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Boundary value problems modeling moisture transport in soils

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ABSTRACT

To model the moisture transport in soil and to better understand physics underneath, we study a boundary value problem for a nonlinear hyperbolic PDE. Using a constructive method for approximation of solutions of the problem, we derive sufficient conditions for existence and uniqueness of its regular solutions and show that these solutions satisfy the sign-preserving inequalities. Additionally, we prove a comparison theorem and a theorem about differential inequalities, and derive an posteriori error of the method. Theoretical results are validated on an illustrative numerical example.

1. Introduction

Water transport in soils plays an important role in agriculture and is highly impacted by the climate change. Due to high temperatures, lack of rainfalls and high evaporation rates, an unsaturated ground layer increases and hinders the necessary in-depth water penetration. This has drastic consequences for quality and quantity of the harvest, specially for countries which Gross Domestic Product (GDP) strongly depends on exports of the agricultural products.

To model the undersurface water transport through a porous medium (such as soil) and to better understand the physics underneath, mathematicians and hydrologists use evolution equations described in terms of time-space PDEs [1–5]. Literature overview shows a broad range of models that are successfully applied to describe such processes. Among them one should name *fractional models for saturated and unsaturated soils* (see [6]), models based on *Richard's equations* (see discussions in [7–9]) and those that are described by a *scalar hyperbolic PDE* of the form

$$m(t,x)\mathcal{D}^{(1.2)}u(t,x) + \alpha(t,x)\mathcal{D}^{(1.1)}u(t,x) + d(t,x)\mathcal{D}^{(0.1)}u(t,x) + \eta(t,x)\mathcal{D}^{(0.2)}u(t,x) + a(t,x)\mathcal{D}^{(1.0)}u(t,x) + b(t,x)u(t,x) = g(t,x), \tag{1}$$

where coefficients $m(t,x)$, $\alpha(t,x)$, $d(t,x)$, $\eta(t,x)$, $a(t,x)$ and $b(t,x)$ are continuous functions in a given bounded domain $D \in \mathbb{R}^2$. The last equation is also applied to describe, among others, fluid infiltration in a double porosity medium and heat transport in a heterogeneous frame, and was analyzed in [10–12].

Since most of the aforementioned models are nonlinear, and thus, in general cannot be solved exactly, it is wise to develop iterative techniques that enable construction of sequences of *approximate solutions* to the problems (analytically or numerically). Indeed, the most commonly used methods are the *numerical techniques* (see [13–16]), which under a chosen initial data set allow visualization of approximate solutions and analysis of their qualitative behavior. However, the downside of these algorithms is in

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their computational cost (which increases with the higher degree of nonlinearity of the model) and availability of measurements for validation. Also the initial guess for simulations might occur to be misleading.

An alternative to numerical methods are *analytical methods* that allow construction of approximate solutions symbolically (see results in [17–20]). The main advantage of these techniques over numerical ones is their coupling with the solvability analysis of the studied initial or boundary value problems (BVPs), improved computational cost and independence of the experimental data for validation. Thus, knowing that the problem under investigation (even being highly nonlinear) has a unique solution, one can construct a sequence of approximations that reflects behavior of the exact solution.

Here one should highlight one particular method belonging to the analytical family, the *sub- and supersolutions method* (sometimes also referred to as *upper and lower solutions*), that has already proven its efficiency in analysis and approximation of solutions to the BVPs for ordinary and partial differential equations (see results in [17,18,20–24]). For example, in [20] authors apply it to study problems with local and non-local boundary constraints for the following nonlinear PDE:

$$\mathcal{D}^{(1.2)}u(t,x) = f\left(t, x, u(t,x), \mathcal{D}^{(1.0)}u(t,x), \mathcal{D}^{(0.1)}u(t,x), \mathcal{D}^{(1.1)}u(t,x),\ \mathcal{D}^{(0.2)}u(t,x)\right). \tag{2}$$

Under certain assumptions, made about the right hand-side of Eq. (2) and about the domain of solutions D, the authors prove sufficient conditions on existence and uniqueness of solutions and show that these solutions satisfy a sign-preserving property. Moreover, authors obtain an a posteriori error of the method and present an approach to accelerate convergence of the constructed iteration scheme.

All these results have motivated us to further investigate applicability of this approach and to present our findings. The layout of the paper is the following. In *Section* 2 we introduce the problem setting and give some auxiliary statements and definitions. *Section* 3 is devoted to construction of the alternating sequences of sub- and supersolutions of the studied problem. A combination of these sequences determines approximate and exact solutions of the problem and is used to prove the main result of the paper; see *Section* 4. Finally, using an illustrative example, we demonstrate a possible simplification and optimization of the algorithm, where only one of the sequences (in our case, a sequence of subsolutions) completely determines the exact solution of the given BVP. Results of our computations are presented in *Section* 5.

2. Problem setting and auxiliary statements

Consider a semilinear hyperbolic PDE:

$$\mathcal{L}_{1,2}u(t,x) := \mathcal{D}^{(1.2)}u(t,x) - a_1(t,x)\mathcal{D}^{(0.2)}u(t,x) + a_2(t,x)\mathcal{D}^{(1.1)}u(t,x) = f\left(t, x, u(t,x), \mathcal{D}^{(1.0)}u(t,x), \mathcal{D}^{(0.1)}u(t,x)\right) := f\left[u(t,x)\right], \tag{3}$$

subject to boundary constraints of the form:

$$\begin{aligned} &u(0,x) = T(x), && x \in [0,a], \\ &\begin{cases} \mathcal{D}^{(0.1)}u(t,0) = \psi(t), \\ u(t,a) = \phi(t), \end{cases} && t \in [0,b], \end{aligned} \tag{4}$$

where $\mathcal{D}^{(\kappa)}u : D_0 \to D_\kappa \subset \mathbb{R}$ $(\kappa = (\kappa_1.\kappa_2), \kappa_2 = 0,1,2; \kappa_1 = 0,1)$ stands for the mixed-order partial derivative of a function $u(t,x)$ with respect to its arguments, with

$$D_0 = \left\{(t,x) |\ t \in (0,b), x \in (0,a)\right\},$$

and the right hand-side function f is such that $f : B \to \mathbb{R}$, with

$$B = D_0 \times \prod_{\kappa_1,\kappa_2} D_\kappa \subset \mathbb{R}^5, \quad \kappa_1 + \kappa_2 < 2.$$

We aim for finding solutions of the BVP (3), (4) in the functional space $C^*(\overline{D}_0) := C^{(1.2)}(D_0) \cap C(\overline{D}_0)$ using the sup- and supersolutions method (see [20,21]).

Throughout the paper we assume that

$$T(x) \in C^2[0,a], \quad \phi(t),\ \psi(t) \in C^1[0,b], \quad 0 \le a_1(t,x) \in C^{(0.1)}(D_0), \quad a_2(t,x) \in C^{(1.0)}(D_0),$$

$$T'(0) = \psi(0), \quad T(a) = \phi(0), \tag{5}$$

and that the right hand-side of the PDE (3) is such that $f\left[u(t,x)\right] \in C(\overline{B})$.

Lemma 1. *Let $a_1(t,x) \in C^{(0.1)}(D_0), a_2(t,x) \in C^{(1.0)}(D_0)$ and*

$$-\mathcal{D}^{(0.1)}a_1(t,x) = \mathcal{D}^{(1.0)}a_2(t,x). \tag{6}$$

Then BVP (3), (4) and the integro-differential equation

$$u(t,x) = \Phi(t,x) + T F[u(\eta,\zeta)] \tag{7}$$

are equivalent, where

$$\Phi(t,x) :=\phi(t) + \int_a^x T'(\xi)exp\left(\int_0^t a_1(\eta,\xi)d\eta\right)d\xi - T'(0)\int_a^x exp\left(\int_0^t a_1(\tau,\xi)d\tau + + \int_\xi^0 a_2(0,\tau)d\tau\right)d\xi +$$
$$+\psi(t)\int_a^x\left(\int_\xi^0 a_2(t,\tau)d\tau\right)d\xi,$$

$$TF[u(\eta,\zeta)] := \int_a^x\int_0^t\int_0^\xi K(t,\xi;\eta,\zeta)F[u(\eta,\zeta)]d\zeta d\eta d\xi,$$

$$F[u(t,x)] := f[u(t,x)] + \left[\mathcal{D}^{(0.1)}a_1(t,x) + a_1(t,x)a_2(t,x)\right]\mathcal{D}^{(0.1)}u(t,x),$$

$$K(t,x;\eta,\xi) := exp\left(\int_\eta^t a_1(\tau,x)d\tau + \int_x^\xi a_2(\eta,\tau)d\tau\right).$$

Proof. Consider BVP (3), (4): integrating differential Eq. (3) and incorporating boundary constraints (4) we obtain integral Eq. (7). This means that if $u(t,x)$ is a solution of the BVP (3), (4), then it is also a solution to the integral Eq. (7).

On the other hand, let $u(t,x)$ solve integral Eq. (7). Differentiating it and taking into account assumption (6), we derive that this function also solves BVP (3), (4).

Hence (3), (4) and (7) are equivalent. □

Note, that function $\Phi(t,x)$ is in $C^*(\overline{D}_0)$, which is easy to check using the definition of continuity of a multivariable function (see [25]), and it also satisfies conditions (4). Thus, we introduce an ansatz

$$z(t,x) := u(t,x) - \Phi(t,x)$$

and re-write the original BVP (3), (4) as a problem with homogeneous boundary restrictions. Without loss of generality we set

$$T(x) = 0,\ \psi(t) = \phi(t) = 0,$$

and write the integro–differential Eq. (7) as

$$u(t,x) = TF[u(\eta,\zeta)]. \tag{8}$$

Definition 1. We say that $F[u(t,x)] \in C_1^*(\overline{B})$, if it satisfies the following conditions (see [26]):

1. $F[u(t,x)] \in C(\overline{B})$;
2. in the functional space $C(\overline{B}_1)$, $\overline{B}_1 \in \mathbb{R}^8$, $Pr_{tOx}\overline{B}_1 = \overline{D}_0$ there exists a function
$$H\left(t,x,u(t,x),\mathcal{D}^{(1.0)}u(t,x),\mathcal{D}^{(0.1)}u(t,x);v(t,x),\mathcal{D}^{(1.0)}v(t,x),\mathcal{D}^{(0.1)}v(t,x)\right) := H[u(t,x);v(t,x)],$$
such that
   (a) $H[u(t,x);u(t,x)] \equiv F[u(t,x)]$;
   (b) for an arbitrary pair of functions $\bar{v}(t,x), \bar{\bar{v}}(t,x) \in \overline{B}_1$ from the space $C^{(\kappa_1.\kappa_2)}(D_0)$ that satisfy conditions:
   $$\mathcal{D}^{(\kappa_1.\kappa_2)}\left[\bar{v}(t,x) - \bar{\bar{v}}(t,x)\right] \ge (\le) 0, \kappa_1 = 0,1; \kappa_2 = 1 (\kappa_2 = 0),$$
   $$\kappa_1 + \kappa_2 < 2, (t,x) \in \overline{D}_0,$$
   an inequality holds:
   $$H\left[\bar{v}(t,x);\bar{\bar{v}}(t,x)\right] - H\left[\bar{\bar{v}}(t,x);\bar{v}(t,x)\right] \le 0; \tag{9}$$
3. function $H[u(t,x);v(t,x)]$ satisfies Lipschitz condition in the domain $\overline{B}_1$, i.e., for all functions $u_r(t,x), v_r(t,x) \in \overline{B}_1 \subset C^*(\overline{D}_0)$ $(r = 1,2)$ it holds that
$$\left|H\left[u_1(t,x);u_2(t,x)\right] - H\left[v_1(t,x);v_2(t,x)\right]\right|$$
$$\le \tfrac{1}{6}L\sum_{r=1}^{2}\left(\left|W_r(t,x)\right| + \left|\mathcal{D}^{(1.0)}W_r(t,x)\right| + \left|\mathcal{D}^{(0.1)}W_r(t,x)\right|\right),$$
where $W_r(t,x) := u_r(t,x) - v_r(t,x)$, and $\frac{1}{6}L$ is the Lipschitz constant.

Clearly, *if* $F[u(t,x)] \in C(\overline{B})$, *then it has bounded first-order partial derivatives with respect to all of its arguments, starting from the third one, and thus,* $F[u(t,x)] \in C_1^*(\overline{B})$ (see discussion in [2]). The inverse statement is not true.

3. Alternating sub- and supersolutions method

In this section we show how a suitable modification of the sub- and supersolutions method can contribute to construction of approximate solutions to the studied BVP (3), (4), and how it can be improved to reduce the computational cost of the method itself.

Let functions $z_p(t,x), v_p(t,x) \in C_1^*(\overline{D}_0)$ $(p \in \mathbb{N}_0)$ be defined in the domain $\overline{B}_1$, and let us introduce the following notations:

$$W_p(t,x) := z_p(t,x) - v_p(t,x), \tag{10}$$

$$\begin{aligned} f^p(t,x) &:= H\left[z_p(t,x); v_p(t,x)\right], \\ f_p(t,x) &:= H\left[v_p(t,x); z_p(t,x)\right], \\ \omega^p(t,x) &:= \int_0^x f^p(t,\xi)K(t,x;t,\xi)d\xi, \\ \omega_p(t,x) &:= \int_0^x f_p(t,\xi)K(t,x;t,\xi)d\xi, \end{aligned}$$

$$\begin{aligned} \alpha_p(t,x) &:= \mathcal{D}^{(1.1)}z_p(t,x) - a_1(t,x)\mathcal{D}^{(0.1)}z_p(t,x) - \omega^p(t,x), \\ \beta_p(t,x) &:= \mathcal{D}^{(1.1)}v_p(t,x) - a_1(t,x)\mathcal{D}^{(0.1)}v_p(t,x) - \omega_p(t,x), \end{aligned} \tag{11}$$

for all $(t,x) \in \overline{D}_0$.

Next, we construct sequences of sub- and supersolutions $\{z_p(t,x)\}$, $\{v_p(t,x)\}$ according to the corresponding recursive formulas:

$$\begin{aligned} z_{p+1}(t,x) &= Tf^p(\eta,\zeta), \\ v_{p+1}(t,x) &= Tf_p(\eta,\zeta), \end{aligned} \qquad (t,x) \in \overline{D}_0. \tag{12}$$

Here functions of the zeroth approximation $z_0(t,x), v_0(t,x) \in C^{(1.1)}(\overline{D}_0)$, which are defined in $\overline{B}_1$, are chosen to satisfy the following sign inequalities [15]:

$$\alpha_0(t,x) \geq 0, \quad \beta_0(t,x) \leq 0, \quad \mathcal{D}^{(\kappa_1.\kappa_2)}W_0(t,x) \geq (\leq)\ 0, \tag{13}$$

where $(t,x) \in \overline{D}_0$, $\kappa_1 = 0,1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \leq 2$.

Definition 2. We call functions $z_0(t,x), v_0(t,x) \in C^{(1.1)}(\overline{D}_0)$, defined in the domain $\overline{B}_1$ and satisfying conditions (4), (5) and (13), *comparison functions* of the BVP (3), (4).

The following lemma is true.

Lemma 2. *Assume that $F[u(t,x)] \in C_1^*(\overline{B})$. Moreover, let integro-differential Eq. (7) have a solution $u(t,x)$ in the functional space $C^{(\kappa_1.\kappa_2)}(\overline{D}_0)$, such that for all $(t,x) \in \overline{D}_0$ and $\kappa_1 = 0,1;\ \kappa_2 = 1\ (\kappa_2 = 0)$, $\kappa_1 + \kappa_2 \leq 2$, inequalities hold:*

$$\mathcal{D}^{(\kappa_1.\kappa_2)}v_0(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}u(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}z_0(t,x). \tag{14}$$

Then in the domain $\overline{B}_1$ sign inequalities (13) are satisfied.

Proof. Indeed, setting $p = 0$ in (10) and (13) we obtain that $W_0^{(\kappa_1.\kappa_2)}(t,x) \geq (\leq)\ 0$ and

$$\alpha_0(t,x) = \mathcal{D}^{(1.1)}\left[z_0(t,x) - u(t,x)\right] - a_1(t,x)\mathcal{D}^{(0.1)}\left[z_0(t,x) - u(t,x)\right] - \left[w^0(t,x) - w(t,x)\right].$$

In view of the boundary conditions (4) and inequalities (14), this yields to

$$I(t,x) := \mathcal{D}^{(0.1)}\left[z_0(t,x) - u(t,x)\right] = \int_0^t \left[\alpha_0(\eta,x) + w^0(\eta,x) - w(\eta,x)\right] \times exp\left(\int_\eta^t a_1(\tau,x)d\tau\right)d\eta \geq 0$$

for all $(t,x) \in \overline{D}_0$.

It is easy to check that $I(0,x) = 0$ and

$$\mathcal{D}^{(1.0)}I(t,x) = \mathcal{D}^{(1.1)}\left[z_0(t,x) - u(t,x)\right] \geq 0,$$

which results in the sign inequalities of the form:

$$\alpha_0(\eta,x) + \omega^0(\eta,x) - \omega(\eta,x) \geq 0,$$

$$\alpha_0(t,x) \geq (\omega(t,x) - \omega^0(t,x)) \geq 0.$$

Using a similar approach we can prove that $\beta_0(t,x) \leq 0$, for all $(t,x) \in \overline{D}_0$.

Next, from the recursive formulas (12) it follows that

$$\mathcal{D}^{(1.1)}z_{p+1}(t,x) - a_1(t,x)\mathcal{D}^{(0.1)}z_{p+1}(t,x) = \omega^p(t,x),$$

$$\mathcal{D}^{(1.1)}v_{p+1}(t,x) - a_1(t,x)\mathcal{D}^{(0.1)}v_{p+1}(t,x) = \omega_p(t,x).$$

Hence, notations (11) and expressions (12) yield to systems of relations:

$$\begin{cases} \alpha_{p+1}(t,x) = \omega^p(t,x) - \omega^{p+1}(t,x), \\ \beta_{p+1}(t,x) = \omega_p(t,x) - \omega_{p+1}(t,x); \end{cases} \tag{15}$$

$$\begin{cases} \alpha_p(t,x) = \mathcal{D}^{(1.1)}\left[z_p(t,x) - z_{p+1}(t,x)\right] - a_1(t,x)\mathcal{D}^{(0.1)}\left[z_p(t,x) - z_{p+1}(t,x)\right], \\ \beta_p(t,x) = \mathcal{D}^{(1.1)}\left[v_p(t,x) - v_{p+1}(t,x)\right] - a_1(t,x)\mathcal{D}^{(0.1)}\left[v_p(t,x) - v_{p+1}(t,x)\right]; \end{cases} \tag{16}$$

$$\begin{cases} W_{p+1}(t,x) = T\left[f^p(\eta,\zeta) - f_p(\eta,\zeta)\right], \\ \mathcal{D}^{(1.1)}W_{p+1}(t,x) - a_1(t,x)\mathcal{D}^{(0.1)}W_{p+1}(t,x) = \omega^p(t,x) - \omega_p(t,x); \end{cases} \tag{17}$$

$$\begin{cases} \mathcal{D}^{(1.1)}\left[z_{p+1}(t,x) - v_p(t,x)\right] - a_1(t,x)\mathcal{D}^{(0.1)}\left[z_{p+1}(t,x) - v_p(t,x)\right] = \omega^p(t,x) - \omega_{p-1}(t,x), \\ \mathcal{D}^{(1.1)}\left[z_p(t,x) - v_{p+1}(t,x)\right] - a_1(t,x)\mathcal{D}^{(0.1)}\left[z_p(t,x) - v_{p+1}(t,x)\right] = \omega^{p-1}(t,x) - \omega_p(t,x). \end{cases} \tag{18}$$

Taking into account the first condition in (4), from equality (16) we derive:

$$\begin{aligned} \mathcal{D}^{(0.1)}\left[z_p(t,x) - z_{p+1}(t,x)\right] &= \int_0^t \alpha_p(\eta,x)exp\left(\int_\eta^t a_1(\tau,x)d\tau\right)d\eta, \\ \mathcal{D}^{(0.1)}\left[v_p(t,x) - v_{p+1}(t,x)\right] &= \int_0^t \beta_p(\eta,x)exp\left(\int_\eta^t a_1(\tau,x)d\tau\right)d\eta, \end{aligned} \tag{19}$$

wherefrom for $p = 0$ and using relations (13) it follows that

$$\mathcal{D}^{(0.1)}\left[z_0(t,x) - z_1(t,x)\right] \geq 0,$$
$$\mathcal{D}^{(0.1)}\left[v_0(t,x) - v_1(t,x)\right] \leq 0.$$

By integrating these inequalities with respect to the x variable over the interval from x to a, we obtain:

$$z_0(t,x) - z_1(t,x) \leq 0,$$
$$v_0(t,x) - v_1(t,x) \geq 0.$$

Thus,

$$\mathcal{D}^{(1.1)}\left[z_0(t,x) - z_1(t,x)\right] = \alpha_0(t,x) + a_1(t,x)\mathcal{D}^{(0.1)}\left[z_0(t,x) - z_1(t,x)\right] \geq 0,$$

what leads to the conclusion that

$$\mathcal{D}^{(1.0)}\left[z_0(t,x) - z_1(t,x)\right] \leq 0,$$

$$\mathcal{D}^{(1.0)}\left[v_0(t,x) - v_1(t,x)\right] \geq 0.$$

Similarly from (17) for $p = 0$ we derive the sign inequalities

$$\mathcal{D}^{(\kappa_1.\kappa_2)}W_1(t,x) \leq (\geq) 0,$$

where $\kappa_1 = 0,1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \leq 2$.

Assume now that conditions

$$\begin{aligned} &\mathcal{D}^{(\kappa_1.\kappa_2)}v_0(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}z_1(t,x), \\ &\mathcal{D}^{(\kappa_1.\kappa_2)}z_0(t,x) \geq (\leq)\mathcal{D}^{(\kappa_1.\kappa_2)}v_1(t,x) \end{aligned} \tag{20}$$

hold. Then from the previous estimates we get that

$$\mathcal{D}^{(\kappa_1.\kappa_2)}v_0(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}z_1(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}v_1(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}z_0(t,x).$$

But then the recursive relation (15) for $p = 0$ results in

$$\alpha_1(t,x) = \omega^0(t,x) - \omega^1(t,x) \leq 0,$$

$$\beta_1(t,x) = \omega_0(t,x) - \omega_1(t,x) \geq 0,$$

which means that $z_1(t,x), v_1(t,x) \in \overline{B}_1$, and that they are the comparison functions if inequalities (20) are satisfied.

By repeating the aforementioned analysis, from (17), (19) for $p = 1$ we derive the inequalities:

$$\mathcal{D}^{(\kappa_1.\kappa_2)}\left[z_1(t,x) - z_2(t,x)\right] \leq (\geq)\ 0,$$
$$\mathcal{D}^{(\kappa_1.\kappa_2)}\left[v_1(t,x) - v_2(t,x)\right] \geq (\leq)\ 0,$$
$$\mathcal{D}^{(\kappa_1.\kappa_2)}W_2(t,x) \geq (\leq)\ 0,$$

for $\kappa_1 = 0,1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \leq 2$.

Next, from (18) for $p = 1$ it follows that

$$\mathcal{D}^{(0.1)}\left[z_2(t,x) - v_1(t,x)\right] = \int_0^t \left[\omega^1(\eta,x) - \omega_0(\eta,x)\right] exp\left(\int_\eta^t a_1(\tau,x)d\tau\right) d\eta < 0,$$
$$\mathcal{D}^{(1.1)}\left[z_1(t,x) - v_2(t,x)\right] = \int_0^t \left[\omega^0(\eta,x) - \omega_1(\eta,x)\right] exp\left(\int_\eta^t a_1(\tau,x)d\tau\right) d\eta \le 0,$$

and thus,

$$\mathcal{D}^{(\kappa_1.\kappa_2)}v_0(t,x) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}z_1(t,x) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}v_2(t,x) \le (\ge) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}z_2(t,x) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}v_1(t,x) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}z_0(t,x),$$

for $\kappa_1 = 0, 1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \le 2$.

On the other hand, from formulas (15) for $p = 1$ we conclude that

$$\alpha_2(t,x) \ge 0,\ \ \beta_2(t,x) \le 0.$$

Using method of mathematical induction we prove that for all $(t,x) \in \overline{D}_0$ and $\kappa_1 = 0, 1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \le 2$ the inequalities

$$\begin{aligned}
&\alpha_{2p}(t,x) \ge 0, \quad \alpha_{2p+1}(t,x) \le 0, \quad \beta_{2p}(t,x) \le 0, \quad \beta_{2p+1}(t,x) \ge 0,\\
&\mathcal{D}^{(\kappa_1.\kappa_2)}v_{2p}(t,x) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}z_{2p+1}(t,x) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}v_{2p+2}(t,x)\\
&\le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}z_{2p+2}(t,x) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}v_{2p+1}(t,x) \le (\ge)\mathcal{D}^{(\kappa_1.\kappa_2)}z_{2p}(t,x)
\end{aligned} \tag{21}$$

are satisfied. □

Results of Lemmas 1 and 2 yield the following theorem:

Theorem 1. *Let $F[u(t,x)] \in C_1(\overline{B})$, $0 \le a_1(t,x) \in C^{(0.1)}(D_0)$, $a_2(t,x) \in C^{(1.0)}(D_0)$ and assume that condition (6) holds.*

Then functions $z_p(t,x), v_p(t,x) \in C^*(\overline{D}_0)$, constructed according to the scheme (12), (13) under constraints (20), satisfy inequalities (21) in the domain $\overline{B}_1$, for all $(t,x) \in \overline{D}_0$ and $p \in \mathbb{N}$.

4. Main results

4.1. Comparison functions for the nonlocal BVP (3), (4)

In this section we demonstrate, how one can find the comparison functions to the BVP (3), (4).

Assume that $h(t,x)$ is an arbitrary function from the space $C^*(\overline{D}_0)$, defined in the domain $\overline{B}$, and denote by $u^*(t,x)$ a function of the form:

$$u^*(t,x) = TF[h(\eta,\zeta)].$$

It is easy to see that $u^*(t,x) \in C^*(\overline{D}_0)$ and that it satisfies restrictions (4). Let us also introduce a function $\alpha^*(t,x)$ such that

$$\alpha^*(t,x) := \mathcal{D}^{(1.1)}u^*(t,x) - a_1(t,x)\mathcal{D}^{(0.1)}u^*(t,x) - \omega^*(t,x),$$

where

$$\omega^*(t,x) := \int_0^x F[u^*(t,\xi)]exp\left(\int_x^\xi a_2(t,\tau)d\tau\right) d\xi.$$

Then

$$u^*(t,x) = TF[u^*(\eta,\zeta)] - \int_x^a \int_0^t \alpha^*(\eta,\xi)exp\left(\int_\eta^t a_1(\tau,\xi)d\tau\right) d\eta d\xi.$$

We want to show that

$$\begin{aligned}
z_0(t,x) &= u^*(t,x) - \int_x^a \int_0^t |\alpha^*(\eta,\xi)|\, exp\left(\int_\eta^t a_1(\tau,\xi)d\tau\right) d\eta d\xi,\\
v_0(t,x) &= u^*(t,x) + \int_x^a \int_0^t |\alpha^*(\eta,\xi)|\, exp\left(\int_\eta^t a_1(\tau,\xi)d\tau\right) d\eta d\xi
\end{aligned} \tag{22}$$

are comparison functions for the BVP (3), (4). For this purpose we are going to simply use Definition 2. Indeed, functions (22) satisfy boundary conditions (4) and are defined in the functional space $C^*(\overline{D}_0)$. At the same time relations hold:

$$\begin{aligned}
W_0(t,x) = &\ -2\int_x^a \int_0^t |\alpha^*(\eta,\xi)|\, exp\left(\int_\eta^t a_1(\tau,\xi)d\tau\right) d\eta d\xi \le 0,\\
\mathcal{D}^{(0.1)}W_0(t,x) = &\ 2\int_0^t |\alpha^*(\eta,x)|\, exp\left(\int_\eta^t a_1(\tau,x)d\tau\right) d\eta \ge 0,\\
\mathcal{D}^{(1.0)}W_0(t,x) = &\ -2\int_x^a |\alpha^*(t,\xi)|\, d\xi - 2\int_0^t \int_x^a a_1(t,\xi)\, |\alpha^*(t,\xi)| \times exp\left(\int_\eta^t a_1(\tau,\xi)d\tau\right) d\xi d\eta \le 0,\\
\mathcal{D}^{(1.1)}W_0(t,x) = &\ 2\,|\alpha^*(t,x)| + 2a_1(t,x)\int_0^t |\alpha^*(t,x)|\, exp\left(\int_\eta^t a_1(\tau,x)d\tau\right) d\eta \ge 0.
\end{aligned}$$

This leads to the following inequalities:

$$\mathcal{D}^{(\kappa_1.\kappa_2)}W_0(t,x) \geq (\leq)\ 0,$$

for $\kappa_1 = 0,1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \leq 2$, and

$$\alpha_0(t,x) = \mathcal{D}^{(1.1)}z_0(t,x) - a_1(t,x)\mathcal{D}^{(0.1)}z_0(t,x) - \omega^0(t,x) = \left|\alpha^*(t,x)\right| + \alpha^*(t,x) + \omega^*(t,x) - \omega^0(t,x) \geq 0.$$

Similarly, we derive that $\beta_0(t,x) \leq 0$, for all $(t,x) \in \overline{D}_0$, which means that functions (21) are indeed the comparison functions for the problem (3), (4), if they are defined in $\overline{B}_1$.

Hence, the following lemma is true:

Lemma 3. *If $F[u(t,x)] \in C^*(\overline{B})$ and $0 \leq a_1(t,x) \in C^{(0.1)}(D_0)$, $a_2(t,x) \in C^{(1.0)}(D_0)$, then a set of comparison functions of the BVP (3), (4) in non-empty.*

4.2. Convergence of the functional sequences (12)

Let us show that the functional sequences $\left\{\mathcal{D}^{(\kappa_1.\kappa_2)}z_p(t,x)\right\}$ and $\left\{\mathcal{D}^{(\kappa_1.\kappa_2)}v_p(t,x)\right\}$ converge uniformly to the same limit in the domain $\overline{D}_0$. In virtue of inequalities (21) it is sufficient to show that

$$\lim_{p\to\infty} \mathcal{D}^{(\kappa_1.\kappa_2)}W_p(t,x) = 0,$$

for all $\kappa_1 = 0,1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \leq 2$.

For simplicity, we first introduce the following notations:

$$\begin{aligned} d &:= \max_{\kappa_1,\kappa_2} \sup_{\overline{D}_0} \left|\mathcal{D}^{(\kappa_1.\kappa_2)}W_0(t,x)\right|, \\ q &:= \sup_{\overline{D}_0} \left|a_1(t,x)\right|, \\ \gamma &:= \max\left\{1, a+b, a(a+b), (a+b)(1+qb)\right\}, \\ K &:= \sup_{\overline{D}_0\times\overline{D}_0} K(t,x;\eta,\xi). \end{aligned}$$

Then from (17) by the method of mathematical induction it is easy to prove that the estimate

$$\left|\mathcal{D}^{(\kappa_1.\kappa_2)}W_p(t,x)\right| \leq \frac{[KL\gamma(a+t-x)]^p}{p!}d \tag{23}$$

is satisfied, for all $(t,x) \in \overline{D}_0$, $p \in \mathbb{N}$ and $\kappa_i = 0,1;\ i = 1,2,\ \kappa_1 + \kappa_2 \leq 2$.

Based on inequalities (23), we can show that

$$\lim_{p\to\infty} \mathcal{D}^{(\kappa_1.\kappa_2)}W_p(t,x) = 0,$$

which yields to the following relations:

$$\lim_{p\to\infty} \mathcal{D}^{(\kappa_1.\kappa_2)}z_p(t,x) = \lim_{p\to\infty} \mathcal{D}^{(\kappa_1.\kappa_2)}v_p(t,x) = u_{\kappa_1,\kappa_2}(t,x).$$

To prove that

$$u_{\kappa_1,\kappa_2}(t,x) = \mathcal{D}^{(\kappa_1.\kappa_2)}u(t,x),$$

where $u(t,x)$ is a regular solution to the integro-differential Eq. (7), it is sufficient to pass to the limit in (11) as $p \to \infty$, and to differentiate the resulting relation κ_1 times with respect to the t variable and κ_2 times with respect to the x variable, where $\kappa_1 + \kappa_2 \leq 2$. According to Lemma 1, the obtained limit function is then a solution to the BVP (3), (4).

Theorem 2. *Let conditions of Theorem 1 to be hold, and assume that the comparison functions of the BVP (3), (4) are chosen to satisfy conditions (20) in the domain $\overline{B}_1$.*

Then sequences of functions $\left\{z_p(t,x)\right\}$, $\left\{v_p(t,x)\right\}$, defined by (12),

1. *converge absolutely and uniformly to the unique regular solution of the BVP (3), (4), for $(t,x) \in \overline{D}_0$;*
2. *satisfy a posteriory estimates (23), and*
3. *satisfy the inequalities:*

$$\begin{aligned} &\mathcal{D}^{(\kappa_1.\kappa_2)}v_{2p}(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}v_{2p+2}(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}u(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}z_{2p+2}(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}z_{2p}(t,x), \\ &\mathcal{D}^{(\kappa_1.\kappa_2)}z_{2p+1}(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}z_{2p+3}(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}u(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}v_{2p+3}(t,x) \leq (\geq)\mathcal{D}^{(\kappa_1.\kappa_2)}v_{2p+1}(t,x), \end{aligned} \tag{24}$$

in the domain $\overline{B}_1$, where $\kappa_1 = 0,1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \leq 2$, $(t,x) \in \overline{D}_0$, $p \in \mathbb{N}_0$.

Proof. Uniqueness of the regular solution of the BVP (3), (4) can be proved by contrary and applying estimates (23). To show that inequalities (24) are true it is sufficient to use (21) and to repeat arguments from [11, p. 211]. We will leave it to the reader. □

Corollary 1. *Assume that conditions of Theorem 1 hold and let $F[u(t,x)] \equiv H[u(t,x);0]$. If $F[0] \le (\ge)\ 0$ in domain B, then for all $(t,x) \in \overline{D}_0$ a solution of the BVP (3), (4) with the homogeneous boundary constraints satisfies the differential inequalities*

$$\mathcal{D}^{(\kappa_1.\kappa_2)}u(t,x) \le (\ge)\ 0, \quad \text{for} \quad \kappa_1 = 0,1; \quad \kappa_2 = 1,$$

and in the case of $\kappa_2 = 0$ it holds that

$$\mathcal{D}^{(\kappa_1.0)}u(t,x) \ge (\le)\ 0.$$

Remark 1. If $F[u(t,x)] \equiv H[u(t,x);0]$, then to construct the lower and upper approximations to the exact solution of the BVP (3), (4) it is sufficient to find only one sequence $\{z_p(t,x)\}$, which substantially reduces the number of computations.

4.3. Comparison theorem

Together with Eq. (3), let us now consider a differential equation

$$L_{1,2}z(t,x) = f_1(t,x,z(t,x),\mathcal{D}^{(1.0)}z(t,x),\mathcal{D}^{(0.1)}z(t,x)) := f_1[z(t,x)], \tag{25}$$

where $f_1 : \overline{B} \to \mathbb{R},\ \overline{B} \subset \mathbb{R}^5$.

Assume that for the right hand-sides of (3) and (25) the following conditions hold:

(i) $F[u(t,x)] \in C_1^*(\overline{B})$;

(ii) $f_1[z(t,x)] \in C(\overline{B})$ and in the domain $\overline{B}$ it has bounded first-order derivatives with respect to all of its arguments, starting from the third one. In addition, it satisfies the inequalities

$$\begin{aligned} &\frac{\partial f_1[z(t,x)]}{\partial \mathcal{D}^{(\kappa_1.0)}z(t,x)} \le 0,\\ &\frac{\partial f_1[z(t,x)]}{\partial \mathcal{D}^{(0.1)}z(t,x)} - \mathcal{D}^{(0.1)}a(t,x) - a_1(t,x)a_2(t,x) \ge 0,\\ &(t,x) \in D_0; \end{aligned} \tag{26}$$

(iii) for any function $v(t,x) \in \overline{B}$ from the space $C^*(\overline{D}_0)$ it holds that

$$f_1[v(t,x)] \ge (\le) f[v(t,x)]. \tag{27}$$

Then one can prove the following Comparison Theorem.

Theorem 3. *Let coefficients $a_i(t,x),\ i = 1,2$, satisfy Theorem 1, and assume that the right hand-sides $f[u(t,x)]$ and $f_1[z(t,x)]$ of differential Eqs. (3) and (25) satisfy the aforementioned conditions (i)–(iii). Additionally, suppose that there exist comparison functions of the problems (3), (4) and (25), (4), (5) in the domain $\overline{B}_1$.*

Then solutions of these problems satisfy the inequalities

$$\begin{aligned} &\mathcal{D}^{(0.1)}[z(t,x) - u(t,x)] \ge (\le) 0,\\ &\mathcal{D}^{(\kappa_1.0)}[z(t,x) - u(t,x)] \le (\ge) 0, \end{aligned} \tag{28}$$

for all $(t,x) \in \overline{D}_0$ and $\kappa_1 = 0,1$.

Proof. According to Theorem 2 and Corollary 1, regular solutions of the BVPs (3), (4) and (25), (4), (5) exist and are unique. Thus, denoting by

$$\mathcal{D}^{(\kappa_1.\kappa_2)}W(t,x) := z(t,x) - u(t,x),$$

and applying the Mean Value Theorem [25] we obtain:

$$L_{1,2}W(t,x) = \sum_{\kappa_1,\kappa_2} b_{\kappa_1,\kappa_2}(t,x)\mathcal{D}^{(\kappa_1.\kappa_2)}W(t,x) + f_1[u(t,x)] - f[u(t,x)],$$

for $\kappa_i = 0,1\ (i = 1,2),\ \kappa_1+\kappa_2 < 2$, where $b_{\kappa_1,\kappa_2}(t,x) := \frac{\partial \widetilde{f}_1[z(t,x)]}{\partial \mathcal{D}^{(\kappa_1.\kappa_2)}z(t,x)}$ are derivatives for the given values of $\mathcal{D}^{(\kappa_1.\kappa_2)}z(t,x) \in \overline{B}$, $(t,x) \in \overline{D}_0$.

It is easy to check that function $W(t,x)$ satisfies homogeneous boundary conditions (4), and that the relation holds:

$$F[W(t,x)] := b_{0,0}(t,x)W(t,x)+b_{1,0}(t,x)\mathcal{D}^{(1.0)}W(t,x)+[b_{0,1}(t,x)-(\mathcal{D}^{(0.1)}a_1(t,x)+a_1(t,x)a_2(t,x))]\mathcal{D}^{(0.1)}W(t,x)+f_1[u(t,x)]-f[u(t,x)].$$

In virtue of (26), (27) we conclude that $F[W(t,x)] \in C_1^*(\overline{B})$ and $F[W(t,x)] \equiv H[F[W(t,x)];0]$, $F[0] \ge (\le)\ 0$. Hence, based on Corollary 1, inequalities (28) are satisfied, for all $(t,x) \in \overline{D}_0$. This finishes the proof. □

Remark 2. To further improve convergence rate of the iterative method (11) one can use ideas from [20] (see p. 228 therein).

5. Illustrative numerical example

Based on the general problem setting (3), (4), let us demonstrate effectiveness and applicability of our method on a numerical example of the illustrative nature.

Example. *In the space of functions* $C^*(\overline{D}_0)$ *with*

$$D_0 = (t,x) | t \in (0,1), x \in (0,1)$$

find a solution to the PDE

$$\begin{aligned} &\mathcal{D}^{(1.2)}u(t,x) - (1+x)^{-1}\mathcal{D}^{(1.1)}u(t,x) \\ &\quad = 0.5(1+x)\left[0.2x\mathcal{D}^{(1.0)}u(t,x) - 0.4(1-t)\mathcal{D}^{(0.1)}u(t,x) - 0.5(t+x)\right], \end{aligned} \tag{29}$$

satisfying homogeneous boundary constraints of the form

$$\begin{aligned} &u(0,x) = 0, \quad x \in [0,1], \\ &\mathcal{D}^{(0.1)}u(t,0) = 0, \quad u(t,1) = 0, \quad t \in [0,1]. \end{aligned} \tag{30}$$

In this case

$$F[u(t,x)] \equiv H[u(t,x); 0]$$

and

$$F[0] = -0.25(t+x)(1+x) \le 0,$$

and thus, according to Corollary 1, solution of the problem (29), (30) complies with the following (differential) inequalities:

$$\mathcal{D}^{(\kappa_1.\kappa_2)}u(t,x) \le (\ge)\ 0,$$

for all $\kappa_1 = 0,1;\ \kappa_2 = 1\ (\kappa_2 = 0)\ \kappa_1 + \kappa_2 \le 2$.

Let us now take the comparison function $z_0(t,x)$ being

$$z_0(t,x) = 0,$$

then $\alpha_0(t,x) \ge 0$. In virtue of (12), for $p = 0$ we have:

$$z_1(t,x) = 0.125t[0.5(1-x^2)(t+0.5(1+x^2)) + 0.33(1-x^3)(1+t)] \ge 0.$$

On the next three iteration steps for $p = 1,2,3$ we obtain that

$$\begin{aligned} z_2(t,x) = &-10^{-4}\cdot 0.65105t(x^8-1) + 0.071430\left[-0.0027084t - 0.0016667t^2\right](x^7-1) \\ &+ 0.083335\left[0.0014583t^2 - 0.0016667t - 0.0041666t^3\right](x^6-1) \\ &+ 0.1\left[-0.0041666t^3 + 0.011458t^2 - 0.0041668t^4\right](x^5-1) \\ &+ 0.125\left[-0.010417t^4 + 0.01875t^2 + 0.0083332t^3 - 0.24271t\right](x^4-1) \\ &+ 0.16666\left[-0.00625t^4 + 0.0083332t^3 - 0.23958t^2 - 0.24271t\right](x^3-1) - 0.0625t^2(x^2-1); \end{aligned}$$

$$\begin{aligned} z_3(t,x) = &10^{-4}\cdot\left[-0.15625t^6 + 0.38193t^5 - 4.0015t^4 - 4.4273t^3 + 11.033t^2\right](x^5-1) \\ &+ 10^{-3}\cdot 0.125\left[-0.069441t^6 + 0.019443t^5 - 10.106t^4 + 8.0973t^3 + 17.984t^2 - 242.88t\right](x^4-1) \\ &+ 10^{-2}\cdot 0.16666\left[-0.5974t^4 + 81.666t^3 - 24.011t^2 - 24.288t\right](x^3-1) - 0.0625t^2(x^2-1) + \mathcal{O}(10^{-5}); \end{aligned}$$

$$\begin{aligned} z_4(t,x) = &10^{-4}\cdot\left[0,1522t^6 - 0,37056t^5 + 4,0142t^4 + 4,4094t^3 - 11,043t^2\right](1-x^5) \\ &+ 10^{-2}\cdot 0,125\left[0,00628t^6 - 0,01811t^5 + 1,0124t^4 - 0,8117t^3 - 1,801t^2 + 24,288t\right](1-x^4) \\ &+ 10^{-2}\cdot 0,16666\left[0,59868t^4 - 0,81786t^3 + 24,009t^2 + 24,288t\right](1-x^3) + 0.0625t^2(1-x^2) + \mathcal{O}(10^{-5}). \end{aligned}$$

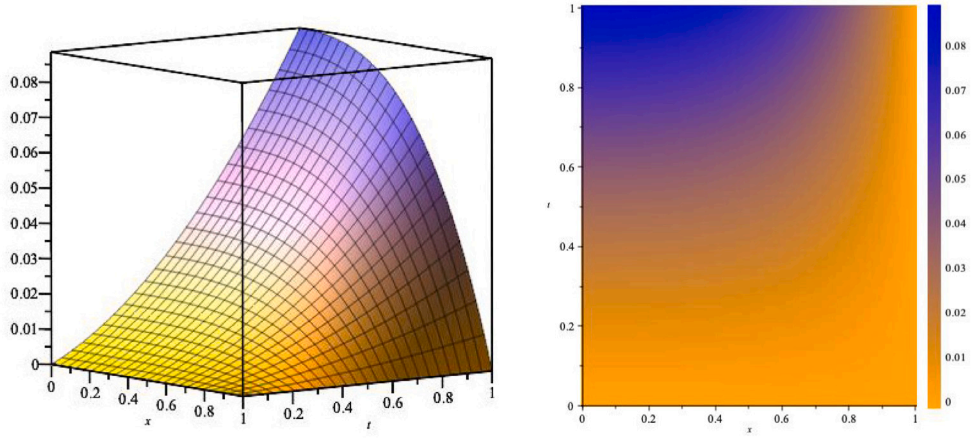
One can check that the constructed approximations $\{z_i(t,x)\}$, $i = \overline{0,4}$ can be ordered as follows:

$$z_0(t,x) \le z_2(t,x) \le z_4(t,x) \le u(t,x) \le z_3(t,x) \le z_1(t,x).$$

Denoting the approximate solution of the BVP (29), (30) as

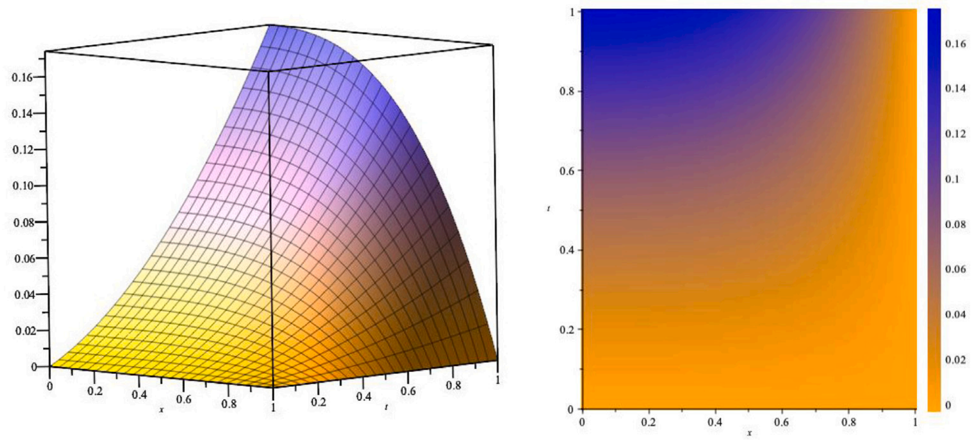
$$\widetilde{u}_n(t,x) := \frac{1}{2}[z_{n-1}(t,x) + z_n(t,x)], \quad n = \overline{1,4},$$

we depict these functions at every iteration step (see Figs. 1–4).



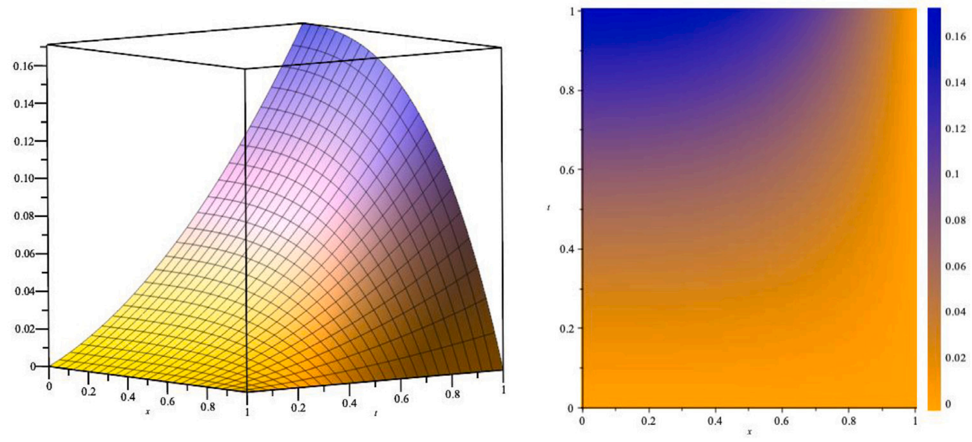
(a) (b)

Fig. 1. Plots of the distance between two consecutive approximations $z_i(t,x)$ and $z_{i+1}(t,x)$, $i=\overline{0,3}$, with the vertical axis denoting values of the distance function $|z_i(t,x)-z_{i+1}(t,x)|$.



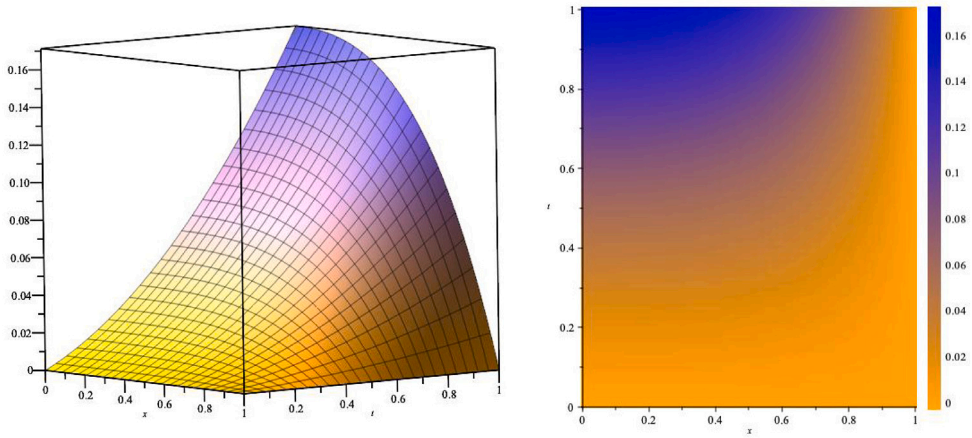
(a) (b)

Fig. 2. **(a)** – First approximation to the exact solution of the BVP (29), (30) and **(b)** – its 2D projection with the color bar corresponding to the value of the function $u_1(t,x)$ at any point $(\tilde{t},\tilde{x})$ within the domain D_0.



(a) (b)

Fig. 3. **(a)** – Second approximation to the exact solution of the BVP (29), (30) and **(b)** – its 2D projection with the color bar corresponding to the value of the function $u_2(t,x)$ at any point $(\tilde{t},\tilde{x})$ within the domain D_0.



(a) (b)

Fig. 4. **(a)** – Third approximation to the exact solution of the BVP (29), (30) and **(b)** – its 2D projection with the color bar corresponding to the value of the function $u_3(t,x)$ at any point $(\tilde{t},\tilde{x})$ within the domain D_0.

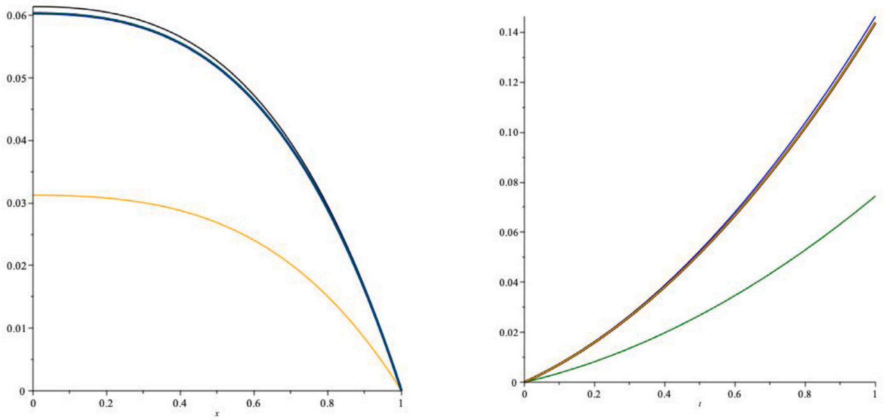
By fixing $x = \frac{1}{2}$ and then $t = \frac{1}{2}$ we also get profiles of all four approximations to the exact solution of the BVP (29), (30), see Fig. 5.

Note, that the behavior of approximate solutions on Fig. 5 aligns with their plots from Figs. 1–4.

From the error analysis of the computational process we can conclude that already on the fourth iteration step the following accuracy is reached:

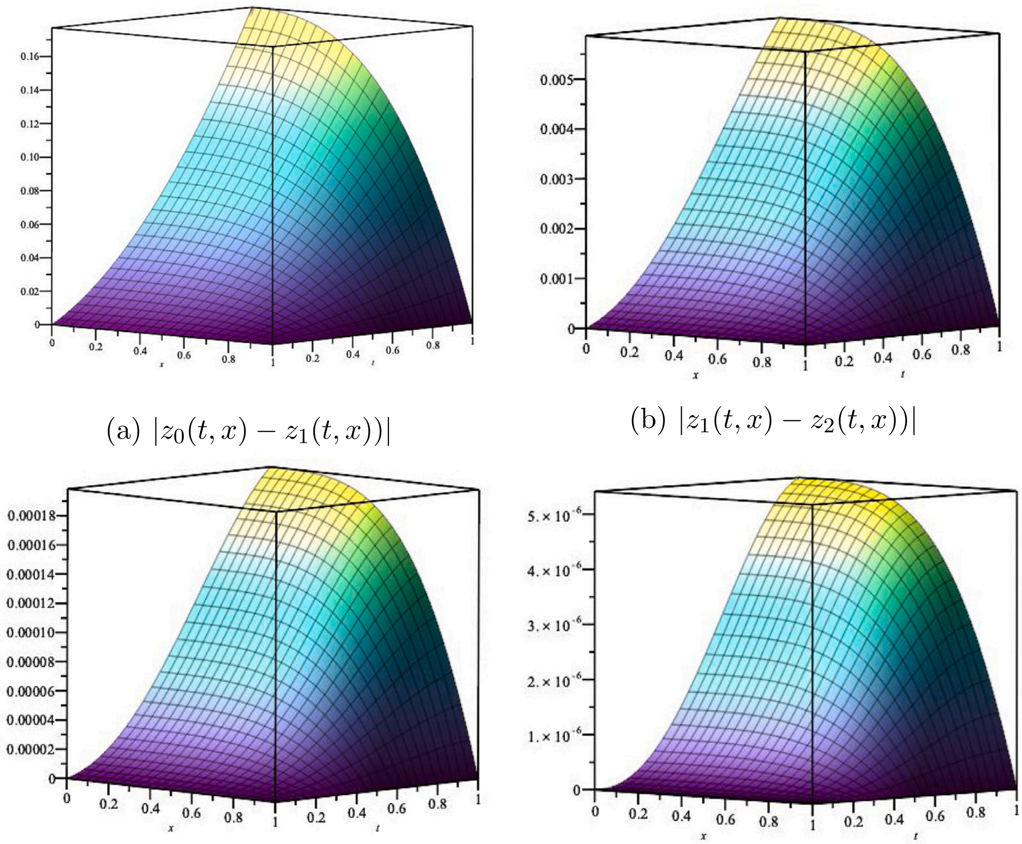
$$\left|u(t,x)-\widetilde{u}_4(t,x)\right| \le \left|z_3(t,x)-z_4(t,x)\right| \le 6\cdot 10^{-6}.$$

Moreover, convergence behavior of the constructed approximations can also be traced using the plotting tool of Maple 2024, where we have compared the neighboring functions of the sequence $\{z_n(t,x)\}$ for $n=\overline{0,4}$ (see Fig. 6).



(a) $u_i(\frac{1}{2}, x)$ (b) $u_i(t, \frac{1}{2})$

Fig. 5. **(a)** – Fourth approximation to the exact solution of the BVP (29), (30) and **(b)** – its 2D projection with the color bar corresponding to the value of the function $u_4(t,x)$ at any point $(\tilde{t},\tilde{x})$ within the domain D_0.

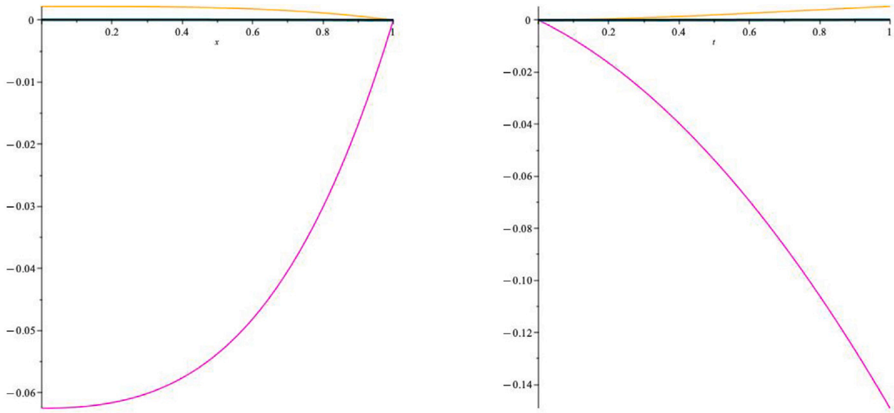


(a) $|z_0(t,x) - z_1(t,x))|$ (b) $|z_1(t,x) - z_2(t,x))|$

(c) $|z_2(t,x) - z_3(t,x))|$ (d) $|z_3(t,x) - z_4(t,x))|$

Fig. 6. **(a)** – Four approximations to the exact solution of the BVP (29), (30) for $t = \frac{1}{2}$; **(b)** – Four approximations to the exact solution of the BVP (29), (30) for $x = \frac{1}{2}$.

Analogically to Fig. 5, we plot 2d profiles of the differences $|z_i(t,x) - z_{i+1}(t,x)|$ when fixing $t = \frac{1}{2}$ and then $x = \frac{1}{2}$. Comparison of these graphs is given on Fig. 7.



(a) $|z_i(\frac{1}{2},x) - z_{i+1}(\frac{1}{2},x)|$ (b) $|z_i(t,\frac{1}{2}) - z_{i+1}(t,\frac{1}{2})|$

Fig. 7. Comparison analysis of the distance functions $|z_i(t,x) - z_{i+1}(t,x)|$ for $t = \overline{0,3}$.

All these facts prove convenience of application of the studied method for approximation of solutions to nonlinear BVPs for hyperbolic PDEs due to its simplicity and high speed of convergence.

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Data availability

No data was used for the research described in the article.

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