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Self-similar oplossingen van de poreuze medium vergelijking

(Engelse titel: Self-similar solutions to the porous medium equation)

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"Self-similar oplossingen van de poreuze medium vergelijking"

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Abstract

The porous medium equation $\frac{\mathrm{d}}{\mathrm{d}t}u = \frac{\mathrm{d}}{\mathrm{d}x}\big(k(u)\frac{\mathrm{d}}{\mathrm{d}x}u\big)$ is a non-linear degenerate parabolic partial differential equation. Consequently, existence and uniqueness of its solutions is not immediately evident. This bachelor thesis presents a detailed discussion of Atkinson's and Peletier's 1971 article "Similarity profiles of flows through porous media" [1] on existence and uniqueness of self-similar solutions to the porous medium equation. First, in chapter 2 the general version of the porous medium equation along with some applications will be discussed. Then, in chapter 3 the proofs and statements of the Picard-Lindelöf theorem, Peano's existence theorem and Gronwall's inequality will be presented. These standard theorems concern differential equations and will be used in the next chapter. Finally, in chapter 4 Atkinson's and Peletier's article will be worked out in detail.

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Chapter 1

Introduction

The goal of this bachelor thesis is to present a more detailed discussion of Atkinson's and Peletier's 1971 article "Similarity profiles of flows through porous media" [1] on existence and uniqueness of self-similar solutions to the porous medium equation

$$\frac{\partial}{\partial t}u = \frac{\partial}{\partial x}\left(k(u)\frac{\partial}{\partial x}u\right),$$

where $k(u)$ is the diffusion coefficient and u denotes the density of a gas in the porous medium depending on position x and time t.

The discussion of this article will start in chapter 2, where the general version of the porous medium equation along with some applications will be explored. It will become clear that the porous medium equation is a degenerate-parabolic and nonlinear partial differential equation. Consequently, the existence and uniqueness of its solutions is not immediately evident. Additionally, in section 2.3 a physical derivation of the porous medium equation will be given.

In chapter 3, definitions and theorems concerning differential equations relevant to the article will be displayed. More specifically, in section 3.2.1 the statement and proof of the Picard-Lindelöf theorem is given and in section 3.2.2 the statement and proof of the Peano existince theorem is presented. These are standard existence theorems for solutions of initial value problems regarding ordinary differential equations. The theorems will be useful for showing existence of solutions to the porous medium equation in non-problematic parts of its domain. Additionally, the proof and statement of Gronwall's inequality can be found in section 3.2.3. Gronwall's inequality will help us in Theorem 4.5.1 to prove existence of solutions around the degeneracy of the porous medium equation.

In chapter 4, Atkinson's and Peletier's 1971 article [1] will be worked out in detail. This is accomplished by introducing self-similar solutions to derive an equivalent problem in section 4.1. Moreover, we will split the self-similar solution in a classical version on $[0, a)$ and weak version on $[0, \infty)$, where $a > 0$ is the point where the porous medium equation degenerates.

First, we focus on the classical self-similar solution on $[0, a)$. In section 4.2, local uniqueness and existence of positive solutions to the porous medium equation will be proved. This proof is executed by using the Peano existence theorem and by proving continuous dependence of the solution on some initial conditions. Then, we look at the same property for solutions near the degeneracy at a in section 4.3. We find an equivalent problem in Lemma 4.3.2, for which we show uniqueness of its solutions by Theorem 3.2.1 (Banach's fixed point theorem). By Lemmas 4.11 and 4.3.2, we conclude that condition $\int_0^1 \frac{k(s)}{s} ds < \infty$ is necessary and sufficient for a unique solution to exist in a deleted left-neighbourhood of a. Afterwards, we continue these solutions backward to 0 in section 4.4. This results in two possibilities. Either the solution can be continued back to 0 as will be shown in Lemma 4.4.1, or the solution will tend to infinity somewhere along the way as will be shown in Lemma 4.4.2. All of these lemmas will provide enough material to present theorem 4.5.1. Theorem 4.5.1 will establish existence and uniqueness of self-similar classical solutions to the porous medium equation on $[0, a)$.

Secondly, Theorem 4.5.4 introduces the weak version of the self-similar solution, which is the classical version of the self-similar solution extended with the constant zero function.

Finally, we will find existence and uniqueness of self-similar weak solutions to the porous medium equation on $[0, \infty)$, which concludes what we wanted to proof.

Chapter 2

About the porous medium equation

In this chapter, properties of the porous medium equation will be discussed. First, in section 2.1 several applications of the porous medium equation will be presented, showing the importance of this equation. Then, in section 2.2 some general mathematical properties of the equation will be briefly touched upon. Finally, in section 2.3 a physical derivation of the porous medium equation will be given.

2.1 Applications of the porous medium equation

There are plenty of applications for the porous medium equation. Since the equation generally describes the way in which a fluid spreads, applications are usually found in the field of fluid dynamics. One of the applications is to describe nonlinear heat transfer that deals with a large variation of temperatures (Vázquez, 2007). This happens in modeling heat propagation in plasmas, for example, after a nuclear explosion (Vázquez, 2007). Another application is found in describing the flow of an incompressible liquid through a homogeneous porous material, for example, in the process of groundwater filtration (Vázquez, 2007). Similarly, the porous medium equation can be used to describe the flow of an ideal gas through a homogeneous porous material, which is the application discussed in Atkinson's and Peletier's article [1].

2.2 Mathematical properties of the porous medium equation

The porous medium equation in its most general form,

$$\frac{\partial u}{\partial t} = \Delta_x(u^m), \quad m > 1,$$

is a non-linear partial differential equation. The non-negative scalar function $u(x, t)$ usually denotes the variable density of the flowing substance, $0 < t < \infty$ is the time coordinate and the Laplacian acts on the spatial coordinate $x \in \mathbb{R}^n$. Rewriting the porous medium equation in the divergence form

$$\frac{\partial u}{\partial t} = \boldsymbol{\nabla} \cdot (D(u)\nabla u) = \boldsymbol{\nabla} \cdot (mu^{m-1}\nabla u),$$

shows that the porous medium equation is a degenerate parabolic equation (Vázquez, 2007). The equation is parabolic if $u \neq 0$ and the equation degenerates if $u = 0$, i.e. if $D(u)$ vanishes (Vázquez, 2007).

Note that the porous medium equation is reminiscent of the famous heat equation, $u_t = \triangle(u)$. Both equations are parabolic partial differential equation, however, the heat equation is linear and has a constant diffusion coefficient. Therefore, the heat equation, in contrast to the porous medium equation, is not degenerate and standard theorems which require continuity of the solution can be applied to ensure existence of solutions.

Atkinson's and Peletier's 1971 article [1] looks at the mathematical theory supporting the existence of solutions of the porous medium equation that describes laminar flow of a gas in one direction. We get

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial}{\partial x}u\right), \tag{2.1}$$

where we denote the density of the gas by the function $u(x,t) > 0$, x denotes the location in the one-dimensional space we are considering, t denotes time and the function $k(u)$ is the diffusion coefficient which contains the properties of the gas. Here, $k(u)$ is defined, real and continuous for $u \geq 0$ with $k(0) = 0$ and $k(u) > 0$ if $u > 0$. In section 2.3, we will derive equation (2.1).

As noted earlier, equation (2.1) is a degenerate parabolic equation. As a result of the definition of the diffusion coefficient $k(u)$, $k(u)$ approaches 0 if u approaches 0. Thus, equation (2.1) is parabolic in a neighbourhood of (x,t) such that $u > 0$ and it is not parabolic in a neighbourhood of (x,t) where $u = 0$. We cannot use standard existence theorems for the solutions of (2.1) around the degeneracy at $u = 0$, because the solution u_x might not be continuous in this neighbourhood. However, in approximating steps to the solution it is permitted to use standard theory, such as Banach's fixed point theorem (see Theorem 3.2.1) and Gronwall's inequality (see Theorem 3.2.5). For solutions $u > 0$, Peano's existence theorem will grant existence. In chapter 4 these arguments will be discussed in more detail.

2.3 Derivation of the porous medium equation

In the previous section, we have introduced the porous medium equation (2.1). In this section, a derivation of equation (2.1) based on physical gas laws will be presented.

We start by looking at the continuity equation for the density of a fluid, $\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho v) = 0$, where v denotes velocity in ms^{-1} and ρ denotes density. In addition, we use Ficks law, $v = -D\nabla p$. Note that the exact value of diffusivity constant D is not of mathematical importance, therefore we can take $D = 1$. Hence, we can equivalently use $v = -\nabla p$ to find that

$$\begin{aligned} 0 = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho v &= \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (-\rho \nabla p) \\ \frac{\partial \rho}{\partial t} &= \boldsymbol{\nabla} \cdot (\rho \nabla p). \end{aligned}$$

Then, we substitute the state equation for an ideal gas, which states that pressure is a function of density of the gas in n/m^3. Thus $p = p(\rho) = c\rho^\gamma$, where $\gamma \geq 1$ is called the polytropic exponent and c is a constant for the reference pressure (Vázquez, 2007). We find

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \boldsymbol{\nabla} \cdot (\rho \nabla (c\rho^\gamma)) \\ \frac{\partial \rho}{\partial t} &= \boldsymbol{\nabla} \cdot (\rho c \gamma \rho^{\gamma-1} \nabla \rho) \\ \frac{\partial \rho}{\partial t} &= \boldsymbol{\nabla} \cdot (k(\rho) \nabla \rho). \end{aligned}$$

Finally, if we substitute u for ρ and use that we are looking in one spatial dimension, we get the porous medium equation (2.1), that is,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial}{\partial x}u\right).$$

Chapter 3

Prerequisite knowledge

In order to properly discuss the article "Similarity Profiles of Flows Through Porous Media" by F.V. Atkinson and L.A. Peletier [1], it is of importance to introduce or recall some definitions and theorems presented in this section. In section 3.1, we will discuss some relevant definitions. In section 3.2, we will move on to state and proof the Picard-Lindelöf theorem, Peano's existence theorem and Gronwall's inequality.

3.1 Defintions

Definition 3.1.1 (Equicontinuouity). *Let X be a metric space; let $C(X) = \{f : X \to \mathbb{R}$ continuous$\}$ the space of continuous functions; let $S \subset C(X)$; and let $x \in X$ be a point. Then S is equicontinuous at x if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $y \in B(x, \delta)$, $f \in S \implies |f(x) - f(y)| < \varepsilon$.*

Definition 3.1.2 (Support). *Let $f : [0, \infty) \to \mathbb{R}$. Then, the support of f is the closure of the set containing all points $\eta \in [0, \infty)$ such that $f(\eta) \neq 0$, i.e.* $\operatorname{supp}(f) = \overline{\{\eta \in [0, \infty) : f(\eta) \neq 0\}}$.

Definition 3.1.3 (Compact support). *$f : [0, \infty) \to \mathbb{R}$ has compact support, if the support set of f is a compact subset of the domain of f.*

Definition 3.1.4 (Fixed point). *A fixed point of a mapping $K : C \subset X \to C$ is an element $x \in C$ such that $K(x) = x$.*

Definition 3.1.5 (Contraction map). *The map $K : C \subset X \to C$ is called a contraction, if there is a contraction constant $\theta \in [0, 1)$ such that*

$$\|K(x) - K(y)\| \leq \theta \|x - y\|, \quad x, y \in C.$$

3.2 Theorems

In this section, we will present and prove Picard-Lindelöf's theorem in subsection 3.2.1, Peano's theorem in subsection 3.2.2 and Gronwall's inequality in subsection 3.2.3.

3.2.1 Picard-Lindelöf

In this section, a more detailed version of the proof of the Picard-Lindelöf theorem from Teschl [2012, p. 38] and of Banach's fixed-point theorem from Teschl [2012, p. 35] will be presented. Let us start by proving Banach's fixed-point theorem, also known as the contraction principle theorem, which will be used in the proof of the Picard-Lindelöf theorem.

Theorem 3.2.1 (Banach's fixed-point theorem)**.** *Let C be a (nonempty) closed subset of a Banach space X and let $K : C \to C$ be a contraction, then K has a unique fixed point $\bar{x} \in C$ such that*

$$\|K^n(x) - \bar{x}\| \leq \frac{\theta^n}{1-\theta}\|K(x) - x\|, \quad x \in C, \tag{3.1}$$

where the notation $K^n(x)$ means $K^n(x) = K(K^{n-1}(x))$ and $K^0(x) = x$.

Proof. First, we will prove that there can be at most one fixed point, then we will prove there exists at least one fixed point with the desired property.

The first part will be a proof by contradiction, so we suppose that K is a contraction with fixed points $\overline{x}, \tilde{x} \in C$ such that $\overline{x} \neq \tilde{x}$, $\overline{x} = K(\overline{x})$ and $\tilde{x} = K(\tilde{x})$. Then $\|\overline{x} - \tilde{x}\| = \|K(\overline{x}) - K(\tilde{x})\| \leq \theta\|\overline{x} - \tilde{x}\|$ by definition of a contraction map. This can only hold if $\theta = 0$, but then $\overline{x} = \tilde{x}$ and we conclude that our assumption was wrong; there can be at most one fixed point.

It remains to be shown that such a fixed point exists. To this end, fix $x_0 \in C$ and consider the sequence $x_n = K^n(x_0)$. Then we find

$$\|x_{n+1} - x_n\| \leq \theta\|x_n - x_{n-1}\| \leq \cdots \leq \theta^n\|x_1 - x_0\|$$

and hence by the triangle inequality (for $n > m$)

$$\begin{aligned}
\|x_n - x_m\| &\leq \sum_{j=m+1}^{n} \|x_j - x_{j-1}\| \\
&\leq \theta^m\|x_1 - x_0\| + \theta^{m+1}\|x_1 - x_0\| + \cdots + \theta^{n-1}\|x_1 - x_0\| \\
&= \theta^m\|x_1 - x_0\| \left(1 + \theta + \cdots + \theta^{n-m-1}\right) \\
&\leq \theta^m\|x_1 - x_0\| \sum_{j=0}^{\infty} \theta^j \\
&= \frac{\theta^m}{1-\theta}\|x_1 - x_0\|,
\end{aligned} \tag{3.2}$$

where we have used that the geometric series $\sum_j \theta^j$ converges because $\theta \in [0, 1)$. Thus, we have found that x_n is a Cauchy sequence and tends to a limit, say $\overline{x}$. Moreover,

$$\begin{aligned}
\|K(\overline{x}) - \overline{x}\| &= \left\|K\left(\lim_{n\to\infty} x_n\right) - \lim_{n\to\infty} x_n\right\| \\
&= \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \theta^n\|x_1 - x_0\| = 0
\end{aligned}$$

shows that $\overline{x}$ is a fixed point. The estimate (3.1) follows after taking the limit $n \to \infty$ in (3.2) in the following way. Since we chose x_0 arbitrary in C, we can choose $x_0 = x$. Then $x_1 = K(x)$, thus we get

$$\begin{aligned}
\|K^m(x) - \bar{x}\| &= \left\|x_m - \lim_{n\to\infty} x_n\right\| = \lim_{n\to\infty} \|x_n - x_m\| \\
&\leq \frac{\theta^m}{1-\theta}\|x_1 - x_0\| = \frac{\theta^m}{1-\theta}\|K(x) - x\|.
\end{aligned}$$

In conclusion, K has a unique fixed point $\bar{x} \in C$ such that $\|K^m(x) - \bar{x}\| \leq \frac{\theta^m}{1-\theta}\|K(x) - x\|$ holds for any $x \in C$. □

Theorem 3.2.2 (Picard-Lindelöf). *Suppose $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, x, and uniformly continuous with respect to the first argument, t, then there exists a unique local solution $\bar{x}(t) \in C^1(I)$ of the IVP*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{3.3}$$

where I is some interval around t_0. More specifically, if $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and M denotes the maximum of $|f|$ on V. Then the solution exists at least for $t \in [t_0, t_0 + T_0]$ and remains in $\overline{B_\delta(x_0)}$, where $T_0 = \min\{T, \frac{\delta}{M}\}$. The analogous result holds for the interval $[t_0 - T, t_0]$.

Proof. Suppose $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$, then integrating both sides of (3.3) with respect to t yields

$$\begin{aligned} \int_{t_0}^{t} \frac{\mathrm{d}x(s)}{\mathrm{d}s} ds &= \int_{t_0}^{t} f(s, x(s)) ds \\ x(t) - x_0 &= \int_{t_0}^{t} f(s, x(s)) ds \\ x(t) &= x_0 + \int_{t_0}^{t} f(s, x(s)) ds. \end{aligned} \tag{3.4}$$

Note that $x_0(t) = x_0$ is, at least for small t, an approximating solution for $x_0(t)$, because the integral becomes negligible if the difference in its boundaries goes to 0. Inserting $x_0(t)$ into our integral equation (3.4) yields another approximating solution, which we call $x_1(t)$. Thus

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s)) ds.$$

Iterating this procedure gives a sequence of approximating solutions

$$x_m(t) = K^m(x_0)(t), \quad K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds.$$

Where $K(x)$ is an operator acting on t, thus $K^m(x)(t)$ means that we apply the operator $K(x)$ m times.

We want to apply the previously discussed contraction principle to the fixed point equation $x = K(x)$, which is precisely our integral equation (3.4). In the rest of this proof we will set $t_0 = 0$ and only consider $t \geq 0$ for notational simplicity. Recall that to apply the contraction principle we need a Banach space and a closed subset $C \subset X$ such that $K : C \to C$. For the Banach space, we choose $X = C([0, T], \mathbb{R}^n)$ for some suitable $T > 0$. For C, we will try a closed ball of radius δ around the constant function x_0 denoted by $\overline{B_\delta(x_0)}$. Thus, before we are allowed to apply the contraction principle, we need to show that $K : C \to C$.

We have assumed that f is locally Lipschitz continuous in the second argument, x, and uniformly continuous with respect to the first argument, t. Furthermore, by assumption U is open and the point $(t_0, x_0) = (0, x_0) \in U$. Thus if we let ϵ be arbitrary and we can choose δ such that $V = [0, T] \times \overline{B_\delta(x_0)} \subset U$, where $\overline{B_\delta(x_0)} = \{x \in \mathbb{R}^n : |x - x_0| < \delta\}$.

Let M be the abbreviation for

$$M = \max_{(t,x) \in V} |f(t, x)|,$$

where the maximum exists because f is continuous and V is compact. Then

$$|x - x_0| = |K(x)(t) - x_0| = \left| x_0 + \int_0^t f(s, x(s))ds - x_0 \right|$$
$$\leq \int_0^t |f(s, x(s))|ds \leq M \int_0^t ds = Mt$$

whenever the graph of $x(t)$ lies within V, that is, $\{(t, x(t)) : t \in [0, T]\} \subset V$.

Hence, for $t \leq T_0$, where

$$T_0 = \min\{T, \frac{\delta}{M}\},$$

we have $T_0 M \leq \delta$ in either case. Consequently, $|K(x)(t) - x_0| < \delta$. As a result, the graph of $K(x)$ restricted to $[0, T_0]$ is again in V.

In the special case $M = 0$, this can be interpreted as $\frac{\delta}{M} = \infty$ such that $T_0 = \min\{T, \frac{\delta}{M}\} = T$. Moreover, note that since $[0, T_0] \subset [0, T]$, we have

$$M = \max_{(t,x) \in V = [0,T] \times \overline{B_\delta(x_0)}} |f(t, x)| \geq \max_{(t,x) \in [0,T_0] \times \overline{B_\delta(x_0)}} |f(t, x)|.$$

Thus the same constant M will bound f on $V_0 = [0, T_0] \times \overline{B_\delta(x_0)} \subset V$.

Now we have shown that if we choose $X = C([0, T_0], \mathbb{R}^n)$ as our Banach space with norm $\|x\| = \max_{0 \leq t \leq T_0} |x(t)|$, and $C = \{x \in X : \|x - x_0\| \leq \delta\}$ as our closed subset, then $K : C \to C$. It remains to be shown that K is a contraction.

In order to show that K is a contraction map, we need to estimate $\|K(x) - K(y)\|$. Observe that

$$|K(x) - K(y)| = \left| x_0 + \int_0^t f(s, x(s))ds - x_0 - \int_0^t f(s, y(s))ds \right|$$
$$\leq \int_0^t |f(s, x(s)) - f(s, y(s))|ds.$$

Furthermore, note that f is locally Lipschitz continuous in the second argument, x, and uniformly continuous in the first argument, t, that is, for every compact set $V_0 \subset U$ the following number

$$L = \sup_{(t,x) \neq (t,y) \in V_0} \frac{|f(t, x) - f(t, y)|}{|x - y|}$$

(which depends on V_0) is finite. Then

$$\int_0^t |f(s, x(s)) - f(s, y(s))|ds = \int_0^t \frac{|f(s, x(s)) - f(s, y(s))|}{|x(s) - y(s)|} |x(s) - y(s)|ds$$
$$\leq L \int_0^t |x(s) - y(s)|ds$$
$$\leq L \sup_{0 \leq s \leq t} |x(s) - y(s)| \int_0^t ds$$
$$= Lt \sup_{0 \leq s \leq t} |x(s) - y(s)|,$$

under the condition that the graphs of $x(t)$ and $y(t)$ are in V_0. Thus we have found that $|K(x) - K(y)| \leq Lt \sup_{0 \leq s \leq t} |x(s) - y(s)|$. Now taking the supremum over $0 \leq t \leq T_0$ on both sides of the inequality, results in

$$\|K(x) - K(y)\| \leq \sup_{0 \leq t \leq T_0} \left(Lt \sup_{0 \leq s \leq t} |x(s) - y(s)| \right) = LT_0 \|x(s) - y(s)\|.$$

In other words

$$\|K(x) - K(y)\| \le LT_0\|x(s) - y(s)\|, \quad x, y \in C.$$

Finally, choosing $T_0 < L^{-1}$ we see that K is a contraction and existence of a unique solution follows from the contraction principle. □

Note that we have now introduced an extra restriction on T_0, $T_0 < L^{-1}$ in addition to $T_0 \le T$ and $T_0 \le \frac{\delta}{M}$. Using Weissinger's Theorem[1] instead of the contraction principle in the proof of Picard Lindelöf will remove the need for this restriction.

3.2.2 Peano existence theorem

In this section, a more detailed version of the proof of Peano's theorem from Teschl [2012, p.56] and of the Arzelà–Ascoli theorem from Teschl [2012, p.55] will be given. Let us start by presenting the Arzelà–Ascoli theorem, which will be used in the proof of Peano's theorem.

Theorem 3.2.3 (Arzelà–Ascoli)**.** *Suppose the sequence of functions $x_m(t) \in C(I, \mathbb{R}^n)$, $m \in \mathbb{N}$, on a compact interval I is (uniformly) equicontinuous. If the sequence x_m is bounded, then there is a uniformly convergent subsequence.*

Proof. Let $\{t_j\}_{j=1}^{\infty} \subset I$ be a dense subset of the compact interval I (for example all rational numbers in I). Since $x_m(t_1)$ is bounded, by the Bolzano-Weierstrass theorem there is a subsequence $x_m^{(1)}(t)$, such that $x_m^{(1)}(t_1)$ converges. Similarly, we can find a subsequence $x_m^{(2)}(t)$ from $x_m^{(1)}(t)$ which converges at t_2. Note that $x_m^{(2)}(t)$ then also converges at t_1, since it is a subsequence of $x_m^{(1)}(t)$. By induction we get a sequence $x_m^{(j)}(t)$ converging at $t_1, \dots, t_j$. The diagonal sequence $\overline{x}_m(t) = x_m^{(m)}(t)$ will by construction converge for all $t = t_j$. We will show that this diagonal sequences converges uniformly for all t, thus that this is the sequence we are looking for.

Since $x_m(t)$ is uniformly equicontinuous, we can fix $\epsilon > 0$ and choose δ such that $|x_m(t) - x_m(s)| \le \frac{\epsilon}{3}$ for $|t - s| < \delta$. Since $\{t_j\}_{j=1}^{\infty} \subset I$ is a dense subset of the compact interval I, finitely many balls $B_\delta(t_j)$ are able to cover I. Say $1 \le j \le p$ balls suffice. Furthermore, using that the diagonal sequence $\overline{x}_m(t)$ converges for all $t = t_j$, we can choose N_ϵ such that $|\overline{x}_m(t_j) - \overline{x}_n(t_j)| \le \frac{\epsilon}{3}$ for $n, m \ge N_\epsilon$ and $1 \le j \le p$.

Now let t be arbitrary and note that $t \in B_\delta(t_j)$ for some j. Thus

$$\begin{aligned} |\overline{x}_m(t) - \overline{x}_n(t)| &\le |\overline{x}_m(t) - \overline{x}_m(t_j)| + |\overline{x}_m(t_j) - \overline{x}_n(t_j)| \\ &\quad + |\overline{x}_n(t_j) - \overline{x}_n(t)| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

for $n, m \ge N_\epsilon$. Consequently, $\overline{x}_m$ is a uniform Cauchy sequence with respect to the maximum norm. By completeness of $C(I, \mathbb{R}^n)$, $\overline{x}_m$ has a limit. Thus, we have found a subsequence of x_m that converges uniformly. □

Theorem 3.2.4 (Peano existince theorem)**.** *Suppose f is continuous on $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and denote the maximum of $|f|$ by M. Then for the initial value problem*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0. \tag{3.5}$$

with $f \in C(U, \mathbb{R}^n)$, where $U \subset \mathbb{R}^{n+1}$ is open and $(t_0, x_0) \in U$, there exists at least one solution for $t \in [t_0, t_0 + T]$ which remains in $\overline{B_\delta(x_0)}$, where $T_0 = \min\{T, \frac{\delta}{M}\}$. An analogous result holds for the interval $[t_0 - T_0, t_0]$.

[1] Weissinger's Theorem and its proof can be found in Teschl [2012, p.39].

Proof. If $\phi(t)$ is a solution of the initial value problem (3.5), then by definition of the derivative, we have

$$\phi(t) = x_0 + \dot{\phi}(t_0)h + o(h) = x_0 + f(t_0, x_0)h + o(h).$$

Now we can define approximating the solution x_h by omitting the error term and applying the procedure iteratively. That is, we set

$$x_h(t_{m+1}) = x_h(t_m) + f(t_m, x_h(t_m))h, \quad t_m = t_0 + mh,$$

and use linear interpolation in between to connect the points, which gives

$$x_h(t) = x_h(t_m) + f(t_m, x_h(t_m))(t - t_m)$$

for $t \in (t_m, t_{m+1})$. This approximation method is also known as the Euler Method.

We want to show that $x_h(t)$ converges to a solution $\phi(t)$ as $h \downarrow 0$. We will use that, since f is continuous, it is bounded by a constant on each compact interval, so here we have $f \leq M$ for some finite constant M on V. Therefore the derivative of x_h, $\frac{\mathrm{d}}{\mathrm{d}t}x_h(t) = f(t_m, x_h(t_m))$, is also bounded by the same constant. Since this constant M is independent of h, we will show that the functions $x_h(t)$ form an equicontinuous family of functions. Then we will apply the Arzelà–Ascoli theorem which will show us that there is a subsequence of $x_h(t)$ that converges uniformly.

We have assumed that f is continuous on the compact set $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)}$, thus f is also uniformly continuous on V. Furthermore, let $M = \max_{(t,x)\in V} |f(t, x)|$.

We need to show that $x_h(t)$ is equicontinuous. Let $\epsilon > 0$ be arbitrary, pick $\delta = \frac{\epsilon}{M} > 0$ and $T > 0$ such that $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$, then $x_h(t) \in \overline{B_\delta(x_0)}$ for $t \in [t_0, t_0 + T_0]$ where $T_0 = \min\{T, \frac{\delta}{M}\}$. Therefore, if $|t - s| < \delta$ and we know that $x_h(t) = x_h(s) + f(s, x_h(s))(t - s)$, then

$$|x_h(t) - x_h(s)| = |f(s, x_h(s))||t - s| < M|t - s| < M\delta < \epsilon.$$

Hence, $x_h(t)$ is equicontinuous. Thus any subsequence of the family $x_h(t)$ is equicontinuous, and by the Arzelà–Ascoli theorem there exists a uniformly convergent subsequence $\phi_m(t)$ that converges to a solution $\phi(t)$, that is, $\phi_m(t) \to \phi(t)$.

Now it remains to be shown that the limit $\phi(t)$ solves the initial value problem (3.5). We will show this by verifying that the corresponding integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \tag{3.6}$$

holds.

Since f is uniformly continuous on V, we can find a sequence $\epsilon(h) \to 0$ as $h \to 0$ such that $|f(s, y) - f(t, x)| \leq \epsilon(h)$ for $|y - x| \leq Mh$, $|s - t| \leq h$. In order to estimate the difference between the left and right-hand side of the equation (3.6) for $x_h(t)$, we choose an m such that $t \leq t_m$ and rewrite

$$\begin{aligned} x_h(t) &= x_h(t_m) + f(t_m, x_h(t_m))(t - t_m) \\ &= x_0 + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \chi(s) f(t_j, x_h(t_j))ds, \end{aligned}$$

where $\chi(s) = 1$ for $s \in [t_0, t]$ and $\chi(s) = 0$ otherwise. Then

$$\begin{aligned}
&\left| x_h(t) - x_0 - \int_{t_0}^{t} f(s, x_h(s))ds \right| \\
&= \left| x_0 + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \chi(s) f(t_j, x_h(t_j))ds - x_0 - \int_{t_0}^{t} f(s, x_h(s))ds \right| \\
&\leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \chi(s) |f(t_j, x_h(t_j)) - f(s, x_h(s))| ds \\
&\leq \epsilon(h) \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \chi(s) ds = |t - t_0| \epsilon(h),
\end{aligned}$$

from which it follows that ϕ is indeed a solution of the equation (3.6), since

$$\phi(t) = \lim_{m\to\infty} \phi_m(t) = x_0 + \lim_{m\to\infty} \int_{t_0}^{t} f(s, \phi_m(s))ds = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds.$$

Note that we are allowed to interchange limit and integral by uniform convergence of ϕ_m by the Arzelà–Ascoli theorem. From this we can conclude that ϕ is also a solution of (3.5). □

3.2.3 Gronwall's inequality

In this section, a more detailed version of the proof of Gronwall's inequality from Teschl [2012, p. 42-43] will be given.

Theorem 3.2.5 (Generalized Gronwall's inequality)**.** *Suppose* $\psi(t)$ *satisfies*

$$\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)ds, \quad t \in [0, T], \tag{3.7}$$

with $\alpha(t) \in \mathbb{R}$ *and* $\beta(t) \geq 0$*. Then*

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)dr\right) ds, \quad t \in [0, T]. \tag{3.8}$$

Moreover, if in addition $\alpha(s) \leq \alpha(t)$ *for* $s \leq t$*, then*

$$\psi(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s)ds\right), \quad t \in [0, T]. \tag{3.9}$$

Proof. Abbreviate $\phi(t) = \exp\left(-\int_0^t \beta(s)ds\right)$, then we find that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi(t) \int_0^t \beta(s)\psi(s)ds\right) &= \phi(t) \cdot \beta(t)\psi(t) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\left(-\int_0^t \beta(s)ds\right)\phi(t) \cdot \int_0^t \beta(s)ds \\
&= \beta(t)\phi(t)\left(\psi(t) - \int_0^t \beta(s)ds\right) \\
&\leq \alpha(t)\beta(t)\phi(t),
\end{aligned}$$

by applying assumption (3.7). Integrating this inequality with respect to t, then dividing by $\phi(t)$ gives

$$\begin{aligned}\phi(t)\int_0^t \beta(s)\psi(s)ds &\le \int_0^t \alpha(s)\beta(s)\phi(s)ds\\ \int_0^t \beta(s)\psi(s)ds &\le \int_0^t \alpha(s)\beta(s)\frac{\phi(s)}{\phi(t)}ds.\end{aligned}$$

Now we add $\alpha(t)$ on both sides and use assumption (3.7) again. This shows that

$$\begin{aligned}\psi(t) &\le \alpha(t)+\int_0^t \beta(s)\psi(s)ds \le \alpha(t)+\int_0^t \alpha(s)\beta(s)\frac{\phi(s)}{\phi(t)}ds\\ \psi(t) &\le \alpha(t)+\int_0^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)dr\right)ds,\end{aligned}$$

which finishes the proof of the first claim.

For the second claim, assume that additionally $\alpha(s) \le \alpha(t)$ for $s \le t$, then inequality (3.8) becomes

$$\begin{aligned}\psi(t) &\le \alpha(t)+\alpha(t)\int_0^t \beta(s)\exp\left(\int_s^t \beta(r)dr\right)ds\\ &= \alpha(t)+\alpha(t)\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\int_t^s \beta(r)dr\right)\exp\left(\int_s^t \beta(r)dr\right)ds\\ &= \alpha(t)-\alpha(t)\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\int_s^t \beta(r)dr\right)\exp\left(\int_s^t \beta(r)dr\right)ds\\ &= \alpha(t)-\alpha(t)\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\exp\left(\int_s^t \beta(r)dr\right)\right)ds\\ &= \alpha(t)-\alpha(t)\exp\left(\int_s^t \beta(r)dr\right)\Bigg|_{s=0}^{s=t}\\ &= \alpha(t)\exp\left(\int_0^t \beta(r)dr\right),\end{aligned}$$

where we have used the fundamental theorem of calculus. □

Chapter 4

Article "Flows Through Porous Media" by F.V. Atkinson and L.A. Peletier

The previous section contained some relevant definitions and theorems that are used in the article "Similarity Profiles of Flows Through Porous Media" by F.V. Atkinson and L.A. Peletier [1]. In this section, I will expand on the content of the article by rewriting it and including a more detailed version of the proofs contained within the article.

4.1 General strategy

As stated before, we will discuss existence and uniqueness of self-similar solutions to the porous medium equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial}{\partial x}u\right). \tag{4.1}$$

That is, we look at solutions $u(x,t)$ in the region $0 < x < \infty$, $0 < t < \infty$. In particular, we look at self-similar solutions of the form $u(x,t) = f(x(t+1)^{-\frac{1}{2}}) = f(\eta)$ with $\eta \in (0,\infty)$. Rewriting the porous medium equation (4.1) using the self-similar form of the solutions gives

$$(k(f)f')' + \frac{1}{2}\eta f' = 0 \text{ for } 0 < \eta < \infty. \tag{4.2}$$

Altogether, we look to solve the initial-boundary value problem (4.2) with

$$f(0) = 0, \quad \lim_{\eta\to\infty} f(\eta) = 0$$

at the boundaries, where self-similar solutions $f(\eta)$ have compact support.

Specifically, our main objective is to proof that if $k(s)$ satisfies

$$\int_0^1 \frac{k(s)}{s} ds < \infty,$$

then for any $U > 0$, equation (4.2) has a unique, non-negative weak solution with compact support in $[0,\infty)$ such that on $f(0) = U$. The function $k(s)$ is assumed to be defined, real and continuous for $s \geq 0$ with $k(0) = 0$ and $k(s) > 0$ if $s > 0$.

Definition 4.1.1 (weak solution). *A weak solution is defined as a function $f(\eta)$ with the following properties:*

1. *f is bounded, continuous and non-negative on $[0, \infty)$;*
2. *$F(f) = \int_0^f k(s)ds$ has a continuous derivative F' with respect to η;*
3. *f satisfies the identity $\int_0^\infty \phi'(F' + \frac{1}{2}\eta f)d\eta + \frac{1}{2}\int_0^\infty \phi f d\eta = 0$ for all $\phi \in C_0^1(0, \infty)$. Note that the subscript in $C_0^1(0, \infty)$ means that ϕ vanishes at ∞.*

In order to achieve this objective, we first prove existence and uniqueness of a classical solution of (4.2) on $[0, a)$ for some $a > 0$ that satisfies the boundary conditions

$$f(0) = U$$
$$f \to 0, \quad k(f)f' \to 0 \text{ as } \eta \to a$$
$$f(\eta) > 0 \quad on \quad 0 \leq \eta < a.$$

Definition 4.1.2 (classical solution). *A classical solution is a continuously differentiable function $f(\eta)$ such that $k(f)f'$ is also continuously differentiable.*

Up to and including Theorem 4.5.1, f will implicitly be assumed to be a classical solution. To be precise, we ask whether for any given $U > 0$, there exists and $a > 0$ such that the aforementioned classical solution exists. This will be shown in Theorem 4.5.1 and in the lemmas leading up to it.

Eventually, we will combine the classical solution on $[0, a)$ with the function $f = 0$ on $[a, \infty)$ and prove in Theorem 4.5.4 that this gives a unique weak solution of (4.2) on $[0, \infty)$, as we set out to show.

4.2 Local uniqueness and monotonicity

In this section, we will proof local uniqueness of the initial value problem (4.2) if $f > 0$ and show that the solution f must be strictly monotonically decreasing.

Lemma 4.2.1. *For arbitrary positive α and real β, and any η_0, equation* (4.2) *has a unique solution in a neighbourhood of η_0, such that*

$$f(\eta_0) = \alpha, \quad k(f(\eta_0))f'(\eta_0) = \beta. \tag{4.3}$$

Proof. By Peano's existence theorem (see Theorem 3.2.4), such a solution as specified in the lemma exists. Now, we still need to proof that the solution is unique. Integrating (4.2) in some neighbourhood of η_0 with the solution $f(\eta)$, that we know exists now, then using integration by parts yields

$$\begin{aligned} k(f)f'|_{\eta_0}^{\eta} &= -\frac{1}{2}\int_{\eta_0}^{\eta} \zeta f'(\zeta)d\zeta \qquad (4.4)\\ &= -\frac{1}{2}\zeta f(\zeta)|_{\eta_0}^{\eta} + \frac{1}{2}\int_{\eta_0}^{\eta} f(\zeta)d\zeta. \end{aligned}$$

Suppose that $f_1(\eta)$ and $f_2(\eta)$ are two solutions of (4.2) such that they satisfy conditions (4.3). Using that $f_1(\eta_0) = f_2(\eta_0) = \alpha$ and $k(f_1(\eta_0))f_1'(\eta_0) = k(f_2(\eta_0))f_2'(\eta_0) = \beta$, we find

$$k(f_2)f_2'|_{\eta_0}^{\eta} - k(f_1)f_1'|_{\eta_0}^{\eta} = k(f_2(\eta))f_2'(\eta)) - k(f_1(\eta))f_1'(\eta) \tag{4.5}$$

and

$$k(f_2)f_2'|_{\eta_0}^{\eta} - k(f_1)f_1'|_{\eta_0}^{\eta} = -\frac{1}{2}\zeta f_2(\zeta)|_{\eta_0}^{\eta} + \frac{1}{2}\int_{\eta_0}^{\eta} f_2(\zeta)d\zeta - \Big(-\frac{1}{2}\zeta f_1(\zeta)|_{\eta_0}^{\eta} + \frac{1}{2}\int_{\eta_0}^{\eta} f_1(\zeta)d\zeta\Big)$$
$$= -\frac{1}{2}\eta(f_2(\eta) - f_1(\eta)) + \frac{1}{2}\int_{\eta_0}^{\eta}(f_2(\zeta) - f_2(\zeta))d\zeta..$$

Now combining (4.5) and (4.5) and taking the absolute value yields

$$\big|k(f_2(\eta))f_2'(\eta) - k(f_1(\eta))f_1'(\eta))\big| \leq \frac{1}{2}|\eta||f_2(\eta) - f_1(\eta)| + \frac{1}{2}\Big|\int_{\eta_0}^{\eta}(f_2(\zeta) - f_2(\zeta))d\zeta\Big|.$$

Recall that η is in a neighbourhood of η_0, so $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$, which implies that $|\eta| \leq \max\{|\eta_0 - \delta|, |\eta_0 + \delta|\}$ and $|\eta - \eta_0| \leq \max\{|-\delta|, |\delta|\} = \delta$, thus

$$\begin{aligned}
&\big|k(f_2(\eta))f_2'(\eta) - k(f_1(\eta))f_1'(\eta))\big| \\
&\leq \frac{1}{2}|\eta||f_2(\eta) - f_1(\eta)| + \frac{1}{2}\Big|\int_{\eta_0}^{\eta}(f_2(\zeta) - f_2(\zeta))d\zeta\Big| \\
&\leq \frac{1}{2}\max\{|\eta_0 - \delta|, |\eta_0 + \delta|\}|f_2(\eta) - f_1(\eta)| + \frac{1}{2}\int_{\eta_0}^{\eta}|f_2(\zeta) - f_2(\zeta)|d\zeta \\
&\leq \frac{1}{2}\max\{|\eta_0 - \delta|, |\eta_0 + \delta|\}|f_2(\eta) - f_1(\eta)| + \frac{1}{2}|\eta - \eta_0|\max_{\zeta\in[\eta_0,\eta]}|f_2(\zeta) - f_1(\zeta)| \\
&\leq \frac{1}{2}\max\{|\eta_0 - \delta|, |\eta_0 + \delta|\}\max_{\zeta\in[\eta_0,\eta]}|f_2(\zeta) - f_1(\zeta)| + \frac{1}{2}\delta\max_{\zeta\in[\eta_0,\eta]}|f_2(\zeta) - f_1(\zeta)| \\
&\leq \frac{1}{2}\max\{|\eta_0 - \delta|, |\eta_0 + \delta|, \delta\}\max_{\zeta\in[\eta_0,\eta]}|f_2(\zeta) - f_1(\zeta)| \\
&\leq A_1 \max_{|\zeta-\eta_0|\leq|\eta-\eta_0|}|f_2(\zeta) - f_1(\zeta)|.
\end{aligned}$$

Accordingly, we have found that

$$\begin{aligned}
\big|k(f_2(\eta))f_2'(\eta) - k(f_1(\eta))f_1'(\eta))\big| &\leq \frac{1}{2}|\eta||f_2(\eta) - f_1(\eta)| \\
&\quad + \frac{1}{2}\Big|\int_{\eta_0}^{\eta}(f_2(\zeta) - f_2(\zeta))d\zeta\Big| \\
&\leq A_1 \max_{|\zeta-\eta_0|\leq|\eta-\eta_0|}|f_2(\zeta) - f_1(\zeta)|
\end{aligned} \tag{4.6}$$

with constant $A_1 = \frac{1}{2}\max\{|\eta_0 - \delta|, |\eta_0 + \delta|, \delta\} > 0$ and η in some neighbourhood of η_0. If we set $F(f) = \int_0^f k(s)ds$, then

$$\begin{aligned}
F(f_2(\eta)) - F(f_1(\eta)) &= \int_0^{f_2(\eta)} k(s)ds - \int_0^{f_1(\eta)} k(s)ds \\
&= \int_0^{\alpha} k(s)ds + \int_{\alpha}^{f_2(\eta)} k(s)ds - \int_{\alpha}^{f_1(\eta)} k(s)ds - \int_0^{\alpha} k(s)ds \\
&= \int_{\alpha}^{f_2(\eta)} k(s)ds - \int_{\alpha}^{f_1(\eta)} k(s)ds
\end{aligned}$$

Now substitute $\alpha = f_j(\eta_0)$ and $s = f_j(\zeta)$ for $j = 1, 2$ in each respective integral.

$$\begin{aligned}
F(f_2(\eta)) - F(f_1(\eta)) &= \int_{f_2(\eta_0)}^{f_2(\eta)} k(s)ds - \int_{f_1(\eta_0)}^{f_1(\eta)} k(s)ds \\
&= \int_{\eta_0}^{\eta} k(f_2(\zeta))f_2'(\zeta) - k(f_1(\zeta))f_1'(\zeta)d\zeta
\end{aligned} \tag{4.7}$$

Now taking the absolute value of (4.7) gives the following inequality

$$\begin{aligned} |F(f_2(\eta)) - F(f_1(\eta))| &\leq \int_{\eta_0}^{\eta} \left| k(f_2(\zeta))f_2'(\zeta) - k(f_1(\zeta))f_1'(\zeta) \right| d\zeta \\ &\leq |\eta - \eta_0| A_1 \max_{|\zeta - \eta_0| \leq |\eta - \eta_0|} |f_2(\zeta) - f_1(\zeta)|. \end{aligned} \tag{4.8}$$

Additionally, note that

$$\begin{aligned} |F(f_2(\eta)) - F(f_1(\eta))| &= \left| \int_0^{f_2(\eta)} k(s)ds - \int_0^{f_1(\eta)} k(s)ds \right| \\ &= \left| \int_{f_1(\eta)}^{f_2(\eta)} k(s)ds \right| \\ &\geq |f_2(\eta) - f_1(\eta)| \min_{s \in D} k(s) = A_2 |f_2(\eta) - f_1(\eta)| \end{aligned} \tag{4.9}$$

with $D = f_1([\eta_0 - \delta, \eta_0 + \delta]) \cup f_2([\eta_0 - \delta, \eta_0 + \delta])$ and $\delta > 0$, holds for a constant A_2 since $f(\eta_0) > 0$ and $k(s)$ is continuous. Combining inequalities (4.8) and (4.9) gives

$$|f_2(\eta) - f_1(\eta)| \leq A_2^{-1} A_1 |\eta - \eta_0| \max_{|\zeta - \eta_0| \leq |\eta - \eta_0|} |f_2(\zeta) - f_1(\zeta)|.$$

Now if we consider $|\eta - \eta_0|$ to be small, since η is in any neighbourhood of η_0, we can for example choose $|\eta - \eta_0| \leq \epsilon < A_2 A_1^{-1}$. Then we find that

$$\begin{aligned} |f_2(\eta) - f_1(\eta)| &\leq A_2^{-1} A_1 \epsilon \max_{|\zeta - \eta_0| \leq |\eta - \eta_0|} |f_2(\zeta) - f_1(\zeta)| \\ &\leq A_2^{-1} A_1 \epsilon \max_{|\zeta - \eta_0| \leq \epsilon} |f_2(\zeta) - f_1(\zeta)| \end{aligned}$$

for all η, thus the inequality also holds if we can take the maximum on both sides. This results in

$$\begin{aligned} \max_{|\zeta - \eta_0| \leq \epsilon} |f_2(\zeta) - f_1(\zeta)| &\leq A_2^{-1} A_1 \epsilon \max_{|\zeta - \eta_0| \leq \epsilon} |f_2(\zeta) - f_1(\zeta)| \\ (1 - A_2^{-1} A_1 \epsilon) \max_{|\zeta - \eta_0| \leq \epsilon} |f_2(\zeta) - f_1(\zeta)| &\leq 0. \end{aligned}$$

Note that $1 - A_2^{-1} A_1 \epsilon > 0$ and the absolute value is non-negative, then we must have that $\max_{|\zeta - \eta_0| \leq \epsilon} |f_2(\zeta) - f_1(\zeta)| = 0$. Thus we have found that $f_2(\zeta) = f_1(\zeta)$ for ζ in a neighbourhood of η_0, which means that the solution as specified in Lemma 4.2.1 is locally unique. □

Lemma 4.2.2. *A solution of equation* (4.2) *which is positive in an interval is either constant or strictly monotonic in that interval.*

Proof. Writing (4.2) as a system of two first order linear equations with $g = k(f)f'$ gives

$$\begin{pmatrix} f \\ g \end{pmatrix}' = \begin{pmatrix} \frac{g}{k(f)} \\ -\frac{1}{2}\eta f' \end{pmatrix} = \begin{pmatrix} \frac{g}{k(f)} \\ -\frac{1}{2}\eta \frac{g}{k(f)} \end{pmatrix} \tag{4.10}$$

Then solving $g' = -\frac{1}{2}\eta f' = -\frac{1}{2}\eta\frac{g}{k(f)}$ for f by integration from η_0 to η gives

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}\eta} &= -\frac{1}{2}\frac{\eta}{k(f)}g \\
\int_{g(\eta_0)}^{g(\eta)} \frac{1}{g}dg &= \int_{\eta_0}^{\eta} -\frac{1}{2}\frac{\zeta}{k(f)}d\zeta \\
\ln(g(\eta)) - \ln(\beta) &= -\frac{1}{2}\int_{\eta_0}^{\eta}\frac{\zeta}{k(f)}d\zeta \\
g(\eta) &= \beta\exp\left(-\frac{1}{2}\int_{\eta_0}^{\eta}\frac{\zeta}{k(f)}d\zeta\right),
\end{aligned}$$

where constant $\beta = g(\eta_0)$. There are 2 cases here; either $\beta = 0$ thus $g(\eta) = 0$ for all $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$ or $\beta \neq 0$. If $\beta = 0$, $g = k(f)f' = 0$ implying $f'(\eta) = 0$, thus f is a constant function. If $\beta \neq 0$, then $g = k(f)f' \neq 0$ for any $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$ since the exponential function is never equal to 0 in that interval. Additionally, the sign of g must agree with the sign of η, thus f is monotonically increasing or decreasing depending on the sign of β. □

4.3 Solutions near $\eta = a$

In the previous section, we have proved local uniqueness if $f > 0$. In this section, we will look at solutions f near $\eta = a$. More specifically, in Lemma 4.3.1 we will show that condition (4.11) is necessary for solutions of (4.2) to exist in a left-neighbourhood of a. In Lemma 4.3.2 we derive integral equation (4.20), which is equivalent to the differential equation (4.2) for solutions f near a. In Lemma 4.3.3 we will proof the existence and uniqueness of solutions to the integral equation (4.20).

Lemma 4.3.1. *The condition*

$$\int_0^1 \frac{k(s)}{s}ds < \infty \tag{4.11}$$

holds, if equation (4.2) *has a solution f in any interval $(a-\epsilon, a)$, where $\epsilon > 0$ such that*

$$\begin{aligned}
&f \to 0, \quad k(f)f' \to 0 \text{ as } \eta \to a, \\
&\text{and } f(\eta) > 0 \text{ on } 0 \le \eta < a.
\end{aligned} \tag{4.12}$$

Proof. Suppose f to be a positive solution of (4.2) in a left-neighbourhood $(a-\epsilon, a)$ of a which satisfies the boundary conditions (4.12). Integration from η to a then yields

$$\begin{aligned}
\int_\eta^a (k(f(\zeta))f'(\zeta))'d\zeta &= -\frac{1}{2}\int_\eta^a \zeta f'(\zeta)d\zeta \\
0 - k(f(\eta))f'(\eta) &= -\frac{1}{2}\int_\eta^a \zeta f'(\zeta)d\zeta \\
k(f(\eta))f'(\eta) &= \frac{1}{2}\int_\eta^a \zeta f'(\zeta)d\zeta
\end{aligned} \tag{4.13}$$

By Lemma 4.2.2, f is monotone and because we have assumed $f > 0$ on $(a-\epsilon, a)$ and $f \to 0$

and $\eta \to a$, we conclude that $f'(\eta)$ is negative. Consequently, with $0 \le \eta < a$ we find

$$\begin{aligned}
\left|k(f(\eta))f'(\eta)\right| &= \frac{1}{2}\left|\int_\eta^a \zeta f'(\zeta)d\zeta\right| \\
&\le \frac{1}{2}\int_\eta^a |\zeta||f'(\zeta)|d\zeta \\
&\le \frac{1}{2}a\int_\eta^a |f'(\zeta)|d\zeta = -\frac{1}{2}a\int_\eta^a f'(\zeta)d\zeta \\
&= -\frac{1}{2}a[f(\zeta)]_\eta^a = \frac{1}{2}a(0 - f(\eta)) = \frac{1}{2}af(\eta).
\end{aligned}$$

Since $k(f) > 0$ if $f > 0$, we find

$$\frac{k(f(\eta))|f'(\eta)|}{f(\eta)} \le \frac{1}{2}a.$$

Thus for any a_1 and a_2 such that $a - \epsilon < a_1 < a_2 < a$, we have for the following integral that

$$\int_{a_1}^{a_2} \frac{k(f(\eta))|f'(\eta)|}{f(\eta)}d\eta \le \int_{a_1}^{a_2} \frac{1}{2}a d\eta = \frac{1}{2}a(a_2 - a_1) \le \frac{1}{2}a^2.$$

Now using that f is monotonically decreasing and substituting $|f'(\eta)|d\eta = dx$ which implies that $x = \int |f'(\eta)|d\eta = -\int f'(\eta)d\eta = -f(\eta)$, then substituting $-f = g$, where g is an auxiliary variable, transforms the integral into

$$\begin{aligned}
\int_{a_1}^{a_2} \frac{k(f(\eta))|f'(\eta)|}{f(\eta)}d\eta &= \int_{-f(a_1)}^{-f(a_2)} \frac{k(-f(\eta))}{-f(\eta)}d(-f'(\eta)) \\
&= \int_{g(a_1)}^{g(a_2)} \frac{k(g))}{g} \cdot (-1)dg \\
&= \int_{f(a_2)}^{f(a_1)} \frac{k(f)}{f}df. \qquad (4.14)
\end{aligned}$$

What follows is

$$\int_{f(a_2)}^{f(a_1)} \frac{k(f)}{f}df \le \frac{1}{2}a^2. \qquad (4.15)$$

If we let a_2 tend to a, then $f(a_2)$ tends to 0. Additionally, note that the integral $\int_c^d \frac{k(f))}{f}df$ exists for $0 < c < d$, since $\frac{k(f)}{f}$ is continuous on $[c, d]$. Thus we have derived equation (4.11). □

Lemma 4.3.2. *If $k(s)$ satisfies condition* (4.11)*, then, for given $a > 0$, there exists an $\epsilon > 0$ such that in $(a - \epsilon, a)$ equation* (4.2) *has a unique solution which is positive and satisfies the boundary conditions*

$$f \to 0, \quad k(f)f' \to 0 \text{ as } \eta \to a \qquad (4.16)$$

Proof. Suppose that such a solution already exists in a left-neighbourhood of $\eta = a$. Later in Lemma 4.3.3 we will proof that this solution indeed exists.

We start in the same way as in Lemma 4.3.1, where we integrate equation (4.2) from η to a resulting in

$$\begin{aligned}
k(f(\eta))f'(\eta) &= \frac{1}{2}\int_\eta^a \zeta f'(\zeta)d\zeta \\
\frac{1}{f'(\eta)} &= \frac{2k(f(\eta))}{\int_\eta^a \zeta f'(\zeta)d\zeta}
\end{aligned}$$

Now using that $f'(\zeta) = \frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}$, we can rewrite the last equation in the folowing way

$$\frac{1}{f'(\eta)} = \frac{\mathrm{d}\eta}{\mathrm{d}f(\eta)} = \frac{2k(f(\eta))}{\int_\eta^a \zeta df(\zeta)}. \tag{4.17}$$

Note that by Lemma 4.2.2 f is monotonic with non-vanishing derivative, since we have assumed f to be positive. The inverse function theorem tells us that we can write η as a function of f. Substituting $\eta = \sigma(f)$ in (4.17) yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}f} = \frac{2k(f(\eta))}{\lim_{x\to a}\int_{f(\eta)}^{f(x)} \sigma(f)df} = -\frac{2k(f(\eta))}{\int_0^{f(\eta)} \sigma(\phi)d\phi} = -\frac{2k(f)}{\int_0^f \sigma(\phi)d\phi}.$$

Thus we have found

$$\frac{\mathrm{d}\sigma(f)}{\mathrm{d}f} = -\frac{2k(f)}{\int_0^f \sigma(\phi)d\phi} \tag{4.18}$$

with solution $\sigma(f)$ such that $\sigma(0) = a$ because $f \to 0$ if $\eta \to a$. Furthermore, the function $\sigma(f)$ is defined and continuous on an interval $[0, f_0)$ for some $f_0 > 0$ and continuously differentiable on $(0, f_0)$. This follows from the inverse function theorem since f is assumed to be a classical solution of equation (4.2), which means that f is a continuously differentiable function.

Integrating equation (4.18) from 0 to f with $0 < f < f_0$ gives

$$\begin{aligned}
\int_0^f \frac{\mathrm{d}\sigma(\phi)}{\mathrm{d}\phi} d\phi &= -2\int_0^f \frac{k(\phi)}{\int_0^\phi \sigma(\psi)d\psi} d\phi \\
\sigma(f) - \sigma(0) &= -2\int_0^f \frac{k(\phi)}{\int_0^\phi \sigma(\psi)d\psi} d\phi \\
\sigma(f) &= a - 2\int_0^f \frac{k(\phi)}{\int_0^\phi \sigma(\psi)d\psi} d\phi.
\end{aligned} \tag{4.19}$$

Now we define

$$\tau(f) = 1 - \frac{\sigma(f)}{a} = 1 - \frac{\eta}{a},$$

then $\sigma(f) = a - a\tau(f) = a(1 - \tau(f))$ and equation (4.19) becomes

$$\begin{aligned}
\tau(f) &= 1 - \frac{1}{a}\left(a - 2\int_0^f \frac{k(\phi)}{\int_0^\phi a(1-\tau(\psi))d\psi} d\phi\right) \\
&= \frac{2}{a^2}\int_0^f \frac{k(\phi)}{\int_0^\phi (1-\tau(\psi))d\psi} d\phi.
\end{aligned} \tag{4.20}$$

By condition (4.11), integrals (4.19) and (4.20) exist.

We look for solutions f of (4.20) defined in a right-neighbourhood of $f = 0$. Retracing the steps of Lemma 4.3.2 yields a solution f of (4.2) in a left-neighbourhood of $\eta = a$, which satisfies condition (4.16). Similarly, if the solution of (4.20) is unique, then the corresponding solution of equation (4.2) is also unique.

Altogether, in order to show that Lemma 4.3.2 holds, it is enough to show that (4.20) has a unique solution in a right neighbourhood of $f = 0$, which is positive if $f > 0$. This will be proved in Lemma 4.3.3.

The proof of Lemma 4.3.3 will share some similarities in structure with the proof of the Picard-Lindelöf theorem (see Theorem 3.2.2). In both proofs we first derive an integral equation, then we apply Banach's fixed point theorem (see Theorem 3.2.1). However, the integral equation (4.20) has a degeneracy on the right hand side if $f = 0$, thus deviations of the proof of the Picard-Lindelöf theorem are necessary. This problem originates from the equivalent problem for f, (4.2), because when it is written as a system of ordinary differential equations, as we did in equation (4.10) of Lemma 4.2.2, we see that on the right hand side of the ordinary differential equations we find the function $k(s)$. The function $k(s)$ does not satisfy a Lipschitz condition. Thus, the conditions of the Picard-Lindelöf theorem are not satisfied and we cannot apply it.

Lemma 4.3.3. *If $k(s)$ satisfies condition*

$$\int_0^1 \frac{k(s)}{s} ds < \infty, \tag{4.21}$$

then there exists a $\gamma > 0$ such that

$$\tau(f) = \frac{2}{a^2} \int_0^f \frac{k(\phi)}{\int_0^\phi (1-\tau(\psi))d\psi} d\phi \tag{4.22}$$

has a unique continuous solution $\tau(f)$ in $0 \leq f \leq \gamma$. If $0 < f \leq \gamma$, then $\tau(0) = 0$ and $\tau(f) > 0$.

Proof. Suppose condition (4.21) holds and suppose there exists a $\gamma > 0$ such that f is a continuous function and $\tau(0) = 0$ and $\tau(f) > 0$ if $0 < f \leq \gamma$. We will choose a specific γ later. Let X denote the set of continuous functions $\tau(f)$ defined on $[0, \gamma]$. Since $\tau(f)$ is continuous and $\tau(0) = 0$ and $\tau(f) > 0$, $0 \leq \tau(f) \leq \frac{1}{2}$ is satisfied for some γ that is small enough. Let $\|.\|$ denote the supremum norm on X, then X is a complete metric space.

It remains to be shown that $\tau(f)$ is a unique solution of (4.22), we will do this by applying Banach's fixed point theorem (see Theorem 3.2.1). To this end, we introduce the map

$$M(\tau)(f) = \frac{2}{a^2} \int_0^f \frac{k(\phi)}{\int_0^\phi (1-\tau(\psi))d\psi} d\phi$$

on X. Then $M(\tau)(f)$ is well-defined, because if we assume that for any f_1 and f_2 in $[0, \gamma]$, $f_1 = f_2$ then we find $M(\tau)(f_1) = M(\tau)(f_2)$. $M(\tau)(f)$ is non-negative and continuous, because $k(s)$ and $1 - \tau(f)$ are on their respective domains. Furthermore, we have since we have $\tau(f)$ defined on $[0, \gamma]$ satisfying $0 \leq \tau(f) \leq \frac{1}{2}$ and that $0 \leq f \leq \gamma$

$$\begin{aligned}
\int_0^\phi (1-\tau(\psi))d\psi &\leq \max_{\psi \in [0,\phi]} \|1-\tau(\psi)\| \int_0^\phi 1 d\psi \\
&\leq \max_{\psi \in [0,\phi]} (\|1\| + \|\tau(\psi)\|)\phi \\
&\leq \max_{\psi \in [0,\gamma]} (1 + \|\tau(\psi)\|)\phi \\
&= \frac{3}{2}\phi
\end{aligned}$$

therefore

$$M(\tau)(f) \leq \frac{4}{3a^2} \int_0^f \frac{k(\phi)}{\phi} d\phi \leq \frac{4}{a^2} \int_0^\gamma \frac{k(\phi)}{\phi} d\phi$$

If $M(\tau)(f) \leq \frac{1}{2}$ then M maps X onto itself. Therefore γ is chosen small enough such that $\frac{4}{a^2} \int_0^\gamma \frac{k(\phi)}{\phi} d\phi \leq \frac{1}{2}$ is satisfied. Now we have found our first restriction on γ, namely $\int_0^\gamma \frac{k(\phi)}{\phi} d\phi \leq \frac{a^2}{8}$

Furthermore, we want M to be a contraction map. M is a contraction map on the metric space X with the supremum norm, if M is a function from X to itself such that there exists a real number k such that $0 \leq k < 1$ and such that for all τ_1 and τ_2 in X, we have $\|M(\tau_1) - M(\tau_2)\| \leq k\|\tau_1 - \tau_2\|$. Let $\tau_1, \tau_2 \in X$ be arbitrary, then

$$\begin{aligned}
\|M(\tau_1) - M(\tau_2)\| &= \left\| \frac{2}{a^2} \int_0^f \frac{k(\phi)}{\int_0^\phi 1 - \tau_1(\psi)d\psi} - \frac{k(\phi)}{\int_0^\phi 1 - \tau_2(\psi)d\psi} d\phi \right\| \\
&= \frac{2}{a^2} \int_0^f k(\phi) \left\| \frac{\int_0^\phi \tau_1(\psi) - \tau_2(\psi)d\psi}{\int_0^\phi 1 - \tau_1(\psi)d\psi \int_0^\phi 1 - \tau_2(\psi)d\psi} \right\| d\phi \\
&\leq \frac{2}{a^2} \int_0^f k(\phi) \frac{\|\tau_1(\psi) - \tau_2(\psi)\|\phi}{(\frac{3}{2}\phi)^2} d\phi \\
&\leq \frac{8}{a^2} \int_0^\gamma \frac{k(\phi)}{\phi} d\phi \|\tau_1(\psi) - \tau_2(\psi)\|.
\end{aligned}$$

Thus for M to be a contraction map we need $\frac{8}{a^2} \int_0^\gamma \frac{k(\phi)}{\phi} d\phi < 1$. This second restriction on γ gives us $\int_0^\gamma \frac{k(\phi)}{\phi} d\phi < \frac{a^2}{8}$, and this restriction implies the first one that we have found in this proof. By Banach's fixed point theorem, see Theorem 3.2.1, if there exists a γ such that $\int_0^\gamma \frac{k(\phi)}{\phi} d\phi < \frac{a^2}{8}$, then M has a unique fixed point $\tau^\star$ in X such that $M(\tau^\star) = \tau^\star$. □

Thus, we know that there exist a unique solution f of (4.2) in a left-neighbourhood of $\eta = a$, which satisfies condition (4.16), since we have found uniqueness and existence of the solutions of the equivalent integral problem in Lemma 4.3.3. This concludes the proof of Lemma 4.3.2. □

4.4 Backward continuation of solutions

In the previous section, we have established existence and uniqueness of a positive solution of (4.2) in a left-neighbourhood of a. In this section, we are looking at the continuation to $\eta = 0$ of the solution described in Lemma 4.3.2. That is, a unique positive solution of (4.2) in any left-neighbourhood $(a - \epsilon, a)$ of $\eta = a$, which satisfies the boundary conditions

$$f \to 0, \quad k(f)f' \to 0 \text{ as } \eta \to a.$$

By Lemma 4.2.1 this solution can be uniquely continued backwards as a function of η. By Lemma 4.2.2 the solution will be strictly monotonically decreasing, thus it will increase monotonically if η decreases. Now, there are two possibilities as f is continued backwards. Either the solution can be continued back to $\eta = 0$ as will be shown in Lemma 4.4.1, or we have $f(\eta) \to \infty$ as η decreases towards some non-negative value as will be shown in Lemma 4.4.2.

Lemma 4.4.1. *In case A, if*

$$\int_1^\infty \frac{k(s)}{s} ds = \infty, \tag{4.23}$$

then the solution described in Lemma 4.3.2 can be continued back to $\eta = 0$ for any $a > 0$.

Proof. Let us consider the solution as described in Lemma 4.3.2, then Lemma 4.3.1 tells us that $\int_0^1 \frac{k(s)}{s} ds < \infty$ must hold. Note that the upper boundary, $h = 1$, is chosen arbitrarily, we could have chosen any finite h such that $0 < h < \infty$. The same applies for the lower boundary of (4.23).

From the proof of Lemma 4.3.1 we can extract (4.15), which is the inequality

$$\int_{f(a_2)}^{f(a_1)} \frac{k(s)}{s} ds \leq \frac{1}{2} a^2.$$

Note that we have $a - \epsilon < a_1 < a_2 < a$. Now, if we take $a_2 \to a$ and let $a_1 \to 0$ in a left-neighbourhood of $\eta = a$ (by Lemma 4.3.2 there exists a unique solution here), we get

$$\int_{\lim_{a_2 \to a} f(a_2)}^{\lim_{a_1 \to 0} f(a_1)} \frac{k(s)}{s} ds \leq \frac{1}{2} a^2 \tag{4.24}$$

Thus (4.24) is a bounded integral and combining this with (4.23), implies that $f(a_1)$ cannot be unbounded. Because if $f(a_1)$ is unbounded, the integrals (4.23) and (4.24) should be equal and they are not. Since $f(a_1)$ is bounded in any left-neighbourhood of $\eta = a$, the solution $f(\eta)$ can be continued back to $\eta = 0$. □

Lemma 4.4.2. *In case B, if*

$$\int_1^\infty \frac{k(s)}{s} ds < \infty,$$

there is an $a^\star$ such that if $a > a^\star$, there exists no positive solution of equation (4.2) *on* $[0, a)$, *which satisfies the boundary conditions*

$$f \to 0, \quad k(f)f' \to 0 \text{ as } \eta \to a.$$

Proof. This will be a proof by contradiction so we assume case B holds and that an $a^\star$ as described in the lemma does not exist, i.e. that for any $a > 0$, there exists a positive solution of equation (4.2) on $[0, a)$, which satisfies the boundary conditions 4.4.2. The proof will mirror some steps as were taken in Lemma 4.3.1. Thus assume that f is a solution of (4.2) in a left-neighbourhood $(a - \epsilon, a)$ of a. By integrating (4.2) from η to a, we obtain (4.13) from Lemma 4.3.1:

$$k(f(\eta))f'(\eta) = \frac{1}{2} \int_\eta^a \zeta f'(\zeta) d\zeta.$$

Since f is monotonically decreasing by Lemma 4.2.2, $f'(\eta) \leq 0$. Consequently, $\eta f'(\eta) \leq 0$. Next, taking the absolute value results in

$$\begin{aligned}
\left|k(f(\eta))f'(\eta)\right| &= \left|\frac{1}{2} \int_\eta^a \zeta f'(\zeta) d\zeta\right| \\
&= (-1) \cdot \frac{1}{2} \int_\eta^a \zeta f'(\zeta) d\zeta \\
&\geq -\eta \frac{1}{2} \int_\eta^a f'(\zeta) d\zeta \\
&= -\eta \frac{1}{2} (\lim_{\zeta \to a} f(\zeta) - f(\eta) \\
&= \frac{1}{2} \eta f(\eta).
\end{aligned}$$

Thus we have found $|k(f(\eta))f'(\eta)| \geq \frac{1}{2} \eta f(\eta)$. Recall (4.14) from the proof of Lemma 4.3.1:

$$\int_{a_1}^{a_2} \frac{k(f(\eta))|f'(\eta)|}{f(\eta)} d\eta = \int_{f(a_2)}^{f(a_1)} \frac{k(s)}{s} ds.$$

Thus, for any a_2 with $a - \epsilon < \eta = a_1 < a_2 < a$, we obtain

$$\begin{aligned}\int_{f(a_2)}^{f(\eta)} \frac{k(s)}{s} ds &= \int_{\eta}^{a_2} \frac{k(f(\zeta))|f'(\zeta)|}{f(\zeta)} d\zeta \\ &\geq \int_{\eta}^{a_2} \frac{\frac{1}{2}\zeta f(\zeta)}{f(\zeta)} d\zeta \\ &= \frac{1}{2}\left[\frac{1}{2}\zeta^2\right]_{\eta}^{a_2} = \frac{1}{4}(a_2^2 - \eta^2).\end{aligned}$$

Next, we let a_2 tend to a, which yields

$$\int_0^{f(\eta)} \frac{k(f)}{f} df \geq \frac{1}{4}(a^2 - \eta^2). \tag{4.25}$$

Hence, if we assume that the solution exists over $[0, a)$ and if we choose a so large that $\frac{a^2}{4} > \int_0^\infty \frac{k(s)}{s} ds$, which we can do by our initial assumption in the beginning of the proof, we find that

$$\frac{a^2}{4} > \int_0^\infty \frac{k(s)}{s} ds \geq \int_0^{f(\eta)} \frac{k(f)}{f} df \geq \frac{1}{4}(a^2 - \eta^2).$$

Now if we let $\eta \to 0$, we find $\frac{a^2}{4} > \frac{a^2}{4}$ thus we have arrived at a contradiction. Hence, the assumption was wrong and the $a^\star$ referred to in the lemma exists. □

4.5 The main result

All of the preceding lemmas enable us to present Theorem 4.5.1. Theorem 4.5.1 establishes existence and uniqueness of self-similar classical solutions to the porous medium equation on $[0, a)$. Recall that we defined classical solutions in definition 4.1.2 as a continuously differentiable function $f(\eta)$ such that $k(f)f'$ is also continuously differentiable.

Theorem 4.5.1. *Let $k(s)$, $s \geq 0$, be continuous with $k(0) = 0$ and $k(s) > 0$ for $s > 0$. Then the condition*

$$\int_0^1 \frac{k(s)}{s} ds < \infty \tag{4.26}$$

is necessary and sufficient for the existence, for any $U > 0$, of a unique $a > 0$ and the existence of a unique classical[1] *solution of*

$$(k(f)f')' + \frac{1}{2}\eta f' = 0 \quad 0 < \eta < \infty \tag{4.27}$$

with the boundary conditions

$$f(0) = U \tag{4.28}$$

$$f \to 0, \quad k(f)f' \to 0 \text{ as } \eta \to a \tag{4.29}$$

$$f(\eta) > 0 \quad \text{on} \quad 0 \leq \eta < a. \tag{4.30}$$

Moreover, in Case A (in which $\int_1^\infty \frac{k(s)}{s} ds = \infty$), $a = a(U) \to \infty$ as $U \to \infty$ and in Case B (in which $\int_1^\infty \frac{k(s)}{s} ds < \infty$), $a(U)$ tends to a finite limit as $U \to \infty$.

[1] See definition 4.1.2 for the exact definition of a classical solution in this context.

Proof. The proof depends on a discussion of the function $b(a)$. We define $b(a) = f(0; a)$, where f is a classical solution of (4.27), which is defined and positive on any left-neighbourhood of $\eta = a$ (thus with $\eta \in (0, a)$) and which satisfies condition (4.29). Note that the function $b(a)$ is defined for all $a \geq 0$ in Case A and $b(a)$ is defined for a sufficiently small in Case B. We claim that $b(a)$ has the following properties:

i $\lim_{a\to 0} b(a) = 0$;

ii $b(a)$ is strictly monotonically increasing;

iii $b(a)$ is continuous;

iv In Case A $\lim_{a\to\infty} b(a) = \infty$;

v In Case B $\lim_{a\to\bar{a}} b(a) = \infty$ for some $\bar{a} < \infty$.

If all of these properties hold for $b(a)$, then the equation $U = b(a)$ has a unique solution for any $U > 0$. Let us call this unique solution $a(U)$. Then, the function $f(\eta; a(U))$ additionally satisfies conditions (4.28) and (4.30). Moreover, because of the uniqueness of $a(U)$ and because of Lemma 4.2.1, which gives uniqueness of the solution for $f > 0$, and Lemma 4.3.2, which gives uniqueness of the solution on a deleted left-neighbourhood of a, we have found that $f(\eta; a(U))$ is the unique solution of problem (4.27), (4.28), (4.29), (4.30). Next, we shall prove (i)-(v) in succession.

(i) *Proof of* $\lim_{a\to 0} b(a) = 0$. From the proof of Lemma 4.3.1 we find (4.15):

$$\int_{f(a_2)}^{f(a_1)} \frac{k(s)}{s} ds \leq \frac{1}{2}a^2.$$

If we let $a_2 \to a$ and $a_1 \to 0$, then we find

$$\int_{0}^{b(a)} \frac{k(f)}{f} df \leq \frac{1}{2}a^2.$$

Since the integrand is non-negative and 0 only if $f = 0$, $b(a)$ tends to 0 as a tends to 0.

(ii) *Proof of* $b(a)$ *is strictly monotonically increasing by contradiction.* Suppose $b(a)$ is not strictly monotonically increasing. Then there exist numbers a_1 and a_2 such that $0 < a_1 < a_2$ and $b(a_2) \leq b(a_1)$. Let us denote $f_i(\eta) = f(\eta; a_i)$ with $i = 1, 2$ and recall that f is continuous and (strictly) monotonically decreasing and by assumption (4.29) $f_i(\eta) \to 0$ if $\eta \to a_i$. Then, by the Intermediate Value Theorem, there exists a number $\eta_0 \in [0, a_1)$ such that $f_1(\eta_0) = f_2(\eta_0)$ and $f_1 < f_2$ on (η_0, a_1).[2]

Since f_1 is a solution of (4.27), we can substitute it into the differential equation. Integrating from η_0 to a_1 yields

$$\begin{aligned} &\int_{\eta_0}^{a_1} (k(f_1(\zeta))f_1'(\zeta))' + \frac{1}{2}\zeta f_1'(\zeta) d\zeta \\ &= k(f_1(\zeta))f_1'(\zeta)|_{\eta_0}^{a_1} + \frac{1}{2}\zeta f_1(\zeta)|_{\eta_0}^{a_1} - \frac{1}{2}\int_{\eta_0}^{a_1} f_1(\zeta)d\zeta = 0. \end{aligned}$$

[2]We will use the negation of this argument, which states that we cannot find such a η_0, in the proof of *(iii) b(a) is continuous.*

Now applying condition (4.29), writing $f_1(\eta_0)$ as $\bar{f}$, and multiplying by -1 results in

$$k(\bar{f})f_1'(\eta_0) + \frac{1}{2}\eta_0\bar{f} + \frac{1}{2}\int_{\eta_0}^{a_1} f_1(\zeta)d\zeta = 0. \tag{4.31}$$

f Integrating f_2, which is also a solution of (4.27), from η_0 to a_2 and writing $f_2(\eta_0)$ also as $\bar{f}$, gives

$$k(\bar{f})f_2'(\eta_0) + \frac{1}{2}\eta_0\bar{f} + \frac{1}{2}\int_{\eta_0}^{a_2} f_2(\zeta)d\zeta = 0. \tag{4.32}$$

Note that at η_0 we have assumed that $\overline{f} = f_1(\eta_0) = f_2(\eta_0)$. Consequently, if we subtract (4.31) from (4.32), we obtain

$$\begin{aligned} &k(\bar{f})\{f_2'(\eta_0) - f_1'(\eta_0)\} - \frac{1}{2}\int_{\eta_0}^{a_1} f_1(\zeta)d\zeta + \frac{1}{2}\int_{\eta_0}^{a_2} f_2(\zeta)d\zeta \\ &= 0. \end{aligned}$$

If we now split the integral form η_0 to a_2 to an integral from η_0 to a_1 and one from a_1 to a_2, we get

$$k(\bar{f})\{f_2'(\eta_0) - f_1'(\eta_0)\} + \frac{1}{2}\int_{\eta_0}^{a_1} f_2(\zeta) - f_1(\zeta)d\zeta + \frac{1}{2}\int_{a_1}^{a_2} f_2(\zeta)d\zeta = 0 \tag{4.33}$$

Recall that $f_2 > f_1$ on (η_0, a_1) and $f_2 > 0$ on $(0, a_2)$ by (4.28) and thus, in particular, $f_2 > 0$ on (a_1, a_2). Consequently, the second and third term in (4.33) are positive. Now we still need to show that $f_2'(\eta_0) - f_1'(\eta_0) \geq 0$. Assuming that the functions are differentiable in η_0, we can take a look at the limit definition of the derivative

$$f_2'(\eta_0) - f_1'(\eta_0) = \lim_{\eta\to\eta_0}\left(\frac{f_2(\eta) - f_2(\eta_0)}{\eta - \eta_0} - \frac{f_1(\eta) - f_1(\eta_0)}{\eta - \eta_0}\right) = \lim_{\eta\to\eta_0}\frac{f_2(\eta) - f_1(\eta)}{\eta - \eta_0}.$$

Since we have assumed that the limit exists, the value for the limit from the right is the same value as for the total limit. Therefore, it is enough to only look at the limit approaching η_0 from the right. Approaching the limit from the right, where $f_2 > f_1$, results in a non-negative value for the limit. That is, $f_2'(\eta_0) - f_1'(\eta_0) \geq 0$. Furthermore, since $k(s) > 0$ for $s > 0$, the first term of (4.33) is non-negative.

Now we have arrived at a contradiction. We have derived that (4.33) is always strictly greater than zero and that it is equal to zero at the same time. Hence, our assumption was false and the function $b(a)$ must be monotonically increasing.

(iii) *Proof of* $b(a)$ *is continuous* The main idea of this proof is to show continuity of the solution $f(\eta; a)$ in a for all $\eta \in [0, a)$, then, in particular, $b(a)$ is continuous in η.

In order to show that the solution $f(\eta; a)$ depends continuously on a for all $\eta \in [0, a)$, we will show in Lemma 4.5.2 that the unique solution $f(\eta; a)$ depends continuously on initial conditions at $\eta = \eta_0$ and that the initial data depends continuously on a. Then, by composition of those two statements, we find that $f(\eta; a)$ depends continuously on a for all $\eta_0 < a$ in a deleted left-neighbourhood of a. In the proof of Lemma 4.5.2 we will need on multiple occasions that τ satisfies a Lipschitz condition in a. The proof of this can be found in Lemma 4.5.3. After proving Lemma 4.5.3, we will show that $b(a)$ must also be continuous in a.

Lemma 4.5.2. *If* $a > 0$, $\eta_0 < a$ *in a deleted left-neighbourhood of* a *and* $f(\eta; a)$ *is a solution of* (4.27) *with* $f(\eta; a)$ *defined and positive on any deleted left-neighbourhood of* $\eta = a$ *and with* f *satisfying condition* (4.29)*, then*

- *the solutions* $a \mapsto f(\eta_0; a)$ *and* $a \mapsto (k(f)f')(\eta_0; a)$ *are continuous in* a *and*
- *the unique solution* $\eta \mapsto f(\eta; a)$ *depends continuously on the following initial conditions given at* $\eta = \eta_0$*:*

$$\begin{aligned} f(\eta_0) &= f(\eta_0; a), \\ (k(f)f')(\eta_0) &= k(f(\eta_0; a))\frac{\mathrm{d}}{\mathrm{d}\eta} f(\eta_0; a). \end{aligned}$$

Proof. Assume that $f(\eta; a)$ is a solution of (4.27) and let $f(\eta; a)$ be defined and positive on a deleted left-neighbourhood of $\eta = a$ and suppose f satisfies condition (4.29). Let $a > 0$ be arbitrary, fix $\eta_0 < a$ close to a, i.e. fix η_0 in a deleted left-neighbourhood of a.

In order to show that $a \mapsto f(\eta_0; a)$ is continuous in a, we will first derive an inequality on $\tau(f; a)$, then we will use the mean value theorem to change the inequality on $\tau(f; a)$ to an inequality on $f(\eta; a)$, from which we can deduce that $a \mapsto f(\eta_0; a)$ is continuous in a by using that the function $\tau(f; a)$ is Lipschitz continuous in a. Then the continuity of $a \mapsto (k(f)f')(\eta_0; a)$ in a follows easily.

Let $0 < a_0 \leq a_1 < a_2 \leq a_3$, where we choose a_3 such that $b(a_3) < \infty$. We set $\tau(f) = \tau(f(\eta; a); a)$[3].

In order to show that $a \mapsto f(\eta_0; a)$ is continuous in a, we first derive an inequality on $\tau(f; a)$, then we use the mean value theorem to transform the inequality on $\tau(f; a)$ to an inequality on $f(\eta; a)$, from which we can deduce that $a \mapsto f(\eta_0; a)$ is continuous in a by using that by Lemma 4.5.3 the function $\tau(f; a)$ satisfies a Lipschtiz condition a.

We know that $\sigma(f; a)$ is continuously differentiable on $(0, f_0)$, for some small f_0 such that $\eta < a$ is in a neighbourhood of a, thus $\frac{\sigma(f(\eta_0; a); a)}{a} = \frac{\eta_0}{a}$ is continuously differentiable on $(0, f_0)$. Then

$$\begin{aligned} \left|\frac{\eta_0}{a_1} - \frac{\eta_0}{a_2}\right| &= |1 - \tau(f(\eta_0; a_1); a_1) - (1 - \tau(f(\eta_0; a_2); a_2))| \\ &= |\tau(f(\eta_0; a_1); a_1) - \tau(f(\eta_0; a_2); a_2)| \\ &= |\tau(f(\eta_0; a_1); a_1) - \tau(f(\eta_0; a_2); a_1) \\ &+ \tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_2); a_2)| \\ &\geq \big||\tau(f(\eta_0; a_1); a_1) - \tau(f(\eta_0; a_2); a_1)| \\ &- |\tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_2); a_2)|\big| \\ &\geq |\tau(f(\eta_0; a_1); a_1) - \tau(f(\eta_0; a_2); a_1)| \\ &- |\tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_2); a_2)|. \end{aligned}$$

Thus we find the following inequality on τ

$$\begin{aligned} |\tau(f(\eta_0; a_1); a_1) - \tau(f(\eta_0; a_2); a_1)| \leq \left|\frac{\eta_0}{a_1} - \frac{\eta_0}{a_2}\right| \\ + |\tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_1); a_1)|, \end{aligned} \tag{4.34}$$

where $\left|\frac{\eta_0}{a_1} - \frac{\eta_0}{a_2}\right| \to 0$ as $a_1 \to a_2$ and $|\tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_1); a_1)| \to 0$ as $a_1 \to a_2$, because τ satisfies a Lipschitz condition in a for $\eta = \eta_0$ by Lemma 4.5.3.

[3]Recall that $\tau(f; a) = 1 - \frac{\sigma(f)}{a}$, where $\sigma(f) = \eta$ is defined as the inverse of f and $\sigma(f)$ is continuous on $[0, f_0)$ and continuously differentiable on $(0, f_0)$ for sufficiently small f_0, i.e. for $f_0 = f(\eta; a)$ such that $\eta < a$ is in a deleted left-neighbourhood of a.

Now, we can use the mean value theorem to rewrite the left-hand side of the inequality (4.34), because τ is continuously differentiable on $(0, f_0)$ for some small f_0, because σ is. We get

$$\left|\tau'(f^\star; a_1)\right| = \frac{|\tau(f(\eta_0; a_1); a_1) - \tau(f(\eta_0; a_2); a_1)|}{|f(\eta_0; a_1) - f(\eta_0; a_2)|}$$

for some $f^\star$ between $f(\eta_0; a_1)$ and $f(\eta_0; a_2)$ by the mean value theorem. Therefore, we can rewrite inequality (4.34) to become an inequality on f in the following way

$$\begin{aligned}\left|\tau'(f^\star; a_1)\right||f(\eta_0; a_1) - f(\eta_0; a_2)| \leq &\left|\frac{\eta_0}{a_1} - \frac{\eta_0}{a_2}\right| \\ &+ |\tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_1); a_1)|.\end{aligned} \tag{4.35}$$

Furthermore, looking at the derivative we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}f}\tau(f) &= \frac{\mathrm{d}}{\mathrm{d}f} 2a^{-2} \int_0^f \frac{k(\phi)}{\int_0^\phi [1 - \tau(\psi)] d\psi} d\phi \\ &= 2a^{-2} \frac{\mathrm{d}}{\mathrm{d}f} \int_0^f \frac{k(\phi)}{\phi} \cdot \frac{\phi}{\int_0^\phi [1 - \tau(\psi)] d\psi} d\phi \\ &= 2a^{-2} \frac{k(f)}{f} \cdot \frac{\phi}{\int_0^f [1 - \tau(\psi)] d\psi} \\ &= 2a^{-2} \frac{k(f)}{f} L^{-1}(f; a) \geq 2a^{-2} \frac{k(f)}{f} \cdot 1,\end{aligned}$$

where we have used the fundamental theorem of calculus. Consequently, τ is bounded from above on $(0, f_0)$ with

$$\frac{\mathrm{d}}{\mathrm{d}f}\tau(f) \geq 2a^{-2}\frac{k(f)}{f}.$$

Now returning to inequality (4.35)

$$\begin{aligned}|f(\eta_0; a_1) - f(\eta_0; a_2)| &\leq \left|\tau'(f^\star; a_1)\right|^{-1} \left(\left|\frac{\eta_0}{a_1} - \frac{\eta_0}{a_2}\right| \right. \\ &\quad \left. + |\tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_1); a_1)| \right) \\ &\leq \frac{1}{2} a_1^2 \frac{f^\star}{k(f^\star)} \left(\left|\frac{\eta_0}{a_1} - \frac{\eta_0}{a_2}\right| \right. \\ &\quad \left. + |\tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_1); a_1)| \right) \\ &\leq \frac{1}{2} a_1^2 \frac{b(a_3)}{k(f^\star)} \left(\left|\frac{\eta_0}{a_1} - \frac{\eta_0}{a_2}\right| \right. \\ &\quad \left. + |\tau(f(\eta_0; a_2); a_1) - \tau(f(\eta_0; a_1); a_1)| \right),\end{aligned} \tag{4.36}$$

since f is strictly decreasing and $b(a)$ is strictly increasing. Therefore, we have found an inequality on f.

Observe that $k(f^\star) > 0$, because is $f^\star$ between $f(\eta_0; a_1)$ and $f(\eta_0; a_2)$, so $f^\star > 0$, thus $\frac{1}{k(f^\star)}$ is defined. Additionally, note that $\frac{1}{k(f^\star)}$ is continuous on the closed interval with boundaries

$f(\eta_0; a_1)$ and $f(\eta_0; a_2)$, thus $\frac{1}{2}a_1^2\frac{b(a_3)}{k(f^\star)}$ is bounded.[4] Earlier, we have found that the right hand side of inequality (4.34) tends to 0 as $a_1 \to a_2$. Consequently, the right hand side of inequality (4.36) goes to 0 as $a_1 \to a_2$. Therefore, $a \mapsto f(\eta_0; a)$ is continuous in a.

It remains to be shown that $a \mapsto (k(f)f')(\eta_0; a)$ is continuous in a. Recall the following definition for τ,

$$\tau(f(\eta; a); a) = 1 - \frac{\sigma(f)}{a} = 1 - \frac{\eta}{a}$$

which holds for small f, i.e. for $f(\eta; a)$ where η is in some neighbourhood of a. In particular the definition holds for $f(\eta_0; a)$. Since τ is continuously differentiable for small f, we can write

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\tau(f(\eta; a); a) = \frac{\mathrm{d}}{\mathrm{d}f}\tau(f(\eta; a); a)\frac{\mathrm{d}}{\mathrm{d}\eta}f(\eta; a) = -\frac{1}{a}$$
$$\frac{\mathrm{d}}{\mathrm{d}\eta}f(\eta; a) = \frac{-1}{a\frac{\mathrm{d}}{\mathrm{d}f}\tau(f(\eta; a); a)}$$

for small f.

Now, recall the other definition for τ,

$$\tau(f(\eta; a); a) = \frac{2}{a}\int_0^f \frac{k(\phi)}{\int_0^\phi (1 - \tau(\psi))d\psi}d\phi$$

which holds for small f, thus this definition also holds for $f(\eta_0; a)$. Since τ is continuously differentiable for small f, we can write for small f

$$\frac{\mathrm{d}}{\mathrm{d}f}\tau(f; a) = \frac{2}{a}\frac{k(f)}{\int_0^f (1 - \tau(\psi))d\psi},$$

where we know that $\tau(\eta; a)$ is Lipschitz continuous by Lemma 4.5.3 and $k(f(\eta; a))$ is continuous in a, with $a > 0$. Thus $\tau'(f(\eta_0; a); a)$ must be continuous in a.

But then $f'(\eta_0; a) = \frac{-1}{a\frac{\mathrm{d}}{\mathrm{d}f}\tau(f(\eta_0;a);a)}$ is continuous in a, since it is a composition of continuous functions in a. Thus, we can conclude that $a \mapsto (k(f)f')(\eta_0; a)$ is continuous is a. This finishes the proof of the first claim.

We still need to show that the second claim holds. We need to show that the unique solution $\eta \mapsto f(\eta; a)$ depends continuously on the initial data given at $\eta = \eta_0$. We do this by using a similar argument as we have used in Lemma 4.2.1. Let $f_1(\eta) = f(\eta; a_1)$ and $f_2(\eta) = f(\eta; a_2)$ be two solutions of (4.27) such that they are defined and positive on any deleted left-neighbourhood of $\eta = a_1$, respectively $\eta = a_2$, and such that each of them satisfies their own set of initial conditions

$$f_1(\eta_0) = \alpha_1 \qquad f_2(\eta_0) = \alpha_2$$
$$(k(f_1)f_1')(\eta_0) = \beta_1 \quad (k(f_2)f_2')(\eta_0) = \beta_2.$$

By Lemma 4.2.1 solutions f_1 and f_2 exists and are unique in a neighbourhood of η_0. Now we will look at their difference and deduce an inequality depending on the initial conditions at η_0 to show that the function $\eta \mapsto f(\eta; a)$ depends continuously on the initial data.

Note that by (4.4) from Lemma 4.2.1 we find that f_1 must satisfy

$$k(f_1)f_1'|_{\eta_0}^{\eta} = -\frac{1}{2}\zeta f_1(\zeta)|_{\eta_0}^{\eta} + \frac{1}{2}\int_{\eta_0}^{\eta} f_1(\zeta)d\zeta$$
$$k(f_1(\eta))f_1'(\eta) = \beta_1 - \frac{1}{2}\eta f_1(\eta) + \frac{1}{2}\eta_0\alpha_0 + \frac{1}{2}\int_{\eta_0}^{\eta} f_1(\zeta)d\zeta,$$

[4] Recall that in Theorem 4.5.1, we assumed that $k(s)$, $s \geq 0$, is continuous with $k(0) = 0$ and $k(s) > 0$ for $s > 0$.

in some neighbourhood of η_0. For f_2 we find the same result.

Now we look at the difference of f_1 and f_2 and apply (4.6) from Lemma 4.2.1, giving

$$\begin{aligned}
&\big|k(f_2(\eta))f_2'(\eta) - k(f_1(\eta))f_1'(\eta)\big| \\
&\le |\beta_1 - \beta_2| + \frac{1}{2}|\eta_0||\alpha_1 - \alpha_2| + \frac{1}{2}|\eta||f_1(\eta) - f_2(\eta)| \\
&+ \frac{1}{2}\left|\int_{\eta_0}^{\eta}(f_2(\zeta) - f_1(\zeta))d\zeta\right| \\
&\le |\beta_1 - \beta_2| + \frac{1}{2}|\eta_0||\alpha_1 - \alpha_2| + A_1 \max_{|\zeta-\eta_0|\le|\eta-\eta_0|}|f_2(\zeta) - f_1(\zeta)|
\end{aligned} \tag{4.37}$$

with constant $A_1 = \frac{1}{2}\max\{|\eta_0 - \delta|, |\eta_0 - \delta|, \delta\} > 0$. Now set $F(f) = \int_0^f k(s)ds$, then

$$\begin{aligned}
F(f_2(\eta)) - F(f_1(\eta)) &= \int_0^{f_2(\eta)} k(s)ds - \int_0^{f_1(\eta)} k(s)ds \\
&= \int_0^{\alpha_2} k(s)ds + \int_{\alpha_2}^{f_2(\eta)} k(s)ds - \int_{\alpha_1}^{f_1(\eta)} k(s)ds - \int_0^{\alpha_1} k(s)ds \\
&= \int_0^{\alpha_2} k(s)ds - \int_0^{\alpha_1} k(s)ds + \int_{f_2(\eta_0)}^{f_2(\eta)} k(s)ds - \int_{f_1(\eta_0)}^{f_1(\eta)} k(s)ds \\
&= F(\alpha_2) - F(\alpha_1) + \int_{\eta_0}^{\eta} k(f_2(\zeta))f_2'(\zeta) - k(f_1(\zeta))f_1'(\zeta)d\zeta.
\end{aligned} \tag{4.38}$$

Now taking the absolute value and using the triangle inequality on (4.38) and combining it with (4.37), we find that

$$\begin{aligned}
|F(f_2(\eta)) - F(f_1(\eta))| &\le |F(\alpha_2) - F(\alpha_1)| \\
&+ \int_{\eta_0}^{\eta} \big|k(f_2(\zeta))f_2'(\zeta) - k(f_1(\zeta))f_1'(\zeta)\big|d\zeta \\
&\le |F(\alpha_2) - F(\alpha_1)| \\
&+ \int_{\eta_0}^{\eta} \Big(|\beta_2 - \beta_1| + \frac{1}{2}|\eta_0||\alpha_2 - \alpha_1| \\
&+ A_1 \max_{|\zeta-\eta_0|\le|\eta-\eta_0|}|f_2(\zeta) - f_1(\zeta)|\Big)d\zeta \\
&\le |F(\alpha_1) - F(\alpha_2)| + |\eta - \eta_0||\beta_1 - \beta_2| \\
&+ |\eta - \eta_0|\frac{1}{2}|\eta_0||\alpha_1 - \alpha_2| \\
&+ |\eta - \eta_0|A_1 \max_{|\zeta-\eta_0|\le|\eta-\eta_0|}|f_1(\zeta) - f_2(\zeta)|.
\end{aligned} \tag{4.39}$$

Finally, we can get an inequality on f by combining (4.9) from Lemma 4.2.1 with (4.39).

This results in

$$\begin{aligned}
A_2|f_2(\eta) - f_1(\eta)| &\le |F(\alpha_1) - F(\alpha_2)| + |\eta - \eta_0||\beta_1 - \beta_2| \\
&\quad + |\eta - \eta_0|\frac{1}{2}|\eta_0||\alpha_1 - \alpha_2| \\
&\quad + A_1|\eta - \eta_0| \max_{|\zeta-\eta_0|\le|\eta-\eta_0|} |f_1(\zeta) - f_2(\zeta)| \\
|f_2(\eta) - f_1(\eta)| &\le A_2^{-1}|F(\alpha_1) - F(\alpha_2)| + A_2^{-1}|\eta - \eta_0||\beta_1 - \beta_2| \\
&\quad + A_2^{-1}|\eta - \eta_0|\frac{1}{2}|\eta_0||\alpha_1 - \alpha_2| \\
&\quad + A_2^{-1}A_1|\eta - \eta_0| \max_{|\zeta-\eta_0|\le|\eta-\eta_0|} |f_1(\zeta) - f_2(\zeta)|, \qquad (4.40)
\end{aligned}$$

for some constant $A_2 > 0$ and for all η. Suppose η is sufficiently close to η_0, say $|\eta - \eta_0| \le \epsilon < A_1^{-1}A_2$, then we can estimate the last term of (4.40) by

$$A_2^{-1}A_1|\eta - \eta_0| \max_{|\zeta-\eta_0|<|\eta-\eta_0|} |f_1(\zeta) - f_2(\zeta)| \le A_2^{-1}A_1\epsilon \max_{|\zeta-\eta_0|<\epsilon} |f_1(\zeta) - f_2(\zeta)|.$$

Since (4.40) holds for all η, we can take the maximum on both sides. This results in

$$\begin{aligned}
\max_{|\zeta-\eta_0|<\epsilon} |f_1(\zeta) - f_2(\zeta)| &\le A_2^{-1}|F(\alpha_1) - F(\alpha_2)| + A_2^{-1}\epsilon|\beta_1 - \beta_2| \\
&\quad + A_2^{-1}\epsilon\frac{|\eta_0|}{2}|\alpha_1 - \alpha_2| \\
&\quad + A_2^{-1}A_1\epsilon \max_{|\zeta-\eta_0|<\epsilon} |f_1(\zeta) - f_2(\zeta)| \\
(1 - A_2^{-1}A_1\epsilon) \max_{|\zeta-\eta_0|<\epsilon} |f_1(\zeta) - f_2(\zeta)| &\le A_2^{-1}|F(\alpha_1) - F(\alpha_2)| + A_2^{-1}\epsilon|\beta_1 - \beta_2| \\
&\quad + A_2^{-1}\epsilon\frac{|\eta_0|}{2}|\alpha_1 - \alpha_2|. \qquad (4.41)
\end{aligned}$$

Note that $1 - A_2^{-1}A_1\epsilon > 0$ and $|F(\alpha_1) - F(\alpha_2)|$ tends to 0 as $\alpha_1 \to \alpha_2$, since F is a continuous function. Then the right hand side of (4.41) tends to 0 as $\alpha_1 \to \alpha_2$ and $\beta_1 \to \beta_2$. Thus, $f_2(\eta) - f_1(\eta)$ vanishes in a small neighbourhood of η_0. Since we have chosen η_0 arbitrarily, we can conclude that $\eta \mapsto f(\eta; a)$ depends continuously on the initial conditions for all η. □

Lemma 4.5.3. *If $f \in (0, f_0)$ with $f_0 = f(\eta_0; a)$ such that $\eta_0 < a$ is in a deleted left-neighbourhood of a, then $a \mapsto \tau(f; a)$ satisfies a Lipschitz condition in a.*

Proof. Let f be small as specified in Lemma 4.5.3. Let $0 < a_0 \le a_1 < a_2 \le a_3$, where we choose a_3 such that $b(a_3) < \infty$.

In order to show that $a \mapsto \tau(f; a)$ satisfies a Lipschitz condition in a, we will start by looking at the difference of $\tau(f; a)$ for different values of a, then we rewrite the inequality by introducing functions $L(f)$ and $w(f)$ and $M(a)$ to transform it into the correct form such that we can apply Gronwall's inequality (see Theorem 3.2.5). Finally, we will apply Grohnwall's inequality and deduce that $\tau(f)$ satisfies a Lipschitz condition in a.

Recall that by Lemma 4.3.3, $\tau(f; a)$ is a unique solution in f of (4.20), $\tau(f; a) > 0$ for $f \in (0, f_0)$, and $\tau(0; a) = 0$. Furthermore, recall that we consider the supremum norm if we consider $\tau(f;a)$. By using definition (4.20) for $\tau(f;a)$ we find

$$
\begin{aligned}
&|\tau(f;a_1)-\tau(f;a_2)| \\
&= \left| 2a_1^{-2}\int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi} d\phi - 2a_2^{-2}\int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_2)]d\psi} d\phi \right| \\
&= \left| 2(a_1^{-2}-a_2^{-2})\int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi} d\phi \right. \\
&\left. + 2a_2^{-2}\left(\int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi} d\phi - \int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_2)]d\psi} d\phi\right) \right| \\
&= \left| 2(a_1^{-2}-a_2^{-2})\int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi} d\phi \right. \\
&\left. + 2a_2^{-2}\left(\int_0^f \frac{k(\phi)\int_0^\phi [1-\tau(\psi;a_2)]d\psi - k(\phi)\int_0^\phi [1-\tau(\psi;a_1)]d\psi}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi \int_0^\phi [1-\tau(\psi;a_2)]d\psi} d\phi\right) \right| \\
&= \left| 2(a_1^{-2}-a_2^{-2})\int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi} d\phi \right. \\
&\left. + 2a_2^{-2}\int_0^f \frac{k(\phi)\int_0^\phi [\tau(\psi;a_1)-\tau(\psi;a_2)]d\psi}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi \int_0^\phi [1-\tau(\psi;a_2)]d\psi} d\phi \right| \\
&\leq 2(a_1^{-2}-a_2^{-2})\int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi} d\phi \\
&+ 2a_2^{-2}\int_0^f \frac{k(\phi)\int_0^\phi |\tau(\psi;a_1)-\tau(\psi;a_2)|d\psi}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi \int_0^\phi [1-\tau(\psi;a_2)]d\psi} d\phi
\end{aligned}
$$

by the triangle and integral inequality and because $k(s) \geq 0$ and $1-\tau(f) \geq 0$.

Now we have found the following inequality,

$$
\begin{aligned}
&|\tau(f;a_1)-\tau(f;a_2)| \\
&\leq 2(a_1^{-2}-a_2^{-2})\int_0^f \frac{k(\phi)}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi} d\phi \\
&+ 2a_2^{-2}\int_0^f \frac{k(\phi)\int_0^\phi |\tau(\psi;a_1)-\tau(\psi;a_2)|d\psi}{\int_0^\phi [1-\tau(\psi;a_1)]d\psi \int_0^\phi [1-\tau(\psi;a_2)]d\psi} d\phi,
\end{aligned}
\tag{4.42}
$$

but it is not yet in the correct form to apply Gronwall's inequality. To aid us with this, we introduce the continuous function

$$
L(\phi;a) = \phi^{-1}\int_0^\phi [1-\tau(\psi;a)]d\psi, \quad 0 < \phi \leq b(a).
$$

We claim that L is a strictly monotonically decreasing function of ϕ and $L \to 1$ as $\phi \to 0$.

In order to show that L is strictly monotonically decreasing, we need the derivative of L to be strictly less than 0. For this reason, let us investigate the derivative of L. We find

$$
\begin{aligned}
\frac{\partial}{\partial\phi}L(\phi;a) &= \frac{\phi\frac{\partial}{\partial\phi}\int_0^\phi [1-\tau(\psi;a)]d\psi - \int_0^\phi [1-\tau(\psi;a)]d\psi}{\phi^2} \\
&= \frac{(1-\tau(\phi,a))\phi - \int_0^\phi [1-\tau(\psi;a)]d\psi}{\phi^2},
\end{aligned}
$$

where we have used the fundamental theorem of calculus. We know that $\phi > 0$, so $\phi^2 > 0$. Furthermore, since $\eta \mapsto f(\eta; a)$ is a strictly decreasing function by Lemma 4.2.2, we know that $\sigma(f) = \eta$, where $\sigma(f)$ is the inverse of $f(\eta; a)$ and is defined for small f, is strictly decreasing by the inverse function theorem, thus $\tau(f; a) = 1 - \frac{\sigma(f)}{a}$ is strictly increasing and $1 - \tau(f; a)$ is strictly decreasing. Then by the definition of the lower sum of the Riemann integral, we find

$$\int_0^\phi [1 - \tau(\psi; a)] d\psi > \inf_{\psi \in [0,\phi]} (1 - \tau(\psi; a))(\phi - 0) = (1 - \tau(\phi))\phi,$$

so the numerator of the derivative of L is non-positive. Therefore, $\frac{\mathrm{d}}{\mathrm{d}\phi} L(\phi; a) < 0$, thus L is strictly monotonically decreasing. Now we show that $L \to 1$ as $\phi \to 0$, as follows

$$\begin{aligned}
\lim_{\phi \to 0} L(\phi; a) &= \lim_{\phi \to 0} \frac{\int_0^\phi [1 - \tau(\psi; a)] d\psi}{\phi} \\
&= \lim_{\phi \to 0} \frac{\int_0^\phi [1 - \tau(\psi; a)] d\psi - \int_0^0 [1 - \tau(\psi; a)] d\psi}{\phi - 0} \\
&= \lim_{\phi \to 0} \frac{\int_0^\phi [1 - \tau(\psi; a)] d\psi - \int_0^0 [1 - \tau(\psi; a)] d\psi}{\phi - 0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\phi} \left(\int_0^\phi [1 - \tau(\psi; a)] d\psi \right) \Bigg|_{\phi = 0} \\
&= 1 - \tau(\phi; a)|_{\phi=0} = 1 - 0 = 1,
\end{aligned}$$

where we have used the definition of the derivative of function $\phi \mapsto \int_0^\phi [1 - \tau(\psi; a)] d\psi$ and the fundamental theorem of calculus. Thus $L \to 1$ as $\phi \to 0$. Since L is strictly monotonically decreasing and $L \to 1$ as $\phi \to 0$, we have that $L(b(a); a) \leq L(\phi; a) \leq 1$. Lastly, note that $L > 0$, so $|L| = L$.

Now, we can begin to rewrite (4.42) such that we can apply Gronwall's inequality to it. First, we rewrite the first term of (4.42)

$$\begin{aligned}
&2(a_1^{-2} - a_2^{-2}) \int_0^f \frac{k(\phi)}{\int_0^\phi [1 - \tau(\psi; a_1)] d\psi} d\phi \\
&= 2 \left(\frac{a_2^2}{a_1^2 a_2^2} - \frac{a_1^2}{a_1^2 a_2^2} \right) \int_0^f \frac{k(\phi)}{\phi} \cdot \frac{\phi}{\int_0^\phi [1 - \tau(\psi; a_1)] d\psi} d\phi \\
&= 2 \left(\frac{a_2^2 - a_1^2}{a_1^2 a_2^2} \right) \int_0^f \frac{k(\phi)}{\phi} \frac{1}{L(\phi; a_1)} d\phi \\
&= 2 \left(\frac{a_2^2 - a_1^2}{a_1^2 a_2^2} \right) \int_0^f \frac{k(\phi)}{\phi} |L(\phi; a_1)|^{-1} d\phi \\
&\leq 2 \frac{(a_2 + a_1)(a_2 - a_1)}{a_1^2 a_2^2} \max_{\phi \in [0,f]} \left(|L(\phi; a_1)|^{-1} \int_0^f \frac{k(\phi)}{\phi} d\phi \right) \\
&= \left(2 \frac{(a_2 + a_1)}{a_1^2 a_2^2} |L(b(a_1); a_1)|^{-1} \int_0^{b(a_1)} \frac{k(\phi)}{\phi} d\phi \right) (a_2 - a_1) \\
&= A(a_2 - a_1),
\end{aligned}$$

where we have defined A as a function of $f(\eta; a)$ such that

$$A = 2 \frac{(a_2 + a_1)}{a_1^2 a_2^2} \left| L^{-1}(b(a_1); a_1) \right| \int_0^{b(a_1)} \frac{k(\phi)}{\phi} d\phi.$$

Secondly, we rewrite the second term of (4.42)

$$
\begin{aligned}
&2a_2^{-2}\int_0^f \frac{k(\phi)\int_0^\phi |\tau(\psi;a_1)-\tau(\psi;a_2)|d\psi}{\int_0^\phi[1-\tau(\psi;a_1)]d\psi\int_0^\phi[1-\tau(\psi;a_2)]d\psi}d\phi \\
&= 2a_2^{-2}\int_0^f\left(\frac{k(\phi)}{\phi^2}\cdot\int_0^\phi |\tau(\psi;a_1)-\tau(\psi;a_2)|d\psi\cdot\frac{\phi}{\int_0^\phi[1-\tau(\psi;a_1)]d\psi}\cdot\frac{\phi}{\int_0^\phi[1-\tau(\psi;a_2)]d\psi}\right)d\phi \\
&= 2a_2^{-2}\int_0^f\left(\frac{k(\phi)}{\phi^2}\cdot\int_0^\phi |\tau(\psi;a_1)-\tau(\psi;a_2)|d\psi\cdot L^{-1}(\phi;a_1)\cdot L^{-1}(\phi;a_2)\right)d\phi \\
&= 2a_2^{-2}\int_0^f\left(\frac{k(\phi)}{\phi^2}\cdot\int_0^\phi |\tau(\psi;a_1)-\tau(\psi;a_2)|d\psi\cdot \left|L^{-1}(\phi;a_1)\right|\cdot \left|L^{-1}(\phi;a_2)\right|\right)d\phi \\
&\le 2a_2^{-2}\max_{\phi\in[0,f]}\left|L^{-1}(\phi;a_1)\right|\cdot\max_{\phi\in[0,f]}\left|L^{-1}(\phi;a_2)\right| \\
&\cdot\int_0^f\left(\frac{k(\phi)}{\phi^2}\cdot\max_{\psi\in[0,\phi]}|\tau(\psi;a_1)-\tau(\psi;a_2)|\cdot\left(\int_0^\phi 1d\psi\right)\right)d\phi \\
&= 2a_2^{-2}L^{-1}(b(a_1);a_1)L^{-1}(b(a_2);a_2)\int_0^f\frac{k(\phi)}{\phi}\max_{\psi\in[0,\phi]}|\tau(\psi;a_1)-\tau(\psi;a_2)|d\phi \\
&= B\int_0^f\frac{k(\phi)}{\phi}\max_{\psi\in[0,\phi]}|\tau(\psi;a_1)-\tau(\psi;a_2)|d\phi,
\end{aligned}
$$

where we have defined B as a function of $f(\eta;a)$ such that

$$B = 2a_2^{-2}L^{-1}(b(a_1);a_1)L^{-1}(b(a_2);a_2).$$

Up to this point, rewriting (4.42) has given us

$$
\begin{aligned}
|\tau(f;a_1)-\tau(f;a_2)| \le{}& A(a_2-a_1) \\
&+ B\int_0^f\frac{k(\phi)}{\phi}\max_{\psi\in[0,\phi]}|\tau(\psi;a_1)-\tau(\psi;a_2)|d\phi.
\end{aligned} \tag{4.43}
$$

Now, we introduce another function $w(f)$ and continue the rewriting process with inequality (4.43). We define

$$w(f) = \max_{\psi\in[0,f]}|\tau(\psi;a_1)-\tau(\psi;a_2)|$$

and substitute it into (4.43) after taking the maximum over ψ on both sides of the inequality. We find

$$
\begin{aligned}
w(f) &= \max_{\psi\in[0,f]}|\tau(\psi;a_1)-\tau(\psi;a_2)| \\
&\le \max_{\psi\in[0,f]}\Big(A(a_2-a_1) \\
&+ B\int_0^\psi\frac{k(\phi)}{\phi}\max_{\chi\in[0,\phi]}|\tau(\chi;a_1)-\tau(\chi;a_2)|d\phi\Big) \\
&\le A(a_2-a_1)+B\int_0^f\frac{k(\phi)}{\phi}w(\phi)d\phi,
\end{aligned} \tag{4.44}
$$

since the integrand is non-negative for all $\phi\in[0,f]$.

We can almost apply Gronwall's inequality to (4.44), but first we need to check whether A and B are bounded functions such that we can conclude that $A(a_2 - a_1) \in \mathbb{R}$ and $B\frac{k(f)}{f} \geq 0$. To aid us in checking this, we define the function

$$M(a) = L\{b(a); a\}.$$

It can be rewritten as

$$\begin{aligned}
M(a) &= L\{b(a); a\} \\
&= \frac{1}{b(a)} \int_0^{b(a)} [1 - \tau(f; a)]df \\
&= \frac{1}{b(a)} \int_0^{b(a)} [1 - \tau(f; a)]df \\
&= \frac{1}{b} \int_0^{b(a)} [1 - (1 - \frac{\sigma(f)}{a})]df \\
&= \frac{1}{ab} \int_0^{b(a)} \sigma(f)df \\
&= \frac{1}{ab} \int_0^{b(a)} \eta df \\
&= (ab)^{-1} \int_0^{a} f(\eta; a)d\eta
\end{aligned}$$

by using the definition of $\tau(f) = 1 - \frac{\sigma(f)}{a}$ and $\sigma(f) = \eta$ from Lemma 4.3.2 and substituting $\eta = f(\eta)$ in the integral.

In the proof of *(ii)* it was already shown that, since $a_i \geq a_0$, $f(\eta; a_i) \geq f(\eta; a_0)$ for $\eta \in [0, a_0)$ and for $i \in \{1, 2\}$. Additionally, by assumption, we also know that $f(\eta; a_i) > 0$ on $[0, a_i)$, so $f(\eta; a_i) > 0$ as well on $[a_0, a_i)$ for $i \in \{1, 2\}$. Therefore, we have

$$M(a_i) = \{a_i b(a_i)\}^{-1} \int_0^{a_i} f(\eta; a_i)d\eta \geq \{a_i b(a_i)\}^{-1} \int_0^{a_0} f(\eta; a_0)d\eta$$

Moreover, $a_i < a_3$ and hence, since $b(a)$ is strictly monotonically increasing by *(ii)*, $b(a_i) \leq b(a_3)$. Therefore,

$$M(a_i) \geq \{a_3 b(a_3)\}^{-1} \int_0^{a_0} f(\eta; a_0)d\eta \quad \text{for } i \in \{1, 2\}.$$

From this, we can conclude that $\frac{1}{M(a_i)} = \frac{1}{L(b(a_i); a_1)} = |L(b(a_i); a_i)|^{-1}$ is bounded for $i \in \{1, 2\}$, so B is bounded. Furthermore, note that $b(a)$ is increasing and we chose $b(a_3) < \infty$, thus $b(a_2) < \infty$. Therefore, A is bounded by assumption (4.26). Now, we can conclude that functions A and B are uniformly bounded functions on $[a_0, a_3]$.

Thus we are finally allowed to apply Gronwall's inequality to (4.44), note that we satisfy the additional condition for (3.9) resulting in

$$w(f) \leq A(a_2 - a_1) \exp\biggl(\int_0^f B\frac{k(\phi)}{\phi}d\phi\biggr).$$

Rewriting that inequality and using that $|\tau(f; a_1) - \tau(f; a_2)| \leq w(f)$, we find

$$|\tau(f; a_1) - \tau(f; a_2)| \leq |a_2 - a_1| A \exp\biggl(\int_0^f B\frac{k(\phi)}{\phi}d\phi\biggr).$$

Combining this with the fact that we let $0 < a_0 \leq a_1 < a_2 \leq a_3$ at the start, we get $f(\eta, a_i) \leq b(a_3)$ for $i \in \{1, 2\}$, because f is strictly monotonically decreasing. In conclusion, we find that $\tau(f)$ satisfies a Lipschitz condition in a which is uniform with respect to $f \in [0, b(a_3))$ and $a \in [a_0, a_3]$. $\square$

Since $f(\eta; a)$ satisfies the conditions of Lemma 4.5.2, we can conclude that $a \mapsto f(\eta; a)$ is continuous in a for all η as explained before.

It remains to be shown that $b(a)$ is continuous in a. Suppose $b(a)$ is not continuous in a. Then there exists an $a^\star \in (0, a)$ such that $b(a^\star) \neq \lim_{a \to a^\star} b(a)$. This means that $f(\eta_0; a^\star) = \lim_{a \uparrow a^\star} f(\eta_0; a)$ at some point $\eta_0 > 0$ in a neighbourhood of 0 and $f(\eta; a^\star) \neq \lim_{a \uparrow a^\star} f(\eta_0; a)$ for $\eta < \eta_0$ (including $\eta = 0$). However, by Lemma (4.5.2) f depends continuously on a for all η. Thus, if the limit and function are the same at η_0 they must be the same in a left-neighbourhood of η_0 as well. Hence, we have arrived at a contradiction. Therefore, we can finally conclude that $b(a)$ is continuous in a.

(iv) *Proof of in Case A we have* $\lim_{a \to \infty} b(a) = 0$. Recall that we defined Case A by the condition

$$\int_1^\infty \frac{k(s)}{s} ds = \infty.$$

Furthermore, we assumed that $k(s)$ is continuous on $[0, \infty)$ and that inequality (4.26) holds, which says

$$\int_0^1 \frac{k(s)}{s} < \infty.$$

Therefore, we obtain that

$$\int_0^\infty \frac{k(s)}{s} = \infty. \tag{4.45}$$

Now consider equation (4.25) from the proof of Lemma 4.4.2. By Lemma 4.4.1 we can continue the solution $f(\eta; a)$ back to $\eta = 0$ for any $a > 0$. Hence, we obtain

$$\lim_{\eta \to 0} \int_0^{f(\eta)} \frac{k(s)}{s} ds \geq \lim_{\eta \to 0} \frac{1}{4}(a^2 - \eta^2)$$

$$\int_0^{b(a)} \frac{k(s)}{s} ds \geq \frac{1}{4} a^2. \tag{4.46}$$

Comparing integral (4.45) with integral (4.46), gives that $b(a)$ must tend to infinity if a tends to infinity. Thus, we have proved that $\lim_{a \to \infty} b(a) = \infty$.

(v) *Proof of in Case B we have* $\lim_{a \to \bar{a}} b(a) = 0$ *for some* $\bar{a} < \infty$. Recall that we have defined Case B by the condition $\int_1^\infty \frac{k(s)}{s} ds < \infty$. Set

$$\int_1^\infty \frac{k(s)}{s} ds = K.$$

In order to show that $\lim_{a \to \bar{a}} b(a) = 0$, we first single out an $\bar{a}$ in $[0, \infty)$ and then we prove that $b(a) = 0$ as $a \to \bar{a}$.

First, we choose a specific $\bar{a}$. To this end, we divide $[0, \infty)$ into two sets A_1 and A_2, where $\bar{a}$ will be the defining number that distinguishes the two sets. Let A_1 be the set of $a \in [0, \infty)$ such that $f(\eta; a)$ can be continued back to $\eta = 0$ i.e. $b(a) < \infty$, thus

$$A_1 = \{a \in [0, \infty) : b(a) < \infty\}.$$

Note that A_1 is an interval, because $b(a)$ is continuous and we can write $A_1 = \{a \in [0,\infty) : b(a) < \infty\} = \bigcup\{[0,a] : b(a) < \infty\}$. Denote by A_2 the complement of A_1, thus

$$A_2 = [0,\infty) \setminus A_1.$$

Then A_2 is also an interval and $A_1 \cup A_2 = [0,\infty)$. Since A_1 and A_2 are intervals and $A_1 \cup A_2 = [0,\infty)$, we find that either A_1 is of the form $[0,a_1)$ and A_2 is of the form $[a_1,\infty)$ for some $a_1 \in [0,\infty)$ or A_1 is of the form $[0,a_2]$ and A_2 is of the form (a_2,∞) for some $a_2 \in [0,\infty)$.

We will show that the interval $[0,\sqrt{2K})$ is a subset of A_1 and that the interval $[2\sqrt{K},\infty)$ is a subset of A_2, then we use the intermediate value theorem on the remaining interval to find the specific $\bar{a}$.

Consider inequality (4.15) from Lemma 4.3.1, if we let $a_1 \to 0$ we get $\int_{f(a_2)}^{b(a)} \frac{k(s)}{s} ds \le \frac{1}{2}a^2$. If $a \in [0,(2K)^{\frac{1}{2}})$, then $\int_{f(a_2)}^{b(a)} \frac{k(s)}{s} ds < K$. Combining this with the fact that in Case B we have $\int_1^\infty \frac{k(s)}{s} ds = K$ and $\frac{k(s)}{s}$ is non-negative on $[0,\infty)$, gives that $\int_{f(a_2)}^{b(a)} \frac{k(s)}{s} ds < K = \int_1^\infty \frac{k(s)}{s} ds$, thus $b(a) < \infty$ and we can conclude that $a \in A_1$.

In order to show that $[2\sqrt{K},\infty)$ is a subset of A_2, we will prove the contrapositive. Therefore, assume that a is not an element of A_2 i.e. $a \in A_1$. We need to show that $a < 2\sqrt{K}$. If $a \in A_1$, then $b(a) < \infty$ and we can estimate (4.46) by

$$\frac{1}{4}a^2 \le \int_0^{b(a)} \frac{k(s)}{s} ds < \int_0^\infty \frac{k(s)}{s} ds = K.$$

Rewriting this to $a < 2\sqrt{K}$ gives the inequality we were looking for. Thus we can conclude that $[2\sqrt{K},\infty) \subset A_2$.

Now we can use the intermediate value theorem to find $\bar{a}$. From (ii) we know that $b(a)$ is monotonically increasing and from (iii) we know $b(a)$ is continuous, so by the intermediate value theorem there must exist a number $\bar{a} \in [\sqrt{2K}, 2\sqrt{K}]$ such that if $a < \bar{a}$, then $a \in A_1$ and if $a > \bar{a}$, then $a \in A_2$.

Now that we have found a specific $\bar{a}$, we have arrived at the second part of the proof. Thus, we need to prove that $\lim_{a\to\bar{a}} b(a) = \infty$. We know that $b(a) \to \infty$ if $a \to \bar{a}$ from above, since $a \in A_2$ if a approaches $\bar{a}$ from above. Hence, it is enough to show that if $a \uparrow \bar{a}$ from below, then $b(a) \to \infty$.

Assume the contrary, so assume that $b(a)$ is bounded as $a \uparrow \bar{a}$ from below. Then there exists a number $b_{\max}$ such that if $a < \bar{a}$ then $b(a) < b_{\max}$. Consider the two solutions $f(\eta;\bar{a}-\delta)$ and $f(\eta;\bar{a}+\delta)$, where $\delta > 0$ is chosen arbitrarily small. From what we have shown earlier, we know that $\bar{a}-\delta \in A_1$ and $\bar{a}+\delta \in A_2$. Then $f(0;\bar{a}-\delta) = b(\bar{a}-\delta) < b_{\max}$ and since f is continuous, there exists some $\eta_0 \ge 0$ in a neighbourhood of 0 such that $f(0;\bar{a}+\delta) \to \infty$ if $\eta \to \eta_0$.

Let $\eta_0 < \eta_1$, then for $\eta \ge \eta_1$ we have by the first result of Lemma 4.5.2 that $a \mapsto f(\eta;a)$ and $a \mapsto (k(f)f')(\eta;a)$ are continuous for all $a > 0$, thus also for $a = \bar{a}$. If we take initial conditions at η_1, we get an initial value problem on $[0,\eta_1]$ of which $f(\eta;\bar{a}-\delta)$ and $f(\eta;\bar{a}+\delta)$ are solutions. Then by the second result of the lemma, for $\eta \in [0,\eta_1]$, $\eta \mapsto f(\eta;a)$ depends continuously on the initial conditions given at η_1. All in all, $f(\eta;a)$ depends continuously on a for all $\eta \in [0,\eta_1]$. In particular, for δ sufficiently small $f(\eta;\bar{a}+\delta)$ can be continued back to $\eta = 0$, implying that $\bar{a}+\delta \in A_1$. Hence, we have arrived at a contradiction, since we had assumed that $\bar{a}+\delta \in A_2$. Thus, we can conclude that $\lim_{a\uparrow\bar{a}} b(a) = \infty$. We already knew that $b(a) \to \infty$ as a approaches $\bar{a}$ from above, so we have deduced that in Case B, $\lim_{a\to\bar{a}} b(a) = \infty$.

The last part of the theorem claims that in Case A, $a = a(U) \to \infty$ as $U \to \infty$ and in Case B, $a(U)$ tends to a finite limit as $U \to \infty$. Suppose that $\lim_{b(a)=U\to\infty} a < \infty$ in Case A, thus if we increase $b(a)$, a stays bounded, but in *(iv)* we proved that if we let $a \to \infty$ then $b(a) \to \infty$. Hence, we have arrived at a contradiction, since if $a \to \infty$, then $b(a) \to \infty$, but then a stays bounded by assumption. Therefore the assumption was wrong and we must have $\lim_{b(a)=U\to\infty} = \infty$. The result for Case B follows similarly from (v). □

Now we take the classical solutions from Theorem 4.5.1 and we combine them with the zero function to create non-negative weak solutions on $[0,\infty)$ that have compact support. Recall that we defined weak solutions in definition 4.1.1 as a function $f(\eta)$ with the following properties:

1. f is bounded, continuous and non-negative on $[0,\infty)$;
2. $F(f) = \int_0^f k(s)ds$ has a continuous derivative F' with respect to η;
3. f satisfies the identity $\int_0^\infty \phi'(F' + \frac{1}{2}\eta f)d\eta + \frac{1}{2}\int_0^\infty \phi f d\eta = 0$ for all $\phi \in C_0^1(0,\infty)$. Note that the subscript in $C_0^1(0,\infty)$ means that ϕ vanishes at ∞.

Theorem 4.5.4. *Let $k(s)$ satisfy the conditions of Theorem 4.5.1, and let U be any positive number. Then the condition*

$$\int_0^1 \frac{k(s)}{s} ds < \infty$$

is necessary and sufficient for the differential equation

$$(k(f)f')' + \frac{1}{2}\eta f' = 0 \quad 0 < \eta < \infty \tag{4.47}$$

to have a unique non-negative weak solution $f(\eta)$ on $[0,\infty)$ with compact support, such that $f(0) = U$.

Proof. We will first show that the function

$$f(\eta) = \begin{cases} f(\eta; a(U)) & \text{for } 0 \le \eta < a(U) \\ 0 & \text{for } a(U) \le \eta < \infty \end{cases} \tag{4.48}$$

is a weak solution with the desired properties, where $f(\eta; a(U))$ is the solution on $[0,a)$ of the problem as described in Theorem 1. Then we will introduce Lemma 4.5.5 to help us show that (4.48) is a unique solution.

Since $f(\eta; a(U))$ is strictly monotonically decreasing on $[0, a(U))$ and $0 \le f(\eta; a(U)) \le b(a)$, $f(\eta)$ is bounded on $[0,\infty)$. Moreover, $f(\eta)$ is continuous on $[0, a(U))$ and $(a(U),\infty)$, because $f(\eta; a(U))$ and 0 are continuous functions on their own domains. Furthermore, since $\lim_{\eta\to a(U)+} f(\eta) = 0 = \lim_{\eta\to a(U)-} f(\eta)$, $f(\eta)$ is also continuous at $a(U)$, therefore $f(\eta)$ is continuous on $[0,\infty)$. Furthermore, $f(\eta)$ is non-negative on $[0,\infty)$, because $f(\eta) = f(\eta; a(U)) \ge 0$ on $[0, a(U))$ and $f(\eta) = 0$ on $(a(U),\infty)$. Thus part 1 of the definition of a weak solution holds for $f(\eta)$.

The second part of the definition of a weak solutions calls for the derivative of F, note that

$$\frac{\mathrm{d}}{\mathrm{d}\eta}F = \frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^f k(s)ds = \frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^\eta \left(k(f(\tilde\eta))\frac{\mathrm{d}}{\mathrm{d}\tilde\eta}f\right)d\tilde\eta.$$

By the fundamental theorem of calculus we get

$$\frac{\mathrm{d}}{\mathrm{d}\eta}F = k(f(\eta))f'(\eta).$$

For $0 \le \eta < a(U)$, $f(\eta) = f(\eta; a(U))$ is a solution of the differential equation where $f(\eta; a(U))$ is continuously differentiable on $[0,a)$. For $a(U) \le \eta < \infty$, $f(\eta) = 0$ is a constant function. Thus, the derivative $\frac{\mathrm{d}}{\mathrm{d}\eta}F$ exists for all $\eta \in [0, \infty)$ and it is continuous. Thereby, we have proven that part 2 of the definition of a weak solution holds for $f(\eta)$.

In order to show that the third part of the definition of a weak solution holds, observe that for $0 \le \eta < a(U)$ by using integration by parts, we get

$$\begin{aligned}
&\int_0^\infty \phi'(F' + \frac{1}{2}\eta f)d\eta + \frac{1}{2}\int_0^\infty \phi f d\eta \\
&= \int_0^\infty \phi'(k(f(\eta))f'(\eta) + \frac{1}{2}\eta f)\eta))d\eta + \frac{1}{2}\int_0^\infty \phi f(\eta)d\eta \\
&= \left[\phi\left(k(f(\eta))f'(\eta) + \frac{1}{2}\eta f\right)\right]_0^\infty \\
&- \int_0^\infty \phi\left((k(f(\eta))f')' + \frac{1}{2}f + \frac{1}{2}\eta f' - \frac{1}{2}f\right)d\eta \\
&= -\int_0^\infty \phi\left((k(f(\eta))f')' + \frac{1}{2}\eta f'\right)d\eta \\
&= -\int_0^{a(U)} \phi\left((k(f(\eta))f')' + \frac{1}{2}\eta f'\right)d\eta \qquad (4.49)\\
&- \int_{a(U)}^\infty \phi\left((k(f(\eta))f')' + \frac{1}{2}\eta f'\right)d\eta \qquad (4.50)\\
&= 0
\end{aligned}$$

for $\phi \in C_0^1(0, \infty)$. Integral (4.49) vanishes, because $f(\eta) = f(\eta; a(U))$ on $0 \le \eta < a(U)$ and $f(\eta; a(U))$ satisfies the differential equation 4.47. Integral (4.50) vanishes, because $f(\eta) = 0$ on $a(U) \le \eta < \infty$. Now it remains to be shown that this solution is unique. We need the following lemma.

Lemma 4.5.5. *Let $u(x)$ and $v(x)$ be defined and continuous on $[0, \infty)$ and let*

$$\int_0^\infty (\phi' u + \phi v)dx = 0 \qquad (4.51)$$

for all $\phi(x) \in C_0^1[0, \infty)$. Then (i) $u \in C^1[0, \infty)$ and (ii) $u' = v$ on $[0, \infty)$.

Proof. If we set $w(x) = \int_0^x v(s)ds$ and integrate the second term in (4.51) by parts, we obtain

$$\begin{aligned}
0 = \int_0^\infty (\phi' u + \phi v)dx &= \int_0^\infty \phi' u dx + \int_0^\infty \phi v dx \\
&= \int_0^\infty \phi' u dx + [\phi w(x)]_0^\infty - \int_0^\infty \phi' w(x) dx \\
&= \int_0^\infty \phi'(u - w)dx + \lim_{x\to\infty} \phi(x)w(x) - \phi(0)w(0) \\
&= \int_0^\infty \phi'(u - w)dx. \qquad (4.52)
\end{aligned}$$

Consequently, $\int_0^\infty \phi'(u - w)dx = 0$ for all $\phi \in C_0^1[0, \infty)$. We want to deduce that $u - w =$ constant, because then it is easy to show (i) and (ii) follow.

Choose $\phi \in C_0^1[0, \infty)$ such that $\phi(0) = 0$ and suppose that $(u - w) \in C^2$.[5] Then we can

[5]Note that general case, $(u - w) \in C^\infty$, follows from a mollification argument.

apply integration by parts to (4.52), giving

$$0 = \int_0^\infty \phi'(u-w)dx = [\phi(u-w)]_0^\infty - \int_0^\infty \phi(u-w)'dx$$
$$\int_0^\infty \phi(u-w)'dx = 0. \tag{4.53}$$

Substituting $\phi = (u-w)'\chi$ into (4.53), where $\chi \in C_0^\infty((0,\infty))$, $\mathrm{supp}\,(\chi) \subset (0,\infty)$ and $\chi \geq 0$, results in

$$0 = \int_0^\infty \chi(u-w)'(u-w)'dx = \int_0^\infty \chi((u-w)')^2 dx = \left\|(u-w)'\right\|^2$$

with the $L^2(0,\infty)$ norm. Hence, we found that $(u-w)' = 0$, thus $u - w =$ constant. It follows that $u = w(x) + c = \int_0^x v(s)ds + c$, then $u \in C^1[0,\infty)$ and by the fundamental theorem of calculus $u' = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x v(s)ds + \frac{\mathrm{d}}{\mathrm{d}x}c = v(x)$ on $[0,\infty)$. □

If we apply Lemma 4.5.5 to weak solutions of the differential equation (4.47), the third part of the definition of weak solutions gives

i $F' + \frac{1}{2}\eta f \in C^1[0,\infty)$,

ii $(F' + \frac{1}{2}\eta f)' = \frac{1}{2}f$.

Let $f > 0$ at some point $\eta = \eta_0$.

Then
$$\frac{\mathrm{d}}{\mathrm{d}f}F = \frac{\mathrm{d}}{\mathrm{d}f}\int_0^f k(s)ds = k(f)$$
is positive and continuous at $f(\eta_0)$ by definition of $k(s)$.

By the definition of weak solutions, F is continuously differentiable with respect to η for $\eta \geq 0$. In particular, F is differentiable in some neighbourhood N of η_0, say $N = (\eta_0 - \delta, \eta_0 + \delta)$ with $\delta > 0$. By the mean value theorem, there exists a $y \in N$ such that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\eta}F\Big|_y &= \frac{F(\eta_0+\delta) - F(\eta_0-\delta)}{\eta_0+\delta-(\eta_0-\delta)} \\
\frac{\mathrm{d}}{\mathrm{d}f}F \cdot \frac{\mathrm{d}}{\mathrm{d}\eta}f\Big|_y &= \frac{F(\eta_0+\delta) - F(\eta_0-\delta)}{2\delta} \\
\frac{\mathrm{d}}{\mathrm{d}\eta}f\Big|_y &= \frac{1}{k(f(y))}\frac{F(\eta_0+\delta) - F(\eta_0-\delta)}{2\delta}.
\end{aligned}$$

This yields that $f \in C^1(N)$, since $k(s)$ and F are continuous on N. Now applying the mean value theorem to F yields that $f \in C^1(N)$ where N is some neighbourhood of η_0. Hence, it follows from (i) that $F' = \frac{\mathrm{d}}{\mathrm{d}\eta}F = k(f(\eta))f'(\eta) \in C^1(N)$, and consequently, that f is a classical solution of (4.47) in N.

Thus in a neighbourhood of any point η_0 where $f > 0$, f is a classical solution of (4.47). Since f is required to have compact support, it follows from Lemma 4.2.2 that f can only be of the form

$$f(\eta) > 0 \quad for \quad 0 \leq \eta < a$$

and

$$f(\eta) = 0 \quad for \quad a \leq \eta < \infty$$

for some $a > 0$. Since f is a weak solution, it is continuous on $[0, \infty)$ by part one of the definition and we have found that $F' = k(f(\eta))f'(\eta) \in C^1(N)$ in a neighbourhood of $\eta > 0$. Therefore, condition

$$f \to 0, \quad k(f)f' \to 0 \quad \text{as} \quad \eta \uparrow a$$

is satisfied. It follows that on $[0, a)$, f must be a classical solution that satisfies the conditions discussed above. By Theorem 4.5.1 such a solution is unique and, for it to exist, we must have $\int_0^1 \frac{k(s)}{s} ds < \infty$. □

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