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Semi-Analytical Sensitivity Analysis for Multibody System Dynamics Described by Differential–Algebraic Equations

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A large number of deployable space structures involve multibody system dynamics, and in order to effectively analyze and optimize dynamic performance, the sensitivity information of multibody systems is often required. At present, the sensitivity analysis methods of multibody system dynamics, which have been widely used, are mainly finite difference method, direct differentiation method, and adjoint variable method. Among them, the finite difference method is an approximate method; the direct differentiation method and the adjoint variable method are analytical methods. Based on the dynamic problems of the multibody system in the form of differential–algebraic equations, the semi-analytical sensitivity analysis method for multibody system dynamics is proposed in this paper, which combines the simplicity of the finite difference method with the accuracy of the analytical methods. It includes the local semi-analytical method based on the element level and the global semi-analytical method based on the system level, of which the latter has higher computational efficiency. Through two numerical examples, the effectiveness and numerical stability of the method are verified. This method not only retains the accuracy and efficiency of the analytical methods, but also simplifies the derivation and coding of analytical formulas by combining with the existing programs. It has stronger versatility and is beneficial to the sensitivity calculation of large-scale complex multibody systems.

Nomenclature

$\boldsymbol{b}$ = design variables
G^0, G^f = initial and final state of system
H = integral item of objective function
$\boldsymbol{M}$, $\boldsymbol{M}_e$ = mass matrix of system and mass matrix of element
m = the number of constraint equations in differential–algebraic equations
n = the number of generalized degrees of freedom
p = the number of design variables
$\boldsymbol{Q}$, $\boldsymbol{Q}_e$ = force vector of system and force vector of element
$\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$ = position coordinate, velocity, and acceleration of system
$\boldsymbol{q}_e$, $\dot{\boldsymbol{q}}_e$, $\ddot{\boldsymbol{q}}_e$ = position coordinate, velocity, and acceleration of element
t^0, t^f = initial and final time of system
γ, η = customization parameters of Baumgarte stabilization method
$\boldsymbol{\lambda}$, $\boldsymbol{\lambda}_e$ = Lagrange multipliers of system and Lagrange multipliers of element
$\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \xi, \boldsymbol{\alpha}, \boldsymbol{\beta}$ = adjoint variables
$\boldsymbol{\Phi}$, $\dot{\boldsymbol{\Phi}}$, $\ddot{\boldsymbol{\Phi}}$ = position constraint, velocity level constraint, and acceleration level constraint of system
$\boldsymbol{\Phi}_e$, $\dot{\boldsymbol{\Phi}}_e$, $\ddot{\boldsymbol{\Phi}}_e$ = position constraint, velocity level constraint, and acceleration level constraint of element
$\overline{\boldsymbol{\Phi}}$, $\overline{\boldsymbol{\Phi}}_e$ = Baumgarte constraint of system and Baumgarte constraint of element
$\boldsymbol{\varphi}^0$, $\overline{\boldsymbol{\varphi}}^0$ = initial position and velocity condition of system
$\boldsymbol{\varphi}_e^0$, $\overline{\boldsymbol{\varphi}}_e^0$ = initial position and velocity condition of element
Ψ = objective function
Ψ_b = derivative of objective function with respect to design variables
Ω^0, Ω^f = initial and final time condition of system

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I. Introduction

In recent years, multibody system dynamics and its optimization analysis are playing an increasingly important role in the aerospace field [1–7]. A large number of space structures involve multibody system dynamics, such as space deployable antennas, spacecraft solar panels, and on-orbit assembly for space station [8–10]. If the optimization algorithm based on gradient description is used to optimize the structures or multidisciplinary optimization for space mission, the sensitivity analysis of multibody systems is necessary. In addition, the sensitivity analysis can be used not only to determine the iteration direction of some optimization algorithms, but also to characterize the influence of design variables on the objective function, which can filter design variables and reduce the number of design variables to improve the optimization efficiency [11,12].

The mathematical model of multibody system dynamics often appears as a set of multidimensional, strongly nonlinear algebraic equations, ordinary differential equations, or differential algebraic mixed equations. Compared with traditional static optimization design, the objective function and the constraint equation of dynamic optimization design of multibody systems contain state variables, and multibody systems are also constrained by state equations. Therefore, the sensitivity of optimization design of multibody system dynamics includes two parts: state sensitivity and design sensitivity. The former is the derivative of state variables with respect to design variables, and the latter is the derivative of the objective function with respect to design variables [12,13]. At present, the main methods for sensitivity analysis of multibody system dynamics include finite difference method (FDM), direct differentiation method, and adjoint variable method [12–18].

FDM is very simple, and it is an approximate sensitivity calculation method. It calculates the sensitivity of design variables by means of difference quotient. But because its computational workload increases proportionally with the number of design variables, the efficiency and accuracy are relatively low. Greene and Haftka [19] used the FDM to calculate the sensitivity of displacement, velocity, acceleration, and stress in linear, structural, and transient response problems. It was found that this method has higher computational cost, and in actual calculation, computing programs are highly demanded for time and accuracy. Dopico et al. [20] used the FDM to approximate the sensitivity of the relevant design variables for index-3 differential–algebraic equations, index-1 differential–algebraic equations, and penalty formulations, respectively. It is concluded that the method is very inaccurate or even completely useless.

Direct differentiation method [21–27] and adjoint variable method [28–32] are analytical methods (AM) for sensitivity calculation. The former calculates the sensitivity by directly deriving the dynamic equations of multibody systems with respect to design variables. The latter calculates the sensitivity of design variables by introducing a series of adjoint variables, eliminating the correlation term involving state sensitivity in the equation and solving a series of adjoint variable equations. The calculation accuracy and efficiency of the AM are relatively high.

The direct differentiation method was first proposed by Haug et al. [21] in 1984. Later, Chang and Nikravesh [22] studied the general method of calculating design sensitivity coefficient matrix of constrained dynamic systems by the direct differentiation method, and proposed a comprehensive optimization design method. The effectiveness of this method was illustrated by two examples. Dias and Pereira [23] used the direct differentiation method to establish the sensitivity equations for sensitivity analysis of the rigid-flexible multibody system, and compared the calculation results with the FDM through two examples of rigid-flexible multibody systems. Serban and Haug [24] deduced the analytical formulas of kinematic and dynamic derivatives for multibody system analysis, including implicit numerical integration, dynamic sensitivity analysis, and kinematic workspace analysis. Compared with the FDM, it is proved that the analytical formulas can accurately and effectively calculate the high-order derivatives for multibody system analysis. Callejo and Dopico [25] applied the direct differentiation method to the state space formulation of rigid body system dynamics simulation, and verified its effectiveness in design sensitivity analysis by vehicle model. Neto et al. [27] applied the direct differential method to solve the sensitivity of flexible multibody systems using composite materials components.

The adjoint variable method was first proposed by Haug and Arora [28] in 1978, which has been widely used in recent years because of its fast calculation speed. Liu [30] derived the first- and second-order sensitivity analysis equations of constrained flexible multibody systems by the adjoint variable method. Ding et al. [12,13], based on the differential–algebraic equation model of multibody system dynamics, gave the formulas of second-order sensitivity analysis of the adjoint variable method and detailed calculation steps. Pi et al. [17] extended the absolute nodal coordinate formulation with emphasis on modeling of beams and plates in large deformation problems to the design sensitivity analysis of flexible multibody systems by using the adjoint variable method; Alexander et al. [31] used floating reference system formulas to model and applied the adjoint variable method to flexible multibody systems with motion loops. Nachbagauer et al. [32] illustrated the potential of the adjoint variable method in multibody dynamics optimization problems, and applies it to inverse dynamics and parameter identification problems.

However, the AM need to analytically solve the derivatives of the multibody system dynamics equations and the objective function with respect to state variables and design variables. When multibody systems are large in scale and complex in structure, the types and numbers of design variables will increase. At this time, the sensitivity calculation formulas by AM will be very complicated, the workload is large, and even it is very difficult to obtain the derivative of some design variables.

In this paper, aiming at the dynamic problems of multibody systems in the form of differential–algebraic equations, the semi-analytical sensitivity analysis method is proposed based on the existing direct differentiation method and adjoint variable method derived by AM. It overcomes the relatively low calculation accuracy and efficiency of the FDM and the complex formula derivation and programming process of the AM. The semi-analytic method proposed in this paper includes the local semi-analytical method (LSAM) and the global semi-analytical method (GSAM). Specifically, in this method, the finite difference is used to replace the derivative with respect to design variables in analytical formulas of sensitivity calculation. Among them, the LSAM is based on the element level, and the GSAM is based on the system level. The validity and numerical stability of the method are verified by numerical examples. By comparing different sensitivity analysis methods for multibody system dynamics, the advantages of the method proposed in this paper are as follows:

1) Compared with the FDM, it does not need to solve the differential–algebraic equations of multibody system dynamics repeatedly, and has higher computing efficiency and accuracy.

2) Compared with the AM, it does not need to analytically derive the derivatives of the multibody system dynamics equation and the objective function with respect to design variables, and can handle a variety of types of design variables flexibly.

3) Compared with the LSAM based on the element level proposed in this paper, the GSAM based on the system level does not need to extract the matrix information of elements before and after perturbation, which simplifies the programming work and has higher computational efficiency.

The rest of this paper is organized as follows: Sec. II describes the mathematical model of multibody system dynamics studied in this paper. Then Sec. III gives the calculation formula of the FDM. The formulas of the AM are given in Sec. IV, including the direct differentiation method and the adjoint variable method. Section V gives the calculation formulas of the semi-analytical sensitivity analysis method, including the LSAM and the GSAM. The effectiveness and numerical stability of the method is verified by two examples in Sec. VI. Finally, the conclusion of this paper is given in Sec. VII.

II. Problem Description

The dynamic optimization design of multibody system dynamics is a design method of selecting design variables, establishing objective function, and obtaining optimal design. It must conform to the laws of dynamics and kinematics of the system and be within the constraints of the system's state, geometric relationship, or other factors. Among them, the state variables are used to describe the dynamic response of the system, which is expressed as $\boldsymbol{q} = [q_1, q_2, \ldots, q_n]^T$. The design variables are independent parameters selected and finally determined in the process of optimization design. It is expressed as $\boldsymbol{b} = [b_1, b_2, \ldots, b_p]^T$, where p is the number of design variables, and the state variables $\boldsymbol{q}$ is related to the design variables $\boldsymbol{b}$. The optimal design problem of multibody system dynamics can be expressed as a general nonlinear constrained optimization problem constrained by equality and inequality, given as

$$\begin{aligned} &\min \psi(\boldsymbol{b}) \\ &\text{s.t. } h_{d_1}(\boldsymbol{b}) = 0 \qquad d_1 = 1, 2, \ldots, c_1 \\ &\qquad g_{d_2}(\boldsymbol{b}) \le 0 \qquad d_2 = 1, 2, \ldots, c_2 \end{aligned} \tag{1}$$

where $\psi(\boldsymbol{b})$ is the objective function, which is the criterion for evaluating the quality of the design scheme; $h_{d_1}(\boldsymbol{b})$ and $g_{d_2}(\boldsymbol{b})$ are the equality and inequality constraint of the system, respectively; and c_1 and c_2 are the number of equality and inequality constraints, respectively. In the optimization problem of multibody system dynamics, the objective function is generally expressed as the following integral form:

$$\begin{aligned} \Psi(\boldsymbol{b}) = {} & G^0(\boldsymbol{q}^0, \dot{\boldsymbol{q}}^0, \boldsymbol{b}, t^0) + G^f(\boldsymbol{q}^f, \dot{\boldsymbol{q}}^f, \boldsymbol{b}, t^f) \\ & + \int_{t^0}^{t^f} H(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \lambda, \boldsymbol{b}, t)\, \mathrm{d}t \end{aligned} \tag{2}$$

where the superscripts 0 and f denote the initial and final times, respectively; the state variables $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and $\ddot{\boldsymbol{q}}$ are the generalized position coordinate, generalized velocity, and generalized acceleration, respectively; and $\boldsymbol{\lambda} \in \mathbb{R}^m$ is the Lagrange multipliers in the dynamic equation. The first two parts G^0 and G^f of Eq. (2) are related to the initial and final state of the multibody system, and the third part H is the integral item, which is related to the intermediate process of the system.

Considering the mathematical model of multibody system dynamics in the form of differential algebraic equations, the dynamic equations are expressed as

$$\boldsymbol{M}(\boldsymbol{q},\boldsymbol{b})\ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_q^T(\boldsymbol{q},\boldsymbol{b},t)\boldsymbol{\lambda}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{b},t) = \boldsymbol{Q}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{b},t) \tag{3}$$

$$\boldsymbol{\Phi}(\boldsymbol{q},\boldsymbol{b},t) = 0 \tag{4}$$

where $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is the generalized mass matrix of the system; $\boldsymbol{\Phi} = [\Phi_1, \Phi_2, \ldots, \Phi_m]^T \in \mathbb{R}^m$ is the position constraint; $\boldsymbol{\Phi}_q = \partial\boldsymbol{\Phi}/\partial\boldsymbol{q} \in \mathbb{R}^{m\times n}$ is the Jacobian matrix of the position constraint; and $\boldsymbol{Q} \in \mathbb{R}^n$ is the generalized force vector, which is related to the design variables of the system.

The initial time t^0 and the final time t^f of the system can be given specific values, or can be determined implicitly by

$$\Omega^i(\dot{\boldsymbol{q}}^i, \boldsymbol{q}^i, \boldsymbol{b}, t^i) = 0, \quad i = 0, f \tag{5}$$

where Ω^0 and Ω^f are related to the initial and final time condition of the multibody system, respectively.

The initial state of the system is related to the design variables, which should satisfy the following compatible additional conditions:

$$\boldsymbol{\varphi}^0(\boldsymbol{q}^0, \boldsymbol{b}, t^0) = \boldsymbol{0} \tag{6}$$

$$\overline{\boldsymbol{\varphi}}^0(\dot{\boldsymbol{q}}^0, \boldsymbol{q}^0, \boldsymbol{b}, t^0) = \boldsymbol{0} \tag{7}$$

where $\boldsymbol{\varphi}^0 \in \mathbb{R}^{n-m}$ and $\overline{\boldsymbol{\varphi}}^0 \in \mathbb{R}^{n-m}$ are the initial position and velocity conditions, respectively, which must make the matrices $(\boldsymbol{\Phi}^0_{q^0}, \boldsymbol{\varphi}^0_{q^0})^T$ and $(\boldsymbol{\Phi}^0_{q^0}, \overline{\boldsymbol{\varphi}}^0_{\dot{q}^0})^T$ nonsingular.

In the next section, sensitivity analysis will be carried out based on the above mathematical model of multibody system dynamics.

III. Finite Difference Method for Sensitivity Analysis of Multibody System Dynamics

Sensitivity is the partial derivative of the objective function with respect to design variables. The formula for sensitivity calculation by FDM is expressed as

$$\psi_{b_i} = \frac{\psi(b_i + \Delta b_i) - \psi(b_i)}{\Delta b_i}, \quad i = 1, 2, \ldots, p \tag{8}$$

where b_i is the ith design variable, and Δb_i is the prescribed small increment of the ith design variable.

For the FDM, it only needs to give a perturbation to the design variables, then recalculate the differential–algebraic equations of multibody system to obtain the objective function, and finally calculate the sensitivity by means of difference quotient.

As can be seen from the above formula, the FDM is very simple, and it needs little knowledge of the interior structure of multibody dynamic equations and does not need to derive the relevant analytical formula. But it is necessary to repeatedly solve the differential–algebraic equations of multibody system. When the system is relatively complex and the number of design variables is large, the calculation efficiency is very low. In addition, the FDM is completely approximate, and its calculation accuracy is poor.

IV. Analytical Methods for Sensitivity Analysis of Multibody System Dynamics

The general sensitivity calculation formula of the multibody system can be described by Eq. (9), which is the derivative of Eq. (2) with respect to design variables:

$$\begin{aligned}\Psi_b = {} & G_{\dot{q}^i}(\dot{\boldsymbol{q}}^i_b + \ddot{\boldsymbol{q}}^i t^i_b) + G_{q^i}(\boldsymbol{q}^i_b + \dot{\boldsymbol{q}}^i t^i_b) + G^i_b + G_{t^i} t^i_b \\ & + \int_{t^0}^{t^f} (H_{\ddot{q}}\ddot{\boldsymbol{q}}_b + H_{\dot{q}}\dot{\boldsymbol{q}}_b + H_q \boldsymbol{q}_b + H_\lambda \boldsymbol{\lambda}_b + H_b)\,\mathrm{d}t \\ & + H^f t^f_b - H^0 t^0_b, \quad i = 0, f\end{aligned} \tag{9}$$

where t^0_b and t^f_b are the derivatives of the initial and final times with respect to design variables, respectively. The constraints of the initial and final times of the system are determined by Eq. (5), and taking the design derivative of Eq. (5), t^0_b and t^f_b can be obtained:

$$t^i_b = -(\Omega^i_{\dot{q}^i}/\dot{\Omega}^i)\dot{\boldsymbol{q}}^i_b - (\Omega^i_{q^i}/\dot{\Omega}^i)\boldsymbol{q}^i_b - \Omega^i_b/\dot{\Omega}^i, \quad i = 0, f \tag{10}$$

In this case, only the derivatives of state variables with respect to design variables, that is, the state sensitivity $\ddot{\boldsymbol{q}}_b$, $\dot{\boldsymbol{q}}_b$, $\boldsymbol{q}_b$, $\dot{\boldsymbol{q}}^0_b$, $\dot{\boldsymbol{q}}^f_b$, $\boldsymbol{q}^0_b$, $\boldsymbol{q}^f_b$, and $\boldsymbol{\lambda}_b$ are unknown in the sensitivity calculation formula described by Eq. (9). The direct differentiation method can directly solve the state sensitivity, and the adjoint variable method can be used to eliminate the state sensitivity by introducing the adjoint variables, thereby calculating the sensitivity of the multibody system. As preliminaries, the basic formulas for sensitivity calculation of multibody systems using the direct differentiation method and the adjoint variable method will be given, respectively, in this section.

A. Direct Differentiation Method

For the direct differentiation method, the state sensitivity of the multibody system is directly obtained by solving matrix differential–algebraic equations given by taking the design derivative of the dynamic equations of the multibody system. Then, by substituting the solved state sensitivity into the sensitivity calculation formula described by Eq. (9), the sensitivity of the objective function with respect to design variables can be obtained.

Firstly, the derivatives of the dynamic equations described by Eqs. (3) and (4) with respect to design variables are

$$\begin{aligned}\boldsymbol{M}\ddot{\boldsymbol{q}}_b + \boldsymbol{\Phi}^T_q\boldsymbol{\lambda}_b + (\boldsymbol{M}_q\ddot{\boldsymbol{q}} + \boldsymbol{\Phi}^T_{qq}\boldsymbol{\lambda} - \boldsymbol{Q}_q)\boldsymbol{q}_b \\ - \boldsymbol{Q}_{\dot{q}}\dot{\boldsymbol{q}}_b + \boldsymbol{M}_b\ddot{\boldsymbol{q}} + \boldsymbol{\Phi}^T_{qb}\boldsymbol{\lambda} - \boldsymbol{Q}_b = 0\end{aligned} \tag{11}$$

$$\boldsymbol{\Phi}_q\boldsymbol{q}_b + \boldsymbol{\Phi}_b = \boldsymbol{0} \tag{12}$$

To ensure the accuracy of calculation results, the Baumgarte stabilization method [33] is used to solve the problem. At this time, the constraint equation of the system includes the position constraint, velocity level constraint, and acceleration level constraint, and Eq. (4) is rewritten as

$$\ddot{\boldsymbol{\Phi}} + 2\gamma\dot{\boldsymbol{\Phi}} + \eta^2\boldsymbol{\Phi} = \boldsymbol{0} \tag{13}$$

where $\dot{\boldsymbol{\Phi}}$ and $\ddot{\boldsymbol{\Phi}}$ are the velocity and acceleration level constraint, respectively, and γ and η are customization parameters.

The derivative of Eq. (13) with respect to design variables is

$$\boldsymbol{\Phi}_q\ddot{\boldsymbol{q}}_b + \boldsymbol{\Phi}_{qq}\dot{\boldsymbol{q}} + 2(\boldsymbol{\Phi}_{qt} + \gamma\boldsymbol{\Phi}_q)\dot{\boldsymbol{q}}_b + \overline{\boldsymbol{\Phi}}_q\boldsymbol{q}_b + \overline{\boldsymbol{\Phi}}_b = \boldsymbol{0} \tag{14}$$

where

$$\overline{\boldsymbol{\Phi}} \triangleq \boldsymbol{\Phi}_q\ddot{\boldsymbol{q}} + (\boldsymbol{\Phi}_{qq}\dot{\boldsymbol{q}} + 2\boldsymbol{\Phi}_{qt} + 2\gamma\boldsymbol{\Phi}_q)\dot{\boldsymbol{q}} + \boldsymbol{\Phi}_{tt} + 2\gamma\boldsymbol{\Phi}_t + \eta^2\boldsymbol{\Phi} = \boldsymbol{0} \tag{15}$$

In the above formulas, to distinguish the position constraint $\boldsymbol{\Phi}$, $\overline{\boldsymbol{\Phi}}$ is defined as Baumgarte constraint. At this time, the state sensitivity $\ddot{\boldsymbol{q}}_b$, $\dot{\boldsymbol{q}}_b$, $\boldsymbol{q}_b$, $\dot{\boldsymbol{q}}^0_b$, $\dot{\boldsymbol{q}}^f_b$, $\boldsymbol{q}^0_b$, $\boldsymbol{q}^f_b$, and $\boldsymbol{\lambda}_b$ can be obtained by solving the differential equations described by Eqs. (11), (14), and (15). To solve the above state sensitivity calculation formulas, the initial values of state sensitivity expressed by $\boldsymbol{q}^0_b$ and $\dot{\boldsymbol{q}}^0_b$ must be given first. Thus, the derivatives of Eqs. (6) and (7) with respect to design variables are

$$\boldsymbol{\varphi}_{q^0}^0 \boldsymbol{q}_b^0 + \boldsymbol{\varphi}_b^0 + \dot{\boldsymbol{\varphi}}^0 t_b^0 = \mathbf{0} \tag{16}$$

$$\overline{\boldsymbol{\varphi}}_{\dot{q}^0}^0 \dot{\boldsymbol{q}}_b^0 + \overline{\boldsymbol{\varphi}}_{q^0}^0 \boldsymbol{q}_b^0 + \overline{\boldsymbol{\varphi}}_b^0 + \dot{\overline{\boldsymbol{\varphi}}}^0 t_b^0 = \mathbf{0} \tag{17}$$

Then, by substituting t_b^0 into Eqs. (16) and (17), and combining the derivative equations of the initial position constraint $\boldsymbol{\Phi}^0$ and velocity level constraint $\dot{\boldsymbol{\Phi}}^0$ with respect to design variables, the initial values of the state sensitivity can be obtained by

$$\boldsymbol{\Phi}_{q^0}^0 \boldsymbol{q}_b^0 + \boldsymbol{\Phi}_b^0 = \mathbf{0} \tag{18}$$

$$\boldsymbol{\Phi}_{q^0}^0 \dot{\boldsymbol{q}}_b^0 + \dot{\boldsymbol{\Phi}}_{q^0}^0 \boldsymbol{q}_b^0 + \dot{\boldsymbol{\Phi}}_b^0 = \mathbf{0} \tag{19}$$

$$(\boldsymbol{\varphi}_{q^0}^0 - \dot{\boldsymbol{\varphi}}^0 \Omega_{q^0}^0 / \dot{\Omega}^0) \boldsymbol{q}_b^0 - (\dot{\boldsymbol{\varphi}}^0 \Omega_{\dot{q}^0}^0 / \dot{\Omega}^0) \dot{\boldsymbol{q}}_b^0 + \boldsymbol{\varphi}_b^0 - \dot{\boldsymbol{\varphi}}^0 \Omega_b^0 / \dot{\Omega}^0 = \mathbf{0} \tag{20}$$

$$(\overline{\boldsymbol{\varphi}}_{q^0}^0 - \dot{\overline{\boldsymbol{\varphi}}}^0 \Omega_{q^0}^0 / \dot{\Omega}^0) \boldsymbol{q}_b^0 + (\overline{\boldsymbol{\varphi}}_{\dot{q}^0}^0 - \dot{\overline{\boldsymbol{\varphi}}}^0 \Omega_{\dot{q}^0}^0 / \dot{\Omega}^0) \dot{\boldsymbol{q}}_b^0 + \overline{\boldsymbol{\varphi}}_b^0 - \dot{\overline{\boldsymbol{\varphi}}}^0 \Omega_b^0 / \dot{\Omega}^0 = \mathbf{0} \tag{21}$$

Finally, by substituting t_b^0, t_b^f, and the solved state sensitivity $\ddot{\boldsymbol{q}}_b$, $\dot{\boldsymbol{q}}_b$, $\boldsymbol{q}_b$, $\dot{\boldsymbol{q}}_b^0$, $\dot{\boldsymbol{q}}_b^f$, $\boldsymbol{q}_b^0$, $\boldsymbol{q}_b^f$, and $\boldsymbol{\lambda}_b$ into Eq. (9), the sensitivity about the objective function of the multibody system with respect to design variables can be determined.

B. Adjoint Variable Method

The adjoint variable method is used to solve the sensitivity of multibody systems. Concretely, the method is to introduce a series of adjoint variables, then transpose and multiply the adjoint variables with the corresponding dynamic equations, and subtract these equations from the sensitivity calculation formula. After that, by eliminating the state sensitivity in the sensitivity calculation formula processed above, a series of adjoint variable equations can be obtained. The corresponding adjoint variables can be acquired by solving these adjoint variable equations. Thus, the sensitivity of the objective function with respect to design variables can be obtained.

Firstly, integrating the terms in the integrals of Eq. (9) involving $\ddot{\boldsymbol{q}}_b$ and $\dot{\boldsymbol{q}}_b$ by parts gives

$$\begin{aligned}\Psi_b = {} & (G_{\dot{q}^i}^i - H_{\ddot{q}^i}^i)\dot{\boldsymbol{q}}_b^i + (G_{q^i} - H_{\dot{q}^i}^i + dH_{\ddot{q}^i}^i/dt^i)\boldsymbol{q}_b^i + G_b^i \\ & + (G_{\dot{q}^i}\ddot{\boldsymbol{q}}^i + G_{q^i}\dot{\boldsymbol{q}}^i + G_{t^i} - H^i)t_b^i \\ & + \int_{t^0}^{t^f} [H_\lambda \boldsymbol{\lambda}_b + H_b + (H_q - dH_{\dot{q}}/dt + d^2 H_{\ddot{q}}/dt^2)\boldsymbol{q}_b]\,\mathrm{d}t, \; i = 0, f\end{aligned} \tag{22}$$

Secondly, the adjoint variables $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\boldsymbol{\nu} \in \mathbb{R}^n$ are introduced, which are transposed and multiplied with Eqs. (11) and (12), respectively, and integrating them from t^0 to t^f yields

$$\begin{aligned}\int_{t^0}^{t^f} & \boldsymbol{\mu}^T[\boldsymbol{M}\ddot{\boldsymbol{q}}_b + \boldsymbol{\Phi}_q^T \boldsymbol{\lambda}_b + (\boldsymbol{M}_q \ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_{qq}^T \boldsymbol{\lambda} - \boldsymbol{Q}_q)\boldsymbol{q}_b - \boldsymbol{Q}_{\dot{q}}\dot{\boldsymbol{q}}_b + \boldsymbol{M}_b \ddot{\boldsymbol{q}} \\ & + \boldsymbol{\Phi}_{qb}^T \boldsymbol{\lambda} - \boldsymbol{Q}_b]\,\mathrm{d}t = \mathbf{0}\end{aligned} \tag{23}$$

$$\int_{t^0}^{t^f} \boldsymbol{\nu}^T(\boldsymbol{\Phi}_q \boldsymbol{q}_b + \boldsymbol{\Phi}_b)\,\mathrm{d}t = 0 \tag{24}$$

Integrating the terms in the integrals of Eq. (23) involving $\ddot{\boldsymbol{q}}_b$ and $\dot{\boldsymbol{q}}_b$ by parts gives

$$\begin{aligned}\int_{t^0}^{t^f} & \{[\ddot{\boldsymbol{\mu}}^T \boldsymbol{M} + \dot{\boldsymbol{\mu}}^T(2\dot{\boldsymbol{M}} + \boldsymbol{Q}_{\dot{q}}) + \boldsymbol{\mu}^T(\ddot{\boldsymbol{M}} + \dot{\boldsymbol{Q}}_{\dot{q}} + \boldsymbol{M}_q \ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_{qq}^T \boldsymbol{\lambda} - \boldsymbol{Q}_q)]\boldsymbol{q}_b \\ & + \boldsymbol{\mu}^T \boldsymbol{\Phi}_q^T \boldsymbol{\lambda}_b + \boldsymbol{\mu}^T(\boldsymbol{M}_b \ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_{qb}^T \boldsymbol{\lambda} - \boldsymbol{Q}_b)\}\,\mathrm{d}t \\ & - [\dot{\boldsymbol{\mu}}^T \boldsymbol{M} + \boldsymbol{\mu}^T(\dot{\boldsymbol{M}} + \boldsymbol{Q}_{\dot{q}})]\boldsymbol{q}_b|_{t^0}^{t^f} + \boldsymbol{\mu}^T \boldsymbol{M}\dot{\boldsymbol{q}}_b|_{t^0}^{t^f} = \mathbf{0}\end{aligned} \tag{25}$$

Introducing adjoint variables $\boldsymbol{\sigma}^i$, $\boldsymbol{\rho}^i \in \mathbb{R}^m$, $\xi^i \in \mathbb{R}^1 (i = 0, f)$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \mathbb{R}^{n-m}$, transposing them separately, and multiplying the derivative equations of the position constraint, velocity level constraint and initial conditions with respect to design variables, yields

$$(\boldsymbol{\sigma}^i)^T \boldsymbol{\Phi}_{q^i}^i \boldsymbol{q}_b^i + (\boldsymbol{\sigma}^i)^T \boldsymbol{\Phi}_b^i = \mathbf{0}, \qquad i = 0, f \tag{26}$$

$$(\boldsymbol{\rho}^i)^T \boldsymbol{\Phi}_{q^i}^i \dot{\boldsymbol{q}}_b^i + (\boldsymbol{\rho}^i)^T \dot{\boldsymbol{\Phi}}_{q^i}^i \boldsymbol{q}_b^i + (\boldsymbol{\rho}^i)^T \dot{\boldsymbol{\Phi}}_b^i = \mathbf{0}, \qquad i = 0, f \tag{27}$$

$$\xi^i \Omega_{\dot{q}^i}^i \dot{\boldsymbol{q}}_b^i + \xi^i \Omega_{q^i}^i \boldsymbol{q}_b^i + \xi^i \Omega_b^i + \xi^i \dot{\Omega}^i t_b^i = \mathbf{0}, \qquad i = 0, f \tag{28}$$

$$\boldsymbol{\alpha}^T \boldsymbol{\varphi}_{q^0}^0 \boldsymbol{q}_b^0 + \boldsymbol{\alpha}^T \boldsymbol{\varphi}_b^0 + \boldsymbol{\alpha}^T \dot{\boldsymbol{\varphi}}^0 t_b^0 = \mathbf{0} \tag{29}$$

$$\boldsymbol{\beta}^T \overline{\boldsymbol{\varphi}}_{\dot{q}^0}^0 \dot{\boldsymbol{q}}_b^0 + \boldsymbol{\beta}^T \overline{\boldsymbol{\varphi}}_{q^0}^0 \boldsymbol{q}_b^0 + \boldsymbol{\beta}^T \overline{\boldsymbol{\varphi}}_b^0 + \boldsymbol{\beta}^T \dot{\overline{\boldsymbol{\varphi}}}^0 t_b^0 = \mathbf{0} \tag{30}$$

Then, subtracting the left parts of Eqs. (24–30) from the sensitivity calculation formula described by Eq. (22), and taking the coefficients of the state sensitivity $\ddot{\boldsymbol{q}}_b$, $\dot{\boldsymbol{q}}_b$, $\boldsymbol{q}_b$, $\dot{\boldsymbol{q}}_b^0$, $\dot{\boldsymbol{q}}_b^f$, $\boldsymbol{q}_b^0$, $\boldsymbol{q}_b^f$, and $\boldsymbol{\lambda}_b$ to be zeros, a series of adjoint variable equations can be obtained, given as

$$\begin{aligned}& \boldsymbol{M}\ddot{\boldsymbol{\mu}} + (2\dot{\boldsymbol{M}} + \boldsymbol{Q}_{\dot{q}})^T \dot{\boldsymbol{\mu}} + (\boldsymbol{M}_q \ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_{qq}^T \boldsymbol{\lambda} - \boldsymbol{Q}_q + \ddot{\boldsymbol{M}} + \dot{\boldsymbol{Q}}_{\dot{q}})^T \boldsymbol{\mu} \\ & \quad + \boldsymbol{\Phi}_q^T \boldsymbol{v} - (H_q + \dot{H}_{\dot{q}} + \ddot{H}_{\ddot{q}})^T = \mathbf{0}\end{aligned} \tag{31}$$

$$\boldsymbol{\Phi}_q \boldsymbol{\mu} - H_\lambda^T = \mathbf{0} \tag{32}$$

$$\boldsymbol{M}^0 \boldsymbol{\mu}^0 - (\boldsymbol{\Phi}_{q^0}^0)^T \boldsymbol{\rho}^0 - \overline{\boldsymbol{\varphi}}_{\dot{q}^0}^0 \boldsymbol{\beta} - (\Omega_{\dot{q}^0}^0)^T \xi^0 + (G_{\dot{q}^0}^0 - H_{\ddot{q}^0}^0)^T = \mathbf{0} \tag{33}$$

$$\boldsymbol{M}^f \boldsymbol{\mu}^f + (\boldsymbol{\Phi}_{q^f}^f)^T \boldsymbol{\rho}^f + (\Omega_{\dot{q}^f}^f)^T \xi^f - (G_{\dot{q}^f}^f + H_{\ddot{q}^f}^f)^T = \mathbf{0} \tag{34}$$

$$\begin{aligned}& \boldsymbol{M}^0 \dot{\boldsymbol{\mu}}^0 + (\dot{\boldsymbol{M}}^0 + \boldsymbol{Q}_{\dot{q}^0}^0)^T \boldsymbol{\mu}^0 + (\boldsymbol{\Phi}_{q^0}^0)^T \boldsymbol{\sigma}^0 + (\dot{\boldsymbol{\Phi}}_{q^0}^0)^T \boldsymbol{\rho}^0 + (\Omega_{q^0}^0)^T \xi^0 \\ & \quad + (\boldsymbol{\varphi}_{q^0}^0)^T \boldsymbol{\alpha} + (\overline{\boldsymbol{\varphi}}_{q^0}^0)^T \boldsymbol{\beta} - (G_{q^0}^0 - H_{\dot{q}^0}^0 + \dot{H}_{\ddot{q}^0}^0)^T = \mathbf{0}\end{aligned} \tag{35}$$

$$\begin{aligned}& \boldsymbol{M}^f \dot{\boldsymbol{\mu}}^f + (\dot{\boldsymbol{M}}^f + \boldsymbol{Q}_{\dot{q}^f}^f)^T \boldsymbol{\mu}^f - (\boldsymbol{\Phi}_{q^f}^f)^T \boldsymbol{\sigma}^f - (\dot{\boldsymbol{\Phi}}_{q^f}^f)^T \boldsymbol{\rho}^f - \Omega_{q^f}^{fT} \xi^f \\ & \quad + (G_{q^f}^f + H_{\dot{q}^f}^f - \ddot{H}_{\ddot{q}^f}^f)^T = \mathbf{0}\end{aligned} \tag{36}$$

$$\dot{\Omega}^0 \xi^0 + (\dot{\boldsymbol{\varphi}}^0)^T \boldsymbol{\alpha} + (\dot{\overline{\boldsymbol{\varphi}}}^0)^T \boldsymbol{\beta} - (\dot{G}^0 - H^0) = 0 \tag{37}$$

$$\dot{\Omega}^f \xi^f - (\dot{G}^f + H^f) = 0 \tag{38}$$

Furthermore, the sensitivity calculation formula of the system described by Eq. (22) is rewritten as Eq. (39). In this case, the sensitivity calculation formula is independent of the state sensitivity, and it is related to the adjoint variables $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, $\boldsymbol{\sigma}^0$, $\boldsymbol{\rho}^0$, ξ^0, $\boldsymbol{\sigma}^f$, $\boldsymbol{\rho}^f$, ξ^f, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, written as

$$\begin{aligned}\Psi_b = {} & G_b^i - (\boldsymbol{\sigma}^i)^T \boldsymbol{\Phi}_b^i - (\boldsymbol{\rho}^i)^T \dot{\boldsymbol{\Phi}}_b^i - \xi^i \Omega_b^i - \boldsymbol{\alpha}^T \boldsymbol{\varphi}_b^0 - \boldsymbol{\beta}^T \overline{\boldsymbol{\varphi}}_b^0 \\ & + \int_{t^0}^{t^f} [H_b - \boldsymbol{\mu}^T(\boldsymbol{M}_b \ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_{qb}^T \boldsymbol{\lambda} - \boldsymbol{Q}_b) - \boldsymbol{\nu}^T \boldsymbol{\Phi}_b]\,\mathrm{d}t, \qquad i = 0, f\end{aligned} \tag{39}$$

Finally, the adjoint variables can be obtained by solving a series of adjoint variable equations expressed by Eqs. (31–38), and substituting them into Eq. (39). And then the sensitivity about the objective function of the multibody system with respect to the design variables can be calculated.

The AM can ensure the accuracy of sensitivity calculation. However, from the above two AMs, it can be seen that the analytical formulas for multibody system sensitivity calculation are complex and lengthy, and strongly depend on the type of design variables. It is necessary to know the detailed information of the multibody system dynamics equation. Moreover, for some particularly complex multibody system dynamics problems, it may be very difficult to obtain the derivative of the dynamics equation with respect to design variables by AM.

V. Semi-Analytical Method for Sensitivity Analysis of Multibody System Dynamics

In the field of structural optimization, considering the complexity of sensitivity calculation by AM, the semi-analytical sensitivity calculation method has been put forward for the static analysis problems for a long time ago [34,35]. At present, this method has been widely used in the field of structural optimization [36,37]. However, for the sensitivity analysis in the field of multibody system optimization, the AM is mostly used at present, which involves the derivation of a large number of complex formulas. Based on the semi-analytical sensitivity analysis method for structural optimization, the semi-analytical method of multibody system dynamics is proposed in this paper.

The semi-analytic sensitivity analysis method combines the simplicity of the FDM with the accuracy of the AM. It replaces the derivative with respect to the design variables in the analytical formulas by using finite differences. Among them, the LSAM uses finite difference at the element level, and the GSAM uses finite difference at the system level.

It should be noted that, for multibody system dynamics, the sensitivity calculation formula mainly includes two parts: the derivative with respect to the state variables and the derivative with respect to the design variables. Among them, the former is very complicated. On the one hand, the differential equations for sensitivity calculation are solved by the numerical integration method. If the derivative of state variables is calculated by the semi-analytical method, it is necessary to use finite difference for all state variables at each time step of the numerical integration. It not only greatly increases the computation load, but also affects the accuracy of calculation results. On the other hand, the number of state variables is related to the modeling method of the multibody system, and the magnitude of the perturbation is difficult to determine. Therefore, the semi-analytical sensitivity analysis method in this paper is aimed at the latter, that is, the derivative of the design variables.

In this section, the sensitivity calculation formulas of the LSAM and the GSAM are given, including the direct differentiation method and the adjoint variable method.

A. Semi-Analytical Direct Differentiation Method

In this subsection, based on the direct differentiation method, the formulas of the GSAM and the LSAM will be given, respectively.

Specifically, the formulas of the GSAM derived by the direct differentiation method will be given below.

Firstly, taking the system level into consideration, the finite difference is used to replace the derivative with respect to design variables in Eqs. (11) and (14) of the direct differentiation method. Then the differential equations are rewritten as

$$\boldsymbol{M}\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \boldsymbol{\Phi}_{\boldsymbol{q}}^T\boldsymbol{\lambda}_{\boldsymbol{b}} + (\boldsymbol{M}_{\boldsymbol{q}}\ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_{\boldsymbol{qq}}^T\boldsymbol{\lambda} - \boldsymbol{Q}_{\boldsymbol{q}})\boldsymbol{q}_{\boldsymbol{b}} - \boldsymbol{Q}_{\dot{\boldsymbol{q}}}\dot{\boldsymbol{q}}_{\boldsymbol{b}} = \bar{\boldsymbol{f}}_1 \tag{40}$$

$$\boldsymbol{\Phi}_{\boldsymbol{q}}\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \boldsymbol{\Phi}_{\boldsymbol{qq}}\dot{\boldsymbol{q}} + 2(\boldsymbol{\Phi}_{\boldsymbol{q}t} + \gamma\boldsymbol{\Phi}_{\boldsymbol{q}})\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \overline{\boldsymbol{\Phi}}_{\boldsymbol{q}}\boldsymbol{q}_{\boldsymbol{b}} = \bar{\boldsymbol{f}}_2 \tag{41}$$

where $\bar{\boldsymbol{f}}_1$ and $\bar{\boldsymbol{f}}_2$ on the right side of the equations are the parts involving finite difference, which are expressed as

$$\bar{\boldsymbol{f}}_1 \approx -\left(\frac{\boldsymbol{M}(\boldsymbol{b}+\Delta\boldsymbol{b}) - \boldsymbol{M}(\boldsymbol{b})}{\Delta\boldsymbol{b}}\ddot{\boldsymbol{q}} + \frac{\boldsymbol{\Phi}_{\boldsymbol{q}}^T(\boldsymbol{b}+\Delta\boldsymbol{b}) - \boldsymbol{\Phi}_{\boldsymbol{q}}^T(\boldsymbol{b})}{\Delta\boldsymbol{b}}\boldsymbol{\lambda} - \frac{\boldsymbol{Q}(\boldsymbol{b}+\Delta\boldsymbol{b}) - \boldsymbol{Q}(\boldsymbol{b})}{\Delta\boldsymbol{b}}\right) \tag{42}$$

$$\bar{\boldsymbol{f}}_2 \approx -\frac{\overline{\boldsymbol{\Phi}}(\boldsymbol{b}+\Delta\boldsymbol{b}) - \overline{\boldsymbol{\Phi}}(\boldsymbol{b})}{\Delta\boldsymbol{b}} \tag{43}$$

In the above formulas, $\Delta\boldsymbol{b}$ is the prescribed small increment of design variables; $\boldsymbol{M}$, $\boldsymbol{\Phi}_{\boldsymbol{q}}$, $\boldsymbol{Q}$, $\overline{\boldsymbol{\Phi}}$, $\ddot{\boldsymbol{q}}$, and $\boldsymbol{\lambda}$ are all at the system level.

Similarly, the system-level finite difference is used to replace the derivative with respect to design variables in formulas for solving the initial values of state sensitivity, then Eqs. (17–20) can be rewritten as

$$\boldsymbol{\Phi}_{\boldsymbol{q}^0}^0\boldsymbol{q}_{\boldsymbol{b}}^0 = \bar{\boldsymbol{f}}_3 \tag{44}$$

$$\boldsymbol{\Phi}_{\boldsymbol{q}^0}^0\dot{\boldsymbol{q}}_{\boldsymbol{b}}^0 + \dot{\boldsymbol{\Phi}}_{\boldsymbol{q}^0}^0\boldsymbol{q}_{\boldsymbol{b}}^0 = \bar{\boldsymbol{f}}_4 \tag{45}$$

$$(\boldsymbol{\varphi}_{\boldsymbol{q}^0}^0 - \dot{\boldsymbol{\varphi}}^0\Omega_{\boldsymbol{q}^0}^0/\dot{\Omega}^0)\boldsymbol{q}_{\boldsymbol{b}}^0 - (\dot{\boldsymbol{\varphi}}^0\Omega_{\dot{\boldsymbol{q}}^0}^0/\dot{\Omega}^0)\dot{\boldsymbol{q}}_{\boldsymbol{b}}^0 = \bar{\boldsymbol{f}}_5 \tag{46}$$

$$(\overline{\boldsymbol{\varphi}}_{\boldsymbol{q}^0}^0 - \dot{\overline{\boldsymbol{\varphi}}}^0\Omega_{\boldsymbol{q}^0}^0/\dot{\Omega}^0)\boldsymbol{q}_{\boldsymbol{b}}^0 + (\overline{\boldsymbol{\varphi}}_{\dot{\boldsymbol{q}}^0}^0 - \dot{\overline{\boldsymbol{\varphi}}}^0\Omega_{\dot{\boldsymbol{q}}^0}^0/\dot{\Omega}^0)\dot{\boldsymbol{q}}_{\boldsymbol{b}}^0 = \bar{\boldsymbol{f}}_6 \tag{47}$$

where $\bar{\boldsymbol{f}}_3, \overline{\boldsymbol{f}}_4, \overline{\boldsymbol{f}}_5$, and $\bar{\boldsymbol{f}}_6$ on the right side of the equations are the parts involving finite difference, which are expressed as

$$\bar{\boldsymbol{f}}_3 \approx -\frac{\boldsymbol{\Phi}^0(\boldsymbol{b}+\Delta\boldsymbol{b}) - \boldsymbol{\Phi}^0(\boldsymbol{b})}{\Delta\boldsymbol{b}} \tag{48}$$

$$\bar{\boldsymbol{f}}_4 \approx -\frac{\dot{\boldsymbol{\Phi}}^0(\boldsymbol{b}+\Delta\boldsymbol{b}) - \dot{\boldsymbol{\Phi}}^0(\boldsymbol{b})}{\Delta\boldsymbol{b}} \tag{49}$$

$$\bar{\boldsymbol{f}}_5 \approx -\left(\frac{\boldsymbol{\varphi}^0(\boldsymbol{b}+\Delta\boldsymbol{b}) - \boldsymbol{\varphi}^0(\boldsymbol{b})}{\Delta\boldsymbol{b}} - \frac{\Omega^0(\boldsymbol{b}+\Delta\boldsymbol{b}) - \Omega^0(\boldsymbol{b})}{\Delta\boldsymbol{b}}\dot{\boldsymbol{\varphi}}^0/\dot{\Omega}^0\right) \tag{50}$$

$$\bar{\boldsymbol{f}}_6 \approx -\left(\frac{\overline{\boldsymbol{\varphi}}^0(\boldsymbol{b}+\Delta\boldsymbol{b}) - \overline{\boldsymbol{\varphi}}^0(\boldsymbol{b})}{\Delta\boldsymbol{b}} - \frac{\Omega^0(\boldsymbol{b}+\Delta\boldsymbol{b}) - \Omega^0(\boldsymbol{b})}{\Delta\boldsymbol{b}}\dot{\overline{\boldsymbol{\varphi}}}^0/\dot{\Omega}^0\right) \tag{51}$$

In the above formulas, $\boldsymbol{\Phi}^0$, $\dot{\boldsymbol{\Phi}}^0$, $\boldsymbol{\varphi}^0$, and $\overline{\boldsymbol{\varphi}}^0$ are also at the system level.

The calculation formula for sensitivity analysis by using the direct differentiation method is described as Eq. (9); in the same way, the finite difference is used in the relevant derivative terms, then Eqs. (9) and (10) can be rewritten as

$$\begin{aligned}\Psi_{\boldsymbol{b}} &= G_{\dot{\boldsymbol{q}}^i}(\dot{\boldsymbol{q}}_{\boldsymbol{b}}^i + \ddot{\boldsymbol{q}}^i t_{\boldsymbol{b}}^i) + G_{\boldsymbol{q}^i}(\boldsymbol{q}_{\boldsymbol{b}}^i + \dot{\boldsymbol{q}}^i t_{\boldsymbol{b}}^i) + G_{t^i}t_{\boldsymbol{b}}^i - \bar{\boldsymbol{f}}_7^i \\ &\quad + \int_{t^0}^{t^f}(H_{\ddot{\boldsymbol{q}}}\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + H_{\dot{\boldsymbol{q}}}\dot{\boldsymbol{q}}_{\boldsymbol{b}} + H_{\boldsymbol{q}}\boldsymbol{q}_{\boldsymbol{b}} + H_{\boldsymbol{\lambda}}\boldsymbol{\lambda}_{\boldsymbol{b}} - \bar{\boldsymbol{f}}_8)\,\mathrm{d}t \\ &\quad + H^f t_{\boldsymbol{b}}^f - H^0 t_{\boldsymbol{b}}^0, \quad i = 0, f\end{aligned} \tag{52}$$

$$t_{\boldsymbol{b}}^i = -(\Omega_{\dot{\boldsymbol{q}}^i}^i/\dot{\Omega}^i)\dot{\boldsymbol{q}}_{\boldsymbol{b}}^i - (\Omega_{\boldsymbol{q}^i}^i/\dot{\Omega}^i)\boldsymbol{q}_{\boldsymbol{b}}^i + \bar{\boldsymbol{f}}_9^i, \quad i = 0, f \tag{53}$$

where $\bar{\boldsymbol{f}}_7^0$, $\bar{\boldsymbol{f}}_7^f$, and $\bar{\boldsymbol{f}}_8$ in Eq. (52) and $\bar{\boldsymbol{f}}_9^0$ and $\bar{\boldsymbol{f}}_9^f$ in Eq. (53) are the parts involving finite difference, which are expressed as

$$\bar{\boldsymbol{f}}_7^i \approx -\frac{G^i(\boldsymbol{b}+\Delta\boldsymbol{b}) - G^i(\boldsymbol{b})}{\Delta\boldsymbol{b}}, \qquad i = 0, f \tag{54}$$

$$\bar{\boldsymbol{f}}_8 \approx -\frac{H(\boldsymbol{b}+\Delta\boldsymbol{b}) - H(\boldsymbol{b})}{\Delta\boldsymbol{b}} \tag{55}$$

$$\bar{\boldsymbol{f}}_9^i \approx -\frac{\Omega^i(\boldsymbol{b}+\Delta\boldsymbol{b}) - \Omega^i(\boldsymbol{b})}{\Delta\boldsymbol{b}}/\dot{\Omega}^i, \quad i = 0, f \tag{56}$$

By using system-level finite difference in the analytical formulas of the direct differentiation method, the sensitivity calculation formulas of the GSAM based on the direct differentiation method can be obtained.

Considering the element level, the formulas of the LSAM derived by the direct differentiation method will be given below.

Specifically, the finite difference terms on the right side of Eqs. (40) and (41), that is, $\bar{\boldsymbol{f}}_1$ and $\bar{\boldsymbol{f}}_2$, are written as $\bar{\boldsymbol{f}}_{e1}$ and $\bar{\boldsymbol{f}}_{e2}$:

$$\bar{f}_{e1} \approx -\left(\sum_e \frac{M_e(b+\Delta b) - M_e(b)}{\Delta b}\ddot{q}_e + \sum_e \frac{\Phi_{qe}^T(b+\Delta b) - \Phi_{qe}^T(b)}{\Delta b}\lambda_e - \sum_e \frac{Q_e(b+\Delta b) - Q_e(b)}{\Delta b}\right) \tag{57}$$

$$\bar{f}_{e2} \approx -\sum_e \frac{\overline{\Phi}_e(b+\Delta b) - \overline{\Phi}_e(b)}{\Delta b} \tag{58}$$

where M_e is the element-level generalized mass matrix; Φ_{qe} is the Jacobian matrix of the element-level position constraint; Q_e is the element-level generalized force vector; $\overline{\Phi}_e$ is the element-level Baumgarte constraint; and $\ddot{q}_e$ and λ_e are the element-level acceleration vector and Lagrange multiplier, respectively.

Similarly, from the element point of view, the finite difference terms $\bar{f}_3$, $\bar{f}_4$, $\bar{f}_5$, and $\bar{f}_6$ on the right side of Eqs. (44–47) are written as $\bar{f}_{e3}$, $\bar{f}_{e4}$, $\bar{f}_{e5}$, and $\bar{f}_{e6}$:

$$\bar{f}_{e3} \approx -\sum_e \frac{\Phi_e^0(b+\Delta b) - \Phi_e^0(b)}{\Delta b} \tag{59}$$

$$\bar{f}_{e4} \approx -\sum_e \frac{\dot{\Phi}_e^0(b+\Delta b) - \dot{\Phi}_e^0(b)}{\Delta b} \tag{60}$$

$$\bar{f}_{e5} \approx -\left(\sum_e \frac{\varphi_e^0(b+\Delta b) - \varphi_e^0(b)}{\Delta b} - \sum_e \frac{\Omega^0(b+\Delta b) - \Omega^0(b)}{\Delta b}\dot{\varphi}_e^0/\dot{\Omega}^0\right) \tag{61}$$

$$\bar{f}_{e6} \approx -\left(\sum_e \frac{\overline{\varphi}_e^0(b+\Delta b) - \overline{\varphi}_e^0(b)}{\Delta b} - \sum_e \frac{\Omega^0(b+\Delta b) - \Omega^0(b)}{\Delta b}\dot{\overline{\varphi}}_e^0/\dot{\Omega}^0\right) \tag{62}$$

where Φ_e^0 and $\dot{\Phi}_e^0$ are initial position and velocity level constraint at the element level, respectively; φ_e^0 and $\overline{\varphi}_e^0$ are initial compatibility additional conditions at the element level.

In the analytical formulas of the direct differentiation method mentioned above, the element-level finite difference is applied to the derivative with respect to design variables. Then the sensitivity calculation formulas of the LSAM based on the direct differential method can be obtained.

B. Semi-Analytical Adjoint Variable Method

In this subsection, based on the adjoint variable method, the formulas of the GSAM and the LSAM will be given, respectively.

As for the adjoint variable method, from the analytical expression deduced in Sec. IV, it can be seen that the derivative terms about the dynamic equation and objective function with respect to design variables only exist in the sensitivity calculation formula described by Eq. (39). The GSAM is used to calculate sensitivity. Considering from the system point of view, the finite difference is used to replace the derivative with respect to design variables, and then Eq. (39) is rewritten as

$$\Psi_b = \bar{f}'^i_3 + \bar{f}'^i_4 + \bar{f}'_5 + \bar{f}'_6 + \bar{f}'^i_7 - \bar{f}'^i_8 + \int_{t^0}^{t^f} (\bar{f}'_1 + \bar{f}'_2 - \bar{f}'_9)\,\mathrm{d}t, \quad i = 0, f \tag{63}$$

where $\bar{f}'^i_j(i = 0, f, j = 3, 4, 7, 8)$ and $\bar{f}'_w(w = 1, 2, 5, 6, 9)$ are the terms involving finite difference, which are, respectively, expressed as

$$\bar{f}'_1 \approx -\mu^T\left(\frac{M(b+\Delta b) - M(b)}{\Delta b}\ddot{q} + \frac{\Phi_q^T(b+\Delta b) - \Phi_q^T(b)}{\Delta b}\lambda - \frac{Q(b+\Delta b) - Q(b)}{\Delta b}\right) \tag{64}$$

$$\bar{f}'_2 \approx -\nu^T \frac{\Phi(b+\Delta b) - \Phi(b)}{\Delta b} \tag{65}$$

$$\bar{f}'^i_3 \approx -(\sigma^i)^T \frac{\Phi^i(b+\Delta b) - \Phi^i(b)}{\Delta b}, \qquad i = 0, f \tag{66}$$

$$\bar{f}'^i_4 \approx -(\rho^i)^T \frac{\dot{\Phi}^i(b+\Delta b) - \dot{\Phi}^i(b)}{\Delta b}, \qquad i = 0, f \tag{67}$$

$$\bar{f}'_5 \approx -\alpha^T \frac{\varphi^0(b+\Delta b) - \varphi^0(b)}{\Delta b} \tag{68}$$

$$\bar{f}'_6 \approx -\beta^T \frac{\overline{\varphi}^0(b+\Delta b) - \overline{\varphi}^0(b)}{\Delta b} \tag{69}$$

$$\bar{f}'^i_7 \approx -\xi^i \frac{\Omega^i(b+\Delta b) - \Omega^i(b)}{\Delta b}, \quad i = 0, f \tag{70}$$

$$\bar{f}'^i_8 \approx -\frac{G^i(b+\Delta b) - G^i(b)}{\Delta b}, \quad i = 0, f \tag{71}$$

$$\bar{f}'_9 \approx -\frac{H(b+\Delta b) - H(b)}{\Delta b} \tag{72}$$

In the above formulas, M, Φ_q, Q, Φ, Φ^i, $\dot{\Phi}^i(i = 0, f)$, φ^0, $\overline{\varphi}^0$, $\ddot{q}$, and λ are all at the system level; μ, ν, α, β, σ^i, ρ^i, and $\xi^i(i = 0, f)$ are adjoint variables.

By using system-level finite difference in the analytical formulas of the adjoint variable method, the sensitivity calculation formulas of the GSAM based on the adjoint variable method can be obtained.

Similar to the LSAM based on the direct differentiation method mentioned above, for the adjoint variable method, the LSAM is used to express the finite difference terms $\bar{f}'^i_j(i = 0, f, j = 3, 4)$ and $\bar{f}'_w(w = 1, 2, 5, 6)$ in Eq. (63), and then they are written as $\bar{f}'^i_{ej}(i = 0, f, j = 3, 4)$ and $\bar{f}'_{ew}(w = 1, 2, 5, 6)$:

$$\bar{f}'_{e1} \approx -\mu^T\left(\sum_e \frac{M_e(b+\Delta b) - M_e(b)}{\Delta b}\ddot{q}_e + \sum_e \frac{\Phi_{qe}^T(b+\Delta b) - \Phi_{qe}^T(b)}{\Delta b}\lambda_e - \sum_e \frac{Q_e(b+\Delta b) - Q_e(b)}{\Delta b}\right) \tag{73}$$

$$\bar{f}'_{e2} \approx -\nu^T \sum_e \frac{\Phi_e(b+\Delta b) - \Phi_e(b)}{\Delta b} \tag{74}$$

$$\bar{f}'^i_{e3} \approx -(\sigma^i)^T \sum_e \frac{\Phi_e^i(b+\Delta b) - \Phi_e^i(b)}{\Delta b}, \quad i = 0, f \tag{75}$$

$$\bar{f}'^i_{e4} \approx -(\rho^i)^T \sum_e \frac{\dot{\Phi}_e^i(b+\Delta b) - \dot{\Phi}_e^i(b)}{\Delta b}, \quad i = 0, f \tag{76}$$

$$\bar{f}'_{e5} \approx -\alpha^T \sum_e \frac{\varphi_e^0(b+\Delta b) - \varphi_e^0(b)}{\Delta b} \tag{77}$$

$$\bar{f}'_{e6} \approx -\beta^T \sum_e \frac{\overline{\varphi}_e^0(b+\Delta b) - \overline{\varphi}_e^0(b)}{\Delta b} \tag{78}$$

where $\boldsymbol{M}_e$, $\boldsymbol{\Phi}_{qe}$, $\boldsymbol{\Phi}_e$, $\boldsymbol{Q}_e$, $\ddot{\boldsymbol{q}}_e$, $\boldsymbol{\lambda}_e$, $\boldsymbol{\Phi}_e^i$, $\dot{\boldsymbol{\Phi}}_e^i (i = 0, f)$, $\boldsymbol{\varphi}_e^0$, and $\overline{\boldsymbol{\varphi}}_e^0$ are all at the element level.

In the analytical formulas of the adjoint variable method mentioned above, the element-level finite difference is applied to the derivative with respect to design variables. Then the sensitivity calculation formulas of the LSAM based on the adjoint variable method can be obtained.

From the formulas of the GSAM and the LSAM, including the direct differentiation method and the adjoint variable method, it can be seen that compared with the LSAM based on element level, the GSAM based on system level does not need to extract the matrix information about multibody system elements before and after perturbation. Therefore, the accuracy problem of effective digital loss caused by the addition or subtraction of similar values is avoided when the element matrix is integrated into the system matrix after finite difference. Moreover, it simplifies the process of program implementation, reduces the workload of data calculation, and has higher computational efficiency.

Compared with the FDM and the AM, the semi-analytical sensitivity analysis method proposed in this paper not only retains the accuracy and efficiency of the AM, but also can be well combined with the existing computational programs, which simplifies the derivation and coding of analytical formulas. Specifically, it does not need to analytically derive the derivatives $\boldsymbol{M_b}\,\ddot{\boldsymbol{q}}$, $\boldsymbol{\Phi}_{qb}^T\,\boldsymbol{\lambda}$, $\boldsymbol{Q_b}$, $\overline{\boldsymbol{\Phi}}_{\boldsymbol{b}}$, $\boldsymbol{\Phi_b}$, $\dot{\boldsymbol{\Phi}}_{\boldsymbol{b}}$, $\boldsymbol{\varphi_b}$, $\overline{\boldsymbol{\varphi}}_{\boldsymbol{b}}$, $\Omega_{\boldsymbol{b}}$, $G_{\boldsymbol{b}}$, and $H_{\boldsymbol{b}}$, and code the corresponding programs. Instead, it regards the programs of matrices $\boldsymbol{M}\,\ddot{\boldsymbol{q}}$, $\boldsymbol{\Phi}_q^T\,\boldsymbol{\lambda}$, $\boldsymbol{Q}$, $\overline{\boldsymbol{\Phi}}$, $\boldsymbol{\Phi}$, $\dot{\boldsymbol{\Phi}}$, $\boldsymbol{\varphi}$, $\overline{\boldsymbol{\varphi}}$, Ω, G, and H as black boxes, and directly replaces the derivative with respect to design variables by finite difference of these programs. So, it does not need to know the functional relationship between the above related programs and design variables in detail. Therefore, the semi-analytical sensitivity analysis method can handle a variety of types of design variables flexibly. For the multibody system with complex structure and many types of design variables, it overcomes the difficulty of deriving the analytical derivative with respect to design variables and is more versatile.

VI. Numerical Results and Discussion

In this section, two numerical examples for sensitivity analysis of multibody system dynamics based on differential–algebraic equations are given. By comparing the numerical results of various sensitivity analysis methods, on the one hand, the accuracy and numerical stability of the semi-analytical sensitivity analysis method proposed in this paper are verified, including the LSAM and the GSAM; on the other hand, the advantages of this method are illustrated.

A. Planar Ten-Bar Mechanism

Figure 1 shows the multibody dynamics model of a planar 10-bar mechanism. The bars of the system are all homogeneous rigid bars with masses of $m_1, m_2, \ldots, m_{10}$ and lengths of $l_1, l_2, \ldots, l_{10}$, respectively. The absolute coordinate method is used to model. The state variables and the design variables of the system are presumed to be $\boldsymbol{q} = [x_i, y_i, \theta_i]^T$ and $\boldsymbol{b} = [l_i, m_i]^T$, $i = 1, 2, \ldots, 10$. When the system is only influenced by gravity, its generalized force vector is expressed as $\boldsymbol{Q} = [0, -m_1 g, 0, 0, -m_2 g, 0, \ldots, 0, -m_{10} g, 0]^T$. The initial and final times of the system are taken as $t^0 = 0$, $t^f = 1$. The objective function is taken as $\psi = \int_0^1 (x^2 + y^2)\,\mathrm{d}t$, where x and y are the coordinates of the endpoint of the end bar. The purpose is to minimize the offset between the movement trajectory of the end bar and the fixed end by optimizing the design variables. The design variables are given as

$$m_i = \begin{cases} 1, i = 1, 3, \ldots, 9 \\ 2, i = 2, 4, \ldots, 10 \end{cases} \quad l_i = \begin{cases} 1, i = 1, 3, \ldots, 9 \\ \sqrt{3}, i = 2, 4, \ldots, 10 \end{cases}$$

and the initial condition is

$$\theta_i^0 = \begin{cases} \pi/3, i = 1, 3, \ldots, 9 \\ 11\pi/6, i = 2, 4, \ldots, 10 \end{cases}$$

For the direct differentiation method and the adjoint variable method, the AM, the LSAM, and the GSAM are used to calculate the sensitivity of the plane 10-bar mechanism, respectively. The perturbation value δ of the semi-analytical method is also taken as 1E-5. The partial sensitivity calculation results and computational cost are shown in Table 1, where $\Psi_{\boldsymbol{b}}$ is the sensitivity of the objective function with respect to design variables $m_i (i = 6, 7, 8, 9, 10)$.

Comparing the numerical results of various sensitivity analysis methods in Table 1, it can be found that the analytical results of the direct differentiation method and the adjoint variable method are consistent, and the results of the LSAM and the GSAM are the same, which are basically consistent with the results of the AM. As far as computational cost is concerned, the computational cost of the adjoint variable method is obviously shorter than that of the direct differentiation method, and the computational cost of the LSAM and the GSAM is longer than that of the AM. Furthermore, the GSAM based on system level has shorter computational cost than the LSAM based on element level, and the cost difference between the GSAM and the AM is small. As a consequence, it proves the accuracy of the semi-analytical sensitivity analysis method proposed in this paper and the high efficiency of the GSAM.

Taking different perturbation $\delta = $ 1E-1, 1E-2, …, 1E-12, the sensitivity of planar 10-bar mechanism is calculated using the GSAM and the FDM, respectively, and the results are compared with those of the AM. The comparisons of the sensitivity calculation results about the objective function with respect to the lengths and masses of the 1st and 10th bars are shown in Fig. 2.

From Fig. 2, it can be found that the calculation results of the FDM are obviously affected by the perturbation. As far as the length of the bar is concerned, the calculation results of the FDM differ greatly from the analytical results. In terms of mass, when the perturbation δ is greater than 1E-7, the calculation results of the FDM are more consistent with the analytical results; when the perturbation δ is less than 1E-7, the calculation error of the FDM increases correspondingly. Instead, the results of the GSAM are less affected by the perturbation, which are consistent with those of the AM with δ varying from 1E-3 to 1E-10. It is proved that the semi-analytical sensitivity analysis method proposed in this paper has good numerical stability.

Figures 3 and 4 show the calculation results of partial design sensitivity and state sensitivity within 1 s when the perturbation δ is taken as 1E-10. From the time history of the sensitivity about the objective function with respect to the masses of the 1st and 10th bars in Fig. 3, it can be found that the results of the GSAM are consistent with those of the AM, whereas the accuracy of the FDM is relatively poor. Figure 4 shows the time history of the state sensitivity about the

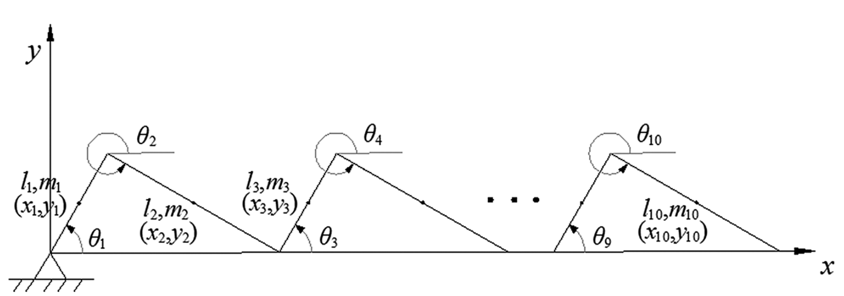

Fig. 1 A planar 10-bar mechanism.

Table 1 Sensitivity and computational cost under various sensitivity analysis methods

Method		Sensitivity Ψ_b	Computational cost (s)
Direct	AM	$[-0.0041\quad -0.0087\quad -0.0014\quad 0.0073\quad -0.0019]$	907.8
	LSAM	$[-0.0042\quad -0.0087\quad -0.0014\quad 0.0071\quad -0.0020]$	1890.0
	GSAM	$[-0.0042\quad -0.0087\quad -0.0014\quad 0.0071\quad -0.0020]$	1070.4
Adjoint	AM	$[-0.0041\quad -0.0087\quad -0.0014\quad 0.0073\quad -0.0019]$	69.6
	LSAM	$[-0.0042\quad -0.0087\quad -0.0014\quad 0.0071\quad -0.0020]$	492.6
	GSAM	$[-0.0042\quad -0.0087\quad -0.0014\quad 0.0071\quad -0.0020]$	124.2

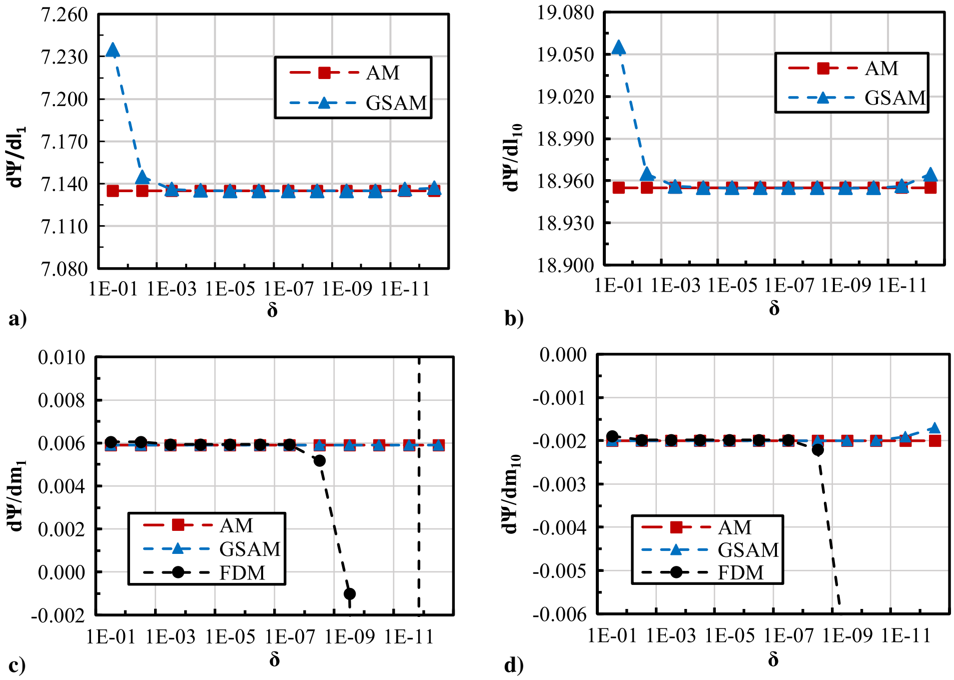

Fig. 2 Comparisons of partial sensitivity calculation results: a) sensitivity of Ψ to l_1, b) sensitivity of Ψ to l_{10}, c) sensitivity of Ψ to m_1, and d) sensitivity of Ψ to m_{10}.

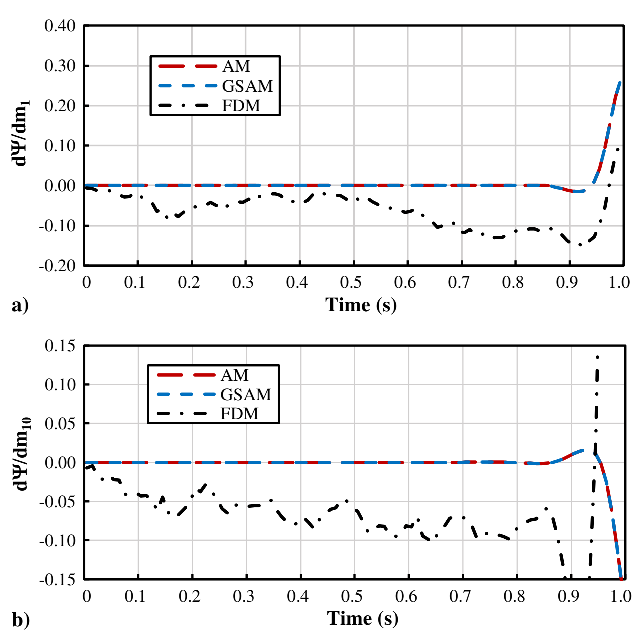

Fig. 3 Time history of partial design sensitivity: a) sensitivity of Ψ to m_1 and b) sensitivity of Ψ to m_{10}.

position coordinates and velocities in the y-axis direction and the relative angle to the x-axis of the 10th bar with respect to its length and mass, from which it can be found that the results of the GSAM are also consistent with those of the AM. By comparing the time history of the design sensitivity and the state sensitivity, it can also be concluded that the semi-analytical sensitivity analysis method has higher accuracy.

B. Space Deployable Antenna Mechanism with Tension Cables

Space deployable structure is the key technology in the field of spaceflight nowadays. With the large-scale of spacecraft and the limitation of effective space of launch vehicle, large-scale structures often need to be contracted into a folding state. When the carrier spacecraft enters orbit, it is deployed according to the instructions and locked into a stable state to start work [38–40]. The folding and working model of the space deployable antenna mechanism with tension cables is shown in Fig. 5, where the blue lines represent the supporting trusses, the green lines represent the cables in the relaxed state (neither tension nor pressure), and the red lines represent the cables in the tension state (tension only, not pressure). The system has a width of 3 m, a height of 2.6 m, and a working length of 60 m. There are 20 basic deployment units, and each deployment unit consists of supporting trusses, hinges, and tension cables. All the cables are ordinary cables, made of silicone rubber, and the trusses are made of steel. The details of material and dimension parameters are listed in Table 2.

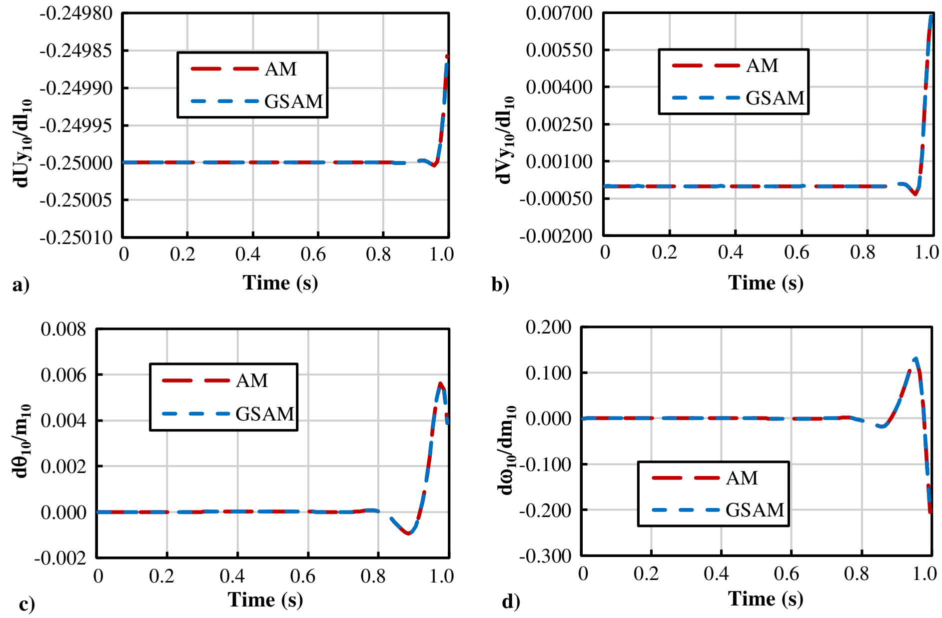

Fig. 4 Time history of partial state sensitivity: a) sensitivity of Uy_{10} to l_{10}, b) sensitivity of Vy_{10} to l_{10}, c) sensitivity of θ_{10} to m_{10}, and d) sensitivity of ω_{10} to m_{10}.

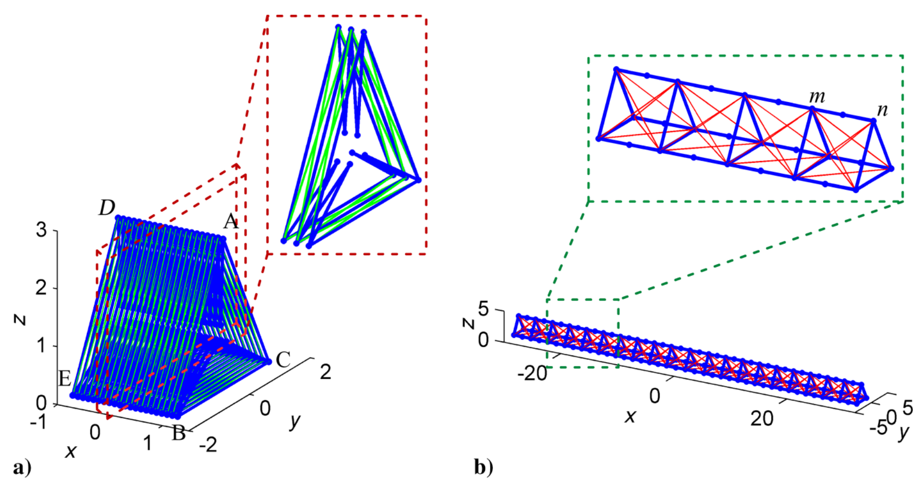

Fig. 5 The space deployable antenna mechanism with tension cables: a) folded state and b) work state.

The deployment process of the space deployable antenna mechanism with tension cables is as follows: under the traction of six initial driving points A, B, C, D, E, and F at both ends of the supporting trusses as shown in Fig. 5, in order to ensure the stability of the deployment process as far as possible, the folded trusses and the slack cables are driven to expand to both sides at the same time, and the driving speed is always in the state of uniform–uniform acceleration–uniform deceleration–uniform. After the basic deployment units of both ends of the system are deployed, the driving points are replaced by six points corresponding to its adjacent basic units to be deployed, and the next pair of basic units is deployed. At the same time, the position, velocity, and acceleration in the y-axis and z-axis directions of the deployed part of the system are constrained. The process is repeated, and the driver points and constraints are replaced in turn until the folded system is fully deployed.

Table 2 Material and dimension parameters of space deployable antenna mechanism

Parameter	Value
Length of triangular trusses	3 m
Length of folded trusses	1.5 m
Elastic modulus of triangular trusses	200 GPa
Elastic modulus of folded trusses	450 GPa
Internal section radius of trusses	1.25 cm
External section radius of trusses	0.75 cm
Density of trusses	4200 kg/m^3
Folding clearance	0.1 m
Length of cables	4.223 m
Elastic modulus of cables	0.78 GPa
Section radius of trusses	5 mm
Density of cables	1000 kg/m^3

The dynamic model of the space deployable antenna mechanism with tension cables is modeled based on the finite element method of position coordinates [41–43]. The state variables of the system are node position coordinates. Considering the storage conditions, folding and working modes of the system, the design variables are taken as truss length L, folding clearance Ls, and initial driving speed v_0. The generalized force is composed of the generalized force vector corresponding to the spring damper actuator describing the trusses and the cables, regardless of the gravity effect of the system.

The deployment distance of each basic deployment unit is defined as the distance of m–n shown in Fig. 5. From the calculation results about the deployment distance of five basic deployment units within 50 s of system deployment in Fig. 6, it can be found that all units have the same distance change, and slight shaking will occur during the deployment process.

Taking the oscillation characteristics of the deployment process into account, the objective function is given by Eq. (79), and the

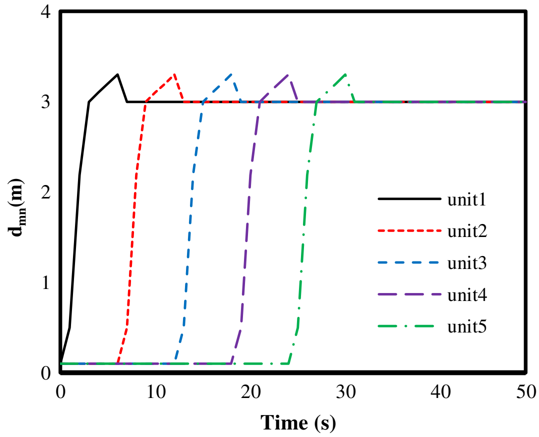

Fig. 6 The distance of m-n of the basic deployment units within 50 s.

purpose is to make the system with minimal oscillation by optimizing the design variables.

$$\psi = \begin{cases} 0, & d_{mn} < L \\ \int_{t^0}^{t^f} (d_{mn} - L)\,\mathrm{d}t, & d_{mn} \geq L \end{cases} \tag{79}$$

The design variables are given as $L = 3$ m, $Ls = 0.1$ m, and $v_0 = 0.1\ \mathrm{m \cdot s^{-1}}$.

It should be noted that, different from the previous example, the deployment motion of the space deployable antenna mechanism with tension cables is a relatively complex multibody system dynamics problem. Because the driving speed and constraint equation of the system change with time, its dynamic modeling is relatively difficult. Considering the engineering practice, the design variables L, Ls, and v_0 are different types, which are the size parameter, position parameter, and motion parameter of the system, respectively. At this time, it is difficult to deduce the derivatives $\boldsymbol{M}_b\ddot{\boldsymbol{q}}$, $\boldsymbol{\Phi}_{qb}^T\boldsymbol{\lambda}$, $\boldsymbol{Q}_b$, $\overline{\boldsymbol{\Phi}}_b$, $\boldsymbol{\Phi}_b$, $\dot{\boldsymbol{\Phi}}_b$, $\boldsymbol{\varphi}_b$, $\overline{\boldsymbol{\varphi}}_b$, Ω_b, G_b, and H_b with respect to design variables by AM, which will greatly increase the workload of deriving analytical formulas and coding computational programs. Considering the complexity of the dynamic equation, the formula derivation is prone to errors. However, by using the semi-analytical sensitivity analysis method proposed in this paper, the programs of terms involving design variables in the dynamic equation can be regarded as black boxes. The analytical derivation of the above-mentioned derivative with respect to design variables is avoided, which facilitates the sensitivity calculation of the multibody system.

Taking different perturbation $\delta = 1\text{E-}1, 1\text{E-}2, \ldots, 1\text{E-}10$, for direct differentiation method and adjoint variable method, respectively, the sensitivity of the system is calculated by using the semi-analytical sensitivity analysis method of multibody system dynamics proposed in this paper, and the calculation results are compared with the FDM.

Figure 7 shows the sensitivity comparisons under different perturbations about the objective function with respect to three design

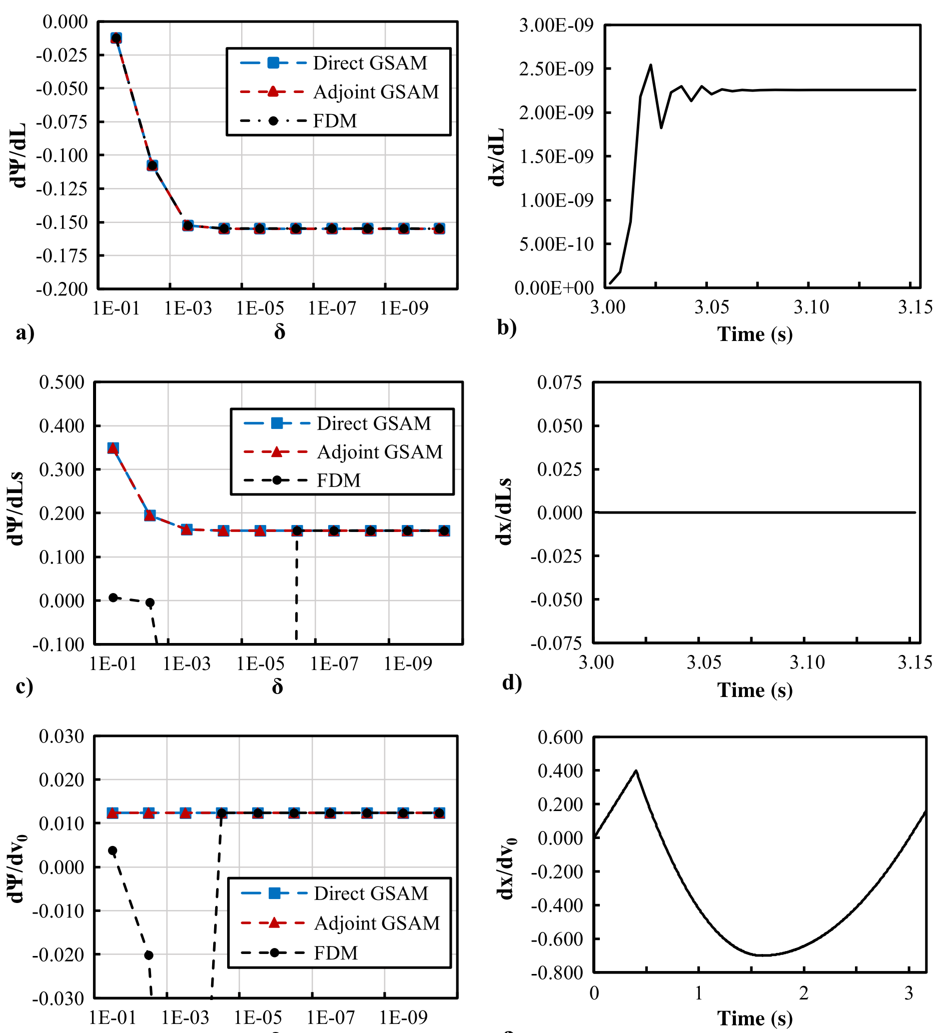

Fig. 7 The design sensitivity comparisons and corresponding time history of the state sensitivity: a) sensitivity of Ψ to L, b) sensitivity of x to L, c) sensitivity of Ψ to Ls, d) sensitivity of x to Ls, e) sensitivity of Ψ to v_0, and f) sensitivity of x to v_0.

variables, truss length, folding clearance, and initial driving speed, and the state sensitivity about the displacement in the x-axis direction of the driving point with respect to design variables. It can be seen from the comparisons that the sensitivity calculation results under different perturbations of the three methods are basically the same for truss length L: when the perturbation δ is less than 1E-4, the sensitivity calculation results tend to be stable and the value is -0.1550; for folding clearance Ls and initial driving speed v_0, the calculation results under different perturbations of the direct differentiation method and the adjoint variable method are basically the same, whereas the results of the FDM are obviously affected by the perturbation. Specifically, for folding clearance Ls, when the perturbation δ is less than 1E-6, the sensitivity calculation results of the three methods are basically consistent and tend to be stable, the value is 0.1600, and the corresponding state sensitivity is always zero; for initial driving speed v_0, when the perturbation δ is less than 1E-4, the sensitivity calculation results of the three methods are basically the same and tend to be stable; the value is 0.0124.

Taking the computational cost into account, when the perturbation δ is 1E-5, the computational cost of three methods for different-scale space deployable antenna mechanism is shown in Table 3, in which the number of basic deployment units of the system is listed on the left side. Through numerical comparisons, it is obvious that the computational cost of the direct differentiation method and the adjoint variable method is shorter, among which the adjoint variable method is the least; compared with the GSAM, the computational cost of the FDM is relatively long, which is 5–8 times as long as that of the GSAM. As a result, it also proves that the GSAM proposed in this paper has higher computational efficiency.

Furthermore, taking the practical engineering problems into consideration, most multibody system dynamics problems are relatively complicated, and there are many factors affecting the system motion, so the types and numbers of design variables will increase accordingly. The semi-analytical sensitivity analysis method proposed in this paper can flexibly handle various types of design variables, greatly reduces the workload of deriving formulas and coding programs, and has stronger generality.

VII. Conclusions

In this paper, based on the problems of multibody system dynamics in the form of differential–algebraic equations, combining the advantages of the FDM and the AM, the semi-analytical sensitivity analysis method is proposed on the basis of the existing direct differentiation method and the adjoint variable method derived by AM. It includes the LSAM based on the element level and the GSAM based on the system level. Compared with the FDM, this method does not need to solve the differential–algebraic equations of multibody system dynamics repeatedly; compared with the AM, it does not need to analytically derive the derivatives about the dynamic equation and the objective function of multibody systems with respect to design variables. Instead, the finite difference is used to replace the derivative with respect to design variables in the analytical formulas. It overcomes the difficulty of deriving the analytical derivative with respect to design variables in some multibody system dynamics equations, and can handle various types of design variables flexibly. In addition, compared with the LSAM, the GSAM proposed in this paper does not need to extract the matrix information of the multibody system elements before and after perturbation. It makes the program implementation simpler and has higher computational efficiency.

Two representative numerical examples for sensitivity calculation of multibody systems based on differential–algebraic equations are given. By using the direct differentiation method and the adjoint variable method, respectively, and taking different perturbation, the numerical results and computational cost of the FDM, the AM, the LSAM, and the GSAM are compared. It proves the accuracy and numerical stability of the semi-analytical sensitivity analysis method for multibody system dynamics proposed in this paper. Moreover, through the more complex dynamic model of the space deployable antenna mechanism with tension cables, it is shown that the semi-analytical sensitivity analysis method can reduce the workload of deriving analytical formulas and coding computational programs. Compared with other sensitivity analysis methods of multibody system dynamics, this method has stronger generality and provides convenience for sensitivity calculation of large-scale complex multibody systems.

Table 3 Computational cost (s) of three methods for different-scale antenna mechanism

Number of basic deployment units	Direct GSAM	Adjoint GSAM	FDM
4	21.0	14.4	101.4
12	62.4	42.6	312.6
20	135.6	81.0	633.6
28	175.8	120.0	864.6
36	222.6	162.6	1123.8
44	274.2	209.4	1383.0
52	322.8	259.8	1645.8

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