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Geyer, Anna; Quirchmayr, Ronald

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WEAKLY NONLINEAR WAVES IN STRATIFIED SHEAR FLOWS

Anna Geyer
Delft University of Technology
Delft Institute of Applied Mathematics, Faculty of EEMCS
Mekelweg 4, 2628 CD Delft, The Netherlands

Ronald Quirchmayr*
University of Vienna
Faculty of Mathematics
Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

Abstract. We develop a Korteweg–De Vries (KdV) theory for weakly nonlinear waves in discontinuously stratified two-layer fluids with a generally prescribed rotational steady current. With the help of a classical asymptotic power series approach, these models are directly derived from the divergence-free incompressible Euler equations for unidirectional free surface and internal waves over a flat bed. Moreover, we derive a Burns condition for the determination of wave propagation speeds. Several examples of currents are given; explicit calculations of the corresponding propagation speeds and KdV coefficients are provided as well.

1. **Introduction.** This work is concerned with the derivation of Korteweg–De Vries (KdV) equations modeling the propagation of weakly nonlinear waves in a two-dimensional stratified incompressible inviscid fluid consisting of two layers with constant densities. The fluid domain is bounded by a free surface and a flat bed; the two layers are separated by an impermeable interface. Additionally, a general steady horizontal current with arbitrary vorticity profile is prescribed. As underlying physical model serve the divergence-free incompressible Euler equations equipped with the usual dynamic and kinematic boundary conditions.

The system under study is motivated by a similar geophysical model for wave-current interactions with the Equatorial Undercurrent (EUC) proposed by Constantin and Johnson [10], in which the presence of a background current and two fluid layers with different constant densities are considered. This type of stratification is mainly caused by a sharp temperature gradient at around 100 meters of depth; the resulting interface is in this context referred to as thermocline. Mathematically, there are only two differences between these two models. Firstly, instead of using Euclidean coordinates, [10] applies the equatorial f-plane approximation of Euler's equations to account for Coriolis effects caused by the Earth's rotation.

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*Corresponding author.

Secondly, the background current in [10] is specifically chosen to mimic the EUC; its profile is piecewise linear to facilitate the analysis and enable explicit solution formulæ. In contrast, the present study considers more general background currents; we only rule out scenarios which entail the formation of critical layers.

Several studies appeared in recent years, which address the derivation of weakly nonlinear model equations for certain variations of the geophysical system in [10]. These derivations are based on Hamiltonian formulations of the governing equations and their boundary conditions. A first step in this direction was made in [9], which establishes a Hamiltonian formulation of the governing equations for the coupling between surface and internal waves (Coriolis effects are not taken into account and the background current consists of a simpler two component piecewise linear flow profile). This Hamiltonian formalism was applied in [7] to derive a weakly nonlinear model equation of KdV-type for the thermocline. Adaptations of this approach to the equatorial f-plane approximation can be found in [16, 8], which include derivations of KdV type equations as models for the thermocline. We refer to [3] for a Hamiltonian formulation of a similar problem with a fixed flat surface; the corresponding KdV approximation for the free interface was derived in [4]; for a KdV and Benjamin-Ono approximation we refer to [11]. The periodic problem and its Hamiltonian formulation for stratified currents in the equatorial f-plane was established in [15]. Furthermore, related problems with an uneven bottom and their Hamiltonian formulations have been addressed in [5] for surface waves over irrotational flows and in [6] for surface waves over certain background currents in the equatorial f-plane; both studies include KdV and Boussinesq approximations for the surface and some numerical solutions.

Johnson [21] proposes an alternative approach for the derivation of (weakly) nonlinear model equations from the underlying geophysical system in [10]. It is based on an asymptotic power series approach, which is directly applied to the governing equations without the detour over Hamiltonian formalisms. This method has the advantage of greater generality: the prescribed background current can in principle be any arbitrary function. The study at hand builds upon the ideas sketched in [21] on the systematic establishment of asymptotic model equations describing nonlinear wave-current interactions with the EUC. We refer to the related derivations in [13, 14], which are based on the same method being applied to far simpler geophysical scenarios of one-layer fluids without the presence of a background current. Let us finally refer to the recent papers [22, 23] concerning exact solutions to the governing equations of geophysical fluid dynamics describing the EUC and Antarctic Circumpolar Current as steady discontinuously stratified flows in spherical coordinates. Explicit solutions of this kind are of great interest and may serve as suitable background currents in future investigations of geophysical wave-current interactions.

The paper is structured as follows. Section 2 introduces the underlying physical system under study; its nondimensionalization is discussed in Section 3. The corresponding setting for steady linear long waves is studied in Section 4. Particularly, we derive the Burns condition from which the wave propagation speeds and corresponding dispersion relations can be derived, and provide several explicit examples. The appropriate setting for weakly nonlinear long waves, whose asymptotic solution yields the desired KdV models, is discussed in Section 5. KdV equations are derived for the surface wave, interface, horizontal velocity and pressure. Some explicit examples are provided as well.

2. The underlying physical model. As underlying physical model we consider the two-dimensional Euler equations for incompressible and divergence-free flows:

$$\bar{\mathfrak{u}}_{\bar{t}} + \bar{\mathfrak{u}}\bar{\mathfrak{u}}_{\bar{x}} + \bar{\mathfrak{w}}\bar{\mathfrak{u}}_{\bar{z}} = -\frac{1}{\bar{\rho}}\bar{\mathfrak{P}}_{\bar{x}}, \tag{2.1}$$

$$\bar{\mathfrak{w}}_{\bar{t}} + \bar{\mathfrak{u}}\bar{\mathfrak{w}}_{\bar{x}} + \bar{\mathfrak{w}}\bar{\mathfrak{w}}_{\bar{z}} = -\frac{1}{\bar{\rho}}\bar{\mathfrak{P}}_{\bar{z}} - \bar{g}, \tag{2.2}$$

$$\bar{\mathfrak{u}}_{\bar{x}} + \bar{\mathfrak{w}}_{\bar{z}} = 0. \tag{2.3}$$

Dimensional variables are indicated by bars. The stratified fluid domain consists of two layers; a sketch is depicted in Fig. 1a. The density $\bar{\rho}$ is piecewise constant; $\bar{\rho}$ takes the value $\bar{\rho}_0$ within the upper fluid layer, whereas $\bar{\rho} = \bar{\rho}_0(1+r)$, $r > 0$, in the denser lower layer. Furthermore, the following boundary conditions are imposed. At the free surface, which is located at $\bar{z} = \bar{\eta}(\bar{x}, \bar{t})$, the pressure obeys

$$\bar{\mathfrak{P}} = \bar{P}_{\text{atm}} \quad \text{on} \quad \bar{z} = \bar{\eta}(\bar{x}, \bar{t}), \tag{2.4}$$

where $\bar{P}_{\text{atm}}$ denotes the constant atmospheric pressure at surface level. This dynamic boundary condition states that the only force exerted by the fluid is due to pressure. Furthermore, the kinematic condition

$$\bar{\mathfrak{w}} = \bar{\eta}_{\bar{t}} + \bar{\mathfrak{u}}\bar{\eta}_{\bar{x}} \quad \text{on} \quad \bar{z} = \bar{\eta}(\bar{x}, \bar{t}) \tag{2.5}$$

holds; it states that particles at the free surface remain there for all times. Similarly, at the interface, which is located at $\bar{z} = -\bar{h} + \bar{H}(\bar{x}, \bar{t})$, it holds that

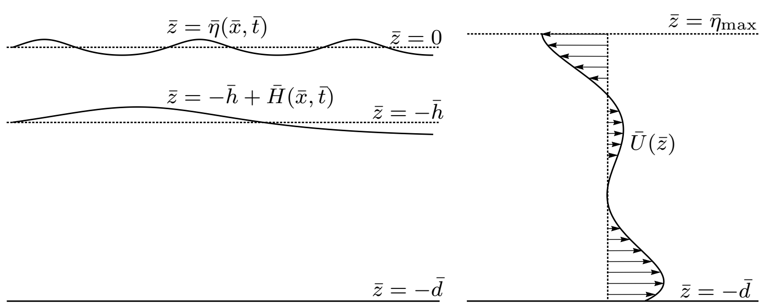
$$\bar{\mathfrak{w}}_{\pm} = \bar{H}_{\bar{t}} + \bar{\mathfrak{u}}_{\pm}\bar{H}_{\bar{x}} \quad \text{on} \quad \bar{z} = -\bar{h} + \bar{H}(\bar{x}, \bar{t}). \tag{2.6}$$

The subscripts "+" and "−" denote the limits at the interface from the upper and lower fluid layer, respectively. At the interface, forces are balanced via

$$\bar{\mathfrak{P}}_{+} = \bar{\mathfrak{P}}_{-} \quad \text{on} \quad \bar{z} = -\bar{h} + \bar{H}(\bar{x}, \bar{t}); \tag{2.7}$$

i.e., the pressure is supposed to be continuous across the interface. The kinematic boundary condition at the flat bed, which is situated at $\bar{z} = -\bar{d}$, states that

$$\bar{\mathfrak{w}} = 0 \quad \text{on} \quad \bar{z} = -\bar{d}. \tag{2.8}$$



(a) fluid domain (b) background current

FIGURE 1. Fig. 1a shows a sketch of the stratified fluid domain bounded by a free surface at $\bar{z} = \bar{\eta}(\bar{x}, \bar{t})$ and a fixed bottom at $\bar{z} = -\bar{d}$ with an interface at $\bar{z} = -\bar{h} + \bar{H}(\bar{x}, \bar{t})$ separating the upper fluid with density $\bar{\rho} = \bar{\rho}_0$ from the denser lower one, where $\bar{\rho} = \bar{\rho}_0(1 + r)$. Fig. 1b illustrates an example of a background current $\bar{U}(\bar{z})$.

Let $\bar{U}\colon [-\bar{d}, \bar{\eta}_{\max}] \to \mathbb{R} \times \{0\}$ be a given steady flow, which we refer to as *background current*; see Fig. 1b for an illustration. The height $\bar{\eta}_{\max}$ denotes the highest elevation a realistic surface wave can reach, thus $\bar{U}$ is defined on the whole *a priori* unknown fluid domain. For simplicity one may assume the background current to be continuously differentiable; we note that our derivations also apply to continuous piecewise smooth profiles with well-defined (finite) one-sided derivatives at each point, such as the piecewise linear profiles considered in [10, 8]. In the following we consider perturbations of $\bar{U}$, which solve the governing equations and boundary conditions (2.1)–(2.8). For this purpose, we write $\bar{u} := \bar{\mathfrak{u}} - \bar{U}$, $\bar{p} := \bar{\mathfrak{P}} - \bar{P}$, $\bar{w} := \bar{\mathfrak{w}}$, where $\bar{P}\colon [-\bar{d}, \bar{\eta}_{\max}] \to \mathbb{R}$ is the pressure necessary to maintain the background current in the absence of waves. With this notation, equations (2.1)–(2.8) are rewritten as follows:

$$\begin{cases} \bar{u}_{\bar{t}} + (\bar{U} + \bar{u})\bar{u}_{\bar{x}} + \bar{w}(\bar{U}' + \bar{u}_{\bar{z}}) = -\frac{1}{\bar{\rho}}\bar{p}_{\bar{x}} \\ \bar{w}_{\bar{t}} + (\bar{U} + \bar{u})\bar{w}_{\bar{x}} + \bar{w}\bar{w}_{\bar{z}} = -\frac{1}{\bar{\rho}}(\bar{P}' + \bar{p}_{\bar{z}}) - \bar{g} \\ \bar{u}_{\bar{x}} + \bar{w}_{\bar{z}} = 0 \end{cases} \quad \text{in} \quad -\bar{d} < z < \bar{\eta}(\bar{x}, \bar{t}); \tag{2.9}$$

$$\begin{cases} \bar{P} + \bar{p} = \bar{P}_{\text{atm}} \\ \bar{w} = \bar{\eta}_{\bar{t}} + (\bar{U} + \bar{u})\bar{\eta}_{\bar{x}} \end{cases} \quad \text{on} \quad \bar{z} = \bar{\eta}(\bar{x}, \bar{t}), \tag{2.10}$$

$$\begin{cases} \bar{P}_+ + \bar{p}_+ = \bar{P}_- + \bar{p}_- \\ \bar{w}_\pm = \bar{H}_{\bar{t}} + (\bar{U} + \bar{u}_\pm)\bar{H}_{\bar{x}} \end{cases} \quad \text{on} \quad \bar{z} = -\bar{h} + \bar{H}(\bar{x}, \bar{t}), \tag{2.11}$$

$$\bar{w} = 0 \quad \text{on} \quad \bar{z} = -\bar{d}. \tag{2.12}$$

3. Nondimensionalization.

Next, we transform the system (2.9)–(2.12) into a dimensionless form. We apply the same nondimensionalization as in [10]. Let therefore $\bar{a}_{\bar{\eta}}$ and $\bar{a}_{\bar{H}}$ denote average amplitudes of $\bar{\eta}$ and $\bar{H}$, respectively, and set $\bar{a} := \max(\bar{a}_{\bar{\eta}}, \bar{a}_{\bar{H}})$. The dimensionless variables x, z, t, u, U, w, p, P, η and H are defined as follows:

$$\begin{gathered} x := \frac{\bar{x}}{\bar{h}}, \quad z := \frac{\bar{z}}{\bar{h}}, \quad t := \frac{\bar{t}}{\bar{h}}\sqrt{\bar{g}\bar{h}}, \quad \eta := \frac{\bar{\eta}}{\bar{a}}, \quad H := \frac{\bar{H}}{\bar{a}}, \\ (U + u, w) := \frac{1}{\sqrt{\bar{g}\bar{h}}}(\bar{U} + \bar{u}, \bar{w}), \quad P + p := \frac{\bar{P} + \bar{p}}{\bar{\rho}_0\bar{g}\bar{h}}. \end{gathered} \tag{3.1}$$

Furthermore, we introduce the dimensionless parameter ε and constants h, P_0:

$$\varepsilon := \frac{\bar{a}}{\bar{h}}, \quad d := \frac{\bar{d}}{\bar{h}}, \quad P_0 := \frac{\bar{P}_{\text{atm}}}{\bar{\rho}_0\bar{g}\bar{h}}. \tag{3.2}$$

Particularly, we infer from (3.1)–(3.2) that

$$P_0 = P(0). \tag{3.3}$$

With these variables, the governing equations and boundary conditions (2.9)–(2.12) take the following dimensionless form:

$$\begin{cases} u_t + (U+u)u_x + w(U' + u_z) = -p_x \\ w_t + (U+u)w_x + ww_z = -p_z \end{cases} \quad \text{in} \; -1 + \varepsilon H(x,t) < z < \varepsilon\eta(x,t), \tag{3.4}$$

$$\begin{cases} u_t + (U + u)u_x + w(U' + u_z) = -\frac{1}{1+r}p_x \\ w_t + (U + u)w_x + ww_z = -\frac{1}{1+r}p_z \end{cases} \quad \text{in} \quad -d < z < -1 + \varepsilon H(x,t), \tag{3.5}$$

$$u_x + w_z = 0 \quad \text{in} \quad -d < z < \varepsilon\eta(x,t); \tag{3.6}$$

$$\begin{cases} P + p = P_0 \\ w = \varepsilon\big(\eta_t + (U+u)\big)\eta_x \end{cases} \quad \text{on} \quad z = \varepsilon\eta(x,t), \tag{3.7}$$

$$\begin{cases} P_+ + p_+ = P_- + p_- \\ w_\pm = \varepsilon\big(H_t + (U+u_\pm)H_x\big) \end{cases} \quad \text{on} \quad z = -1 + \varepsilon H(x,t), \tag{3.8}$$

$$w = 0 \quad \text{on} \quad z = -d. \tag{3.9}$$

In (3.4)–(3.5) we already employed that the "background pressure" P necessarily satisfies

$$P'(z) = -1 \qquad \text{in the upper fluid layer,} \tag{3.10}$$

$$P'(z) = -(1+r) \quad \text{in the lower fluid layer.} \tag{3.11}$$

Indeed, the Euler equations would otherwise be violated in the absence of waves, i.e., when u, w, p, η and H vanish. Moreover, we infer that P is continuous at $z = -1$.

From (3.7)–(3.8) it follows that w is proportional to ε. Furthermore, Taylor expanding P about $z = 0$ in conjunction with (3.3) and (3.7) yields the same for p. Thus u must be proportional to ε as well. This motivates the scaling

$$(u, w, p) \mapsto \varepsilon(u, w, p), \tag{3.12}$$

so that (3.4)–(3.9) and (3.12) yields the following set of governing equations and corresponding boundary conditions in dimensionless scaled variables:

$$\begin{cases} u_t + (U+\varepsilon u)u_x + w(U' + \varepsilon u_z) = -p_x \\ w_t + (U+\varepsilon u)w_x + \varepsilon w w_z = -p_z \end{cases} \quad \text{in} \; -1+\varepsilon H(x,t) < z < \varepsilon\eta(x,t), \tag{3.13}$$

$$\begin{cases} u_t + (U+\varepsilon u)u_x + w(U' + \varepsilon u_z) = -\frac{1}{1+r}p_x \\ w_t + (U+\varepsilon u)w_x + \varepsilon w w_z = -\frac{1}{1+r}p_z \end{cases} \quad \text{in} \; -d < z < -1+\varepsilon H(x,t), \tag{3.14}$$

$$u_x + w_z = 0 \quad \text{in} \quad -d < z < \varepsilon\eta(x,t); \tag{3.15}$$

$$\begin{cases} P + \varepsilon p = P_0, \\ w = \eta_t + (U+\varepsilon u)\eta_x, \end{cases} \quad \text{on} \quad z = \varepsilon\eta, \tag{3.16}$$

$$\begin{cases} P_+ + \varepsilon p_+ = P_- + \varepsilon p_- \\ w_\pm = H_t + (U+\varepsilon u_\pm)H_x \end{cases} \quad \text{on} \quad z = -1+\varepsilon H, \tag{3.17}$$

$$w = 0 \quad \text{on} \quad z = -d. \tag{3.18}$$

This system is the foundation for all subsequent considerations and formal asymptotic derivations. These apply to scenarios satisfying $\varepsilon \ll 1$, i.e., to stratified flows where the average amplitude of surface and internal waves is small in comparison to the upper fluid layer's thickness, cf. (3.2).

3.1. **Transformation of the boundary conditions.** For later use we transform the boundary conditions (3.16)–(3.17) to the fixed horizontal lines at $z = 0$ and $z = -1$ via Taylor approximations. In view of (3.3) and (3.10) it holds that

$$P(\varepsilon\eta) + \varepsilon p(\varepsilon\eta) - P_0 = -\varepsilon\eta + \varepsilon p(0) + \varepsilon^2 p_z(0)\eta + \mathcal{O}(\varepsilon^3). \tag{3.19}$$

Similarly, employing (3.10)–(3.11) and the fact that P is continuous at $z = -1$ yields that

$$(P_+ + \varepsilon p_+ - P_- - \varepsilon p_-)\big|_{z=-1+\varepsilon H}$$

$$= \varepsilon\big(p_+(-1) - p_-(-1)\big) + \varepsilon^2\big(p_{z+}(-1) - p_{z-}(-1)\big)H + \varepsilon r H + \mathcal{O}(\varepsilon^3). \tag{3.20}$$

To transform the kinematic boundary conditions, we use that

$$\begin{aligned}&(\eta_t + [U + \varepsilon u]\eta_x - w)\big|_{z=\varepsilon\eta} \\ =&\eta_t + \big[U_0 + \varepsilon U_0'\eta + \varepsilon\big(u(0) + \varepsilon u_z(0)\eta\big)\big]\eta_x - (w(0) + \varepsilon w_z(0)\eta) + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{3.21}$$

$$\begin{aligned}&(H_t + [U + \varepsilon u_\pm]H_x - w_\pm)\big|_{z=-1+\varepsilon H} \\ =&H_t + \big[U_1 + \varepsilon U_{1\pm}' H + \varepsilon\big(u_\pm(-1) + \varepsilon u_{z\pm}(-1)H\big)\big]H_x \\ &\quad - \big(w_\pm(-1) + \varepsilon w_{z\pm}(-1)H\big) + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{3.22}$$

where $U_l := U(-l)$. In view of (3.16)–(3.17) and the approximations (3.19)–(3.22) we obtain the following approximated boundary conditions:

$$\begin{cases} p = \eta - \varepsilon p_z \eta \\ w + \varepsilon w_z \eta = \eta_t + [U + \varepsilon U'\eta + \varepsilon(u + \varepsilon u_z \eta)]\eta_x \end{cases} \quad \text{on} \quad z = 0, \tag{3.23}$$

$$\begin{cases} p_+ - p_- = -rH - \varepsilon(p_{z+} - p_{z-})H \\ w_\pm + \varepsilon w_{z\pm} H = H_t + [U + \varepsilon U_\pm' H + \varepsilon(u_\pm + \varepsilon u_{z\pm} H)]H_x \end{cases} \quad \text{on} \quad z = -1. \tag{3.24}$$

4. **Linear long waves.**

In concordance with the classical theory for shallow water waves on shear flows [12], we first discuss a suitable linearization of (3.13)–(3.18) to infer the correct setup for nonlinear generalizations. We are interested in waves propagating at a specific speed $c \in \mathbb{R}$, which is *a priori* unknown due to stratification and the presence of a background current. To determine c, we consider the *steady* version of (3.13)–(3.18) for linear *long waves*, which is obtained via $\partial_t \mapsto -c\partial_x$, the assumption that p does not change in z-direction, and by taking the limit $\varepsilon \to 0$:

$$\begin{cases} (U - c)u_x + wU' = -p_x \\ 0 = p_z \end{cases} \quad \text{in} \quad -1 < z < 0, \tag{4.1}$$

$$\begin{cases} (U - c)u_x + wU' = -\frac{p_x}{1+r} \\ 0 = p_z \end{cases} \quad \text{in} \quad -d < z < -1, \tag{4.2}$$

$$u_x + w_z = 0 \quad \text{in} \quad -d < z < 0; \tag{4.3}$$

$$\begin{cases} p = \eta \\ w = (U - c)\eta_x \end{cases} \quad \text{on} \quad z = 0, \tag{4.4}$$

$$\begin{cases} p_+ - p_- = -rH \\ w = (U - c)H_x \end{cases} \quad \text{on} \quad z = -1, \tag{4.5}$$

$$w = 0 \quad \text{on} \quad z = -d. \tag{4.6}$$

4.1. **Solution, Burns condition and dispersion relations.**

In the following we derive the solution of (4.1)–(4.6) in terms of an arbitrary function $\eta(x)$. In the upper layer, we obtain from (4.1) and (4.4) that

$$p = \eta, \quad z \in [-1, 0]. \tag{4.7}$$

Thus we infer from (4.1) in combination with (4.3) that

$$\frac{\partial}{\partial z}\left(\frac{w}{U - c}\right) = \frac{\eta_x}{(U - c)^2}, \quad z \in [-1, 0].$$

Integrating over $[z,0]$ and employing (4.4) yields that

$$w = -(U-c)(I_2-1)\eta_x, \quad z \in [-1,0], \tag{4.8}$$

where

$$I_n(z) := \int_z^0 \frac{\mathrm{d}s}{(U(s)-c)^n}, \quad z \in [-1,0].$$

Differentiating with respect to z, invoking (4.3), and integrating with respect to x yields that

$$u = \big[(U-c)(I_2-1)\big]'\eta, \quad z \in [-1,0],$$

where we have set the integration constant to zero since we are assuming that perturbations of the velocity field are caused only by the passage of waves and therefore, $u=0$ if $\eta=0$, cf. [19]. From (4.5), (4.8) and by setting $I_{21} := I_2(-1)$, we obtain the following proportionality between η and H:

$$H = (1-I_{21})\eta. \tag{4.9}$$

Next, we investigate the lower layer. Since p is independent of z, we may write

$$p = (1+r)A, \quad z \in [-d,-1],$$

for some function $A = A(x)$, which will be determined later. Analogously as in the upper layer we deduce that

$$w = (U-c)(H_x - J_2 A_x), \quad z \in [-d,-1], \tag{4.10}$$

where

$$J_n(z) := \int_z^{-1} \frac{\mathrm{d}s}{(U(s)-c)^n}, \quad z \in [-d,-1].$$

With the help of (4.3) we infer from (4.10) that

$$u = \big[(U-c)J_2\big]'A - U'H, \quad z \in [-d,-1]. \tag{4.11}$$

Since $(1+r)A = p_- = p_+ + rH = \eta + rH$ at $z=-1$ by (4.5), (4.9) implies

$$A(x) = L\,\eta(x), \quad L := 1 - \frac{rI_{21}}{1+r}. \tag{4.12}$$

Thus, by (4.10), (4.11) and (4.12), u and w can be rewritten in the lower layer as follows:

$$\begin{cases} u = \big[(U-c)(I_{21}+LJ_2-1)\big]'\eta \\ w = -(U-c)\big(I_{21}+LJ_2-1\big)\eta_x \end{cases} \qquad z \in [-d,-1].$$

Finally, the above identity for w in conjunction with (4.6) implies

$$I_{21} + LJ_{2d} = 1, \tag{4.13}$$

where $J_{2d} := J_2(-d)$. This is the so-called *Burns condition* [2] from which c, and hence the dispersion relations, can be determined. In full detail this condition reads

$$\int_{-d}^0 \frac{\mathrm{d}s}{(U-c)^2} - \frac{r}{1+r}\int_{-d}^{-1}\frac{\mathrm{d}s}{(U-c)^2}\cdot\int_{-1}^0\frac{\mathrm{d}s}{(U-c)^2} = 1. \tag{4.14}$$

It is clear from (4.14) that dispersion relations are generally non-explicit. Only very specific background currents U will allow for exact formulæ; some examples are considered below in Section 4.2.

Throughout the above derivations we implicitly used that

$$U(z) \neq c \quad \text{for all} \quad z \in [-d,0] \tag{4.15}$$

to avoid singularities. It is known that scenarios, which violate (4.15), entail the formation of critical layers—even in the case of arbitrarily small linear waves [17, 18]. To rule out critical layers we henceforth assume the validity of (4.15). We call c *non-critical* if it satisfies (4.15).

4.2. **Examples of explicit dispersion relations.** In general, wave propagation speeds c can only be computed numerically via (4.14). However, for certain simple background currents U, explicit formulæ can be found; some examples are provided in the following subsections. For unstratified flows, they coincide with the classical results (cf. [19, ch. 3.4] and the references therein), while for the stratified cases they turn out to be in agreement with the considerations made in [21] for $\Omega = 0$, $d = O(1)$ and the particular examples of background currents U given below.

4.2.1. *Uniform flows without stratification.* Let U be a uniform background current, say

$$U(z) \coloneqq md, \qquad z \in [-d, 0], \tag{4.16}$$

for some $m \in \mathbb{R}$, and let $r \coloneqq 0$, i.e., the fluid is not stratified. Then (4.14) yields the two non-critical wave speeds

$$c = md \pm \sqrt{d},$$

which correspond to upstream and downstream propagation. For the special case $d = 1$ and $m = 0$ (absence of any current) we obtain the two wave speeds ± 1 corresponding to right and left moving waves.

4.2.2. *Constant vorticity shear flows without stratification.* Let $r = 0$ and U be a shear flow with constant vorticity, i.e.,

$$U(z) \coloneqq \gamma(z + d) + md, \qquad z \in [-d, 0], \tag{4.17}$$

with $\gamma, m \in \mathbb{R}$. Then (4.14) yields the two wave speeds

$$c = \frac{d(\gamma + 2m) \pm \sqrt{d\left(\gamma^2 d + 4\right)}}{2},$$

which are both non-critical.

4.2.3. *Uniform flows with stratification.* Let $r > 0$ and U be given by (4.16). Then (4.14) implies the following four non-critical propagation speeds:

$$c = md \pm \sqrt{\frac{d}{2} \pm \frac{\sqrt{d^2 + r(d-2)^2}}{2\sqrt{1+r}}}. \tag{4.18}$$

4.2.4. *Constant vorticity shear flows with stratification.* Let $r > 0$ and U be given by (4.17). Then the Burns condition (4.14) reads

$$\frac{d - \frac{(d-1)r}{(r+1)(c+\gamma-d(\gamma+m))^2}}{(c - dm)\big(c - d(\gamma+m)\big)} = 1, \tag{4.19}$$

which can still be solved explicitly for c. However, the four roots of equation (4.19) are analytically intractable—at least for the most general setting. Therefore, we consider the special case where both fluid layers are equally thick, i.e., we set $d \coloneqq 2$. With this choice the four solutions of (4.19) are given by

$$c = \gamma + 2m \pm \frac{1}{\sqrt{2}}\sqrt{\gamma^2 + 2 \pm \frac{\sqrt{(r+1)\left(\left(\gamma^2+4\right)\gamma^2(r+1)+4\right)}}{r+1}}. \tag{4.20}$$

Generally, not all of the four values in (4.20) satisfy (4.15). Let us demonstrate this numerically by means of the following choice of parameters: $r \coloneqq 1/100$, $\gamma \coloneqq 1$, $d \coloneqq 2$, $m \coloneqq 0$. Then the four solutions of (4.19) are

$$c = 2.7310\dots, \quad c = -0.7310\dots, \quad c = 1.0574\dots, \quad c = 0.9425\dots.$$

The latter two cause a formation of critical layers situated at $z_c \approx -1 \pm 0.0573$ close to the interface. For comparison let us also consider the corresponding setting without stratification ($r = 0$), which yields the two non-critical wave speeds

$$c = 1 - \sqrt{3} \approx -0.732 \quad \text{and} \quad c = 1 + \sqrt{3} \approx 2.732.$$

5. Weakly nonlinear long waves.

To obtain an appropriate nonlinear extension of the linear system in Section 4, we follow the classical approach in [12], which is a generalization of the seminal work [1] on solitary waves over shear flows, as well as the more recent adaptations to discontinuously stratified equatorial flows in [21].

5.1. Far field variables.

Let c be a non-critical wave propagation speed; i.e., c is supposed to satisfy both (4.14) and (4.15). Guided by the considerations in [21, Sec. 6.2], we employ the following spatial and temporal variables:

$$\xi \coloneqq \sqrt{\varepsilon}(x - ct), \quad \tau \coloneqq \varepsilon^{3/2} t, \tag{5.1}$$

thus,

$$\partial_x = \sqrt{\varepsilon}\partial_\xi, \qquad \partial_t = \sqrt{\varepsilon}(\varepsilon\partial_\tau - c\partial_\xi).$$

Applying (5.1) and the scaling $w \mapsto \sqrt{\varepsilon} w$ (to maintain mass conservation) to the equations (3.13)–(3.18), which are considered on the fixed fluid domain whose boundaries/interface are located at $z = -d$, $z = -1$ and $z = 0$ (recalling that (3.16)–(3.17) are approximated by (3.23)–(3.24)), yields the following system:

$$\begin{cases} \varepsilon u_\tau + [U - c + \varepsilon u]u_\xi + w[U' + \varepsilon u_z] = -p_\xi \\ \varepsilon\{\varepsilon w_\tau + [U - c + \varepsilon u]w_\xi + \varepsilon w w_z\} = -p_z \end{cases} \quad \text{in } -1 < z < 0, \tag{5.2}$$

$$\begin{cases} \varepsilon u_\tau + [U - c + \varepsilon u]u_\xi + w[U' + \varepsilon u_z] = -\frac{p_\xi}{1+r} \\ \varepsilon\{\varepsilon w_\tau + [U - c + \varepsilon u]w_\xi + \varepsilon w w_z\} = -\frac{p_z}{1+r} \end{cases} \quad \text{in } -d(\varepsilon) < z < -1, \tag{5.3}$$

$$u_\xi + w_z = 0 \quad \text{in } -d(\varepsilon) < z < 0, \tag{5.4}$$

$$\begin{cases} p = \eta - \varepsilon p_z \eta \\ w + \varepsilon w_z \eta = \varepsilon \eta_\tau + [U - c + \varepsilon U'\eta + \varepsilon(u + \varepsilon u_z \eta)]\eta_\xi \end{cases} \quad \text{on } z = 0, \tag{5.5}$$

$$\begin{cases} p_+ - p_- = -rH - \varepsilon(p_{z+} - p_{z-})H \\ w_\pm + \varepsilon w_{z\pm} H \\ \quad = \varepsilon H_\tau + [U - c + \varepsilon U'_\pm H + \varepsilon(u_\pm + \varepsilon u_{z\pm} H)]H_\xi \end{cases} \quad \text{on } z = -1, \tag{5.6}$$

$$w = 0 \quad \text{on } z = -d. \tag{5.7}$$

In the limit $\varepsilon \to 0$, equations (5.2)–(5.7) reduce to (4.1)–(4.6), which explains the particular choice of far field variables in (5.1).

To obtain an asymptotic solution of this system in terms of η and H, we formally expand the variables u, w, p, η and H near $\varepsilon = 0$ as follows:

$$\left.\begin{aligned} u(\xi,\tau,z) &\sim \textstyle\sum_{n=0}^{\infty} \varepsilon^n u_n(\xi,\tau,z),\\ w(\xi,\tau,z) &\sim \textstyle\sum_{n=0}^{\infty} \varepsilon^n w_n(\xi,\tau,z),\\ p(\xi,\tau,z) &\sim \textstyle\sum_{n=0}^{\infty} \varepsilon^n p_n(\xi,\tau,z),\\ \eta(\xi,\tau) &\sim \textstyle\sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi,\tau),\\ H(\xi,\tau) &\sim \textstyle\sum_{n=0}^{\infty} \varepsilon^n H_n(\xi,\tau), \end{aligned}\right\} \quad \text{as} \quad \varepsilon \to 0.$$

Plugging this ansatz into (5.2)–(5.7) and collecting the terms of order ε^n, $0 \le n < \infty$, yields an infinite hierarchy of systems, which can be solved recursively. To obtain KdV equations describing the weakly nonlinear wave propagation of η_0, H_0, u_0 and p_0, it is sufficient to consider the orders ε^0 and ε^1.

5.2. **The zero order system.** At order ε^0 we obtain precisely the linear system (4.1)–(4.6), but with x, u, w, p, η and H being replaced by ξ, u_0, w_0, p_0, η_0 and H_0, respectively. The τ-dependence does not play a role at this stage and will become relevant at the next order. Therefore, we have at this order that

$$p_0 = \eta_0, \qquad z \in [-1,0], \tag{5.8}$$

$$p_0 = (1+r)A, \qquad z \in [-d,-1], \tag{5.9}$$

$$w_0 = -(U-c)(I_2-1)\eta_{0\xi}, \qquad z \in [-1,0], \tag{5.10}$$

$$w_0 = -(U-c)\big(I_{21}+LJ_2-1\big)\eta_{0\xi}, \qquad z \in [-d,-1], \tag{5.11}$$

$$u_0 = \big[(U-c)(I_2-1)\big]'\eta_0, \qquad z \in [-1,0], \tag{5.12}$$

$$u_0 = \big[(U-c)\big(I_{21}+LJ_2-1\big)\big]'\eta_0, \qquad z \in [-d,-1], \tag{5.13}$$

$$H_0 = (1-I_{21})\eta_0, \tag{5.14}$$

with $A(\xi,\tau) \coloneqq L\,\eta(\xi,\tau)$; we recall that

$$I_n(z) \coloneqq \int_z^0 \frac{\mathrm{d}s}{(U(s)-c)^n}, \quad J_n(z) \coloneqq \int_z^{-1} \frac{\mathrm{d}s}{(U(s)-c)^n},$$
$$I_{21} \coloneqq I_2(-1), \quad L \coloneqq 1 - \frac{rI_{21}}{1+r}.$$

5.3. **The first order system.** At order ε^1 we obtain the following system:

$$\begin{cases} u_{0\tau} + (U-c)u_{1\xi} + u_0u_{0\xi} + w_1U' + w_0u_{0z} = -p_{1\xi} \\ (U-c)w_{0\xi} = -p_{1z} \end{cases} \quad \text{in} \quad -1 < z < 0, \tag{5.15}$$

$$\begin{cases} u_{0\tau} + (U-c)u_{1\xi} + u_0u_{0\xi} + w_1U' + w_0u_{0z} = -\frac{p_{1\xi}}{1+r} \\ (U-c)w_{0\xi} = -\frac{p_{1z}}{1+r} \end{cases} \quad \text{in} \quad -d < z < -1, \tag{5.16}$$

$$u_{1\xi} + w_{1z} = 0 \quad \text{in} \quad -d < z < 0, \tag{5.17}$$

$$\begin{cases} p_1 = \eta_1 \\ w_1 + w_{0z}\eta_0 = \eta_{0\tau} + (U-c)\eta_{1\xi} + (U'\eta_0 + u_0)\eta_{0\xi} \end{cases} \quad \text{on} \quad z = 0, \tag{5.18}$$

$$\begin{cases} p_{1+} - p_{1-} = -rH_1 \\ w_{1\pm} + w_{0z\pm}H_0 \\ \quad = H_{0\tau} + (U-c)H_{1\xi} + (U'_\pm H_0 + u_{0\pm})H_{0\xi} \end{cases} \quad \text{on} \quad z = -1, \tag{5.19}$$

$$w_1 = 0 \quad \text{on} \quad z = -d. \tag{5.20}$$

We aim to derive a KdV equation for the free surface approximation η_0 from (5.15)–(5.20). The related model for H_0 can then be obtained via (5.14); similarly, the equations for p_0 in the upper and lower fluid layer can then be deduced via (5.8)–(5.9); the equations for $u_0(\cdot,\cdot,z)$, where $z \in [-d,0]$ is fixed, will follow from (5.12)–(5.13). For this purpose we will derive expressions for w_1 in both fluid layers to obtain the limits w_{1+} and w_{1-} at $z=-1$. A comparison with the interface condition for $w_{1\pm}$ in (5.19) will yield the desired KdV equation for η_0.

Let us begin with the upper fluid layer. Employing (5.10), (5.15) and (5.18) yields that

$$p_1 = \int_z^0 (U-c)^2\big(1-I_2\big)\,\mathrm{d}s\;\eta_{0\xi\xi} + \eta_1, \qquad z \in [-1,0]. \tag{5.21}$$

By differentiating (5.21) with respect to ξ and applying (5.10), (5.12), (5.15) and (5.17) we obtain the equation

$$\begin{aligned}
&\int_z^0 (U(s)-c)^2(I_2(s)-1)\,\mathrm{d}s\;\eta_{0\xi\xi\xi} - \eta_{1\xi}\\
=&[(U-c)(I_2-1)]'\eta_{0\tau} - (U-c)w_{1z} + U'w_1\\
&+\Big[\big([(U-c)(I_2-1)]'\big)^2 - (U-c)(I_2-1)[(U-c)(I_2-1)]''\Big]\eta_0\eta_{0\xi},
\end{aligned}$$

which we recast in the form

$$\begin{aligned}
&(U-c)^2\frac{\partial}{\partial z}\left(\frac{w_1}{U-c}\right)\\
=&[(U-c)(I_2-1)]'\eta_{0\tau} - \int_z^0 (U(s)-c)^2(I_2(s)-1)\,\mathrm{d}s\;\eta_{0\xi\xi\xi} + \eta_{1\xi}\\
&+\Big[\big([(U-c)(I_2-1)]'\big)^2 - [(U-c)(I_2-1)]''(U-c)(I_2-1)\Big]\eta_0\eta_{0\xi}.
\end{aligned}$$

Integrating both sides of the above equation over $[z,0]$ yields that

$$\begin{aligned}
w_1 = &\frac{U-c}{U_0-c}w_1(0) - (U-c)\Bigg\{\int_z^0 \frac{U'(I_2-1)-(U-c)^{-1}}{(U-c)^2}\;\eta_{0\tau}\\
&+\int_z^0 \frac{\big([(U-c)(I_2-1)]'\big)^2 - [(U-c)(I_2-1)]''(U-c)(I_2-1)}{(U-c)^2}\,\mathrm{d}s\;\eta_0\eta_{0\xi}\\
&-\int_z^0 \frac{\int_{z'}^0 (U-c)^2(I_2-1)\,\mathrm{d}s}{(U-c)^2}\,\mathrm{d}z'\;\eta_{0\xi\xi\xi} + I_2\eta_{1\xi}\Bigg\}, \qquad z\in[-1,0],
\end{aligned} \tag{5.22}$$

where

$$w_1(0) = \eta_{0\tau} + (U_0-c)\eta_{1\xi} + \big([(U-c)(2I_2-1)]'\big)\big|_{z=0}\,\eta_0\eta_{0\xi}$$

due to (5.10) and (5.18). Evaluating (5.22) at $z=-1$ yields that

$$w_{1+} = a_1\eta_{0\tau} + a_2\eta_0\eta_{0\xi} + a_3\eta_{0\xi\xi\xi} + a_4\eta_{1\xi} \quad \text{on} \quad z=-1, \tag{5.23}$$

where

$$\begin{aligned}
a_1 &= 2(U_1-c)I_{31} - (I_{21}-1),\\
a_2 &= -3I_{41}(U_1-c) - U'_{1+}(I_{21}-1)^2 + 2\frac{I_{21}-1}{U_1-c},
\end{aligned}$$

$$a_3 = (U_1 - c) \int_{-1}^{0} \frac{\int_z^0 (U-c)^2 (I_2 - 1)\, \mathrm{d}s}{(U-c)^2}\, \mathrm{d}z,$$
$$a_4 = (U_1 - c)(1 - I_{21}). \tag{5.24}$$

Due to (5.19) it holds that

$$w_{1+} = c_1 \eta_{0\tau} + c_{2+} \eta_0 \eta_{0\xi} + c_5 H_{1\xi} \quad \text{on} \quad z = -1, \tag{5.25}$$

where

$$\begin{aligned} c_1 &= 1 - I_{21}, \\ c_{2+} &= -U'_{1+}(I_{21} - 1)^2 + 2\frac{I_{21} - 1}{U_1 - c}, \\ c_5 &= U_1 - c. \end{aligned} \tag{5.26}$$

Combining (5.23) and (5.25) yields

$$H_{1\xi} = (1 - I_{21})\eta_{1\xi} + 2I_{31}\eta_{0\tau} - 3I_{41}\eta_0\eta_{0\xi} + \frac{a_3}{c_5}\eta_{0\xi\xi\xi}. \tag{5.27}$$

Next, we consider the lower fluid layer. By (5.11) and (5.16),

$$p_1 = -(1+r) \int_z^{-1} (U-c)^2 (I_{21} + LJ_2 - 1)\, \mathrm{d}s\; \eta_{0\xi\xi} + p_{1-}, \quad z \in [-d, -1],$$

where

$$p_{1-} = p_{1+} + rH_1 = -\int_{-1}^{0} (U-c)^2 (I_2 - 1)\, \mathrm{d}s\; \eta_{0\xi\xi} + \eta_1 + rH_1$$

due to (5.19) and (5.21). Therefore,

$$\frac{p_{1\xi}}{1+r} = -\left[\int_z^{-1} (U-c)^2 \big(I_{21} + LJ_2 - 1\big)\, \mathrm{d}s + \int_{-1}^{0} \frac{(U-c)^2}{1+r}(I_2 - 1)\, \mathrm{d}s\right]\eta_{0\xi\xi\xi} + \frac{\eta_{1\xi} + rH_{1\xi}}{1+r}$$

for $z \in [-d, -1]$. With the help of (5.11), (5.13), (5.16) and (5.20) we compute that

$$\begin{aligned} w_1 = (U-c) \int_{-d}^{z} \frac{1}{(U-c)^2} \Bigg\{ &\Big[(U-c)(I_{21} + LJ_2 - 1)\Big]' \eta_{0\tau} + \frac{\eta_{1\xi} + rH_{1\xi}}{1+r} \\ &- \left[\int_{z'}^{-1} (U-c)^2 \big(I_{21} + LJ_2 - 1\big)\, \mathrm{d}s + \int_{-1}^{0} \frac{(U-c)^2}{1+r}(I_2 - 1)\, \mathrm{d}s\right]\eta_{0\xi\xi\xi} \\ &+ \Bigg[\bigg(\Big[(U-c)(I_{21} + LJ_2 - 1)\Big]'\bigg)^2 \\ &\quad - \Big[(U-c)(I_{21} + LJ_2 - 1)\Big]'' (U-c)\big(I_{21} + LJ_2 - 1\big)\Bigg]\eta_0\eta_{0\xi} \Bigg\}\, \mathrm{d}z' \end{aligned}$$

for $z \in [-d, -1]$. Evaluating this expression at $z = -1$, we find that

$$w_{1-} = b_1\eta_{0\tau} + b_2\eta_0\eta_{0\xi} + b_3\eta_{0\xi\xi\xi} + b_4\eta_{1\xi} + b_5 H_{1\xi} \quad \text{on} \quad z = -1, \tag{5.28}$$

where

$$b_1 = 1 - 2(U_1 - c)LJ_{3d} - I_{21},$$
$$b_2 = 3L^2 J_{4d}(U_1 - c) + 2L\frac{I_{21} - 1}{U_1 - c} - U'_{1-}(I_{21} - 1)^2,$$

$$
\begin{aligned}
b_3 &= \int_{-d}^{-1} \frac{U_1 - c}{(U-c)^2} \left[\int_z^{-1} (U-c)^2 \big(1 - I_{21} - LJ_2\big)\,\mathrm{d}s + \int_{-1}^{0} \frac{(U-c)^2}{1+r}(1-I_2)\,\mathrm{d}s \right] \mathrm{d}z,\\
b_4 &= \frac{U_1 - c}{1+r} J_{2d},\\
b_5 &= r b_4.
\end{aligned} \tag{5.29}
$$

Due to (5.19) it holds that

$$
w_{1-} = c_1 \eta_{0\tau} + c_{2-} \eta_0 \eta_{0\xi} + c_5 H_{1\xi} \quad \text{on} \quad z = -1, \tag{5.30}
$$

with c_i, $i = 1, 5$, according to (5.26) and

$$
c_{2-} = -U'_{1-}(I_{21} - 1)^2 + 2L \frac{I_{21} - 1}{U_1 - c}. \tag{5.31}
$$

Combining (5.28) and (5.30) in conjunction with the Burns condition (4.13) yields

$$
H_{1\xi} = (1 - I_{21}) \eta_{1\xi} + \frac{b_1 - c_1}{c_5 - b_5} \eta_{0\tau} + \frac{b_2 - c_{2-}}{c_5 - b_5} \eta_0 \eta_{0\xi} + \frac{b_3}{c_5 - b_5} \eta_{0\xi\xi\xi}. \tag{5.32}
$$

5.4. **KdV models.** We are now in the position to directly deduce the desired KdV model equations for the asymptotic approximations of the free surface η_0 and interface H_0, the horizontal velocity component u_0 (evaluated at a fixed depth $z \in [-d, 0]$) as well as the pressure p_0 for both fluid layers.

5.4.1. *The free surface.* By subtracting (5.27) from (5.32), we deduce that η_0 satisfies the following KdV equation:

$$
\alpha_1 \eta_{0\tau} + \alpha_2 \eta_0 \eta_{0\xi} + \alpha_3 \eta_{0\xi\xi\xi} = 0 \tag{5.33}
$$

with

$$
\begin{aligned}
\alpha_1 &= -2 \left[I_{31} + J_{3d} \frac{1 + r - rI_{21}}{1 + r - rJ_{2d}} \right],\\
\alpha_2 &= 3 \left[I_{41} + J_{4d} \frac{(1 + r - rI_{21})^2}{(r+1)(1 + r - rJ_{2d})} \right],\\
\alpha_3 &= \frac{a_3}{c - U_1} + \frac{b_3 (r+1)}{(c - U_1)(r(J_{2d} - 1) - 1)}\\
&= \int_{-1}^{0} \int_z^{0} \int_{-d}^{s} V(z,s,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}s\,\mathrm{d}z\\
&\quad - \frac{1 + r - rI_{21}}{1 + r - rJ_{2d}} \int_{-d}^{-1} \int_z^{-1} \int_s^{-d} V(z,s,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}s\,\mathrm{d}z\\
&\quad + \frac{1}{1 + r - rJ_{2d}} \int_{-d}^{-1} \int_{-1}^{0} \int_{-d}^{s} V(z,s,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}s\,\mathrm{d}z\\
&\quad - \frac{rI_{21}}{1+r} \int_{-1}^{0} \int_z^{0} \int_{-d}^{-1} V(z,s,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}s\,\mathrm{d}z\\
&\quad - \frac{rI_{21}}{(1+r)(1 + r - rJ_{2d})} \int_{-d}^{-1} \int_{-1}^{0} \int_{-d}^{-1} V(z,s,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}s\,\mathrm{d}z,
\end{aligned} \tag{5.34}
$$

where

$$
V(z,s,\zeta) := \frac{(U(s) - c)^2}{(U(z) - c)^2 (U(\zeta) - c)^2}.
$$

In particular, we recover the KdV equation in [12, (4.15)] for the unstratified case, where $r = 0$ and $d := -1$.

5.4.2. *The interface.* From (5.14) and (5.33) we deduce that H_0 satisfies the following KdV equation:

$$\beta_1 H_{0\tau} + \beta_2 H_0 H_{0\xi} + \beta_3 H_{0\xi\xi\xi} = 0 \tag{5.35}$$

with

$$\beta_i = \alpha_i/(1 - I_{21}), \quad \beta_2 = \alpha_2/(1 - I_{21})^2, \quad i = 1, 3. \tag{5.36}$$

We observe that if $I_{21} \in (0, 1)$, then H_0 and η_0 are in phase and the amplitude of the interface is smaller than that of the surface. If, on the other hand, $I_{21} > 1$, then the surface and interface are out of phase (H_0 and η_0 have opposite signs) and the internal wave can be larger than the corresponding surface wave. This happens if $|U - c|$ is small (on average) in the upper fluid layer. Similar findings were made in [10] (cf. Sections 5.2.1, 5.2.2 and Fig. 5 therein) for linear waves in the EUC.

5.4.3. *The velocity field and pressure.* From (5.12), (5.13) and (5.33) we deduce that the horizontal velocity component u_0 evaluated at $z \in [-d, 0]$ satisfies

$$\gamma_1 u_{0\tau} + \gamma_2 u_0 u_{0\xi} + \gamma_3 u_{0\xi\xi\xi} = 0 \tag{5.37}$$

with

$$\gamma_i(z) = \begin{cases} \alpha_i/[(U - c)I - U]', & z \in (-1, 0], \\ \alpha_i/\big(L\,[(U - c)J]' - U'(1 - I_{21})\big), & z \in [-d, -1), \end{cases} \qquad i = 1, 3,$$

$$\gamma_2(z) = \begin{cases} \alpha_2/\big([(U - c)I - U]'\big)^2, & z \in (-1, 0], \\ \alpha_2/\big(L\,[(U - c)J]' - U'(1 - I_{21})\big)^2, & z \in [-d, -1). \end{cases}$$

From (5.8), (5.9) and (5.33) we deduce that the pressure p_0 evaluated at $z \in [-d, 0]$ satisfies

$$\delta_1 p_{0\tau} + \delta_2 p_0 p_{0\xi} + \delta_3 p_{0\xi\xi\xi} = 0 \tag{5.38}$$

with

$$\delta_i(z) = \begin{cases} \alpha_i, & z \in (-1, 0], \\ \alpha_i/\big(1 + r(1 - I_{21})\big), & z \in [-d, -1), \end{cases} \qquad i = 1, 3,$$

$$\delta_2(z) = \begin{cases} \alpha_2, & z \in (-1, 0], \\ \alpha_2/\big(1 + r(1 - I_{21})\big)^2, & z \in [-d, -1). \end{cases}$$

5.5. **Examples for specific background currents.** In the following subsections we calculate the KdV coefficients for the particular background currents considered in Section 4.2.

5.5.1. *Uniform flows without stratification.* Let $r \coloneqq 0$, $U(z) \coloneqq md$ for some $m \in \mathbb{R}$ and $z \in [-d, 0]$, and $c \coloneqq md + \sqrt{d}$. Then the coefficients α_i in (5.34) for the surface equation (5.33) take the values

$$\alpha_1 = \frac{2}{\sqrt{d}}, \quad \alpha_2 = \frac{3}{d}, \quad \alpha_3 = \frac{d^2}{3}.$$

The case $d \coloneqq 1$ yields precisely the standard KdV equation with coefficient ratio $2\colon 3\colon \frac{1}{3}$ for shallow water long waves, cf. [19].

5.5.2. *Constant vorticity shear flows without stratification.* Let $r := 0$, $U(z) := \gamma(z + d) + md$, $z \in [-d, 0]$, for some $m \in \mathbb{R}$, and set $c := 2^{-1}[d(\gamma + 2m) + \sqrt{d(\gamma^2 d + 4)}]$. Then the coefficients α_i in (5.34) for the surface equation (5.33) read

$$\alpha_1 = \frac{2\gamma^2\sqrt{d^4\Delta} - 2\gamma^3 d^3 + 2(\gamma^2 - 4)\gamma d^2 - 2\gamma^2 d\sqrt{\Delta} + 4d\sqrt{\Delta} + 8\gamma d}{d(\gamma(d-2) - \sqrt{\Delta})(\gamma d - \sqrt{\Delta})},$$

$$\alpha_2 = \frac{16d\Big(\gamma^6 d^4 - 6\gamma\sqrt{d^2\Delta} + 2\left(7 - 3\gamma^2\right)\gamma^2 d^2 + 3\gamma\sqrt{\Delta} + \left(6 - 9\gamma^2\right)d\Big)}{(\sqrt{\Delta} - \gamma d)^4((\gamma^2 + 2)d + \gamma\sqrt{\Delta})}$$
$$+ \frac{16d\Big(\gamma^3(4\sqrt{d^2\Delta} - 5\sqrt{d^4\Delta}) - \gamma^4(\gamma^2 - 7)d^3 + \gamma^5(\sqrt{d^4\Delta} - \sqrt{d^6\Delta})\Big)}{(\sqrt{\Delta} - \gamma d)^4((\gamma^2 + 2)d + \gamma\sqrt{\Delta})},$$

$$\alpha_3 = \frac{(\gamma^2 d - \gamma\sqrt{\Delta} + 2)\Big[\sqrt{d^8\Delta} + \gamma\Big(\gamma(\sqrt{d^8\Delta} - \sqrt{d^6\Delta}) - \tilde{d}d^4 - \gamma^2\tilde{d}(d-1)d^3\Big)\Big]}{6d\big(\gamma^2(d-1) + d\big)\big(\sqrt{\Delta} - \gamma\tilde{d}\big)},$$

where $\Delta := d(\gamma^2 d + 4)$ and $\tilde{d} := d - 2$. In the case $d := 1$ these coefficients coincide with those of the classical KdV model equation for surface waves over a shear flow with constant vorticity, cf. [20], i.e.,

$$\alpha_1 = \sqrt{\gamma^2 + 4}, \quad \alpha_2 = \gamma^2 + 3, \quad \alpha_3 = \frac{1}{6}\Big(\gamma^2 + \gamma\sqrt{\gamma^2 + 4} + 2\Big).$$

5.5.3. *Uniform flows with stratification.* Let $r > 0$, $U(z) := md$ for all $z \in [-d, 0]$ and some $m \in \mathbb{R}$. In the following we demonstrate that the particular wave speeds

$$c = md \pm \sqrt{\frac{d}{2} - \frac{\sqrt{d^2 + r(d-2)^2}}{2\sqrt{1+r}}}, \tag{5.39}$$

which reflect the slower propagation of internal waves, entail scenarios in which α_2 vanishes (thus also β_2, γ_2 and δ_2). The remaining two wave speeds in (4.18) do not show this property as we will see below. Without loss of generality let (5.39) be satisfied with a plus sign. Then the coefficients α_i in (5.34) for the surface equation (5.33) are given by

$$\alpha_1 = \frac{\sqrt{2}(\sqrt{\Delta_2\Delta_1} + d - dr + 2r)(d^2\Delta_2^{3/2} - d(\sqrt{\Delta_1} + r(\sqrt{\Delta_1} + 4\sqrt{\Delta_2})) + 4r\sqrt{\Delta_2})}{(d-1)^2 r\sqrt{\Delta_2}\left(\sqrt{\frac{4(d-1)}{\Delta_2} + (d-2)^2} - d\right)^{3/2}},$$

$$\alpha_2 = 12\left[1 - \frac{(\sqrt{\Delta_3} - d + 2)\left(\frac{2r}{\sqrt{\Delta_3} - d} + \Delta_2\right)^2}{2\Delta_2}\right]\left(d - \sqrt{\Delta_3}\right)^{-2},$$

$$\alpha_3 = \frac{2(d-1)^2 r}{d(\sqrt{\Delta_2(d^2 + (d-2)^2 r)} + r\Delta_2) - d^2\Delta_2 + r(\sqrt{\Delta_2(d^2 + (d-2)^2 r)} - 2\Delta_2)}$$
$$- \frac{2(d-1)r}{(\sqrt{d^2 + (d-2)^2 r} - d\sqrt{\Delta_2})^2} - \frac{2(d-1)^3\left(\frac{2r}{\sqrt{\Delta_3} - d} + \Delta_2\right)}{3(d(r-1) + r(\sqrt{\Delta_3} - 2) + \sqrt{\Delta_3})}$$
$$+ \frac{(1 - 2d)(d-1)}{d(r-1) + r(\sqrt{\Delta_3} - 2) + \sqrt{\Delta_3}} + \frac{\frac{1}{3} - d}{\sqrt{\Delta_3} - d},$$

where $\Delta_1 := d^2 + (d-2)^2 r$, $\Delta_2 := 1 + r$ and $\Delta_3 := \frac{4(d-1)}{r+1} + (d-2)^2$. From these explicit expressions it follows that $\alpha_i \neq 0$ for $i = 1, 3$ and all $r > 0$, $d > 1$, whereas

$\alpha_2 = 0$ if and only if

$$d = \frac{1}{6\Delta_2}\left(2^{2/3}\sqrt[3]{9\sqrt{3}\sqrt{r(27r+32)}+3r\left(-9r+\sqrt{3}\sqrt{r(27r+32)}+51\right)+16}\right.$$
$$\left.+\frac{2\sqrt[3]{2}(4-15r)}{\sqrt[3]{9\sqrt{3}\sqrt{r(27r+32)}+3r\left(-9r+\sqrt{3}\sqrt{r(27r+32)}+51\right)+16}}+12r+4\right).$$

If the wave propagation speed satisfies

$$c = md \pm \sqrt{\frac{d}{2}+\frac{\sqrt{d^2+r(d-2)^2}}{2\sqrt{1+r}}},$$

it holds that $\alpha_i \neq 0$ for all $i = 1,2,3$, $r > 0$ and $d > 1$.

5.5.4. *Constant vorticity shear flows with stratification.* Despite the relative simplicity of this scenario (from an application point of view), exact computations—although possible—are no longer practicable. For this reason we consider the concrete example in Section 4.2.4, compute the coefficients α_i numerically, and compare their values with those of the corresponding unstratified case.

Let $r \coloneqq 1/100$, $U(z) \coloneqq \gamma(z+d)$ for $z \in [-d, 0]$ with $\gamma \coloneqq 1$, $d \coloneqq 2$ and the non-critical propagation speed

$$c \coloneqq \gamma + \frac{1}{\sqrt{2}}\sqrt{\gamma^2+2+\frac{\sqrt{(r+1)\left((\gamma^2+4)\gamma^2(r+1)+4\right)}}{r+1}} = 2.7310\ldots;$$

cf. (4.20). Then the coefficients α_i in (5.34) of the surface equation (5.33) take the values

$$\alpha_1 = 1.7356\ldots, \quad \alpha_2 = 2.5079\ldots, \quad \alpha_3 = 0.3551\ldots.$$

The coefficients β_i in (5.36) of the interface equation (5.35) take the values

$$\beta_1 = 8.2706\ldots, \quad \beta_2 = 56.9463\ldots, \quad \beta_3 = 1.6922\ldots.$$

The corresponding unstratified setting, i.e., $r \coloneqq 0$, $\gamma \coloneqq 1$, $d \coloneqq 2$, $c \coloneqq 1+\sqrt{3}$, yields the values

$$\alpha_1 = 1.7320\ldots, \quad \alpha_2 = 2.5, \quad \alpha_3 = 0.3572\ldots.$$

Remark 1. We have seen in 5.5.3 that the KdV coefficient α_2 vanishes for certain configurations of the wave propagation speed c, the depth d and the stratification parameter r (the unstratified examples in 5.5.1–5.5.2 generally yield $\alpha_i \neq 0$, $i = 1,2,3$). To obtain a nonlinear model equation for this scenario, a higher order asymptotic approximation is required, so that e.g. the cubic order term $\eta_0^2\eta_{0\xi}$ comes into play.

Let us note that the examples in 5.5.1–5.5.3 do not allow for a vanishing KdV coefficient α_3. Thus, a nonlinear model equation without dispersive terms requires a nonuniform background current U. Determining analytically whether α_2 or α_3 in (5.34) can vanish for such flows becomes very challenging (see 5.5.4) due to the intricate relation between U, c, r and d, which are merely implicitly linked by the Burns condition. In this context we refer to the derivation of an inviscid Burgers equation in [8] for equatorial internal waves, which is based on a Hamiltonian approach and uses a particular scaling.

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E-mail address: a.geyer@tudelft.nl
E-mail address: ronald.quirchmayr@univie.ac.at