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ORIGINAL PAPER

An impossibility theorem for verisimilitude

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Abstract In this paper, we show that Arrow's well-known impossibility theorem is instrumental in bringing the ongoing discussion about verisimilitude to a more general level of abstraction. After some preparatory technical steps, we show that Arrow's requirements for voting procedures in social choice are also natural desiderata for a general verisimilitude definition that places content and likeness considerations on the same footing. Our main result states that no qualitative unifying procedure of a functional form can simultaneously satisfy the requirements of *Unanimity, Independence of irrelevant alternatives and Non-dictatorship* at the level of sentence variables. By giving a formal account of the incompatibility of the considerations of content and likeness, our impossibility result makes it possible to systematize the discussion about verisimilitude, and to understand it in more general terms.

Keywords Verisimilitude · Truthlikeness · Truth-content · Arrow's theorem · Preference aggregation · Min-sum measure

1 Introduction

Since its publication, Arrow's (1950) impossibility theorem is a cornerstone of the theory of social choice. In his article, the economist Arrow showed that no procedure exists that, under reasonable constraints, will faithfully merge any collection of individual preference orders into a single global order. Arrow's mathematical result is not restricted to questions of social choice, but applies to many situations in which a collection of orders is to be combined into one order, and where Arrow's requirements are reasonable constraints. In the philosophy of science, any situation of merging different

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strands of qualitative reasoning potentially qualifies as a suitable application. In this paper, we apply Arrow's result to the discussion of the precise characterization of verisimilitude, thereby bringing this discussion to a general level.

The notion of verisimilitude was introduced by Popper (1963, Chap. 10) in an attempt to extend his falsificationist methodology with a measure of the relative merit of scientific theories to capture at least part of the truth. A decade later, Tichý and Miller independently published their well-known proofs that Popper's definition of verisimilitude is incapable of ordering false theories (Tichý, 1974; Miller, 1974). As most scientific statements are strictly speaking false, this result made Popper's definition useless, something he explicitly acknowledged (Popper, 1979, p. 371). Since then, researchers have come up with a plethora of alternative verisimilitude definitions that do not suffer from the drawback of Popper's proposal (for an overview, see Niiniluoto, 1998). Three decades after the publications of Tichý and Miller, we have more than a dozen different verisimilitude definitions, each of which is claimed by its author(s) to be the most adequate one. The problem, which of these definitions is to be preferred is seldom discussed on a general level. It is bogged down in the details of the pros and cons of the specific proposals, the proverbial exception to the rule being van Benthem's (1987).[1] As a result, we are still lacking a generally accepted set of desiderata, let alone axioms, for the notion of verisimilitude. In this paper, we intend to lift the discussion of the adequacy of the various definitions to a more general level. This means that we will refrain from ad hoc discussions about the ranking of specific sentences and instead focus on the ordering relations as such.

In this paper, we will investigate in detail the extent to which Arrow's theorem applies to the problem of defining verisimilitude. We will argue that the social-choice situation differs from the verisimilitude case in several aspects, discussed in Sections 4 and 5. For example, verisimilitude orders are incomplete, whereas Arrow's theorem supposes the orders of alternatives to be complete. Nevertheless, we show that the unification of content and likeness definitions is still subject to an impossibility result similar to Arrow's result. The structure of our paper is as follows. In the next section, we formally present general versions of the content and likeness definitions underlying the various verisimilitude proposals. In Sect. 3, we introduce Arrow's theorem. Next, we discuss the relevance of the Arrovian requirements for the problem of combining the content and likeness intuitions, and we argue that the social-choice case and the verisimilitude case are similar in structure. In the fifth section, we show that the domain of content and likeness orders does not cover all logical possibilities. Nevertheless, the domain remains rich enough to allow an Arrovian impossibility theorem for the unification of the likeness and content orders. Finally, in the last section, we discuss possible conclusions, including the role of quantitative verisimilitude measures and make suggestions for future research.

2 Two aspects of verisimilitude

With the collapse of Popper's definition in 1974, the two proposed alternatives, the one by Miller and the other by Tichý, reflected important general differences. Miller's proposal ordered sentences using *truth-value* and *content*, whereas Tichý based his similarity relation between sentences on *likeness between possible worlds*.

[1] Miller's 'language-dependence' argument and Oddie's 'child's play' argument, though, published in 1974 and to be discussed below, can be interpreted as general constraints of adequacy.

Both authors formulated difficulties for each other's approach. Using his '*language dependence*' argument, Miller pointed out that Tichý's likeness proposal violated the general requirement of being independent of the conceptual framework.[2] Tichý argued, by way of the '*child's play objection*', that Miller's content proposal violated the requirement of non-triviality. According to Miller's definition, one only has to add an independent false statement to a false theory for their deductive closure to improve the original theory.[3] Both *Language independence* and *Nontriviality* can be seen as general adequacy conditions that any verisimilitude definition has to meet, and until today no qualitative definition of verisimilitude straightforwardly satisfies both requirements.[4] With the analysis of Oddie (1986, 2001), the basic distinction between the content and likeness aspects of verisimilitude has found its way in the literature (see Niiniluoto, 2003, and Burger & Heidema, 2005).

As will be shown below, the content and likeness orders are not universally in conflict; they mainly disagree about the ordering of false theories. The question arises, therefore, whether some overarching verisimilitude definition faithfully combines content and likeness considerations. This is where Arrow's impossibility theorem becomes relevant for the verisimilitude project, provided that Arrow's axioms are reasonable requirements for the constitution of such an overarching definition.[5] The new Arrovian perspective allows us to compare different verisimilitude constraints and helps us to break from the deadlock of conflicting intuitions that has hampered the research project of verisimilitude for too long. In order to introduce Arrow's framework, however, we first have to define exactly the two verisimilitude orders involved, as the distinction between content and likeness definitions is generally recognized but attempts to define them are hard to find.

2.1 The definitions

All verisimilitude definitions that have the next definition as a necessary ingredient, we will call *content* definitions.

Definition 1 (*Content*) φ has at least as much truth-content as ψ iff

$$\text{Mod}(\varphi) \subseteq \text{Mod}(\psi) \cup \text{Mod}(\tau).$$

Notation: $\varphi \sqsupseteq \psi$.

In this definition, $\text{Mod}(\varphi)$ denotes the set of models of a sentence φ. We say that sentence φ has as much truth-content as $\psi, \varphi \simeq \psi$, iff $\varphi \sqsupseteq \psi$ and $\psi \sqsupseteq \varphi$, whereas φ has more truth-content than $\psi, \varphi \sqsupset \psi$, iff $\varphi \sqsupseteq \psi$ and not $\psi \sqsupseteq \varphi$. According to this characterization, Popper's original definition (1963, p. 233) is a content definition,

2 The 'language dependence' argument is cast as a reductio, but Zwart (2001) makes clear that Miller's argument only shows that likeness between possible worlds is not an absolute but a relative concept, and does not make the notion of truthlikeness meaningless at all. Gorham (1996, p. S227) has expressed a similar opinion.

3 Preferring logically weak false statements to strong ones, however, is hardly reconcilable with a Popperian form of realism.

4 See Britton (2004) and Miller (2006) for recent discussions of the persistent problem of language dependence.

5 The relevance of Arrow's theorem to epistemological issues has been argued for earlier by Darmstadter (1975), but only informally and staying much closer to the theorem's original scope in addressing the problem of aggregating the opinions of different scientists concerning the relative scientific worth of competing theories.

since Definition 1 equals its clause of truth-content.[6] Moreover, Definition 1 is also part of Miller's (1978) and Kuipers' (1982, 2000) symmetric difference definitions.

In the same vein, we will call all definitions that have the next constraint as a necessary ingredient *likeness* definitions.

Definition 2 (*Likeness*) φ is at least as truth-like as ψ iff

$$\forall \mathfrak{M} \in \mathrm{Mod}^{+}(\psi)\exists \mathfrak{N} \in \mathrm{Mod}(\varphi) : \mathfrak{N} \geq \mathfrak{M}, \text{ and}$$
$$\forall \mathfrak{N} \in \mathrm{Mod}^{-}(\varphi)\exists \mathfrak{M} \in \mathrm{Mod}(\psi) : \mathfrak{N} \geq \mathfrak{M}$$
$$\text{Notation: } \varphi \succcurlyeq \psi.$$

In this definition, $\mathrm{Mod}^{+}(\varphi)$ is the set of the 'best' models of φ with regard to our actual world $\mathfrak{T}$, and is defined as $\{\mathfrak{M} \in \mathrm{Mod}(\varphi) | \neg\exists \mathfrak{N} \in \mathrm{Mod}(\varphi) : \mathfrak{N} > \mathfrak{M}\}$. Here, $\mathfrak{N} > \mathfrak{M}$ means that $\mathfrak{N}$ is more like the actual world than $\mathfrak{M}$. Analogously, $\mathrm{Mod}^{-}(\varphi)$, the set of the 'worst' models of φ, is defined as $\{\mathfrak{M} \in \mathrm{Mod}(\varphi) | \neg\exists \mathfrak{N} \in \mathrm{Mod}(\varphi) : \mathfrak{N} < \mathfrak{M}\}$. Again sentence φ, is equally truth-like as $\psi, \varphi \sim \psi$, iff $\varphi \succcurlyeq \psi$ and $\psi \succcurlyeq \varphi$, and φ is more truth-like than $\psi, \varphi \succ \psi$, iff $\varphi \succcurlyeq \psi$ and not $\psi \succcurlyeq \varphi$ It is not difficult to prove that the proposals of Hilpinen (1976), Tichý and Oddie (Oddie, 1986), Heidema et al. (Burger & Heidema, 1994, 2002) and the refined qualitative proposal of Kuipers (2000) are likeness definitions in the above sense. They are all based on a partial order $>$ of all possible worlds with regard to their similarity to $\mathfrak{T}$, the possible world that represents our actual world.

It is perhaps needless to say that neither Definition 1 nor Definition 2 has been proposed as a complete definition of verisimilitude; they are at least implicitly included in all influential verisimilitude definitions mentioned. In what follows, we will call any verisimilitude definition that has Definition 1 as a necessary ingredient *a* content definition. By *the* content definition we mean Definition 1 itself. Likewise, we call the ordering of a set of sentences generated by Definition 1 *the* content order of these sentences, and any order of which the content order is a subset *a* content order. The distinction between *a* likeness definition or order, and *the* likeness definition or order is to be understood analogously. We refer to Zwart (2001) for proofs that the verisimilitude definitions mentioned above are content or likeness definitions in the sense here defined. Note that the Definitions 1 and 2 are weak in the sense that little is required for the definition to compare sentences. They are, therefore, strong in the sense that they order a considerable number of sentences compared to the verisimilitude definitions found in the literature. Definitions 1 and 2 result in little partiality and accordingly cover many sentences. This will prove important in the rest of the paper.

2.2 Some technicalities

In our presentation, the entities to be ordered are theories, which are deductively closed sets of $\mathcal{L}$-sentences, represented by their axioms. We assume $\mathcal{L}$ to include the Boolean connectives and to contain at least two independent sentences, and we will consider sentences modulo logical equivalence. Furthermore, 'the truth' in verisimilitude theory is *not* to be equated with the total true theory of everything. We define the true theory conservatively as the set of all true $\mathcal{L}$-sentences, which is purged of every metaphysical connotation. Throughout our paper, we assume the complete $\mathcal{L}$-sentence τ to axiomatize all empirically true $\mathcal{L}$-sentences.

[6] This does not mean that the content relation $\sqsupseteq$ is a subset of Popper's ordering relation of sentences, because Popper's definition fails to order false sentences.

Every set of possible worlds includes a set of best worlds and a set of worst world with respect to $<$. Leaving out the subscript $\mathfrak{T}$, in our representation $\mathfrak{N} \geq \mathfrak{M}$ denotes the fact that possible world $\mathfrak{N}$ is at least as much like the actual world, the 'true world' $\mathfrak{T}$, as $\mathfrak{M}$ is, and accordingly $\mathfrak{N} > \mathfrak{M}$ means that $\mathfrak{N}$ is more like the actual world than $\mathfrak{M}$.[7] Regarding the application of Arrow's theorem, we assume this basic order to be *partial* and *irreflexive*, and all worlds to be comparable to $\mathfrak{T}$ and to the single worst possible world $\mathfrak{T}^*$ Of course, $\mathfrak{T} \models \tau$ and a sentence φ is false iff $\varphi \models \neg\tau$. We write τ^* for the only sentence that is true in $\mathfrak{T}^*$ and false in all other, non-elementary equivalent, models of $\mathcal{L}$.

2.3 Conflict

Let us turn to an intuitive description of the differences between the content and likeness definitions. Regarding true sentences, the content and likeness orders do not deviate all that drastically. According to both definitions, for true sentences an increase of logical strength always leads to a theory that is as least as good as the weaker theory, and often to a better theory. The important difference between the two definitions concerns false sentences. According to content definitions, the negation of the truth, $\neg\tau$, is most distant from the truth, since it bears minimal information content, it gives false information, and it is not the consequence of any true theory. According to likeness definitions, however, $\neg\tau$ is not all that bad, since it has only one false consequence, namely itself, and its lack of information content is not considered a serious drawback. With regard to τ^* the roles are exactly reversed. Likeness definitions despise τ^*, not only because it has many false consequences, but also because it describes the possible world that is the least like $\mathfrak{T}$. Content definitions, in contrast, allow the many false consequences of τ^* to be counterbalanced by its enormous information content, that is, its many true and false consequences, since this means that τ^* runs the risk of being easily falsified, which is to be welcomed from a Popperian perspective. The above remarks are illustrated by the following example.

Example 1 Our standard example is a finite propositional language $\mathcal{L}[p_1, \ldots, p_n]$, where we choose $p_1 \wedge \cdots \wedge p_n$ to be τ, the complete truth of $\mathcal{L}$. Each of the following sets, then, is linearly ordered by the content and likeness definitions:

$$\left\{\varphi | \varphi := \bigwedge_{i=1}^{k} p_i \text{ for all } k, 1 \leq k \leq n\right\},$$

$$\left\{\psi | \psi := \bigwedge_{i=1}^{l} \neg p_i \text{ for all } l, 1 \leq l \leq n\right\}.$$

The content and likeness orders agree on the order of all pairs in the first set, but they disagree on the order of any pair of sentences in the second set. ⊣

3 Arrow's impossibility theorem

We now turn to the formal apparatus Arrow developed to merge several orders into one in the theory of social choice, and investigate the prospects for the reconciliation

[7] Of course, the models $\mathfrak{T}, \mathfrak{M}\, \mathfrak{N}$, etc. are actually representatives of sets of isomorphic models.

of the ordering principles of content and likeness. For an overview of social-choice theory see (Arrow, Sen, & Suzumura, 2002).

3.1 The requirements

The basic problem of social choice is to find a procedure that 'faithfully' or 'fairly' merges the preferences of several voters, represented by linear orders of a set of options, into a single linear 'collective' order. Arrow approached this problem by considering general procedures that map collections of individual orders to 'collective' orders. He formulated axioms that such a procedure must satisfy if the resulting order is to be called a faithful representation of the individual orders. In 1950 he published his famous theorem about functions attaching to all sets of m orders ($m \geq 2$) over n options ($n \geq 3$) a collective order of the same n options. Such functions always violate the requirement of *Unanimity*, of *Independence of irrelevant alternatives*, or of *Non-dictatorship* (Arrow, 1950, 1963). Let us consider Arrow's theorem in more detail.

Let X be a finite set of *alternatives* $\{a, b, c, \ldots\}$. A ranking $\gtrsim$ is a transitive, reflexive and connected order of all alternatives. The interpretation of $a \gtrsim b$ is that *a is at least as preferred as b*. If $a \gtrsim b$ and not $b \gtrsim a$, then a is strictly preferred to $b, a > b$. If $a \gtrsim b$ and $b \gtrsim a$, then the ranking is *indifferent* regarding a and $b, a \approx b$. We take $\mathcal{O}$ to be the set of all logically possible rankings O of the alternatives, and $\mathcal{P}$ to be the set of all profiles, where a *profile* $P = \{O_i\}$ is an i-indexed set, with $i \in \{1, 2, \ldots, n\}, n \geq 2$, of rankings of the same length. Intuitively, a ranking represents the way an individual 'voter' orders the alternatives, and the index i points to the various voters, such that a profile consists of the rankings held by the voters in a particular population.

Example 2 For the case of three individuals, the ordered triple $\langle a > b > c, b > c > a, a \approx b > c \rangle$ is an example of a profile.⊣

In the theory of social choice, an *order-unifying* function F maps *all* profiles of $\mathcal{P}$ into $\mathcal{O}$. It represents the way in which 'the collective' or 'society' makes its choice among the alternative options on the basis of the preferences of its citizens. We now introduce three requirements that specify what it means for an order-unifying function F to be fair or faithful.

Definition 3 An order-unifying function F respects:

1. *Unanimity* iff, for all alternatives a and b, F's image strictly prefers a to b whenever in all rankings a is strictly preferred to b.[8]
2. *Independence of irrelevant alternatives* iff, for all alternatives a and b, the way F's image ranks a with respect to b depends exclusively on how a is ranked with respect to b in the individual rankings.
3. *Non-dictatorship* iff, for all alternatives a and b, there is no individual i such that F's image strictly prefers a to b whenever individual i strictly prefers a to b.

3.2 Unrestricted domain and Collective rationality

Occasionally, the constraint stating that the accommodating procedure must be a function mapping *all* profiles of $\mathcal{P}$ into $\mathcal{O}$ is presented in the form of two demands,

[8] In social-choice theory this requirement is usually called the *Pareto* requirement or, more precisely, the *weak Pareto* requirement. For reasons of clarity we prefer to call it the *Unanimity* requirement.

called *Collective rationality* and *Unrestricted domain*, on a par with the above three requirements (e.g. Arrow, 1963; Sen, 1970). *Collective rationality* requires the procedure to attach to every profile an *order* of the alternatives; for instance, it does not merely single out the collectively best alternative. *Unrestricted domain* requires the social-choice procedure to be able to accommodate all logically possible profiles. These demands address the ability of the accommodating procedure to give an answer in the first place, rather than the faithfulness of the procedure. We prefer to distinguish between *Collective rationality* and *Unrestricted domain* as formal constraints on the unifying function, and *Unanimity, Independence of irrelevant alternatives* and *Non-dictatorship* as requirements governing its faithfulness.

With the formal apparatus in place and the relevant requirements introduced, we can now state Arrow's result precisely.

Theorem (Arrow) *Any order-unifying function, mapping the domain $\mathcal{P}$ of profiles containing n rankings ($n \geq 2$) of m options ($m \geq 3$) into the domain $\mathcal{O}$ of orders of m options, that respects Unanimity and Independence of irrelevant alternatives is Dictatorial.* $\dashv$

Since, the literature abounds with proofs of Arrow's theorem, we will not present one here. For a particularly short and elegant proof, the reader is referred to Geanakoplos (2001).

4 How Arrow's theorem fits verisimilitude

Before, we get to our main result in the next section, we first have to specify in more detail how Arrow's conceptual framework fits the problem of reconciling the content and likeness intuitions. We start with considering some superficial differences between the two; then we explain that in the verisimilitude case we have to concentrate on sentence variables; next we show that Arrow's framework and requirements fit the verisimilitude case; and finally we show that the requirement of *Invariance under language expansion* in verisimilitude is connected to Arrow's requirement of *Independence of irrelevant alternatives*.

4.1 Differences and similarities

Let us first consider the main similarities and differences between the Arrovian framework and that of verisimilitude. In situations of social choice as well as in the case of the merger of the likeness and content orders, conflicting orders of an arbitrary number of items have to be merged. Moreover, this merger must somehow, and to a sufficiently large extent, reflect the contributing orders. Furthermore, what is sought is not a solution for a particular profile of orders of particular sentences, but a unifying procedure that is able to take care of all possible profiles.

Turning to the differences, we first note that verisimilitude concerns ternary relations, where social choice theory focuses on dyadic relations. This problem evapourates, however, if we take one of the relata, the truth τ, to be a constant. Second, in the case of verisimilitude, the cardinality of the profiles is fixed to two, content and likeness, whereas the theory of social choice allows any finite number of voters.[9]

[9] If, besides content and likeness, more kinds of orders are to be distinguished, the Arrovian approach becomes even more relevant.

Verisimilitude can be seen as a special case, however, since Arrow's theorem covers the combination of only two rankings. Finally, Arrow's theorem concerns connected or complete orders whereas almost all qualitative verisimilitude definitions are partial relations. As Arrow's theorem is a negative result, it suffices to show that the theorem holds in the content-likeness case for linear suborders. Such suborders do exist, witness Example 1. Even for a language of two sentences, the content and likeness definition linearly orders subsets of up to six sentences. What matters here is that if the unified content-likeness order fails to order these subsets of $\mathcal{L}$-sentences, then it will fail for the set of all $\mathcal{L}$-sentences in general.

4.2 Sentences and sentence variables

The content and likeness orders of a specific set of sentences are completely fixed and mutually exclusive as soon as a target sentence is given. Consider the following example:

Example 3 If $\tau := p \wedge q$, then according to the content order $\neg p \wedge \neg q$ is more verisimilar than $\neg p \vee \neg q$, i.e. $\neg p \wedge \neg q \sqsupset \neg p \vee \neg q$, while the likeness definition ranks these sentences the other way around: $\neg p \vee \neg q \succ \neg p \wedge \neg q$. $\dashv$

One easily proves that for all propositional languages with more than one atomic proposition, the strict content order contradicts the strict likeness order regarding some sentence pairs. This leads to the question whether a general unification procedure exists that faithfully reconciles the contradictions between likeness and content into a single overall order, in a consistent and general way. Our aim is to discuss the possibility of merging the content and likeness orders on a general level, the latter meaning that the procedure takes the sentences to be ordered as indexed black boxes. It uses only the *position* of the sentences in the content and likeness orders and nothing else. In social choice, voting procedures are often investigated on the same level of generality. A voting procedure satisfying the requirement of *Neutrality* does not consider the content of the various alternatives. We will, therefore, apply the Arrovian framework at the level of sentence *variables* rather than specific sentences. Consequently, in the remainder of this paper, the term 'profile' must be read as 'profile of content-likeness orders of sentence *variables*', and 'order' abbreviates 'order of sentence *variables*'. Accordingly, in this paper φ, ψ and χ are sentence variables, and $\alpha, \beta, \zeta, \eta, \theta, \iota, \kappa$ are sentence constants. Likewise, from now on $\mathcal{O}$ refers to the set of all orders of sentence variables, and $\mathcal{P}$ denotes the set of all profiles of $\mathcal{O}$-elements.

Our aim of merging the content and likeness orders of $\mathcal{L}$-sentences may be achieved by a procedure mapping profiles in $\mathcal{P}$ to rankings in $\mathcal{O}$. To see why we require this unifying procedure to be a *function* on the domain of profiles of sentence variables, consider a procedure that fails to be a function. This would mean that, for at least one instantiation, it fails to map a profile onto a unique order of sentence variables. Consider the following example.

Example 4 Let a unifying procedure applied to instantiation $\langle \alpha \sqsupset \beta, \beta \succ \alpha \rangle$ of profile $\langle \varphi \sqsupset \psi, \psi \succ \varphi \rangle$ result in $\alpha > \beta$, and let the outcome of the same procedure on the instantiation $\langle \zeta \sqsupset \eta, \eta \succ \zeta \rangle$ of the same profile be $\eta \approx \zeta$. This procedure fails to be a function, because the profile $\langle \varphi \sqsupset \psi, \psi \succ \varphi \rangle$ is mapped sometimes to $\varphi > \psi$ and sometimes to $\psi \approx \varphi$. $\dashv$

In this example, the content and the likeness definition mutually rank sentences α and β exactly as they rank sentences ζ and η. According to the non-functional procedure, however, the mutual α and β ranking in the overall verisimilitude differs from the mutual ranking of ζ and η. Since, the unifying procedure is supposed to be a faithful unification of the content and likeness orders, if it fails to be a function, it fails to treat the ordered sentences as indexed black boxes. Either the procedure is entirely arbitrary, or it uses additional information about these sentences to come to the overall order. This is exactly the consideration of particular sentences we wish to avoid when discussing the problem at a general level.

4.3 Non-dictatorship, Unanimity and Independence

In this section, we explain why it is reasonable to require the unifying function on the level of sentence variables to satisfy *Unanimity, Non-dictatorship*, and *Independence of irrelevant alternatives. Non-dictatorship* and *Unanimity* strike us as necessary requirements to put on a faithful unifying function. *Non-dictatorship* expresses a quintessential aspect of what it means for a unification to be faithful or fair on the subset of comparable sentences, which is that both parties have at least some 'share' in the resulting order. A violation of the *Unanimity* principle would affect the rationality of the unifying function; if the content and likeness orders agree, no compelling reason can be found for the unified order to deviate from this preference order.

This leaves us with *Independence of irrelevant alternatives*. In the content-likeness case, this requirement means that the way the unifying function orders two sentences does not depend on the other alternatives in the rankings. To judge the reasonableness of this condition, let us have a look at a case where it is violated.

Example 5 Let φ, ψ and χ be $\mathcal{L}$-sentence variables. Suppose that $F(\langle\varphi \sqsupset \psi, \psi \succ \varphi\rangle) = \varphi > \psi$ and $F(\langle\varphi \sqsupset \psi \sqsupset \chi, \chi \succ \psi \succ \varphi\rangle) = \chi > \psi > \varphi$. This means that, although in both profiles $\varphi \sqsupset \psi$ and $\psi \succ \varphi$, the mutual ranking of φ and ψ in the overall order depends on whether sentence χ is taken into consideration. $\dashv$

If a function violates *Independence*, the consistency of the union of the overall orders on the subsets is not guaranteed. A possible way out would be to consider only the set of all sentences, but due to the partiality of the content and likeness orders, the profile covering all sentences does not exist. *Independence* is therefore an unassailable requirement for a content-likeness unifying function.[10]

4.4 Language expansions

A separate important issue concerns the relation between the requirements of *Independence of irrelevant alternatives* and in verisimilitude *invariance* under *language expansion*. Comparing the way two or more sentences are ordered in two different profiles can be done within a single, fixed language or across different languages, which, of course, must share some sentences. An expansion of a language, however, may change the order relations between sentences of the language. In the content order, for instance, under a conservative expansion of the language with additional

[10] Note that a violation of *Independence* does not imply path-dependence, which means that the order attached to a profile depends on the order in which the sentences making up the profile have been taken into account. Path independence is guaranteed by the unifying procedure being a function on the domain of profiles.

atomic sentences, the truth remaining complete, certain strict orderings between sentences drop out and certain equivalences change into strict orderings.

Let $\mathcal{L}'$ be an expansion of language $\mathcal{L}$. As defined by Zwart (2001), a verisimilitude definition is strongly invariant under language expansion if all strict verisimilitude orders in $\mathcal{L}$ are preserved in $\mathcal{L}'$.[11] All likeness orders mentioned in Sect. 2.1 are strongly invariant. Only the definitions of Miller (1978) and Kuipers (1982, 2000) are not invariant in the sense that some strict preferences in the poorer language vanish in the richer one. None of the eight proposals reviewed by Zwart (2001) allows for the inversion of strict orderings when the language is expanded. The relation between *Invariance under language expansion* and *Independence of irrelevant alternatives* is obvious. Expansion of the language introduces new options in the verisimilitude order, and for definitions that are invariant under language expansion this does not affect the rankings of the sentences. If new options do not change the individual content or likeness ranking of two alternatives, then, for definitions invariant under language expansions, these options do not change the new overall ranking of these alternatives. To prevent confusions between the invariance issue and the unification of the content and likeness orders, in the rest of the paper, we assume sentences to be taken from one, fixed, overarching language.

5 An impossibility result

By now, we have seen that the problem of constructing a collective verisimilitude order on a subset of sentence variables is similar to Arrow's problem of social-choice; i.e. a function is to be found that maps all occurring profiles of several orders into a single order. Moreover, Arrow's requirements are perfectly reasonable for the unification of the content and likeness intuitions. Consequently, Arrow's theorem seems to imply the impossibility of constructing an overall verisimilitude order that incorporates the content and likeness orders while meeting the three requirements of *Unanimity, Independence of irrelevant alternatives* and *Non-dictatorship*. Yet, this is not so. The reason is the *Unrestricted domain* condition of Sect. 3.2. When applied to profiles of orders of sentence variables, this condition implies that all logically possible profiles are instantiated by sentences of the language. This, however, turns out not to be the case. Since the proof of Arrow's theorem depends crucially on this *Unrestricted domain* restriction, the impossibility of unifying the content and likeness orders does not follow straightforwardly from the theorem.

5.1 Domain restrictions in verisimilitude setting

In this section, we prove that even for profiles of length three, not all logically possible verisimilitude profiles have instances, which implies domain restrictions for any number of ranked sentences greater than two. To prove this, we use two lemmas. The first lemma states that if the strict content order of φ and ψ contradicts the likeness order of φ and ψ, then the sentence that has the most truth-content is false.

Lemma 1 *For all $\mathcal{L}$-sentences φ and ψ, if $\varphi \sqsupset \psi$ and $\psi \succ \varphi$, then φ is false.*

[11] In his (2001), Zwart uses the term *context independence* instead of *invariance under language expansion*. In this paper, we prefer to use the latter terminology.

Proof Let $\varphi \sqsupseteq \psi$ and $\psi \succ \varphi$. Suppose φ is true; this implies $\text{Mod}^+(\varphi) = \{\mathfrak{T}\}$. Since $\psi \succ \varphi$ means that $\text{Mod}^+(\varphi) \subseteq \text{Mod}(\psi)$, it follows that ψ is also true. For true φ and $\psi, \varphi \sqsupseteq \psi$ is equivalent to $\text{Mod}(\varphi) \subseteq \text{Mod}(\psi)$, which implies $\text{Mod}^-(\varphi) \subseteq \text{Mod}(\psi)$. Now $\text{Mod}^+(\varphi) = \{\mathfrak{T}\} = \text{Mod}^+(\psi)$ and $\text{Mod}^-(\varphi) \subseteq \text{Mod}(\psi)$ together imply that $\varphi \succcurlyeq \psi$, which contradicts the assumption $\psi \succ \varphi$. Therefore φ is false. □

Our second lemma shows that under the same circumstances as Lemma 1 the sets of worst models of both sentences are the same.

Lemma 2 *For all $\mathcal{L}$-sentences φ and ψ, if $\varphi \sqsupseteq \psi$ and $\psi \succ \varphi$, then* $\text{Mod}^-(\varphi) = \text{Mod}^-(\psi)$.

Proof Let $\varphi \sqsupseteq \psi$ and $\psi \succ \varphi$. From Lemma 1 it follows that φ is false, and therefore the condition $\text{Mod}(\varphi) \subseteq \text{Mod}(\psi) \cup \{\mathfrak{T}\}$ required by $\varphi \sqsupseteq \psi$ reduces to $\text{Mod}(\varphi) \subseteq \text{Mod}(\psi)(*)$. Now, suppose that $\text{Mod}^-(\varphi) \neq \text{Mod}^-(\psi)$. This means that (1) there is an $\mathfrak{N}$ in $\text{Mod}^-(\varphi)\backslash\text{Mod}^-(\psi)$, or (2) there is an $\mathfrak{M}$ in $\text{Mod}^-(\psi)\backslash\text{Mod}^-(\varphi)$. Let (1) be the case. Due to (*), $\mathfrak{N}$ in $\text{Mod}^-(\varphi)\backslash\text{Mod}^-(\psi)$ is in $\text{Mod}(\psi)$. This implies there is an $\mathfrak{M}$ in $\text{Mod}(\psi)$ such that $\mathfrak{N} > \mathfrak{M}$, which contradicts $\psi \succ \varphi$. Let (2) be the case. Suppose $\mathfrak{M}$ in $\text{Mod}^-(\psi)\backslash\text{Mod}^-(\varphi)$ is in $\text{Mod}(\varphi)$. Then there is an $\mathfrak{N}$ in $\text{Mod}(\varphi)$ such that $\mathfrak{M} > \mathfrak{N}$. Due to (*), $\mathfrak{M}$ is also in $\text{Mod}(\psi)$, which, together with $\mathfrak{M} > \mathfrak{N}$, contradicts $\mathfrak{M}$ in $\text{Mod}^-(\psi)$. So $\mathfrak{M}$ is not in $\text{Mod}(\varphi)$, which implies there is no $\mathfrak{N}$ in $\text{Mod}(\varphi)$ such that $\mathfrak{M} \geq \mathfrak{N}$, which again contradicts $\psi \succ \varphi$. Hence, $\text{Mod}^-(\varphi) = \text{Mod}^-(\psi)$. □

Lemmas 1 and 2 enable us to prove a domain restriction on the domain of content-likeness profiles.

Proposition 1 *For all $\mathcal{L}$-sentences φ, ψ and χ, if $\chi \succ \varphi \succ \psi$ then not $\varphi \sqsupseteq \psi \sqsupseteq \chi$.*

Proof Let $\chi \succ \varphi \succ \psi$. First suppose $\varphi \sqsupseteq \psi \sqsupset \chi$. By applying Lemma 2 to (φ, χ) and (ψ, χ), we see that $\text{Mod}^-(\varphi) = \text{Mod}^-(\psi) = \text{Mod}^-(\chi)(\dagger)$. In the same way, Lemma 1 implies that φ and ψ are false. Now as (†) holds, $\varphi \succ \psi$ implies that for at least one model $\mathfrak{M}$ in $\text{Mod}^+(\psi)$ there is a model $\mathfrak{N}$ in $\text{Mod}(\varphi)$ such that $\mathfrak{N} > \mathfrak{M}$. So $\mathfrak{N}$ is in $\text{Mod}(\varphi)\backslash\text{Mod}(\psi)$, and additionally $\mathfrak{N} \neq \mathfrak{T}$ because φ is false. This contradicts $\text{Mod}(\varphi) \subseteq \text{Mod}(\psi) \cup \{\mathfrak{T}\}$ as required by $\varphi \sqsupseteq \psi$. Hence, not $\varphi \sqsupseteq \psi \sqsupset \chi$. Next suppose $\varphi \sqsupseteq \psi \simeq \chi$. By applying Lemma 1 to $(\varphi, \chi), \varphi$ is false, and by applying Lemma 2 to $(\varphi, \chi), \text{Mod}^-(\varphi) = \text{Mod}^-(\chi)$. Since φ is false and $\varphi \succ \psi, \psi$ is also false. Now $\psi \simeq \chi$ is by definition equivalent to, $\text{Mod}(\chi) \subseteq \text{Mod}(\psi) \cup \{\mathfrak{T}\}$ and $\text{Mod}(\psi) \subseteq \text{Mod}(\chi) \cup \{\mathfrak{T}\}$. For false ψ, the latter part is equivalent to $\text{Mod}(\psi) \subseteq \text{Mod}(\chi)$. Suppose χ is false; then in the same way $\text{Mod}(\chi) \subseteq \text{Mod}(\psi)$. It follows that χ and ψ are logically equivalent, contradicting $\chi \succ \psi$. So χ is true, and $\text{Mod}(\psi) \subseteq \text{Mod}(\chi)$ is equivalent to $\text{Mod}(\psi) \cup \{\mathfrak{T}\} \subseteq \text{Mod}(\chi)$. Therefore, $\text{Mod}(\chi) = \text{Mod}(\psi) \cup \{\mathfrak{T}\}$. Consequently, $\text{Mod}^-(\psi) = \text{Mod}^-(\chi) = \text{Mod}^-(\varphi)$, in which case $\varphi \succ \psi$ implies that there is an $\mathfrak{M}$ in $\text{Mod}^+(\varphi)\backslash\text{Mod}(\psi)$. This contradicts $\text{Mod}(\varphi) \subseteq \text{Mod}(\psi)$, which is, for false φ and ψ, equivalent to $\varphi \sqsupseteq \psi$. Hence, not $\varphi \sqsupseteq \psi \simeq \chi$. □

Note that Proposition 3 does not imply that there are instances of the profiles $\langle \varphi \sqsupset \psi \sqsupset \chi, \psi \succ \varphi \succ \chi \rangle$ and $\langle \varphi \sqsupset \psi \sqsupset \chi, \chi \succ \psi \succ \varphi \rangle$; nor, of course, does it entail that all remaining logically possible profiles have instances. In Table 1, we give instances of all profiles, for arbitrary languages $\mathcal{L}$, in which exclusively strict orderings occur and that Proposition 1 does not exclude. Additionally, twenty-four logically possible profiles combine strict orderings and equivalences. Not all of them

Table 1 Instances of content-likeness profiles in $\mathcal{L}[p_1,\ldots,p_n]$, where $\tau = p_1 \wedge p_2 \wedge \cdots \wedge p_n$

	φ	ψ	χ	profile
Instances of strict profiles in $\mathcal{L}[p_1,\ldots,p_n]$, $n \geq 2$				
1	τ	$\tau \vee \zeta$	$\tau \vee \zeta \vee \eta$	$\{\varphi \sqsupset \psi \sqsupset \chi, \varphi \succ \psi \succ \chi\}$
2	τ^*	$\tau^* \vee \theta$	$\tau^* \vee \theta \vee \iota$	$\{\varphi \sqsupset \psi \sqsupset \chi, \chi \succ \psi \succ \varphi\}$
3	τ	τ^*	$\tau^* \vee \theta$	$\{\varphi \sqsupset \psi \sqsupset \chi, \varphi \succ \chi \succ \psi\}$
4	τ^*	$\tau \vee \tau^* \vee \zeta$	$\tau^* \vee \zeta \vee \kappa$	$\{\varphi \sqsupset \psi \sqsupset \chi, \psi \succ \chi \succ \varphi\}$
Instances of strict profiles in $\mathcal{L}[p_1,\ldots,p_n]$, $n \geq 4$				
5	$\zeta \vee \theta$	$\tau \vee \zeta \vee \theta \vee \kappa$	$\tau^* \vee \zeta \vee \theta \vee \kappa$	$\{\varphi \sqsupset \psi \sqsupset \chi, \psi \succ \varphi \succ \chi\}$
Instances of profiles with equivalences in $\mathcal{L}[p_1,\ldots,p_n]$, $n \geq 2$				
6	$\tau \vee \tau^*$	$\tau^* \vee \theta$	$\tau \vee \tau^* \vee \theta \vee \eta$	$\{\varphi \sqsupset \psi \sqsupset \chi, \varphi \sim \chi \succ \psi\}$
7	$\tau \vee \tau^*$	τ^*	$\tau^* \vee \theta$	$\{\varphi \simeq \psi \sqsupset \chi, \varphi \succ \chi \succ \psi\}$

Here $\zeta := \neg p_1 \wedge p_2 \wedge \cdots \wedge p_n, \eta := p_1 \wedge \neg p_2 \wedge p_3 \wedge \cdots \wedge p_n, \theta := \neg p_1 \wedge p_2 \wedge \neg p_3 \wedge \cdots \wedge \neg p_n$, $\iota := p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$, and $\kappa := \neg p_1 \wedge p_2 \wedge \neg p_3 \wedge p_4 \wedge \cdots \wedge p_n$.

have instances in the domain of profiles either. Proposition 3 excludes three of them. Besides, it follows easily from the definitions that two non-equivalent sentences differ regarding their truthlikeness or have different truth-content. In Table 1, we also give instances of two mixed content-likeness profiles that we will need later in the paper.

5.2 The theorem

For the adherents of a comprehensive verisimilitude concept who take courage from the domain restriction proved in the previous section, the message is that the situation remains critical. We now prove our main result:

Theorem *Let F be a unifying function from the domain of content-likeness profiles into the set $\mathcal{O}$ of all reflexive orders. If F satisfies Unanimity and Independence of irrelevant alternatives, then F is Dictatorial.*

Proof The proof is an examination of all logical possibilities. Let *Unanimity* and *Independence* hold. The latter implies that F is fixed once it is defined for all profiles of two sentence variables, φ and ψ, incorporating the orders of two 'voters', called *Ct* and *Lk*. First, we consider the profiles where *Ct* and *Lk* agree, viz. $\langle \varphi \sqsupset \psi, \varphi \succ \psi \rangle$ and $\langle \varphi \simeq \psi, \varphi \sim \psi \rangle$. Since, the latter has no instances, we need not take it into consideration. The first complies with the dictatorship of either *Ct* or *Lk*, since by *Unanimity* $F(\langle \varphi \sqsupset \psi, \varphi \succ \psi \rangle) = \varphi > \psi$. Next, we examine the cases in which *Ct* and *Lk* disagree. Due to *Independence*, there are three two-sentence profiles to be considered:

$$\text{A. } \langle \varphi \sqsupset \psi, \psi \succ \varphi \rangle, \quad \text{B. } \langle \varphi \simeq \psi, \psi \succ \varphi \rangle, \quad \text{C. } \langle \varphi \sqsupset \psi, \psi \sim \varphi \rangle.$$

In each of these cases, F may map the profile to one of the following three:

$$1.\ \varphi \approx \psi, \quad 2.\ \varphi > \psi, \quad 3.\ \psi > \varphi.$$

We have to consider all 27 possible two-sentence profiles combining A1, A2, and A3 with $B1, B2, \ldots, C3$, where A1 means $F(\langle \varphi \sqsupset \psi, \psi \succ \varphi \rangle) = \varphi \approx \psi$, and so forth. First, we consider combination A1 and define $F^{\approx}(\langle \varphi \sqsupset \psi, \psi \succ \varphi \rangle) = \varphi \approx \psi$. Now, consider $\langle \varphi \sqsupset \psi \sqsupset \chi, \psi \succ \chi \succ \varphi \rangle$, which, according to Table 1, is instantiated in $\mathcal{L}$. Since, $F^{\approx}(\langle \varphi \sqsupset \psi, \psi \succ \varphi \rangle) = \varphi \approx \psi$ and $F^{\approx}(\langle \varphi \sqsupset \chi, \chi \succ \varphi \rangle) = \chi \approx \varphi$, the transitivity of the $\approx$-relation implies $F^{\approx}(\langle \varphi \sqsupset \psi \sqsupset \chi, \psi \succ \chi \succ \varphi \rangle) = \chi \approx \varphi \approx \psi$. According to

Unanimity, however, $F^{\approx}(\langle\varphi \sqsupset \psi \sqsupset \chi, \psi \succ \chi \succ \varphi\rangle) \supset \psi > \chi$. Contradiction. So, with A1, its nine possible combinations with $B1, B2, \ldots, C3$ are excluded.

Let us call the A2 combination, defining $F^{\mathrm{C}}(\langle\varphi \sqsupset \psi, \psi \succ \varphi\rangle) = \varphi > \psi(^{*})$, the *content combination*, and the A3 combination, defining $F^{\mathrm{L}}(\langle\varphi \sqsupset \psi, \psi \succ \varphi\rangle) = \psi > \varphi(\dagger)$, the *likeness combination*. We claim (1) that all consistent combinations of F^{C} with any pair out of the combinations B1, B2, B3 and C1, C2, C3 are content dictators; (2) likewise, that all consistent combination of F^{L} with any pair out of the combinations B1, B2, B3 and C1, C2, C3 are likeness dictators.

Ad (1). Note first that the combination of F^{C} with either B1 or B2 or B3 complies with the dictatorship of *Ct*. Let $F^{\mathrm{C}}_{\mathrm{B}}$ refer to any of these three combinations. Turning to C1, C2, C3, we see that C2 complies with a dictatorship of *Ct*. Therefore the three functions $F^{\mathrm{C}}_{\mathrm{B\cup C2}}$ are each content-dictators. This leaves the six content combinations $F^{\mathrm{C}}_{\mathrm{B\cup C1}}$ and $F^{\mathrm{C}}_{\mathrm{B\cup C3}}$ to be considered. For these functions we have $F^{\mathrm{C}}_{\mathrm{B\cup C1/3}}(\langle\varphi \sqsupset \psi, \psi \sim \varphi\rangle) = \psi \gtrsim \varphi$. These content combinations, however, all lead to a contradiction, since $F^{\mathrm{C}}_{\mathrm{B\cup C1/3}}(\langle\varphi \sqsupset \psi \sqsupset \chi, \varphi \sim \chi \succ \psi\rangle) \supset \chi \gtrsim \varphi$. Because of (*), however, $F^{\mathrm{C}}_{\mathrm{B\cup C1/3}}(\langle\varphi \sqsupset \psi \sqsupset \chi, \varphi \sim \chi \succ \psi\rangle) \supset \psi > \chi$, and because of *Unanimity* $F^{\mathrm{C}}_{\mathrm{B\cup C1/3}}(\langle\varphi \sqsupset \psi \sqsupset \chi, \varphi \sim \chi \succ \psi\rangle) \supset \varphi > \psi$. Together $\varphi > \psi$ and $\psi > \chi$ imply $\varphi > \chi$ by transitivity, which contradicts $\chi \gtrsim \varphi$. Table 1 shows that $\langle\varphi \sqsupset \psi \sqsupset \chi, \varphi \sim \chi \succ \psi\rangle$ is instantiated in $\mathcal{L}$. Consequently, $F^{\mathrm{C}}_{\mathrm{B\cup C2}}$ are the only possible functions covering A2, and they all make *Ct* a dictator.

Ad (2). The line of reasoning is similar to the case of A2. First, all extensions of F^{L} with C1 or C2 or C3, denoted by $F^{\mathrm{L}}_{\mathrm{C}}$, comply with the dictatorship of *Lk*. Next, the three combinations $F^{\mathrm{L}}_{\mathrm{C\cup B3}}$ indeed all make *Lk* a dictator. The only cases left to be considered are $F^{\mathrm{L}}_{\mathrm{C\cup B1/2}}$, such that $F^{\mathrm{L}}_{\mathrm{C\cup B1/2}}(\langle\varphi \simeq \psi, \psi \succ \varphi\rangle) = \varphi \gtrsim \psi$. These six possible functions again all contradict the conjunction of *Unanimity* and *Neutrality*. For consider

$F^{\mathrm{L}}_{\mathrm{C\cup B1/2}}(\langle\varphi \simeq \psi \sqsupset \chi, \varphi \succ \chi \succ \psi\rangle)$. Respecting *Unanimity*,
$F^{\mathrm{L}}_{\mathrm{C\cup B1/2}}(\langle\varphi \simeq \psi \sqsupset \chi, \varphi \succ \chi \succ \psi\rangle) \supset \varphi > \chi$, and according to (†),
$F^{\mathrm{L}}_{\mathrm{C\cup B1/2}}(\langle\varphi \simeq \psi \sqsupset \chi, \varphi \succ \chi \succ \psi\rangle) \supset \chi > \psi$. Together $\varphi > \chi$ and $\chi > \psi$ imply $\varphi > \psi$

because of transitivity. On the basis of its definition, however, $F^{\mathrm{L}}_{\mathrm{C\cup B1/2}}(\langle\varphi \simeq \psi \sqsupset \chi, \varphi \succ \chi \succ \psi\rangle) \supset \psi \gtrsim \varphi$. Again we have a contradiction. Table 1 shows that the profile $\langle\varphi \simeq \psi \sqsupset \chi, \varphi \succ \chi \succ \psi\rangle$ is instantiated in $\mathcal{L}$. The only possible unifying functions among the likeness combination, therefore, are the three functions $F^{\mathrm{L}}_{\mathrm{C\cup B3}}$, and these all make *Lk* a dictator. Consequently, all logically possible combinations of A1, A2, A3 with B1, B2, B3 and C1, C2, C3 lead either to a *Ct*-dictatorship or to a *Lk*-dictatorship □

Note that the exclusion of $F^{\approx}$ means that unification on the basis of a *consensus rule* is impossible. Such a rule states that two sentences are strictly ordered only if both *Ct* and *Lk* agree about their order, and considers sentences equivalent in all other cases.

6 Discussion

In formal philosophy of science, the significance of a formal result often depends on its interpretation. The most important question regarding the previous formal result therefore reads: What are the philosophically implications of the theorem?[12] In the next three subsections, we discuss three possible reactions.

6.1 Two fundamentally different concepts

Taken at face value, our impossibility theorem can be interpreted as a no-go theorem for a global and fair likeness-content verisimilitude axiomatization. It excludes any fair or faithful overall combination of qualitative content and likeness explications of Popper's verisimilitude notion, provided the acceptation of the Arrovian requirements and our explications of the content and likeness intuitions. If these are agreed to, we may safely conclude that the verisimilitude idea covers at least two radically different and incompatible intuitions about content and likeness. On this interpretation, the theorem strengthens the 'language-dependence' and 'child's play' arguments of Miller and Tichý, which already stressed the difference between the two similarity notions.

Acknowledging this incompatibility, one may adopt the fundamentalist point of view stating that we need a single unique explanation of verisimilitude for all situations and contexts. The best we can do, then, will be to single out either content or likeness as the one basic notion underlying Popper's notion of verisimilitude, and fully discard the other one. If this route is taken, our theorem will only deepen the gap between the supporters of likeness and content, and it will have failed to make a positive contribution to the discussion about verisimilitude. Things, however, are often not as black as they seem. Let us therefore consider another point of view.

6.2 Context-dependent combinations

The second possible point of view is to follow Popper's suggestion in *Objective knowledge* (1979, p. 372), where he says, after acknowledging the mistake in his definition:

> Perhaps it [the problem of defining verisimilitude] cannot be solved by purely logical means but only by a relativization to problems or even by bringing in the historical problem situation.

The second route, then, is to admit that content and likeness are different similarity notions, but that nonetheless, for many contexts, they can be combined in a reasonable way into an adequate verisimilitude concept. From this perspective, it is the 'problem situation' or context that determines what combination of likeness and content deliberations makes a reasonable verisimilitude definition. This takes the sting out of Miller's 'language dependence' argument, since if similarity judgements are relativized to a set of relevant properties, a change of this set may cause a change

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[12] Notice that the consequences of our theorem go even beyond the theory of verisimilitude proper and concern any field where theories or hypotheses are compared regarding their similarity to an ideal. An important example is the notion of empirical adequacy. We completely agree with the "fair conjecture" made by Sarkar (1998, p. 385) stating that "if 'verisimilitude' cannot be defined, then 'is empirically less inadequate than' cannot be defined either." The implications of our impossibility result do not distinguish between a realist and an anti-realist point of view. The theorem leaves open the question whether the instantiations of the variables are directly measurable values or theoretical values. Moreover, any measure of empirical adequacy comparing false empirical hypotheses of different logical strength must balance the putting forward of incorrect answers and the refusal to give any answer at all. Since, the former corresponds to likeness and the latter to content considerations, no measure of empirical adequacy is a fair merger of content and similarity considerations.

of similarity judgements. Similarly, the introduction of contextual considerations also takes the sting out of the child's play argument. Originally, the addition of an arbitrary independent falsehood to a false theory brought this theory closer to the truth. When we bring in the context, however, only those sentences are acceptable that are relevant in the specific contexts. This makes theory improvement much less a case of child's play.

In some contexts, content considerations are far less important than considerations of likeness. For instance, in a context where the number of false consequences is less important than being close to the target, an overall likeness definition will suffice. An example of such a context is the following. The statement '20 students are in that classroom' seems nearer to the truth than 'exactly 19 students are in that classroom' if in fact 21 students are in that classroom. This, however, does not imply that the answers to the question that are incomparable regarding content cannot be mixed with considerations of content. Let the first answer be extended with 'and they are all male' and the second with 'and they are all female', where the 21 students in fact are all female. Then, the second instead of the first answer may be considered the most verisimilar one.

Burger and Heidema (1994, 2002) have chosen to define verisimilitude by combining intuitions of content and likeness. Their overall comparison of theories is based on likeness; sentences left out of the overall likeness order are occasionally ordered by considerations of content. The refined verisimilitude proposal of Zwart (2001) goes the opposite way.[13] It takes the content order as the primary order, and only content-incomparable sentences are candidates for being ordered by likeness considerations of similarity between possible worlds. Clearly, these reasonable combinations of content and likeness notions give up the requirement of *Non-dictatorship*. The important point is, however, that if we take verisimilitude to be a context-dependent notion, the context determines whether the dictatorship of the one or the other similarity notion is acceptable. The partiality of the content and likeness orders, and the various ways in which individual models can be compared, leave room for various content-likeness combinations.

Looked upon in this way, our theorem gets a positive flavour. It shapes the discussion and may guide the process of determining which context is served best with which content or likeness considerations. On top of this, in the next section, we will see that the theorem may also tighten the link between the various qualitative definitions of verisimilitude and the quantitative min-sum measure.

6.3 Going quantitative

Niiniluoto (1987) has comprehensively elaborated quantitative truthlikeness measures. He favours one particular measure, the min-sum measure $>_{\mathrm{ms}}$, which is specifically constructed as a normalized and γ/γ'-weighted mean of the *minimal distance*, $\gtrsim_{\min}$, and the *sum distance*, $\gtrsim_{\mathrm{sum}}$, between sets of models. These two distances are closely related to considerations of likeness and content, respectively, the ratio γ/γ' weighing the content against the likeness behaviour of the definition.[14] We can show that for all $\mathcal{L}$-sentences φ and $\psi, \varphi \succcurlyeq \psi$ only if $\varphi \gtrsim_{\min} \psi$, and $\varphi \sqsupseteq \psi$ iff $\varphi \gtrsim_{\mathrm{sum}} \psi$ and φ, ψ $\sqsupseteq$-comparable. Moreover, we can prove that for all $\mathcal{L}$-sentences φ and ψ, if $\varphi \sqsupset \psi$ and $\varphi \succ \psi$, then $\varphi >_{\mathrm{ms}} \psi$, which means that the qualitative consequences of the

[13] See Zwart (2001).

[14] See, apart from the source text (Niiniluoto, 1987), Chap. 3 of Zwart (2001) for a concise overview. Note that Niiniluoto (1987, p. 204) presents his proposal as a quantitative version of Hilpinen (1976).

min-sum measure satisfy *Unanimity*. Given the similarity between, on the one hand, the min and sum measures and, on the other hand, the content and likeness orders, choosing the min-sum measure does not safeguard one against the consequences of our theorem. It curtails the min-sum measure, too, in the following way, a result that we state here without proof. Let P be the unifying procedure that applies only the qualitative consequences of the min-sum measure. Then for all values of γ and γ' for which P is a *function* on the domain of profiles, P grants dictatorship to the sum measure and therefore to the content definition.

The last observation shows that if one opts for context-relative combinations of content and likeness notions, the link between the quantitative min-sum measure and qualitative orders is even stronger than pointed out previously by Zwart (2001) and acknowledged by Niiniluoto (2003). Not only are content and likeness similar to the min and sum measure; *the same Arrovian restrictions* apply to both pairs of orders. Moreover, if we agree to let the context determine whether, for example, in the case where $\varphi \sqsupset \psi$ and $\psi \succ \varphi$, overall $\varphi > \psi$ or $\psi > \varphi$ holds, the qualitative approach is even more in accordance with the quantitative approach. The context determines the tuning of γ and γ', which in its turn determines whether the min-sum measure ranks φ and ψ as $\varphi >_{\mathrm{ms}} \psi$ or as $\psi >_{\mathrm{ms}} \varphi$.

If we follow Popper's suggestion to consider verisimilitude not as an absolute but as a relative notion, and therefore accept that it depends on the context which formal explication, which content-likeness combination, captures best our intuitions about better or worse theories, and if we take the min-sum measure as the formal explication of verisimilitude in quantitative contexts, then our Arrovian impossibility theorem has two positive sides. First, it prohibits a context-independent and faithful combination of the content and likeness considerations that is valid in all circumstances, and thus it forces us to accept different combinations in different circumstances. Second, it enables us to see much more unity between the qualitative and quantitative verisimilitude proposals than previously recognized. The theorem shows similar impossibilities for the qualitative definitions as for their quantitative counterparts, and doing so it may be said to fulfil a catalytic function.

7 Conclusion

In this paper, we have set out to break open the deadlock in the verisimilitude discussion on a general level. We abstracted from the content of specific sentences, and took as our starting point the distinction between verisimilitude notions based on content and those based on likeness. By proving an Arrovian impossibility result for the restricted verisimilitude domain, we have shown, on this general level, the impossibility of a faithful combination of likeness and content considerations. Our impossibility theorem can therefore be seen to substantiate the view that the likeness and content considerations are radically different similarity notions. Acknowledging the difference of similarity notions based on content and on likeness, we have shown that Popper's suggestion of bringing in the context and to discard the idea that verisimilitude is an absolute notion is a useful one. Accepting, then, the point of view that for some contexts particular combinations are reasonable, our impossibility theorem adds to the coherence of verisimilitude research. Not only does it underline the distinction between the content and likeness intuitions, it also stresses the similarity between qualitative combinations of content and likeness and the quantitative combination of min and sum measures.

The implications of our theorem for future research are at least threefold. First, it should be investigated in more detail how the qualitative verisimilitude combinations and the quantitative min-sum measure are related. Second, approaches that we have not discussed here should be investigated to see how they fit into the general scheme set out in this paper. Finally, and perhaps most importantly, an investigation is due of the various relevant contexts and the way they favour particular verisimilitude combinations.

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