues are given by $\frac{N^2\pi^2}{4}$ for the domain $[-1, 1]$. The corresponding eigen spaces are spanned by the eigen functions $\sin(\frac{N\pi x}{2})$. In Boffi's paper, the domain is $[0, \pi]$. Hence The exact eigenvalues are given by N^2 because the domain is $[0, \pi]$. The corresponding eigen spaces are spanned by the eigen functions $\sin(Nx)$.
5. We used 'eigs' MATLAB function.
6. The analysis has started from polynomial degree N=8 because we require minimum six eigen values for single element.
7. The polynomial degree is varied from $N = 8$ to $N = 13$.

3.3.1 Exact solution

Exact solution for Poisson problem is solved to compare it with the calculated solutions which are obtained from previous section. The function used for exact solution is given by $\phi^{(1)} = sin(2\pi x)dx$. Exact solution of Poisson equation in 1D.

$$\frac{d^2\phi^{(1)}}{dx^2} = f^{(1)}. \tag{3.36}$$

$$dd^*\phi^{(1)} = f^{(1)}. \tag{3.37}$$

$$\phi^{(1)} = \sin(2\pi x)dx, \tag{3.38}$$

$$*\phi^{(1)} = \sin(2\pi x), \tag{3.39}$$

$$d * \phi^{(1)} = 2\pi\cos(2\pi x)dx, \tag{3.40}$$

$$*d * \phi^{(1)} = 2\pi\cos(2\pi x), \tag{3.41}$$

$$d * d * \phi^{(1)} = -4\pi^2\sin(2\pi x)dx = f^{(1)}. \tag{3.42}$$

3.4 Direct method

Solve the Poisson equation problem in 0-form in this section. This problem has only one codifferential which is easy to change to exterior derivative by doing integration by parts.

This can be done directly with the same original equation unlike mixed method where the equation has to be broken down to two different parts. The domain is $\Omega = [-1, 1]$.

$$d^{\star} d\phi^{(0)} = \lambda f^{(0)}. \tag{3.43}$$

Multiplying the equation (3.43) with the trail function $(\phi^{\tilde{(0)}})$. $\Sigma = H\Lambda^0(\Omega)$

$$(d^{\star} d\phi^{(0)}, \tilde{\phi}^{(0)})_{\Omega} = \lambda(f^{(0)}, (\phi^{\tilde{(0)}}))_{\Omega}, \forall(\phi^{\tilde{(0)}}) \in \Sigma. \tag{3.44}$$

by doing integration by parts the codifferential is changed into exterior derivative

$$(d\phi, d(\tilde{\phi}))_{\Omega} = \lambda(f, \tilde{(}\phi))_{\Omega}, \forall(\phi^{\tilde{(0)}}) \in \Sigma. \tag{3.45}$$

The above equation (3.45) is written in form of basis functions

$$\phi^{\tilde{(0)}} E^{(1,0)T} e_i^T e_j E^{(1,0)} \phi^{(0)} = \lambda \phi^{\tilde{(0)}} h_j^T h_i f^{(0)}, \forall(\phi^{\tilde{(0)}}) \in \Sigma. \tag{3.46}$$

The incidence matrix $E^{(1,0)T}$ connects the test function $\phi^{\tilde{(0)}}$ to e_i^T and the incidence matrix $E^{(1,0)}$ connects e_j and $\phi^{(0)}$.

After removing test functions

$$-E^{(1,0)T} M^{(1)} E^{(1,0)} \phi^{(0)} = \lambda M^{(0)} f^{(0)} \tag{3.47}$$

The above equation 3.47 is in similar format as generalized eigenvalue problem $Av = \lambda Bv$. $'A'$ matrix is $-E^{(1,0)T} M^1 E^{(1,0)}$ and $'B'$ matrix is M^0.A and B are both symmetric matrices.(The dimensions of A and B matrices is equal to polynomial degree).

1. The domain consisdered in this program is $[-1, 1]$.
2. It gives N eigenvalues.The exact eigenvalues are given by $\frac{N^2\pi^2}{4}$ because the domain is $[-1, 1]$.
3. The corresponding eigen spaces are spanned by the eigen functions $\sin(\frac{N\pi x}{2})$.
4. We used 'eigs' MATLAB function.
5. The first six eigen values are tabulated in the table starting from the polynomial degree $N = 8$. The polynomial degree is varied from $N = 8$ to $N = 13$.
6. The homogeneous dirchlet boundary conditions are enforced, so that the matrices A,B and M do not include boundary conditions.

3.4.1 Exact solution

Exact solution for Poisson problem is solved to compare it with the calculated solutions which are obtained from previous section. The function used for exact solution is given by $\phi^{(0)} = sin(2\pi x)$. Exact solution of Poisson equation in 0-form.

$$\frac{d^2\phi^{(0)}}{dx^2} = f^{(0)}. \tag{3.48}$$

$$d^\star d\phi^{(0)} = \lambda f^{(0)}. \tag{3.49}$$

$$*d*d\phi^{(0)} = \lambda f^{(0)}. \tag{3.50}$$

$$\phi^{(0)} = \sin(2\pi x), \tag{3.51}$$

$$d\phi^{(0)} = 2\pi\cos(2\pi x)dx, \tag{3.52}$$

$$*d\phi^{(0)} = 2\pi\cos(2\pi x), \tag{3.53}$$

$$d*d\phi^{(0)} = -4\pi^2\sin(2\pi x)dx, \tag{3.54}$$

$$*d*d\phi^{(0)} = -4\pi^2\sin(2\pi x) = f^{(0)}. \tag{3.55}$$

3.5 Mixed method in 2D

2-form is applied to Poisson equation. It is n-form hence mixed method approach is used to solve the poisson equation in 2-form. The domain is $\Omega = [-1,1]\times[-1,1]$

$$\frac{d^2\phi^{(2)}}{dx^2} = f^{(2)}. \tag{3.56}$$

$$dd^*\phi^{(2)} = f^{(2)}. \tag{3.57}$$

It is mixed method approach hence the above equation (3.56) is rewritten in to two different equations.

$$u^{(1)} - d^\star\phi^{(2)} = 0, \tag{3.58}$$

$$du^{(1)} = f^{(2)}. \tag{3.59}$$

mutiplying equations (3.58) and (3.59) with trial functions and integrating over the domain.

$\Sigma = H\Lambda^1(\Omega)$, $U = L^2\Lambda^2(\Omega)$

$$(u^{(1)} - d^\star\phi^{(2)})\tilde{u}^{(1)} = 0, \forall\tilde{u}^{(1)} \in \Sigma, \tag{3.60}$$

$$(du^{(1)})\tilde{u}^{(2)} = f^{(2)} * \tilde{u}^{(2)}, \forall\tilde{u}^{(2)} \in U. \tag{3.61}$$

After integration by parts for the equations (3.60) and (3.61)

$$(u^{(1)}, \tilde{u}^{(1)}) + (d^\star\phi^{(2)}, d\tilde{u}^{(1)}) = 0, \forall\tilde{u}^{(1)} \in \Sigma, \tag{3.62}$$

$$((du^{(1)}, \tilde{u}^{(2)}) = (f^{(2)}, \tilde{u}^{(2)})\forall\tilde{u}^{(1)} \in U. \tag{3.63}$$

The boundary integral vanishes because of Dirichlet boundary conditions.

The above equations (3.62) and (3.63) are written in terms of basis functions as

$$\tilde{u^T} e_i^T e_i u + \tilde{u^T} E^{(2,1)T} e_i^T e_i e_j^T e_j \phi = 0, \forall\tilde{u}^{(1)} \in \Sigma, \tag{3.64}$$

$$\tilde{u^T} e_i^T e_i e_j^T e_j E^{(2,1)} u = \tilde{u^T} e_i^T e_i e_j^T e_j f \forall\tilde{u}^{(2)} \in U, . \tag{3.65}$$

The product of edge functions integrated over the domain produces an entry in mass matrix. The above equations (3.64) and (3.65) are written in the form of mass matrices as follows,

$$M^{(1)}\vec{u^{(1)}} + E^{(2,1)T}M^{(2)}\vec{\phi^{(2)}} = 0 \tag{3.66}$$

$$M^{(2)}E^{(2,1)}\vec{u^{(1)}} = M^{(2)}\vec{f^{(2)}} \tag{3.67}$$

The equations (3.66) and (3.67) are in the form of mixed laplace eigen value problem which is similar to the algebraic structure as mentioned in 1D case. The equations (3.66) and (3.67) are in the following format as mentioned in Boffi's paper

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix}$$

where 'A' is a mass matrix in zero form is M^1 and 'B' is $M^2E^{(2,1)}$ and M is M^2 mass matrix in one form.The homogeneous dirchlet boundary conditions are enforced.So that the matrices A,B and M do not include boundary conditions.

1. The domain taken is $[-1, 1] \times [-1, 1]$ in the program, in Boffi's paper the domain is $[0, \pi] \times [0, \pi]$
2. The number of points is equal to the $polynomialdegree(N) + 1$.
3. It gives N eigenvalues.

4. The exact eigenvalues are given by

$$\frac{((N^2+M^2)\pi^2)}{4} \tag{3.68}$$

because the domain is $[-1,1]\times[-1,1]$. The corresponding eigen spaces are spanned by the eigen functions $\sin(\frac{N\pi x}{2})\times\ sin(\frac{M\pi x}{2})$. In Boffi's paper the domain is $[0,\pi]\times[0,\pi]$. Hence The exact eigenvalues are given by N^2+M^2. The corresponding eigen spaces are spanned by the eigen functions $\sin(Nx)\times\sin(Mx)$.

5. We used 'eigs' MATLAB function.

6. The analysis has started from polynomial degree N=8 because i require minimum six eigen values.

7. The polynomial degree is varied from $N=8$ to $N=13$.

3.6 Multi element analysis 1D

Until now the analysis is using by considering the entire domain as single element. Now in this section entire domain is divided into $'k'$ number of elements

It is already known from the previous Mixed 1d section the format of final equation to solve which is given by

$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}\times\begin{bmatrix} X \\ Y \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}\times\begin{bmatrix} X \\ Y \end{bmatrix}$

where 'A' is a mass matrix in zero form is M^0 and 'B' is $M^1E^{(1,0)}$ and M is M^1 mass matrix in one form.The homogeneous dirchlet boundary conditions are enforced.So that the matrices A,B and M do not include boundary conditions.

Instead of one single element as the entire domain , the domain is divided into more than one element i.e $[0,a]$ is entire domain in multielement analysis it becomes $[0,a/2],[a/2,a]$. In single element case Legendre polynomials N=8 is used for the entire domain $[0,a]$. In multi element case Legendre polynomial N=8 is used for $[0,a/2],[a/2,a]$.

Now the domain is changed in multielement case. Hence

$\zeta\in[-1,1]\longrightarrow x\in[x_0,x_1]$

$'x'$ is related to $'\zeta'$

$$x=\frac{1}{2}(1-\zeta)x_0+\frac{1}{2}(1+\zeta)x_1 \tag{3.69}$$

$$x=\frac{x_0+x_1}{2}+\frac{x_1-x_0}{2} \tag{3.70}$$

$$\zeta=\frac{2}{x_1-x_0}[x-\frac{x_0+x_1}{2}] \tag{3.71}$$

$$u(\zeta)=u(\zeta(x)) \tag{3.72}$$

$$\phi(\zeta)d\zeta = \phi(\zeta(x))\frac{d\zeta}{dx}dx \tag{3.73}$$

$$\phi(\zeta)d\zeta = \phi(\zeta(x))\frac{2}{x_1 - x_0}dx \tag{3.74}$$

'A' matrix is mutiplied with $\frac{x_1-x_0}{2}$ and 'M' matrix is multiplied with $\frac{x_1-x_0}{2}$. where $\frac{x_1-x_0}{2}$ denotes the size of element divided by 2. This term is multiplied to mass matrices because domain is changed.

The mass matrices are same for all elements because element size is same.

At first the gathering matrix is created which is in the format of

Table 3.1: Gathering matrix for multielements 1D case

k=1	k=2	k=3	k=4	k=5	k=6
1	3	5	7	9	11
2	4	6	8	10	12
3	5	7	9	11	13
14	16	18	20	22	24
15	17	19	21	23	25

1. The value k is number of element.
2. Total number of elements is 6.
3. Velocity points are from 1 to 13.
4. Pressure points are from 14 to 25.

At boundaries of the elements the velocity points overlap. The pressure elements wont overlap at the boundaries of the elements.

1. The single element domain consisdered in this program is $[-1, 1]$. For multielements the domain is $[x_0, x_1]$.
2. It gives N eigen values.The exact eigen values are given by $\frac{N^2\pi^2}{4}$ because the domain is $[-1, 1]$.
3. The corresponding eigen spaces are spanned by the eigen functions sin$(\frac{N\pi x}{2}$.
4. We used 'eigs' MATLAB function.
5. The first five eigenvalues are tabulated in the table with polynomial degree $N = 8$. Number of elements is varied from $N = 1$ to $N = 8$.
6. The homogeneous dirchlet boundary conditions are enforced.So that the matrices A,B and M do not include boundary conditions.

Chapter 4

Results

The code setup is discussed in the previous section. The code for mixed method (i.e Poisson equation in k-forms) and direct method (i.e Poisson equation in 0-form) in both single element and multielement cases has been exectued for eigenvalues. Eigenvalues are obtained at different polynomial degrees for mixed and direct methods. The results obtained are compared with [13] paper. The rate of convergence analysis is discussed and compared with [13] paper.

4.1 Edge and nodal basis functions

Programming of this project has been done in MATLAB language.

The programs already developed used in this project are

1. GLL nodes are obtained from the program GLL NODES already written. The input to give for this program is polynomial degree.This program uses the principle of Gauss-Legendre-Lobatto.
2. The program MIMETIC POLYVAL already written has been used to obtain nodal basis functions and edge functions. This program uses lagrangian polynomials as interpolation functions. The input to give for this program are polynomial degree and GLL nodes and which method has to be used.GLL nodes are obtained from the program mentioned above. This program has three different basis functions procedures.Which are,
   (a) Gauss-Lobbato-Legendre
   (b) Gauss
   (c) Extended gauss

   The first method Gauss-Lobatto-Legendre is used in this project which uses legendre polynomials and GLL nodes.

4.2 Rate of convergence for codifferential problem

Results presented in this section are for the codifferential problem

$$d^* u^{(1)} = f^{(0)} \tag{4.1}$$

$f^{(0)}$ is the known data and $u^{(1)}$ is the unknown data.

$$-M^{(1)} E^{(1,0)} M^{(0)^{-1}} E^{(1,0)T} M^{(1)} u = M^{(1)} E^{(1,0)} f. \tag{4.2}$$

$u^{(1)}$ is obtained by solving (4.2)

At first the discrete solution is obtained. Then it is changed to approximate continous solution by using mass matrices.The basis functions play keyrole in this operation. Results obtained from these calculations are compared with exact solutions.

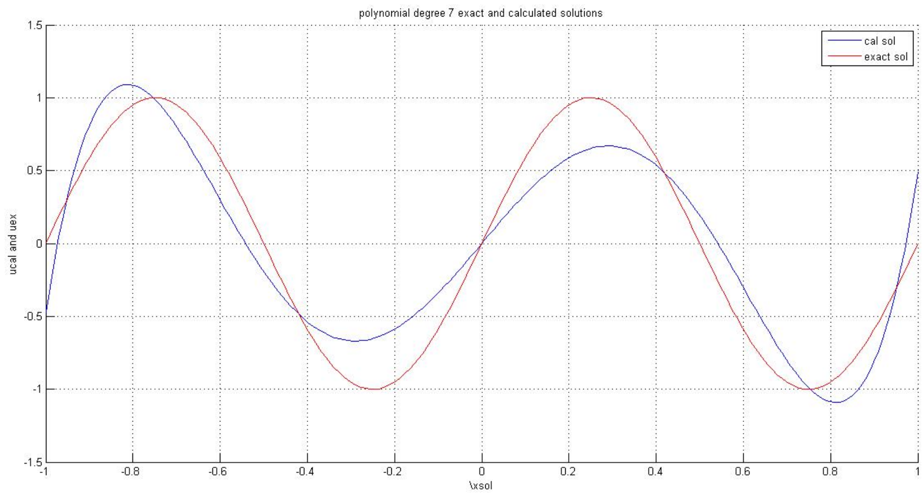


Figure 4.1: exact and calculated solutions for polynomial degree 7.

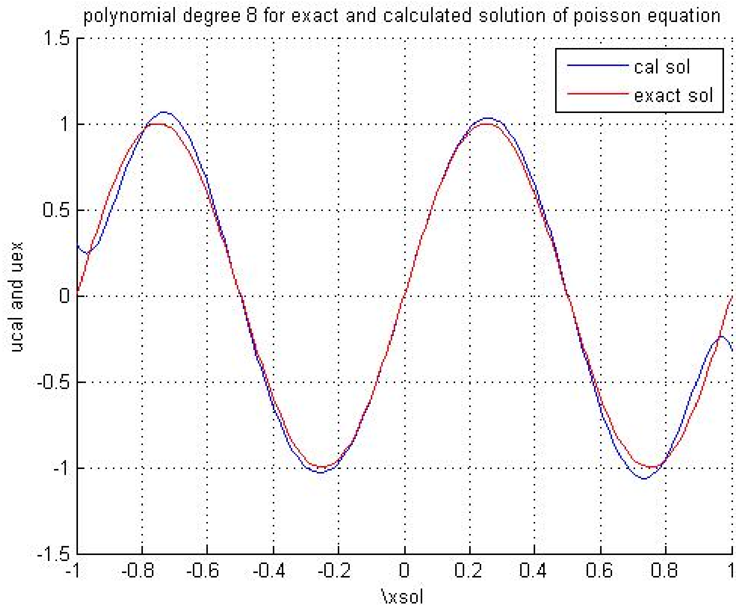


Figure 4.2: exact and calculated solutions for polynomial degree 8.

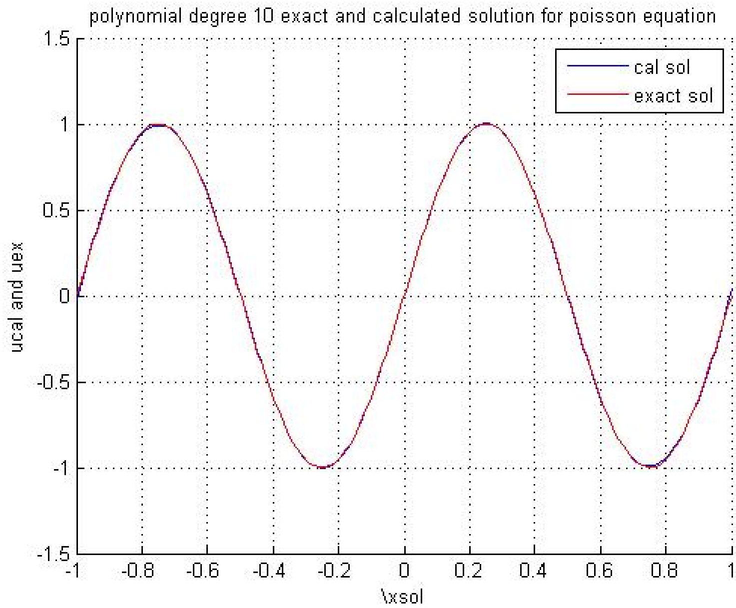


Figure 4.3: exact and calculated solutions for polynomial degree 10.

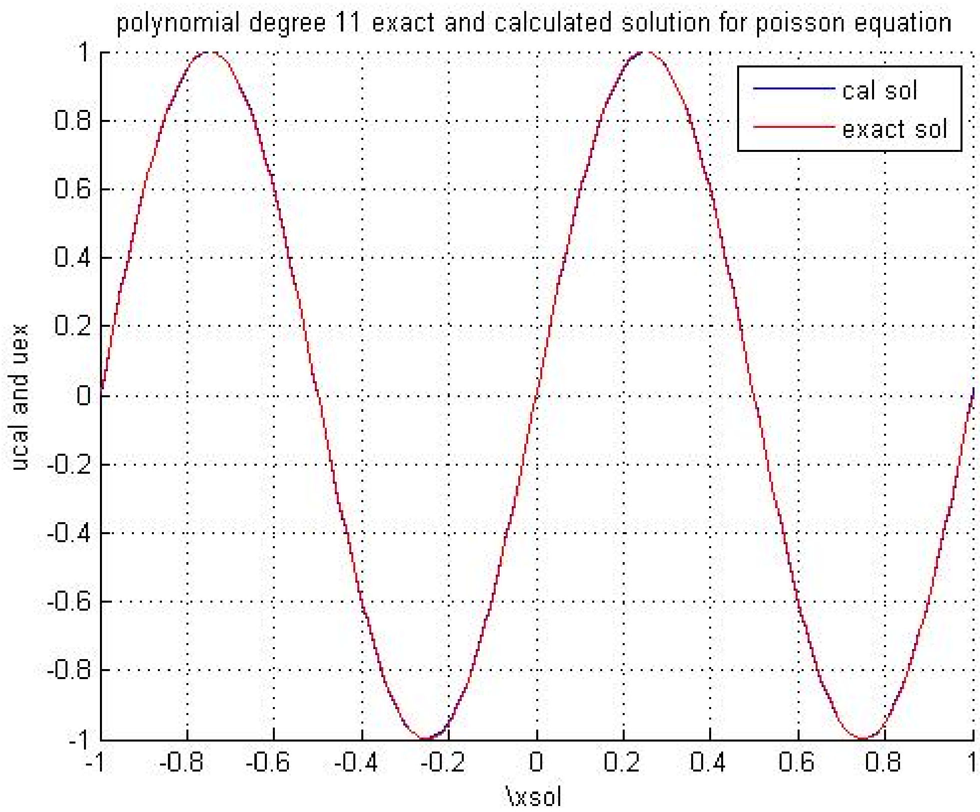


Figure 4.4: exact and calculated solutions for polynomial degree 11 .

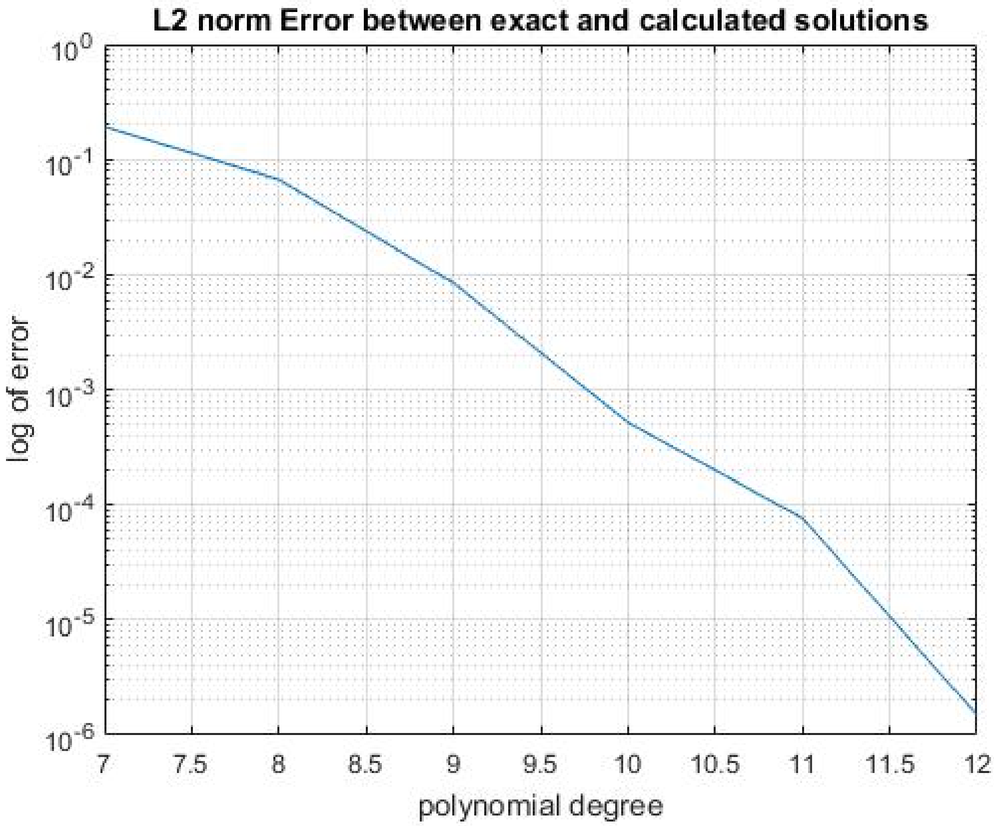


Figure 4.5: loglog error plot.

From above plots it is seen that as polynomial degree is increased, the error between exact solution and calculated solution is reduced.

The rate of convergence is exponential.

In spectral methods convergence is achieved by raising the polynomial degree order k of the approximation rather than by reducing the mesh width h, as is done in finite difference or classical finite element methods. In finite difference/finite volume (FD/FV) methods on the contrary convergence is achieved by refining mesh or by reducing the order of the scheme near discontinuities using limiters. Unlike FD/FV methods the convergence order of spectral methods is limited only by the regularity of the solution.

4.3 Results of mixed method for 1d case

Eigen values for mixed form poisson equation in 1D are obtained by solving the equation

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix}$$

which is a generalized eigenvalue problem

Calculated and exact eigenvalues for one form of poisson equation are given in the following table.

Table 4.1: Exact and Calculated eigenvalues mix form

Exact	N=8	N=9	N=10	N=11	N=12	N=13
2.4674	2.4674	2.4674	2.4674	2.4674	2.4674	2.4674
9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
22.2066	22.2159	22.2057	22.2067	22.2066	22.2066	22.2066
39.4784	38.8190	39.5638	39.4654	39.4795	39.4783	39.4784
61.6850	68.7424	59.4639	62.1116	61.5910	61.6957	61.6836
88.8264	78.5633	105.705	83.6866	90.2946	88.3966	88.8894

1. The value N is polynomial degree.
2. The eigenvalues are real and positive since Laplace operator is a self adjoint operator.
3. The eigenvalues are integer multiples of π^2 i.e $N^2\frac{\pi^2}{4}$.
4. The first two eigenvalues are similar to exact values for all polynomial degrees ranging from N=8 to 13. We had started with N=8 because minimum six eigenvalues are required for analysis.
5. As the eigenvalue rank increases the error between exact and calculated value decreases as the polynomial degree is increased.
6. No spurious modes are seen in the eigenvalue solutions.

Boffi's paper[13] has eigenvalues in the domain of $[0, \pi]$. Hence inorder to compare eigenvalues for convergence analysis. We changed the domain from [-1,1] to $[0, \pi]$. (How to change the domain is mentioned in section 3.6)

Calculated and exact eigenvalues for one form of poisson equation in the domain $[0, \pi]$ are given in the following table.

Table 4.2: Exact and Calculated eigenvalues mix form of my method

Exact	N=8	N=9	N=10	N=11	N=12	N=13
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
9	9.0038	8.9996	9.0000	9.0000	9.0000	9.0000
16	15.7327	16.0346	15.9947	16.0004	16.0000	16.0000
25	27.9413	24.0998	25.1729	24.9619	25.0043	24.99994
36	31.8405	42.8407	33.9169	36.5950	35.8258	36.0255

1. The value N is polynomial degree. The domain is $[0, \pi]$.
2. The eigenvalues are real and positive.
3. The eigenvalues are multiples i.e of N^2 because the domain is $[0, \pi]$.

4. The first two eigenvalues are similar to exact values for all polynomial degrees ranging from N=8 to 13.
5. As the eigenvalue rank increases the error between exact and calculated value decreases as the polynomial degree is increased.
6. No spurious modes are seen in the eigenvalue solutions.

Calculated and exact eigenvalues for one form of poisson equation of Boffi's[13] paper are given in the following table.

Table 4.3: Exact and Calculated eigenvalues mix form of daniel boffi

Exact	N=8	N=9	N=10	N=11	N=12	N=13
1	1.0129	1.0102	1.0083	1.0068	1.0057	1.0049
4	4.2095	4.1650	4.1333	4.1099	4.0922	4.0785
9	10.0803	9.8484	9.6838	9.5629	9.4715	9.4007
16	19.4537	18.7203	18.1925	17.8030	17.5083	17.2803
25	33.2628	31.6439	30.3964	29.4497	28.7248	28.1609
36	51.3724	49.2421	47.0606	45.2325	43.7708	42.6099

1. The value N is number of subdivisions.
2. The eigenvalues are real and positive.
3. The eigenvalues are multiples of i.e N^2.
4. No eigenvalues are similar to exact values for all polynomial degrees ranging from N=8 to 13.
5. As the eigenvalue rank increases the error between exact and calculated value decreases as the polynomial degree is increased.
6. No spurious modes are seen in the eigenvalue solutions.
7. In Boffi case the eigenvalues are approximated only from above and the convergence is quadratic.

Error between exact and calculated eigen values for mixed form one dimensional Poisson equation are given in the following table.(The values considered for this table are the values we obtained for the domain $[-1, 1]$)

The relative error increases with the rank of the eigenvalues in the spectrum. For instance the error for sixth eigenvalue is more than 16 times the error for the first one for polynomial degree N=9. For each eigenvalue at different polynomial degree a pattern can be noticed, the error values has alternative plus and minus signs. At same polynomial degree for different eigenvalues the error values has alternative plus and minus signs.

Table 4.4: error between exact and calculated eigenvalues for mixed method

poly degree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
8	$8.3710e^{-11}$	$-1.6169e^{-05}$	0.0093	-0.6594	7.2574	-10.2631
9	$-7.2520e^{-13}$	$4.5589e^{-07}$	$-8.9140e^{-04}$	0.0854	-2.2211	16.8787
10	$-4.4409e^{-15}$	$-1.3278e^{-08}$	$4.7914e^{-05}$	-0.0131	0.4266	-5.1398
11	$3.9968e^{-15}$	$2.5388e^{-10}$	-2.7399^{-06}	0.0011	-0.0941	1.4681
12	$5.3291e^{-15}$	$-4.9560e^{-12}$	1.0263^{-07}	$-9.8578e^{-05}$	0.0107	-0.4298
13	$4.8850e^{-15}$	$7.2831e^{-14}$	$-3.9671e^{-09}$	$5.8108e^{-06}$	-0.0014	0.0630

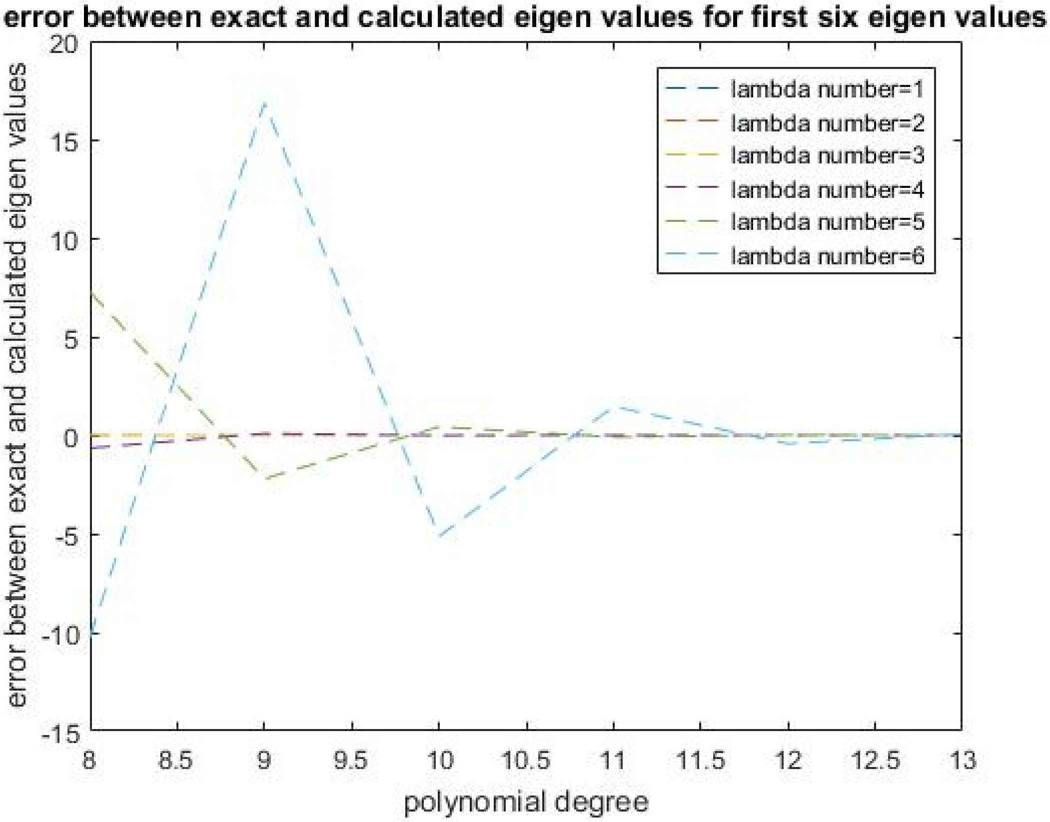


Figure 4.6: Error between exact and calculated solutions.

From above plot 4.6 alternative plus minus pattern can be seen for error values of eigenvalues of Poisson equation in 0-form. The alternative plus minus error pattern is not clearly seen until λ_3. It is little bit clearly seen for λ_4 . The pattern is clear for λ_5 and λ_6. All eigenvalues for different polynomial degree are approximated from above and below alternatively. Whereas in Boffi case the eigenvalues are approximated only from above and the convergence is quadratic.

The convergence analysis for eigenvalue problems usually consists of two parts.

1. In the first step one shows that there are no spurious modes in eigenvalues.

2. In the second step error estimates are looked for which provide the order of convergence for eigenvalues.

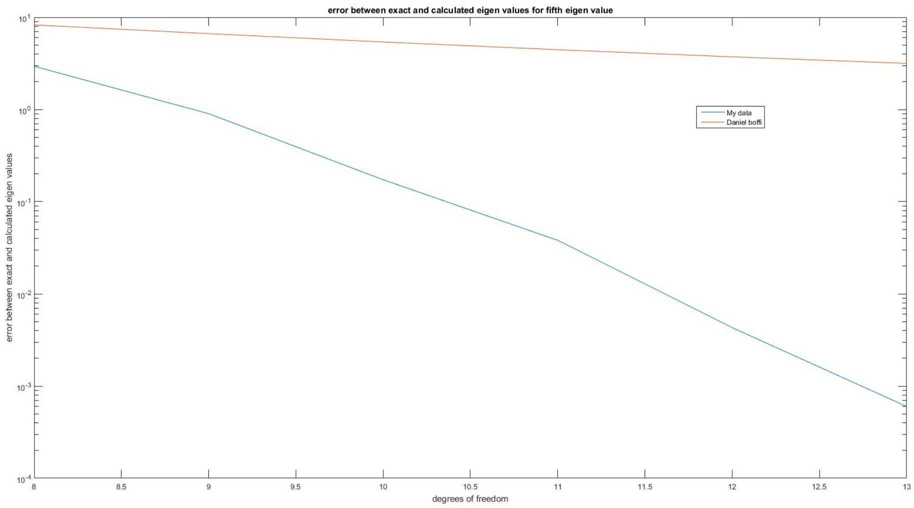


Figure 4.7: Error between exact and calculated solutions fifth eigen value mixed method 1d.

1. Boffi at first uses p1-p1 element method but it gives spurious modes. Next he uses p1-p0 element method which is nicely convergent but lateron for more convergence he uses p2-p0 element method. For p2-p0 method he uses continous piecewise linear elements and discontinous piecewise constants.

2. The results from Boffi case converges slower than my calculated results.

3. The convergence of Boffi's case is almost quadratic and the convergence of my results is exponential.

4. In my case the convergence is exponential because spectral methods have exponential convergence. The amount of reduction in error by changing the order of polynomial is way higher than the the amount of error by changing the order of polynomial in Daniel boffi's case.

5. At all points of degrees of freedom Boffi has more error compared to my results.

6. In both cases of my method and Boffi's case we considered first six eigenvalues. In my case we used spectral element method whereas Daniel boffi uses finite element method. As from the plot it is seen that the convergence rate is really less when the number of subdivisions is increased at the same rate as by increasing the number of order in my method. For example when the order of polynomial is increased by 1 from 9 to 10 the amount of error is still less than the amount of error at N=13 for Boffis method.

4.4 Direct method

Eigenvalues for direct form Poisson equation are obtained by solving the equation

$$-E^{(1,0)T} M^1 E^{(1,0)} \phi^0 = \lambda M^0 f^0 \tag{4.3}$$

which is a generalized eigenvalue problem.

Exact and calculated eigenvalues for 0-form Poisson equation are given in the following table

Table 4.5: Exact and Calculated eigenvalues for direct form

Exact	N=8	N=9	N=10	N=11	N=12	N=13
2.4674	2.4674	2.4674	2.4674	2.4674	2.4674	2.4674
9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
22.2066	22.2159	22.2057	22.2067	22.2066	22.2066	22.2066
39.4784	38.8190	39.5638	39.4654	39.4795	39.4783	39.4784
61.6850	68.7424	59.4639	62.1116	61.5910	61.6957	61.6836
88.8264	78.5633	105.705	83.6866	90.2946	88.3966	88.8894

1. The value N is polynomial degree.
2. The domain is [-1,1].
3. The eigenvalues are real and positive.
4. The eigenvalues are integer multiples of π^2 i.e $N^2 \frac{\pi^2}{4}$.
5. The first two eigenvalues are similar to exact values for all polynomial degrees ranging from N=8 to 13.
6. As the eigenvalue rank increases the error between exact and calculated value decreases as the polynomial degree is increased.
7. No spurious modes are seen in the eigenvalue solutions.

For mixed and direct methods the eigenvalues are almost similar for polynomial degrees N=8 to 13.

Error between exact and calculated eigenvalues for direct method are given below.

For each eigenvalue at different polynomial degree a pattern can be noticed, The error values has alternative plus and minus signs. At same polynomial degree for different eigenvalues the error values has alternative plus and minus signs. The error increases as the rank of eigenvalue increases. The error values look almost similar for both mixed and direct methods.

Table 4.6: error between exact and calculated eigenvalues for direct method

poly degree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
8	$-8.3786e^{-11}$	$1.6169e^{-05}$	-0.0093	0.6594	-16.8783	19.8840
9	$7.2609e^{-13}$	$-4.5589e^{-07}$	$8.9140e^{-04}$	-0.0854	2.2211	-16.8787
10	$-1.0214e^{-14}$	$1.3278e^{-08}$	$-4.7914e^{-05}$	0.0131	-0.4266	5.1398
11	$4.8850e^{-15}$	$-2.5386e^{-10}$	2.7399^{-06}	-0.0011	0.0941	-1.4681
12	$2.3093e^{-14}$	$4.8992e^{-12}$	-1.0263^{-07}	$9.8578e^{-05}$	-0.0107	0.4298
13	$2.0428e^{-14}$	$-4.9738e^{-14}$	$3.9671e^{-09}$	$-5.8108e^{-06}$	0.0014	-0.0630

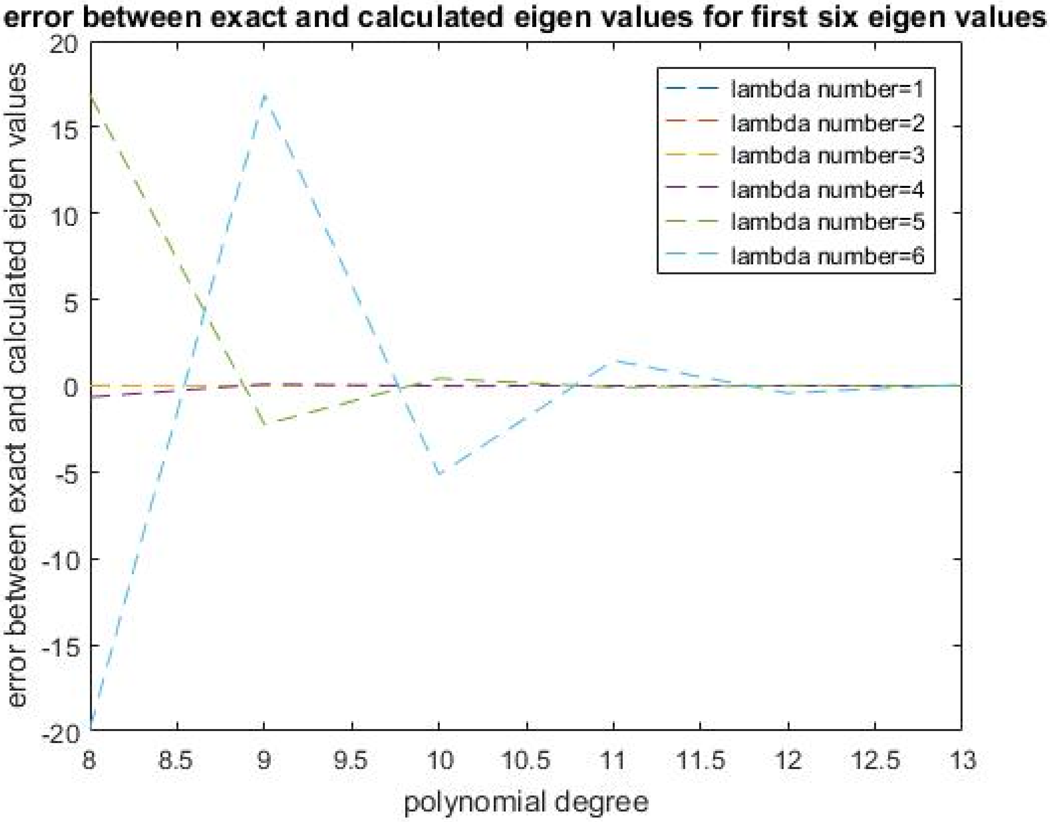


Figure 4.8: Error between exact and calculated solutions.

From above plot 4.8 alternative plus minus pattern can be seen for error values of eigenvalues of Poisson equation in 0-form. The alternative plus minus error pattern is not clearly seen until λ_3. It is little bit clearly seen for λ_4 . The pattern is clear for λ_5 and λ_6.

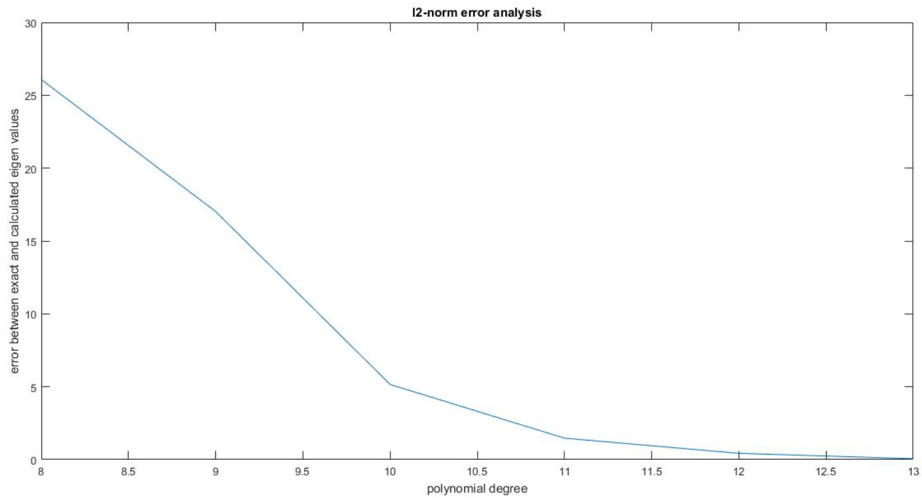


Figure 4.9: Error analysis for direct method.

The above plot 4.9 shows the L2 norm error analysis. L2 error analysis is done for all 6 eigenvalues together for polynomial degree 8 to 13. The convergence is exponential since we used spectral element method.

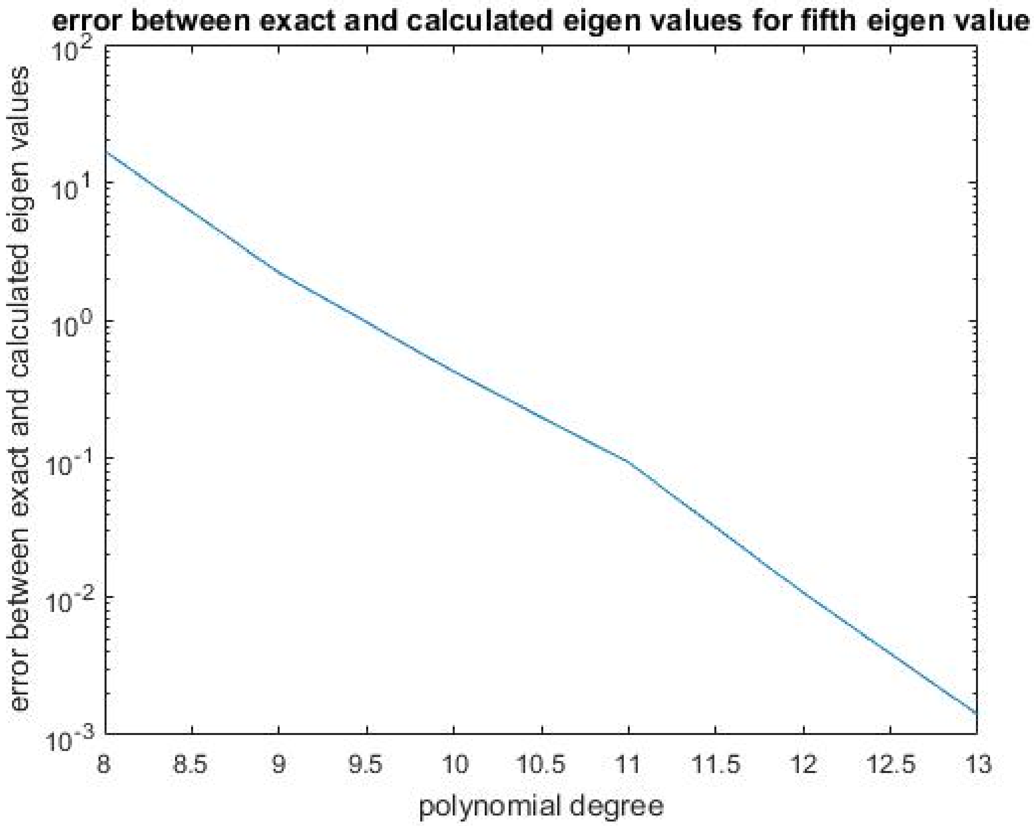


Figure 4.10: log-lin Error between exact and calculated solutions for fifth eigenvalue direct method.

The above plot 4.10 shows the L1 norm error analysis for λ_5. L1 error analysis is done for 5th eigenvalue because variation of error is seen clearly for this eigenvalue for polynomial degree 8 to 13. The convergence is exponential since we used spectral element method.

1. The convergence is exponential for both overall L2 error analysis and individual eigenvalue error analysis. Individual eigenvalue error analysis is done for 5th eigenvalue.

2. The convergence is exponential because we used spectral method.

4.5 2d case

Eigenvalues for mixed form poisson equation in 2D are obtained by solving the equation

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix}$$

which is a generalized eigenvalue problem

Calculated and exact eigenvalues for 2-form of poisson equation are given in the following table.

Table 4.7: Exact and Calculated eigenvalues for 2-dimensional mixed case

Lambda	Exact	N=8	N=9	N=10	N=11	N=12	N=13
λ_1	4.9348	4.9348	4.9348	4.9348	4.9348	4.9348	4.9348
λ_2	12.3370	12.3370	12.3370	12.3370	12.3370	12.3370	12.3370
λ_3	12.3370	12.3370	12.3370	12.3370	12.3370	12.3370	12.3370
λ_4	19.7392	19.7392	19.7392	19.7392	19.7392	19.7392	19.7392
λ_5	24.6740	24.6834	24.6731	24.6741	24.6740	24.6740	24.6740
λ_6	24.6740	24.6834	24.6731	24.6741	24.6740	24.6740	24.6740
λ_7	32.0762	32.0855	32.0753	32.0763	32.0762	32.0762	32.0762
λ_8	32.0762	32.0855	32.0753	32.0763	32.0762	32.0762	32.0762
λ_9	41.9458	41.2864	42.0312	41.9328	41.9469	41.9457	41.9458
λ_{10}	41.9458	41.2864	42.0312	41.9328	41.9469	41.9457	41.9458
λ_{11}	44.4132	44.4319	44.4114	44.4133	44.4132	44.4132	44.4132
λ_{12}	49.3480	48.6886	49.4334	49.3350	49.3491	49.3479	49.3480
λ_{13}	49.3480	48.6886	49.4334	49.3350	49.3491	49.3479	49.3480
λ_{14}	61.6850	61.0350	61.7695	61.6720	61.6861	61.6849	61.6850

1. N represents polynomial degree.
2. The eigenvalues are real and positive.
3. The eigenvalues are integer multiples of π^2 i.e $N^2\frac{\pi^2}{4}$.
4. The first four eigenvalues are similar to exact values for all polynomial degrees ranging from N=8 to 13.
5. We had considered 14 eigenvalues because there are some repeated eigenvalues.
6. As the eigenvalue rank increases the error between exact and calculated value decreases as the polynomial degree is increased.
7. No spurious modes are seen in the eigen value solutions.
8. All eigenvalues for different polynomial degree are approximated from above and below alternatively.

Error between exact and calculated eigenvalues for mixed form poisson equation 2D are given below.

The relative error increases with the rank of the eigenvalues in the spectrum. For instance the error for fourteen eigenvalue is more than 0.65 times the error for the first one for polynomial degree N=8. For each eigenvalue at different polynomial degree a pattern can be noticed, The error values has alternative plus and minus signs. At same polynomial degree for different eigenvalues the error values has alternative plus and minus signs.

Table 4.8: error between exact and calculated eigenvalues for mixed method 2 dimensional case from 1 to 8 eigenvalues

poly degree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
8	0.0000	-0.0000	-0.0000	-0.0000	-0.0094	-0.0094	-0.0093	-0.0093
9	-0.0000	0.0000	0.0000	0.0000	$9e^{-4}$	$9e^{-4}$	$9e^{-4}$	$9e^{-4}$
10	0.0000	-0.0000	-0.0000	-0.0000	$-1e^{-4}$	$-1e^{-4}$	$-1e^{-4}$	$-1e^{-4}$
11	-0.0000	0.0000	0.0000	0.0000	0.000	0.0000	0.0000	0.0000
12	0.0000	-0.0000	-0.0000	-0.0000	-0.000	-0.0000	-0.0000	-0.0000
13	-0.0000	0.0000	0.0000	0.0000	0.000	0.0000	0.0000	0.0000

Table 4.9: error between exact and calculated eigenvalues for mixed method 2 dimensional case from 9 to 14 eigenvalues

poly degree	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}
8	0.6594	0.6594	-0.0187	0.6594	0.6594	0.6500
9	-0.0854	-0.0854	0.0018	-0.0854	-0.0854	-0.0845
10	0.0130	0.0130	$-1e^{-4}$	0.0130	0.0130	0.0130
11	-0.0011	-0.0011	0.0000	-0.0011	-0.0011	-0.0011
12	$1e^{-4}$	$1e^{-4}$	-0.0000	$1e^{-4}$	$1e^{-4}$	$1e^{-4}$
13	-0.0000	-0.0000	0.0000	-0.0000	-0.0000	-0.0000

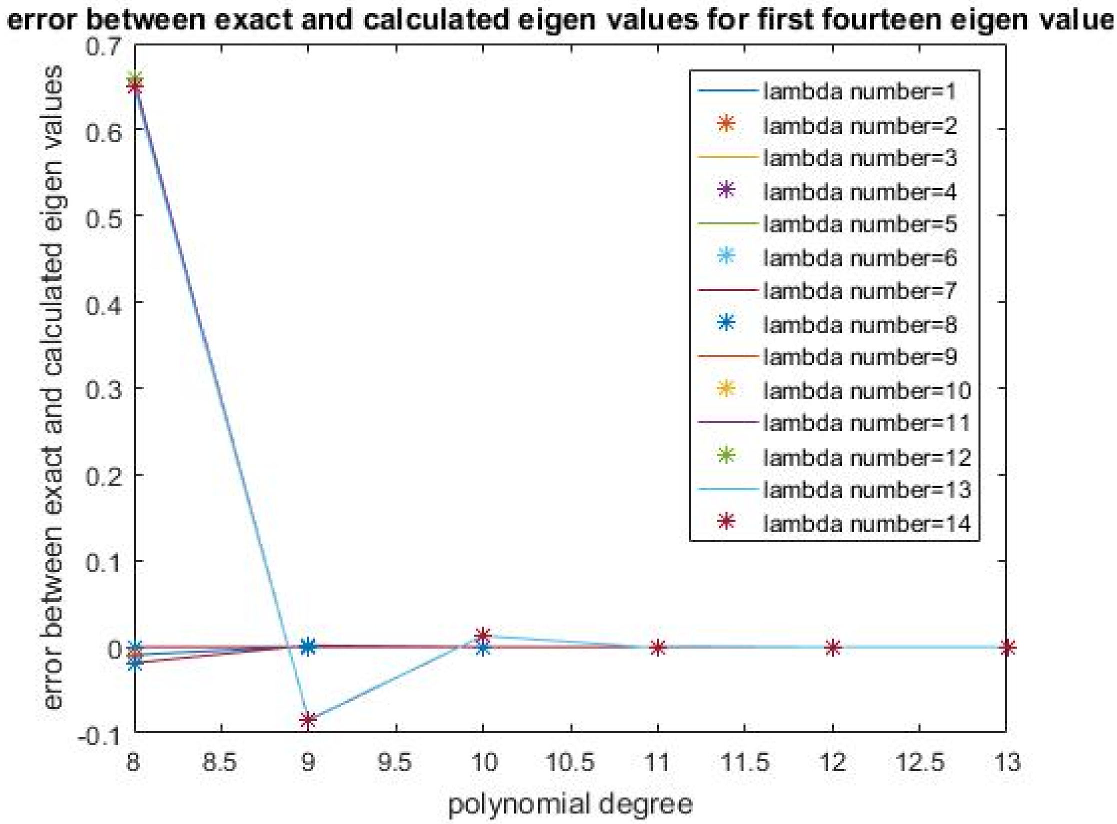


Figure 4.11: error vs polynomial degree for 2d case.

The above plot 4.11 shows the error values for all eigenvalues. The alternative plus minus error pattern is not clearly seen until λ_8. It is clearly seen after λ_8 eigenvalue. The alternative plus minus pattern is noticed clearly from λ_{10}.

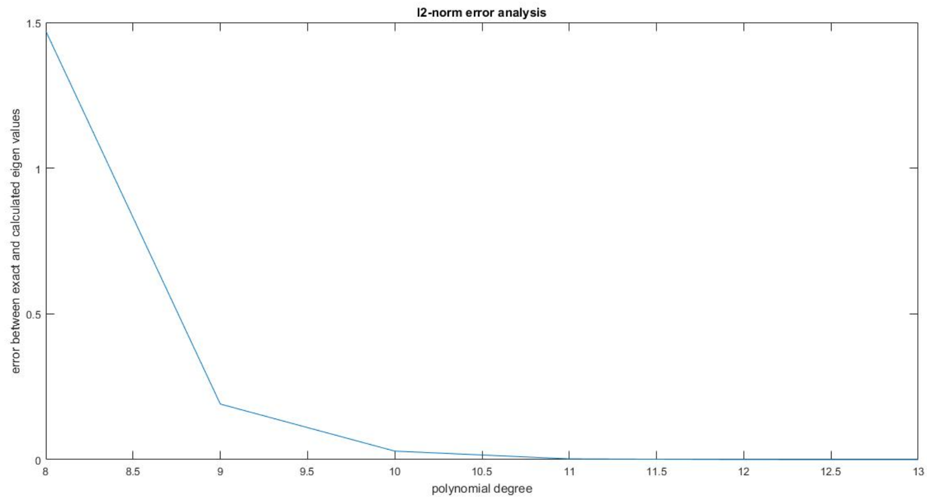


Figure 4.12: Error between exact and calculated solutions for 2d case for different eigenvalues.

The above plot 4.12 shows the L2 error norm analysis.

L2 norm error analysis is done for 14 eigenvalues from polynomial degree 8 to 13. The convergence is exponential because we used spectral method.

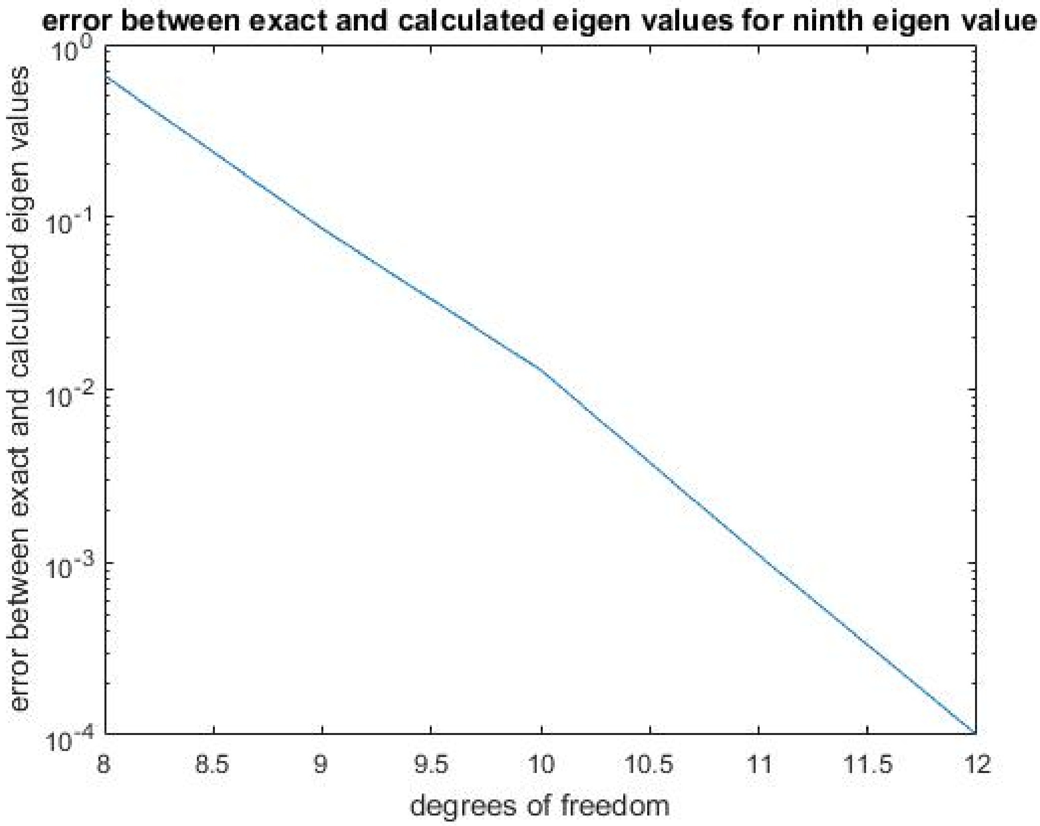


Figure 4.13: log error vs polynomial degree for 2d case.

The above plot 4.13 shows the L1 norm error for λ_9. It is a linear-logarithm plot. L1 norm error analysis is done for individual ninth eigenvalue from polynomial degree 8 to 13. The convergence is exponential because we used spectral method. The error reduces as polynomial degree is increased.

4.6 Multi element analysis in 1-D

Until now the results are obtained for single element case i.e domain $[-1, 1]$ is considered as one single element. 1-form of mixed Poisson equation in multi elements is solved in this section.

Calculated and exact eigen values for 1-form of Poisson equation are given in the following table.

1. We had taken 6 eigenvalues.
2. The eigenvalues are real and positive.
3. The eigenvalues are integer multiples of π^2 i.e $N^2\frac{pi^2}{4}$.
4. K denotes number of elements.

Table 4.10: Exact and Calculated eigenvalues multielement analysis for polynomial degree 8

Exact	k=1	k=2	k=3	k=4	k=5	k=6
2.4674	2.4674	2.4674	2.4674	2.4674	2.4674	2.4674
9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
22.2066	22.2159	22.2066	22.2066	22.2066	22.2066	22.2066
39.4784	38.8190	39.4784	39.4784	39.4784	39.4784	39.4784
61.6850	68.7424	61.6848	61.6850		61.6850	61.6850
88.8264	78.5633	88.8638	88.8263	88.8264	88.8264	88.8264

5. The first two eigenvalues are similar to exact values for all number of elements ranging from k=1 to 6 for polynomial degree N=8. K=1 means single element case.
6. As the eigenvalue rank increases the error between exact and calculated value decreases as thenumber of elements is increased.
7. No spurious modes are seen in the eigenvalue solutions.

Table 4.11: Error between Exact and Calculated eigen values for mixed form multi elements 1D

number of elements	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
1	0.0000	-0.0003	0.0093	-0.6594	7.2574	-10.2631
2	0.0000	0.0000	-0.0000	-0.0000	-0.0000	0.0374
3	0.0000	-0.0000	0.0000	-0.0000	-0.0000	-0.0000
4	0.0000	- 0.0000	-0.0000	0.0000	-0.0000	-0.0000
5	0.0000	-0.0000	-0.0000	-0.0000	-0.0000	0.0000

In single case, for each eigen value at different polynomial degree a pattern can be noticed, The error values has alternative plus and minus signs. At same polynomial degree for different eigen values the error values has alternative plus and minus signs. But in multielement case the alternative plus minus pattern is not seen.

1. L1 norm error is calculated for different number of elements and polynomial degree=8. k denotes the number of elements.
2. For one particular number of elements, the error reduces exponentially as number of elements is increased. The error approximation is exponential because it is spectral element method.
3. As number of elements is increased the amount of error decreases. The size of element reduces as the number of elements is increased since the interval is same. $[-1, 1]$.

As it can be seen from the above table that all error values are close to zero. Inorder to notice error values clearly and to see how rate of convergence is with respect to p refinement. Hence the error values are plotted for N=2 and number of elements is changed from 1 to 70.

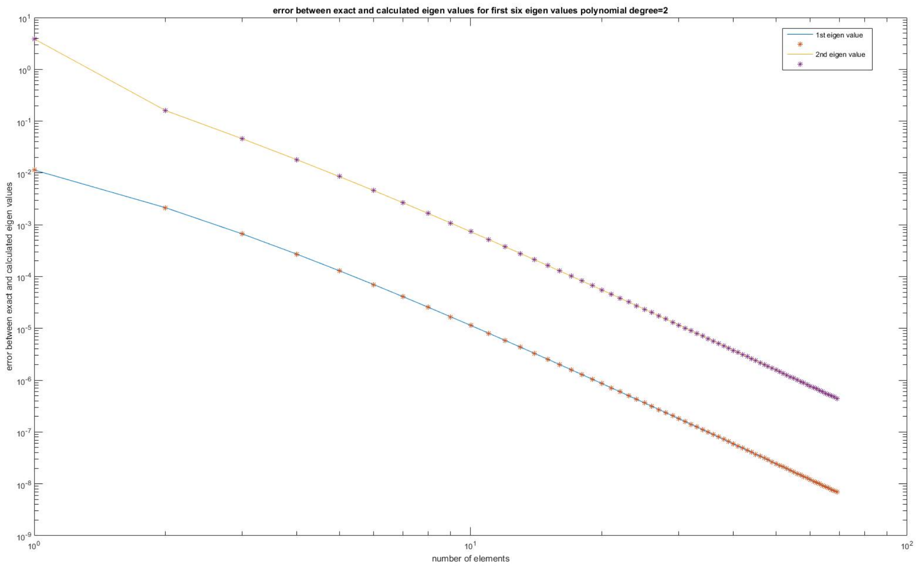


Figure 4.14: Error between exact and calculated solutions for 1d case for two eigenvalues.

The above plot 4.14 shows the error between calculated and exact eigen values. The above Rate of convergence is given by the equation

$$y = ax^k \tag{4.4}$$

taking the logarithm of the equation gives

$$\log y = k \log x + \log a \tag{4.5}$$

The above equation (4.5) is in the form of $y = mx+b$ where m=k,$b = \log a$ is the intercept on the ($\log y$ axis). where m is given by

$$m = \frac{\log y_2 - \log y_1}{\log x_2 - \log x_1} \tag{4.6}$$

From above graph the rate of convergence is -6.8433. The number of elements is changed from k=1 to k=70 for polynomial degree of N=2. The obtained two eigen values are plotted.

Chapter 5

Conclusion and Recommendations

5.1 Conclusion

In chapter 2 of this report we focussed on exterior derivative, codifferential operator, wedge product and hodge operator which constitute the building blocks that enable the representation of most of the physical field laws. We also focussed on Laplace operator and how different boundary conditions gives different set of eigenvalues. By using differential geometry and the link to its discrete counterpart algebraic topology we derived a mimtic discretization of Laplace operator. Hence two different formulations of Laplace operator depending on the use of differential forms are obtained, which are mixed and direct. mixed and direct formulations are solved in 1D and 2D domains in single element. Later on we will extend to solve mixed formulation in 1D domain for multi element case. The methodology for single element and multi element cases are solved in chapter 3. The eigenvalues error analysis of Laplace operator for these cases are discussed in chapter 4. The following conclusions are made based on the results discussed in chapter 4.

1. For 1D mixed formulation single element case in section 4.3 it is shown that eigenvalues obtained through spectral element method has faster convergence rate than Boffi's finite element method.

2. There is an alternative plus minus pattern for error values of eigenvalues. This pattern is profoundly noticed when error is of substantial value. This pattern holds for mixed and direct formulations in 1D and 2D cases of single element case. This pattern is not noticed anymore in multielement 1D mixed formulation case.

3. There are no spurious modes in results of all single element 1D, 2D and multielement 1D cases.

4. For multielement lesser polynomial degree can be used to obtain same accurate results compared to single element case.

5. The convergence rate for multi element case is 6.8433.

In this thesis we set out to obtain rate of convergence of eigenvalues of Laplace operator to be exponential which has been obtained by using spectral element methods. There are no spurious modes in eigenvalues.

5.2 Recommendations

1. To calculate CPU time for spectral methods and compare them with traditional finite element methods. Analyze between the CPU time and accuracy in detail.
2. To analyze why there is alternative plus minus sign pattern occurs in error values of eigenvalues.
3. To analyze why alternative plus minus sign pattern won't hold for multielement case.
4. To analyze if this alterntive plus minus sign pattern holds for dual grid method.

Bibliography

[1] Gerritsma,M.I, *An Introduction to a Compatible Spectral Discretization Method,* Mechanics of Advanced Materials and Structures, 19:1-3, 48-67, 2012 DOI: 10.1080/15376494.2011.572237

[2] Gerritsma.M.I, *Edge functions for spectral elements.* Lect.Notes Comput.Sci.Eng.,Vol.76, Springer, 2011, pp.199-207.

[3] Palha, A, Gerritsma.M.I. *Spectral Element Approximation of the Hodge operator in Curved Domains,in*:Lect.Notes comput.sci.eng.,vol.76,springer,2011,pp. 283-292.

[4] Palha,A, Bouman,M, Kreeft,J and Gerritsma.M.I, *A conservative spectral element method for arbitrary domains,in*:Lect.Notes comput.sci.eng., vol. 76, Springer, 2011, pp.111-119

[5] Kreeft,J, Gerritsma.M.I, *Mixed mimetic spectral element method for Stokes flow:A pointwise divergence-free solution* , J.Comput.Phys.240(2013) , 284-309.

[6] Hesthaven.J, Rnquist.E (Eds.), *Spectral and High Order Methods for Partial DifferentialEquations,* Springer, 2011, pp. 199–208.

[7] Palha.A, *High order mimetic discretization, Development and application to Laplace and advection problems in arbitrary quadrilaterals*,PhD thesis, 2013

[8] Reuter.M, Biasoti.S, Giorgi,D., Patane,G., Spanghenoto,M.,*Discrete Laplace beltrami operators for shape analysis and segmentation,* 2009.

[9] *"Spectral Methods in Fluid Dynamics", Annual Review of Fluid Mechanics,* 01/1987 Publication.

[10] Wang, Z.J., Krzysztof Fidkowski, Rémi Abgrall,Francesco Bassi, Doru Caraeni, Andrew Cary,Herman Deconinck, Ralf Hartmann, Koen Hillewaert, H.T. Huynh, Norbert Kroll, GeorgMay, Per-Olof Persson, Bram van Leer, andMiguel Visbal.*"High-order CFD methods:current status and perspective : HIGH-ORDER CFD METHODS", International Journal forNumerical Methods in Fluids,* 2013.Publicat ie

[11] Catherine.M, *"The Challenges of High Order Methods in Numerical Weather Prediction",* Lecture Notes in Computational Science and Engineering, 2011

[12] Bochev.P and Gunzburger.M. *Least-Squares finite element methods, volume 166 of Applied Mathematical Sciences.* Springer Verlag, 2009.

[13] Boffi.D, Gardini.F and Gastaldi.L, *Lecture notes in computational science and engineering,* 2010.

[14] Blair.J, Perot.C, zusi.J *Differential forms for scientists and engineers,* January 2014.

[15] Tonti.E, *why starting from differential equations for computational physics,* January 2014.

[16] Babuska.I, Narasimhan.R, *The Babuska-Brezzi condition and the patch test:an example.* Comput.Methods Appl.Mech.Engrg., 140(1-2):183-199,1997.

[17] Babuska.I.,Osborn.J., *Eigenvalue problems.In handbook of numerical analysis,* vol 2, Handb.Numer.Anal.,pages.North Holland,Amsterdam, 1991.

[18] Nica,M., *Eigenvalues and eigenfunctions of the Laplacian* ,Lecture notes, university of waterloo.

[19] C,Mattiussi,The finite volume, finite element, and finite difference methods for physical field problems, Adv.Imaging Electron phys. 113 ,1-147 2000.

[20] Flanders.H, *Differential forms with applications to the physical sciences.* Dover publications,inc.New york,1962.

[21] I.Babuska .,Rheinboldt,W.c.,*A posteriori error estimates for the finite element method,* 1978.

[22] D.N.Arnold,R.s.Falk,R.winther*Finite element exterior calculus from hodge theory to numerical stability*, Bull.Am.Math.Soc.47(2) (2010) 281-354.

[23] D.Boffi,F.Brezzi,and L.Gastaldi*on the convergence of eigenvalues for mixed formulations,* 1997.

[24] Bochev.P, Hyman.J,*Principles of mimetic discretizations of differential eqquations*,IMA Vol.Math.Appl. 142 (2006)89.

[25] Brezzi.F,Fortin.M,*Mixed and Hybrid finite element methods*,Springer Ser.Comput.Math.,Vol. 15, Springer-Verlag, New York,1991.

[26] Sherwin.S.J, Karniadakis.G.M, *Spectral hp element methods for cfd*, Oxford university press,1999.

[27] Raviart.P.A, Thomas.M.J, *A mixed finite element method for second order elliptic problems.* Mathematical aspects of the finite element method, Volume 606 of lecture notes in math., pp 291-315, Newyork,1977.

Appendix A

Appendix

The programs of the mixed method developed in this thesis and Boffi mixed method in 1D single element case are compared for the time taken by the CPU to compute. The programs are iterated for 1000 times then the amount of time is in seconds.

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
thesis4_9	1	7.434 s	3.907 s	
eigs	1000	2.263 s	0.243 s	
eigs>checkInputs	1000	1.669 s	1.559 s	
MimeticpolyVal	1000	1.054 s	0.579 s	
roundn	36000	0.360 s	0.360 s	
eigs>fullEig	1000	0.350 s	0.319 s	
GLLnodes	3000	0.270 s	0.227 s	
LegendreVal	2000	0.055 s	0.055 s	
cell.ismember	1000	0.053 s	0.053 s	

Figure A.1: mixed method N=8

The above figure A.1 shows the amount of time taken by mixed method which uses spectral element method for polynomial degree 8.

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
thesis4_9	1	10.690 s	5.956 s	
eigs	1000	2.715 s	0.314 s	
eigs>checkInputs	1000	1.889 s	1.756 s	
MimeticpolyVal	1000	1.726 s	0.983 s	
roundn	40000	0.571 s	0.571 s	
eigs>fullEig	1000	0.511 s	0.465 s	
GLLnodes	3000	0.385 s	0.326 s	
LegendreVal	2000	0.080 s	0.080 s	
cell.ismember	1000	0.067 s	0.067 s	
eigs>checkInputs/CHOLfactorB	1000	0.060 s	0.060 s	

Figure A.2: mixed method N=9

The above figure A.2 shows the amount of time taken by mixed method which uses spectral element method for polynomial degree 9.

Profile Summary

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Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
Untitled2	1	7.840 s	0.553 s	
eigs	1000	6.953 s	0.306 s	
eigs>checkInputs	1000	4.764 s	4.212 s	
eigs>fullEig	1000	1.883 s	1.845 s	
eigs>checkInputs/LUfactorAminusSigmaB	1000	0.296 s	0.296 s	
spdiags	2000	0.277 s	0.277 s	
eigs>checkInputs/CHOLfactorB	1000	0.172 s	0.172 s	
cell.ismember	1000	0.077 s	0.077 s	
eigs>Bmtimes	1000	0.038 s	0.038 s	
close	1	0.030 s	0.003 s	

Figure A.3: Boffi mixed method N=8

The above figure A.3 shows the amount of time taken by Boffi mixed method which uses finite element method for number of subdivisions 8.

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
Untitled2	1	32.562 s	0.263 s	
eigs	1000	32.015 s	14.145 s	
eigs>checkIpntr	244944	6.488 s	6.488 s	
eigs>@(v)Bmtimes(v)	242944	6.196 s	2.529 s	
eigs>Bmtimes	323944	4.823 s	4.823 s	
eigs>@(v)AminusSigmaBsolve(Bmtimes(v))	61000	3.307 s	0.797 s	
eigs>AminusSigmaBsolve	61000	1.671 s	1.671 s	
eigs>checkInputs	1000	0.927 s	0.353 s	
...(arraytoconvert)int64(arraytoconvert)	252944	0.596 s	0.596 s	

Figure A.4: Boffi mixed method N=200

The above figure A.4 shows the amount of time taken by Boffi mixed method which uses finite element method for number of subdivisions 200.

Table A.1: Exact and Calculated eigenvalues mix form of my method and Boffi method

Exact	Boffi no. of subdivisions=200	N=9
1	1.0000	1.0000
4	4.0003	4.0000
9	9.0017	8.9996
16	16.0053	16.0346
25	25.0129	24.0998

The above table A.1 shows the eigenvalues obtained for Boffi mixed method 200 subdivisions and my mixed method for polynomial degree 9. The eigenvalues are almost similar for both methods. But the amount of time taken for Boffi finite element method is 32.562s where as for my method it is 10.690s. The amount of time for Boffi method is almost thrice than my method. But still detail discussion of these results and different scenarios have to be analyzed to have proper conclusions.