

Verhagen

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Tides and Tidal Currents



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Tides and Tidal Currents

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1. BASIC PHENOMENA

1.1 Introduction

In most seas and estuaries, a periodic rise and fall of the water surface can be observed (see Figure 1.1). It is known as the *vertical astronomical tide*.

The period of the vertical movement is 12 h 25 min. This is called the tidal period T . The highest level is called the High Water level (HW), the lowest level is called the Low Water level (LW), whereas the difference between HW and LW is called the Tidal Range.

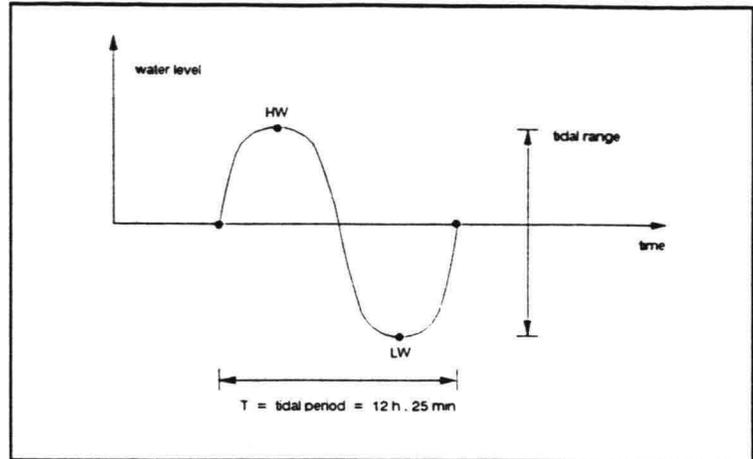


Figure 1.1 Example of a vertical astronomic tide

When the vertical movement of the water level is measured for about one day (say 25 hours), than it can be observed that the second HW and LW differ from the first HW and LW (see Figure 1.2). This difference in HW's and LW's is called the *daily inequality*.

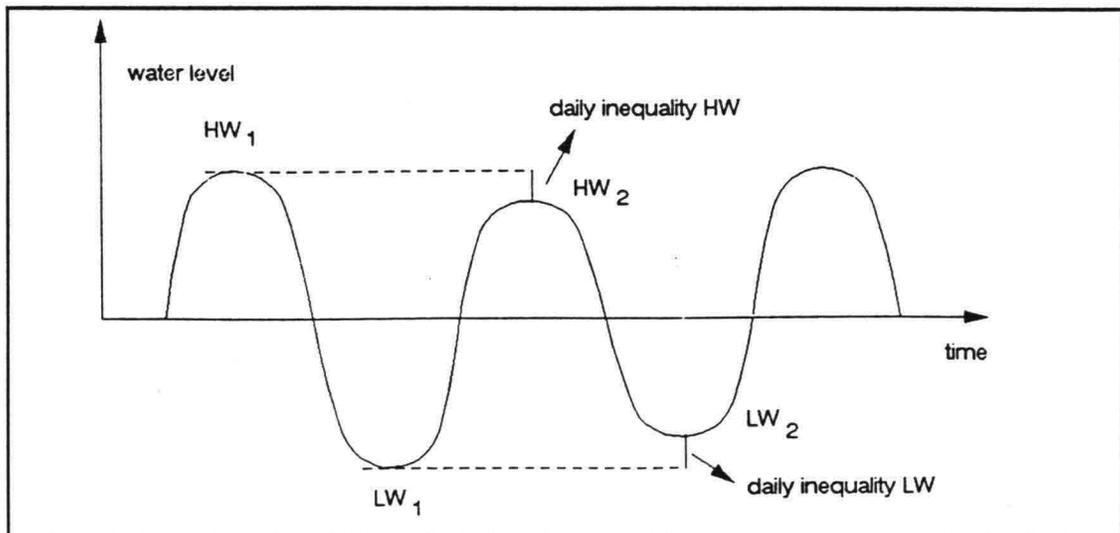


Figure 1.2 Daily inequality of a tide

When the tide is observed for a longer period (about one month), it can be seen that the tidal range varies in time (see Figure 1.3). Periods occur with relatively large tidal ranges, and periods with smaller tidal ranges. The period with the large tidal ranges is called *spring tide*, whereas the period

with the smaller tidal ranges is called *neap tide*. The time between two successive periods of spring tide is about 15 days (half a month).

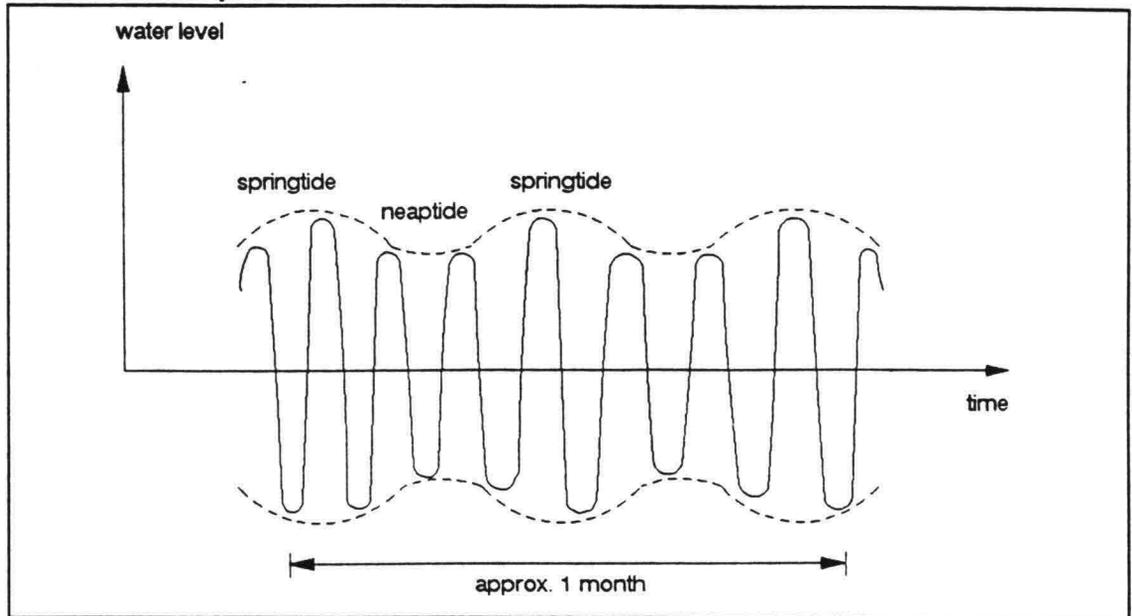


Figure 1.3 Occurrence of spring and neap tides during approximately one month

The above phenomena concern the water level variation in one location. What actually happens, however, is that a long wave (a tidal wave) is passing along the location, where the observations are made (see Figure 1.4). The length of such a tidal wave can be several hundreds of kilometres (depending on the depth).

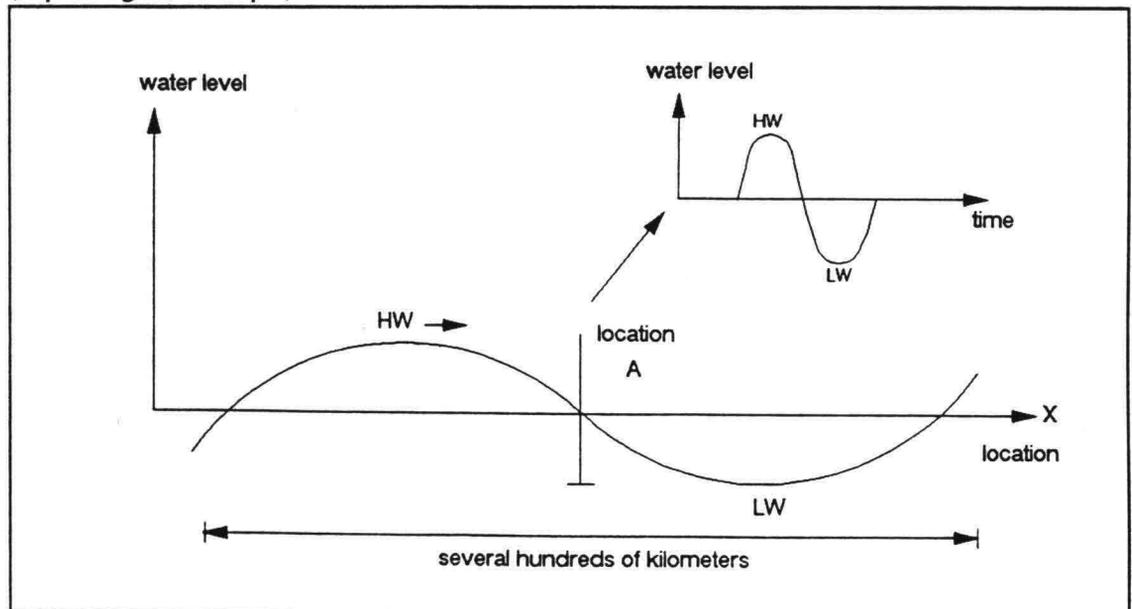


Figure 1.4 Schematic presentation of a tidal wave

When the water level is measured at location A and the wave moves to the right, a periodic rise and fall of the water level can be observed. So, associated with the vertical movement of the water surface, there are also horizontal movements of the water particles.

When the water level is measured at location A and the wave moves to the right, a periodic rise and fall of the water level can be observed. So, associated with the vertical movement of the water surface, there are also horizontal movements of the water particles.

This periodic movement of the water level is a fascinating phenomenon. The study on tides started as a scientific interest on how tides are generated. Why they are so periodic? For many centuries, people tried to understand and explain the observed phenomena.

There are also more practical interests in the tide:

- ships that want to enter a harbour. The captains want to know if there will be enough keel clearance (water under the ship's hull). They want to know the time of occurrence of HW and LW and also the water levels at HW and LW. Therefore, they need a prediction of the tide. These predictions are needed for *one location* (for instance a harbour or its access channel);
- since important civil engineering works are carried out, it becomes necessary to predict what the effect of such works will be on the tidal motion (like water levels and velocities) in the relevant area. Therefore tidal calculations have to be carried out, based on the equations for fluid flow. The calculations are made for a *certain area* of interest, which will be influenced by the civil engineering works;

1.2 Framework of the lectures

The lectures on tides are divided into parts, which follow the different interests as discussed above:

- Chapter 2 deals with the *origin and generation of tides*;
- Chapter 3 deals with the *analysis and prediction of tides*, which concerns the analysis of the tidal curve at one location. The purpose of the analysis is to be able to predict the tide in future at that location;
- Chapters 4,5,6,7 and 8 deal with the *tidal computations*. Chapter 4 discusses the derivation of the basic equations and some types of long waves. In chapter 5 addresses some considerations on tidal propagation in one dimension. Chapter 6 concerns tidal propagation in two dimensions. In chapter 7 shows some analytical solutions are. Since the computer became a tool for calculations, numerical computations techniques were developed. Numerical models are common used now. The aspects of numerical calculations are therefore discussed in Chapter 8.

2 ORIGIN AND GENERATION OF TIDES

2.1 Introduction

Tides are generated by mutual attraction forces between Earth, moon and sun. The influence of other celestial bodies can be disregarded. The attraction force between two bodies is determined by Newton's law of gravity.

When only two bodies, with masses m_1 and m_2 , are considered, an attraction force F will occur. When the distance between the bodies is denoted as x (see Figure 2.1), then Newton's law of gravity becomes:

$$F = a \frac{m_1 \cdot m_2}{x^2}$$

where a is the universal gravity constant.

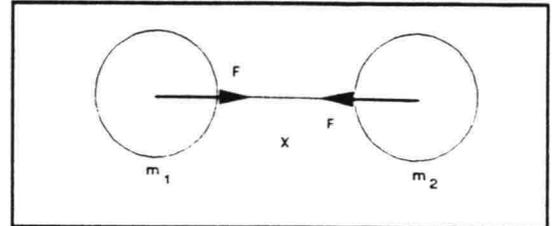


Figure 2.1 Attraction forces between two celestial bodies

The universal gravity constant a can be expressed in terms of the acceleration due to gravity g . Consider a body with mass m_p on the surface of the Earth. The mass of the Earth is m_E and the radius is r . The weight of the body is equal to the attraction force between the body and the Earth:

$$F = m_p g = a \frac{m_p m_E}{r^2}$$

From this it follows that: $a = \frac{gr^2}{m_E}$

Now the system Earth-moon is considered (see Figure 2.2). The distance between the Earth and moon is denoted as $K.r$. The mass of the moon is denoted as $M.m_E$, and the attraction force between Earth and moon is expressed by:

$$F = a \frac{m_E M m_E}{(Kr)^2}$$

Substitution of:

$$a = \frac{gr^2}{m_e}$$

gives:

$$F = \frac{gr^2}{m_E} \cdot \frac{m_E M m_E}{K^2 r^2} = g \frac{M m_E}{K^2}$$

which is an expression for the attraction force between Earth and moon.

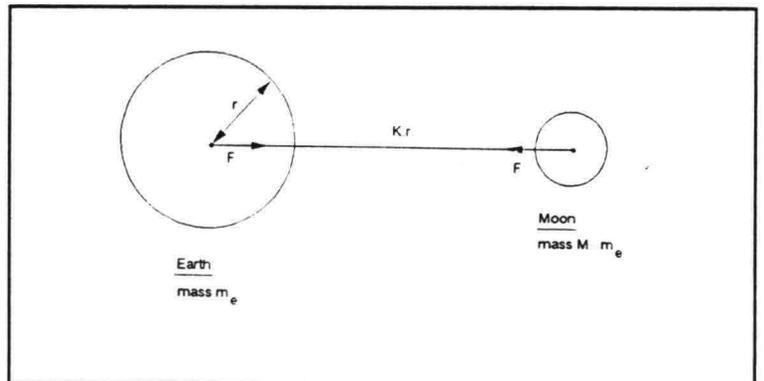


Figure 2.2 Earth-moon system

Origin and generation of tides

This attraction force is counteracted by the centrifugal force due to the rotation of the Earth and moon system around their common centre of gravity. The location of this common centre of gravity can be derived as follows (see Figure 2.3):

$$m_E x r = M m_E (K r - x r)$$

$$x = M k - M x$$

$$x = \frac{MK}{M + 1}$$

Substitution of the values for M and K gives:

$$M = 0.0123$$

$$K = 60.3$$

So $x \approx 0.73$, which means that the common centre of gravity is inside the Earth.

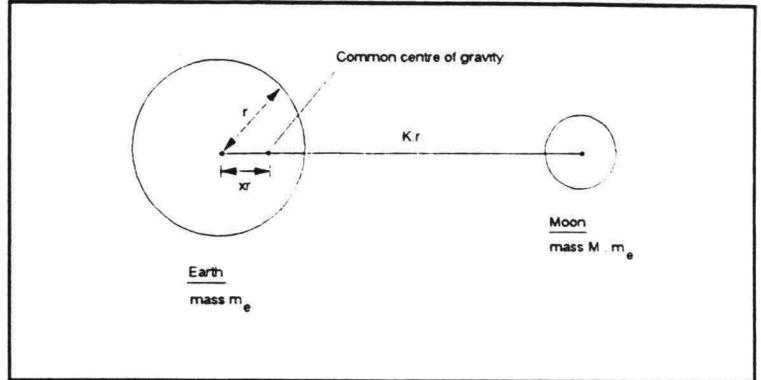


Figure 2.3 Location of common centre of gravity

Now the rotation of the Earth-moon system around the common centre of gravity is considered. First an expression for the centrifugal force is derived. The centrifugal force acting on a body with mass m equals to:

$$\vec{F}_a = m \vec{a}_c$$

To find an expression for a_c , point P with a circular orbit is considered (see Figure 2.4). At time t the velocity is v_t . At time $t + dt$ the velocity is v_{t+dt} . The difference in v in time dt is denoted as dv . Acceleration is the change of velocity per unit of time:

$$\vec{a}_c = \frac{d\vec{v}}{dt} = \frac{v d\theta}{dt} \quad \text{and} \quad \frac{d\theta}{dt} = \omega \quad (\text{angular speed}),$$

gives

$$\hat{a}_c = v \omega$$

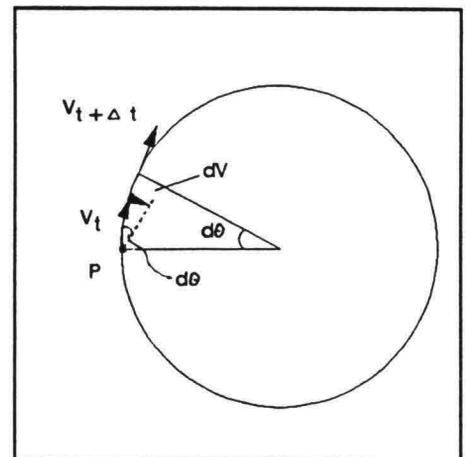


Figure 2.4 Point P with circular orbit

Figure 2.5 illustrates the expression for v in that formula:

$$V = \frac{ds}{dt} = \frac{rd\theta}{dt} = r\omega$$

Substitution in the expression $a_c = v\omega$ yields:

$$a_c = r\omega^2$$

The expression for the centrifugal forces \vec{F}_a now becomes:

$$F_a = mr\omega^2$$

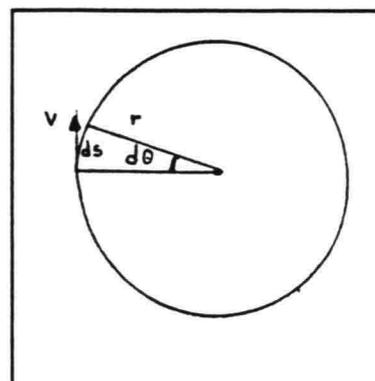


Figure 2.5 Expression for v

To find the angular speed of the rotation of the Earth-moon system around their common centre of gravity denoted as ω_m (see Figure 2.6).

Attraction Force = Centrifugal Force

$$g \cdot \frac{M \cdot m_e}{K^2} = m_e \cdot \omega_m^2 \cdot \frac{M \cdot K}{M+1} \cdot r$$

$$\omega_m^2 = \frac{g}{K^2} \cdot \frac{M+1}{Kr} = \frac{g(M+1)}{K^3 r}$$

Substitution of:

$$g = 9.81 \text{ m/s}^2$$

$$M = 0.0123$$

$$K = 60.3$$

$$r = 6.38 \times 10^6 \text{ m}$$

gives:

$$\omega_m = 2.66 \times 10^{-6} \text{ rad/s or } T = \text{time for one revolution} = 27.32 \text{ days}$$

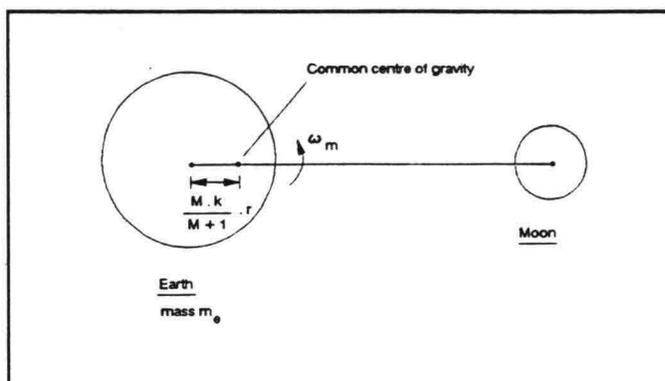


Figure 2.6 Angular speed

To investigate what the rotation around the common centre of gravity means for an arbitrary point P on the Earth surface, only the *translation* of the Earth is considered; the *rotation* around its own axis is neglected (see Figure 2.7):

- the Earth and moon are sketched in position (1); 14 days later the Earth and moon are in position (2). The orbit of the centre of the Earth follows circle *a*. The centrifugal force is directed from the centre of the circle;
- point P at the Earth surface (top of the head) follows the same circle as the centre of the Earth (circle *b*). The centrifugal force is directed parallel to the force in the centre of the Earth and has the same magnitude (per unit of mass), because the circles have the same radii;
- the same holds for the left and right ears.

From the above it can be concluded that:

- every point on Earth revolves through a circle with the same radius;
- the centrifugal force in every point P is directed parallel to the line that connects the centres of Earth and moon;
- the centrifugal force (per unit mass) is equal for all points on the Earth surface.

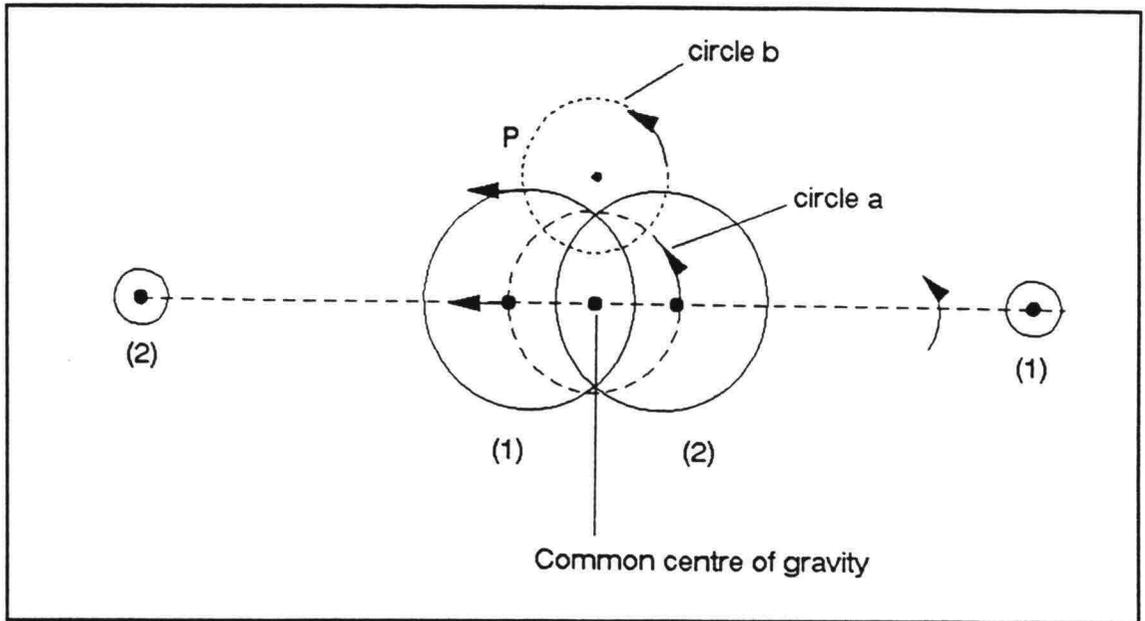


Figure 2.7 Translation of the Earth due to rotation of the moon

2.2 Tide generating force

Tides are caused by forces acting on the water particles on the surface of the Earth. The forces acting on point P at the surface of the Earth are now considered (see Figure 2.8).

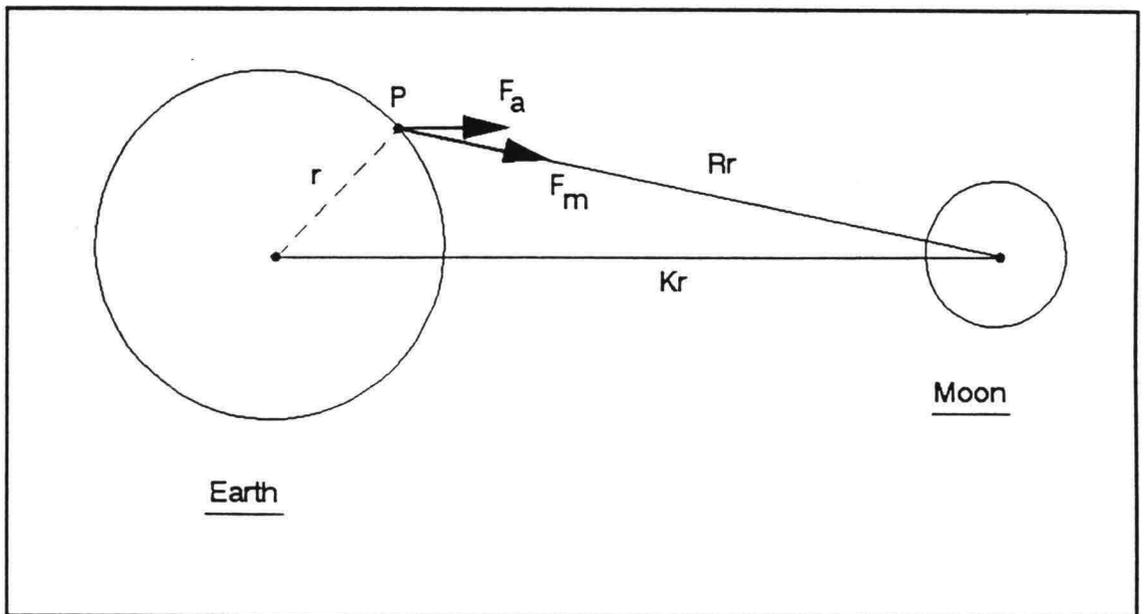


Figure 2.8 Forces acting on water particles

The distance from point P to the centre of the moon is $R.r$. The attraction force between Earth and moon is denoted as:

$$F = g \frac{M \cdot m_e}{K^2}$$

When the attraction force in point P per unit mass is considered (dividing by m_e and replacing K by R), then:

$$F_m = g \frac{M}{R^2}$$

The *acceleration force* is equal to the attraction force if the entire Earth is considered. The attraction force between Earth and moon is:

$$F = g \frac{M \cdot m_e}{K^2}$$

which equals the acceleration force for the entire Earth. The acceleration force per unit mass is found by dividing by m_e .

$$F_a = g \frac{M}{K^2}$$

The acceleration force is equal for any point on Earth, and directed parallel to the line that connects the centres of Earth and moon. The attraction force F_m can be decomposed into the acceleration force and F_a and F_t which is the residual force that causes the tides on Earth. F_t is called the *tide generating force* (see Figure 2.9).

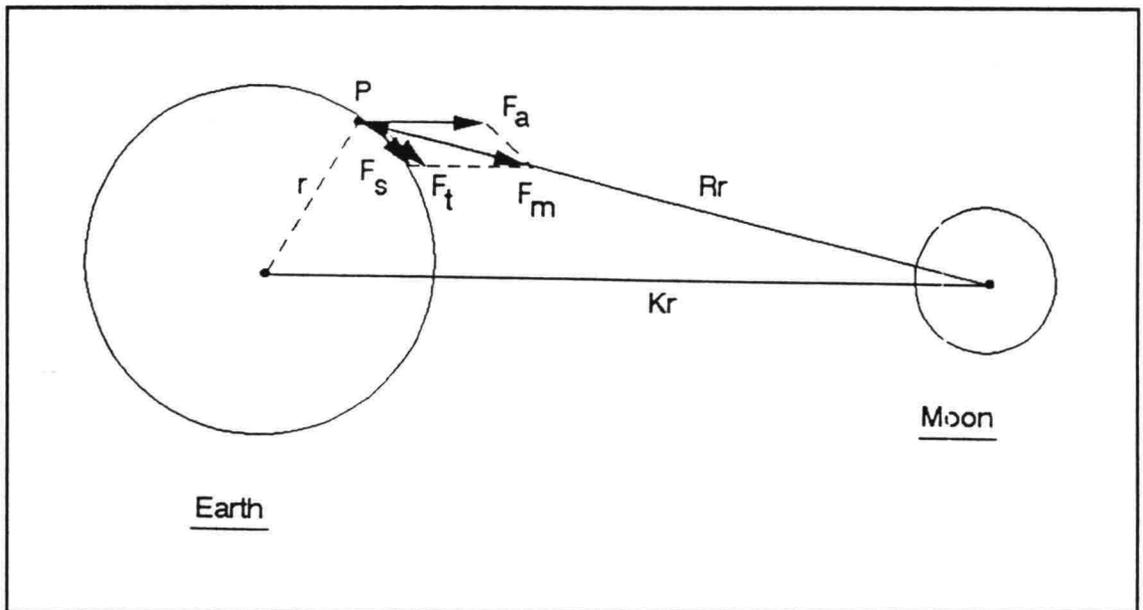


Figure 2.9 Direction of the acceleration force F_a and the tide generation force F_t

Figure 2.10 gives a schematic presentation of the distribution of the tide generating force on the Earth surface. The forces in locations A and B are opposite and almost equal. These forces are very small, compared to g . The centrifugal forces due to the rotation of the Earth around its own axis are neglected: these are also very small compared to g . For the actual motion of the water masses, only the component of F_t is important, which is directed along the Earth surface. This force is denoted as F_s and is called *the tractive force*.

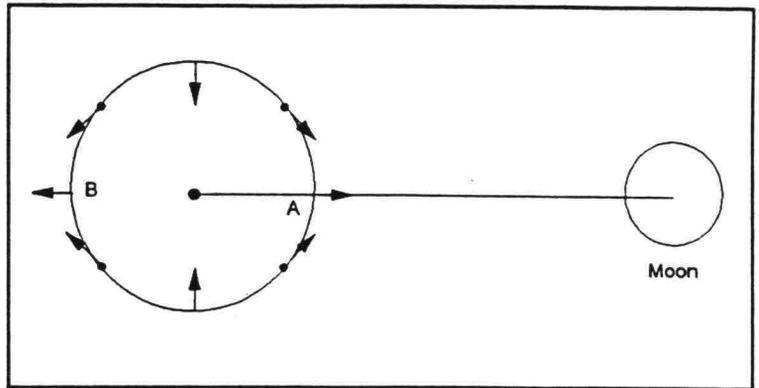


Figure 2.10 Distribution of the tide generation forces

For deriving an expression for the *tractive force* F_s , it can be related to the location on the Earth by considering the angle θ (see Figure 2.11).

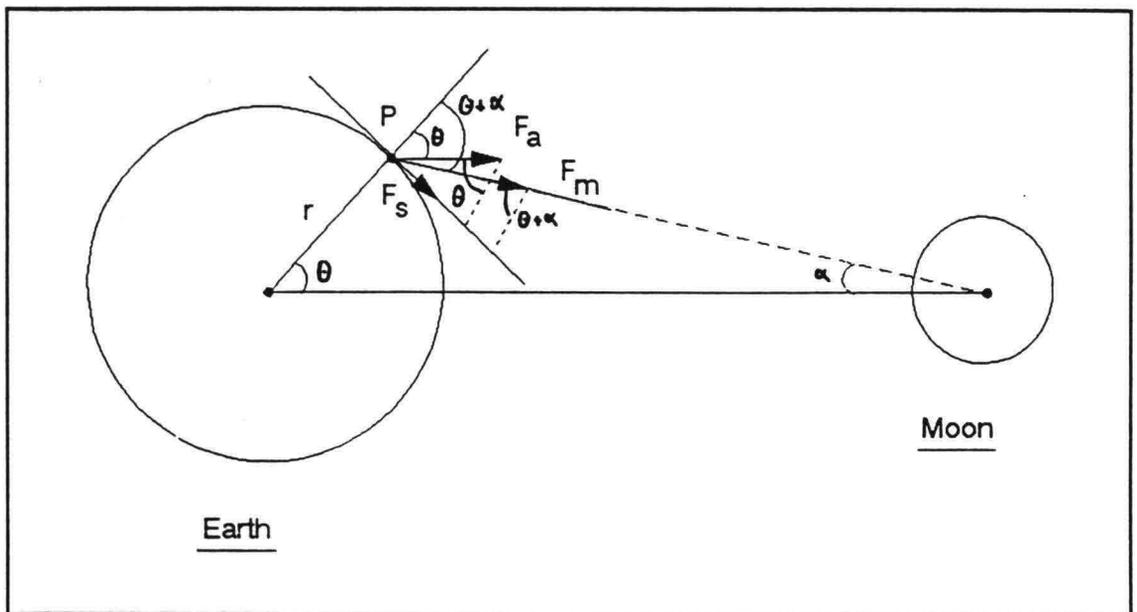


Figure 2.11 Derivation of the tractive force F_s

The expression for F_s becomes:

$$F_s = F_m \sin(\theta + \alpha) - F_a \sin \theta$$

Substituting the expressions for F_m and F_a gives:

$$F_m = g \frac{M}{R^2} ; F_a = g \frac{M}{K^2}$$

so that the tractive force F_s can be expressed by:

$$F_s = \frac{gM}{R^2} \sin(\theta + \alpha) - \frac{gM}{K^2} \sin \theta$$

This formula can be simplified by considering the following geometric relations (see Figure 2.12):

- (1) $Rr \approx Kr - r \cos \theta$
(because α is very small).
 $R \approx K - \cos \theta$
- (2) $\cos \alpha \approx 1$
(because α is small)
- (3) $Rr \sin \alpha = r \sin \theta$
 $(K - \cos \theta) \sin \alpha \approx \sin \theta$
 $K \gg \cos \theta$
 $K \sin \alpha \approx \sin \theta$
 $\sin \alpha \approx \frac{\sin \theta}{K}$

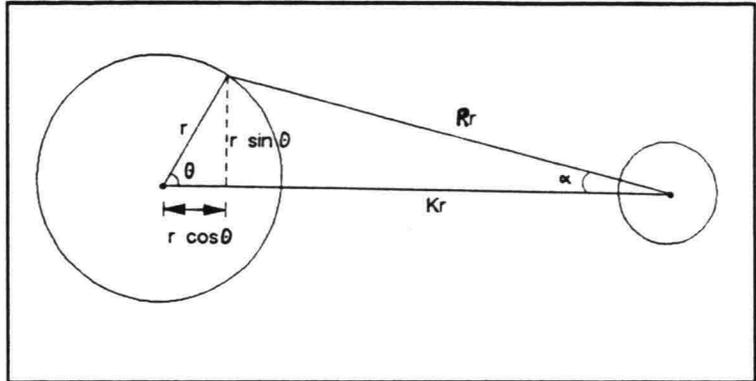


Figure 2.12 Illustration of the geometric relations

Substituting the expressions for R and α into the equation for the tractive force F_s , gives:

$$F_s = \frac{gM}{R^2} (\sin \theta \cdot \cos \alpha + \cos \theta \sin \alpha) - \frac{gM}{K^2} \sin \theta$$

Substituting the following relations:

$$R \approx (K - \cos \theta) ; \quad \cos \alpha \approx 1 ; \quad \sin \alpha \approx \frac{\sin \theta}{K}$$

gives:

$$F_s = \frac{gM}{(K - \cos \theta)^2} \left(\sin \theta + \frac{\sin \theta \cdot \cos \theta}{K} \right) - \frac{gM}{K^2} \sin \theta$$

$$F_s = \frac{gM}{K^2} \left[\frac{\sin \theta + \frac{\sin \theta \cdot \cos \theta}{K}}{\left(1 - \frac{\cos \theta}{K} \right)^2} - \sin \theta \right]$$

For the term $\left(1 - \frac{\cos \theta}{K} \right)^{-2}$ the binomial theorem can be applied. The general formula is:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

Applying the first and second term of this series on the expression gives:

$$\left(1 - \frac{\cos \theta}{K} \right)^{-2} \approx 1 + 2 \frac{\cos \theta}{K} + \dots$$

Substituting this result into the equation for the tractive force F_s , gives finally:

$$F_s \approx \frac{gM}{K^2} \left[\left(1 + 2 \frac{\cos \theta}{K} \right) \left(\sin \theta + \frac{\sin \theta \cos \theta}{K} \right) - \sin \theta \right]$$

$$\approx \frac{gM}{K^2} \left[\sin \theta + \frac{\sin \theta \cos \theta}{K} + \frac{2 \sin \theta \cos \theta}{K} + 2 \frac{\sin \theta \cos^2 \theta}{K^2} - \sin \theta \right]$$

$$F_s \approx \frac{gM}{K^2} \left[\frac{3 \sin \theta \cos \theta}{K} + 2 \frac{\sin \theta \cos^2 \theta}{K^2} \right]$$

Because $K \gg \cos \theta$ the term $2 \frac{\sin \theta \cos^2 \theta}{K^2}$ can be neglected. So:

$$F_s \approx \frac{gM}{K^2} \frac{3 \sin \theta \cos \theta}{K}$$

Substituting $\sin \theta \cos \theta = \frac{1}{2} \cos 2\theta$ gives:

$$F_s \approx \frac{3 gM}{2 K^3} \sin 2 \theta$$

The distribution of the tractive force over the surface of the Earth can be found by:

$$F_s = 0 \quad \text{if } 2\theta = 0, \pi, 2\pi \quad \text{so if } \theta = 0, \pi/2, \pi$$

$$F_s = \text{max} \quad \text{if } 2\theta = \pi/2, 3\pi/2 \quad \text{so if } \theta = \pi/4, 3\pi/4$$

The other values are in between. Figure 2.13 gives the result.

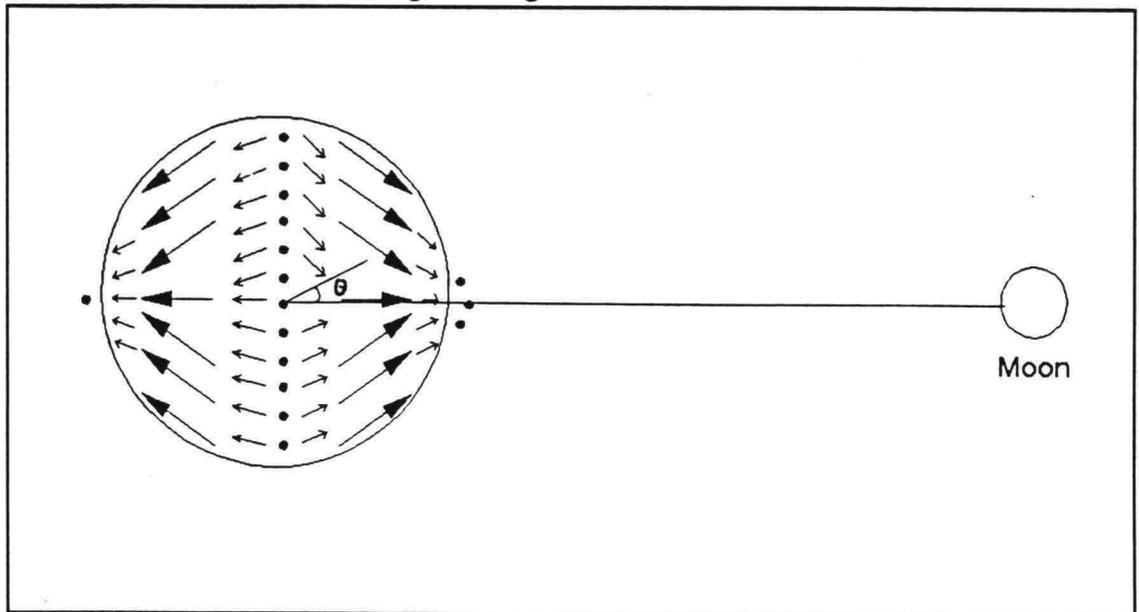


Figure 2.13 Distribution of the tractive force over the Earth surface

So far the tractive force due to the moon was considered. The same, however, holds for the sun. The ratio of the tractive forces caused by the moon and the sun can now be derived as follows. For this the magnitude of the term

$$\frac{3 gM}{2 K^3} \quad (\text{force per unit mass})$$

is considered. The ratio can be found in Table 2.1.

Table 2.1 Ratio of the tractive forces of the moon and the sun

Symbol	Moon	Sun	Dimension
M	0.0123	333,000	(-)
K	60.3	23,500	(-)
$\frac{3gM}{2K^3}$	$0.82 \cdot 10^{-6}$	$0.38 \cdot 10^{-6}$	(m/s ²)

Remarks about the values from Table 2.1 are:

- The forces per unit mass are very small (compared it to $g = 9.8 \text{ m/s}^2$).
- The ratio of the tractive forces caused by moon and sun is about 2 to 1.
- So the effect of the sun on the tide can not be neglected.

2.3 The equilibrium theory

The previous Section explained the influence of the tractive force on water particles on the Earth. This Section considers the influence of the tractive force on the water masses on the Earth. For this, it is firstly assumed that the Earth is fully covered with water and how the shape of the water surface is influenced by the tractive force. This was the assumption of *Newton* when he derived his *equilibrium theory*.

When the inertia forces are neglected, the tractive force has to balance the force from the slope or gradient of the water level. Now a water element with length dx is considered at the surface of the Earth (see Figure 2.14).

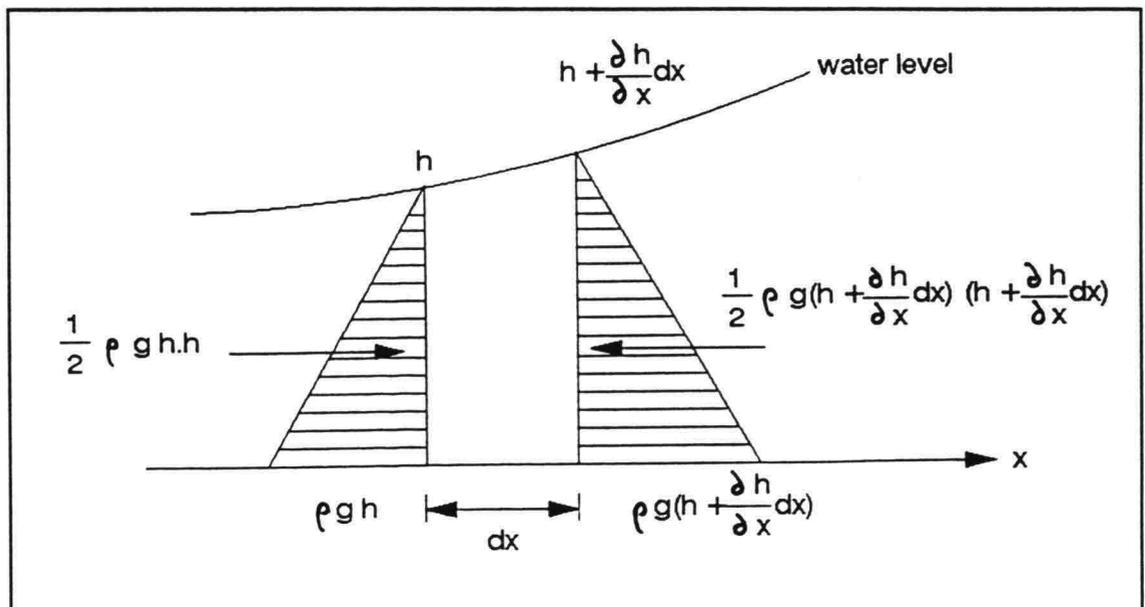


Figure 2.14 Schematic presentation of a water element at the surface of the Earth

At the left hand side the water level is h . At the right hand side the water level is:

$$h + \frac{\partial h}{\partial x} dx$$

When a hydrostatic pressure distribution is assumed, then the pressure at the left hand side has its maximum value at the bottom ρgh , in which ρ = density of water.

At the right hand the maximum pressure is

$$\rho g \left(h + \frac{\partial h}{\partial x} dx \right)$$

The force acting on the left hand side is:

$$\frac{1}{2} \rho g h \cdot h$$

The force acting on the right hand side is:

$$\frac{1}{2} \rho g \left(h + \frac{\partial h}{\partial x} dx \right) \left(h + \frac{\partial h}{\partial x} dx \right)$$

So the net force acting on the water element (in x-direction) is:

$$\begin{aligned} & \frac{1}{2} \rho g h^2 - \frac{1}{2} \rho g \left(h + \frac{\partial h}{\partial x} dx \right)^2 \\ & \frac{1}{2} \rho g h^2 - \frac{1}{2} \rho g \left(h^2 + 2h \frac{\partial h}{\partial x} dx + \left(\frac{\partial h}{\partial x} dx \right)^2 \right) \end{aligned}$$

The term $\left(\frac{\partial h}{\partial x} dx \right)^2$ is small and can be neglected, so the net force becomes:

$$- \rho g h \frac{\partial h}{\partial x} dx$$

The force per unit mass is found by dividing the net force by $\rho h dx$ (the mass of the considered water element). So the net force per unit mass is:

$$- \frac{\rho g \frac{\partial h}{\partial x} dx}{\rho h dx} = - g \frac{\partial h}{\partial x}$$

That force has to balance the tractive force per unit mass:

$$F_s = \frac{3gM}{2K^3} \sin 2\theta$$

at all locations on the Earth surface. The resultant water level is an *ellipsoid* and is sketched in Figure 2.15. Increased water levels occur at the side of the moon and at the opposite side. A similar ellipsoid results from the attraction of the sun.

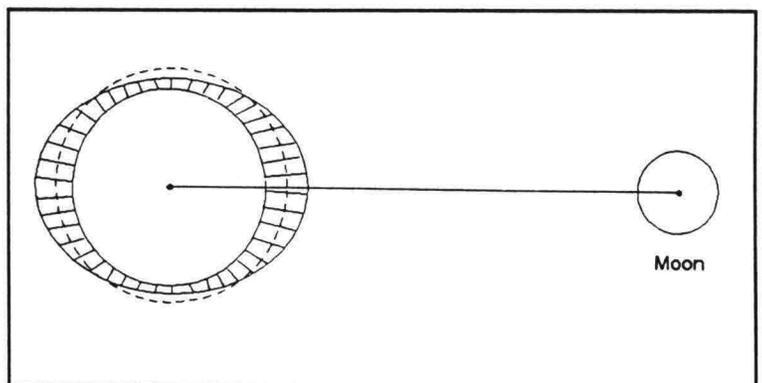


Figure 2.15 Deformation of the water surface due to the tractive forces

Origin and generation of tides

Now the rotation of the Earth is introduced. First the simple (not correct) situation is considered that the moon is positioned in the plane of the equator (see Figure 2.16).

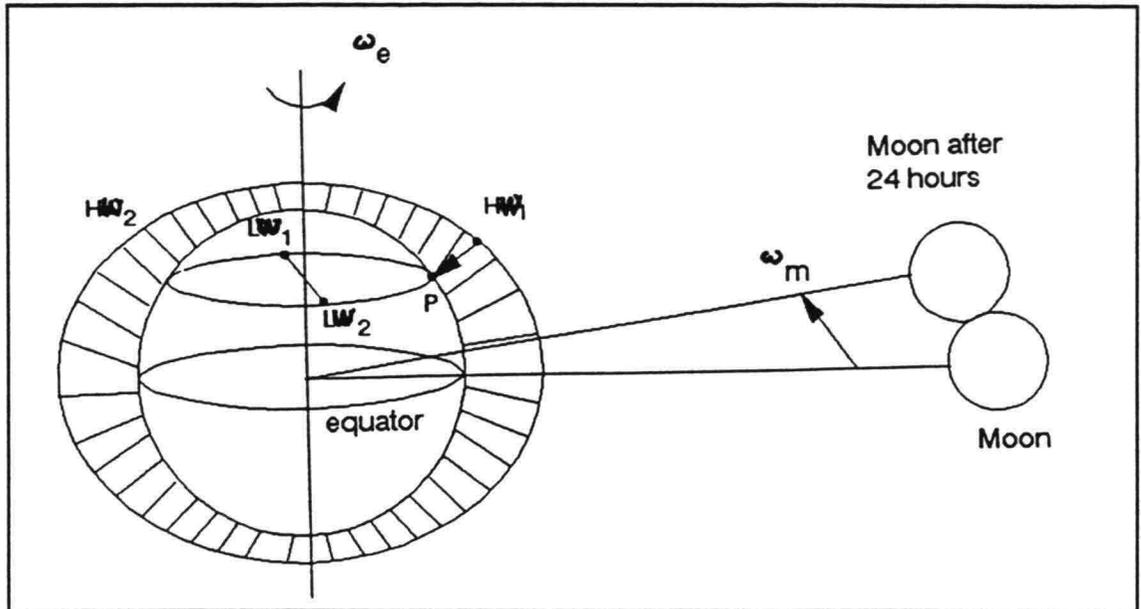


Figure 2.16 Moon positioned in the plane of the Earth's equator

The angular speed of the Earth is ω_e . In one revolution of the Earth point P meets HW_1 , LW_1 , HW_2 , LW_2 and again HW_1 . The time history of the water level in point P during one revolution of the Earth can be recorded as indicated in Figure 2.17.

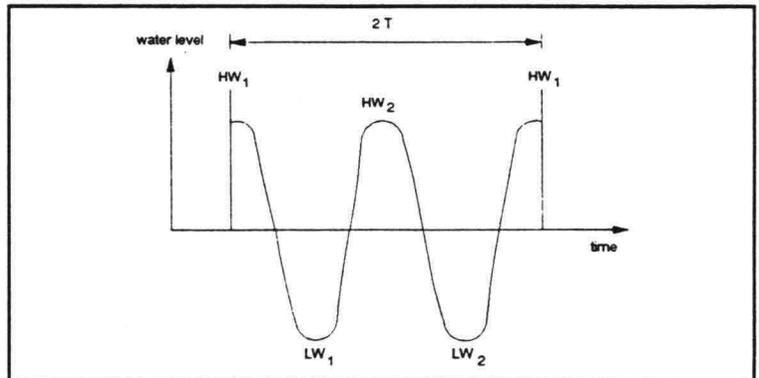


Figure 2.17 Recording water levels in time at P

Two high waters and two low waters occur per day. This is called a *semi-diurnal tide*.

After 24 hours point P is back at its original position. In that time the moon moved along its orbit to another position. The angular speed of the moon is ω_m .

Also the ellipsoid turned somewhat, because it follows the position of the moon. Point P meet the next HW_1 not after 24 hours but somewhat later. This time can be calculated from the angular speeds of Earth and moon, as is presented in see Table 2.2.

Origin and generation of tides

TABLE 2.2 Angular speeds of Earth and moon

Earth	Moon
$T_E = 24 \text{ hrs}$	$T_m = 27.32 \text{ days}$
$\omega_E = 15.041^\circ/\text{h}$	$\omega_m = 0.549^\circ/\text{h}$

In a little more than 24 hrs, *two periods* of the tidal cycle of a semi-diurnal tide occur. Therefore:

$$2T = \frac{2\pi}{(\omega_e - \omega_m)} = \frac{360^\circ}{14.49^\circ/\text{h}} = 24.84 \text{ h}$$

$$T = 12.42 \text{ h} = 12 \text{ h } 25 \text{ min.}$$

This is the basic period of the tide due to the *moon*. The basic period of the tide caused by the *sun* is 12 hours.

Now the assumption that the moon was located in the plane of the Earth equator should be corrected. In reality, the plane of the orbit of the moon makes an angle with the plane of the equator. This angle is called the *moon's inclination* (see Figure 2.18).

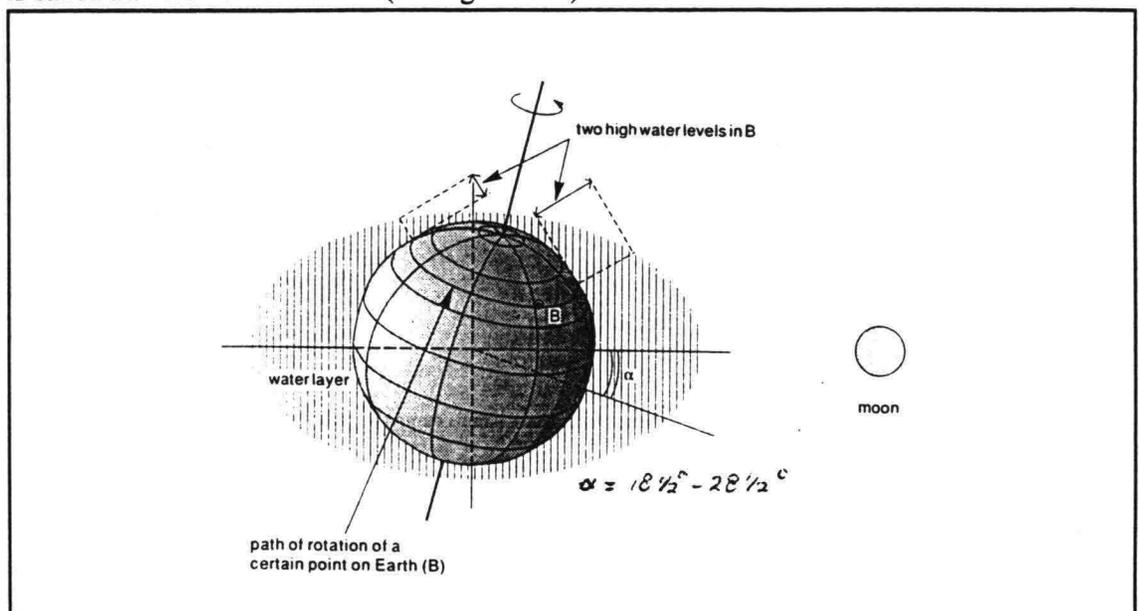


Figure 2.18 Inclination of the moon

In one revolution of the Earth point P meets now different high and low waters: HW_1 , LW_1 , HW_2 , LW_2 . The time history of the water level in point P is presented in Figure 2.19. The high waters HW_1 and HW_2 are different. Also the low waters LW_1 and LW_2 are different. This is called the *daily inequality*.

Origin and generation of tides

So far, attention was paid to the tractive force of the moon. However, both the moon and the sun have their effect on the tide.

When sun, Earth and moon are in one line, the solar bulge and the moon bulge are working together (they are in phase). That is the case during New Moon and Full Moon (see Figure 2.20) and is called *spring tide*. The high waters are extra high, the low waters are extra low: the tidal range is therefore large.

When the moon is perpendicular to the line of sun and Earth, the bulges of moon and sun are out of phase. This is the case during First Quarter and Last Quarter of the moon (see Figure 2.21) and is called *neap tide*. During neap tide, the high waters are extra low, the low waters are extra high: the tidal range is small.

Observing a tide during a month results in a time history as sketched in Figure 2.22. A periodic variation of the tidal range can be observed, where the spring tides and neap tides can be clearly distinguished. The period T of this phenomenon can be derived from the angular speeds of moon and sun (relative to the Earth; see Figure 2.23).

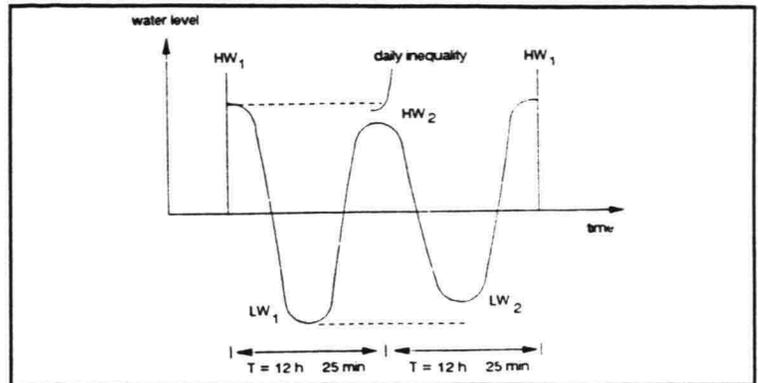


Figure 2.19 Time history of water level at point P

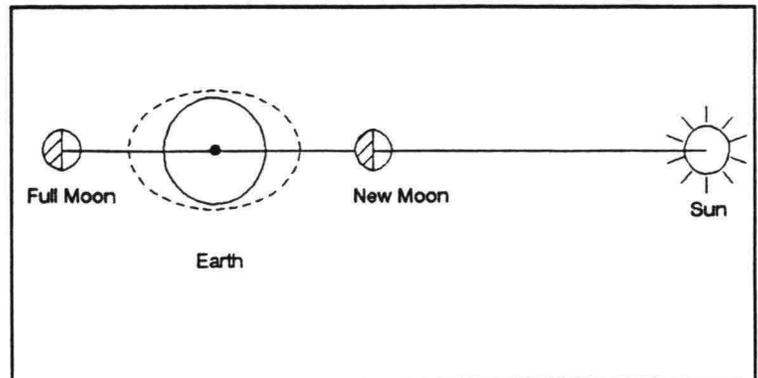


Figure 2.20 Sun, Earth and moon in one line: Spring tide

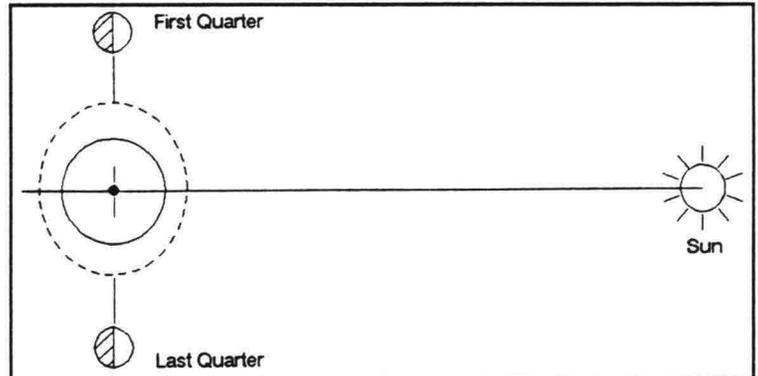


Figure 2.21 Moon perpendicular to the line sun Earth: Neap tide

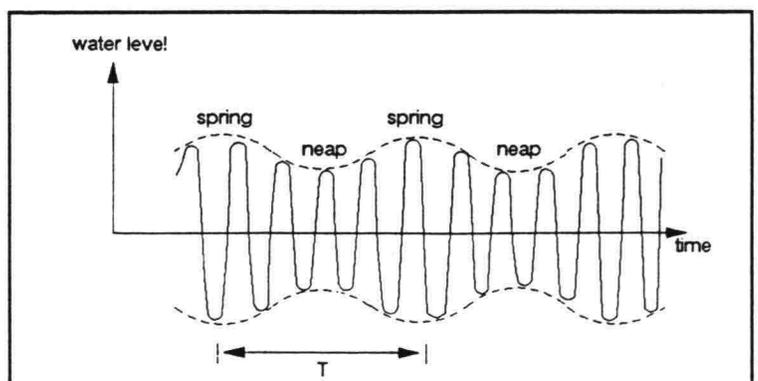


Figure 2.22 Tidal recordings during one month

Origin and generation of tides

The angular speed of the moon is $\omega_m = 0.549^\circ/h$

The angular speed of the sun is $\omega_s = 0.041^\circ/h$

A full revolution of the moon around the Earth will last about $2T$:

$$\frac{2\pi}{\omega_m - \omega_s}$$

$$2T = \frac{360^\circ}{0.508^\circ/h} = 708 \text{ h}$$

$$T = 354 \text{ h} = 14.8 \text{ days}$$

So, spring tides occur about twice a month.

Until now, the Earth was considered to be fully covered with water and inertia of the water masses was neglected. In reality tides will propagate in the oceans of the globe, encountering reflection, damping and distortion.

According to the equilibrium theory, HW would be expected

at a certain location at the moment that the moon crosses the meridian of that location. In reality, however, HW lags behind the moment that the moon crosses the meridian (see Figure 2.24). That time lag is called *high water full and change (HWF&C)* or *port establishment*. It is caused by the inertia of the tidal system.

The same holds for spring tide, which would be expected to occur when sun and moon are in the same line (relative to the Earth), which is at New Moon and Full Moon. Generally, spring (and neap) tides occur 1 to 3 days later (see Figure 2.25). That time lag of about 1-3 days is called the *age of the tide*.

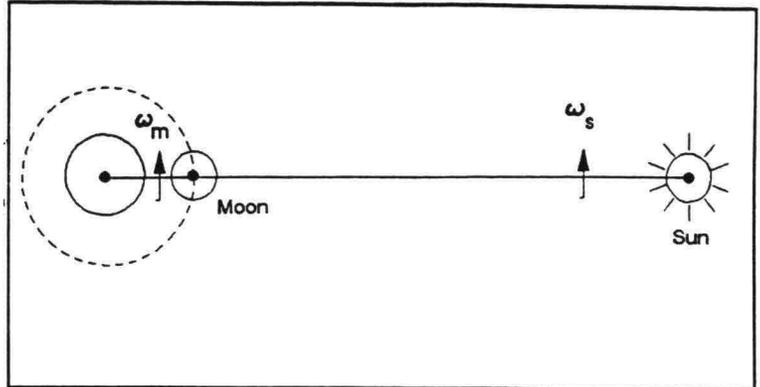


Figure 2.23 Angular speeds of sun and moon

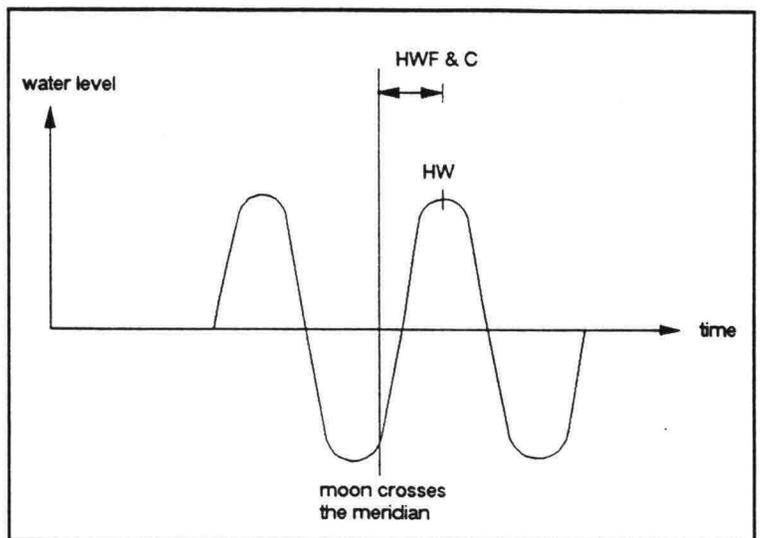


Figure 2.24 Age of the tide

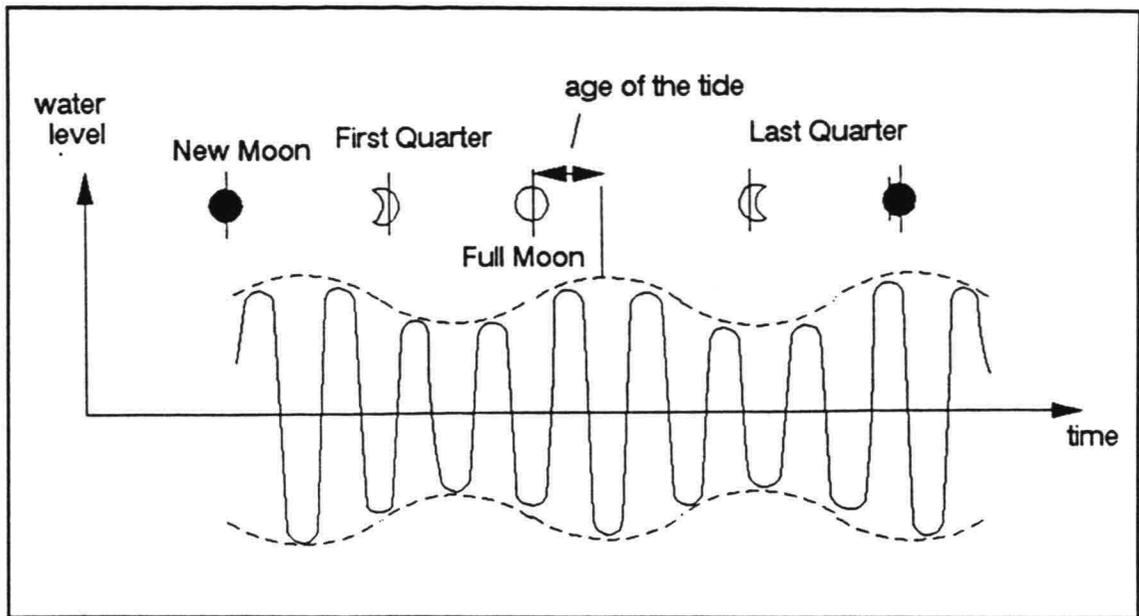


Figure 2.25 Occurrence of spring and neap tides

A complicating factor is that the distances between Earth, moon and sun are not constant. Actually, the orbit around the sun is an ellipse (see Figure 2.26). So, the distance between Earth and sun varies. The tide generating force contains K^3 . The force varies $\pm 5\%$ from the mean value. The sun is nearest to the Earth in January (that is winter in the Northern hemisphere). It is farthest away in July (summer at the Northern hemisphere). The tides caused by the sun, the solar tides are stronger in January and weaker in July. The effect of the changing distance can be described by adding an extra tide, called the solar elliptic tide.

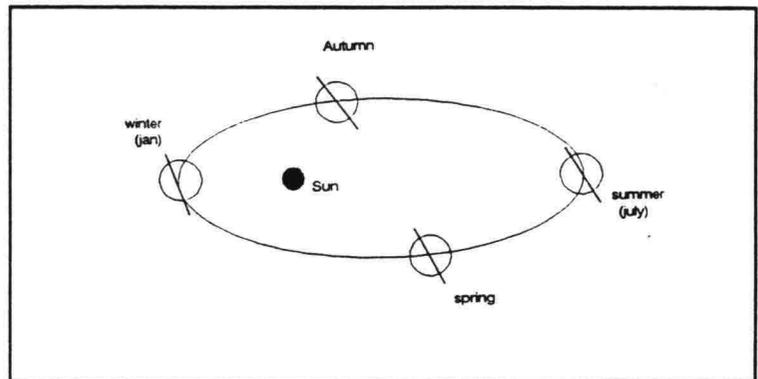


Figure 2.26 Elliptical orbit of the Earth

The orbit of the moon around the Earth is also an ellipse. So the distance to the Earth is also varying. The tidal force varies by $\pm 16\%$ from the mean value. In a similar way as for the sun, this effect can be described by adding an extra tide, called the lunar elliptic tide. Tides caused by the moon are also called lunar tides.

Besides the astronomical complications there are more phenomena, which influence the tides:

- reflections of water masses against irregular coast of oceans;
- frictional resistance of the bottom of shallow seas;
- rotation of the Earth around its axis, which causes deviations of the tidal waves;
- wind effecting tidal water levels.

2.4 Astronomical analysis of the tide generating force

The component of the tide generating force which is directed along the Earth surface is the *tractive force* F_s ,

$$F_s = \frac{3gM}{2K^3} \sin 2\theta$$

In this formula K and θ are not constant; they depend on motions of the moon and the sun. It is known that:

- the motions of moon and sun have a periodic character;
- each motion has its own characteristic mean angular speed.

The next step is to assume that:

The phenomenon, generated by the tide generating force (which is the tide), contains the same frequencies as the force itself.

This assumption is essential, because the analysis and prediction of tides (which will be discussed in Chapter 3) is based on this assumption.

For the analysis of the tidal signal (which is the water level versus time), it is important to know the important frequencies. Those frequencies can be found from the decomposition of the tractive force into its components. Investigators like *Doodson* and *Darwin* have succeeded to decompose the tractive force into its sinus components. This astronomical analysis gives as a result the frequencies and relative importance of each component. This decomposition is not discussed further. It is illustrated *how* the tractive force can be decomposed, and what can be learned from it.

The motions resulting from the Earth, moon and sun can be described by looking at the *celestial sphere*. This is a non-rotating sphere, which moves along with the Earth. The relative motions of the moon and the sun, as they appear to the celestial sphere are projected on this sphere (see Figure 2.27).

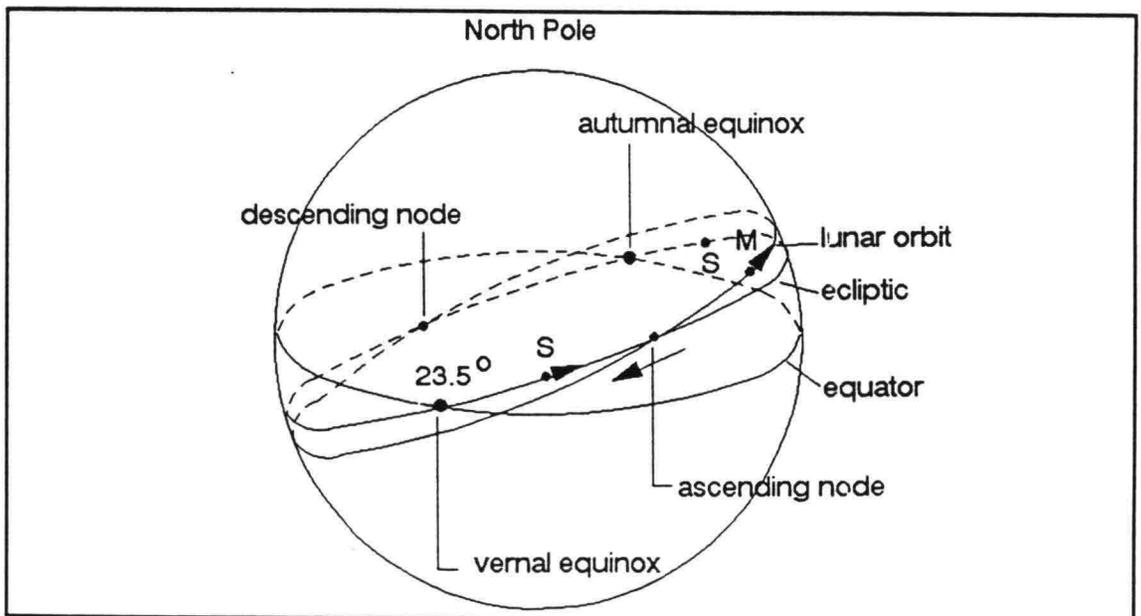


Figure 2.27 Relative motions of moon and sun in relation to the celestial sphere

First the projection of the *sun* on the celestial sphere is considered. It is a circle, which is called the *ecliptic*. The angle with the equator is constant, about 23.5° . The ecliptic intersects the equator at two places: the *vernal equinox* and the *autumnal equinox*.

The vernal equinox is used as a point of reference for the description of the motions of the celestial bodies moon and sun on the sphere. When the sun is in the vernal equinox, *spring* starts in the Northern hemisphere. The position of the vernal equinox is not constant; it makes one revolution around the equator in about 26,000 years. That motion can be ignored on the tide.

The Earth rotates around the sun in 365.24 days. That is the period between two successive crossings of the sun through the vernal equinox. The mean angular speed of the sun is $\omega_s = 0.041^\circ/\text{h}$.

The next step is to consider the projection of the *moon* on the celestial sphere. The motion of the moon is a more complicated one. The *lunar orbit* intersects the ecliptic at two points: the *ascending node* and the *descending node*. The lunar orbit makes an angle with the ecliptic of about 5° . The location of the nodes is not constant. They move along the ecliptic with a period of 18.6 years in westward direction.

The *declination* is the angle between the plane of the equator and the line that connects the centre of the Earth with a certain point on the sphere. The *maximum declination* of the moon occurs when the ascending node is at the vernal equinox. It is $23.5^\circ + 5^\circ = 28.5^\circ$.

The *minimum declination* of the moon occurs when descending node is at the vernal equinox. It is $23.5^\circ - 5^\circ = 18.5^\circ$.

It is known that the moon moves in an ellipse around the Earth. The position of that ellipse is not constant. The *perigeum* (which is the point closest to the Earth) rotates once in 8.85 years (see Figure 2.28).

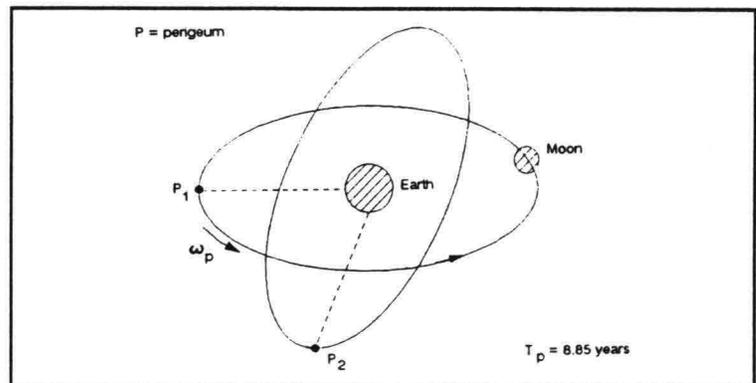


Figure 2.28 Moving position of the perigeum

Other periods are:

- the moon completes one revolution around the Earth in about one month (27.32 days);
- the Earth rotates around its axis in one day.

The relevant periods and angular speeds are given in Table 2.3.

Table 2.3 Periods and angular speeds

Origin	Angular speed in °/hour	Period
Rotation Earth	15.041069	0.997 day
Moon around Earth	0.549016	27.32 day
Earth around sun	0.041069	365.24 day
Perigeum moon	0.004642	8.85
Nodes lunar orbit	0.002206	18.60 year

To describe the decomposition of the tractive force, the expression of F_s is decomposed systematically, where all the frequencies are represented by sinus components:

$$F_s = \frac{3gM}{2K^3} \sin 2\theta = \frac{3gM}{2K^3} \left[A_o + \sum_{i=1}^n A_i \cos \omega_i t + \phi_i \right]$$

where:

A_o constant

A_i amplitude of component i

ω_i angular speed of component i

$$= j \cdot \omega_e + k \cdot \omega_m + l \omega_s + m \omega_p$$

in which $\omega_e, \omega_m, \omega_s, \omega_p$ are the angular speeds of Earth, moon, sun, perigeum of the moon

ϕ_i phase of component i at $t = 0$.

The effect of the nodes of the moon is left out. They are taken into account in a different way which will be shown further on.

To give an impression of how the decomposition can be carried out, the celestial sphere is considered again (see Figure 2.29).

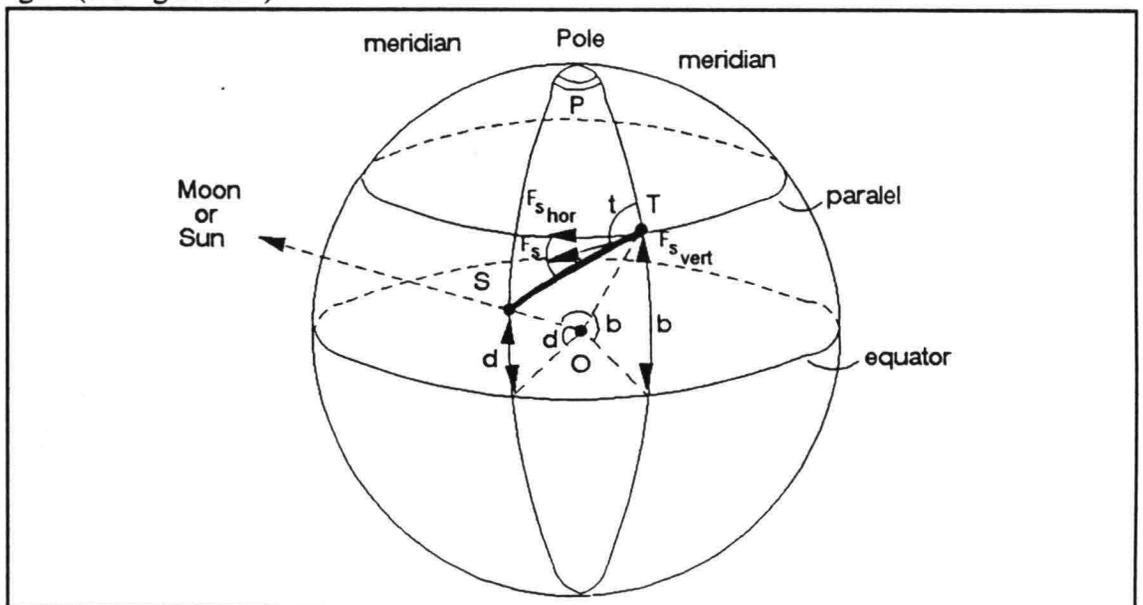


Figure 2.29 Celestial sphere

The symbols used in Figure 2.29 have the following meaning:

- S* position of a celestial body (which can be moon or sun); it has a declination *d*;
- T* position of an observer at latitude *b*;
- P* Angle of intersections between points *S* and *T* (both located on meridians);
- O* centre of the Earth.

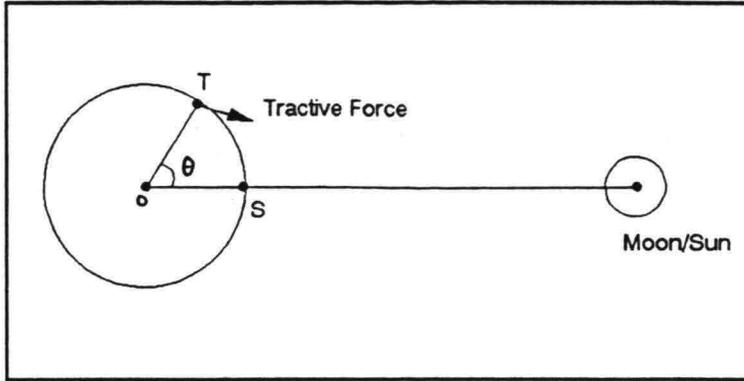


Figure 2.30 Locations of *O*, *S* and *T*

In the expression of the tractive force, θ is the angle between the lines which connect;

- the centre of the Earth and the moon or sun *OS*, and
- the centre of the Earth and location on the Earth surface *OT* (see Figure 2.30).

Thus the angle *SOT* equals to θ . *ST* is part of a circle. The

tractive force in *T* is directed along the circle *TS*.

Looking at Figure 2.30, it can be seen that F_s makes an angle *t* with the meridian. F_s can be decomposed in a horizontal and a vertical component. F_s horizontal is directed along the parallel, whereas F_s vertical is directed along the meridian:

$$F_{s_{hor}} = F_s \sin t$$

$$F_{s_{vert}} = F_s \cos t$$

Now the expression for F_s can be substituted in these equations:

$$F_{s_{hor}} = \frac{3gM}{2K^3} \sin 2\theta \sin t = \frac{3gM}{K^3} \sin \theta \cos \theta \sin t$$

$$F_{s_{vert}} = \frac{3gM}{2K^3} \sin 2\theta \cos t = \frac{3gM}{K^3} \sin \theta \cos \theta \cos t$$

To express θ and *t* into *d*, *b* and *p*, the following geometric relations can be used (they are not derived here):

$$\cos \theta = \sin b \sin d + \cos b \cos d \cos p$$

$$\sin \theta \sin t = \cos d \sin p$$

$$\sin \theta \cos t = \sin d \cos b + \sin b \cos d \cos p$$

These geometric relations can be substituted in the expressions for $F_{s_{hor}}$ and $F_{s_{vert}}$:

$$F_{s_{vert}} = \frac{3gM}{K^3} (\sin d \cos b + \sin b \cos d \cos p) (\sin b \sin d + \cos b \cos d \cos p)$$

After some elaboration the following result can be obtained:

$$F_{s_{hor}} = \frac{3gM}{2K^3} [(\sin b \sin 2d \sin p) + (\cos b \cos^2 d \sin 2p)]$$

$$F_{s_{vert}} = \frac{3gM}{2K^3} \left[\frac{1}{2}(3\sin^2 d - 1)\sin 2b + (\cos 2b \sin 2d \cos p) + \left(\frac{1}{2}\sin 2b \cos^2 d \cos 2p\right) \right]$$

Numbers 1 to 5 are put at the terms of the equations. The formulae contain b , d and p , where:

- b latitude of the location of the observer on Earth;
- d declination of the moon or sun (the declination of the celestial bodies varies with time)
- p angle between the meridian of the observer and the meridian of the position of the moon or sun (see Figure 2.31).

To find the angular speed of p , it should be realized that the location of the observer relative to the sphere rotates with an angular speed ω_e , whereas the position of the moon or sun rotates with ω_m or ω_s . So the angular speed of p is:

$$\frac{dp}{dt} = \omega_e - \omega_m \quad \text{for the moon}$$

$$\frac{dp}{dt} = \omega_e - \omega_s \quad \text{for the sun}$$

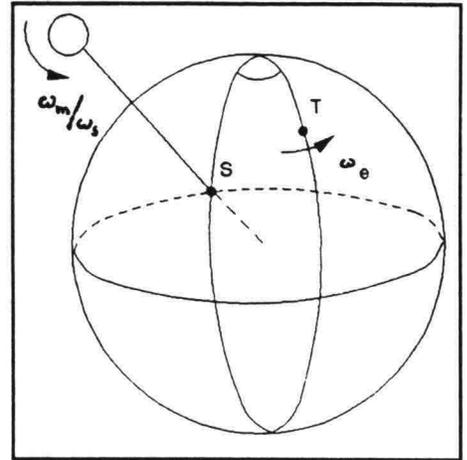


Figure 2.31 Angle between observer and moon or sun

Considering the formulae for $F_{s_{hor}}$ and $F_{s_{vert}}$ with a focus on p :

- terms 4 and 5 contain $\sin 2p$ and $\cos 2p$. The angular speed of $2p$ is:
 $2(\omega_e - \omega_m)$ for the moon
 $2(\omega_e - \omega_s)$ for the sun.

This means that *semi-diurnal tides* are involved;

- terms 2 and 3 contain $\sin p$ and $\cos p$. The angular speed of p is:
 $(\omega_e - \omega_m)$ for the moon
 $(\omega_e - \omega_s)$ for the sun.

This means that the *diurnal tides* are involved;

- looking closer to terms 2 and 3, it can be seen that they contain $\sin 2d$. So if the declination = 0 (if Moon and Sun are in the vernal equinox), the diurnal components are 0. Therefore those diurnal components are called *declination tides*;
- term 1 contains only the declination d . The declination varies with the angular speed of moon and sun. Typical angular speeds (which are called *long period tides*) are:
 ω_m for the moon
 ω_s for the sun.

To extend the decomposition, the declination d is expressed in terms of the motion of moon and sun. For the sun, d is expressed in the longitude of the sun. Figure 2.32 shows the celestial sphere. The longitude of the sun with respect to the vernal equinox is denoted as h .

For the moon, longitude d is expressed in:

- the longitude of the moon;
- the longitude of the ascending node;
- the longitude of the perigee.

Figure 2.33 shows the celestial sphere, where:

- the longitude of the moon with respect to the vernal equinox is denoted as s ;
- the longitude of the ascending node with respect to the vernal equinox is denoted as N ;
- the longitude of the perigee with respect to the vernal equinox is denoted as p .

The variation in time of those variables are known. As an example, the expressions for the longitudes with reference time $t = 0$ at 1 January 1900 at 0.00 hours is given:

$$\begin{aligned}
 h &= 280.190 + \omega_s t, && \text{(longitude of the sun)} \\
 s &= 277.026 + \omega_m t, && \text{(longitude of the moon)} \\
 N &= 259.156 + \omega_n t, && \text{(longitude of the node of} \\
 &&& \text{the moon)} \\
 p &= 334.385 + \omega_p t, && \text{(longitude of the perigee)} \\
 &&& \text{(dimension in degrees).}
 \end{aligned}$$

If these relations are substituted in the general equations of the tractive force, expressions composed of the sum of numerous harmonic components are obtained. Each *harmonic component*, which is found from that elaboration, has its own *amplitude* and *angular speed* (or frequency).

In the above, the effect of the moving nodes of the moon is not taken into account as a separate component. The effect of the nodes is taken into account in a different way (see also Figure 2.34). The angle of the ecliptic with the equator is

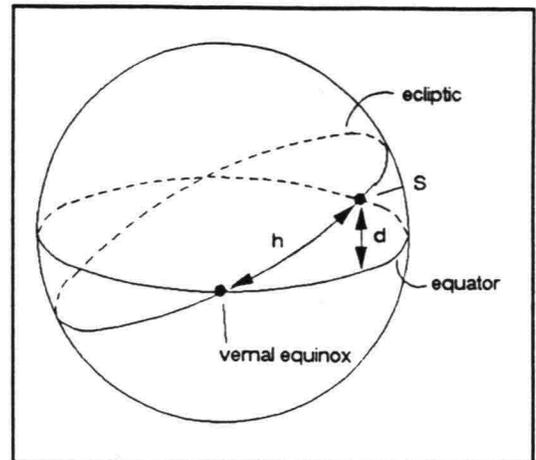


Figure 2.32 Celestial sphere showing the longitude d of the sun

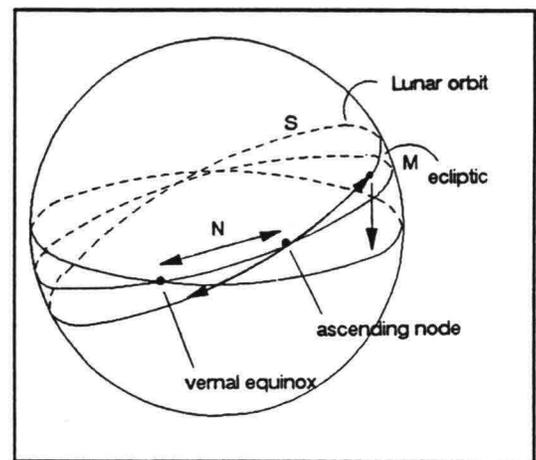


Figure 2.33 Celestial sphere, showing the ascending node

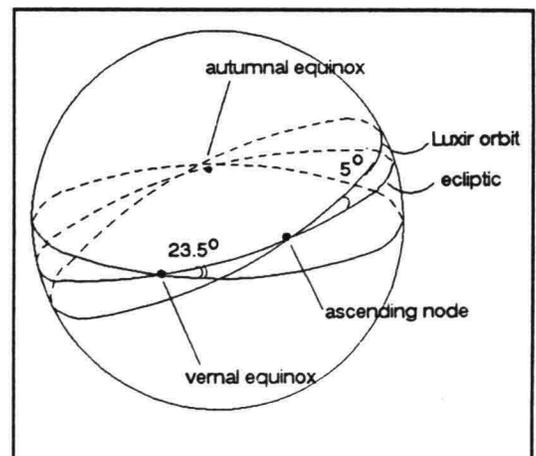


Figure 2.34 Effect of the nodes of the moon

constant: 23.5° . The moon's orbit makes an angle of about 5° with the ecliptic. The maximum angle between the moon's orbit and the equator varies from $18\frac{1}{2}^\circ$ - $28\frac{1}{2}^\circ$, depending on the location of the nodes:

- $28\frac{1}{2}^\circ$ if ascending node is in vernal equinox;
- $18\frac{1}{2}^\circ$ if ascending node is in autumnal equinox.

The nodes make one revolution in 18.6 years.

The most convenient way is to substitute $23\frac{1}{2}^\circ$ for the angle between the moon's orbit and the equator in the expressions of the tractive force. The variation due to the revolution of the nodes is taken into account by multiplying the amplitude with a factor f_i and adding a phase shift u_i to the harmonic terms. The general equation for the tractive force is:

$$F_s = \frac{3gM}{2K^3} \left[A_o + \sum_{i=1}^n A_i \cos(\omega_i t + \phi_i) \right]$$

With the corrections the expression becomes:

$$F_s = \frac{3gM}{2K^3} \left[A_o + \sum f_i A_i \cos(\omega_i t + \phi_i + u_i) \right]$$

The node factors f_i and u_i are known from astronomical data for each component. The node factor f_i is considered to be *constant per calendar year*.

The harmonic terms that we obtain from the astronomical analysis can be regarded as tide generating forces due to ideal stars. For example, $M_2 \cos(2\omega_e - 2\omega_m)$ is the force that would be exerted by a moon with a circular orbit in the plane of the equator. A second example is $S_2 \cos(2\omega_e - 2\omega_s)$, which is the force that would be exerted by a sun with a circular orbit in the plane of the equator.

The number of harmonic terms is large because:

- the orbits are not in the plane of the equator, which cause *declination tides* (mainly diurnal);
- the distance between the Earth and the moon and sun are not constant, as their orbits are ellipses. The distances vary and also the angular speed. That cause the *elliptical tides* (both diurnal and semi-diurnal).

2.5 Main constituents of the tide

The most important tidal constituents are given in Table 2.4. They are called the main astronomic constituents of the tide. Five groups can be distinguished; their meaning will be explained later. In Table 2.4, the following information can be found:

- the symbol of each constituent (like M_2 , S_2);
- the angular speed, expressed in the angular speeds of the Earth, the moon, the sun, the perigee of the orbit of the moon and also the numerical values;
- the astronomic coefficient, which gives some information about the relative strength of the component. This will also be discussed later;
- the last column gives the type of the constituents.

Origin and generation of tides

Table 2.4 Main astronomic constituents of the tide

Group	Symbol	Frequency	Period (hours)	Angular speed (deg/hour)	Astronomic coefficients	Type of constituent
I	M2	$2\omega_e - 2\omega_m$	12.42	28.9841	0.908	semi-diurnal principle lunar tide
	S2	$2\omega_e - 2\omega_s$	12.00	30.0000	0.423	semi-diurnal principle solar tide
	K1	ω_e	23.94	15.0411	0.531	diurnal lunar-solar declination tide
	O1	$\omega_e - 2\omega_m$	25.80	13.9430	0.377	diurnal lunar declination tide
II	P1	$\omega_e - 2\omega_s$	24.07	14.9589	0.176	diurnal solar declination tide
	N2	$2\omega_e - 3\omega_m + \omega_p$	12.66	28.4397	0.174	semi-diurnal lunar elliptic tide
	K2	$2\omega_e$	11.97	30.0821	0.115	semi-diurnal lunar-solar declination tide
III	Q1	$\omega_e - 3\omega_m + \omega_p$	26.87	13.3987	0.072	diurnal lunar elliptic tide
	L2	$2\omega_e - \omega_m - \omega_p$	12.19	29.5285	0.026	semi diurnal lunar elliptic tide
IV	Mf	$2\omega_m$	328	1.0980	0.156	long periodic lunar tide
	Mm	$\omega_m - \omega_p$	661	0.5444	0.083	long periodic lunar tide
	Ssa	$2\omega_s$	4383	0.0821	0.026	long periodic solar tide
V	Sa	ω_s	8759	0.0411	0.012	long periodic solar tide
	Msm	$\omega_m - 2\omega_s + \omega_p$	764	0.4715	0.012	
	Msf	$2\omega_m - 2\omega_s$	354	1.0159	0.008	
	Mtm	$3\omega_m - \omega_p$	219	1.6424	0.030	
	M1	$\omega_e - \omega_m + \omega_p$	24.83	14.4967	0.030	
	$\pi 1$	$\omega_e - 3\omega_s$	24.13	14.9179	0.010	
	$\phi 1$	$\omega_e + 2\omega_s$	23.80	15.1232	0.008	
	J1	$\omega_e + \omega_m - \omega_p$	23.10	15.5854	0.030	
	001	$\omega_e + 2\omega_m$	22.31	16.1391	0.016	
	2N2	$2\omega_e - 4\omega_m + 2\omega_p$	12.91	27.8954	0.024	
	$\mu 2$	$2\omega_e - 4\omega_m + 2\omega_s$	12.87	27.9682	0.022	
	v2	$2\omega_e - 3\omega_m + 2\omega_s - \omega_p$	12.63	28.5126	0.034	
	$\lambda 2$	$2\omega_e - \omega_m - 2\omega_s + \omega_p$	12.22	29.4556	0.007	
T2	$2\omega_e - 3\omega_s$	12.02	29.9590	0.025		

ω_e = angular speed of Earth

ω_m = angular speed of moon

ω_s = angular speed of sun

ω_p = angular speed of perigeum of moon's orbit

In a tidal analysis, the tidal signal (= the observed water level versus time) is decomposed into its constituents. When the constituents have been determined, a prediction of the tide can be made (for a week, a month, a year in advance). For the analysis and prediction of the tide, a distinction can be made between the important and less important constituents. For this, Table 2.4 shows 5 groups:

- group I is always needed for a tidal prediction;
- group II is also taken into account;
- group III is theoretically of minor importance. In several seas they are stronger than the astronomic coefficients indicate;
- group IV reflects the tides with longer periods. They have to be taken into account if accurate predictions are needed for a longer time;
- the constituents of group V will be considered if an accurate prediction is needed.

Table 2.5 presents the most important astronomic constituents.

Table 2.5 Most important constituents

Type of tide	Constituent
Principle tides, semi-diurnal	M2, moon S2, sun
Declination tides, diurnal	K1, moon and sun O1, moon P1, sun
Declination tides, semi-diurnal	K2, moon and sun
Elliptical tides, diurnal	Q1, moon
Elliptical tides, semi-diurnal	N2, moon L2, moon

In Table 2.4, the column of the astronomic coefficient gives the value of A_i in the expression:

$$F_s = \frac{3gM}{2K^3} [A_o + \sum f_i A_i \cos(\omega_i t + \phi_i + u_i)]$$

Multiplying A_i by $\frac{3gM}{2K^3}$ gives the amplitude of the tractive force for that component.

So, A_i or the astronomic coefficient indicates the relative importance of the component.

When the tide is measured at a certain location on Earth, the relative magnitudes of the components can differ considerably from the astronomic ones. This is caused by the irregularities in the oceans and seas.

The tide can completely be described by the sum of the astronomic components in deep oceans. Those tides are observed on ocean islands.

In shallow coastal shelf seas the tide is effected by:

- bottom friction;
- variable propagation speed of the tidal wave.

I Bottom friction

The bottom friction is proportional to the water velocity squared:

$$F \text{ proportional to } u^2$$

in which

F friction force

u water velocity.

For alternating flows, the flow direction should be taken into account. The expression becomes:

$$F \text{ proportional to } u | u |$$

If the direction of flow changes, the direction of the friction force must change as well.

If u positive, than F negative.

If u negative, than F positive.

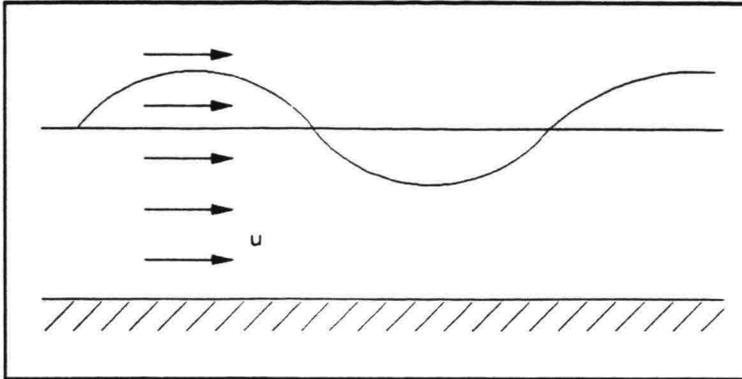


Figure 2.35 Sinusoidal tidal wave

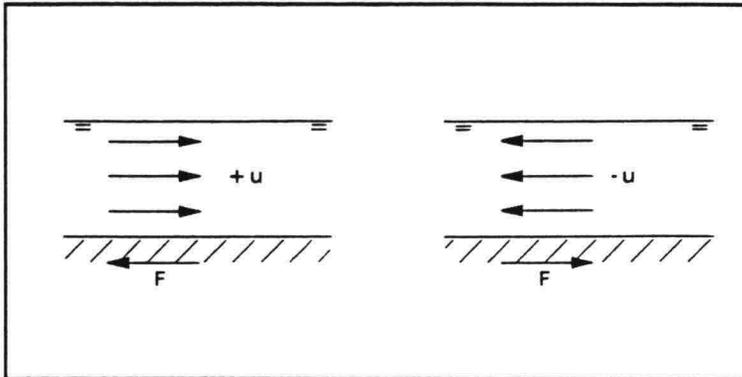


Figure 2.36 Velocities in a tidal wave

Consider a tidal wave with a sinusoidal shape.

The velocities in a tidal wave can be described by (see also Figure 2.36):

$$u = \hat{u} \sin \omega t$$

where:

\hat{u} maximum velocity

ω angular speed of the wave

This means that F is proportional to:

$$u |u| = \hat{u}^2 \sin \omega t |\sin \omega t|.$$

This relation can be expressed in a Fourier series:

F is proportional to

$$\hat{u}^2 \left[\frac{8}{3\pi} \sin \omega t + \frac{8}{15} \sin 3\omega t + \dots \right]$$

It means that the friction generates terms with a frequency 3 times the basic frequency. If the basic frequency of M_2 (semi-diurnal component) is taken, than a M_6 tidal component will be generated. M_6 has six oscillations per day, and is called a *sixth-diurnal* component. It is clear that the M_6 -component does not have an astronomic origin.

II Variable propagation speed of a tidal wave in shallow water

A purely sinusoidal wave is considered, which enters from the ocean into a shallow sea. In this

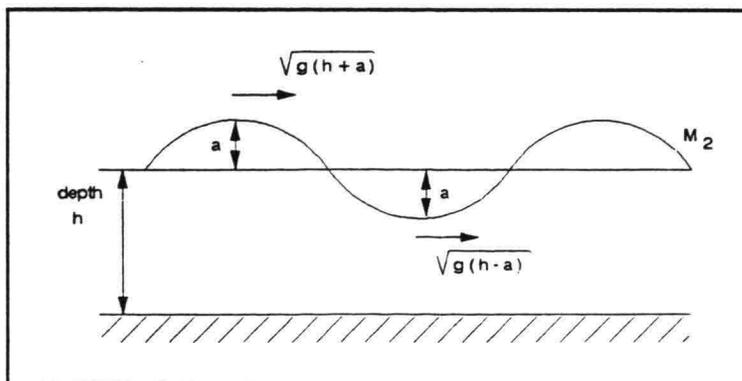


Figure 2.37 M_2 tidal component

example, an M_2 -tidal component is taken (see Figure 2.37). The propagation speed of a disturbance in water with depth h is \sqrt{gh} . The amplitude of the wave is a . This means that the propagation speed of the top is $\sqrt{g(h+a)}$. The propagation speed of the through is $\sqrt{g(h-a)}$. In deep oceans h

is much larger than a . The propagation speeds are equal. In shallow seas there is a difference. If the tidal wave propagates into a shallow sea after some time the shape will be distorted (see Figure 2.38).

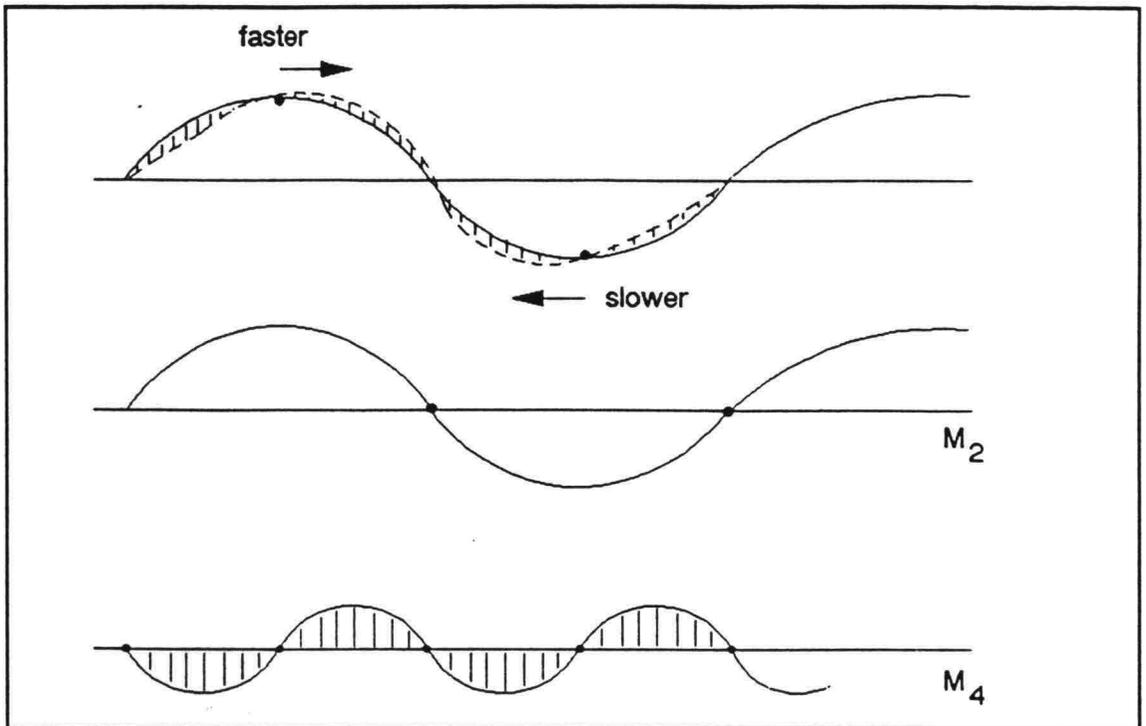


Figure 2.38 Influence of the water depth on a tidal wave

When this distorted wave is decomposed in its components, then the original M_2 -component plus a component with double frequency is found. It has four oscillations per day and is called the M_4 tide. In fact, from the M_2 -tide a series of super-harmonics is generated in shallow seas, like M_4 , M_6 , M_8 . The same holds for the S_2 -tide: S_4 , S_6 , S_8 . Those tides are also called *over tides*. The distortion of the tides with a period of one day, like K_1 and O_1 is very small. It is not necessary to take their super-harmonics into consideration.

It never occurs that one partial tide enters a shallow sea. There are always more tidal components that interact. This interaction gives rise to new components with frequencies deviating from the original ones. We call them *compound tides*. They are derived from M_2 , S_2 , N_2 , K_1 , O_1 . The interaction between M_2 and S_2 yields MS_4 . The interaction between M_2 and N_2 yields MN_4 and so on. The symbol is denoted by the symbols of the original tides. The subscript denotes the period, expressed in parts of the diurnal tide (or the number of oscillations per day).

The tidal components, generated by non-linear effects in shallow water are called *shallow water tides*, and consist of:

- super-harmonic tides or over-tides;
- compound tides.

Origin and generation of tides

The most important components, also called the *shallow water tides*, are listed in Table 2.6. This table contains groups, which are semi-diurnal, ter-diurnal, quarter-diurnal, sixth-diurnal and eighth-diurnal tides. The table does not give astronomic coefficients, because the components do not have an astronomic origin. The amplitudes of the components depend on the shape of the sea in which they are generated. Experience has learned that from the shallow water tides at least the M_4 , M_6 , M_8 , MS_4 , MN_4 have to be considered.

Table 2.6 Shallow water tides

Symbol	Origin	Frequency	Period (hrs)	Angular speed (deg./h)	Remarks
MNS2	M2+N2-S2	$2\omega_e - 5\omega_m + 2\omega_s + \omega_p$	13.13	27.4238	semi-diurnal
2MS2	2M2-S2	$2\omega_e - 4\omega_m + 2\omega_s$	12.87	27.9682	
2SM2	2S2-M2	$2\omega_e + 2\omega_m - 4\omega_s$	11.61	31.0159	
MK3	M2+K1	$3\omega_e - 2\omega_m$	8.18	44.0252	terdiurnal
2MK3	2M2-K1	$3\omega_e - 4\omega_m$	8.39	42.9271	
SK3	S2+K1	$3\omega_e - 2\omega_s$	7.99	45.0411	
SO3	S2+O1	$3\omega_e - 2\omega_m - 2\omega_s$	8.19	43.9430	
M4	2M2	$4\omega_e - 4\omega_m$	6.21	57.9682	quarter diurnal
MS4	M2+S2	$4\omega_e - 2\omega_m - 2\omega_s$	6.10	58.9841	
MN4	M2+N2	$4\omega_e - 5\omega_m + \omega_p$	6.27	57.4238	
MK4	M2+K2	$4\omega_e - 2\omega_m$	6.09	59.0662	
S4	2S2	$4\omega_e - 4\omega_s$	6.00	60.0000	
M6	3M2	$6\omega_e - 6\omega_m$	4.14	86.9523	sixth diurnal
2MS6	2M2+S2	$6\omega_e - 4\omega_m - 2\omega_s$	4.09	87.9682	
2MN6	2M2+N2	$6\omega_e - 7\omega_m + \omega_p$	4.17	86.4079	
2SM6	2S2+M2	$6\omega_e - 2\omega_m - 4\omega_s$	4.05	88.9841	
MSN6	M2+S2+N2	$6\omega_e - 5\omega_m - 2\omega_s + \omega_p$	4.12	87.4238	
S6	3S2	$6\omega_e - 6\omega_s$	4.00	90.0000	
M8	4M2	$8\omega_e - 8\omega_m$	3.11	115.9364	eighth diurnal
3MS8	3M2+S2	$8\omega_e - 6\omega_m - 2\omega_s$	3.08	116.9523	
2(MS)8	2M2+2S2	$8\omega_e - 4\omega_m - 4\omega_s$	3.05	117.9682	
2MSN8	2M2+S2+N2	$8\omega_e - 7\omega_m - 2\omega_s + \omega_p$	3.07	117.4079	
S8	4S2	$8\omega_e - 8\omega_s$	3.00	120.0000	

ω_e = angular speed of Earth

ω_m = angular speed of moon

ω_s = angular speed of sun

ω_p = angular speed of perigeum of moon's orbit

Meteorological tides

In addition to the main constituents of the tide, the *meteorological tides* should be addressed. Most meteorological phenomena are unpredictable and not harmonic. There are, however, two exceptions on that rule:

- *Monsoons*, which blow in one direction during half of the year and in the opposite direction during the other half. This causes an annual variation of the water level. That variation can be described by a harmonic wave, with a period of one year;
- A wave period of one day, which is caused by *alternating land- and sea wind*. In some tropical regions that should be included in the tidal analysis.

3 Analysis and prediction of tides

3.1 Introduction

Chapter 2 discussed the generation of tides and the main tidal constituents. This Chapter discusses the analysis and prediction of the tide at a certain location, based on a *measured tidal signal*.

Figure 3.1 shows an observed tidal signal. The decomposition of the tide generating forces provides accurate information about the *frequencies* of the harmonic components of the tidal signal. The *magnitudes* and *phase lags* of the components do not follow from theoretical considerations. They must be calculated from the observed tide at a given location. The derivation of the characteristics of the components from the observed tide is called *tidal analysis*.

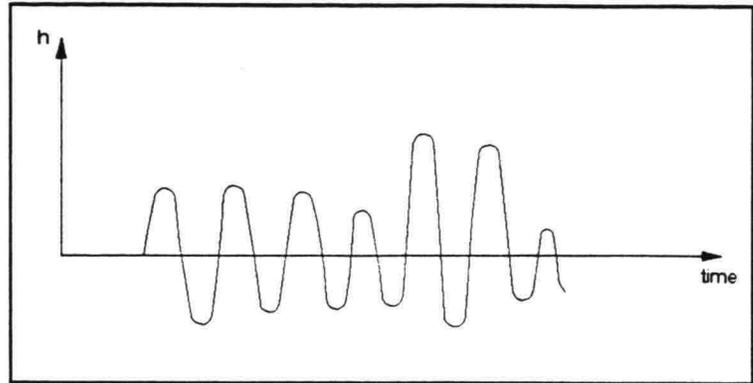


Figure 3.1 Observed tidal signal

3.2 Harmonic analysis of the tide

3.2.1 Formula used in tidal analysis

The tidal analysis is based on the general formula for the tractive force:

$$F_s = \frac{3gM}{2K^3} [A_o + \sum_{i=1}^n A_i \cos(\omega_i t + \phi_i)] \quad (3.1)$$

The analysis of the observed tide is based on a similar relation:

$$h(t) = h_o + \sum_{i=1}^n h_i \cos(\omega_i t - \alpha_i) \quad (3.2)$$

where:

$h(t)$ water level at time t

h_o mean water level

ω_i angular frequency of component i (known)

h_i amplitude of component i (unknown)

α_i phase lag of component i , related to the time base of the observation (unknown)

Equation 3.2 can be rewritten, by introducing the corrections due to the revolution of the moon's nodes:

f_i multiplying factor for the amplitude;
 u_i phase correction for the phase angle.

Further, the phase angle can be related to the equilibrium tide in Greenwich (England). Then Equation 3.2 becomes:

$$h(t) = h_o + \sum_{i=1}^n f_i H_i \cos(\omega_i t - v_i + u_i) \quad (3.3)$$

where:

H_i amplitude of component i $\left(= \frac{h_i}{f_i} \right)$

$v_i + u_i$ phase angle of the equilibrium tide in Greenwich of constituent i at $t = 0$ (astronomical argument)

v_i uniform changing part

t GMT (Greenwich mean time)

Equation 3.3 holds for the equilibrium tide, which is observed at the meridian of Greenwich (England; see Figure 3.2).

For analysing the tidal signal at an arbitrary location on the globe, indicated as P (see Figure 3.3), its relative location to Greenwich should be taken into account. Location P is L degrees west of Greenwich, and it is there S hours earlier. To include these, the following corrections can be made:

- phase correction for the location: $-pL$;
 $p = 0$ for long period tides;
 $p = 1$ for diurnal tides;
 $p = 2$ for semi-diurnal tides;
- phase correction for the time: $+\omega S$.

Thus, the correction in phase angle is $-pL + \omega S$

The formula for the *equilibrium tide* at an *arbitrary location* becomes:

$$h(t) = h_o + \sum_{i=1}^n f_i H_i \cos(\omega_i t + v_i + u_i - pL + \omega_i S) \quad (3.4)$$

Equation 3.4 holds for the analysis of the equilibrium tide.

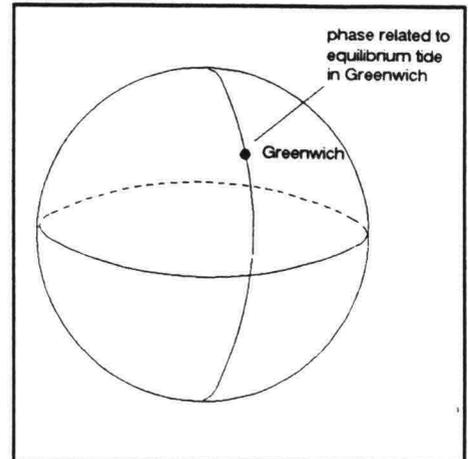


Figure 3.2 Meridian of Greenwich

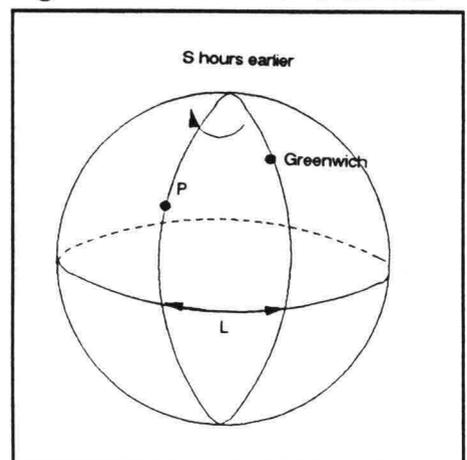


Figure 3.3 Location P related to Greenwich

The next step is to come to the real tide. The phase of the components for the real tide will differ from those of the equilibrium tide. This phase difference is called kappa K_i (for each component i). So the equation for the analysis of the *real tide* at an *arbitrary location* becomes:

$$h(t) = h_o + \sum_{i=1}^n f_i H_i \cos(\omega_i t + v_i + u_i - pL + \omega_i S - K_i) \quad (3.5)$$

In tidal analyses it is usual to use the *corrected kappa number* g_i , which is expressed by:

$$-g_i = -pL + \omega_i S - K_i \quad (3.6)$$

Substituting the corrected kappa number in Equation 3.5 gives:

$$h(t) = h_o + \sum_{i=1}^n f_i H_i \cos(\omega_i t + v_i + u_i - g_i) \quad (3.7)$$

In Equation 3.7 H_i and g_i are the tidal constants which have to be determined from the observed tidal signal. The other factors f_i , ω_i , $(v_i + u_i)$ are known from astronomical data.

Equation 3.7 is used for the analysis of the tide and also for the inverse operation, the prediction of the tide. When the tidal constants are known for a location, the astronomical tide (the tide without meteorological influences) can be predicted for any period in future at that location. The method can be used for water levels (which is most used), but also for velocities.

After this introduction into tidal analysis, the procedure how to determine tidal components from an observed signal is discussed. For this, a closer look is taken at the basic formula (Equation 3.7) for the tidal analysis. The observed tidal signal is composed of many sinusoidal functions, each with its own:

- amplitude;
- angular speed;
- phase (at $t = 0$).

The unknowns are the tidal constants H_i and g_i . The other factors are known from astronomical analysis, that are f_i , ω_i , v_i , u_i . For the analysis, Equation 3.7 is simplified into:

$$h(t) = h_o + \sum_{i=1}^n h_i \cos(\omega_i t - \alpha_i) \quad (3.8)$$

in which h_i and α_i are the unknowns. The real tidal constants can be derived from them by:

$$H_i = \frac{h_i}{f_i}$$

$$g_i = \alpha_i + v_i + u_i$$

Two methods are commonly applied for tidal analysis:

- method of least squares;
- Fourier analysis.

In these lecture notes, the method of least squares is further elaborated.

An example of the result of a tidal analysis is presented in Figure 3.4, which shows the tidal amplitude (H_i) in m as a function of the angular speed (ω_i) for location Hook of Holland along the Dutch coast.

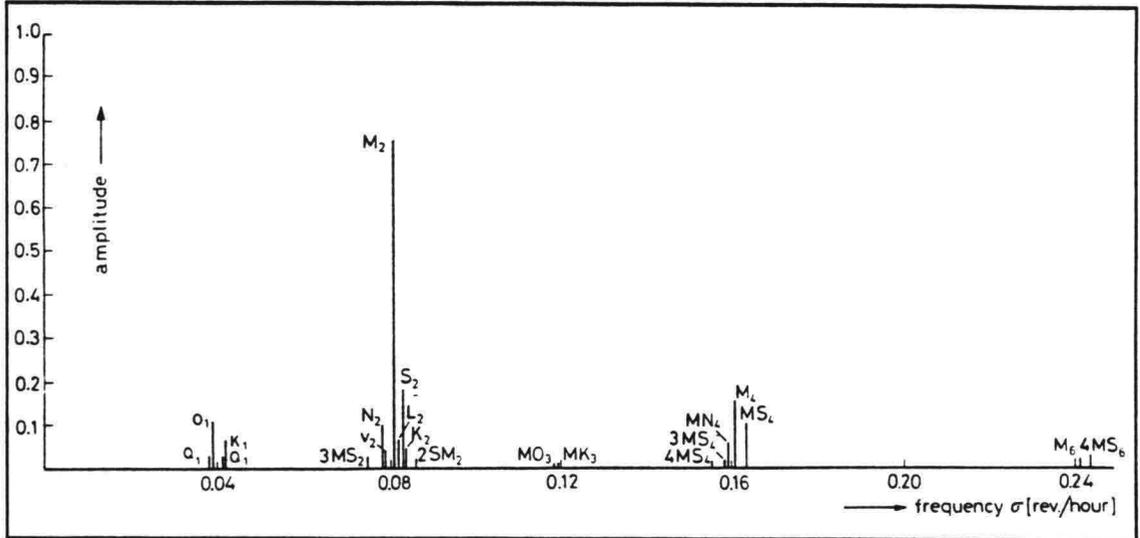


Figure 3.4 Results of a tidal analysis for Hook of Holland (The Netherlands), showing the amplitudes H_i as function of the angular speeds ω_i

In Figure 3.4 the following groups of components can be distinguished:

- the first group around 0.04 rev./hour, which consist of diurnal components (once a day);
- the second group around 0.08 rev./hour, which consist of semi-diurnal components (twice a day). The M_2 (semi-diurnal lunar tide) and S_2 (semi-diurnal solar tide) are the most important components;
- the third group around 0.12 rev./hour, consist of ter-diurnal components (less important);
- the fourth group around 0.16 rev./hour, being the quarter-diurnal components. In this group, M_4 is a rather important component; it is a shallow water component;
- the last group, which is composed of the sixth-diurnal components.

Figure 3.4 shows that within a group the differences in angular speed (or frequency) are very small.

3.2.2 Method of least squares

The aim of a tidal analysis is to determine amplitudes and phases for a series of sinus functions from an observed tidal signal. The determination of amplitudes and phases is a problem of best fit, for which the method of least squares

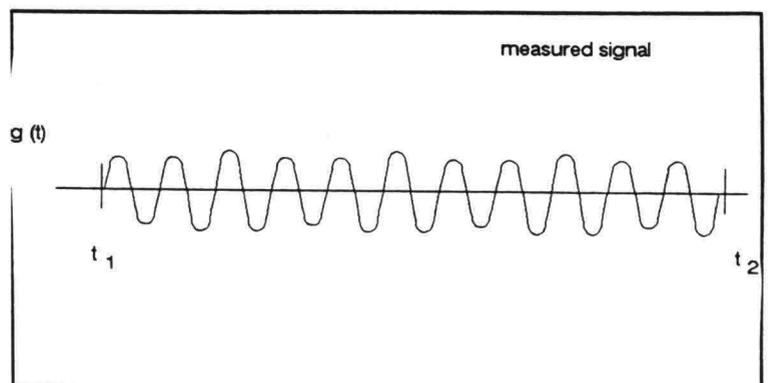


Figure 3.5 Measured tidal signal

can be applied. Suppose a tide is observed as is shown by the measured signal $g(t)$ over a time interval t_1-t_2 (see Figure 3.5)

Then a function $h(t)$ can be found, which is an approximation of the measured signal (see Figure 3.6). The function $h(t)$ contains four parameters A_1, A_2, B_1, B_2 . The measured signal $g(t)$ and the approximation $h(t)$ are not equal. The above four parameters should be determined such that the best fit with the measured signal can be found.

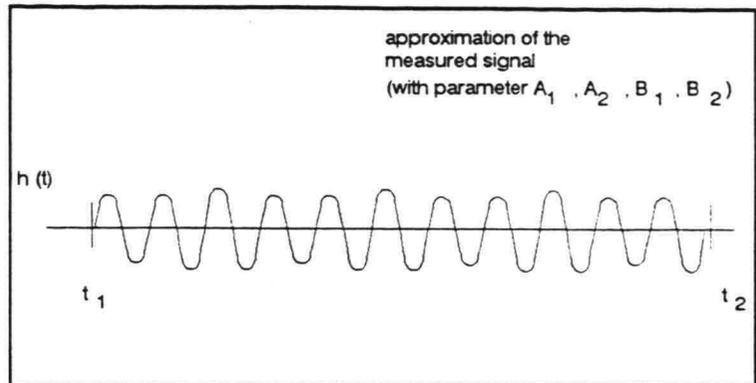


Figure 3.6 Approximation of the measured tide

There is a small difference or error $\epsilon(t)$ between the two functions:

$$\epsilon(t) = h(t) - g(t) \tag{3.9}$$

The method of least squares requires that the error $\epsilon(t)^2$, integrated over the time interval $t_1 - t_2$ is minimum:

$$\int_{t_1}^{t_2} \epsilon(t)^2 dt = \text{minimum} \tag{3.10}$$

The parameters A_1, A_2, B_1, B_2 are parameters of approximation $h(t)$ to minimize the error. So, Equation 3.10 can be rewritten as:

$$\int_{t_1}^{t_2} \epsilon(t)^2 dt = \int_{t_1}^{t_2} [h(t) - g(t)]^2 dt = F(A_1, A_2, B_1, B_2) = \text{minimum} \tag{3.11}$$

The function F will be minimum, when the derivatives to A_1, A_2, B_1, B_2 are 0:

$$\frac{\delta F}{\delta A_1} = 0, \quad \frac{\delta F}{\delta A_2} = 0, \quad \frac{\delta F}{\delta B_1} = 0, \quad \frac{\delta F}{\delta B_2} = 0$$

These are four equations with four unknowns A_1, A_2, B_1, B_2 . So the unknowns can be solved from these equations and the best fit approximation is found.

The method of least squares can be demonstrated for the case of two sinusoidal functions (or tidal components). In a real tidal analysis, more sinusoidal functions (or tidal components) are involved, for which the help of a computer is required to solve all equations. The general simplified expression for a tidal component is:

$$h \cos(\omega t - \alpha)$$

For two components, $h(t)$ can be written as:

$$h(t) = h_1 \cos(\omega_1 t - \alpha_1) + h_2 \cos(\omega_2 t - \alpha_2) \quad (3.12)$$

where $h_1, \alpha_1, h_2, \alpha_2$ have to be determined.

Expression 3.12 can be elaborated as:

$$h(t) = h_1 \cos(\omega_1 t) \cos \alpha_1 + h_1 \sin(\omega_1 t) \sin \alpha_1 + h_2 \cos(\omega_2 t) \cos \alpha_2 + h_2 \sin(\omega_2 t) \sin \alpha_2 \quad (3.13)$$

To simplify the procedure, the following relations are introduced:

$$h_1 \cos \alpha_1 = A_1 \quad h_2 \cos \alpha_2 = A_2$$

$$h_1 \sin \alpha_1 = B_1 \quad h_2 \sin \alpha_2 = B_2$$

Substitution in Equation 3.13 gives:

$$h(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)$$

For determining parameters A_1, B_1, A_2, B_2 , values from the measured signal $g(t)$ of Figure 3.5 at time instants $t_0, t_0 + \Delta t, t_0 + 2\Delta t, \dots, t_0 + k\Delta t$ can be taken

(see Figure 3.7). For $t_0 + i\Delta t = t_i$, the corresponding value is $g(t_i)$. For $g(t_0)$ to $g(t_k)$, the integrated error (which should be minimum), can be written as:

$$F(A_1, B_1, A_2, B_2) = \sum_{i=0}^k [h(t_i) - g(t_i)]^2 \Delta t$$

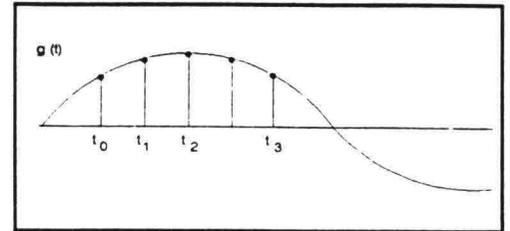


Figure 3.7 Values of $g(t_i)$

Substituting the relation for $h(t_i)$ gives:

$$F = \sum_{i=0}^k [A_1 \cos(\omega_1 t_i) + B_1 \sin(\omega_1 t_i) + A_2 \cos(\omega_2 t_i) + B_2 \sin(\omega_2 t_i) - g(t_i)]^2 \Delta t$$

The derivatives of F to the parameters A_1, B_1, A_2, B_2 should be 0.

$$\frac{\delta F}{\delta A_1} = 0, \quad \frac{\delta F}{\delta B_1} = 0, \quad \frac{\delta F}{\delta A_2} = 0, \quad \frac{\delta F}{\delta B_2} = 0$$

The derivatives become:

$$\frac{\delta F}{\delta A_1} = \sum_{i=0}^k 2[A_1 \cos(\omega_1 t_i) + B_1 \sin(\omega_1 t_i) + A_2 \cos(\omega_2 t_i) + B_2 \sin(\omega_2 t_i) - g(t_i)] \cos(\omega_1 t_i) \Delta t = 0$$

$$\frac{\delta F}{\delta B_1} = \sum_{i=0}^k 2[A_1 \cos(\omega_1 t_i) + B_1 \sin(\omega_1 t_i) + A_2 \cos(\omega_2 t_i) + B_2 \sin(\omega_2 t_i) - g(t_i)] \sin(\omega_1 t_i) \Delta t = 0$$

$$\frac{\delta F}{\delta A_2} = \sum_{i=0}^k 2[A_1 \cos(\omega_1 t_i) + B_1 \sin(\omega_1 t_i) + A_2 \cos(\omega_2 t_i) + B_2 \sin(\omega_2 t_i) - g(t_i)] \cos(\omega_2 t_i) \Delta t = 0$$

$$\frac{\delta F}{\delta B_2} = \sum_{i=0}^k 2[A_1 \cos(\omega_1 t_i) + B_1 \sin(\omega_1 t_i) + A_2 \cos(\omega_2 t_i) + B_2 \sin(\omega_2 t_i) - g(t_i)] \sin(\omega_2 t_i) \Delta t = 0$$

Elaborating the first equation by dividing both sides with $2\Delta t$ yields:

$$\sum_{i=0}^k [A_1 \cos(\omega_1 t_i) + B_1 \sin(\omega_1 t_i) + A_2 \cos(\omega_2 t_i) + B_2 \sin(\omega_2 t_i) - g(t_i)] \cos(\omega_1 t_i) = 0$$

Further elaboration of this relation gives:

$$\begin{aligned} A_1 \sum_{i=0}^k \cos(\omega_1 t_i) \cos(\omega_1 t_i) + B_1 \sum_{i=0}^k \sin(\omega_1 t_i) \cos(\omega_1 t_i) + A_2 \sum_{i=0}^k \cos(\omega_2 t_i) \cos(\omega_1 t_i) + B_2 \sum_{i=0}^k \sin(\omega_2 t_i) \cos(\omega_1 t_i) = \\ = \sum_{i=0}^k g(t_i) \cos(\omega_1 t_i) \end{aligned}$$

Doing a similar derivation for the other three equations gives a set of equations of the following form:

$$\begin{aligned} A_1 a_{11} + B_1 a_{12} + A_2 a_{13} + B_2 a_{14} &= b_1 \\ A_1 a_{21} + B_1 a_{22} + A_2 a_{23} + B_2 a_{24} &= b_2 \\ A_1 a_{31} + B_1 a_{32} + A_2 a_{33} + B_2 a_{34} &= b_3 \\ A_1 a_{41} + B_1 a_{42} + A_2 a_{43} + B_2 a_{44} &= b_4 \end{aligned}$$

Here, A_1, B_1, A_2, B_2 are unknowns and $a_{11} - a_{44}$ and $b_1 - b_4$ are known. The four linear equation with the 4 unknowns can be solved by mathematical techniques. When parameters A_1, B_1, A_2, B_2 have been found, the parameters $h_1, \alpha_1, h_2, \alpha_2$ can be determined by:

$$\begin{aligned} h_1 \cos \alpha_1 &= A_1 & h_2 \cos \alpha_2 &= A_2 \\ h_1 \sin \alpha_1 &= B_1 & h_2 \sin \alpha_2 &= B_2 \end{aligned}$$

So the parameters of the function $h(t) = h_1 \cos(\omega_1 t - \alpha_1) + h_2 \cos(\omega_2 t - \alpha_2)$ are determined, thereby approximating the measured signal $g(t)$.

Now the residual function $\epsilon(t) = h(t) - g(t)$ remains. In some cases, this residual is examined to see whether some components are overlooked. It also contains meteorological effects (like wind set-up). If we want to analyse the residual we make use of *spectral analysis*, which is not further discussed here.

3.2.3 Sample interval

To make a good estimation of the amplitudes and phases of tidal components in a tidal analysis, values are taken from the measured signal at a certain time interval Δt (see Figure 3.8). These are called samples. The time between two samples is called sample interval, and is usually taken constant.

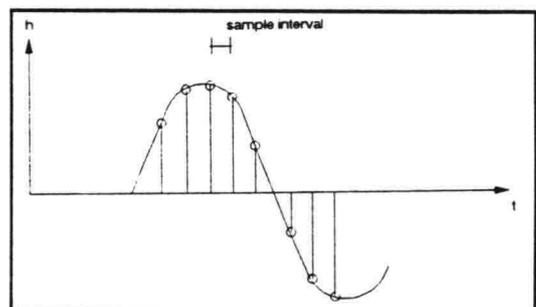


Figure 3.8 Values at sample interval Δt

It can be proved that a sinus function must be sampled at least 2 times per period. For example, when the tidal component M_2 is important, with a period of 3 hours, the sample interval should be less than $1\frac{1}{2}$ hours. A common sample interval is 1 hour in tidal analysis.

3.2.4 Duration of the tidal measurement

Figure 3.5 showed the results of the tidal analysis for Hook of Holland. Within the groups, components with angular speeds vary close to each other. Take for instance M_2 and S_2 . They have angular speeds of $29^\circ/\text{hour}$ and $30^\circ/\text{hour}$ respectively. If these components need to be separated in a tidal analysis, a certain length of the observation is required.

Consider two components with slightly different angular speeds ω_1 and ω_2 and same amplitude (1). Then the sum becomes:

$$h(t) = \sin(\omega_1 t) + \sin(\omega_2 t)$$

with:

$$\omega_1 = n\Delta\omega$$

$$\omega_2 = (n - 1)\Delta\omega$$

n is large, so $(\omega_1 - \omega_2) = \Delta\omega$ is small.

Substitution yields:

$$h(t) = 2\cos\left(\frac{1}{2}(\omega_1 - \omega_2)t\right) \times \sin\left(\frac{1}{2}(\omega_1 + \omega_2)t\right) = 2\cos\left(\frac{1}{2}\Delta\omega t\right) \times \sin\left(\omega_1 - \frac{1}{2}\Delta\omega\right)t$$

The sinus function represents the fast oscillating part of the combined wave; the cosine function represents the slow oscillating part. The cosine function is the slowly varying amplitude of the sinus function (see Figure 3.9).

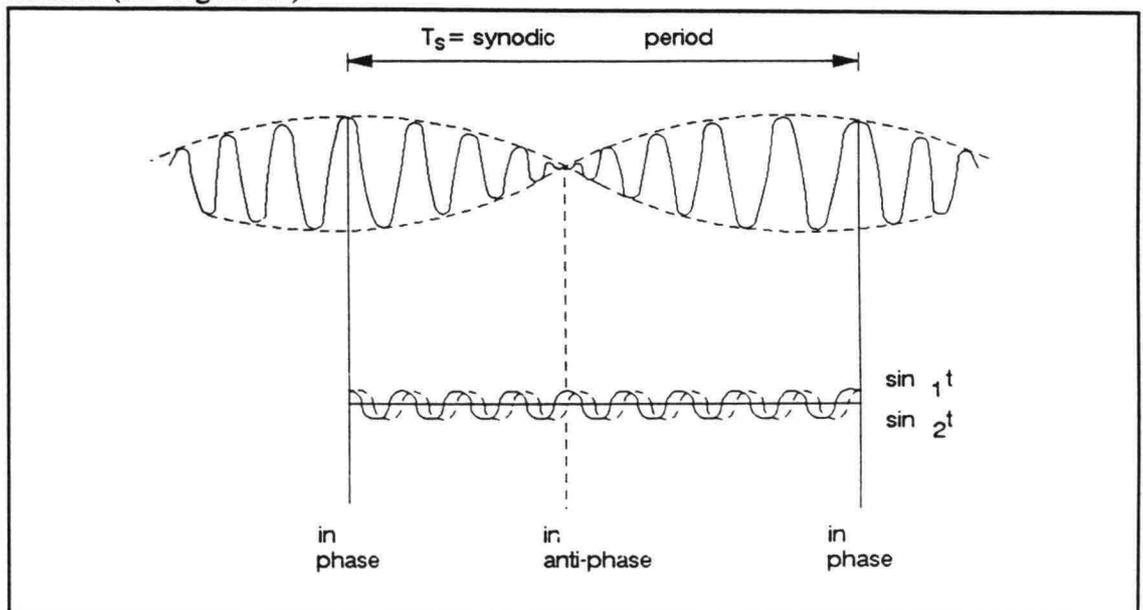


Figure 3.9 Resulting tidal curve consisting of two components

From Figure 3.9, the following conclusions can be drawn:

- the combined amplitude is maximum when $\sin\omega_1 t$ and $\sin\omega_2 t$ are in phase;
- the amplitude is minimum (= 0), when $\sin\omega_1 t$ and $\sin\omega_2 t$ are in anti-phase;
- the function is periodic with T_s , which is called the *synodic period* of the two components.

Period T_s can be derived from the equation for $h(t)$, in which:

$$\frac{\Delta\omega}{2} T_s = \pi \quad \rightarrow \quad T_s = \frac{2\pi}{\Delta\omega}$$

In the synodic period:

$$\sin(\omega_1 t) = \sin(n\Delta\omega t) \quad \text{makes } n \text{ oscillations, and}$$

$$\sin(\omega_2 t) = \sin(n - 1)\Delta\omega t \quad \text{makes } n + 1 \text{ oscillations.}$$

To separate two components in a tidal analysis, at least *one synodic period* must be measured. Thus, to separate diurnal and semi-diurnal components, their angular speeds should be considered:

$$\omega_{diurnal} \approx 15^\circ/h \quad (=360^\circ/24 \text{ h})$$

$$\omega_{semi-diurnal} \approx 30^\circ/h \quad (=360^\circ/12 \text{ h})$$

The criterion for separation is:

$$T = \frac{2\pi}{\Delta\omega} = \frac{360^\circ}{30^\circ/h - 15^\circ/h} = \frac{360^\circ}{15^\circ/h} = 24 \text{ h} = 1 \text{ day}$$

To separate M_2 and S_2 , the minimum observation period should be at least:

$$(\omega_{M_2} = 29^\circ/h, \omega_{S_2} = 30^\circ/h)$$

$$T = \frac{2\pi}{\Delta\omega} = \frac{360^\circ}{30^\circ/h - 29^\circ/h} = \frac{360^\circ}{1^\circ/h} = 360 \text{ h} = 15 \text{ days}$$

Table 3.1 gives the minimum observation periods to separate tidal components P_1 , K_1 , O_1 and Q_1 . For instance, 13.7 days of observation are required to separate diurnal tides K_1 and O_1 .

Table 3.1 Minimum observation periods (in days) for four diurnal tidal components

Tidal component	Minimum observation period			
	P_1	K_1	O_1	Q_1
P_1	x	182.6	14.8	9.6
K_1		x	13.7	9.1
O_1			x	27.6
Q_1				x

Table 3.2 shows the required observation periods to separate the semi-diurnal tides S_2 , K_2 , M_2 , N_2 , L_2 and $2MS_2$. To separate the important semi-diurnal tides M_2 and S_2 , a minimum observation period of 14.8 days is required. Other components need a period of about 30 days.

Table 3.2 Minimum observation periods (in days) for six semi-diurnal tidal components

Tidal component	Minimum observation period (days)					
	S ₂	K ₂	M ₂	N ₂	L ₂	2MS ₂
S ₂	x	182.6	14.8	9.6	31.7	7.4
K ₂		x	13.7	9.1	27.1	7.1
M ₂			x	27.6	27.6	14.8
N ₂				x	13.8	9.6
L ₂					x	9.6
2MS ₂						x

The period of 30 days is more or less accepted as the standard observation period for a minimum tidal analysis. To separate S₂ and K₂ and P₁ and K₁, an observation period of half a year is required. This is displayed in the Tables as 182.6 days. A period of 369 days (about 1 year) is very nearly a multiple of all values of the synodic periods. Therefore, 369 days is considered as the standard length for a tidal analysis.

3.3 Tidal prediction

A tidal prediction is the *inverse process* of the tidal analysis. When the harmonic constants (= the amplitudes and phase angles) at a given location are known (which are always valid), the tide can be predicted for any time in future. There is, however, one condition. The physical condition of the sea or the river must not change. In cases where important civil engineering works have been implemented (like the Delta Works in The Netherlands) the tidal components will be affected by the morphological changes.

The prediction of the tide is carried out by using Equation 3.7:

$$h(t) = h_o + \sum_{i=1}^n f_i H_i \cos(\omega_i t + v_i + u_i - g_i)$$

From the harmonic analysis, the harmonic constituents H_i and g_i are known. The mean sea level h_o is also derived from the harmonic analysis of the tide. The angular speed ω_i of each constituent is known from the astronomical analysis. Table 2.4 presents the astronomic components, whereas Table 2.6 gives the shallow water components. The nodal factor f_i and the astronomic argument or equilibrium argument ($v_i + u_i$) have been computed for many years in advance. The values for the node factor f_i are given in Table 3.3. These node factors are taken at the middle of each year from 1970 to 1999 for a range of tidal components, and are considered to be constant over one year! Finally, Tables 3.4, 3.5 and 3.6 a, b, c give the values for the astronomic argument ($v_i + u_i$).

Table 3.3 Node factor f for the middle of each year for the period 1970 to 1999 (P. Schureman: *Manual of harmonic analysis and prediction of tides. U.S. Department of Commerce, Coast and Geodetic Survey, 1941*)

Constituent	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979
J ₁	1.155	1.132	1.097	1.051	0.995	0.936	0.881	0.842	0.827	0.839
K ₁	1.105	1.088	1.063	1.029	0.991	0.951	0.916	0.891	0.882	0.890
K ₂	1.289	1.232	1.150	1.055	0.957	0.871	0.804	0.763	0.748	0.760
L ₂	0.889	0.868	1.118	1.270	1.014	0.898	0.988	1.179	1.169	0.904
M ₁	1.987	2.178	1.503	1.012	1.535	1.777	1.428	0.870	0.874	1.361
M ₂ ^o , N ₂ , 2N, λ ₂ , μ ₂ , ν ₂	0.986	0.973	0.983	0.995	1.008	1.020	1.029	1.035	1.038	1.036
M ₃	0.950	0.980	0.975	0.983	1.012	1.029	1.044	1.054	1.057	1.054
M ₄ , MN	0.934	0.948	0.967	0.981	1.016	1.039	1.059	1.072	1.077	1.073
M ₆	0.903	0.922	0.951	0.986	1.024	1.069	1.090	1.110	1.118	1.112
M ₈	0.873	0.898	0.935	0.981	1.032	1.081	1.122	1.149	1.160	1.151
O ₁ , Q ₁ , 2Q, ρ ₁	1.170	1.143	1.101	1.047	0.984	0.920	0.863	0.822	0.806	0.819
OO	1.716	1.875	1.390	1.159	0.940	0.750	0.607	0.517	0.485	0.512
MK	1.088	1.059	1.045	1.024	0.998	0.970	0.943	0.923	0.915	0.922
2MK	1.082	1.031	1.028	1.020	1.006	0.989	0.970	0.956	0.950	0.955
Mf	1.417	1.341	1.233	1.102	0.952	0.831	0.723	0.652	0.625	0.647
Mm	0.883	0.906	0.940	0.982	1.025	1.067	1.100	1.123	1.131	1.124

Constituent	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989
J ₁	0.877	0.830	0.869	1.045	1.093	1.130	1.163	1.164	1.163	1.148
K ₁	0.912	0.948	0.987	1.026	1.080	1.086	1.104	1.112	1.111	1.100
K ₂	0.790	0.864	0.949	1.045	1.142	1.226	1.285	1.315	1.310	1.270
L ₂	0.868	1.001	1.238	1.157	0.745	0.811	1.263	1.244	0.749	0.746
M ₁	1.656	1.468	0.974	1.323	2.050	2.082	1.292	1.367	2.142	2.122
M ₂ ^o , N ₂ , 2N, λ ₂ , μ ₂ , ν ₂	1.030	1.021	1.009	0.997	0.984	0.974	0.967	0.964	0.964	0.960
M ₃	1.045	1.031	1.013	0.994	0.977	0.962	0.951	0.946	0.947	0.954
M ₄ , MN	1.061	1.042	1.018	0.993	0.969	0.949	0.935	0.928	0.930	0.939
M ₆	1.092	1.063	1.027	0.999	0.954	0.924	0.904	0.894	0.896	0.910
M ₈	1.125	1.085	1.036	0.996	0.939	0.901	0.874	0.862	0.864	0.881
O ₁ , Q ₁ , 2Q, ρ ₁	0.858	0.915	0.979	1.041	1.096	1.140	1.168	1.182	1.180	1.161
OO	0.596	0.735	0.921	1.137	1.361	1.560	1.706	1.778	1.786	1.668
MK	0.941	0.967	0.996	1.022	1.043	1.058	1.068	1.072	1.071	1.065
2MK	0.960	0.967	1.005	1.019	1.027	1.031	1.032	1.032	1.032	1.032
Mf	0.715	0.820	0.949	1.088	1.221	1.333	1.412	1.450	1.443	1.392
Mm	1.103	1.070	1.029	0.986	0.944	0.909	0.884	0.872	0.874	0.891

Constituent	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
J ₁	1.120	1.080	1.030	0.972	0.914	0.864	0.833	0.829	0.852	0.896
K ₁	1.079	1.051	1.015	0.976	0.937	0.905	0.886	0.883	0.897	0.926
K ₂	1.203	1.115	1.016	0.922	0.842	0.785	0.754	0.750	0.772	0.821
L ₂	1.216	1.248	0.898	0.801	1.077	1.208	1.107	0.921	0.893	1.096
M ₁	1.284	1.156	1.778	1.829	1.282	0.900	1.063	1.487	1.590	1.214
M ₂ ^o , N ₂ , 2N, λ ₂ , μ ₂ , ν ₂	0.977	0.988	1.000	1.013	1.024	1.032	1.037	1.038	1.034	1.027
M ₃	0.966	0.982	1.000	1.019	1.036	1.048	1.056	1.057	1.051	1.040
M ₄ , MN	0.955	0.976	1.000	1.025	1.048	1.065	1.075	1.076	1.069	1.054
M ₆	0.932	0.964	1.000	1.038	1.072	1.099	1.115	1.117	1.105	1.082
M ₈	0.911	0.952	1.000	1.051	1.098	1.134	1.156	1.159	1.143	1.111
O ₁ , Q ₁ , 2Q, ρ ₁	1.128	1.081	1.024	0.960	0.897	0.844	0.812	0.808	0.832	0.879
OO	1.505	1.296	1.072	0.843	0.688	0.565	0.498	0.489	0.538	0.543
MK	1.054	1.038	1.015	0.988	0.959	0.934	0.918	0.916	0.928	0.950
2MK	1.030	1.025	1.015	1.000	0.982	0.964	0.952	0.951	0.959	0.976
Mf	1.303	1.184	1.048	0.910	0.786	0.691	0.636	0.629	0.669	0.752
Mm	0.918	0.956	0.998	1.042	1.081	1.110	1.128	1.130	1.117	1.091

Note Factor f of P₁, R₂, S₁, S₂, S₄, S₆, T₂, S_a and S_{sa} are each unity

Table 3.4 Equilibrium argument (V_0+u) for meridian of Greenwich at the start of each calendar year (P. Schureman: Manual of harmonic analysis and prediction of tides. U.S. Department of Commerce, Coast and Geodetic Survey, 1941)

Constituent	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989
J ₁	270.0	1.8	92.1	197.4	286.6	14.3	100.2	198.6	281.8	4.9	89.1	188.9	276.5	5.6	95.8	201.0	292.7	24.7	114.8	222.8
K ₁	13.4	15.5	17.1	19.0	19.2	18.2	16.2	14.1	10.5	6.8	3.7	2.5	1.5	1.5	2.4	4.9	6.9	9.2	11.6	14.8
K ₂	207.2	211.6	214.8	218.4	218.2	215.8	211.5	207.6	200.9	194.2	188.2	185.6	183.1	182.6	184.2	189.3	193.5	198.3	203.3	210.0
L ₂	262.8	60.3	345.2	81.8	282.4	99.0	276.4	96.7	297.9	130.9	310.2	116.6	313.4	159.2	350.3	135.8	334.5	188.0	25.5	166.0
M ₁	156.2	48.6	307.8	234.9	158.7	51.8	302.5	198.6	135.7	41.8	291.5	170.8	87.9	30.0	291.6	171.7	82.0	31.2	299.1	179.8
M ₂	165.4	268.8	8.0	84.5	185.3	285.8	26.0	101.8	201.8	301.9	42.0	117.9	218.4	319.1	60.0	136.8	238.1	339.6	81.0	158.1
M ₃	68.1	40.2	11.9	306.8	277.9	248.6	219.0	152.7	122.7	92.8	63.0	356.8	327.6	298.6	270.0	205.2	177.2	149.3	121.6	57.1
M ₄	330.9	173.6	15.9	169.1	10.5	211.5	52.1	203.6	43.7	243.8	84.0	235.8	76.7	278.1	120.0	273.6	116.2	319.1	162.1	316.2
M ₅	136.3	80.4	28.9	253.6	195.8	137.2	78.1	305.3	245.5	185.7	129.0	352.7	295.1	237.2	180.0	50.3	354.3	298.7	243.1	114.2
M ₆	301.7	347.1	31.8	338.2	21.0	63.0	104.1	47.1	87.3	127.5	168.0	111.6	153.4	196.2	240.0	187.1	232.4	278.2	324.1	272.3
N ₁	270.1	262.7	295.2	270.0	282.0	293.8	305.3	279.3	290.6	301.9	313.3	287.4	290.2	311.2	323.4	298.4	311.0	323.7	336.4	311.7
2N ₁	14.8	298.7	224.4	95.4	18.7	301.8	224.6	96.8	19.4	302.0	224.7	97.0	20.0	303.2	226.7	100.0	23.8	307.8	231.9	105.3
O ₁	180.7	249.0	347.8	61.7	162.0	263.5	6.5	85.7	191.3	296.9	41.6	119.4	221.1	321.4	60.8	134.2	232.6	330.7	68.8	141.6
OO	68.7	326.4	232.6	183.9	64.6	321.1	212.6	126.5	9.8	253.0	139.2	57.5	313.6	213.9	117.6	51.0	318.5	227.0	185.6	71.2
P ₁	349.8	350.0	350.2	349.5	349.7	350.0	350.2	349.5	349.7	349.9	350.2	349.4	349.7	349.9	350.2	349.4	349.6	349.9	350.1	349.4
Q ₁	256.4	265.0	276.0	247.2	258.7	271.5	285.8	293.2	280.1	297.0	312.9	289.0	301.9	313.5	324.2	295.8	305.4	314.8	324.2	295.3
2Q ₁	0.1	281.0	262.2	78.6	345.4	279.5	305.1	80.7	8.8	207.0	224.3	96.5	22.7	305.6	227.5	97.4	18.3	299.0	219.6	88.9
R ₁	177.8	177.6	177.3	178.0	177.8	177.5	177.3	178.0	177.7	177.5	177.2	178.0	177.7	177.4	177.2	177.9	177.7	177.4	177.2	177.9
S ₁	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0
S ₂	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
T ₁	2.2	2.4	2.7	2.0	2.2	2.6	2.7	2.0	2.3	2.5	2.8	2.0	2.3	2.6	2.8	2.1	2.3	2.6	2.8	2.1
U ₁	265.6	197.4	109.1	7.6	278.8	189.8	100.6	358.2	288.8	179.4	90.0	347.8	258.8	170.6	81.4	240.0	251.9	163.9	75.9	324.8
U ₂	330.0	172.1	14.0	167.0	8.4	209.7	50.7	202.8	43.6	244.4	85.3	237.6	78.8	280.2	121.9	275.1	117.2	319.4	161.6	316.0
U ₃	225.3	156.1	86.8	341.5	271.7	131.4	25.3	314.9	244.4	174.0	68.0	358.0	298.2	218.6	113.5	44.3	335.2	266.2	161.4	145.0
U ₄	210.6	136.4	66.6	318.7	248.4	179.4	111.9	9.2	304.3	239.4	173.6	69.6	0.7	280.5	219.4	110.9	28.8	326.4	254.0	145.0
MK	178.9	262.3	25.1	103.6	204.4	304.0	42.2	115.9	212.3	308.7	45.7	120.4	219.8	320.5	62.4	141.7	245.0	348.8	92.6	172.8
2MK	317.4	158.1	358.8	150.0	351.4	193.3	35.9	199.4	33.2	237.0	80.4	233.3	73.3	276.6	117.6	268.6	109.3	309.9	150.5	301.4
MN	75.6	199.5	308.2	354.5	107.2	219.5	331.3	31.1	132.4	243.8	355.3	45.3	157.5	242.2	23.4	75.1	189.1	303.2	57.5	109.8
M8	165.4	266.8	8.0	84.5	185.3	285.8	26.0	101.8	201.8	301.9	42.0	117.9	218.4	319.1	60.0	136.8	238.1	339.6	81.0	158.1
2SM	194.6	98.2	352.0	275.5	174.7	74.2	334.0	258.2	158.2	58.1	318.0	242.1	141.6	40.9	200.0	222.2	131.9	20.4	279.0	201.9
Mf	44.0	308.7	212.4	141.1	41.3	298.8	193.0	110.4	359.3	248.0	138.8	59.0	316.2	216.2	118.4	48.4	313.0	218.1	123.4	54.8
MSf	194.6	93.2	352.0	275.5	174.7	74.2	334.0	258.2	158.2	58.1	318.0	242.1	141.6	40.9	200.0	222.2	131.9	20.4	279.0	201.9
Mm	255.3	344.0	72.8	174.6	263.3	352.0	80.7	182.5	271.2	0.0	88.7	190.5	279.2	7.9	96.6	198.4	287.1	18.9	104.6	206.4
Sa	280.2	280.0	279.8	280.5	280.3	280.0	279.8	280.5	280.3	280.1	279.8	280.6	280.3	280.1	279.8	280.6	280.4	280.4	280.2	280.6
Saa	200.5	200.0	199.5	201.0	200.5	200.0	199.6	201.1	200.6	200.1	199.6	201.1	200.6	200.2	199.7	201.2	200.7	200.2	199.8	201.3

Constituent	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
J ₁	314.4	45.2	185.2	227.9	325.0	50.2	134.0	231.1	322.5	39.3	125.8
K ₁	16.7	18.0	18.7	19.4	18.0	15.6	12.2	9.5	6.0	3.2	1.5
K ₂	213.9	216.6	217.6	218.5	215.4	210.3	204.0	196.2	192.7	187.2	183.4
L ₂	2.2	212.4	49.0	205.8	80.4	239.4	64.7	242.7	59.3	244.2	82.7
M ₁	85.9	33.5	306.0	184.1	79.3	4.9	283.5	178.9	64.8	319.0	251.4
M ₂	259.4	0.5	101.3	177.6	278.0	18.2	118.3	194.0	294.0	34.2	134.5
M ₃	29.1	0.7	332.0	296.4	237.0	207.3	177.4	110.9	81.0	51.8	21.8
M ₄	158.7	0.9	202.7	355.2	196.0	36.4	236.6	27.9	228.0	66.4	269.1
M ₅	58.1	1.4	304.0	172.8	114.0	54.6	354.9	221.9	162.1	102.6	43.6
M ₆	317.5	1.9	45.3	350.3	31.9	72.8	113.2	55.8	96.1	136.8	178.1
N ₁	324.3	336.7	348.8	323.3	334.9	346.4	357.8	331.7	83.0	354.5	6.1
2N ₁	29.2	312.8	236.3	104.9	31.9	314.6	237.3	109.4	32.0	314.7	237.6
O ₁	240.1	339.0	78.7	154.0	256.1	359.7	104.8	185.2	290.5	34.6	137.2
OO	338.4	243.8	146.6	73.2	327.7	217.2	102.3	12.5	256.5	144.4	37.4
P ₁	349.6	349.8	350.1	349.3	349.6	349.8	350.1	349.3	349.6	349.8	350.0
Q ₁	305.0	315.2	326.1	299.7	313.0	327.9	344.3	322.9	339.5	354.9	8.8
2Q ₁	9.9	291.4	213.6	85.4	10.0	299.2	223.8	100.6	28.5	315.2	240.3
R ₁	177.6	177.4	177.1	177.8	177.6	177.3	177.1	177.8	177.5	177.3	177.0
S ₁	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0	180.0
S ₂	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
T ₁	2.4	2.6	2.9	2.2	2.4	2.7	2.9	2.2	2.5	2.7	3.0
U ₁	246.6	158.2	69.6	327.7	288.6	149.4	60.0	317.5	228.1	138.8	49.6
U ₂	157.1	358.9	200.5	353.2	194.3	85.3	236.1	28.2	229.0	69.9	271.0
U ₃	92.2	22.7	313.1	207.5	187.4	67.0	356.6	250.4	180.0	109.6	39.4
U ₄	72.9	1.3	290.4	183.9	115.4	48.6	343.1	241.6	176.5	110.1	42.1
MK	276.0	18.5	120.0	197.0	296.0	33.8	130.5	303.4	37.4	136.0	136.0
2MK	142.0	342.0	184.0	335.8	177.9	20.8	224.4	18.4	222.1	65.2	267.6
MN	223.6	337.1	90.1	140.8	252.9	4.6	116.1	165.6	277.0	28.7	140.6
M8	269.4	0.5	101.3	177.6	278.0	18.2	118.3	194.0	294.0	34.2	134.5
2SM	100.6	359.5	258.7	182.4	82.0	241.8	241.7	166.0	66.0	325.8	225.5
Mf	319.2	222.4	124.0	49.6	305.9	198.8	88.8	3.7	253.0	144.9	40.1
MSf	100.6	359.5	258.7	182.4	82.0	241.8	241.7	166.0	66.0	325.8	225.5
Mm	295.1	23.8	112.5	214.3	203.0	31.8	120.5	222.3	311.0	39.7	128.4
Sa	280.4	280.2	279.9	280.7	280.4	280.2	279.9	280.7	280.4	280.2	280.7
Saa	200.8	200.3	199.8	201.3	200.8	200.4	200.9	201.4	200.9	200.4	200.9

Table 3.5 Differences to adapt the values of Table 3.4 to the beginning of each calendar month in any calendar year (*P. Schureman: Manual of harmonic analysis and prediction of tides. U.S. Department of Commerce, Coast and Geodetic Survey, 1941*)

	Month of year*											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
J ₁	0.00	75.57	109.99	184.56	246.06	321.65	23.17	98.74	174.31	235.82	311.39	12.91
K ₁	0.00	30.56	54.15	85.71	118.28	148.83	178.40	208.96	239.51	269.08	299.64	329.21
K ₂	0.00	61.11	116.31	177.42	236.56	297.66	356.80	37.91	119.02	178.16	239.27	298.41
L ₁	0.00	9.19	52.33	61.54	82.02	91.21	111.71	120.90	130.09	150.59	159.78	180.29
M ₁ f	0.00	345.54	7.32	352.86	350.48	336.02	333.64	319.18	304.72	302.34	287.88	285.50
M ₁ s	0.00	342.06	0.75	342.83	337.11	319.30	313.47	295.56	277.65	271.93	264.01	248.29
M ₂	0.00	324.17	1.49	325.66	314.22	278.39	266.95	231.12	195.30	183.85	145.02	136.58
M ₃	0.00	306.26	2.24	308.50	291.33	237.59	220.42	196.68	112.94	95.78	42.04	24.57
M ₄	0.00	298.35	2.98	291.33	268.44	196.79	173.99	102.24	30.59	7.70	296.05	273.10
M ₅	0.00	252.52	4.48	257.00	222.66	115.18	80.85	333.37	225.89	191.55	84.07	49.74
M ₆	0.00	216.69	5.97	222.66	176.89	33.58	247.80	204.49	61.18	15.40	232.10	156.32
N ₁	0.00	279.16	310.66	229.82	186.42	105.88	62.18	341.34	260.60	217.11	136.27	92.87
2N ₁	0.00	234.14	259.82	133.97	58.62	292.77	217.42	91.57	325.71	250.36	124.31	49.16
O ₁	0.00	293.82	303.34	236.96	195.94	129.56	88.65	22.16	315.78	274.77	208.39	167.37
OO	0.00	127.49	172.97	300.48	40.61	198.10	268.28	35.75	163.24	263.39	30.89	131.04
P ₁	0.00	329.44	301.85	271.29	241.72	211.17	181.60	151.04	120.49	90.92	60.36	30.79
Q ₁	0.00	249.80	252.50	141.11	68.14	316.75	243.78	132.39	20.99	308.03	196.63	123.67
2Q ₁	0.00	203.89	201.67	45.26	300.34	143.93	39.02	242.61	86.20	341.28	184.87	79.96
R ₁	0.00	30.56	54.15	85.71	118.28	148.83	178.40	208.96	239.51	269.08	299.64	329.21
S ₁	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
T ₁	0.00	329.44	301.85	271.29	241.72	211.17	181.60	151.04	120.49	90.92	60.36	30.79
A ₁	0.00	314.98	309.16	264.15	232.20	187.19	155.24	110.22	65.21	33.26	348.24	316.29
a ₁	0.00	288.35	2.98	291.33	268.44	196.79	173.99	102.24	30.59	7.70	296.05	273.16
a ₂	0.00	333.36	53.82	27.18	36.24	9.00	18.66	352.02	325.38	334.44	307.81	316.87
a ₃	0.00	302.81	355.66	296.47	277.96	220.77	200.26	143.07	85.87	65.36	8.17	347.66
MK	0.00	354.73	50.64	54.37	72.50	67.23	85.35	80.08	74.81	92.93	87.66	105.79
2MK	0.00	257.79	304.83	202.62	150.16	47.96	356.50	253.29	151.08	98.62	356.41	303.95
MN	0.00	243.33	312.15	195.48	140.64	28.97	329.13	212.47	95.80	40.96	284.29	229.45
MS	0.00	324.17	1.49	325.66	314.22	278.39	266.95	231.12	195.30	183.85	146.02	136.58
2SM	0.00	35.83	358.51	34.34	45.78	81.61	93.05	128.88	164.70	176.15	211.96	223.42
Mf	0.00	98.94	114.82	211.75	282.94	19.27	89.86	186.79	283.73	354.31	91.25	161.83
Msf	0.00	35.83	358.51	34.34	45.78	81.61	93.05	128.88	164.70	176.15	211.96	223.42
Mm	0.00	46.02	60.84	95.85	127.80	172.81	204.76	249.78	294.79	326.74	11.76	43.71
Sa	0.00	30.56	54.15	85.71	118.28	148.83	178.40	208.96	239.51	269.08	299.64	329.21
Ssa	0.00	61.11	116.81	177.42	236.56	297.66	356.80	57.91	119.02	178.16	239.27	298.41

Table 3.6a Differences to adapt Table 3.5 to the beginning of each day of a month (*P. Schureman: Manual of harmonic analysis and prediction of tides. U.S. Department of Commerce, Coast and Geodetic Survey, 1941*)

	Day of month*										
	1	2	3	4	5	6	7	8	9	10	11
J ₁	0.00	14.06	28.10	42.15	56.20	70.25	84.30	98.35	112.41	126.46	140.51
K ₁	0.00	0.99	1.97	2.96	3.94	4.93	5.91	6.90	7.88	8.87	9.86
K ₂	0.00	1.97	3.94	5.91	7.88	9.86	11.83	13.80	15.77	17.74	19.71
L ₁	0.00	348.08	337.37	326.06	314.73	303.42	292.10	280.78	269.47	258.15	246.84
M ₁ f	0.00	347.92	335.84	323.76	311.68	299.60	287.52	275.44	263.37	251.29	239.21
M ₁ s	0.00	347.81	335.62	323.43	311.24	299.05	286.86	274.68	262.47	250.28	238.09
M ₂	0.00	325.62	311.24	296.86	282.47	268.09	253.71	186.33	164.96	140.57	116.18
M ₃	0.00	323.43	286.86	250.28	213.71	177.14	140.57	103.99	67.42	30.85	354.28
M ₄	0.00	311.24	262.47	213.71	164.96	116.18	67.42	18.66	329.90	281.13	232.37
M ₅	0.00	298.86	213.71	140.57	67.42	354.28	281.13	207.99	134.84	61.70	348.56
M ₆	0.00	262.47	164.96	67.42	329.90	223.37	134.84	37.32	299.79	202.27	104.74
N ₁	0.00	322.55	285.11	247.66	210.21	172.77	135.32	97.88	60.43	22.98	345.54
2N ₁	0.00	308.40	258.98	208.47	157.95	107.44	56.93	6.42	315.91	265.40	214.89
O ₁	0.00	334.63	309.27	283.90	258.53	233.16	207.80	132.43	157.06	131.70	106.33
OO	0.00	27.34	54.68	82.02	109.35	136.68	164.02	191.37	218.71	246.05	273.38
P ₁	0.00	-0.99	-1.97	-2.96	-3.94	-4.93	-5.91	-6.90	-7.88	-8.87	-9.86
Q ₁	0.00	321.57	283.14	244.70	206.27	167.84	129.41	90.98	52.54	14.11	335.68
2Q ₁	0.00	308.50	257.01	205.51	154.01	102.51	51.02	259.52	308.02	256.53	205.03
R ₁	0.00	0.99	1.97	2.96	3.94	4.93	5.91	6.90	7.88	8.87	9.86
S ₁	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
T ₁	0.00	-0.99	-1.97	-2.96	-3.94	-4.93	-5.91	-6.90	-7.88	-8.87	-9.86
A ₁	0.00	346.93	333.87	320.80	307.74	294.68	281.61	268.54	255.48	242.42	229.35
a ₁	0.00	311.24	262.47	213.71	164.96	116.18	67.42	18.66	329.90	281.13	232.37
a ₂	0.00	324.30	288.00	252.91	217.21	181.51	145.81	110.11	74.42	38.72	1.02
a ₃	0.00	323.32	286.63	249.95	213.27	176.58	139.90	103.21	66.53	29.85	353.18
MK	0.00	336.60	313.21	289.81	266.42	243.02	219.63	196.23	172.83	149.44	126.04
2MK	0.00	310.25	260.50	210.75	161.00	111.25	51.51	11.76	322.01	272.26	222.51
MN	0.00	298.17	236.34	174.52	112.69	50.86	349.03	287.20	225.38	163.55	101.72
MS	0.00	335.62	311.24	286.86	262.47	238.09	213.71	189.33	164.95	140.57	116.18
2SM	0.00	24.38	48.76	73.14	97.53	121.91	146.29	170.67	195.05	219.43	243.82
Mf	0.00	26.35	52.71	79.06	105.41	131.76	158.12	184.47	210.82	237.18	263.53
Msf	0.00	24.38	48.76	73.14	97.53	121.91	146.29	170.67	195.05	219.43	243.82
Mm	0.00	13.07	26.13	39.20	52.26	65.32	78.39	91.46	104.52	117.58	130.65
Sa	0.00	0.99	1.97	2.96	3.94	4.93	5.91	6.90	7.88	8.87	9.86
Ssa	0.00	1.97	3.94	5.91	7.88	9.86	11.83	13.80	15.77	17.74	19.71

Analysis and prediction of tides

Table 3.6b Differences to adapt the values of Table 3.4 to the beginning of each calendar month in any calendar year (P. Schureman: Manual of harmonic analysis and prediction of tides. U.S. Department of Commerce, Coast and Geodetic Survey, 1941)

	Day of month*										
	12	13	14	15	16	17	18	19	20	21	22
J ₁	•	•	•	•	•	•	•	•	•	•	•
K ₁	154.86	166.61	182.66	196.71	210.76	224.81	238.86	252.91	266.96	281.01	295.06
L ₁	10.84	11.53	12.81	13.60	14.78	15.77	16.76	17.74	18.73	19.71	20.70
M ₁	21.68	23.66	25.63	27.60	29.57	31.54	33.51	35.48	37.45	39.43	41.40
M ₂	235.52	224.20	212.88	201.57	190.25	178.94	167.62	156.30	144.99	133.67	122.35
M ₃	227.13	215.06	202.97	190.89	178.81	166.73	154.65	142.57	130.49	118.41	106.33
M ₄	225.90	213.71	201.52	189.33	177.14	164.95	152.76	140.57	128.38	116.19	103.99
M ₅	91.80	67.42	43.04	18.66	354.28	329.90	305.52	281.13	256.75	232.37	207.99
M ₆	317.70	281.13	244.56	207.99	171.42	134.84	98.27	61.70	25.13	348.56	311.98
M ₇	183.61	134.84	86.06	37.22	348.56	299.79	251.03	202.27	153.50	104.74	55.98
M ₈	275.41	202.27	129.12	55.98	342.83	288.62	234.41	180.20	126.00	71.79	17.58
M ₉	7.21	269.69	172.15	74.64	337.11	236.58	136.05	36.52	307.01	209.46	111.91
N ₁	309.09	270.64	233.20	195.75	158.30	120.86	83.41	45.96	8.52	331.07	293.62
N ₂	184.37	113.86	63.35	12.84	322.33	271.82	221.30	170.79	120.28	69.77	19.26
O ₁	80.96	58.59	30.23	4.86	339.49	314.13	288.76	263.39	238.02	212.66	187.29
O ₂	300.72	328.08	355.40	22.74	60.08	77.42	104.75	132.09	169.43	186.77	214.11
P ₁	-10.84	-11.83	-12.81	-13.80	-14.78	-15.77	-16.76	-17.74	-18.73	-19.71	-20.70
Q ₁	297.25	308.61	320.98	181.95	143.52	105.09	66.65	28.22	349.79	311.36	272.92
Q ₂	183.53	102.03	50.54	359.04	307.54	256.05	204.55	153.05	101.55	50.05	358.56
R ₁	10.84	11.83	12.81	13.80	14.78	15.77	16.76	17.74	18.73	19.71	20.70
S ₁	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
T ₁	-10.84	-11.83	-12.81	-13.80	-14.78	-15.77	-16.76	-17.74	-18.73	-19.71	-20.70
U ₁	216.28	203.22	190.16	177.09	164.02	150.96	137.90	124.83	111.76	98.70	85.64
U ₂	183.61	134.84	86.06	37.22	348.56	299.79	251.03	202.27	153.50	104.74	55.98
U ₃	327.22	291.62	255.03	220.23	184.53	148.83	113.13	77.44	41.74	6.04	330.24
U ₄	316.48	279.60	243.11	206.43	169.75	133.06	96.38	59.69	23.01	346.33	309.64
MK	102.65	79.25	55.85	32.45	9.06	345.57	322.27	298.98	275.68	252.38	229.09
2MK	172.78	123.02	73.27	23.52	333.77	284.02	234.27	184.52	134.78	85.03	35.28
MN	39.89	338.06	276.24	214.41	152.58	90.75	28.92	327.10	265.27	203.44	141.61
MR	91.80	67.42	43.04	18.66	354.28	329.90	305.52	281.13	256.75	232.37	207.99
2SM	268.20	292.58	316.96	341.34	6.72	80.10	64.48	78.87	103.25	127.63	152.01
Mf	289.88	316.23	342.59	8.94	35.29	61.64	88.00	114.35	140.70	167.06	193.41
Msf	268.20	292.58	316.96	341.34	5.72	80.10	64.48	78.87	103.25	127.63	152.01
Mm	143.72	156.76	169.84	182.91	195.98	209.04	222.10	235.17	248.24	261.30	274.36
Sa	10.84	11.83	12.81	13.80	14.78	15.77	16.76	17.74	18.73	19.71	20.70
Ssa	21.68	23.66	25.63	27.60	29.57	31.54	33.51	35.48	37.46	39.43	41.40

Table 3.6c Differences to adapt Table 3.5 to the beginning of each day of a month (P. Schureman: Manual of harmonic analysis and prediction of tides. U.S. Department of Commerce, Coast and Geodetic Survey, 1941)

	Day of month*									
	23	24	25	26	27	28	29	30	31	32
J ₁	•	•	•	•	•	•	•	•	•	•
K ₁	309.11	323.16	337.22	351.27	365.32	379.37	393.42	407.47	421.52	435.57
L ₁	21.68	22.67	23.66	24.64	25.63	26.61	27.60	28.58	29.57	30.56
M ₁	43.37	45.34	47.31	49.28	51.25	53.22	55.20	57.17	59.14	61.11
M ₂	111.04	99.72	88.40	77.08	65.77	54.45	43.14	31.82	20.50	9.19
M ₃	94.25	82.18	70.10	58.02	45.94	33.86	21.78	9.70	357.62	345.54
M ₄	91.80	79.61	67.42	55.23	43.04	30.85	18.66	6.47	354.28	342.09
M ₅	183.61	158.23	134.84	110.46	86.08	61.70	37.32	12.94	348.56	324.17
M ₆	275.41	238.84	202.27	165.69	128.52	92.35	56.18	19.40	342.83	306.26
M ₇	7.21	318.45	269.69	220.92	172.16	123.40	74.64	25.87	357.11	298.35
M ₈	190.82	117.68	44.53	33.39	228.24	185.10	111.95	38.81	325.66	252.52
M ₉	14.43	278.90	179.38	81.85	344.22	246.80	149.27	51.75	414.22	216.69
N ₁	256.18	218.73	181.28	143.84	106.39	68.94	31.50	354.05	316.60	279.15
N ₂	328.75	278.24	227.73	177.21	126.70	76.19	25.68	335.17	284.66	234.15
O ₁	161.92	136.56	111.19	85.82	60.45	35.09	9.72	344.25	318.99	293.62
O ₂	241.45	268.78	296.12	323.46	350.80	378.14	405.48	432.81	460.15	487.49
P ₁	-21.68	-22.67	-23.66	-24.64	-25.63	-26.61	-27.60	-28.58	-29.57	-30.56
Q ₁	234.49	196.06	157.63	119.20	80.78	42.35	3.90	325.47	287.04	248.61
Q ₂	307.09	253.57	204.07	152.57	101.07	49.58	358.09	306.58	255.09	203.59
R ₁	21.68	22.67	23.66	24.64	25.63	26.61	27.60	28.58	29.57	30.56
S ₁	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
T ₁	-21.68	-22.67	-23.66	-24.64	-25.63	-26.61	-27.60	-28.58	-29.57	-30.56
U ₁	72.57	59.50	46.44	33.38	20.31	7.24	354.18	341.12	328.05	314.98
U ₂	7.21	318.45	269.69	220.92	172.16	123.40	74.64	25.87	337.11	298.35
U ₃	294.64	258.95	223.25	187.56	151.85	116.15	80.46	44.76	9.06	333.36
U ₄	272.90	236.28	199.59	162.91	126.22	89.54	52.86	16.17	339.49	302.81
MK	305.29	181.90	158.50	135.10	111.71	88.31	64.92	41.52	18.12	354.73
2MK	345.53	295.78	246.03	196.28	146.54	96.79	47.04	357.29	307.54	257.79
MN	79.78	17.96	318.13	254.30	192.47	130.64	68.82	6.99	305.16	243.33
MS	181.61	159.23	134.84	110.46	86.08	61.70	37.32	12.94	348.56	324.17
2SM	176.39	300.77	225.16	249.84	273.92	298.30	322.68	347.06	11.44	35.83
Mf	219.76	246.11	272.47	298.82	325.17	351.52	377.88	404.23	70.58	96.94
Msf	176.39	300.77	225.16	249.84	273.92	298.30	322.68	347.06	11.44	35.83
Mm	287.43	300.50	313.56	326.62	339.69	352.76	365.82	378.88	391.95	405.02
Sa	21.68	22.67	23.66	24.64	25.63	26.61	27.60	28.58	29.57	30.56
Ssa	43.37	45.34	47.31	49.28	51.25	53.22	55.20	57.17	59.14	61.11

Table 3.4 shows the astronomic arguments at the begin of each calendar year from 1970 to 2000 for a range of tidal components. The astronomic argument varies over the year. So it must be corrected for the month and the day within that month.

Table 3.5 shows the corrections to adjust the astronomic argument for a certain month within a year. The correction is 0 for January.

Table 3.6 a,b,c gives the corrections to adjust the astronomic argument for a specific day within a month. The correction is 0 for the first day in the month.

To get more feeling for carrying out a tidal prediction, a practical example is given. The question is to predict the water level in Hook of Holland (The Netherlands) at 23 April 1990, 12.00 o'clock (at noon). The prediction is carried out by the Formula 3.7:

$$h(t) = h_o + \sum_{i=1}^n f_i H_i \cos(\omega_i t + (v_i + u_i) - g_i)$$

For simplicity, a restricted number of components (with an amplitude of 0.1 m or more) are taken into account:

- A_o mean water level (= 0.06 m for Hook of Holland);
- O₁ diurnal lunar declination tide;
- N₂ semi-diurnal lunar elliptic tide;
- M₂ semi-diurnal principle lunar tide;
- S₂ semi-diurnal principle solar tide.

Table 3.7 presents the data on tidal components of Hook of Holland.

Table 3.7 Tidal components of Hook of Holland

Component	ω_i (°/h)	H _i (m)	g _i (°)
O ₁	13.943	0.10	187
N ₂	28.440	0.12	59
M ₂	28.984	0.79	85
S ₂	30.000	0.19	145

Other data needed from Table 3.4-3.6 are displayed in Table 3.8.

Table 3.8 Other data needed for tidal prediction of Hook of Holland

Component	f _i	1 Jan 1990	1 Apr 1990	23 Apr 1990	23 Apr 1990
		0 h	0 h	0 h	12 h
		$\Delta(v_i+u_i)$	$\Delta(v_i+u_i)$	$\Delta(v_i+u_i)$	$\Delta(v_i+u_i)$
O ₁	1.128	240	236	161	167
N ₂	0.977	324	229	256	341
M ₂	0.977	259	325	183	347
S ₂	1	0	0	0	360

Combining Tables 3.7 and 3.8 gives:

Table 3.9 Combining data of Table 3.7 and 3.8

Component	f_i	H_i	$\cos(\omega_i t + (v_i + u_i) - g_i)$
O_1	1.128	10	$\cos(167 + 240 + 236 + 161 - 187) = -0.22$
N_2	0.977	12	$\cos(341 + 324 + 229 + 256 - 59) = +0.98$
M_2	0.977	79	$\cos(347 + 259 + 325 + 183 - 85) = +0.62$
S_2	1	19	$\cos(360 + 0 + 0 + 0 - 145) = -0.81$

Now the contribution of each component to the water level can be determined as follows:

$$A_o = +0.06 \text{ m}$$

$$O_1 = -0.024 \text{ m}$$

$$N_2 = +0.114 \text{ m}$$

$$M_2 = +0.478 \text{ m}$$

$$S_2 = -0.153 \text{ m} +$$

+0.475 m is water level in Hook of Holland at 23 April 1990, 12.00 hours.

A more accurate prediction can be obtained by including more components.

3.4 Type of tides

The tide can be classified by the so-called form-number:

$$F = \frac{H_{K1} + H_{O1}}{H_{M2} + H_{S2}}$$

$K1$ and $O1$ are the main diurnal components; $M2$ and $S2$ are the main semi-diurnal components.

Four types of tides can be distinguished (see also Figure 3.10):

- *fully semi-diurnal* ($F < 0.25$). Such a tide can be found at Immingham in England. There are two HW's and two LW's per day of about the same height. The mean tidal range at springtide is $2(H_{M2} + H_{S2})$;
- *mixed, mainly semi-diurnal* ($0.25 < F < 1.5$). Such a tide can be found at San Francisco in the U.S. There are two HW's and two LW's per day which are different in height and time. The mean tidal range at springtide is $2(H_{M2} + H_{S2})$;
- *mixed, mainly diurnal* ($1.5 < F < 3$). Such a tide can be found in Manila in the Philippines. Most of the time there is one HW per day, for a short time there are two HW's with a strong inequality in height and time. The mean tidal range at springtide is $2(H_{K1} + H_{O1})$;
- *fully diurnal* ($F > 3$). Such a tide can be found in Do-Son in Vietnam. There is only one HW and one LW per day. The mean tidal range at springtide is $2(H_{K1} + H_{O1})$.

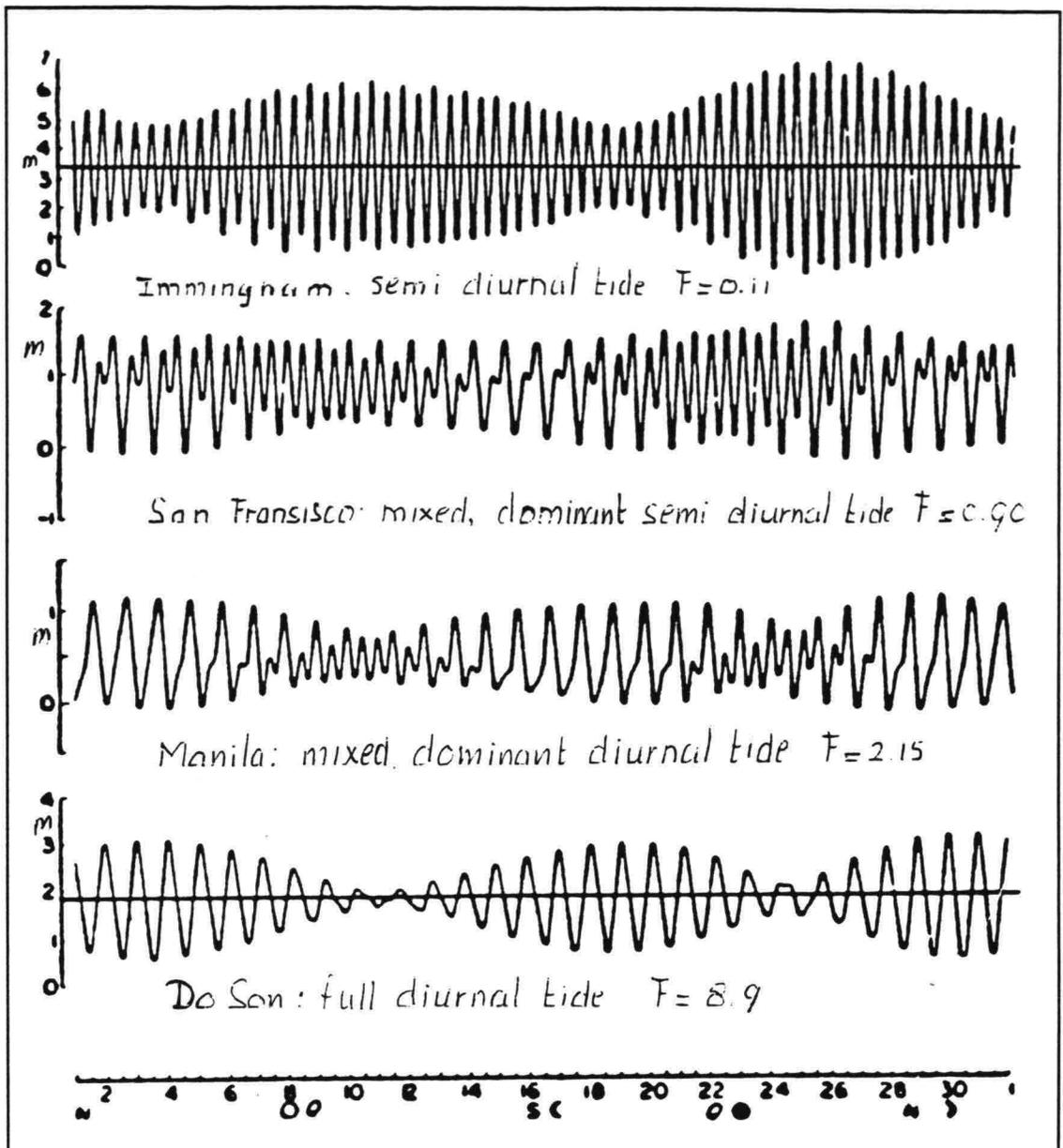


Figure 3.10 Examples of four main tide types

Finally a definition of the most commonly used tidal terms is given in Table 3.10.

Table 3.10 Most commonly used tidal terms

Abbreviation	Full name	Explanation
MSL	Mean Sea Level	The average sea level over a long period.
MHW	Mean High Water	The average of all high water levels.
MLW	Mean Low Water	The average of all low water levels
MHWS	Mean High Water Spring	The average of two successive high water levels at springtide.
MLWS	Mean Low Water Spring	The average of two successive low water levels at springtide.
MHWN	Mean High Water Neap	The average of the two successive high water levels at neap tide.
MLWN	Mean Low Water Neap	The average of two successive low water levels at neap tide.

4 Basic equations and types of long waves in one dimension

4.1 Introduction

The tidal analysis and prediction concentrated on the tide at *one location*. In this Chapter, the propagation of a *tidal wave* is discussed. Equations are derived, which describe the propagation of tidal waves. With these equations, tidal computations can be made. The first methods for tidal computations are *analytical methods* (Chapter 7). In later stages, *numerical methods* have been developed (Chapter 8).

One of the main reasons to make tidal computations is to see how water levels and velocities will change when civil engineering works will be carried out in tidal regions. First, *some arbitrary examples* are presented, showing the changes in tidal motion.

Figure 4.1 shows a river that flows in a sea, where a tide is present. For the planned dam in the river, it is necessary to compute the changes in the tidal motion (water levels and velocities) downstream of the dam in prior to construction.

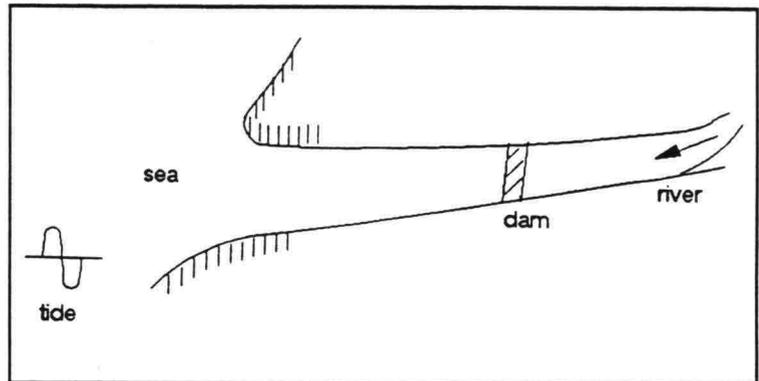


Figure 4.1 Dam, planned in the tidal reach of a river

Figure 4.2 shows an estuary, where plans are to reclaim a certain area. The water levels and velocities can change considerably due to the change in geometry. Therefore, prior to the start of the project it should be known how the tide will change in the estuary. For this, tidal computations must be carried out.

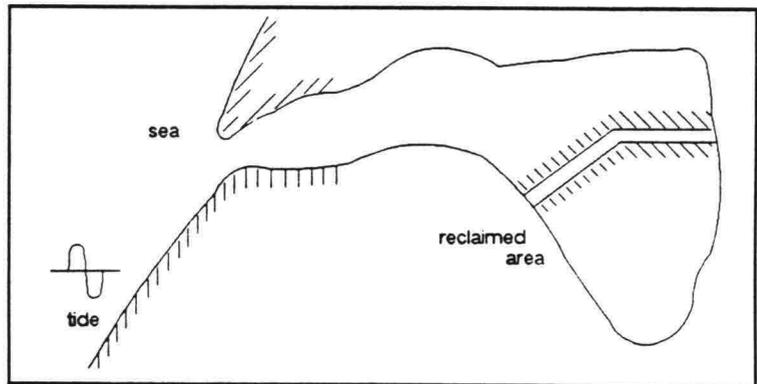


Figure 4.2 Plans for reclamation in an estuary

Before deriving the equations, first a survey of the *various types of flow* is given, to show in which category *tidal waves* belong.

A *flow motion* can be characterized by the water level η and the velocity (with components: u, v, w). The parameters are functions of time t and space coordinates x, y, z . The space

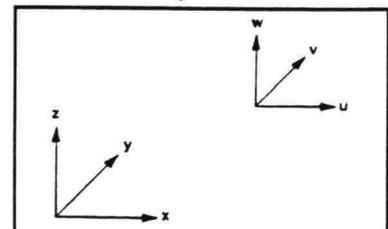


Figure 4.3 Directions of u, v, w and x, y, z

coordinates and the components of the velocity are presented in Figure 4.3, where:

- u is directed in the x -direction;
- v is directed in the y -direction;
- w is directed in the z -direction.

An overview of the various types of flow motion is given in Figure 4.4. From this Figure it can be concluded that tidal waves can be considered as long waves.

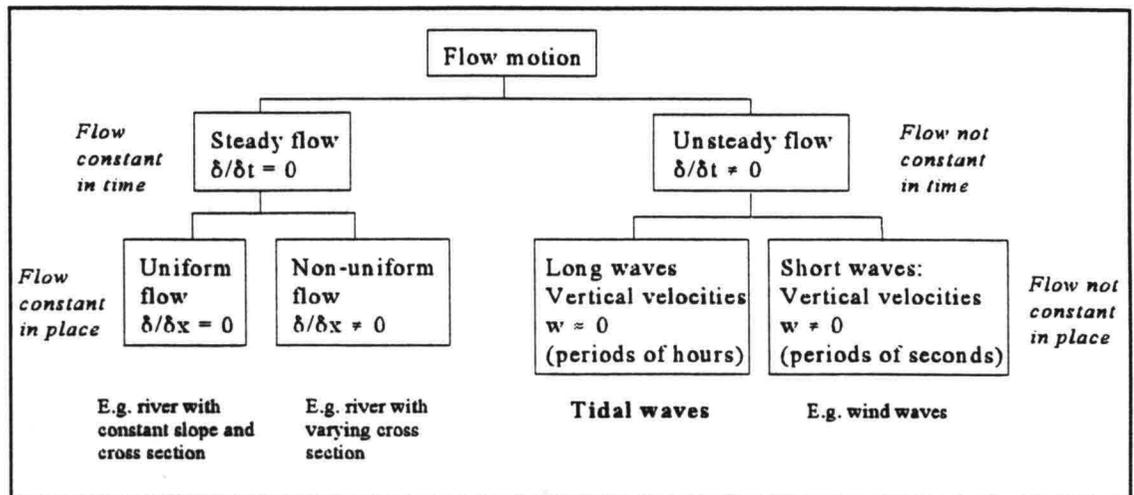


Figure 4.4 Overview of various types of motion

4.2 Basic equations for long waves in one dimension

The basic equations for long waves in one dimension are derived for one-dimensional problems (like a tidal river). The first step is to make the following assumptions:

- *vertical velocities are small.* The flow is nearly horizontal, which means that the pressure in the water is proportional to the depth. This implies a *hydrostatic pressure distribution*;
- *width of the channel is small.* This implies that the water level in cross-direction is horizontal. To give an idea, the width should be less than 10 km;
- *density of the water is constant.*

The equations which describe the water motion in long waves are:

- equation of continuity;
- equation of motion.

4.2.1 Equation of continuity

Consider an element of the channel of Figure 4.5.

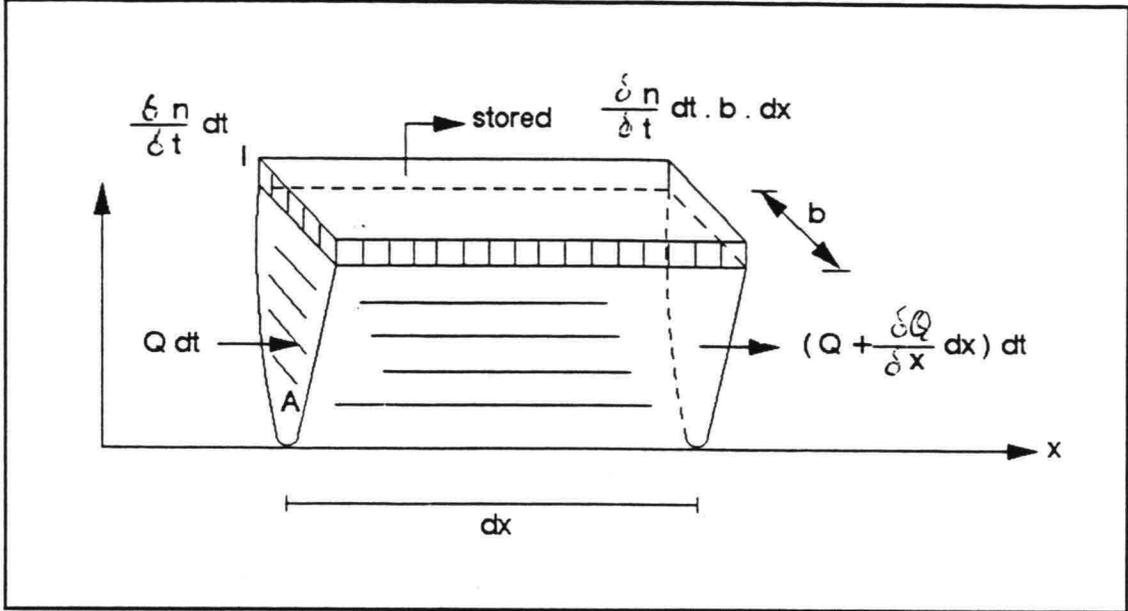


Figure 4.5 Element of a channel with length Δx

The symbols of Figure 4.5 have the following meaning:

- Δx length of element;
- A cross section;
- b width at still water surface;
- Q discharge in the channel.

The inflow at the left-hand side in a time increment dt is Qdt .

The outflow at the right-hand side in a time increment dt is:

$$Q + \left(\frac{\partial Q}{\partial x} dx \right) dt.$$

In the time increment dt , water is stored in the element. The water level increases with:

$$\frac{\partial \eta}{\partial t} dt.$$

The width at the water surface is b . So the stored amount of water is:

$$\frac{\partial \eta}{\partial t} dt b dx.$$

Conservation of mass, over a time increment dt means:

$$\text{Inflow} - \text{Outflow} = \text{Storage}$$

Thus:

$$Qdt - \left(Q + \frac{\partial Q}{\partial x} dx \right) dt = \frac{\partial \eta}{\partial t} dt b dx.$$

$$Qdt - Qdt - \frac{\partial Q}{\partial x} dx dt = \frac{\partial \eta}{\partial t} dt b dx.$$

$$\boxed{\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0} \quad (4.1)$$

Equation 4.1 is the *equation of continuity*.

4.2.2 Equation of motion

The equation of motion can be derived by considering an element of the channel with a length Δx and applying *Newton's law*:

$$F = ma$$

By definition, acceleration is the change of the velocity per unit of time:

$$a = \frac{du}{dt}$$

Consider $u = f(x, t)$. Then, a change in u can be written as:

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx$$

Dividing by dt , yields:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt}$$

The water particle has to be followed, that means that the change in x per unit of time is the velocity:

$$\frac{dx}{dt} = u$$

So the acceleration can also be written as:

$$a = \frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}$$

According $F = ma$, the acceleration equals the force per unit mass:

$$a = \frac{F}{m}$$

The forces acting on the considered element originate from *gravity*, *pressure* and *bottom friction*. For further elaboration, an element of the channel per unit width is considered.

A. Gravity force

Figure 4.6 shows the coordinate system in the channel. x is directed along the river. η is directed upwards. h is the local depth, so:

$$h = h_0 + \eta.$$

I is the slope of the river bed.

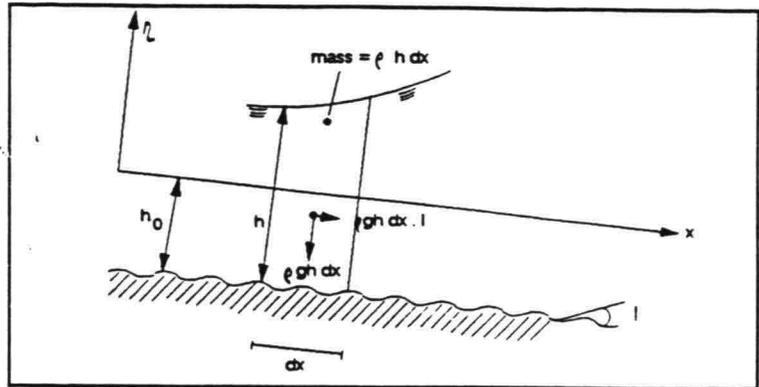


Figure 4.6 System of coordinates with element Δx

Consider an element with length Δx . The mass m of the element is $\rho h \Delta x$. The weight of the element is $\rho g h \Delta x$.

The force component in the x -direction F is $\rho g h \Delta x I$ (in fact it is $\sin I$, but $\sin I \approx I$, because I is small). The gravity force in the x -direction per unit of mass is:

$$\frac{F}{m} = \frac{\rho g h \Delta x I}{\rho h \Delta x} = g I$$

B. Pressure force

For the pressure force, the same element with length Δx is considered (see Figure 4.7).

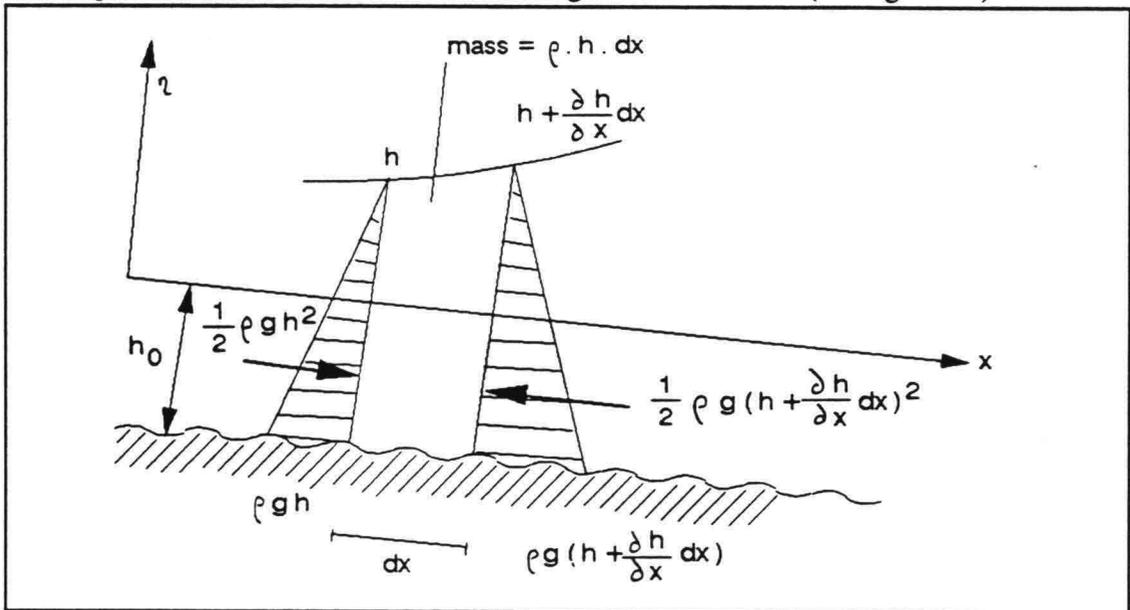


Figure 4.7 Element dx with symbols for pressure force

The water level at the left-hand side is h . The pressure force is proportional to the water depth. At the bottom the pressure force is $\rho g h$. The resultant pressure force at the left-hand side is:

$$\frac{1}{2} \times \rho g h \times h = \frac{1}{2} \rho g h^2$$

At the right-hand side, the water level is $h + \frac{\partial h}{\partial x} \Delta x$. So, the pressure at the bottom is:

$$\rho g \left(h + \frac{\partial h}{\partial x} dx \right).$$

The resultant pressure force at the right hand side is:

$$\frac{1}{2} \rho g \left(h + \frac{\partial h}{\partial x} dx \right) \left(h + \frac{\partial h}{\partial x} dx \right) = \frac{1}{2} \rho g \left(h + \frac{\partial h}{\partial x} dx \right)^2.$$

The net force in the x-direction is:

$$\frac{1}{2} \rho g h^2 - \frac{1}{2} \rho g \left(h^2 + 2h \frac{\partial h}{\partial x} dx + \left(\frac{\partial h}{\partial x} dx \right)^2 \right) \approx \rho g h \frac{\partial h}{\partial x} dx$$

The mass of the element is $\rho h dx$, so the net force in the x-direction per unit mass is:

$$\frac{F}{m} = - \frac{\rho g h \frac{\partial h}{\partial x} dx}{\rho h dx} = - g \frac{\partial h}{\partial x} \tag{4.2}$$

Because $h = h_0 + \eta$, Equation 4.2 can be written as:

$$\frac{F}{m} = - g \frac{\partial \eta}{\partial x} \tag{4.3}$$

C. Bottom friction

For the bottom friction, the same element with length Δx is considered (see Figure 4.8). In the element, water flows with velocity u in the positive x-direction. Due to that flow, along the bottom a *shear stress* τ acts in the opposite direction on the water element, *per unit length* of the channel and *per unit width*. The force on the element $F = -\tau dx$.

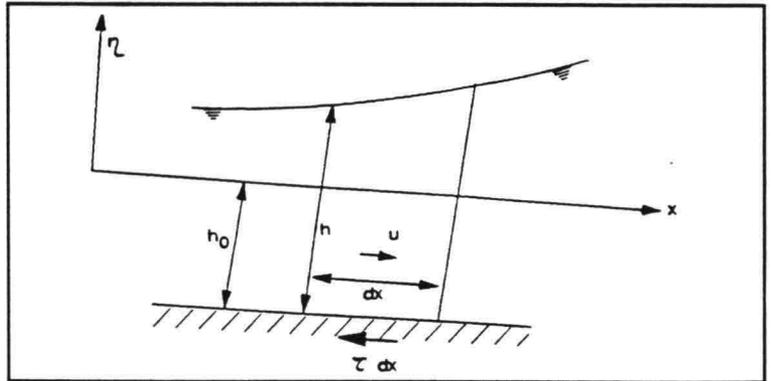


Figure 4.8 Element Δx with symbols for bottom friction

Now, an expression must be found for τ . Therefore, a closer look is taken at a river with *uniform flow* (see Figure 4.9). The slope of the river is I . The cross section of the river is A , whereas the length along the bank is O .

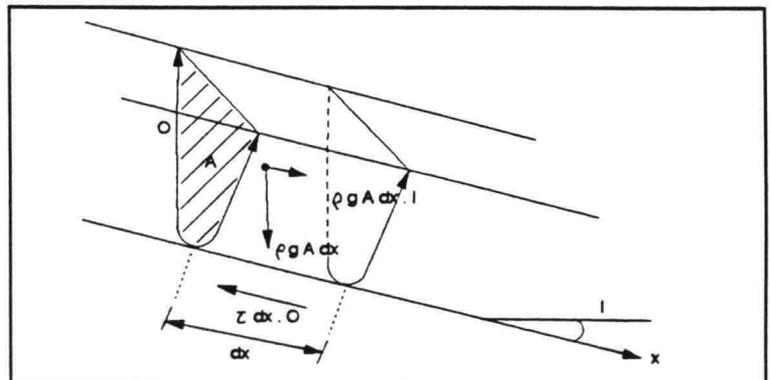


Figure 4.9 River with uniform flow

For uniform flow, the gravity force due to the slope of the river and the friction force, will balance. The weight of the element is $\rho g A dx$. So the gravity force in x-direction $F_x = \rho g A dx I$ (assuming that $\sin I \approx I$). The friction force along the bank of the river is $-\tau dx O$.

There is a balance between both forces, so:

$$\rho g A dx I = \tau dx O \rightarrow \tau = \rho g I \frac{A}{O}$$

In this expression A/O is the *hydraulic radius* R .

The hydraulic radius can be expressed in terms of width and depth of a river (see Figure 4.10).

$$R = \frac{A}{O} = \frac{wh}{w + 2h}$$

For a side river, than $w \gg h$, so

$$R \approx \frac{wh}{w} = h.$$

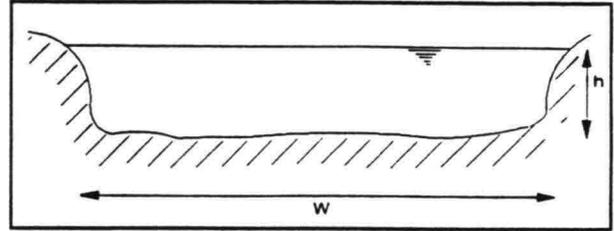


Figure 4.10 Cross section of a river

The hydraulic radius R is about equal to the depth. For smaller rivers (in relation to the depth), the hydraulic radius R can be computed from:

$$R = \frac{A}{O}$$

The equation for the shear stress becomes now:

$$\tau = \rho g I R \tag{4.4}$$

For *uniform flow*, Chézy's law holds:

$$u = C \sqrt{RI}$$

The velocity is proportional to the square root of the slope and the square root of the hydraulic radius. C is Chézy's constant, and depends on the roughness of the river bed. For rivers in the tidal area in Holland, $C = 50$ is a common value.

The *dimension* of C can be derived from:

$$C = \frac{u}{\sqrt{RI}} \rightarrow \left[\frac{m/s}{\sqrt{m}} = m^{1/2}/s \right]$$

Re-writing Chézy's law $u = C \sqrt{RI}$ gives:

$$I = \frac{u^2}{C^2 R}$$

Substitution in Relation 4.4 yields:

$$\tau = \rho g \frac{u^2}{C^2 R} R = \rho g \frac{u^2}{C^2} \tag{4.5}$$

Substitution of Equation 4.5 in the general expression of the friction force per unit width ($F = -\tau dx$) gives:

$$-\rho g \frac{u^2}{C^2} dx$$

The mass of the considered element is ρdxh , so the friction force per unit mass is:

$$\frac{F}{m} = -\frac{\rho g \frac{u^2}{C^2} dx}{\rho dxh} = -g \frac{u^2}{C^2 h} \quad (4.6)$$

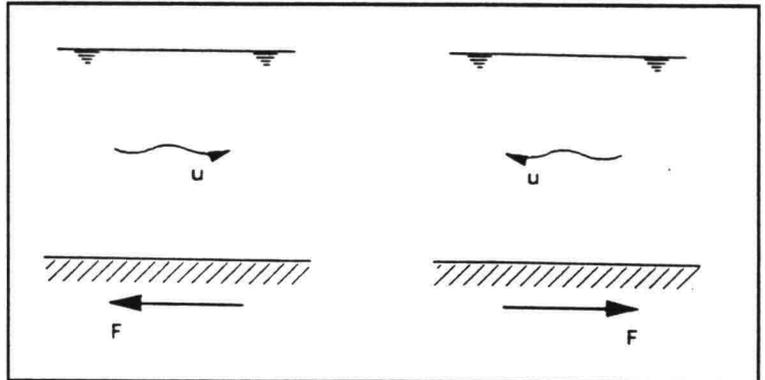
Considering a *unit width* of the channel means that the hydraulic radius $R \approx h$. Considering an *arbitrary cross section* of the channel, R can better be taken instead of h . So a more general expression for the friction force per unit mass is then:

$$\frac{F}{m} = -g \frac{u^2}{C^2 R} \quad (4.6)$$

The friction force holds for a uniform flow, and can be applied for tidal flow. A justification for that assumption is that the flow oscillates slow due to the tide.

The direction of the tidal flow alternates. Ebb flow and flood flow are in opposite directions. So the friction force, directed opposite to the flow direction is alternating too (see Figure 4.11):

- u positive $\rightarrow F$ negative;
- u negative $\rightarrow F$ positive.



This is taken into account by writing u^2 as $u|u|$.

So for tidal flows the term for the friction force can be written as :

$$\frac{F}{m} = -g \frac{u|u|}{C^2 R} \quad (4.7)$$

For the force per unit of mass, the expressions for *gravity*, *pressure* and *bottom friction* have been derived as follows:

$$\begin{array}{ccc} \frac{F}{m} = gI & - & g \frac{\partial h}{\partial x} & - & g \frac{u}{C^2 R} \\ \text{gravity} & & \text{pressure} & & \text{bottom friction} \end{array}$$

The equation of motion becomes now:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = gI - g \frac{\partial h}{\partial x} - g \frac{u|u|}{C^2 R}$$

or:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - gI + g \frac{\partial h}{\partial x} + g \frac{u|u|}{C^2 R} = 0$$

This equation does not hold only for tidal flow, but for flow in all kinds of long waves. For *tidal computations*, the x-axis is mostly considered to be horizontal, so that $I = 0$. The term $-gI$ disappears from the equation. So it reduces to :

$$\boxed{\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} + g \frac{u|u|}{C^2 R} = 0} \quad (4.7)$$

So, two equations are now available for describing the fluid motion in tidal waves:

- Equation of continuity (4.1): $\frac{\partial Q}{\partial x} + b \frac{\partial h}{\partial t} = 0$;
- Equation of motion (4.7): $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} + g \frac{u|u|}{C^2 R} = 0$

For tidal calculations it is more convenient to use the discharge Q instead of the velocity u . Figure 4.12 shows a schematization of a river, which splits up in two branches. Considering sections (1), (2) and (3) of the schematization, the condition at the *node* is that the water levels are equal and that there is conservation of mass:

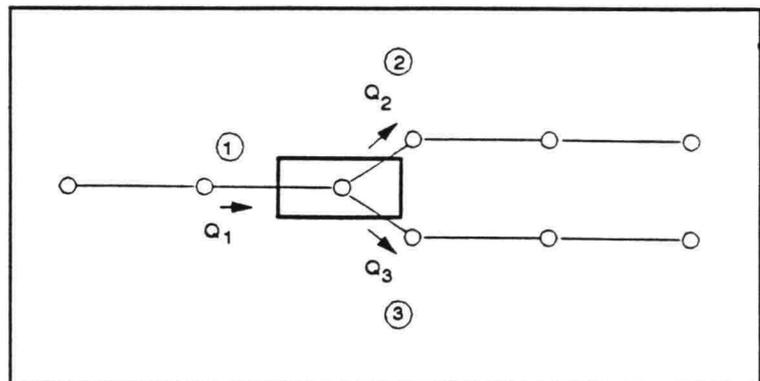


Figure 4.12 Schematization of a river with two branches

- $\eta_1 = \eta_2 = \eta_3$;
- $Q_1 = Q_2 + Q_3$.

The equation of motion can be re-written by introducing $u = \frac{Q}{A}$.

The first and second term of the equation of motion (Equation 4.7) become:

$$\frac{\partial u}{\partial t} = \frac{\frac{\partial Q}{\partial t} A - \frac{\partial A}{\partial t} Q}{A^2} = \frac{1}{A} \frac{\partial Q}{\partial t} - \frac{Q}{A^2} \frac{\partial A}{\partial t}$$

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial Q}{\partial x} A - \frac{\partial A}{\partial x} Q}{A^2} = \frac{1}{A} \frac{\partial Q}{\partial x} - \frac{Q}{A^2} \frac{\partial A}{\partial x}$$

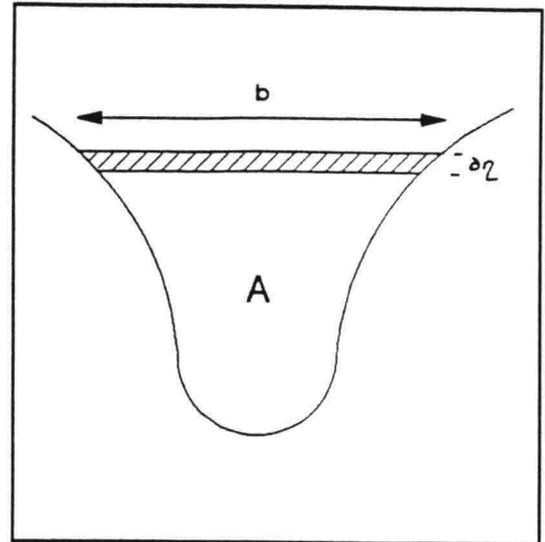


Figure 4.13 Changing cross section

Figure 4.13 shows that the change in cross section dA can be written as:

$$dA = b d\eta$$

so:

$$\frac{\partial A}{\partial x} = b \frac{\partial \eta}{\partial x}$$

$$\frac{\partial A}{\partial t} = b \frac{\partial \eta}{\partial t}$$

Substitution in the first two terms of Equation 4.7 gives:

$$\frac{\partial u}{\partial t} = \frac{1}{A} \frac{\partial Q}{\partial t} - \frac{Qb}{A^2} \frac{\partial \eta}{\partial t}$$

$$\frac{\partial u}{\partial x} = \frac{1}{A} \frac{\partial Q}{\partial x} - \frac{Qb}{A^2} \frac{\partial \eta}{\partial x}$$

So, the equation of motion becomes (also replacing u with Q/A):

$$\left[\frac{1}{A} \frac{\partial Q}{\partial t} - \frac{Qb}{A^2} \frac{\partial \eta}{\partial t} \right] + \frac{Q}{A} \left[\frac{1}{A} \frac{\partial Q}{\partial x} - \frac{Qb}{A^2} \frac{\partial \eta}{\partial x} \right] + g \frac{\partial \eta}{\partial x} + g \frac{Q|Q|}{C^2 A^2 R} = 0$$

Substitution of the equation of continuity $\frac{\partial Q}{\partial x} = -b \frac{\partial \eta}{\partial t}$ gives:

$$\frac{1}{A} \frac{\partial Q}{\partial t} - \frac{Qb}{A^2} \frac{\partial \eta}{\partial t} - \frac{Qb}{A^2} \frac{\partial \eta}{\partial t} - \frac{Q^2 b}{A^3} \frac{\partial \eta}{\partial x} + g \frac{\partial \eta}{\partial x} + g \frac{Q|Q|}{C^2 A^2 R} =$$

One step further:

$$\frac{1}{A} \frac{\partial Q}{\partial t} - \frac{2Qb}{A^2} \frac{\partial \eta}{\partial t} + g \left(1 - \frac{Q^2 b}{g A^3} \right) \frac{\partial \eta}{\partial x} + g \frac{Q|Q|}{C^2 A^2 R} =$$

term (2)
term (3)

Taking a closer look to term 2:

$$- \frac{2Q}{A^2} b \frac{\partial \eta}{\partial t}$$

The equation of continuity can be written as:

$$-b \frac{\partial \eta}{\partial t} = -\frac{\partial Q}{\partial x}$$

Substitution gives for term (2):

$$\frac{2Q}{A^2} \frac{\partial Q}{\partial x}$$

To elaborate term (2) further, the following expression is considered:

$$\frac{\partial}{\partial x} \left(\frac{Q}{A} \right)^2 = 2 \frac{Q}{A} \frac{\frac{\partial Q}{\partial x} A - \frac{\partial A}{\partial x} Q}{A^2} = 2 \frac{Q}{A^2} \frac{\partial Q}{\partial x} - 2 \frac{Q^2}{A^3} \frac{\partial A}{\partial x}$$

Substitution of $\frac{\partial A}{\partial x} = b \frac{\partial \eta}{\partial x}$ gives:

$$\frac{\partial}{\partial x} \left(\frac{Q}{A} \right)^2 = \frac{2Q}{A^2} \frac{\partial Q}{\partial x} - 2 \frac{Q^2 b}{A^3} \frac{\partial \eta}{\partial x}$$

So term (2) becomes:

$$\frac{2Q}{A^2} \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \left(\frac{Q}{A} \right)^2 + 2 \frac{Q^2 b}{A^3} \frac{\partial \eta}{\partial x}$$

Now considering terms (2) and (3) together:

$$\underbrace{\frac{\partial}{\partial x} \left(\frac{Q}{A} \right)^2}_{\text{term (2)}} + \underbrace{2 \frac{Q^2 b}{A^3} \frac{\partial \eta}{\partial x} + g \left(1 - \frac{Q^2 b}{gA^3} \right) \frac{\partial \eta}{\partial x}}_{\text{term (3)}}$$

This can be simplified to:

$$\frac{\partial}{\partial x} \left(\frac{Q}{A} \right)^2 + g \left(1 - \frac{Q^2 b}{gA^3} \right) \frac{\partial \eta}{\partial x}$$

Expression $\frac{Q^2 b}{gA^3}$ can be written as:

$$\frac{Q^2 b}{gA^3} = \frac{Q^2}{A^2} \frac{b}{gA} = u^2 \frac{1}{g A/b} \approx \frac{u^2}{gh}$$

This is called the *Froude number*. For tidal situations this number is small compared to 1. This can be illustrated with the next example. Take $u = 1$ m/s, $g = 10$ m/s², and $h = 10$ m.

The Froude number $\frac{u^2}{gh} = \frac{1}{100} = 0.01$, which is small compared to 1.

In different types of long waves, the different terms play a more or less important role. Three types of long waves will be considered:

- *Translation waves*, where inertia terms are most important and the friction term can be neglected;
- *Flood waves (river)*, where friction is the most important term, and inertia terms can be neglected;
- *Tidal waves*, where both the inertia terms and the friction term are of importance (in some cases, friction can be neglected to obtain simple solutions).

4.3.2 Translation waves

A translation wave is a disturbance that *propagates without any significant deformation*. It is caused by a sudden relief or *withdraw* of water in a channel, for instance by opening of sluices. A translation wave can be positive (increase by water level) in the case of a relief (see Figure 4.14). Figure 4.15 shows a negative translation wave, which occurs in case of a withdrawal.

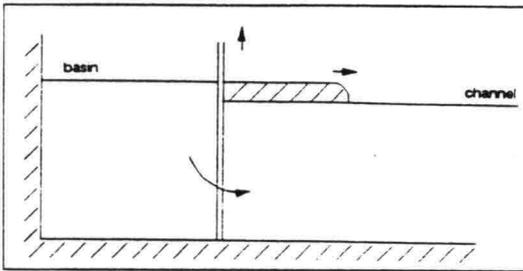


Figure 4.14 Positive translation wave resulting from opening a sluice

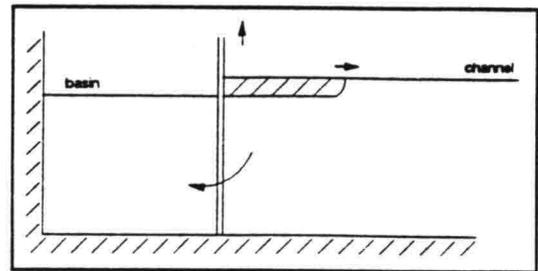


Figure 4.15 Negative translation wave, in case of a withdrawal

Now consider the situation as sketched in Figure 4.16.

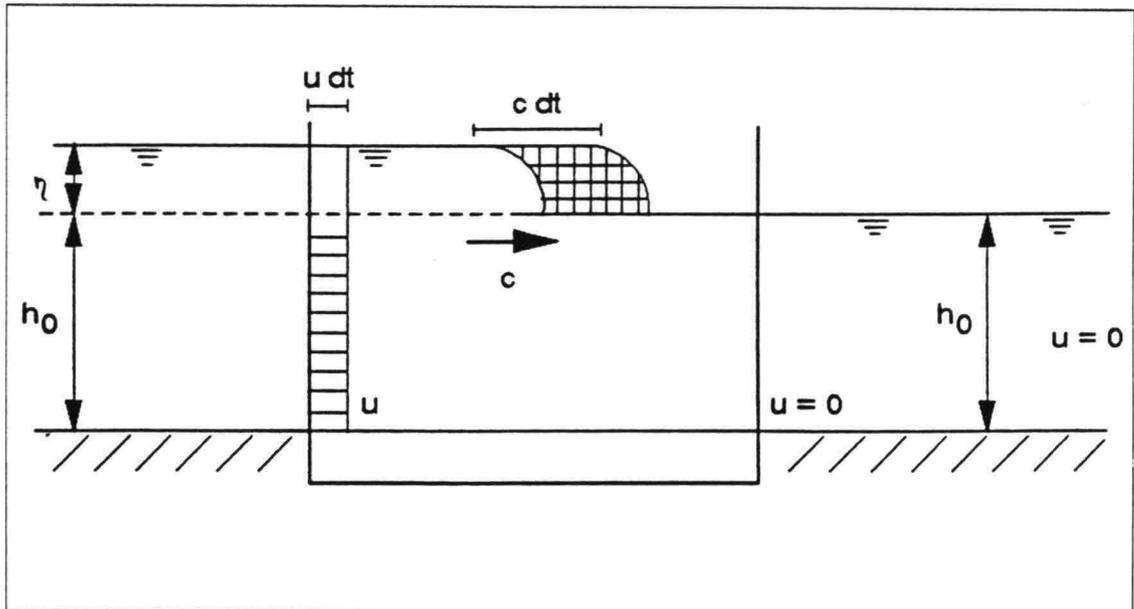


Figure 4.16 Symbols used for translation waves

Figure 4.16 shows a part of a channel with depth h_0 . The height of the translation wave is η . The wave has a steep front. The propagation speed of the wave front is c , which is called the *celerity*. The velocities at the right hand side of the wave front are zero; the velocities at the left hand side are small. Therefore, bottom friction. Can be neglected.

Equation for this specific case can be derived by considering the box, indicated in Figure 4.16. At the left hand side of the box, the water velocity is u , whereas at the right hand side the water velocity is zero.

I. Continuity

The first equation that can be derived is the *equation of continuity*. Continuity for the box in Figure 4.16 means:

$$\text{Inflow} - \text{Outflow} = \text{Storage}$$

So:

$$\rho g u dt (h_0 + \eta) - 0 = \rho g c dt \rho$$

From this relation, the following expression for velocity u can be found:

$$u = c \frac{\eta}{h_0 + \eta}$$

For $\eta \ll h_0$:

$u = c \frac{\eta}{h_0}$	Equation of continuity
--------------------------	------------------------

II. Equation of motion

Applying *Newton's law* ($F = ma$) to the box in Figure 4.16 gives:

$$F = ma = m \frac{du}{dt} = \frac{d(mu)}{dt}$$

or

$$F dt = d(mu)$$

Here, term mu is the momentum. Newton's law in this form says:

$$\begin{aligned} \text{Force during } dt &= \text{Net increase of momentum in time } dt \\ \text{(on the box)} & \qquad \qquad \qquad \text{(inside the box)} \\ &= \text{Increase of momentum inside the box} \\ &= \text{inflow of momentum} - \text{outflow of momentum} \end{aligned}$$

Applying this to the box in Figure 4.17 gives (neglecting the friction force):

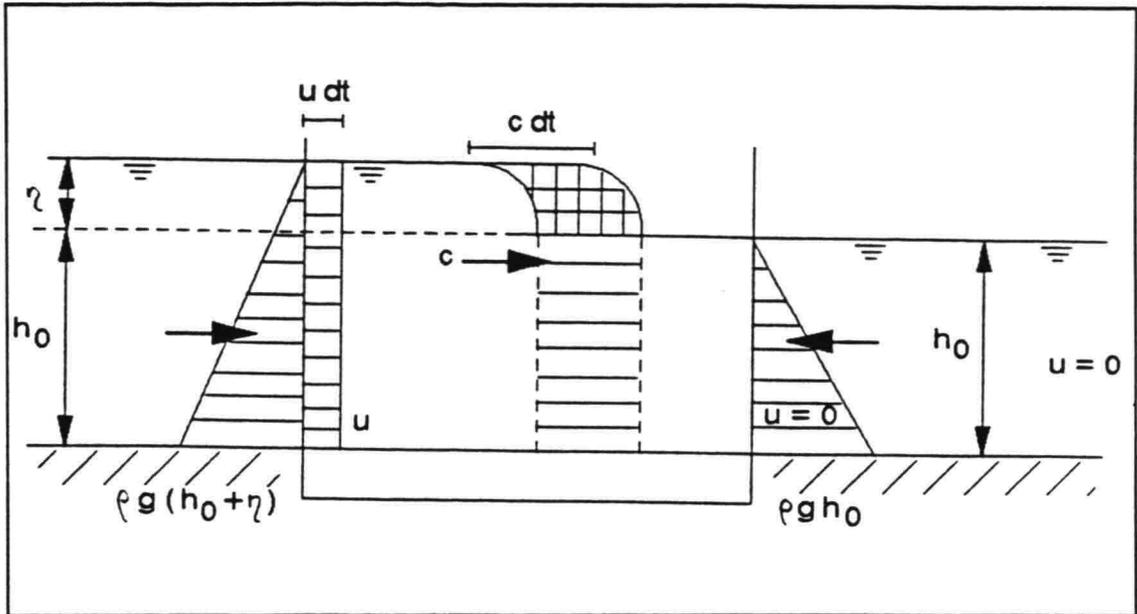


Figure 4.17 Increase of momentum inside the box

Pressure at the left hand side : $\frac{1}{2}\rho g(h_0 + \eta)^2$

Pressure at the right hand side: $\frac{1}{2}\rho gh_0^2$

The net pressure force is equal to: $\frac{1}{2}\rho g(h_0 + \eta)^2 - \frac{1}{2}\rho gh_0^2 = \rho gh_0 \eta \left(1 + \frac{\eta}{2h_0}\right)$

The increase of momentum inside the box is: $\rho \cdot c dt (h_0 + \eta)u$

Inflow of momentum - outflow of momentum is: $\rho u dt (h_0 + \eta)u - 0$

The net increase of momentum in the box is the difference between both terms:

$$\rho (c - u) dt (h_0 + \eta)u$$

Now $F dt = d(mu)$ can be written as:

$$\rho gh_0 \eta \left(1 + \frac{\eta}{2h_0}\right) dt = \rho (c - u) dt (h_0 + \eta)u$$

Assuming that $\eta \ll h_0$, then this expression reduces to:

$$\rho gh_0 \eta dt = \rho (c - u) h_0 u dt$$

or

$g\eta = (c - u)u$	<i>Equation of motion</i>
--------------------	---------------------------

III. Combining equations of continuity and motion

The equations of continuity and motion are:

$$u = c \frac{\eta}{h_0} \quad \text{Continuity}$$

$$g\eta = (c - u)u \quad \text{Motion}$$

Elimination u by substituting u from the equation of continuity in the equation of motion gives an expression for the celerity c :

$$g\eta = \left(c - c \frac{\eta}{h_0} \right) \frac{c\eta}{h_0}$$

$$c^2 = \frac{gh_0}{1 - \frac{\eta}{h_0}}$$

$$c = \frac{\sqrt{gh_0}}{\sqrt{1 - \frac{\eta}{h_0}}}$$

When $\eta \ll h_0$, then $c \approx \sqrt{gh_0}$

Substitution of c into the equation of continuity gives for the velocity u :

$$u = \frac{\eta}{h_0} \sqrt{gh_0} = \eta \sqrt{\frac{g}{h_0}}$$

From the equation of continuity, it can be seen that:

$$uh_0 = c\eta$$

This is the discharge through the plane at the left hand side of the box, denoted as q in Figure 4.18.

So:

$$q = uh_0 = c\eta \quad \text{and} \quad \eta = \frac{q}{c}$$

Some important properties of the translation waves are:

- Reflection against a vertical wall. The height of the reflected wave is equal to the height of the original wave (see Figure 4.19);
- "Dying out" at a boundary where the water depth or width largely increases (see Figure 4.20);
- Partially reflecting and partially transmitting at sudden changes in width and depth (see Figure 4.21).

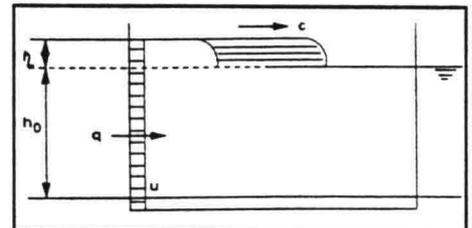


Figure 4.18 Discharge q

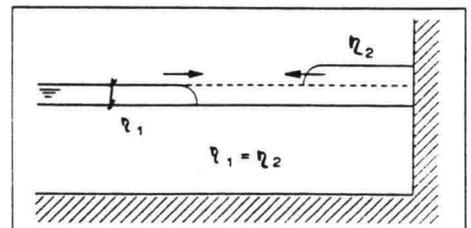


Figure 4.19 Reflection against a vertical wall

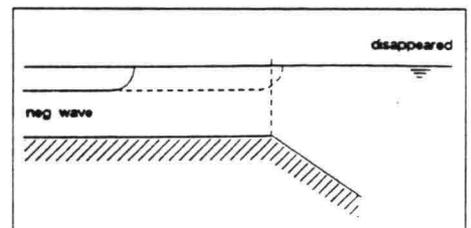


Figure 4.20 Dying out of transl. waves

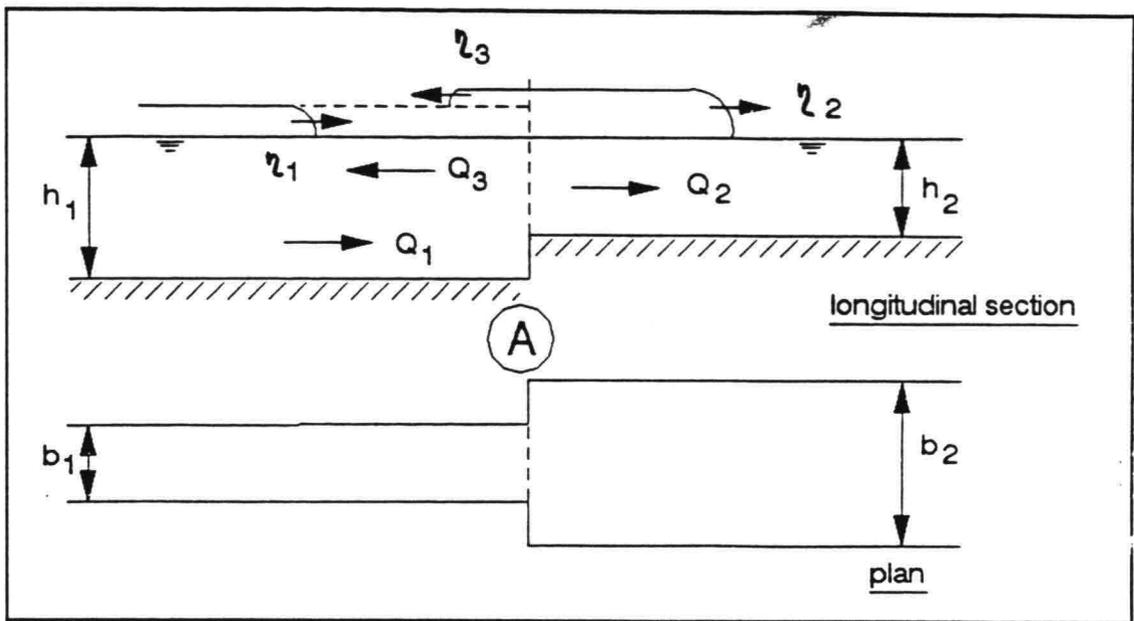


Figure 4.21 Sudden changes in width and depth of a channel

The conditions in cross section A are:

- $\eta_1 + \eta_3 = \eta_2$ (same water levels left and right of A);
- $Q_1 - Q_3 = Q_2$ (continuity in A)

The general expression for Q is:

$$Q = c\eta b \text{ in which } c = \sqrt{gh}$$

Substitution in the continuity relation $Q_1 - Q_3 = Q_2$ gives:

$$\eta_1 b_1 \sqrt{gh_1} - \eta_3 b_1 \sqrt{gh_1} = \eta_2 b_2 \sqrt{gh_2}$$

$$(\eta_1 - \eta_3) b_1 \sqrt{gh_1} = \eta_2 b_2 \sqrt{gh_2}$$

When η_1 is known, η_2 and η_3 can be solved:

$$\eta_2 = \eta_1 \frac{2b_1 \sqrt{gh_1}}{b_1 \sqrt{gh_1} + b_2 \sqrt{gh_2}}$$

$$\eta_3 = \eta_1 \frac{b_1 \sqrt{gh_1} - b_2 \sqrt{gh_2}}{b_1 \sqrt{gh_1} + b_2 \sqrt{gh_2}}$$

4.3.3 Flood wave

In this type of wave, the friction term is most important. The inertia terms can be neglected. The changes of u in x and t are small and the changes of η and h in x and t are small.

The flood wave in a river has the shape as sketched in Figure 4.22. The flood wave has a propagation speed c .

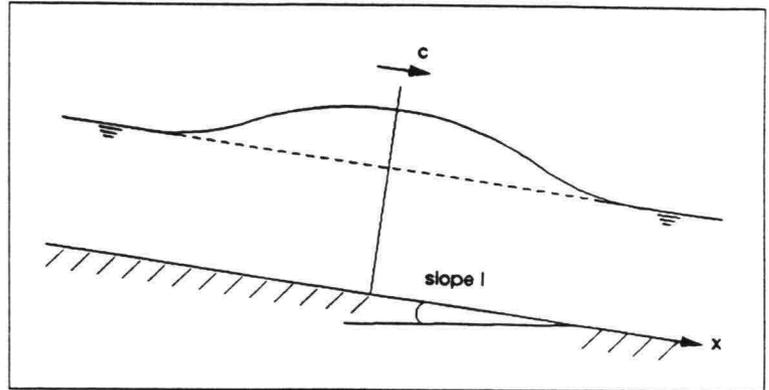


Figure 4.22 Flood wave in a river

The equations for this wave are:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0 \quad (\text{continuity})$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} + g \frac{\partial \eta}{\partial x} - gI + g \frac{Q|Q|}{C^2 A^2 R} = 0 \quad (\text{motion})$$

The inertia term $\frac{1}{A} \frac{\partial Q}{\partial t}$ and the slope of wave $g \frac{\partial \eta}{\partial x}$ are small and can be neglected.

In general, rivers have a difference between the discharge width and the storage width (Figure 4.23).

The water is stored over a width b , which is term b in the equation of continuity. The part of the river that is transporting the discharge has a width b_s .

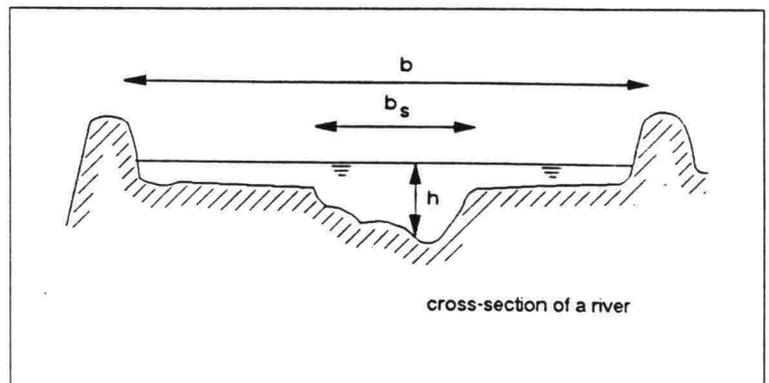


Figure 4.23 Cross section of a river

Neglecting the inertia and the slope of the wave in the *equation of motion* gives:

$$g \frac{Q|Q|}{C^2 A^2 R} = gI$$

$$\frac{Q^2}{A^2} = C^2 R I$$

As $u = \frac{Q}{A}$, $u^2 = C^2 R I$,

or $u = C\sqrt{RI}$ which is Chézy's law, which holds for uniform steady flow.

Chézy's law can also be written in terms of Q : $Q = CA\sqrt{RI}$

Introducing $A = b_s h$, (A is the cross section that transports the discharge Q) and when $b_s \gg h$ then:

$$R \approx h$$

So:

$$Q = C b_s h \sqrt{hI}$$

or

$$Q = C b_s h^{3/2} I^{1/2}$$

The propagation speed (or celerity) c of the flood wave is associated with the progress of a certain discharge Q . So the celerity (speed of the top of the wave) is the speed of the property $dQ = 0$ (no change in Q).

$$\text{As } Q = f(x, t), \quad dQ = \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial t} dt = 0$$

The celerity of the wave is :

$$c = \frac{dx}{dt} = - \frac{\partial Q / \partial t}{\partial Q / \partial x}$$

Expressions for $\frac{\partial Q}{\partial t}$ and $\frac{\partial Q}{\partial x}$ can be found by:

$$\text{Continuity:} \quad \frac{\partial Q}{\partial x} = -b \frac{\partial \eta}{\partial t} \quad \text{or} \quad \frac{\partial Q}{\partial x} = -b_s \frac{\partial h}{\partial t}$$

$$\begin{aligned} \text{Motion (reduced to Chézy's law):} \quad \frac{\partial Q}{\partial t} &= \frac{\partial}{\partial t} (C b_s h^{3/2} I^{1/2}) = C b_s \frac{3}{2} h^{1/2} \frac{\partial h}{\partial t} I^{1/2} \\ &= \frac{3}{2} b_s C \sqrt{hI} \frac{\partial h}{\partial t} = \frac{3}{2} b_s u \frac{\partial h}{\partial t} \end{aligned}$$

The celerity of the flood wave becomes:

$$c = - \frac{\partial Q / \partial t}{\partial Q / \partial x} = - \frac{\frac{3}{2} b_s u \frac{\partial h}{\partial t}}{-b \frac{\partial h}{\partial t}} = + \frac{3}{2} \frac{b_s}{b} u$$

In the expression $b > b_s$, so $\frac{b_s}{b} < 1$.

When for instance $b = 2b_s$, then $c = \frac{3}{4} u$.

The propagation speed of the flood wave is less than the velocity of the water which is often the case. The ratio b_s/b has a large influence on the propagation speed of the flood wave. Important from this consideration is that the propagation speed is not equal to \sqrt{gh} .

4.3.4 Tidal wave

Tidal waves are described by the equation of continuity and the equation of motion. As both *inertia* and *friction* are important, the complete equations should be considered:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} + g \frac{\partial \eta}{\partial x} + g \frac{Q|Q|}{C^2 A^2 R} = 0$$

It is, however, impossible to solve the complete equations analytically. Analytical solutions can only be found for simplified equations, which will be treated in Chapter 7. It is possible to solve the complete equation numerically by using computers Chapter 8).

To get insight in the behavior of tidal waves, first some simplified equations will be discussed which describe the so-called *harmonic waves*.

4.4 Harmonic waves

Harmonic waves are:

- periodic in time;
- sinusoidally shaped;
- and have amplitudes which are much smaller compared to the water depth.

The small amplitude, relative to the water depth, means that the current velocities are small. Therefore the friction term in the equation of motion can be neglected. The equations reduce to:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

$$\frac{\partial Q}{\partial t} + gA \frac{\partial \eta}{\partial x} = 0$$

The objective is to derive an equation for the *water elevation* η . Therefore, the first equation is differentiated to t :

$$\frac{\partial^2 Q}{\partial x \partial t} + b \frac{\partial^2 \eta}{\partial t^2} = 0$$

and the second equation is differentiated to x :

$$\frac{\partial^2 Q}{\partial x \partial t} + gA \frac{\partial^2 \eta}{\partial x^2} = 0$$

Subtracting these equations gives:

$$b \frac{\partial^2 \eta}{\partial t^2} - gA \frac{\partial^2 \eta}{\partial x^2} = 0 \text{ or } \frac{\partial^2 \eta}{\partial t^2} - \frac{gA}{b} \frac{\partial^2 \eta}{\partial x^2} = 0$$

The general solution of this equation for harmonic waves is :

$$\eta = \hat{\eta} \cos(\omega t - kx)$$

So the water elevation η is a function of x and t , which means that it varies in place and time. When $\omega t = \text{constant} = \alpha$, it means that wave η is a function of x for a certain time (see Figure 4.24):

$$\eta(x,t) = \hat{\eta} \cos(\alpha - kx) = \hat{\eta} \cos(kx - \alpha),$$

where:

$$k = \frac{2\pi}{L} \quad (L = \text{wave length}).$$

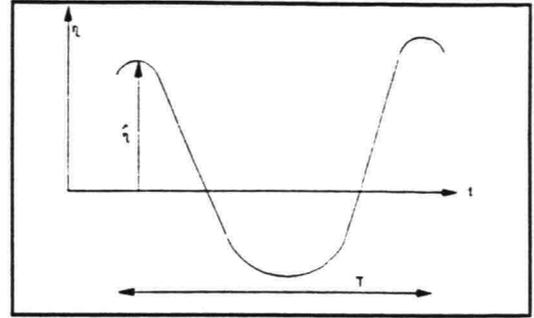


Figure 4.24 Sketch of η for $\omega t = \text{constant}$

When $kx = \text{constant} = \beta$, the wave is a function of time at a certain location (see Figure 4.25):

$$\eta(x,t) = \hat{\eta} \cos(\omega t - \beta)$$

where:

$$\omega = \frac{2\pi}{T} \quad (T = \text{wave period})$$

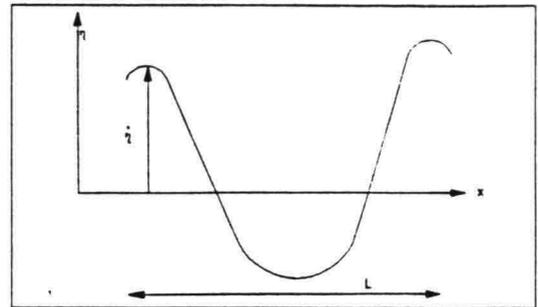


Figure 4.25 Sketch of η for $kx = \text{constant}$

Looking at the wave in time shows that the top (or the total shape) has a certain *propagation speed* or *celerity* c , see Figure 4.26.

During period T the top of the wave travels over a distance L , so:

$$c = \frac{L}{T} \quad \text{or} \quad L = cT$$

This is the *well-known relation* for wave phenomena.

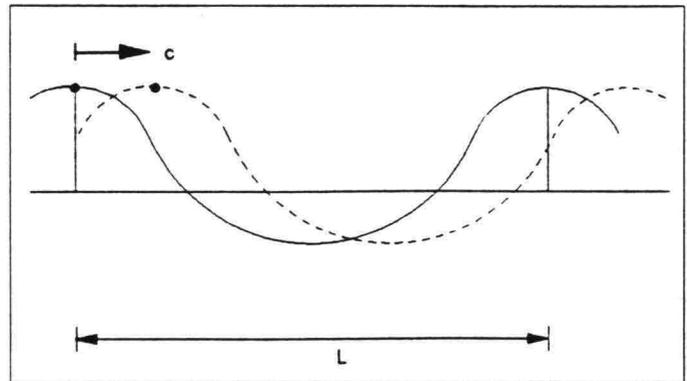


Figure 4.26 Celerity c of a tidal wave

The celerity c can be expressed in terms of ω and k :

$$c = \frac{L}{T} \quad \text{and} \quad \omega = \frac{2\pi}{T}$$

so
$$\frac{\omega}{k} = \frac{2\pi/T}{2\pi/L} = \frac{L}{T} \quad \text{or} \quad c = \frac{\omega}{k}$$

thus
$$k = \frac{2\pi}{L}$$

The first and second order derivatives of the general equation $\eta = \hat{\eta} \cos(\omega t - kx)$ to x and t are:

$$\frac{\partial \eta}{\partial t} = \omega \hat{\eta} \sin(\omega t - kx) \quad \text{and} \quad \frac{\partial^2 \eta}{\partial t^2} = -\omega^2 \hat{\eta} \cos(\omega t - kx)$$

$$\frac{\partial \eta}{\partial x} = -k \hat{\eta} \sin(\omega t - kx) \quad \text{and} \quad \frac{\partial^2 \eta}{\partial x^2} = -k^2 \hat{\eta} \cos(\omega t - kx)$$

Substitution of these derivatives in the equation:

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{gA}{b} \frac{\partial^2 \eta}{\partial x^2} = 0$$

gives:

$$-\omega^2 \hat{\eta} \cos(\omega t - kx) - \frac{gA}{b} \times -k^2 \times \hat{\eta} \cos(\omega t - kx) = 0$$

or
$$-\omega^2 + k^2 \frac{gA}{b} = 0$$

Thus:
$$\frac{\omega^2}{k^2} = \frac{gA}{b}$$

As the celerity of the wave is $c = \frac{\omega}{k}$:

$$c^2 = \frac{gA}{b} \text{ or } c = \pm \sqrt{\frac{gA}{b}}$$

This *general solution* represents *two harmonic waves*, one propagating in the *positive x-direction* and one propagating in the *negative x-direction*, each with a propagation speed or celerity

$$c = \sqrt{\frac{gA}{b}}$$

Note that c = propagation speed of the wave shape and not the velocity of the water particles!

The general solution can also be written as:

$$\begin{aligned} \eta &= \hat{\eta} \cos(\omega t - kx) \\ &= \hat{\eta} \cos \omega \left(t - \frac{k}{\omega} x \right) \text{ with } c = \frac{\omega}{k} \\ &= \hat{\eta} \cos \omega \left(t - \frac{x}{c} \right) \text{ with } c = \pm \sqrt{\frac{gA}{b}} \end{aligned}$$

So the *complete solution* with the two waves propagation in the positive and negative x -direction is:

$$\eta = \hat{\eta} \cos \omega \left(t - \frac{x}{c} \right) + \hat{\eta} \cos \omega \left(t + \frac{x}{c} \right), \text{ in which } c = \sqrt{\frac{gA}{b}}$$

Now the solution of the *discharge* Q for the harmonic wave will be discussed. This solution is similar to the solution of the water elevation η :

$$Q = \hat{Q} \cos(\omega t - kx + \alpha)$$

For substitution in the equation of continuity:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

the first order derivative of Q to x and the first order derivative of η to t are needed:

$$\frac{\partial Q}{\partial x} = \hat{Q} k \sin(\omega t - kx + \alpha), \text{ and } \frac{\partial \eta}{\partial t} = \hat{\eta} \omega \sin(\omega t - kx).$$

Substitution in the equation of continuity gives:

$$\hat{Q}k\sin(\omega t - kx + \alpha) - b\hat{\eta}\omega\sin(\omega t - kx) = 0$$

This means that

$$\alpha = 0 \text{ and}$$

$$\hat{Q}k - b\hat{\eta}\omega = 0$$

So: $\hat{Q} = b\hat{\eta}c$

The general solution for discharge Q becomes:

$$Q = b\hat{\eta}c\cos\omega\left(t - \frac{x}{c}\right) \text{ with } c = \pm\sqrt{\frac{gA}{b}}$$

The complete solution for the two waves, propagating in the positive and negative x -direction, is:

$$Q = b\hat{\eta}c\cos\omega\left(t - \frac{x}{c}\right) - b\hat{\eta}c\cos\omega\left(t + \frac{x}{c}\right) \text{ with } c = \sqrt{\frac{gA}{b}}$$

This can also be written as:

$$Q = \hat{Q}\cos\omega\left(t - \frac{x}{c}\right) - \hat{Q}\cos\omega\left(t + \frac{x}{c}\right) \text{ with } \hat{Q} = b\hat{\eta}c.$$

4.4.1 Single progressive harmonic wave

A specific solution is obtained by introducing boundary conditions. For example:

$$\text{at } x = 0 : \eta = \hat{\eta}\cos\omega t$$

The general solution becomes:

$$\eta = \hat{\eta}\cos(\omega t - kx) + \hat{\eta}\cos(\omega t + kx)$$

for $x = 0$:

$\eta = \hat{\eta}\cos\omega t$	+	$\hat{\eta}\cos\omega t$
<i>wave travelling</i>		<i>wave travelling</i>
<i>in the positive</i>		<i>in the negative</i>
<i>x-direction</i>		<i>x-direction</i>

The boundary condition can hold for one of the two waves: for the single progressive wave in the positive x -direction *or* the single progressive wave in the negative x -direction. Consider the *single progressive harmonic wave*, propagating in the positive x -direction, which is described by:

$$\eta = \hat{\eta}\cos\omega\left(t - \frac{x}{c}\right) \text{ or } \eta = \hat{\eta}\cos(\omega t - kx)$$

To illustrate that this formula describes, a wave is considered which propagates in the positive x -direction (see Figure 4.27). The starting point is at $x = x_0$. Some further to the right (at $x_0 + \Delta x$) the phase is earlier.

This means that after some time the top will arrive there. So for a small positive Δx and constant t , the phase difference:

$$\Delta \left[\omega \left(t - \frac{x}{c} \right) \right]$$

should be *negative*.

With $c = \sqrt{\frac{gA}{b}}$ = positive,

this is true for the wave:

$$\eta = \hat{\eta} \cos \omega \left(t - \frac{x}{c} \right)$$

The second wave

$$\eta = \hat{\eta} \cos \omega \left(t + \frac{x}{c} \right)$$

is the other wave, travelling in the *negative* x-direction.

Length of a tidal wave

The wave length of a tidal wave is large. The wave length is given by:

$$L = cT$$

Take for instance the wave corresponding with the M_2 -tidal component:

$$T = 12 \text{ h } 25 \text{ min} = 44700 \text{ sec.}$$

$$\text{For a water depth } h = 10 \text{ m, } c = \sqrt{\frac{gA}{b}} \approx \sqrt{gh} = \sqrt{100} = 10 \text{ m/s}$$

$$\text{so: } L = 10 * 44700 = 447.000 \text{ m} \approx 450 \text{ km.}$$

In deep oceans the wave length is even much larger, because the wave length L is proportional to the square root of the depth.

A harmonic wave propagates over large distances, thereby keeping its original shape. The fact that the wave does not deform is a consequence of neglecting the friction. In nature we will never find pure harmonic waves. In shallow seas estuaries the following effects occur:

- friction;
- damping;
- reflection, and so on.

Discharge in a single progressive harmonic wave

Consider the motion of the water particles in a single progressive harmonic wave that propagates in the positive x-direction.

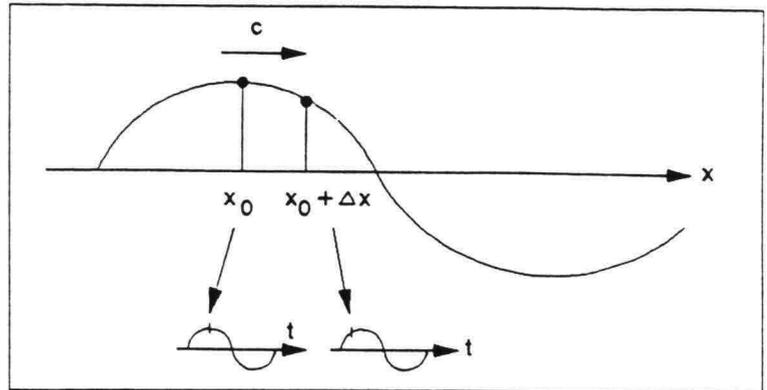


Figure 4.27 Wave propagating in positive x-direction

The water elevation is given by: $\eta = \hat{\eta} \cos \omega t$

The discharge is given by: $Q = \hat{Q} \cos(\omega t - kx)$ with $\hat{Q} = bc\hat{\eta}$

So η and Q are in phase. This means that:

- when $\eta = \text{max. positive}$, then $Q = \text{max. positive}$;
- when $\eta = 0$ then $Q = 0$;
- when $\eta = \text{max. negative}$, then $Q = \text{max. negative}$.

The relation between η and Q is given in Figure 4.28.

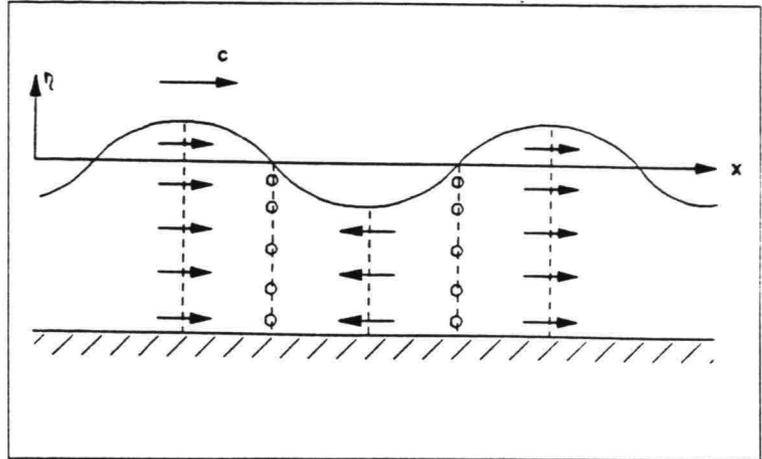


Figure 4.28 Relation between wave η and discharge Q

The discharge (or velocities) of the water particles are in phase with the water elevation. The velocities are *maximum positive* under the top or crest of the wave. The velocities are *maximum negative* under the trough of the wave.

Consider now the ratio of the velocities of the water particles and the propagation speed of the wave.

$$\hat{Q} = bc\hat{\eta}$$

and: $\hat{Q} = \hat{u}A$

where:

$$\hat{u} = \text{maximum velocity water particles};$$

$$A = bh \text{ (cross section).}$$

So: $\hat{u}bh = bc\hat{\eta}$ or $\frac{\hat{u}}{c} = \frac{\hat{\eta}}{h}$

The amplitude $\hat{\eta}$ is small compared to the water depth h . So the velocity u is small compared to the propagation speed c . This means that the *assumption to neglect the friction term* in the equation of motion was correct. The velocities of the water particles appear to be small.

It was derived mathematically that the velocities are maximum positive under the crest of the wave, and maximum negative under the trough of the wave. This can be illustrated by considering continuity or the mass balance for a wave, propagating in positive x -direction (Figure 4.29). Consider a wave at time t_0 and some time later at $t_0 + \Delta t$. In this time the wave shape propagates. The water particles *move back and forth*, but in the mean, they stay in the box.

If case of complete reflection, the amplitude of the reflected wave is equal to the amplitude of the incoming wave. The resultant wave is the sum of the two progressive waves.

$$\eta = \hat{\eta} \cos(\omega t - kx) + \hat{\eta} \cos(\omega t + kx)$$

Using the relation:

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

and: $c = \omega t - kx, b = \omega t + kx$

so that: $\frac{a+b}{2} = \frac{2\omega t}{2} = \omega t$ and $\frac{a-b}{2} = \frac{-2kx}{2} = -kx$

gives: $\eta = 2\hat{\eta} \cos \omega t \cos(-kx)$

Consider the expression for the complete solution of the discharge:

$$Q = \hat{Q} \cos(\omega t - kx) - \hat{Q} \cos(\omega t + kx)$$

Applying the above relations gives:

$$Q = \hat{Q} \times (-2) \sin \omega t \sin(-kx) \text{ or}$$

$$Q = 2\hat{Q} \sin kx \sin \omega t \text{ with } \hat{Q} = b\hat{\eta}c$$

To interpret these results, first a closer look is taken to the expression for the *water elevation*:

$$\eta = 2\hat{\eta} \underbrace{\cos kx}_{\text{amplitude is a function of } x} \cos \omega t$$

amplitude is a function of x

The water elevations have the same phase ωt for all values of x . The amplitude of the water elevation is a function of x . There are locations where the amplitude is always 0:

$$\cos kx = 0 \text{ for } kx = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

$$k = \frac{2\pi}{L}, \text{ substitution yields } x = \frac{1}{4}L, \frac{3}{4}L, \dots$$

These locations are called *nodes*.

There are locations where the amplitude is maximum: $\cos kx = \pm 1$ for $kx = 0, \pi, 2\pi, \dots$

These locations are called *antinodes*.

At certain instances, all elevations are 0, namely when $\cos \omega t = 0$.

At certain instances all elevations are maximum, namely when $\cos \omega t = \pm 1$.

The resulting water elevation in the standing harmonic wave are indicated in Figure 4.31.

At a certain moment in time, the wave has the slope given

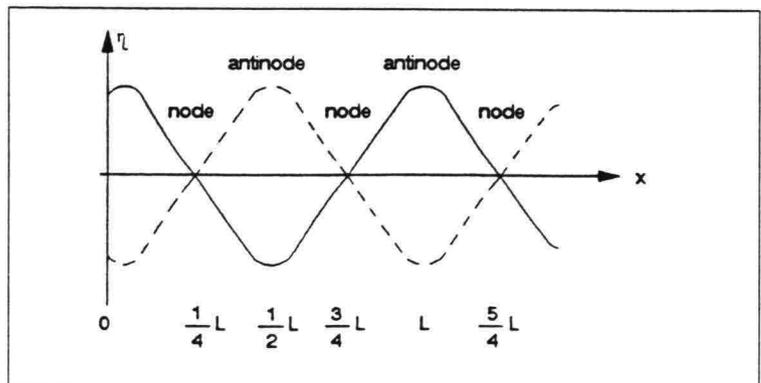


Figure 4.31 Standing wave with nodes and antinodes

by the drawn line. A quarter period later all elevations are 0. After a half period the wave has the shape of the dashed line. After three quarter of the period all elevations are 0 again, and so on.

Considering the expression for the *discharge* gives:

$$Q = 2\hat{Q}\sin kx \sin \omega t$$

amplitude is a function of x

The discharges have the same phase ωt for all values of x . The amplitude of this discharge is a function of x (see Figure 4.32).

There are locations where the amplitude is always 0:

$$\begin{aligned} \sin kx &= 0 \\ \text{for } kx &= 0, \pi, 2\pi, \dots \end{aligned}$$

As $k = 2\pi/L$, it means that $x = 0, \frac{1}{2}L, L, \dots$ which is in the *antinodes*.

Locations where the amplitude is always maximum are:

$$\begin{aligned} \sin kx &= \pm 1 \\ \text{for } kx &= \pi/2, 3\pi/2, \dots \end{aligned}$$

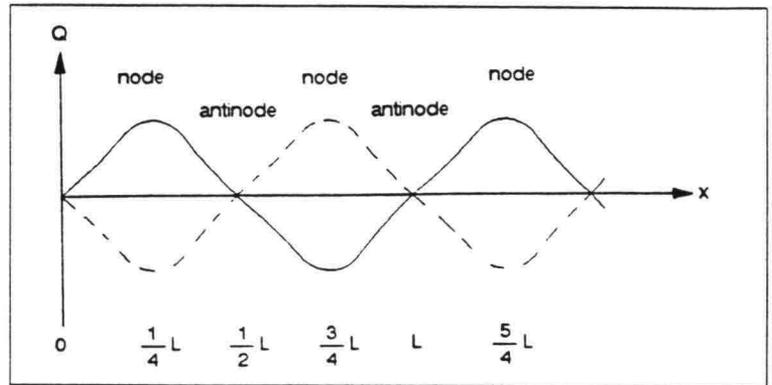


Figure 4.32 Nodes and antinodes of the discharge

As $k = 2\pi/L$, it means that amplitude is maximum for $x = 0, \frac{1}{4}L, \frac{3}{4}L$, which is in the *nodes*.

As said before, standing waves can be observed in rivers and estuaries which are closed at one end. It is logical that the *antinode* (where $Q = 0$ all the time) is located at the closed end.

Figure 4.33 shows the time history of a standing wave.

Long waves in one dimension

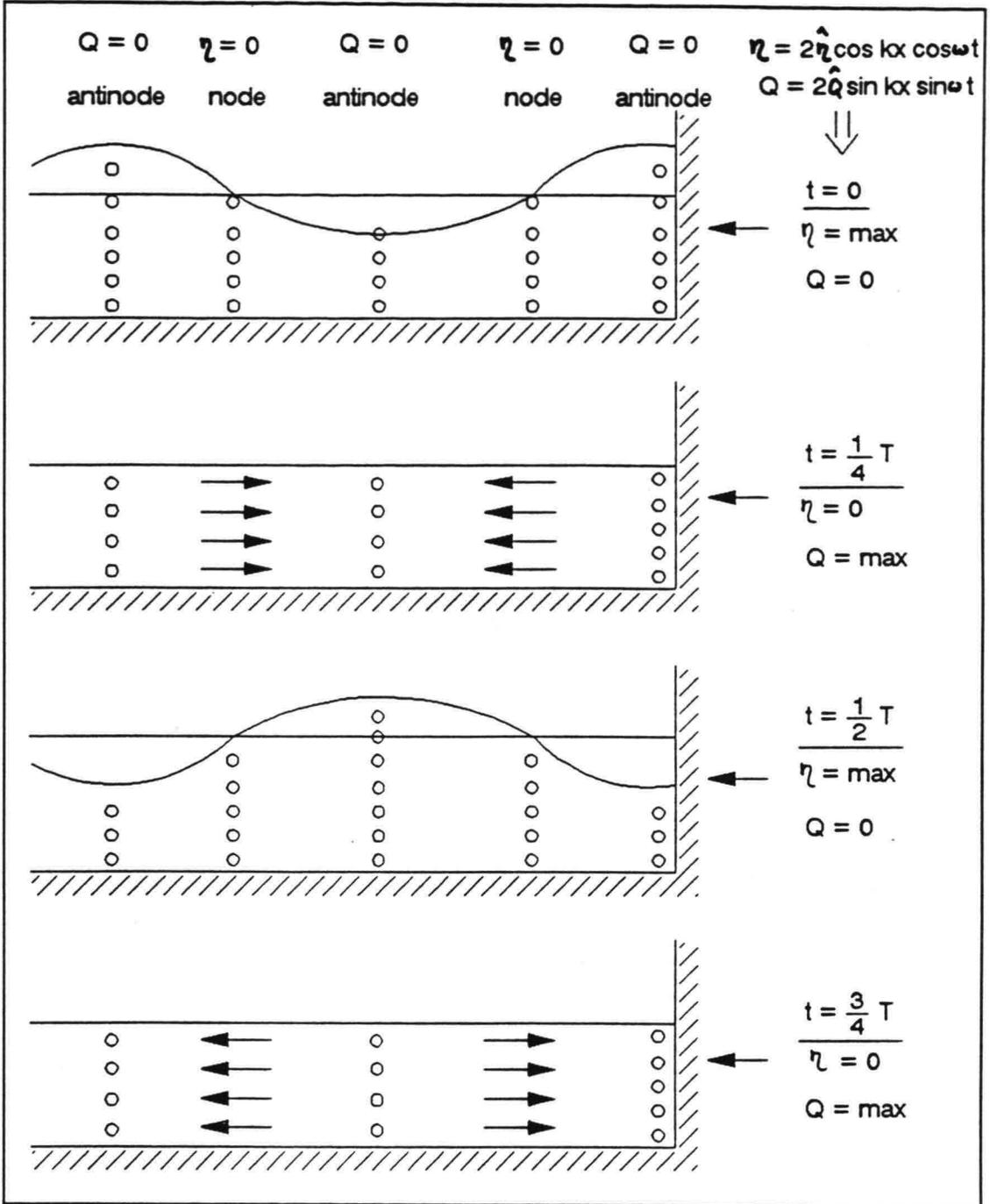


Figure 4.33 Time history of a standing wave

Resonance of a standing wave
 The mode of oscillation in a bay or estuary, which is closed at one end, is governed by the ratio of the length of the tidal wave and the length of the bay. Consider a bay with a length l (see Figure 4.34).

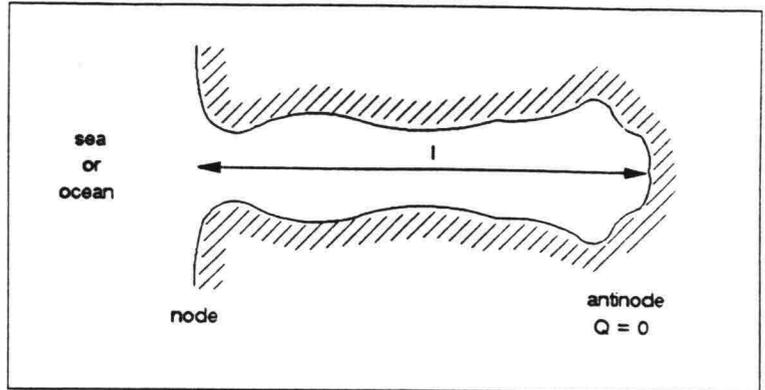


Figure 4.34 Bay with length l

If the length of the bay approaches $\frac{1}{4}L$, $\frac{3}{4}L$, $\frac{5}{4}L$ (where L is the length of the tidal wave), then a node occurs at the sea entrance and an antinode at the closed end (see Figure 4.35).

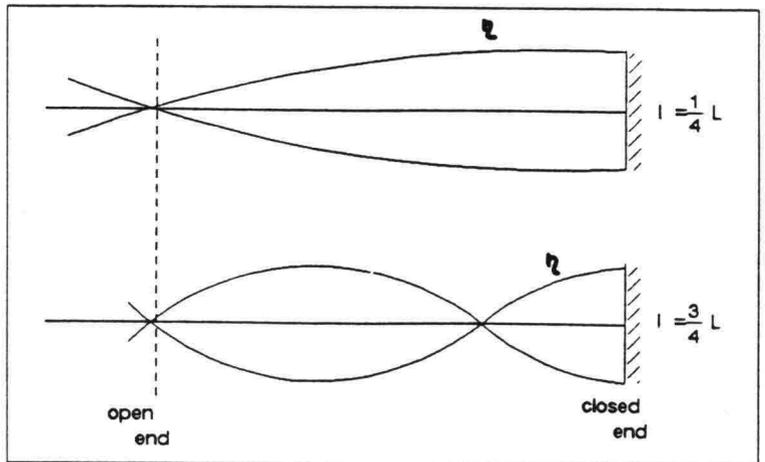


Figure 4.35 Resonance in the bay of Figure 4.34

In that case, resonance occurs. The water levels in the antinodes become very large (theoretically infinite). In nature, friction will prevent that the amplitude of the water levels becomes infinite.

Thus the friction should be taken into account with the computation of the resulting wave height. For this, mathematical models are available.

Resonance will take place if $l = \frac{1}{4}L, \frac{3}{4}L, \frac{5}{4}L, \dots$. A general expression is:

$$l = \frac{2n + 1}{4}L \text{ with } n = 0, 1, 2, 3, 4, \dots$$

The *natural period of oscillation* of a bay with length l can be found from:

$$L = c.T \rightarrow T = \frac{L}{c} = \frac{L}{\sqrt{gh}}$$

$$T = \frac{4l}{(2n + 1)\sqrt{gh}}, \text{ with } n = 0, 1, 2, 3, \dots$$

$$l = \frac{2n + 1}{4}L \rightarrow L = \frac{4l}{2n + 1}$$

In this way it can be checked if a bay or estuary will have resonance problems for certain periods from the tide at sea.

An *example of a standing wave*, in which the length of the bay is about one quarter of the wave length ($\frac{1}{4}L$) is found in the *Bay of Fundy* at the east coast of Canada (see Figure 4.36).

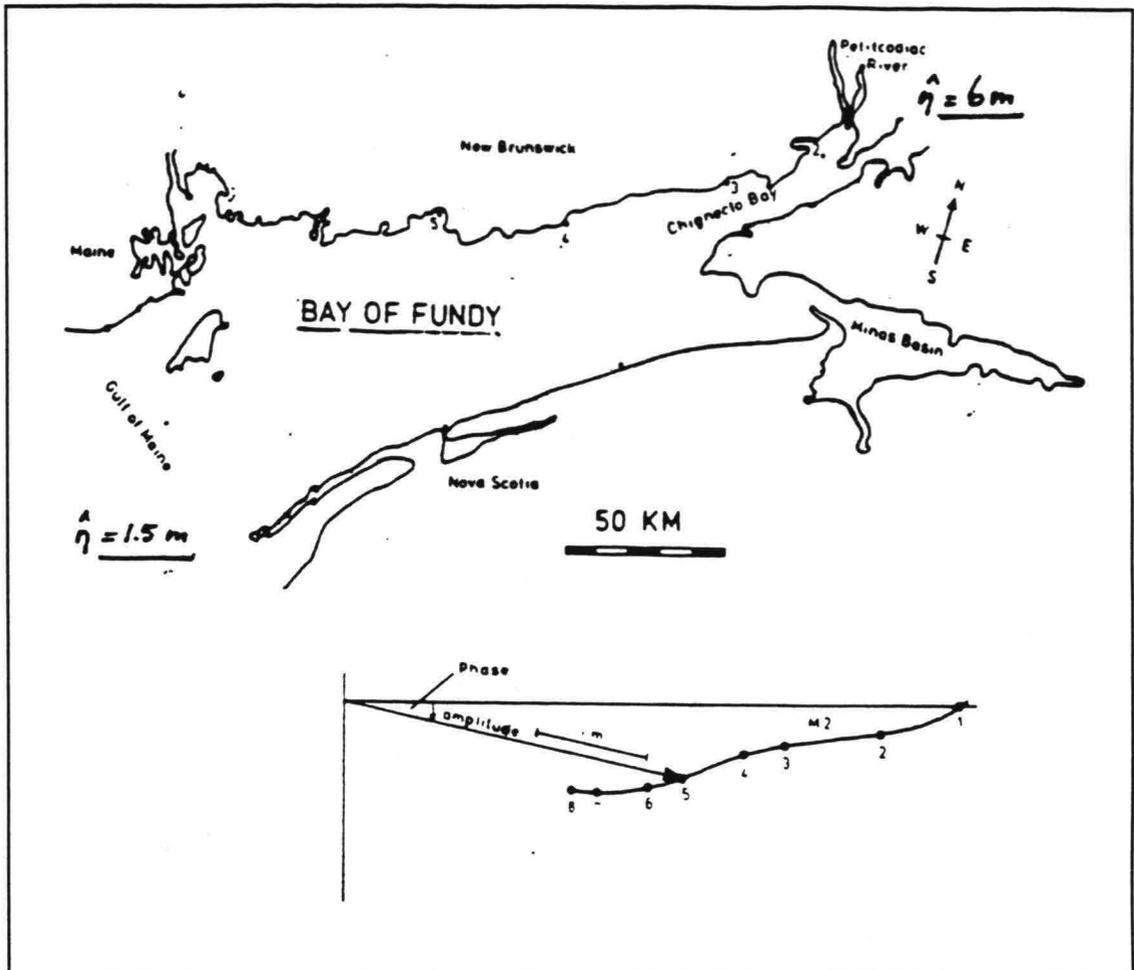


Figure 4.36 Bay of Fundy, Canada

The bay is connected to the open sea and has a closed end. The tidal amplitude increases considerably between the open end and the closed end of the bay.

At the open end the amplitude is about 1.5 m. At the closed end the amplitude is about 6 m. The high waters and low waters occur in the whole bay almost simultaneously. Both features indicate a standing wave in resonance. This can be checked as follows:

- length of the bay: $l = 300 \text{ km}$;
- average depth: $h = 75 \text{ m}$;
- principal tide: M_2 -tide (period of 12 h 25 min, so $T = 44700 \text{ s}$).

$$L = cT = \sqrt{gh}T$$

$$\sqrt{gh} = \sqrt{10 \times 75} = 27 \text{ m/s}$$

$$T = 44700 \text{ s}$$

$$\text{so: } L = 27 \times 44700 = 120600 \text{ m} \approx 1200 \text{ km.}$$

The length of the bay is about a quarter of the wave length ($1/4L$), so resonance occurs.

Seiches

Resonance does not happen only due to the tidal wave with periods of hours (like the M_2 -component with 12 h 25 min.), there are also periodic long waves of relative short periods, ranging from 5-30 minutes, which are called *seiches*. Seiches are caused by meteorological phenomena, like moving depressions. The amplitudes of seiches are small compared to the water depth, which means that the friction can be neglected and that the harmonic wave solution can be applied. Seiches are especially of relevance for harbour basins and should be taken into account.

As example, consider the harbour area of Figure 4.37.

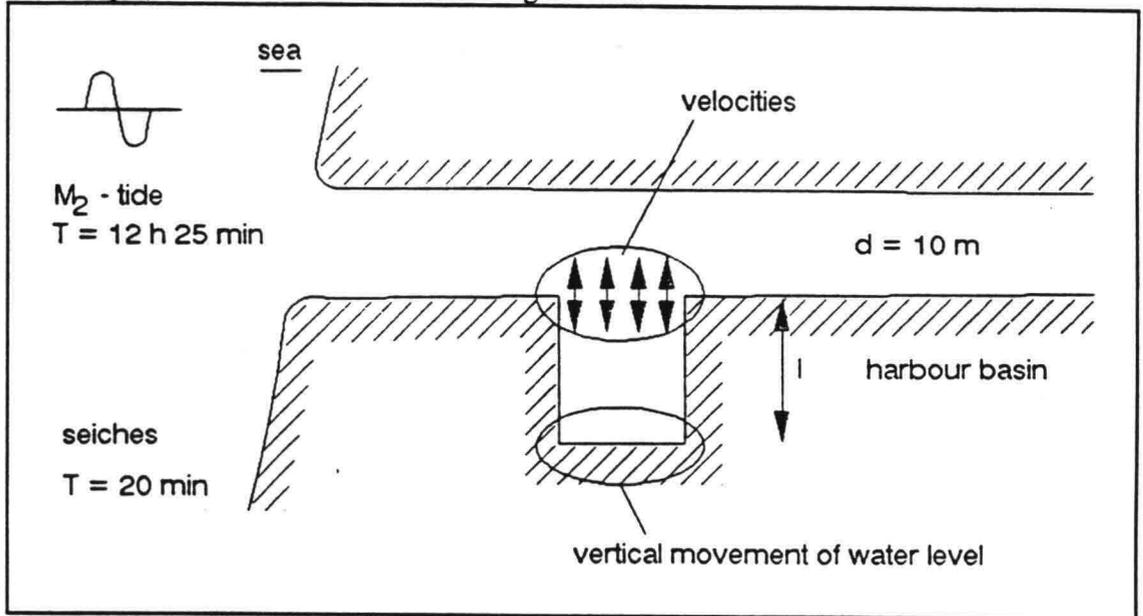


Figure 4.37 Harbor area along a tidal river

Seiches can occur with periods of about 20 minutes. The depth of the harbour area is 10 m. The critical length of the harbour basin can be determined as follows:

Length of the seiches wave (superimposed on the tidal wave) is:

$$L = cT = \sqrt{gh}T = \sqrt{10 \times 10} \times 20 \times 60 = 12000 \text{ m} = 12 \text{ km}$$

So the *critical length* of the basin $l = \frac{1}{4}L = 3 \text{ km}$, which means that with such a length resonance due to seiches can be expected. Practical consequences can be:

- high velocities at the mouth of the harbour basin and cross currents on the river;
- important vertical movements of moored ships.

Transverse oscillation

For completeness also a third mode of oscillation, a transverse oscillation in seas or bays, is considered (see Figure 3.38). This occurs between two closed ends. The length between the two closed ends should be $\frac{1}{2}L, L, \frac{3}{2}L$. In such a case, antinodes are located at both ends and resonance may occur, with large amplitudes of the water levels in the antinodes.

Resonance will take place if $l = \frac{1}{2}L, L, \frac{3}{2}L, \dots$. A general expression is:

$$l = \frac{n}{2}L \text{ with } n = 1, 2, 3, \dots$$

The natural period of oscillation of a basin with length l can be derived through:

$$L = cT \rightarrow T = \frac{L}{c} = \frac{L}{\sqrt{gh}}$$

$$l = \frac{n}{2}L \rightarrow L = \frac{2l}{n}$$

$$\text{so: } T = \frac{2L}{n\sqrt{gh}}$$

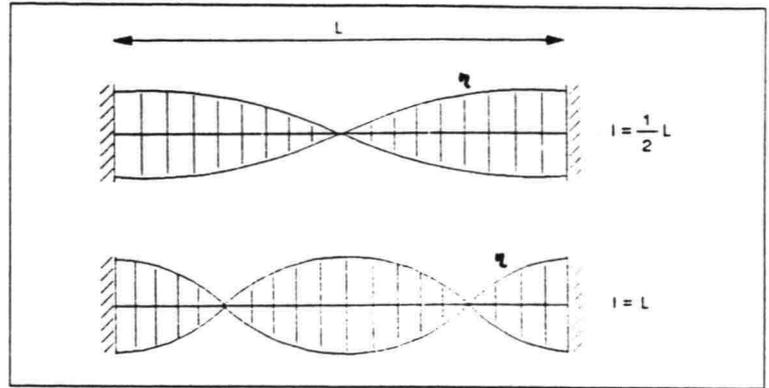


Figure 4.38 Transverse oscillation

It is also possible that the length of a bay or estuary, closed at one end equals $\frac{1}{2}L, L, \dots$ (see Figure 4.39). In this case also resonance will take place, caused by a standing wave. In this case, however, no amplification of the tide occurs.

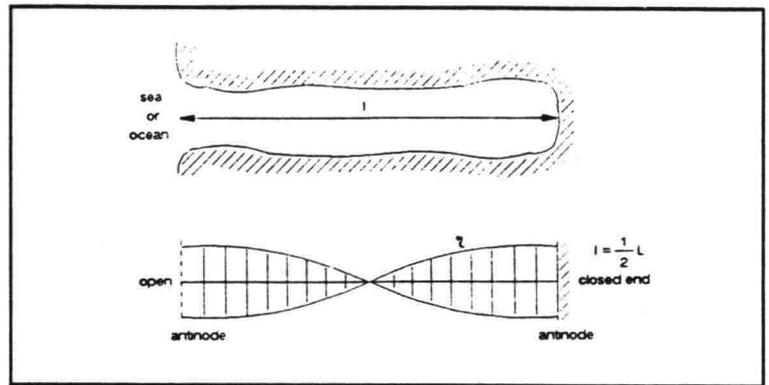


Figure 4.39 Oscillation for $l = 0.5 L$

The tide at the closed end has the same amplitude as the tide at the open end. It may become *somewhat smaller due to friction*.

An application of resonance between two closed ends is a lake, which is exposed to a heavy wind for a certain time and suddenly the wind stops (see Figure 4.40).

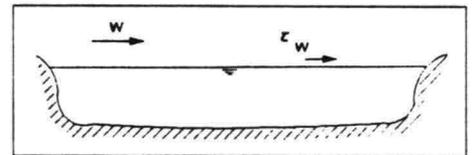


Figure 4.40 Lake exposed to wind

The wind causes a shear stress over the water surface:

$$\tau_w = f_a \rho_a w^2$$

in which:

- τ_w shear stress;
- f_a friction coefficient;
- ρ_a density of air;
- w wind velocity (m/s).

As reaction to the wind a slope in the water level of the lake occurs. Therefore, consider an element of the lake with length dx , see Figure 4.41.

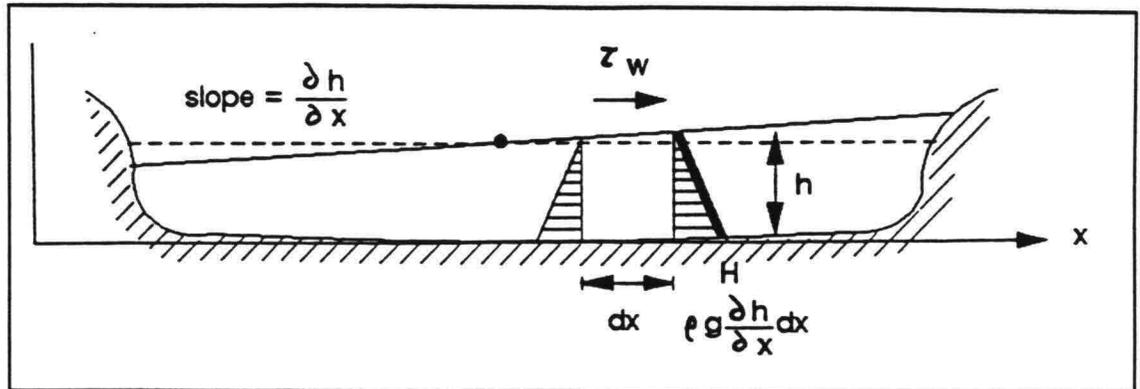


Figure 4.41 Sloping water level as reaction to the wind

The shear stress will balance with a pressure gradient due to the slope in the water surface. The shear force (per unit width) is: $\tau_w dx$

The net pressure force (per unit width) is: $\rho_w g \frac{\delta \eta}{\delta x} dx \times h$

Equilibrium of the forces means:

$$\rho_w g h \frac{\delta \eta}{\delta x} dx = f_a \rho_a w^2 dx$$

So the slope of the water level is:

$$\frac{\delta \eta}{\delta x} = \frac{f_a \rho_a w^2}{\rho_w g h}$$

Note that the slope is reverse proportional to the depth ($1/h$), which means that:

- in shallow lakes the wind can cause an important set-up;
- in deep lakes (or seas, or oceans) the slope will be very small.

This means that storm wind will lead to storm surges (= set-up due to storm) in shallow seas, and not in deep oceans. Figure 4.4 shows the equilibrium situation during heavy wind, which is a more or less constant slope.

When the wind stops, the shear force at the surface has ceased. The pressure gradient forces the water to flow back. Now the effect of inertia becomes visible: the lake starts to oscillate (see Figure 4.42). Of course this motion will be damped due to friction.

The period of oscillation can be computed from the discussed theory:

- length of the lake l ;
- depth of the lake h .

The oscillation is performed by a half wave length:

$$l = \frac{1}{2}L, L = cT = T\sqrt{gh}$$

$$\text{So: } T = \frac{L}{\sqrt{gh}} = \frac{2l}{\sqrt{gh}}$$

Long waves in one dimension

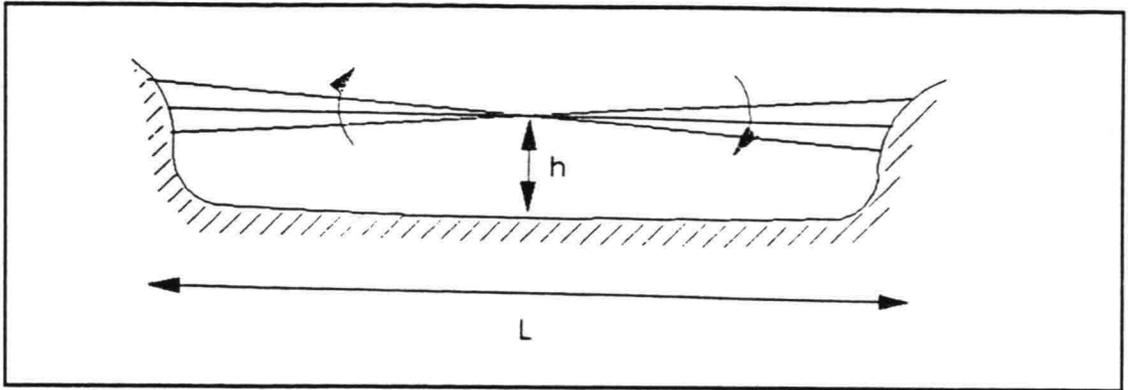


Figure 4.42 Oscillation in a lake after a storm

5 Tidal propagation in one dimension

5.1 Introduction

The discussions on the harmonic wave gave some insight in the behaviour of a tidal wave. This Chapter treats some types of tidal propagation in one dimension, namely a tidal wave travelling on a river in upstream direction and a tidal bore. Tidal waves in seas and oceans are discussed in Chapter 6.

5.2 Tidal wave on a river

A tidal river is the lower reach of a river, which is under influence of the tide. Here, interaction takes place between the oscillating flow caused by the tide at sea and the run-off of the river. One of the effects is that the amplitude of the tide decreases in the upstream direction.

Consider the situation of Figure 5.1. The assumptions for the river are:

- a uniform cross section;
- constant slope I ;
- discharge $Q_0 = \text{constant}$.

The x -axis is parallel to the bed of the river and positive in the downstream direction. The discharges Q_0 is positive; the slope I is also positive.

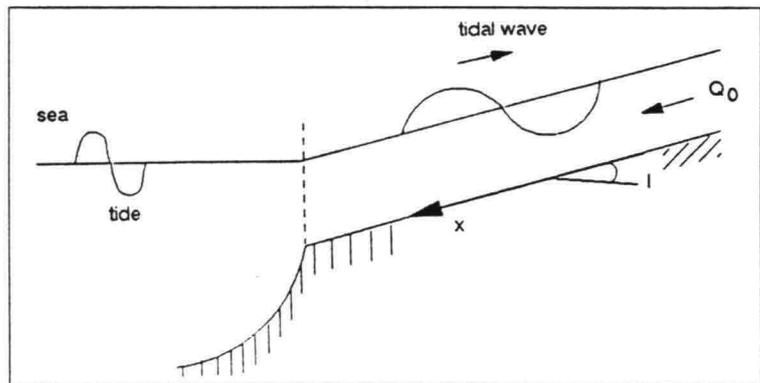


Figure 5.1

The run-off flow in the river is basically balanced by the friction force (Chézy's law holds). Thus the equation of motion for this problem should contain the friction term. The equations describing this problem are:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0 \quad \text{Equation of continuity}$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} + g \frac{\partial \eta}{\partial x} - gI + g \frac{Q|Q|}{C^2 A^2} \quad \text{Equation of motion}$$

Assume that the deviations from the mean water level are small, implies that C , A , R can be considered as constants. The friction term is non-linear, which causes problems in the analytical solution. Therefore, this term is linearized:

$$g \frac{Q|Q|}{C^2 A^2 R} = \frac{g|Q|}{C^2 A^2 R} \times Q = mQ$$

where $m = \frac{g|Q|}{C^2 A^2 R}$

As g, C, A, R are considered as constants, m is a mean of discharge $|Q|$ over a tidal cycle. So m is a kind of mean value of the friction over a tidal cycle. As a consequence of the linearization, the solution will not show any distortion of the tidal wave due to the quadratic friction.

The linearized equations now become:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0 \quad \text{Equation of continuity}$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} + g \frac{\partial \eta}{\partial x} - gI + mQ = 0 \quad \text{Equation of motion}$$

From these equations, an equation for the water elevation η can be derived. For that, the first equation is differentiated to t and the second to x :

$$\therefore \frac{\partial^2 Q}{\partial x \partial t} + b \frac{\partial^2 \eta}{\partial t^2} = 0 \quad (5.1)$$

$$\frac{1}{A} \frac{\partial^2 Q}{\partial x \partial t} + g \frac{\partial^2 \eta}{\partial x^2} + m \frac{\partial Q}{\partial x} = 0, \text{ or } \frac{\partial^2 Q}{\partial x \partial t} + gA \frac{\partial^2 \eta}{\partial x^2} + mA \frac{\partial Q}{\partial x} = 0 \quad (5.2)$$

Subtracting Equations 5.1 and 5.2 gives:

$$b \frac{\partial^2 \eta}{\partial t^2} - gA \frac{\partial^2 \eta}{\partial x^2} - mA \frac{\partial Q}{\partial x} = 0$$

Replacing $\frac{\partial Q}{\partial x}$ by $(-b \frac{\partial \eta}{\partial t})$ (equation of continuity) gives a relation with independent variable η :

$$b \frac{\partial^2 \eta}{\partial t^2} - gA \frac{\partial^2 \eta}{\partial x^2} + mA b \frac{\partial \eta}{\partial t} = 0 \text{ or (dividing by } b) \frac{\partial^2 \eta}{\partial t^2} - \frac{gA}{b} \frac{\partial^2 \eta}{\partial x^2} + mA \frac{\partial \eta}{\partial t} = 0$$

As $\frac{gA}{b} = c_o^2$ (propagation speed), this equation becomes:

$$\frac{\partial^2 \eta}{\partial t^2} - c_o^2 \frac{\partial^2 \eta}{\partial x^2} + mA \frac{\partial \eta}{\partial t} = 0, \text{ which is known as the telegraph equation.} \quad (5.3)$$

Oscillating solutions of this linear equation are of the form:

$$\eta = \hat{\eta}_o e^{\pm \lambda x} \cos(\omega t \pm kx)$$

in which:

$\hat{\eta}_o$: reference wave amplitude for $x = 0$

$\hat{\eta}_o e^{\pm \lambda x}$: amplitude of the wave, function of x

λ, k : unknown.

It is not likely that a wave with a very high amplitude occurs for $x = -\infty$ (up the river), so only $e^{+\lambda x}$ satisfies. The wave on the river, is a progressive wave in the negative x-direction, so only $\cos(\omega t + kx)$ should be considered.

The solution of the equation is :

$$\eta = \hat{\eta}_o e^{\lambda x} \cos(\omega t + kx)$$

This is a *single progressive wave, propagating in the upstream direction, with decreasing amplitude.*

Factors λ and k can be obtained by substituting the solution in the *telegraph equation*. The derivatives of η are:

$$\frac{\partial \eta}{\partial t} = \hat{\eta}_o e^{\lambda x} \cdot -\sin(\omega t + kx) \cdot \omega = -\hat{\eta}_o \omega e^{\lambda x} \sin(\omega t + kx)$$

$$\frac{\partial^2 \eta}{\partial t^2} = -\hat{\eta}_o \omega \cdot e^{\lambda x} \cdot \cos(\omega t + kx) = -\hat{\eta}_o \omega^2 e^{\lambda x} \cos(\omega t + kx)$$

$$\frac{\partial \eta}{\partial x} = \hat{\eta}_o [\lambda e^{\lambda x} \cdot \cos(\omega t + kx) + e^{\lambda x} \cdot -\sin(\omega t + kx) \cdot k] = \hat{\eta}_o \lambda e^{\lambda x} \cos(\omega t + kx) - \hat{\eta}_o k e^{\lambda x} \sin(\omega t + kx)$$

$$\begin{aligned} \frac{\partial^2 \eta}{\partial x^2} &= \hat{\eta}_o \lambda [\lambda e^{\lambda x} \cdot \cos(\omega t + kx) + e^{\lambda x} \cdot -\sin(\omega t + kx) \cdot k] - \hat{\eta}_o k [\lambda \cdot e^{\lambda x} \cdot \sin(\omega t + kx) + e^{\lambda x} \cdot \cos(\omega t + kx) \cdot k] \\ &= \hat{\eta}_o \lambda^2 e^{\lambda x} \cos(\omega t + kx) - \hat{\eta}_o \lambda k e^{\lambda x} \sin(\omega t + kx) - \hat{\eta}_o k \lambda e^{\lambda x} \sin(\omega t + kx) - \hat{\eta}_o k^2 e^{\lambda x} \cos(\omega t + kx) \\ &= \hat{\eta}_o \lambda^2 e^{\lambda x} \cos(\omega t + kx) - 2\hat{\eta}_o k \lambda e^{\lambda x} \sin(\omega t + kx) - \hat{\eta}_o k^2 e^{\lambda x} \cos(\omega t + kx) \end{aligned}$$

Substitution of these derivatives in Equation 5.3 yields:

$$-\hat{\eta}_o \omega^2 e^{\lambda x} \cos(\omega t + kx) - c_o^2 [\hat{\eta}_o \lambda^2 e^{\lambda x} \cos(\omega t + kx) - 2\hat{\eta}_o k \lambda e^{\lambda x} \sin(\omega t + kx) - \hat{\eta}_o k^2 e^{\lambda x} \cos(\omega t + kx)] - m A \hat{\eta}_o \omega e^{\lambda x} \sin(\omega t + kx) = 0$$

Dividing by $\hat{\eta}_o$ and $e^{\lambda x}$ gives:

$$\cos(\omega t + kx) [-\omega^2 - c_o^2 \lambda^2 + c_o^2 k^2] + \sin(\omega t + kx) [2c_o^2 k \lambda - m A \omega] = 0$$

This equation can only be fulfilled when:

$$(-\omega^2 - \lambda^2 c_o^2 + k^2 c_o^2) = 0 \quad (1)$$

$$(2k\lambda c_o^2 - m A \omega) = 0 \quad (2)$$

Now λ and k can be solved:

From equation (2): $k = \frac{m A \omega}{2\lambda c_o^2}$

Equation (1) can be written as $c_o^2 \lambda^2 - c_o^2 k^2 + \omega^2 = 0$

Substitution of k gives: $c_o^2 \lambda^2 - c_o^2 \frac{m^2 A^2 \omega^2}{4\lambda^2 c_o^4} + \omega^2 = 0$

Multiplying with λ^2 :
$$c_o^2[\lambda^2]^2 + \omega^2[\lambda^2] - \frac{m^2 A^2 \omega^2}{4c_o^2} = 0$$

This is a *quadratic equation* of the type :

$$ax^2 + bx + c = 0 \text{ with roots } x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Applying this:

$$[\lambda^2]_{1,2} = \frac{-\omega^2 \pm \sqrt{\omega^4 + 4c_o^2 \frac{m^2 A^2 \omega^2}{4c_o^2}}}{2c_o^2} = \frac{\omega^2 \left[-1 \pm \sqrt{1 + \frac{m^2 A^2}{\omega^2}} \right]}{2c_o^2}$$

λ^2 is positive, which means that only the positive root holds:

$$\lambda^2 = \frac{\omega^2 \left[-1 + \sqrt{1 + \frac{m^2 A^2}{\omega^2}} \right]}{2c_o^2}, \text{ so} \quad \lambda = \frac{\omega \sqrt{-1 + \sqrt{1 + \frac{m^2 A^2}{\omega^2}}}}{c_o \sqrt{2}} \quad (5.4)$$

In a similar way, the solution for k becomes:

$$k = \frac{\omega \sqrt{+1 + \sqrt{1 + \frac{m^2 A^2}{\omega^2}}}}{c_o \sqrt{2}} \quad (5.5)$$

5.2.1 Celerity of the wave

For investigating the celerity of a wave, the above solutions can be used. Substitution of k (Equation 5.5) in the expression for celerity c gives:

$$c = \frac{\omega}{k} = \frac{c_o \sqrt{2}}{\sqrt{+1 + \sqrt{1 + \frac{m^2 A^2}{\omega^2}}}}$$

From this relation it can be concluded that:

- $m = 0$ (no friction): $c = \frac{c_o \sqrt{2}}{\sqrt{2}} = c_o = \frac{\sqrt{gA}}{b}$
- $m \neq 0$ (friction): $c = \frac{c_o \sqrt{2}}{\text{term} > \sqrt{2}} < c_o$

So in case of friction, the celerity of the wave is smaller than without friction.

Example

$$m = \frac{g|Q|}{C^2 A^2 R} = \frac{g}{C^2 A R} \frac{|Q|}{A} = \frac{g|u|}{C^2 A R} \quad (\text{mean value over a tidal cycle})$$

So: $m A = \frac{g|u|}{C^2 R}$

Substitution of:

$$g = 10 \text{ m/s}^2$$

$$|u| = 1 \text{ m/s}$$

$$C = 50 \text{ m}^{1/2}/\text{s}$$

$$R = 5 \text{ m}$$

$$\omega_{M_2} = 0.00014 \text{ rad/s}$$

gives:

$$m A = \frac{g|u|}{C^2 R} = \frac{10 \times 1}{(50)^2 \times 5} = 0.0008$$

$$\frac{m^2 A^2}{\omega^2} = \frac{0.0008^2}{0.00014^2} = 32$$

Substitution in the expression for c gives:

$$c = \frac{c_o \sqrt{2}}{\sqrt{1 + \sqrt{1 + \frac{m^2 A^2}{\omega^2}}}} = \frac{c_o \sqrt{2}}{\sqrt{1 + \sqrt{33}}} = 0.54 c_o$$

This shows that the celerity or propagation speed of a tidal wave is to a large extent reduced by the friction.

5.2.2 Attenuation of the wave

The attenuation of a tidal wave is determined by the factor $e^{\lambda x}$ in the general solution:

$$\eta = \hat{\eta}_o e^{\lambda x} \cos(\omega t + kx)$$

Substituting the numerical values of the above example in Equation 5.4 gives for λ :

$$\frac{m^2 A^2}{\omega^2} = 32$$

$$c_o = \sqrt{gh} = \sqrt{50}$$

$$\lambda = \frac{0.00014 \sqrt{-1 + \sqrt{33}}}{\sqrt{100}} = 3 \times 10^{-5} \text{ m}^{-1}$$

The attenuation of the amplitude is indicated in Figure 5.2.

At $x = 0$ (mouth of the river; see Figure 5.3), the amplitude is η_0

The decrease in amplitude in in upstream direction can be determined as follows:

- $x = -33$ km
 $\lambda_x x = -1$, so $e^{-1} = 0.37$;
- $x = -67$ km
 $\lambda_x x = -2$, so $e^{-2} = 0.14$;
- $x = -100$ km
 $\lambda_x x = -3$, so $e^{-3} = 0.05$

Thus, after 100 km only 5% of the amplitude is left.

Figure 5.4 gives an example of tidal wave propagation in one of the branches of the river Rhine. The amplitude decrease from location 1 to 7 (see water level curves), which confirms the theory.

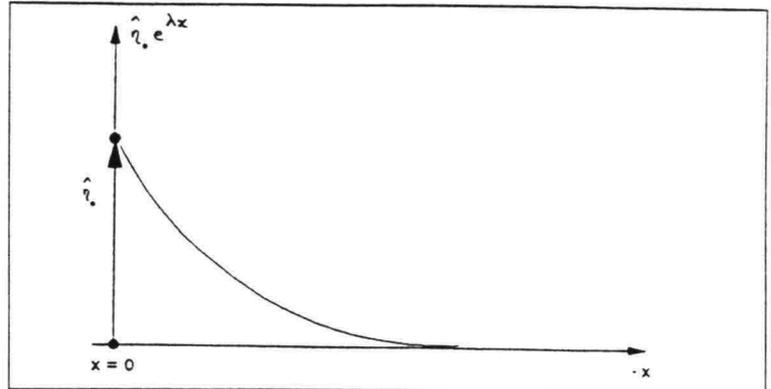


Figure 5.2 Development of the amplitude of a wave traveling in upstream direction

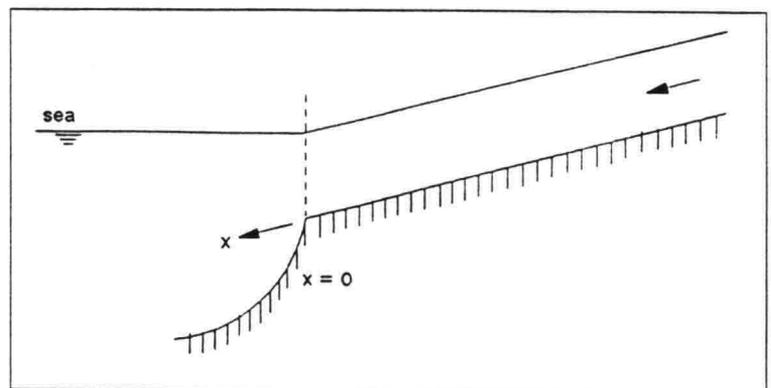


Figure 5.3 Mouth of the river

5.2.3 Discharge of tidal river

The discharge of a tidal can be derived from the equation of continuity:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

in which:

$$\eta = \hat{\eta}_0 e^{\lambda x} \cos(\omega t + kx)$$

$$\frac{\partial \eta}{\partial t} = -\hat{\eta}_0 \omega e^{\lambda x} \sin(\omega t + kx)$$

Substitution in the equation of continuity yields:

$$\frac{\partial Q}{\partial x} = -b \frac{\partial \eta}{\partial t} = b \hat{\eta}_0 \omega e^{\lambda x} \sin(\omega t + kx)$$

Integrating this equation gives:

$$\int \partial Q = b \hat{\eta}_0 \omega \int e^{\lambda x} \sin(\omega t + kx) dx$$

The solution of this integral can be found in standard books:

Tidal propagation in one dimension

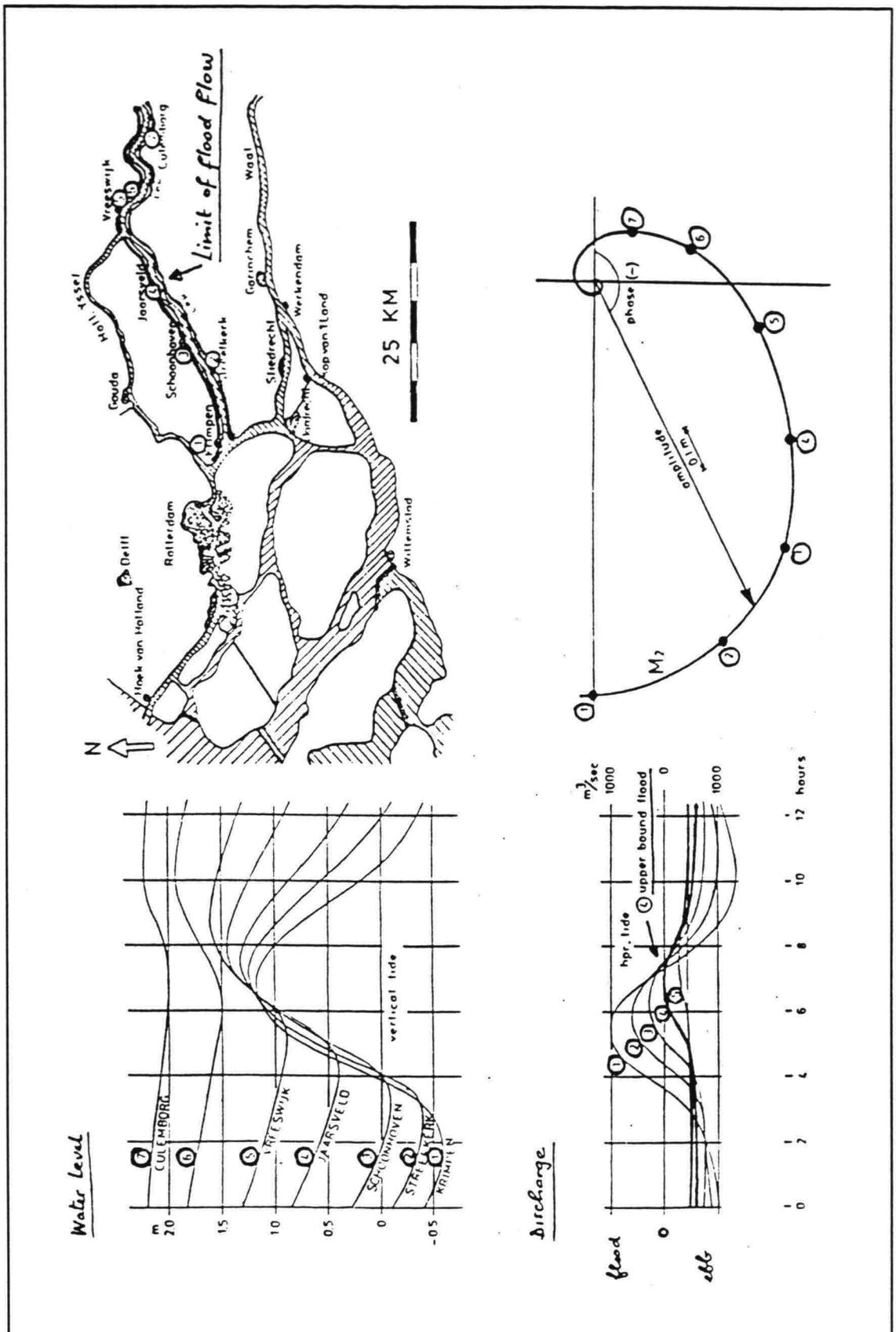


Figure 5.4 Tidal wave, propagating in upstream direction in a branch of the river Rhine

$$\int e^{\lambda x} \sin(\omega t + kx) dx = \frac{e^{\lambda x}}{\lambda^2 + k^2} [\lambda \sin(\omega t + kx) - k \cos(\omega t + kx)]$$

The integration step gives :

$$Q = Q_o + b \hat{\eta}_o \omega \frac{e^{\lambda x}}{\lambda^2 + k^2} [\lambda \sin(\omega t + kx) - k \cos(\omega t + kx)]$$

in which Q_o is the integration constant, being the constant run-off discharge of the river. The second term is the oscillating part, which is superimposed on the constant discharge. The solution of the discharge for a given point x along the river can be found as follows:

A. Neglecting the friction

When it is assumed that the friction can be neglected, then factor $m = 0$, which means that $\lambda = 0$. The expression for Q becomes:

$$Q = Q_o - \frac{b \hat{\eta}_o \omega}{k} \cos(\omega t + kx)$$

This expression shows that the oscillating part of the discharge is in phase with the amplitude of the water level:

$$\eta = \hat{\eta}_o e^{\lambda x} \cos(\omega t + kx)$$

The solutions for the discharge and the water level for a given point x along the river are given in Figure 5.5. These solutions correspond with the behaviour of a single progressive wave. The moments, during which the currents are 0, are called *slack water*. We see that the moments of slack water move to the moments of *HW*, because of the constant discharge.

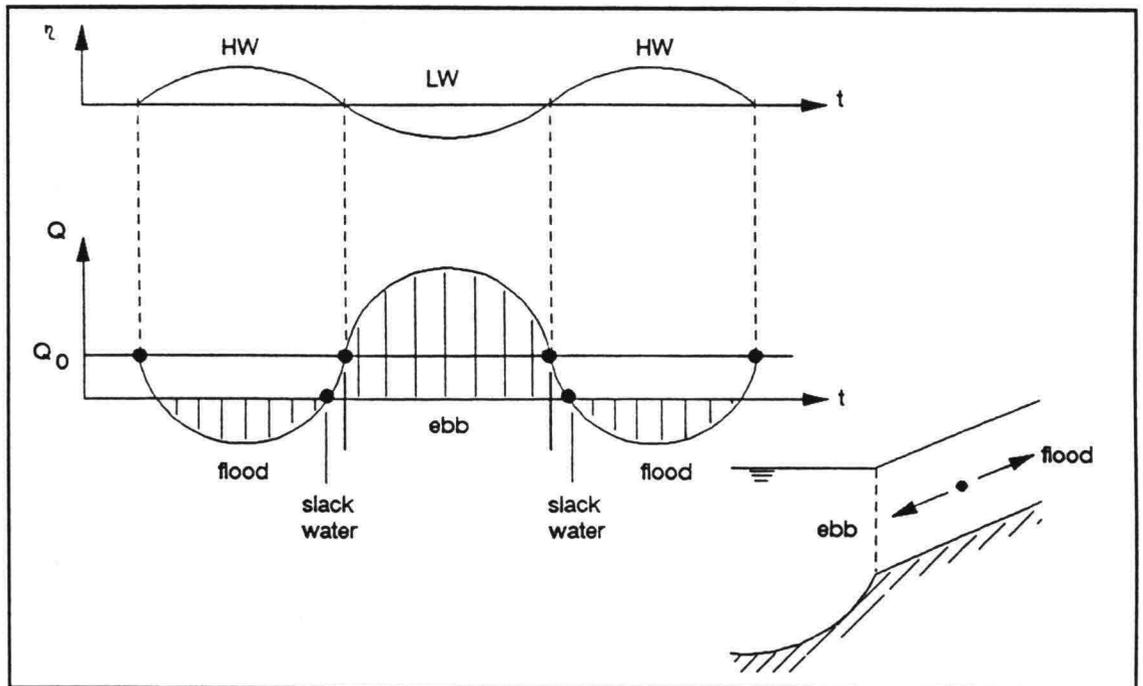


Figure 5.5 Discharge and water level for a given location x along a river

B. Friction taken into account

Taking friction into account implies that both terms of the oscillating part should be considered. Now the expression for Q is:

$$Q = Q_0 - b\hat{\eta}_0\omega \frac{e^{\lambda x}}{\lambda^2 + k^2} k \cos(\omega t + kx) + b\hat{\eta}_0\omega \frac{e^{\lambda x}}{\lambda^2 + k^2} \lambda \sin(\omega t + kx)$$

The different terms of the discharge are given in Figure 5.6. The term with the sinus function is due to the friction. The effect of that term is that slack water occurs earlier.

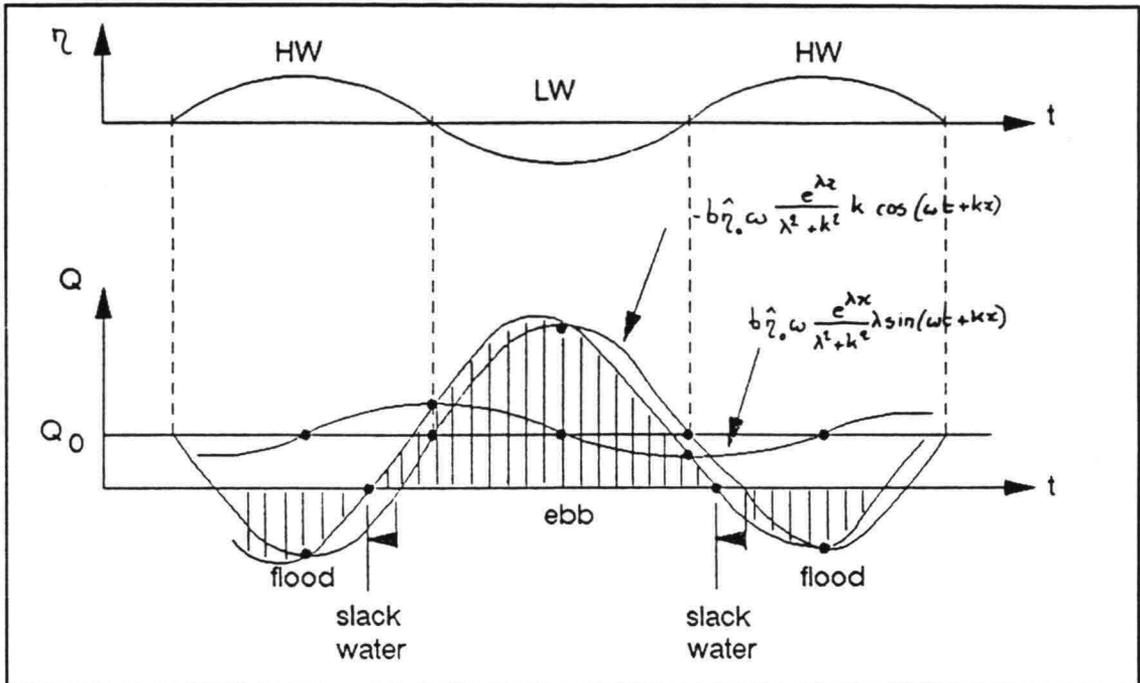


Figure 5.6 Discharge when friction is taken into account

When the amplitude of the oscillating part is equal to Q_0 , slack water occurs at HW (see Figure 5.7). This means that there is no flood flow.

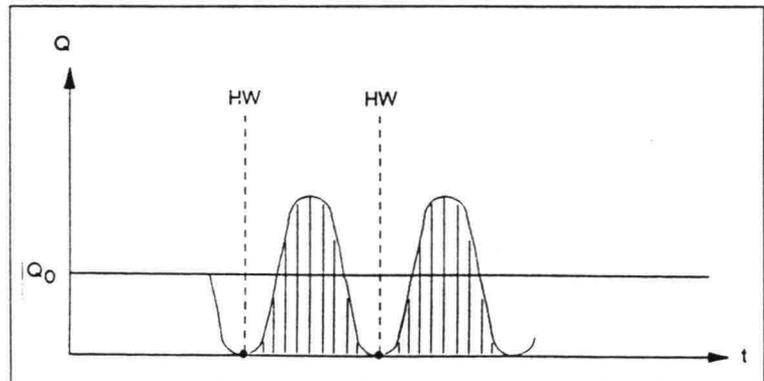


Figure 5.7 Amplitude of oscillating part = Q_0

When the amplitude of the oscillating part is smaller than Q_0 , slack waters do not occur (see Figure 5.8). In this case, only ebb flow can be observed.

Looking to the discharge of tidal wave propagation of Figure 5.4, it can be seen that at location (4) only ebb flow occurs. This location is called the *limit of the flood flow*. Downstream of this location flood low and ebb flow occur, whereas upstream only ebb flow can be observed.

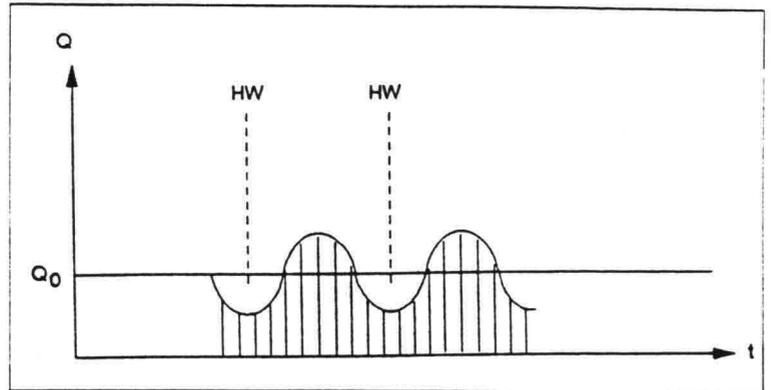


Figure 5.8 Amplitude of the oscillating part $< Q_0$

5.3 Tidal bore

For the discussion of tidal bores, a gradually varied steady flow situation is considered. For the element of Figure 5.9, Newton's law holds:

$$F = ma \text{ or } a = \frac{F}{m}$$

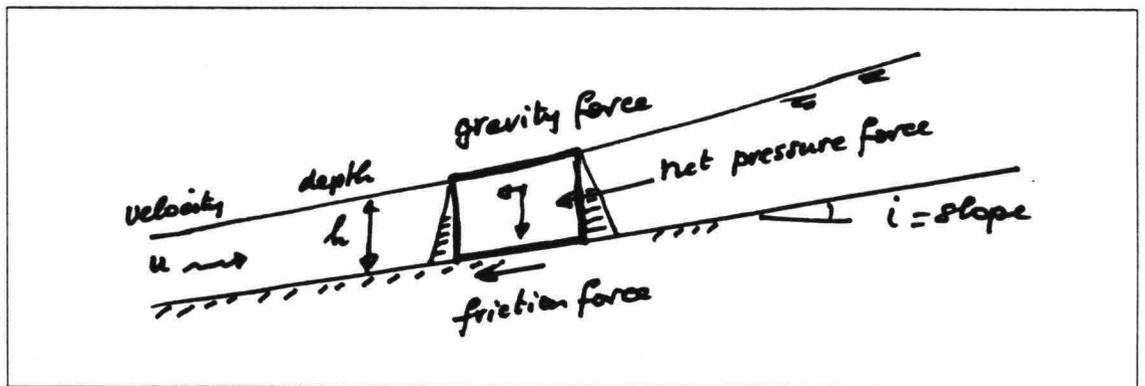


Figure 5.9 Element for a gradually varied flow situation

For steady flow, the acceleration is:

$$a = \frac{du}{dt} = \frac{du}{dx} \times \frac{dx}{dt} = u \frac{du}{dx}$$

The force per unit mass can be expressed by (see also Figure 5.9):

$$\frac{F}{m} = -\frac{dh}{dx} - gi - g\frac{u^2}{C^2h} = \text{pressure} - \text{gravity} - \text{friction}$$

The equation that describes steady flow is:

$$u \frac{du}{dx} + g \frac{dh}{dx} + gi + g\frac{u^2}{C^2h} = 0 \text{ (equation of motion)} \quad (5.6)$$

The continuity equation shows that for constant q , $\frac{d(uh)}{dx} = 0$.

This can be written as:

$$u \frac{dh}{dx} + h \frac{du}{dx} = 0 \rightarrow \frac{du}{dx} = -\frac{u}{h} \frac{dh}{dx}$$

Substitution in Equation 5.6 gives:

$$-\frac{u^2}{h} \frac{dh}{dx} + g \frac{dh}{dx} + gi + g \frac{u^2}{C^2 h} = 0$$

Dividing by g and rearranging leads to:

$$\frac{dh}{dx} = -\frac{i + \frac{u^2}{C^2 h}}{1 - \frac{u^2}{gh}}$$

Expression of a tidal bore (5.7)

Note that $\frac{u}{\sqrt{gh}}$ is the Froude number, with $c = \sqrt{gh}$ (celerity of the wave).

Figure 5.10 shows the development of a tidal bore.

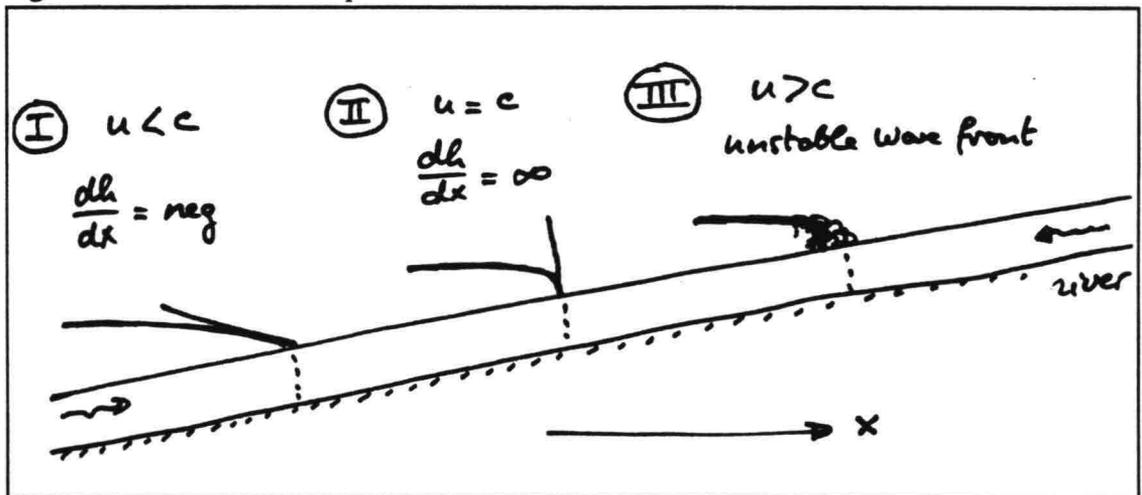


Figure 5.10 Development of a tidal bore

The development can be explained with Equation 5.7:

- I For a current in upstream direction (subcritical flow: $u < c$ or $Fr < 1$): $\frac{dh}{dx}$ is negative;
- II Slope $\frac{dh}{dx}$ can become ∞ when $1 - \frac{u^2}{gh} = 0 \rightarrow u = \sqrt{gh} \rightarrow u = c$;

- III When the front propagates into the estuary, generally the depth will decrease. This means that:
- velocity u will increase;
 - celerity c will decrease.
- Now the wave front becomes unstable: a bore is formed ($u > c$).

5.4 Resonance

Text not yet available.

6 Tidal propagation in two dimensions

6.1 Basic equations for waves in two dimensions

The tidal motion in seas differs from the cases of the previous Chapters, as:

- The flow is two-dimensional;
- The rotation of the Earth has effect on the flow.

To describe the influences of these effects, first the water motions in two-dimensions are described.

The water motion in two dimensions can be described by three equations:

- A continuity;
- B motion in x-direction;
- C motion in z-direction.

6.1.1 Equation of continuity

For the Equation of continuity, a square box with lengths dx and dy is considered (see Figure 6.1).

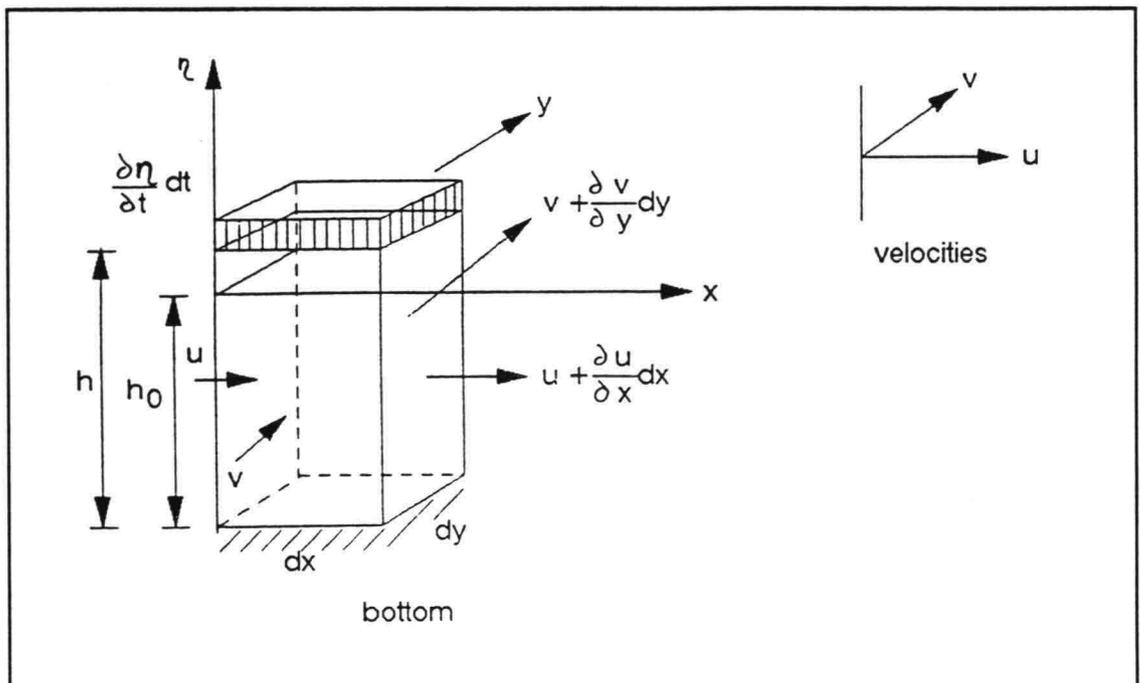


Figure 6.1 Box in sea, with length dx and dy

The symbols used in the box have the following meaning:

- η water level
- h local depth ($h = h_0 + \eta$)

The inflow at the left-hand side and the front in time increment dt is:

$$uh dy dt + v h dx dt$$

The outflow at the right-hand side and the back in a time increment dt is:

$$\left(uh + \frac{\partial(uh)}{\partial x} dx \right) dydt + \left(vh + \frac{\partial(vh)}{\partial y} dy \right) dxdt$$

During time increment dt , water is stored in the box at the water surface:

$$\frac{\partial \eta}{\partial t} dt dx dy$$

Conservation of mass, over a time increment dt , means:

$$\text{Inflow} - \text{Outflow} = \text{Storage}$$

So:

$$(uh dydt + vh dxdt) - \left(uh dydt + \frac{\partial(uh)}{\partial x} dx dy dt + vh dxdt + \frac{\partial(vh)}{\partial y} dy dx dt \right) = \frac{\partial \eta}{\partial t} dt dx dy$$

After some elaboration:

$$\frac{\partial \eta}{\partial t} + \frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y} = 0 \quad \text{Equation of continuity for two-dimensional flow} \quad (6.1)$$

6.1.2 Equation of motion in x- and y-direction

The equation of motion in x-direction can be derived by considering a box of water with horizontal dimensions dx and dy , and applying Newton's law in x-direction (see Figure 6.2):

Force = mass * acceleration (in x-direction), thus

$$F_x = ma_x = m \frac{du}{dt}$$

$$u = f(x, y, t)$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial t} dt$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} + \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial t}$$

As $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, it follows that

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}$$

Substitution in the relation for F_x , gives:

$$\frac{F_x}{m} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}$$

The forces F_x which act on the box are pressure, bottom friction, fluid friction along the walls (called turbulent viscosity), wind force (shear stress over the surface), Coriolis force (due to rotation of the earth), and tractive force.

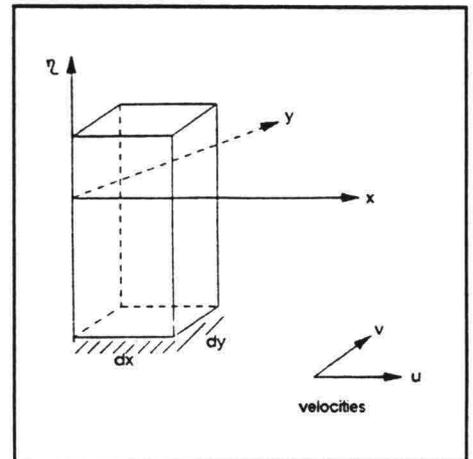


Figure 6.2 Box with x, y and u, v

For a first approximation it is sufficient to pay attention to:

- Pressure force;
- Bottom friction;
- Coriolis force.

a. Pressure force

The pressure force can be derived in a similar way as for the one-dimensional case (see Figure 6.3).

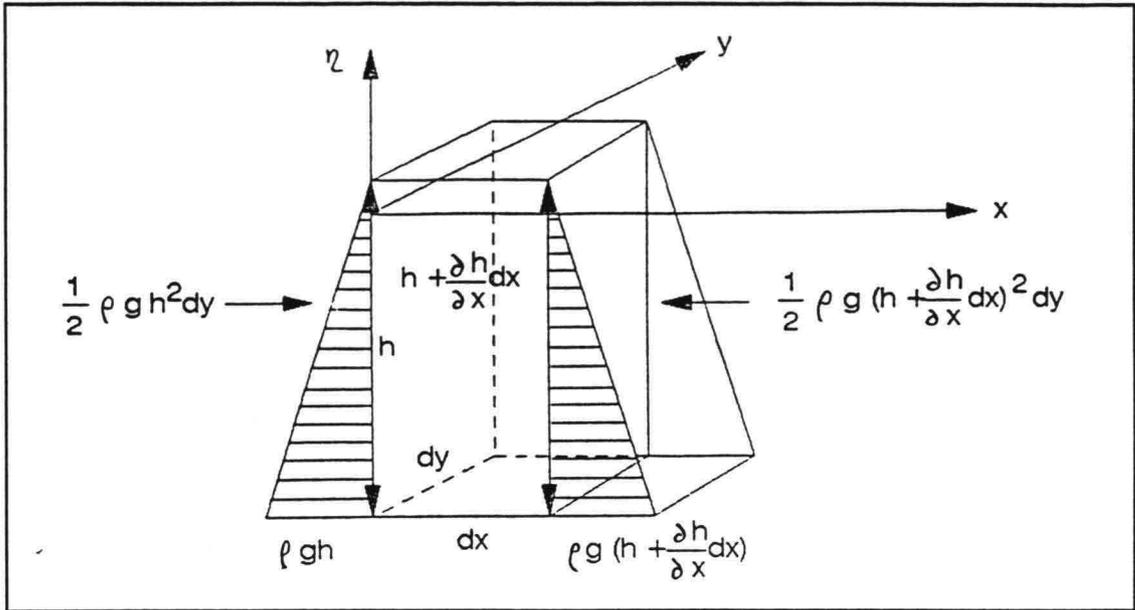


Figure 6.3 Symbols used for pressure force

Left-hand side

- Water depth: h .
- Pressure at the bottom: $\rho g h$.
- Resultant pressure force in x -direction: $\frac{1}{2} \rho g h^2 dy$.

Right-hand side

- Water depth: $h + \frac{\partial h}{\partial x} dx$.
- Pressure at the bottom: $\rho g (h + \frac{\partial h}{\partial x} dx)$.
- Resultant pressure force in x -direction: $\frac{1}{2} \rho g (h + \frac{\partial h}{\partial x} dx)^2 dy$.

Net force in x -direction

$$\frac{1}{2} \rho g h^2 dy - \frac{1}{2} \rho g (h^2 + 2h \frac{\partial h}{\partial x} dx + (\frac{\partial h}{\partial x} dx)^2) dy = -\rho g h \frac{\partial h}{\partial x} dx dy$$

The mass of the box is $\rho h dx dy$, so the net force per unit of mass is:

$$\frac{F_x}{m} = \frac{-\rho g h \frac{\partial h}{\partial x} dx dy}{\rho h dx dy} = -g \frac{\partial h}{\partial x}$$

Because $h = h_0 + \eta$, this can also be written as:

$$\frac{F_x}{m} = -g \frac{\partial \eta}{\partial x} \tag{6.2}$$

b. Bottom friction

Consider the same box of water (see Figure 6.4). The velocity of the water in the box is V with components u and v . The friction force τ (per unit of surface) is opposite to the direction of the velocity. The components are τ_x and τ_y .

Consider the friction force in x-direction:
 $-\tau_x dx dy$

The derivation for the one-dimensional case showed that τ can be expressed in terms of velocity V and the coefficient of Chézy C by:

$$\tau = \rho g \frac{V^2}{C^2}$$

The component in x-direction is:

$$\tau_x = \tau \cos \phi = \rho g \frac{V^2}{C^2} \cos \phi = \rho g \frac{V \cos \phi V}{C^2} = \rho g \frac{u \sqrt{u^2 + v^2}}{C^2}$$

The friction force in x-direction is:

$$\tau_x dx dy = -\rho g \frac{u \sqrt{u^2 + v^2}}{C^2} dx dy$$

The mass of the box is $\rho h dx dy$

The net force per unit of mass is:

$$\frac{F_x}{m} = \frac{-\rho g \frac{u \sqrt{u^2 + v^2}}{C^2} dx dy}{\rho h dx dy} = -g \frac{u \sqrt{u^2 + v^2}}{C^2 h} \tag{6.3}$$

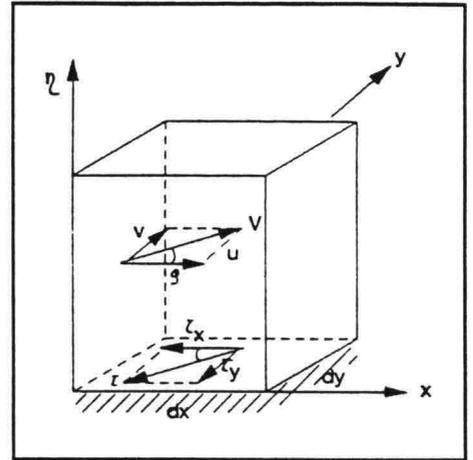


Figure 6.4 Symbols used for bottom friction

c. Coriolis force

The Coriolis force is caused by the rotation of the Earth. It is significant in oceans, seas and wide estuaries. A rotating coordinate system introduces additional acceleration forces, which can be illustrated with the example of Figure 6.5. This Figure shows a rotating disk with angular speed ω .

At location A, a person throws a ball (with mass m and velocity v) to a person in location B. If the disk would not rotate, the ball would arrive in location B after Δt s. From $S = vt$ it follows that $AB = v\Delta t$. However, as the disk rotates, the ball will actually arrive in location C after Δt s. Apparently an acceleration a_{Cor} causes the ball to arrive in C and not in B.

From $S = \frac{1}{2}at^2$ it follows that $B^1C = \frac{1}{2}a_{Cor}(\Delta t)^2$.

So: $B^1C = A^1C\Delta\alpha = AB\Delta\alpha = (v\Delta t)(\omega\Delta t)$

Thus: $\frac{1}{2}\alpha_{Cor}(\Delta t)^2 = v\omega(\Delta t)^2 - \alpha_{Cor} = 2v\omega$

Therefore, the Coriolis force becomes:

$$F_{Cor} = m\alpha_{Cor} = 2mv\omega$$

The Coriolis force is directed perpendicular to the direction of motion (see Figure 6.6). It is directed to the right, if the plane is turning anticlockwise

Consider a point P moving on the Earth surface (see Figure 6.7). Point P has a latitude φ . Particle P has velocity V in the tangent plane. The rotation of the Earth is ω . The angular speed of the tangent plane is $\omega \sin\varphi$.

This means that the force acting on particle P is:

$$F_{cor} = 2mv\omega \sin\varphi$$

From this relation it can be concluded that:

- For $\sin\varphi = \text{positive}$ (Northern Hemisphere), the Coriolis force is directed to the right;
- For $\sin\varphi = \text{negative}$ (Southern Hemisphere) the Coriolis force is directed to the left;
- At the poles $\sin\varphi = \text{maximum}$. Here the Coriolis force is maximum;
- At the equator $\sin\varphi = 0$. Thus, here the Coriolis force is zero.

In x-direction, the Coriolis force F_x is directed to the right on velocity $+v$: $F_x = 2m\omega v \sin\varphi$

The Coriolis force per unit mass in x-direction is:

$$\frac{F_x}{m} = 2\omega v \sin\varphi = fv \tag{6.4}$$

in which $f = 2\omega \sin\varphi$, which is called the *Coriolis parameter*.

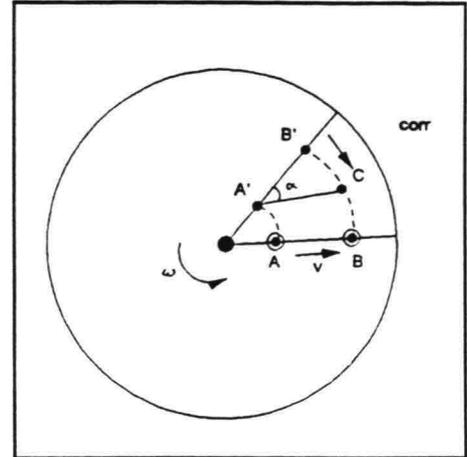


Figure 6.5 Symbols used for the Coriolis force

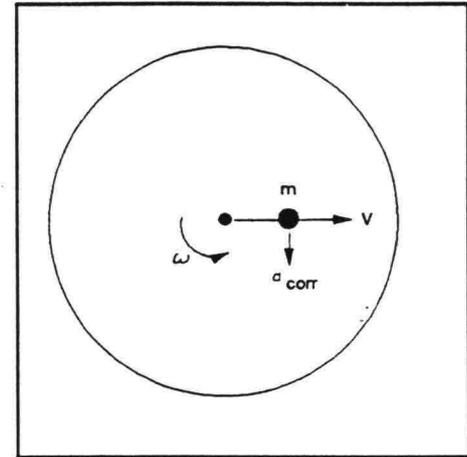


Figure 6.6 Direction of Coriolis force

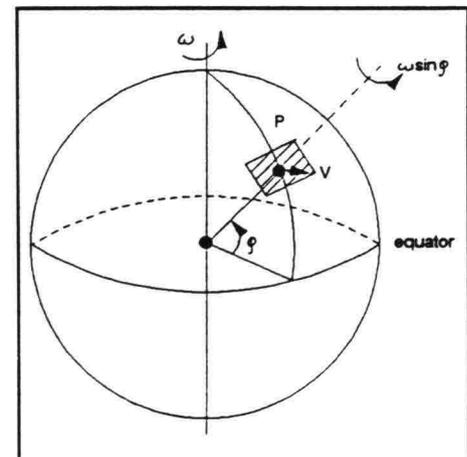


Figure 6.7 Effect of Coriolis force on Point P

Equation of motion

The Equation of motion in x-direction (6.1) can now be written in terms of pressure force (6.2), bottom friction (6.3) and Coriolis force (6.4) per unit of mass:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -g \frac{\partial \eta}{\partial x} - g \frac{u \sqrt{u^2 + v^2}}{C^2 h} + f$$

or
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial \eta}{\partial x} + g \frac{u \sqrt{u^2 + v^2}}{C^2 h} - fv =$$
 (6.5)

In a similar way, the *equation of motion in y-direction* can be written as:

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial \eta}{\partial y} + g \frac{v \sqrt{u^2 + v^2}}{C^2 h} + fu =$$
 (6.6)

Figure 6.8 shows the direction of u and v in relation to x and y .

Equations 6.5 and 6.6, together with the Equation of continuity (6.1) describe the water motion in two dimensions. These can be used for describing tidal waves in large estuaries, seas and oceans.

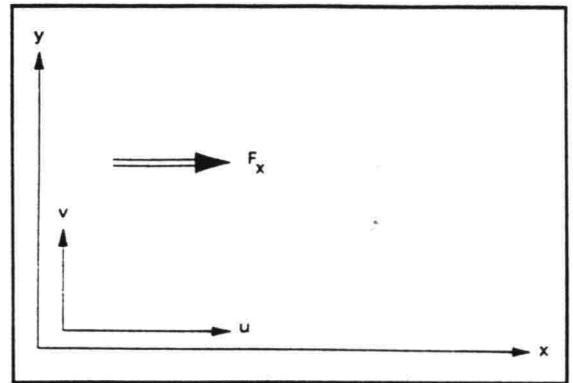


Figure 6.8 Directions of u , v and x , y

6.2 Effect of the Coriolis force

The effect of the Coriolis force on the tidal system in seas and oceans can be illustrated by considering a river (see Figure 6.9).

The equation of motion in the y-direction is:

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial \eta}{\partial y} + g \frac{v \sqrt{u^2 + v^2}}{C^2 h} + fu =$$

In the river: $v = 0$, $\frac{\partial v}{\partial t} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

So the equation of motion reduces to:

$$g \frac{\partial \eta}{\partial y} + fu = 0$$

It can be concluded that the Coriolis force is balanced by the cross-slope of the water level in the river:

$$\frac{\partial \eta}{\partial y} = -\frac{f}{g} u = -\frac{2\omega \sin \phi}{g} u$$

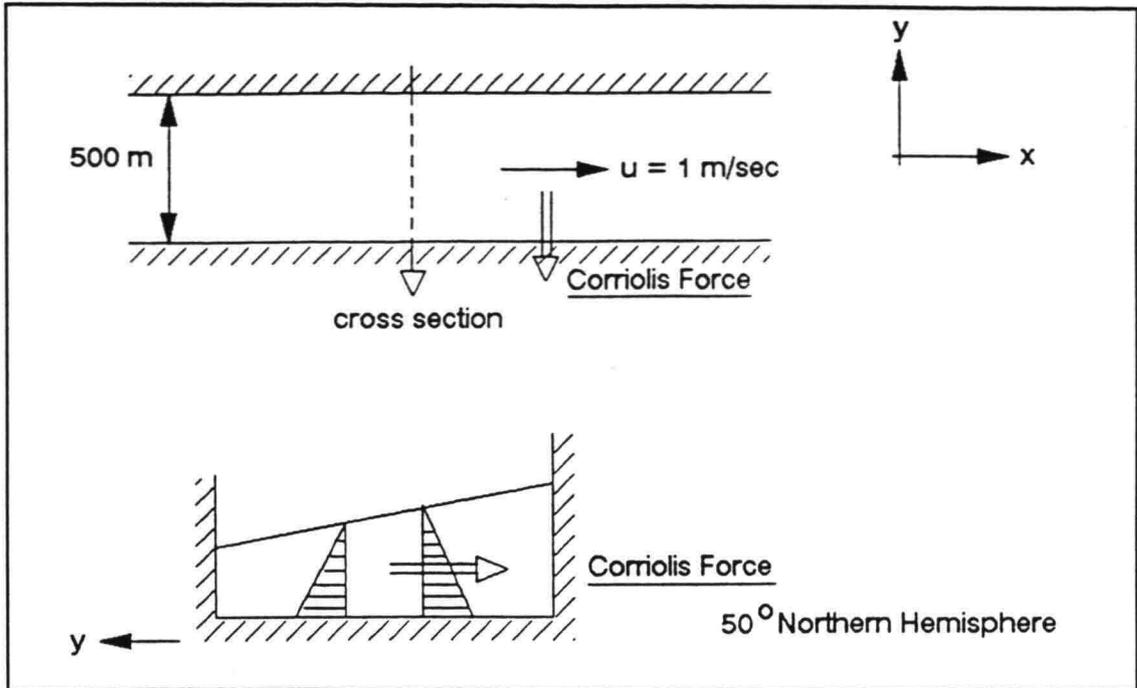


Figure 6.9 Effect of Coriolis force in a river

Example

- Width of the river: 500 m;
- Velocity: 1 m/s;
- Latitude: 50° (Northern Hemisphere).

Substitution gives:
$$\frac{\partial \eta}{\partial y} = -\frac{2 \times 0.73 \times 10^{-4} \times 0.71}{10} \times 1 = 10^{-5}$$

So the difference in water level in cross direction is: $\Delta \eta = 10^{-5} \times 500 \text{ m} = 5 \text{ mm}$. The influence on the main flow in the river can thus be neglected. The situation, however, changes if the width increases to many kilometers. Then also cross velocities may occur, caused by the Coriolis forces.

If large water bodies are considered (like the North Sea), the gravity terms and the Coriolis terms in the Equations of motion can be of the same order of magnitude. Then so-called *Kelvin waves* can be observed. For arbitrarily configurations no analytical solutions are known. However, numerical solutions can be obtained very accurately.

An example of such an analytical solution is presented in Figure 6.10, which shows a rectangular channel of constant depth, rotating around a vertical axis with angular speed ω . The x-axis is located along the bank of the channel; the y-axis perpendicular to it.

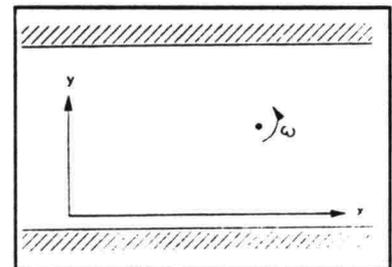


Figure 6.10 Rectangular channel with rotation ω

Assuming small water level elevations, compared to the depth, a harmonic solution of the two-dimensional equations is:

$$\eta = \hat{\eta} e^{-\frac{f y}{c}} \cos \omega \left(t - \frac{x}{c} \right)$$

in which:

- $\hat{\eta}$: amplitude at the wall (for $y = 0$)
- f : Coriolis parameter
- c : celerity (\sqrt{gh})

Further:

$$v = 0$$

$$\frac{u}{c} = \frac{\eta}{h}$$

The amplitude of this wave is an exponential function of y . At the wall the amplitude is highest.

Assume a wide channel, with the following data:

- Latitude 50° (Northern Hemisphere);
- Depth 100 m;
- Wave amplitude for $y = 0$ $\hat{\eta} = 1$ m

The wave amplitude as a function of y . Table 6.1 shows some values, whereas Figure 6.11 shows this Kelvin wave.

Table 6.1 Wave amplitude as a function of distance y

Distance y from the wall (km)	Wave amplitude $\hat{\eta}$ (m)
0	1
1	0.997
10	0.966
100	0.71
1000	0.03

6.3 Amphidromic systems

Kelvin waves can be reflected as Kelvin waves too. This is illustrated in Figure 6.12, which shows the effect of the Coriolis force on a standing wave. To explain this Figure, Figure 6.13 is needed, which shows the motion of a standing wave *without rotation*. This is also indicated at the left-hand side of Figure 6.13.

The effect of rotation can be explained as follows. The Coriolis force is acting to the right on the moving water particles (see Figure 6.13).

$$t = 0$$

The velocities in the length direction are zero. So the Coriolis force in cross direction is also zero, which means that the water surface in cross direction is horizontal.

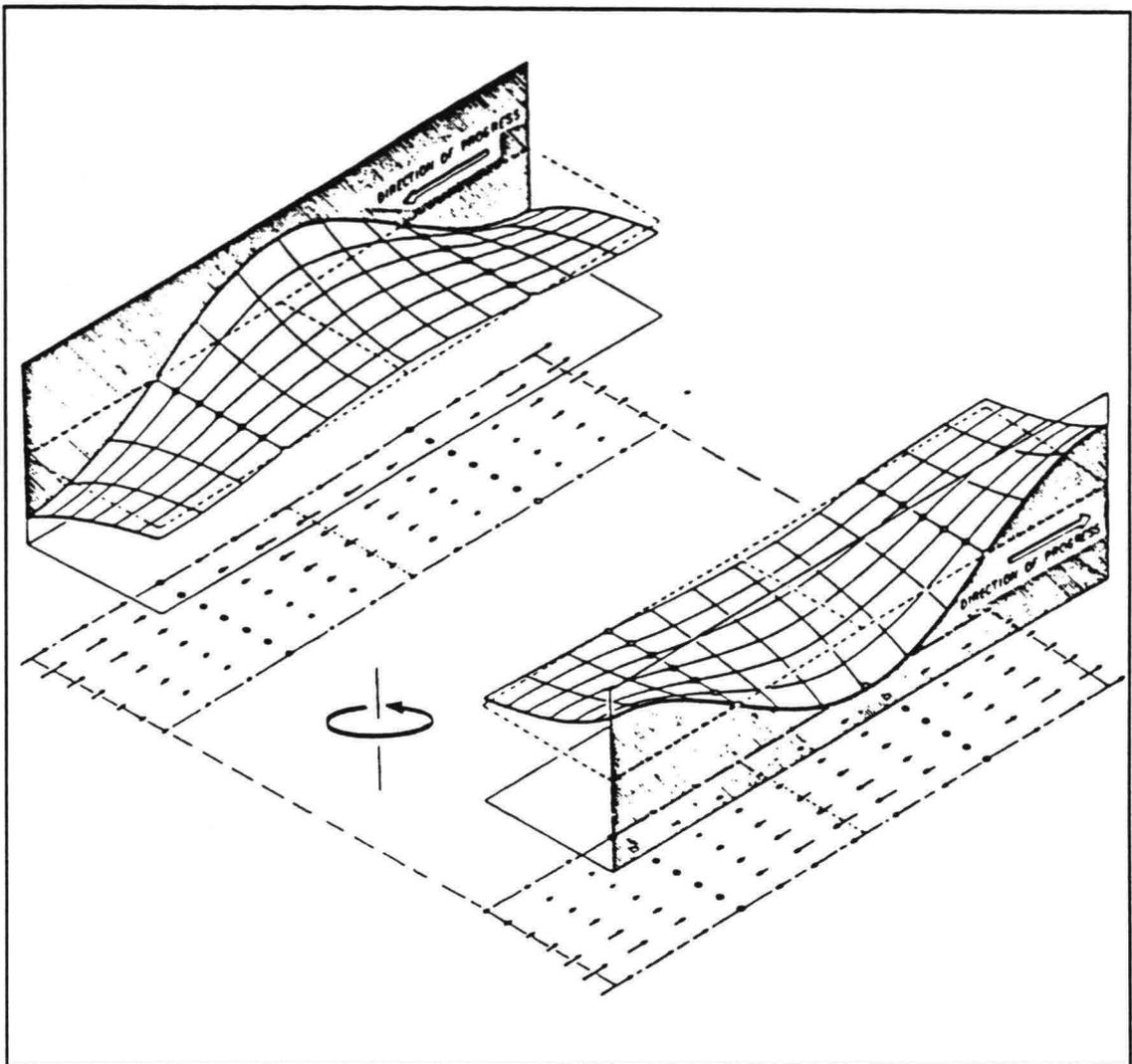


Figure 6.11 Kelvin wave

$$t = \frac{1}{4} \text{ 3 hours}$$

The velocities in length direction at the node are maximum. The Coriolis force in cross direction is acting toward the wall at the left-hand side. HW occurs at that wall.

$$t = \frac{1}{2} \text{ (6 hours)}$$

The velocities in the length direction are again zero. So the Coriolis force in cross direction is also zero and the water surface in cross direction is horizontal.

$$t = \frac{3}{4} \text{ (9 hours)}$$

The velocities in length direction at the node are maximum. The Coriolis force in direction is acting toward the wall at the right-hand side. HW occurs at that wall.

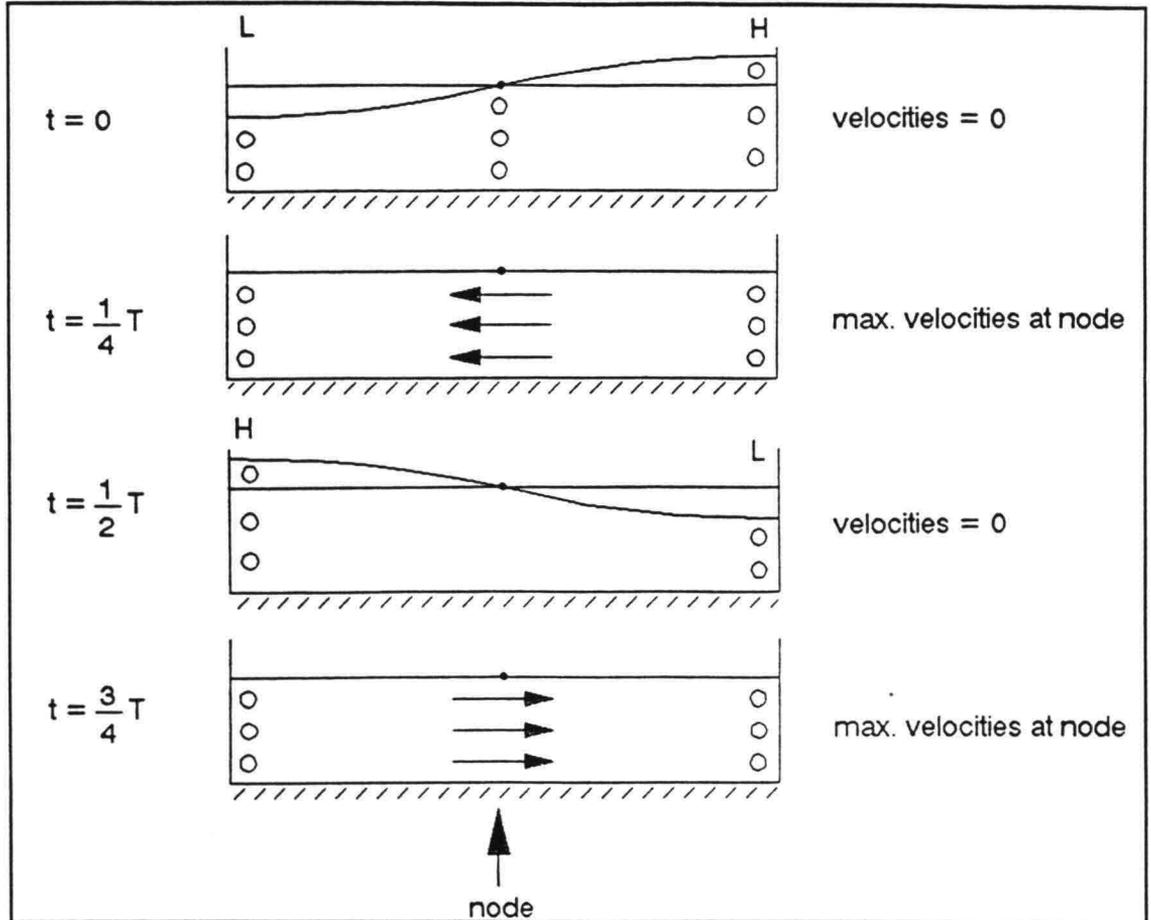


Figure 6.12 Effect of Coriolis force on a standing wave

So HW and LW are turning around in the basin:

- This rotation is *anticlockwise* on the *Northern* hemisphere;
- This rotation is *clockwise* on the *Southern* hemisphere.

It can also be observed that cross currents are introduced by the waves in cross-direction, on which also the Coriolis force is acting.

The effect of the rotation of the earth on such a standing oscillation is that the nodal line is reduced to a nodal point. At the nodal point the water level is constant. That point is called the *Amphidromic point*. The wave system is called the *Amphidromic system*.

Around Amphidromic points, lines of equal *phases* can be seen (for instance for the M_2 -tide). These are called the *co-tidal lines*. Also lines of equal tidal ranges can be drawn, which are called the *co-range lines*.

In nature, many Amphidromic systems can be found. In the North Sea, three Amphidromic points for the M_2 -tide occur (see Figure 6.14). One point is located in the Southern part of the North Sea, one near the coast of Denmark and one near the Norwegian coast. The co-tidal lines (equal phases) are shown as solid lines. The co-range lines (equal range) are indicated as dashed lines.

Figure 6.15 shows the M_2 co-tidal and co-range lines for the entire world.

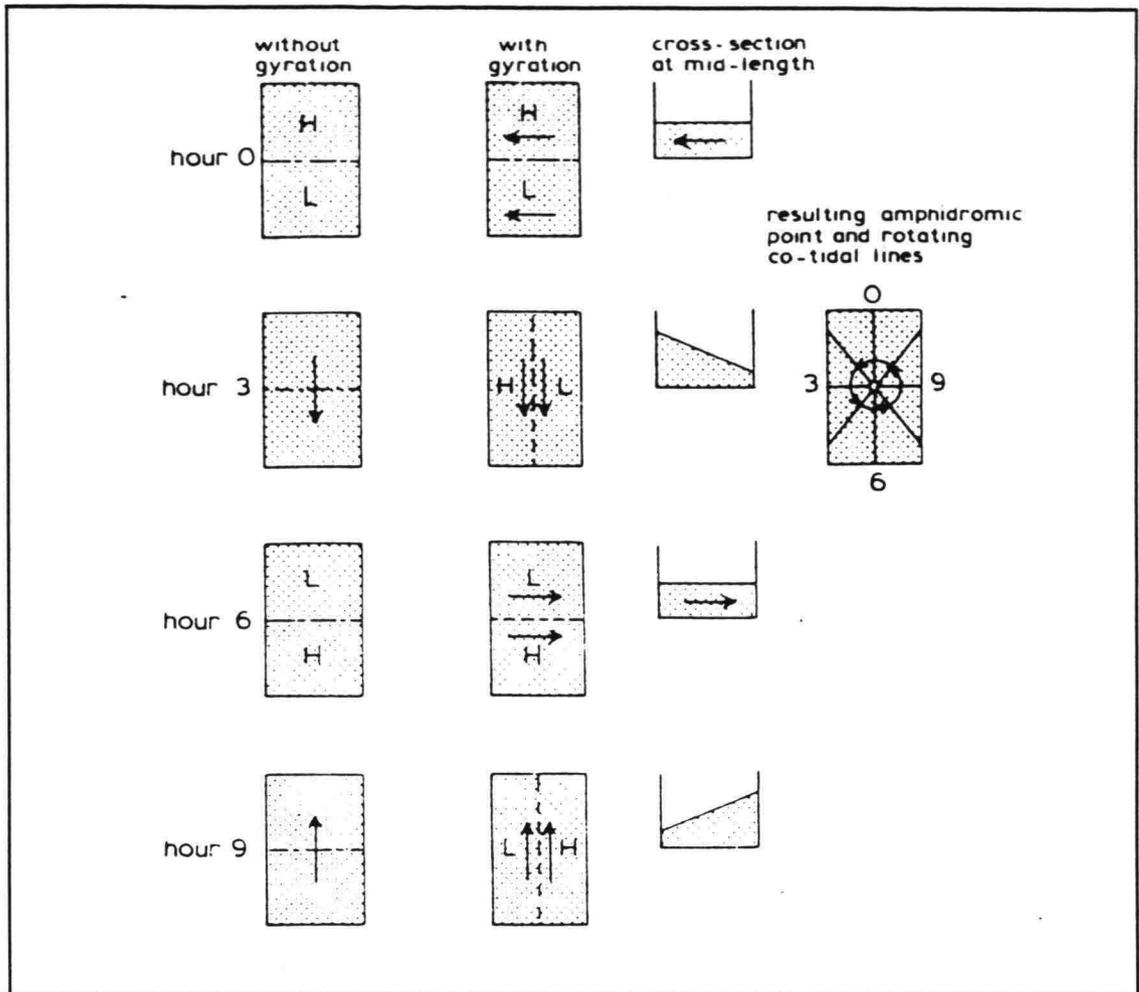


Figure 6.13 Standing wave without rotation

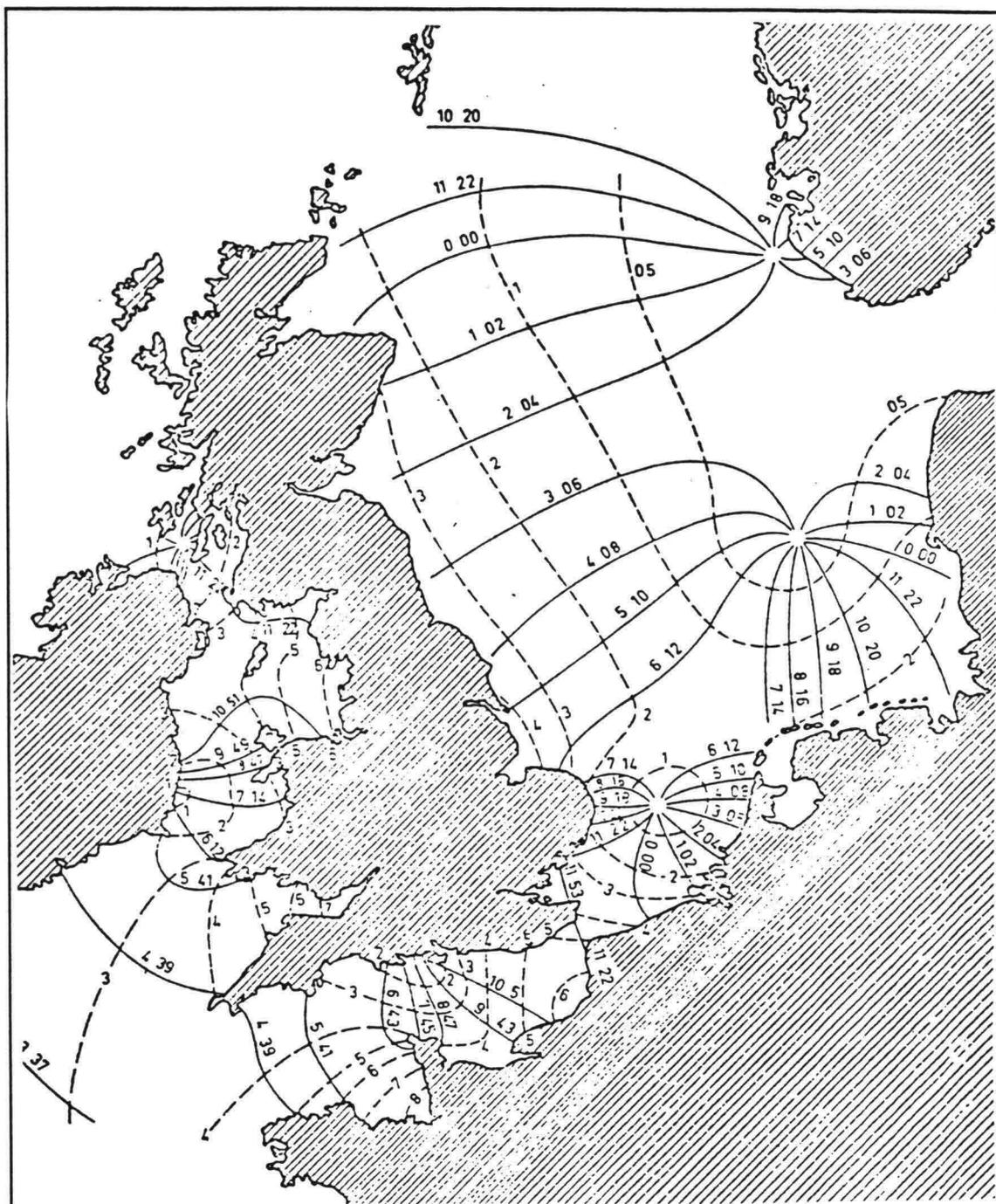


Figure 6.14 Amphidromic points for the M₂ tide

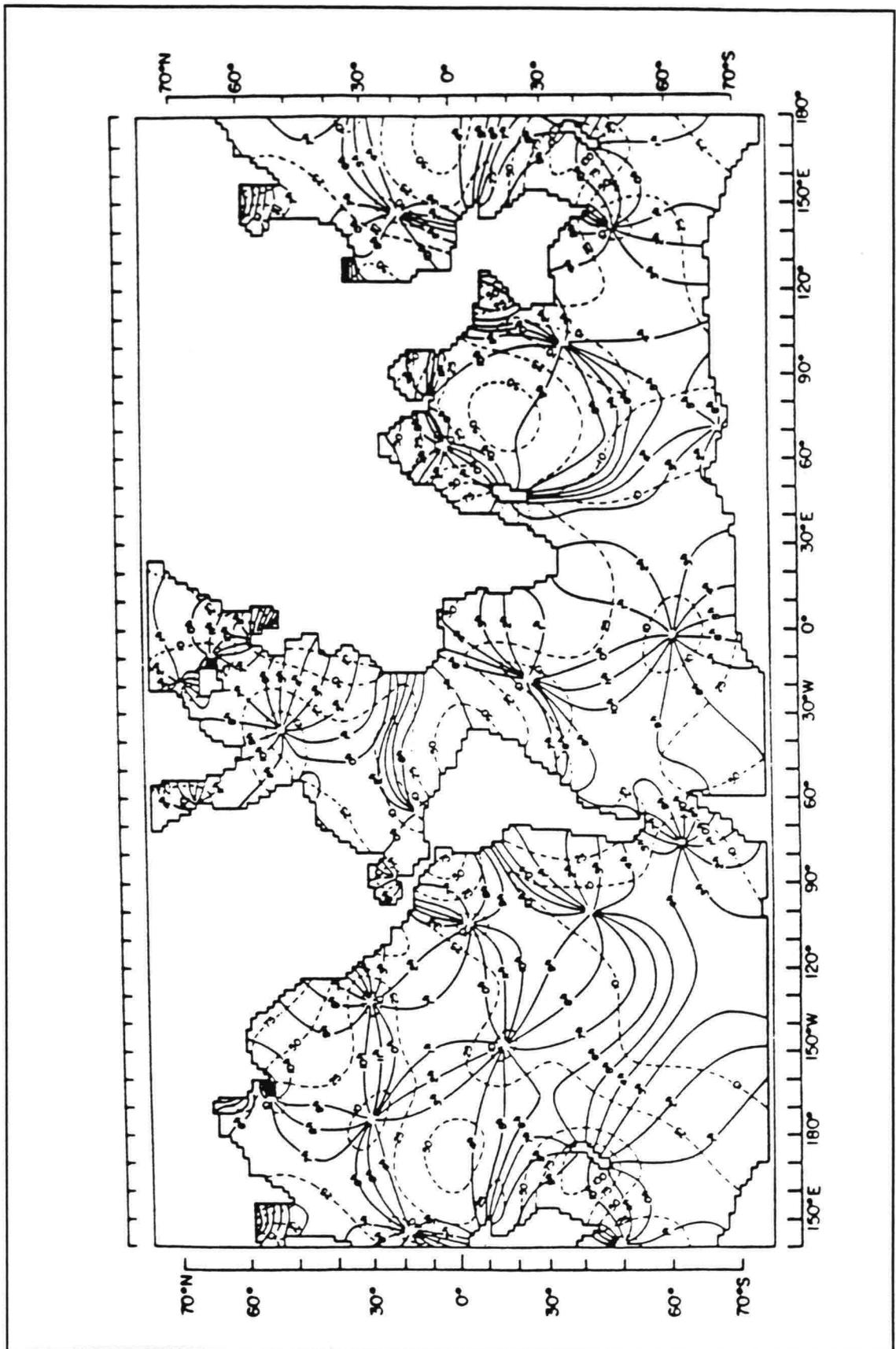


Figure 6.15 M₂ co-tidal and co-range lines for the world

7. Analytical tidal computations

So far, properties of tidal propagation were considered, for which simplified equations were used. In addition the geometry and the boundary conditions were simplified as well. In this way, insight was obtained of the fundamental aspects of tidal motion.

Engineering problems, however, are not that simple. For concrete and complex situations, predictions should be made as accurately as possible. This also refers to the prediction of effects of civil engineering works in the coastal area on the tidal motion. Several approaches are available:

1. For first guesses and insight in a problem:
 - Simplified analytical computations;
2. For accurate predictions in complex situations:
 - Hydraulic scale models;
 - Numerical tidal computations (mathematical models).

In *hydraulic scale models* the water in the models provides the solution. These models have been used extensively in the past. However, they are expensive and it is time consuming to investigate alternatives, as it takes a lot of effort to make changes in the geometry.

In *mathematical models* the equations are not simplified. The advantages are:

- The geometry is schematized accurately;
- Boundary conditions can be used from measurements;
- The computations are made by computers;
- Depending on the problem, one-dimensional, two-dimensional and even three-dimensional models can be utilized.

Nowadays, tidal problems are mainly investigated with mathematical models. Hydraulic scale models are an exception; they are still used to study complicated three-dimensional current situations.

This Chapter, however, focuses on simplified analytical computations, which gives a first guess approach for an engineering problem.

7.1 Small basin

The first problem concerns the tidal motion in a *short basin*, which can be a harbor along a tidal river, or a relative short bay along a sea or ocean. Figure 7.1 shows such a small basin, with the following characteristics:

- l length of the basin;
- b width;
- h depth h .

The x -axis is directed positive to the right: $x = 0$ is located at the mouth; $x = l$ at the end of the basin. At the mouth there is a sinusoidal tide, with amplitude $\hat{\eta}$ and period T . The length of a tidal wave can be computed by:

$$L = cT$$

in which $c = \sqrt{gh}$

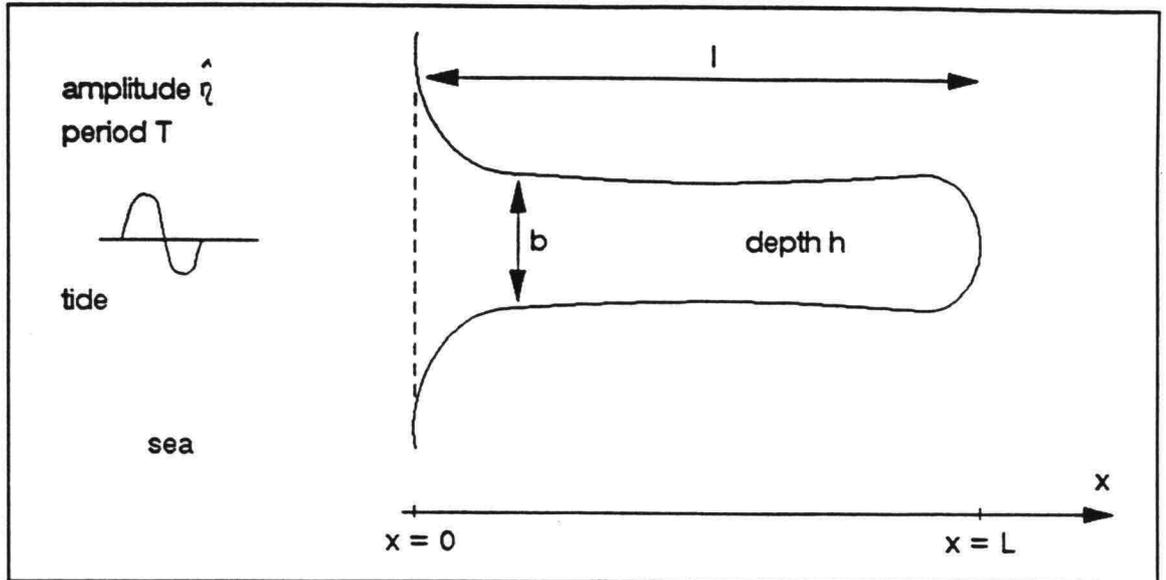


Figure 7.1 Short basin

The term short basin is related to the parameter:

$$\frac{l}{L} = \frac{\text{length basin}}{\text{length tidal wave}}$$

If this ratio is small (smaller than 0.02) the basin will be filled in horizontal layers. A small increase of the water level at the mouth can be considered as a disturbance. So a small translation wave will run through the basin. The time to propagate to the end of the bay and back to the mouth is:

$$2\tau = \frac{2l}{c}$$

As $l = c\tau$ and $L = cT$, the ratio $l/L (= \tau/T)$ becomes:

$$\frac{\tau}{T} = \frac{l}{L} < 0.02$$

This means that the time for a disturbance to propagate to the end of the basin and back is small compared to the period of the tidal wave.

As the basin is filled in horizontal layers, the equation of motion is reduced to:

$$\frac{\partial \eta}{\partial x} = 0$$

Figure 7.2 shows a sketch of the longitudinal section of the basin. To compute the discharge at the mouth of the basin, the equation of continuity can be used:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

As $\frac{\partial \eta}{\partial x} = 0$, $\frac{\partial \eta}{\partial t} = \frac{d\eta}{dt}$.

So: $\frac{\partial Q}{\partial x} = -b \frac{d\eta}{dt}$

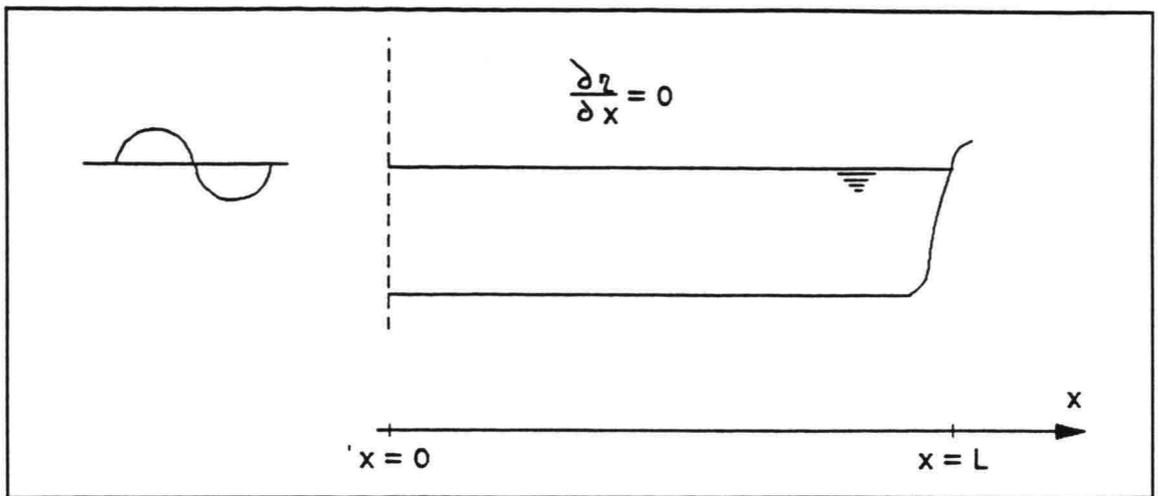


Figure 7.2 Longitudinal section of the short basin of Figure 7.1

To obtain the discharge at the mouth of the basin, this equation should be integrated with respect to x :

$$\int_0^l \frac{\partial \varphi}{\partial x} dx = \int_0^l -b \frac{d\eta}{dt} dx \rightarrow Q_1 - Q_0 = -b \frac{d\eta}{dt} l$$

For a closed basin at the end, $Q_1 = 0$

$$\text{So: } Q_0 = bl \frac{d\eta}{dt}$$

(Note that bl is the surface area of the basin).

Take $\eta = \hat{\eta} \cos \omega t$ then:

$$Q_0 = -\hat{Q} \sin \omega t, \text{ with } \hat{Q} = bl \hat{\eta} \omega$$

Figure 7.3 presents this relation between η and Q at the mouth of the short basin. From this Figure it can be observed that:

- A decrease in water level means the emptying of the basin;
- An increase in water level means the filling of the basin;
- The phase difference between η and Q is 90° .

This relatively simple approach can often be used for basins of short length.

7.2 Lorentz method

The second analytical computation method is the *Lorentz Method*. This method has been named after Prof. Lorentz, who was a famous physicist at the beginning of the 20th century. He developed the method for tidal computations, to be implemented before the former Zuiderzee (now Lake IJssel) was closed off from the Waddenzee in 1932. At that time, no computers were available, so all the computations had to be carried out by hand.

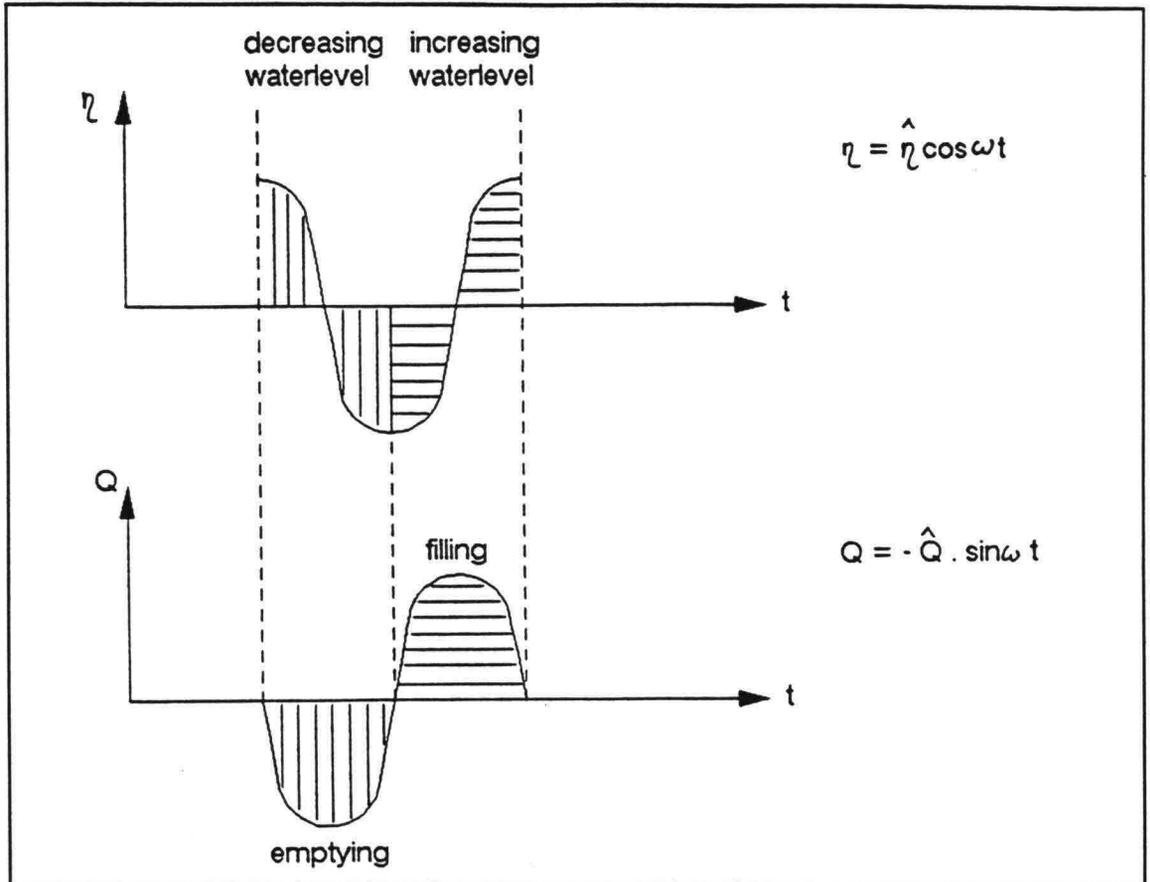


Figure 7.3 Relation between water level and discharge at the mouth of a short basin

The description of the Lorentz method starts with the basic equations for long waves. As was shown in previous Chapters, the equation of motion can be simplified for small Froude numbers. The Froude number was:

$$Fr = \frac{u^2}{gh} \ll 1$$

This is often the case. For example:

$$h = 10 \text{ m}, \quad u = 1 \text{ m/s} \quad \rightarrow \quad Fr = 0.01.$$

The equations for small Froude numbers are:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} + g \frac{\partial \eta}{\partial x} + g \frac{Q|Q|}{C^2 A^2 R} = 0$$

These equations can be applied for tidal calculations. The equations look simple, but analytical solutions do not exist, because of the quadratic friction term. However, for small deviations in the mean water level of a uniform horizontal channel, than b , A , R , C can be considered as constants. Now, analytical solutions can be formulated when the friction term is linearised. This linearisation of the friction term was introduced by Lorentz.

7.2.1 Linearization of the friction

The idea of Lorentz was that the calculation would not be too inaccurate if a linear friction term would dissipate the same amount of energy during a tidal cycle, as the original quadratic term does:

$$E_q = E_l$$

in which:

E_q energy dissipated by the *quadratic friction* term during one tidal cycle;

E_l energy dissipated by the *linear friction* term during one tidal cycle.

The terms in the equation of motion are forces per unit mass. They can be seen as forces acting on a water particle with velocity u . The work did by a force F during period dt equals to:

$$\text{Work} = \text{Force} \times \text{Distance}$$

$$\text{So: } dE = Fdx = Fudt = F \frac{Q}{A} dt$$

The original quadratic friction term was written as:

$$g \frac{Q|Q|}{C^2 A^2 R}$$

This term is now replaced by the linear term mQ .

So:

$$qdE = \frac{gQ|Q|}{C^2 A^2 R} \frac{Q}{A} dt = \frac{gQ^2|Q|}{C^2 A^3 R} dt$$

$$dE_l = mQ \frac{Q}{A} dt = \frac{mQ^2}{A} dt$$

Assume a sinusoidal flow with $Q = \hat{Q} \cos \omega t$. The energy dissipated during a tidal cycle by the friction terms is:

$$E_q = \int_0^T \frac{g\hat{Q}^3 \cos^2 \omega t |\cos \omega t|}{C^2 A^3 R} dt$$

$$E_l = \int_0^T \frac{m\hat{Q}^2 \cos^2 \omega t}{A} dt$$

First the energy dissipated by the *quadratic friction* term is further elaborated.

$$E_q = \int_{t=0}^T \frac{g\hat{Q}^3 \cos^2 \omega t |\cos \omega t|}{C^2 A^3 R} dt = \int_{\omega t=0}^{2\pi} \frac{g\hat{Q}^3}{C^2 A^3 R} \frac{1}{\omega} \cos^2 \omega t |\cos \omega t| d(\omega t) = 4 \frac{g\hat{Q}^3}{C^2 A^3 R} \frac{1}{\omega} \int_{\omega t=0}^{\pi/2} \cos^3 \omega t d(\omega t)$$

This integral can be solved as follows:

$$\int_{\omega t=0}^{\pi/2} \cos^3 \omega t d(\omega t) = \int_{\omega t=0}^{\pi/2} (1 - \sin^2 \omega t) \cos \omega t d(\omega t) = \int_{\omega t=0}^{\pi/2} \cos \omega t d(\omega t) - \int_{\omega t=0}^{\pi/2} \sin^2 \omega t d(\sin \omega t) = \sin \omega t \Big|_{\omega t=0}^{\pi/2} - \frac{1}{3} \sin^3 \omega t \Big|_{\omega t=0}^{\pi/2} = (1 - 0) - \frac{1}{3}(1 - 0) = \frac{2}{3}$$

So the energy dissipated by the quadratic friction term becomes:

$$E_q = \frac{4}{\omega} \frac{g\hat{Q}^3}{C^2 A^3 R} \frac{2}{3} = \frac{8}{3\omega} \frac{g\hat{Q}^3}{C^2 A^3 R}$$

The energy dissipated by the linear friction term can be elaborated as follows:

$$E_l = \int_{t=0}^T \frac{m\hat{Q}^2 \cos^2 \omega t}{A} dt = \int_{\omega t=0}^{2\pi} \frac{m\hat{Q}^2}{A} \frac{1}{\omega} \cos^2 \omega t d(\omega t) = 4 \frac{m\hat{Q}^2}{A} \frac{1}{\omega} \int_{\omega t=0}^{\pi/2} \cos^2 \omega t d(\omega t)$$

Solving the integral gives:

$$\begin{aligned} \int_{\omega t=0}^{\pi/2} \cos^2 \omega t d(\omega t) &= \int_{\omega t=0}^{\pi/2} \frac{1}{2} (1 + \cos 2\omega t) d(\omega t) = \int_{\omega t=0}^{\pi/2} \frac{1}{2} d(\omega t) + \int_{\omega t=0}^{\pi/2} \frac{1}{4} \cos 2\omega t d(2\omega t) \\ &= \frac{1}{2} \omega t \Big|_{\omega t=0}^{\pi/2} + \frac{1}{4} \sin 2\omega t \Big|_{\omega t=0}^{\pi/2} = \frac{1}{2} \frac{\pi}{2} + 0 = \frac{\pi}{4} \end{aligned}$$

So the energy dissipated by the linear friction term becomes:

$$E_l = \frac{4}{\omega} \frac{m\hat{Q}^2}{A} \frac{\pi}{4} = \frac{\pi}{\omega} \frac{m\hat{Q}^2}{A}$$

The assumption is: $E_q = E_l$

$$\text{Thus: } \frac{8}{3\omega} \frac{g\hat{Q}^3}{C^2 A^3 R} = \frac{\pi}{\omega} \frac{m\hat{Q}^2}{A}$$

Dividing with $\frac{1}{\omega}$, $\frac{1}{A}$, \hat{Q}^2 gives:

$$m = \frac{8}{3\pi} \frac{g\hat{Q}}{C^2 A^2 R}$$

The coefficient $\frac{8}{3\pi}$ is called the *number of Lorentz*.

Two remarks should be made about the linearised friction:

1. With the value for m the correct amount of energy is taken out of the system during a tidal cycle. During maximum velocities, the force is underestimated. During low velocities, the force is overestimated. So the distribution of the energy loss over a tidal cycle is not correct. The correct distribution (by the quadratic friction term) of the energy loss over the tidal cycle causes a distortion of the tidal wave.
The result of introducing the linear frictions is:
 - The damping of the tidal wave is correct;
 - The distortion of the tidal wave is not taken into account.
2. In the expression for m , \hat{Q} is a variable to be calculated. The procedure is to estimate \hat{Q} , and afterwards check this estimate. In case of deviation in \hat{Q} , the calculation has to be repeated with a better guess. A small difference between the computed \hat{Q} and the estimated \hat{Q} is acceptable.

7.2.2 Harmonic solution

The equations with the linearize friction term linearize are:

$$\frac{\partial Q}{\partial x} + b \frac{\partial \eta}{\partial t} = 0$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} + g \frac{\partial \eta}{\partial x} + mQ = 0$$

As was shown in a previous Chapter, differentiation of the first equation to t and of the second equation to x , and eliminating Q results in the *telegraph equation*:

$$\frac{\partial^2 \eta}{\partial t^2} - c_0^2 \frac{\partial^2 \eta}{\partial x^2} + mA \frac{\partial \eta}{\partial t} = 0 \quad \text{with} \quad c_0 = \sqrt{\frac{gA}{b}}$$

This is a linear differential equation of the second order. A harmonic solution of the equation is one in which the amplitude and the phase depend on x :

$$\eta(x,t) = \hat{\eta}(x) \cos(\omega t + \phi(x))$$

To determine the unknown parameters $\hat{\eta}(x)$ and $\phi(x)$, the derivations of $\eta(x,t)$ can be substituted in the telegraph equation, as was done for the tidal wave travelling on a river. A simpler solution procedure is introduced by Lorentz using complex numbers.

First, a short review of some relevant characteristics of *complex numbers* is given:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta, \quad \text{in which } i = \sqrt{-1}$$

in which:

$$\begin{aligned} \cos\theta & \quad \text{real part;} \\ i \sin\theta & \quad \text{imaginary part.} \end{aligned}$$

Figure 7.4 shows the complex plane can be defined, with the real axis displayed horizontally and the imaginary axis vertically. In Figure 7.4, $e^{i\theta}$ is represented as a vector in the complex plane with:

- real component $\cos\theta$,
- imaginary component $i \sin\theta$.

The same holds for $e^{-i\theta}$.

A complex number w can be written as:

$$w = g e^{i\theta}$$

with θ real and g complex..

g can be written as: $g = |g| e^{i\alpha} = a + ib$

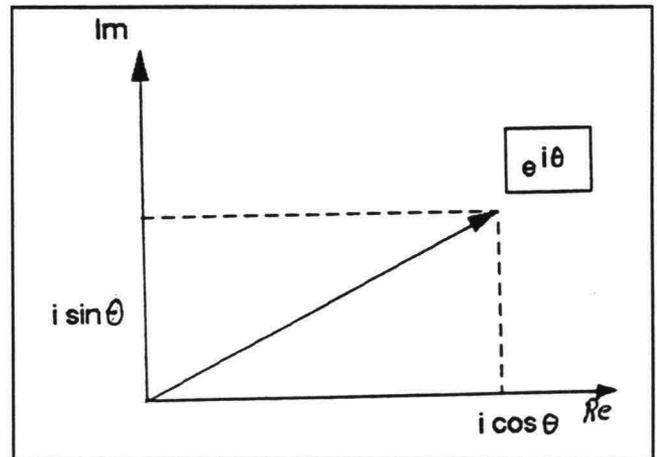


Figure 7.4 Complex plane

Figure 7.5 shows the vector g with length $|g|$ and an angle with the real axis α .

The same holds for the complex number w (see Figure 7.6): $w = g e^{i\theta} = |g| e^{i\alpha} e^{i\theta} = |g| e^{i(\alpha+\theta)}$

When the angle θ varies uniformly in time ($\theta = \omega t$) then:

$$\begin{aligned} w &= |g| e^{i(\alpha+\omega t)} \\ &= |g| \cos(\omega t + \alpha) + |g| I \sin(\omega t + \alpha) \end{aligned}$$

The real part of w is:

$$\text{Re}\{w\} = |g| \cos(\omega t + \alpha)$$

It represents a harmonic motion with:

- $|g|$ amplitude;
- ω angular speed;
- α phase lag.

This is similar to:

$$\eta(x,t) = \hat{\eta}(x) \cos(\omega t + \phi(x))$$

This can be written in complex form as:

$$\eta(x,t) = \text{Re} \left(\hat{\eta}(x) e^{i(\omega t + \phi(x))} \right) = \text{Re} \left(\hat{\eta}(x) e^{i\phi(x)} \times e^{i\omega t} \right)$$

Term $\hat{\eta}(x) e^{i\phi(x)}$ can be represented in the complex plane as shown in Figure 7.7. This Figure shows the *phase - amplitude diagram* for a certain location x with:

- $\hat{\eta}(x)$ amplitude;
- $\phi(x)$ phase.

For substitution in the telegraph equation, $\eta(x,t)$ can be written as:

$$\eta(x,t) = \hat{\eta}(x) e^{i(\omega t + \phi(x))} = \hat{\eta}(x) e^{i\phi(x)} e^{i\omega t}$$

which includes the imaginary part:

$$\hat{\eta}(x) i \sin(\omega t + \phi(x))$$

For $\hat{\eta}(x) e^{i\phi(x)}$ can be written as $C e^{rx}$, in which r is a complex number, which must be solved. So the expression to be substituted in the telegraph equation is:

$$\underline{\underline{\eta(x,t) = C e^{rx} e^{i\omega t} = C e^{i\omega t + rx}}}$$

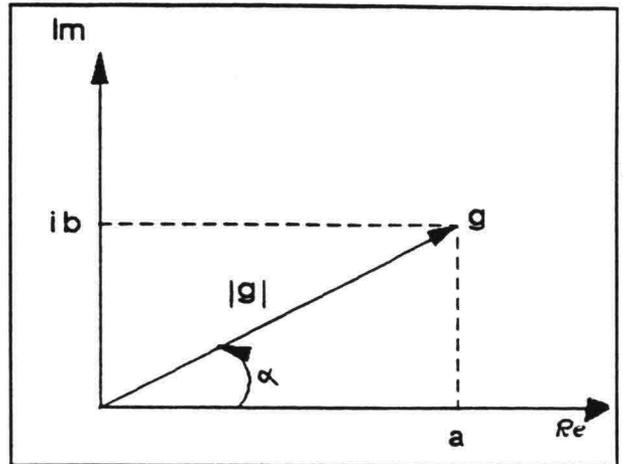


Figure 7.5 Vector g in the complex plane

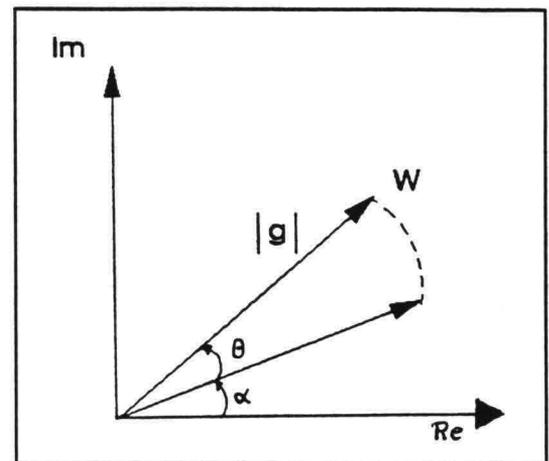


Figure 7.6 Vector g in the complex plane

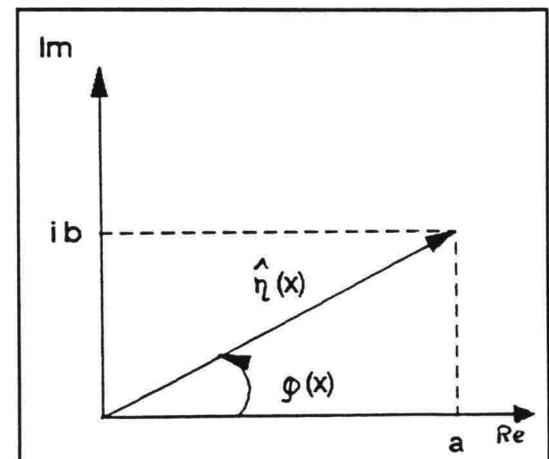


Figure 7.7 Presentation of $\hat{\eta}(x)$ in the complex plane

In complex notation, the derivatives of $\eta(x, t)$ are relatively simple:

$$\frac{\partial \eta}{\partial t} = i\omega C e^{i\omega t + rx}, \quad \frac{\partial \eta}{\partial x} = r C e^{i\omega t + rx}$$

$$\frac{\partial^2 \eta}{\partial t^2} = -\omega^2 C e^{i\omega t + rx}, \quad \frac{\partial^2 \eta}{\partial x^2} = r^2 C e^{i\omega t + rx}$$

Substituting these derivatives in the telegraph equation gives an equation for the unknown r . The telegraph equation is given by:

$$\frac{\partial^2 \eta}{\partial t^2} - c_0^2 \frac{\partial^2 \eta}{\partial x^2} + mA \frac{\partial \eta}{\partial t} = 0$$

Substitution of the derivatives yields:

$$(-\omega^2 C - c_0^2 r^2 C + mA i \omega C) e^{i\omega t + rx} = 0$$

$$-\omega^2 - c_0^2 r^2 + imA\omega = 0$$

$$r^2 = \frac{1}{c_0^2} (-\omega^2 + imA\omega)$$

This equation for r yields two complex roots. The complex number r can be written as:

$$r = p + iq$$

$$r^2 = p^2 - q^2 + 2ipq = -\frac{\omega^2}{c_0^2} + \frac{imA\omega}{c_0^2}$$

The real parts are equal to:

$$p^2 - q^2 = -\frac{\omega^2}{c_0^2}$$

The imaginary parts are equal to:

$$2pq = \frac{mA\omega}{c_0^2}$$

By eliminating p or q , the following quadratic equations can be derived::

$$c_0^2 [p^2] + \omega^2 [p^2] - \frac{m^2 A^2 \omega^2}{4c_0^2} = 0$$

$$c_0^2 [q^2] - \omega^2 [q^2] - \frac{m^2 A^2 \omega^2}{4c_0^2} = 0$$

The roots from these equations are:

$$p_{1,2} = \pm \frac{\omega \sqrt{-1 + \sqrt{1 + (mA/\omega)^2}}}{c_0 \sqrt{2}}, \quad \text{and} \quad q_{1,2} = \pm \frac{\omega \sqrt{1 + \sqrt{1 + (mA/\omega)^2}}}{c_0 \sqrt{2}}$$

Similar expressions were found for the tidal wave on the river in Section 5.2, with λ instead of p and k of q .

The roots of r are $r_{1,2} = \pm (p + iq)$.

The final result for the solution of $\eta(x,t)$ is composed of a linear combination of terms with the positive and negative root.

$$\eta(x,t) = C_1 e^{i\omega t + rx} C_2 e^{i\omega t - rx}$$

$$\eta(x,t) = C_1 e^{i\omega t + px + iqx} + C_2 e^{i\omega t - px - iqx} = C_1 e^{px} \times e^{i(\omega t + qx)} + C_2 e^{-px} \times e^{i(\omega t - qx)}$$

The real part of the first term is:

$$C_1 e^{px} \cos(\omega t + qx)$$

which is a wave propagating in the negative x -direction, with increasing amplitude for increasing x .

The real part of the second term is:

$$C_2 e^{-px} \cos(\omega t - qx)$$

which is a wave propagating in the positive x -direction, with decreasing amplitude for increasing x .

Both parts of the solution are drawn in Figure 7.8.

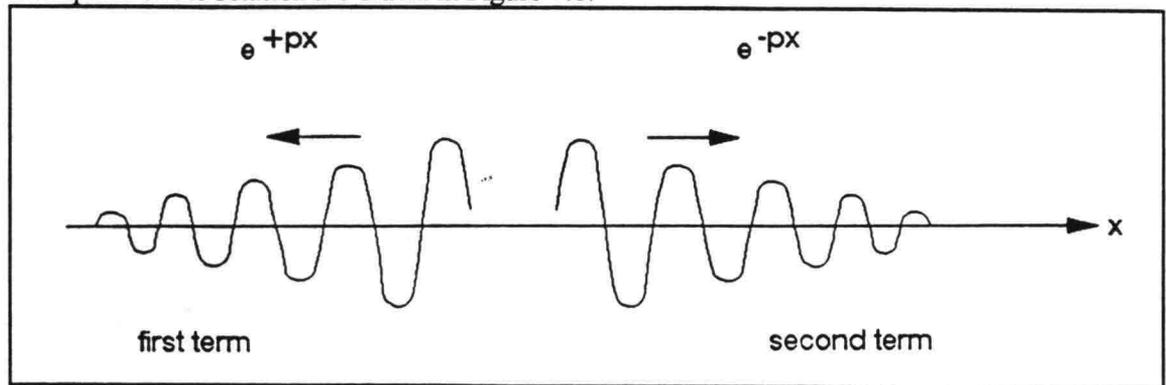


Figure 7.8 Solution for $\eta(x,t)$

The term e^{+px} and e^{-px} represent the damping or attenuation of the tidal wave. The celerity of the wave is $c = \frac{\omega}{q}$ (substitution of k instead of q gives the known $c = \frac{\omega}{k}$).

So far the water level η was considered. A formula for the discharge Q can be found too with the equation of continuity:

$$\frac{\partial Q}{\partial x} = -b \frac{\partial \eta}{\partial t}$$

Substitution of the solution for $\eta(x,t)$ ($= C_1 e^{i\omega t + rx} + C_2 e^{i\omega t - rx}$) gives:

$$\frac{\partial Q}{\partial x} = -bi\omega e^{i\omega t} (C_1 e^{rx} + C_2 e^{-rx})$$

Integration yields:

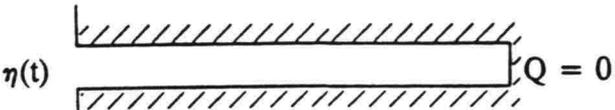
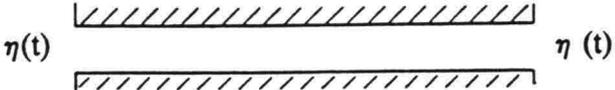
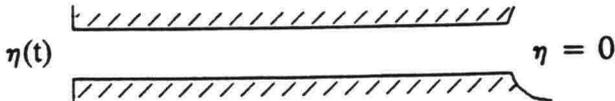
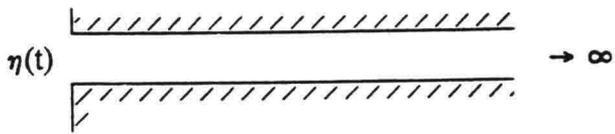
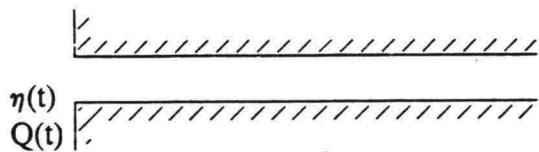
$$Q(x,t) = -\frac{bi\omega}{r} e^{i\omega t} (C_1 e^{rx} - C_2 e^{-rx})$$

This result is similar to $\eta(x,t)$.

Harmonic solutions can be found for various boundary conditions, through which the integration constants C_1 and C_2 can be solved. Two integration constants must be solved, so two boundary conditions are needed. Different combinations of boundary conditions are possible:

- 2 water levels at different locations;
- 1 water level and 1 discharge;
- 2 discharges at different locations.

Some examples where the Lorentz method can be applied:

1. Channel closed at one end. 
2. Channel connecting two oceans. 
3. Channel connecting an ocean, with tideless large lake. 
4. Channel at one side to infinity. (There is only a wave propagating in the positive x-direction. One integration constant has to be solved. So one boundary condition is sufficient). 
5. Channel with known water level and discharge at one location 

Case 5 is considered as the *standard case*. The other cases can be derived from it. Case 4 represents the tidal wave on a river. Special attention is paid to Case 1 in Section 7.3.3.

First *standard case* 5 is considered with the following equations:

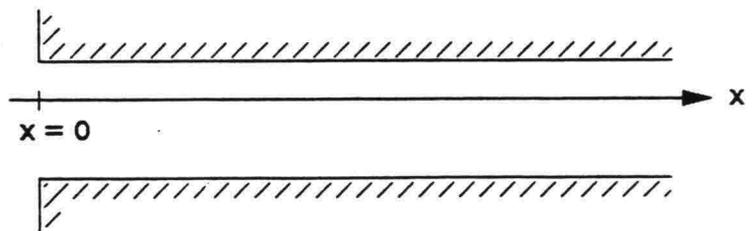
$$\eta(x, t) = C_1 e^{i\omega t + rx} + C_2 e^{i\omega t - rx}$$

$$Q(x, t) = -\frac{bi\omega}{r} e^{i\omega t} (C_1 e^{rx} - C_2 e^{-rx})$$

The boundary conditions at $x = 0$ are:

$$\eta(0, t) = \eta(0) e^{i\omega t}$$

$$Q(0, t) = Q(0) e^{i\omega t}$$



$\eta(0)$ and $Q(0)$ are complex quantities, representing the amplitudes and phases. Substitution of these boundary conditions gives:

$$\begin{aligned}\eta(0,t) &\rightarrow C_1 e^{i\omega t} + C_2 e^{-i\omega t} = \eta(0) e^{i\omega t}, \text{ so} & C_1 + C_2 &= \eta(0) \\ Q(0,t) &\rightarrow -\frac{bi\omega}{r} e^{i\omega t} (C_1 - C_2) = Q(0) e^{i\omega t}, \text{ so} & -\frac{bi\omega}{r} (C_1 - C_2) &= Q(0)\end{aligned}$$

C_1 and C_2 therefore become:

$$\begin{aligned}C_1 &= \frac{1}{2}\eta(0) - \frac{1}{2}r \frac{Q(0)}{bi\omega} \\ C_2 &= \frac{1}{2}\eta(0) + \frac{1}{2}r \frac{Q(0)}{bi\omega}\end{aligned}$$

A. Water level

Knowing that:

$$\eta(x, t) = C_1 e^{i\omega t + rx} + C_2 e^{i\omega t - rx}, \text{ and } \eta(x,t) = \eta(x) e^{i\omega t}, \text{ thus } \eta(x) = C_1 e^{rx} + C_2 e^{-rx},$$

and substituting the values for C_1 and C_2 gives:

$$\begin{aligned}\eta(x) &= \frac{1}{2}\eta(0)e^{rx} - \frac{1}{2}\frac{r}{bi\omega}Q(0)e^{rx} + \frac{1}{2}\eta(0)e^{-rx} + \frac{1}{2}\frac{r}{bi\omega}Q(0)e^{-rx} \\ &= \eta(0)\frac{e^{rx} + e^{-rx}}{2} - \frac{r}{bi\omega}Q(0)\frac{e^{rx} - e^{-rx}}{2}\end{aligned}$$

As:

$$\frac{e^{rx} + e^{-rx}}{2} = \cosh(rx), \text{ and } \frac{e^{rx} - e^{-rx}}{2} = \sinh(rx)$$

the water level amplitude can be written as function of x :

$$\eta(x) = \eta(0)\cosh(rx) - \frac{r}{bi\omega}Q(0)\sinh(rx)$$

B. Discharge

Knowing that:

$$Q(x,t) = -\frac{bi\omega}{r} e^{i\omega t} (C_1 e^{2x} - C_2 e^{-2x}), \text{ and } Q(x,t) = Q(x) e^{i\omega t}, \text{ thus } Q(x) = -\frac{bi\omega}{r} (C_1 e^{-rx} - C_2 e^{rx})$$

and substituting the values for C_1 and C_2 gives:

$$\begin{aligned}Q(x) &= -\frac{bi\omega}{r} \left[\left(\frac{1}{2}\eta(0)e^{rx} - \frac{1}{2}\frac{r}{bi\omega}Q(0)e^{rx} - \frac{1}{2}\eta(0)e^{-rx} - \frac{1}{2}\frac{r}{bi\omega}Q(0)e^{-rx} \right) \right] \\ &= -\frac{bi\omega}{r} \left[\eta(0)\frac{e^{rx} - e^{-rx}}{2} - \frac{2}{bi\omega}Q(0)\frac{e^{rx} + e^{-rx}}{2} \right]\end{aligned}$$

This formula can be written as:

$$Q(x) = -\frac{bi\omega}{r} \eta(0)\sinh(rx) + Q(0)\cosh(rx)$$

So the formulas for $\eta(x)$ and $Q(x)$ for the standard case are:

$$\eta(x) = \eta(0)\cosh(rx) - \frac{r}{bi\omega}Q(0)\sinh(rx)$$

$$Q(x) = -\frac{bi\omega}{r}\eta(0)\sinh(rx) + Q(0)\cosh(rx)$$

7.2.3 Tide in a channel which is closed at one end

The case of a channel, closed at one end, is of special interest, because under certain circumstances high amplitudes can occur at the closed end and high discharges can occur at the mouth. The following approach applies to tides (with periods of hours), but also to seiches in harbours (with periods of 5-30 minutes). Consider the prismatic channel of Figure 7.9 with length l .

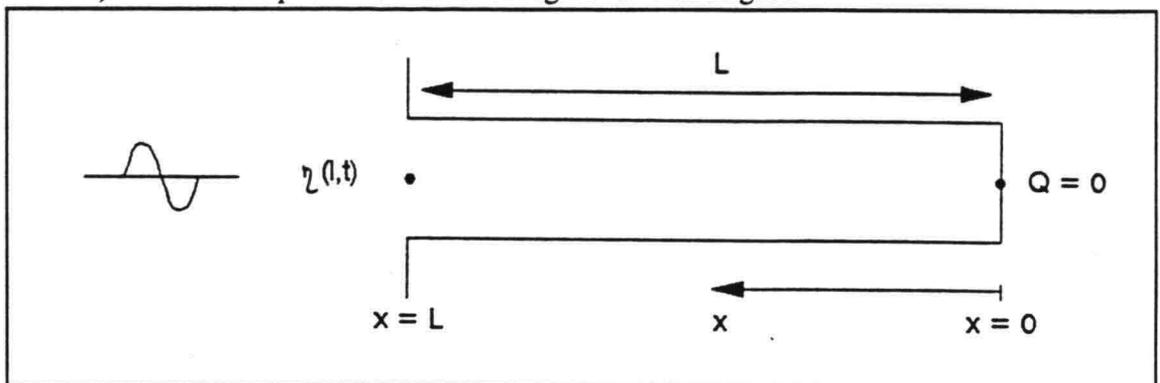


Figure 7.9 Prismatic channel

The location of $x=0$ is at the end of the channel. The x -axis is directed to the sea. The vertical tide is given at the mouth by $\eta(l,t)$. The discharge Q at the end is zero. To find the water levels and discharges, the equations for the standard case can be used:

$$\eta(x) = \eta(0)\cosh(rx) - \frac{r}{bi\omega}Q(0)\sinh(rx)$$

$$Q(x) = -\frac{bi\omega}{r}\eta(0)\sinh(rx) + Q(0)\cosh(rx)$$

The boundary condition for $x=0$ is given by $Q(0) = 0$. The equations now reduce to:

$$\eta(x) = \eta(0)\cosh(rx)$$

$$Q(x) = -\frac{bi\omega}{r}\eta(0)\sinh(rx)$$

The second boundary condition is formed by the water level at $x=l$:

$$\eta(l) = \eta(0)\cosh(rl)$$

Looking closer to the behaviour of the amplitude of the water level at the closed end as a function of the length of the channel. The amplitude of the water level is given by the modules of the complex number η (see Figure 7.10).

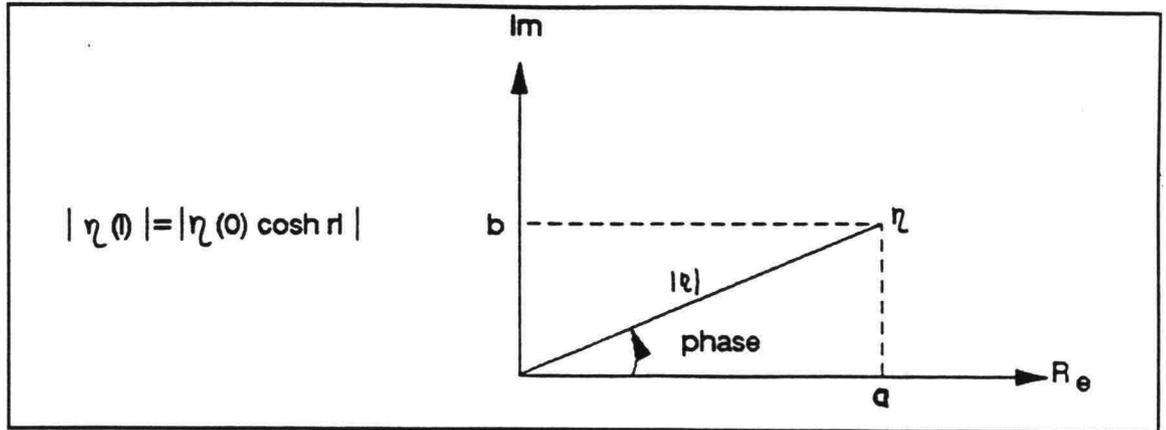


Figure 7.10 Behaviour of the amplitude of the water level at the closed end of the channel

For finding the ratio of the amplitudes, the phases are taken equal to 0 (see Figure 7.11). Then:

$$\begin{aligned}\hat{\eta}(l) &= \hat{\eta}(0) |\cosh(rl)| = \hat{\eta}(0) |\cosh(p + iq)l| \\ &= \hat{\eta}(0) |\cosh(pl)\cos ql + i\sinh(pl)\sin ql|\end{aligned}$$

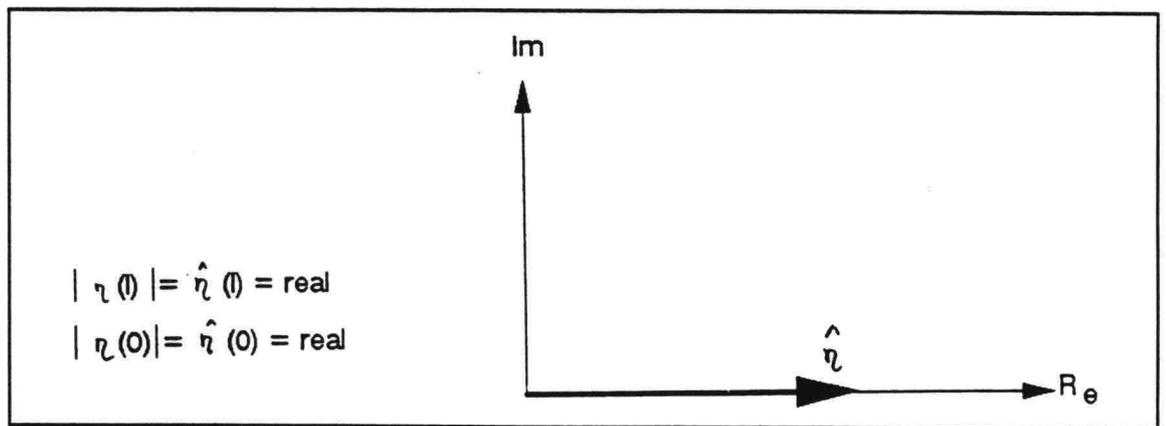


Figure 7.11 Ratio of amplitudes

The value of the modulus of the complex number is the square root of the squared real and imaginary parts ($c = \sqrt{a^2 + b^2}$):

$$\begin{aligned}\hat{\eta}(l) &= \hat{\eta}(0) \sqrt{\cosh^2(pl)\cos^2 ql + \sinh^2(pl)\sin^2 ql} \\ &= \hat{\eta}(0) \sqrt{\cosh^2(pl)\cos^2 ql + (\cosh^2(pl) - 1)(1 - \cos^2 ql)} \\ &= \hat{\eta}(0) \sqrt{\cos^2 ql + \cosh^2(pl) - 1} \\ &= \hat{\eta}(0) \sqrt{\cos^2 ql + \sinh^2(pl)}\end{aligned}$$

Defining the *amplification factor* as the ratio of the amplitude of the water level at the end of the channel and at the mouth, then:

$$\frac{\hat{\eta}(0)}{\hat{\eta}(l)} = \frac{1}{\sqrt{\cos^2 ql + \sinh^2(pl)}}$$

in which:

$$pl = \frac{\omega l}{c_0 \sqrt{2}} \sqrt{1 + \sqrt{1 + \left(\frac{mA}{\omega}\right)^2}}$$

$$ql = \frac{\omega l}{c_0 \sqrt{2}} \sqrt{1 + \sqrt{1 + \left(\frac{mA}{\omega}\right)^2}}$$

The amplification factor is determined by the parameters

$\frac{\omega l}{c_0}$: parameter for the length of the channel;

$\frac{mA}{\omega}$: parameter for the friction.

Figure 7.12 shows a plot of the amplification factor:

$$\frac{\hat{\eta}(0)}{\hat{\eta}(l)} = f\left(\frac{\omega l}{c_0}, \frac{mA}{\omega}\right).$$

Along the vertical axis the amplification factor is given. Along the horizontal axis the length parameter $\frac{\omega l}{c_0}$. The friction parameter $\frac{mA}{\omega}$ is found as parameter in Figure 7.12.

From Figure 7.12 it can be observed that the friction has a reducing effect on the amplification factor. The amplification factor tends to go to infinity if the friction term becomes zero for certain values of $\frac{\omega l}{c_0}$.

If $m = 0$ (no friction) then $pl = 0$ and $ql = \frac{\omega l}{c_0}$

The amplification factor in case of no friction is:

$$\frac{\hat{\eta}(0)}{\hat{\eta}(l)} = \frac{1}{\cos \frac{\omega l}{c_0}}$$

The amplification factor is ∞ if $\cos \frac{\omega l}{c_0} = 0$, which is the case for:

$$\frac{\omega l}{c_0} = \pi/2, 3\pi/2, 5\pi/2, \dots = (2n + 1)\frac{\pi}{2} \text{ with } n = 0, 1, 2, 3, \dots$$

Knowing that:

$$\omega = \frac{2\pi}{T}, \text{ and } L = c_0 T, \text{ or } c_0 = \frac{L}{T} \text{ gives } \frac{\omega}{c_0} = \frac{2\pi/T}{L/T} = \frac{2\pi}{L}$$

$$\text{So: } \frac{\omega l}{c_0} = 2\pi \frac{l}{L}$$

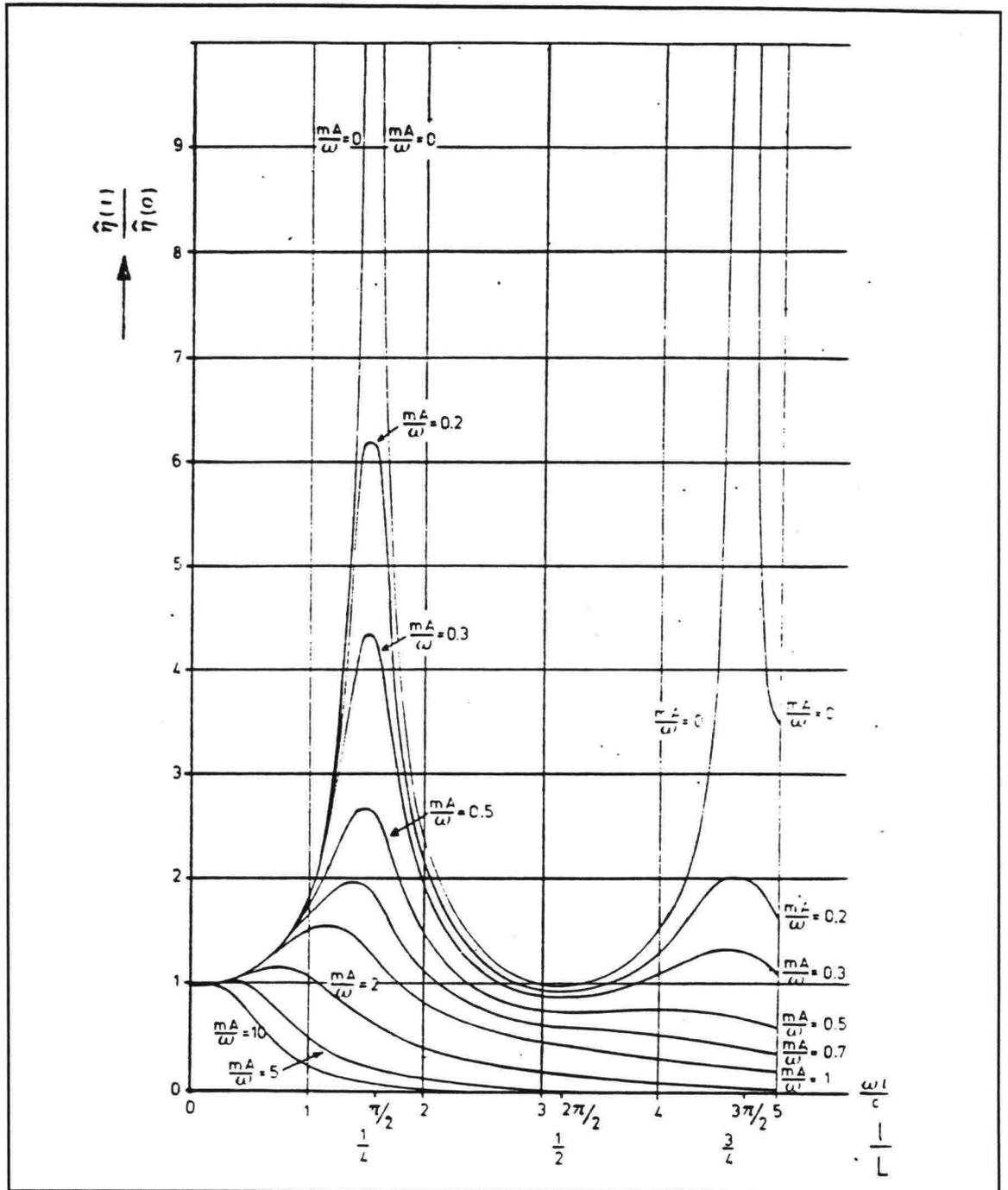


Figure 7.12 Amplification factor

The amplification factor is ∞ if:

$$2\pi \frac{l}{L} = (2n + 1) \frac{\pi}{2} \text{ or } \frac{l}{L} = \frac{2n + 1}{4}, \text{ for } n = 0, 1, 2, 3, \dots$$

So resonance occurs for $\frac{l}{L} = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \dots$

Infinite amplitudes will never occur, as the friction will prevent that.

Now the second equation for the discharge will be looked in further detail:

$$Q(x) = -\frac{bi\omega}{r}\eta(0)\sinh(rx)$$

The discharge at the mouth is noted as $Q(l)$. In the above equation, $\eta(0)$ is noted as $\hat{\eta}(0)$. Further $x = l$ and $r = p + iq$. So:

$$Q(l) = -\frac{bi\omega}{p + iq}\hat{\eta}(0)\sinh(p + qi)l$$

As $\hat{\eta}(0)$ is related to $\hat{\eta}(l)$, the amplitude of the discharge can be computed. Consider the case with *no friction*. Take $m = 0$, so $p = 0$. Then $Q(l)$ can be written as:

$$Q(l) = -\frac{b\omega}{q}\hat{\eta}(0)\sinh(iql)$$

Because $\sinh(ix) = I \sin x$, $Q(l)$ becomes:

$$Q(l) = -\frac{ib\omega}{q}\hat{\eta}(0)\sin ql.$$

For the case without friction it was found that $q = \frac{\omega}{c_0}$

Substitution gives:

$$Q(l) = -ibc_0\hat{\eta}(0)\sin\frac{\omega l}{c_0}$$

The amplitude for the complex number $Q(l)$ is:

$$\hat{Q}(l) = |Q(l)| = bc_0\hat{\eta}(0)\sin\frac{\omega l}{c_0}$$

For the case without friction, the amplitude of the water level $\hat{\eta}(0)$ was determined as:

$$\hat{\eta}(0) = \hat{\eta}(l)\frac{1}{\cos\frac{\omega l}{c_0}}$$

Substitution in $\hat{Q}(l)$ yields:

$$\hat{Q}(l) = bc_0\hat{\eta}(l)\tan\frac{\omega l}{c_0}$$

This result shows that high discharges can be expected if:

$$\frac{\omega l}{c_0} = \pi/2, 3\pi/2, 5\pi/2 \dots = (2n + 1)\frac{\pi}{2} \text{ with } n = 0, 1, 2, 3, 4$$

$$\text{or } \frac{l}{L} = \frac{1}{4}, \frac{3}{4}, \frac{5}{4} \dots$$

When the length of an estuary is changed (for instance by building a dam at $x = \frac{1}{2}l$) the flow regime can change considerably. The possibility of resonance should therefore always be checked.

8 Numerical tidal computations

8.1 Introduction

To solve engineering problems, different tools can be used:

1. Simplified analytical computations for first guess/insight in the problem;
2. Numerical tidal computations for accurate prediction in complex geometries.

In the second group also hydraulic scale models should be mentioned, but their use is an exception nowadays. In the simplified analytical computations:

- the equations are simplified;
- the geometry is simplified.

In the numerical tidal computations:

- the complete equations are used (without simplifications);
- the complete complex geometry is schematized accurately.

For numerical tidal computations, mathematical models are used. The following types of models can be distinguished:

1. The computer program, which is general and can solve all kinds of problems (from small scale flow computations to complete continental shelf seas);
2. The area considered, which is schematized and for which data have been gathered like depth's, Chézy values (bottom friction) etc.

The term mathematical model is often used for both elements, which may be confusing. In these lecture notes, the term mathematical model concerns the area considered, with all the data that characterize the geometry and which is used as input for the computer program.

Different types of mathematical models can be distinguished:

- 1D-models for rivers, canals;
- 2D-models-horizontal for sea-areas;
- 3D-models when the vertical current distribution is important, e.g. in areas with density currents.

Experiences with numerical computations depend on the type of engineer:

1. Engineers, who are interested in numerical computation procedures, like stability, accuracy, efficiency of the computation schemes. They build the computer programs;
2. Model engineers, who schematize the area considered and build the mathematical model. They are responsible for the computations;
3. Engineers, who use the results of the mathematical models. They must be able to judge if their questions are solved with the correct type of mathematical model. They should have a general knowledge of the characteristics of mathematical models.

This part of the lectures is directed to the second and third categories of engineers (model engineers). First, some general aspects related to numerical computations are made, followed by the illustration of some aspects with some practical examples.

The numerical computations of this chapter focus on the complete equations, without simplifications. Thus, the examples on 2D-models consider the 2D-equations as derived in Chapter 6:

$$\frac{\partial \eta}{\partial t} + \frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y} = 0$$

Continuity

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial \eta}{\partial x} + g \frac{u \sqrt{u^2 + v^2}}{C^2 h} - fv =$$

Motion in x-direction

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial \eta}{\partial y} + g \frac{v \sqrt{u^2 + v^2}}{C^2 h} + fu =$$

Motion in y-direction

For numerical computations the space steps and time step are made discrete:

$$\begin{aligned} dx &- \Delta x \\ dy &- \Delta y \\ dt &- \Delta t \end{aligned}$$

The partial derivatives are replaced by difference quotients (see Figure 8.1).

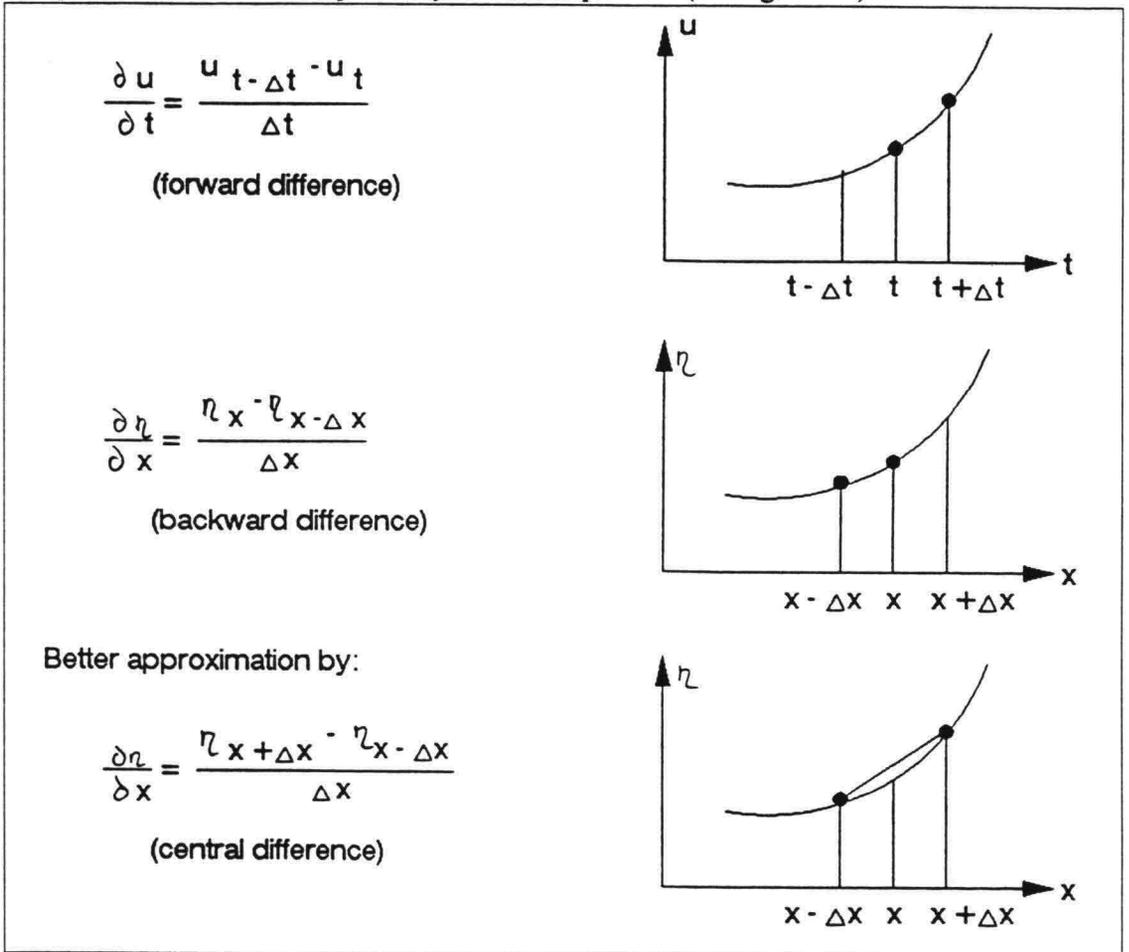


Figure 8.1 Examples of discrete time steps

With the different possible combinations of forward, backward and central differences, the equations can be transformed into many difference schemes. With that difference schemes, u , v , and η are solved for discrete space steps Δx , Δy and time steps Δt (see Figure 8.2).

The following schemes can be distinguished:

- explicit schemes;
- implicit schemes.

This can be illustrated for the 1-dimensional case, where η (and u) is a function of x , (see Figure 8.3). A variable, like η , is computed at $t + \Delta t$.

To summarize:

The equations are transformed into a difference scheme. With that scheme, u , v , η are computed for every location on the Δx - Δy grid for every time step Δt . In most cases a rectangular grid is chosen with $\Delta x = \Delta y$. The computation of the variables on the grid is performed by a computer.

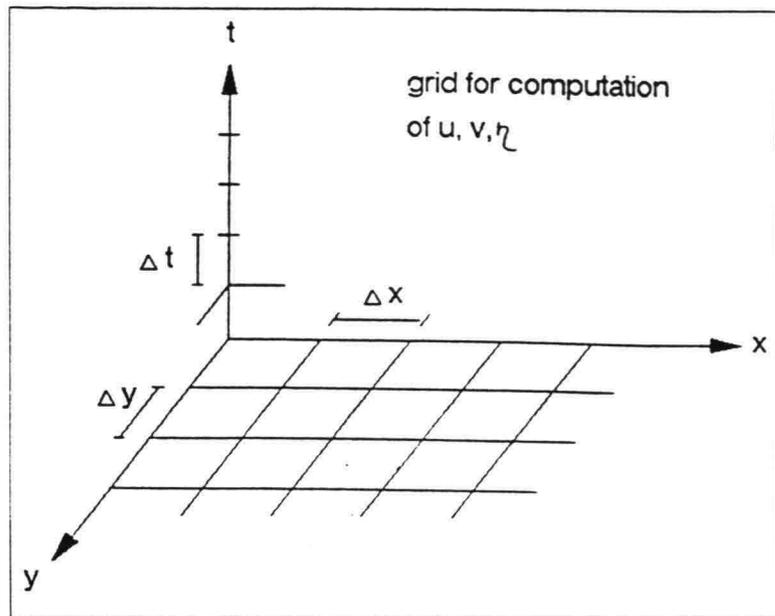


Figure 8.2 Grid for computation

8.2 Set-up of a mathematical model

This Section focuses on the building of a mathematical model. For this, first the purpose of the model must be formulated. Therefore, a sound description of the required output is needed.

Aspects to be included are:

- area considered (location of the boundaries);
- rate of detail (grid size);
- type of reliable output (water levels, velocities);
- wind as parameter to be included or not;
- what data are available for boundary conditions, for calibration and validation?

An *important remark* is that the right type of model should be selected. For instance, do not use a 3D-model when the problem can simply be solved by a manual calculation.

Location of boundaries

The boundaries of the model follow from the area under consideration. Sometimes the boundaries of the model must be chosen outside the area of interest.

Two examples:

1. If a model is used to study the effect of civil engineering works on the tidal motion, the boundaries should be located, where the change in geometry due to the civil engineering works has no effect on the water levels (when the water level is used as boundary condition);

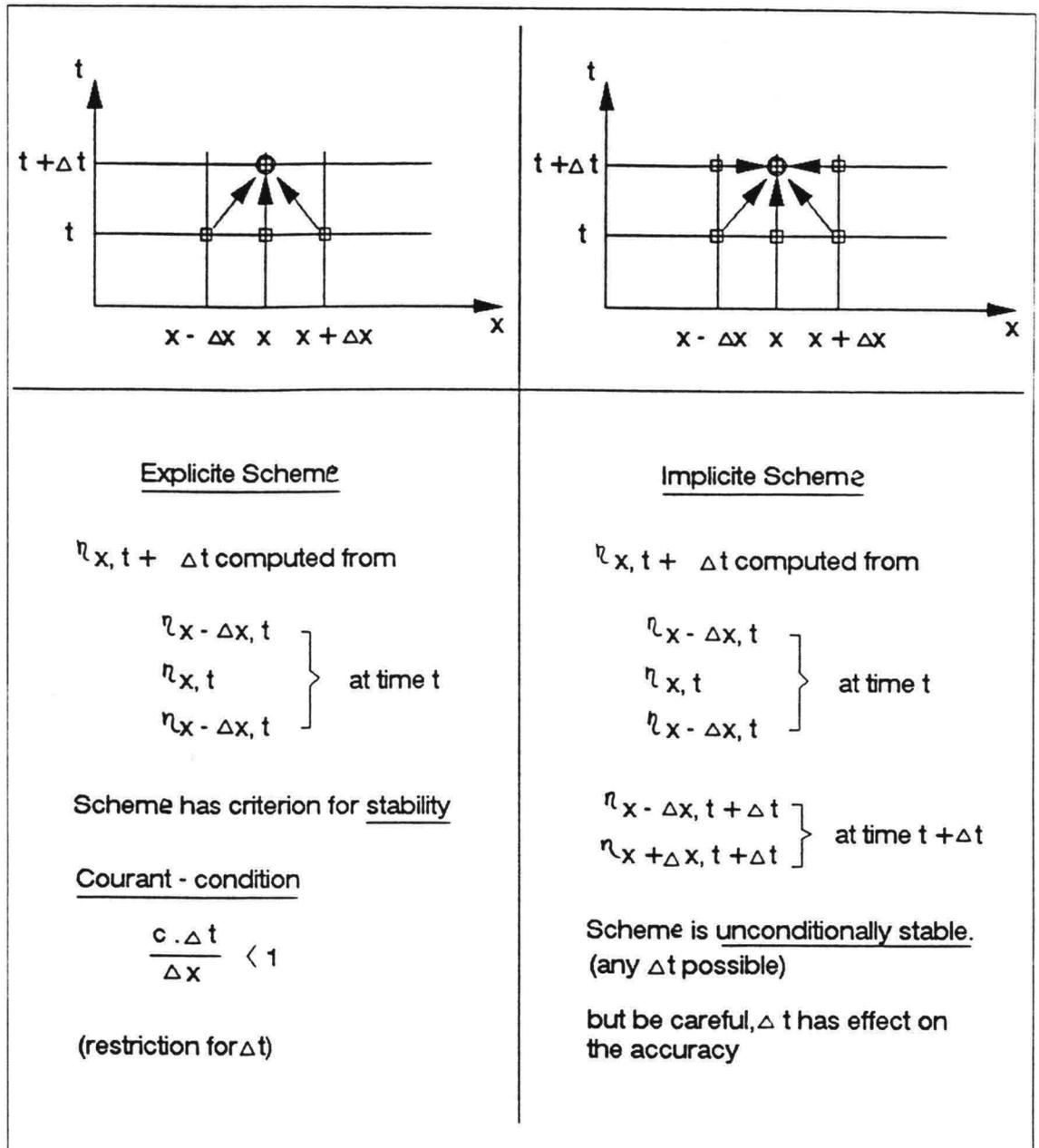


Figure 8.3 Explicit versus implicit schemes

2. If a model has to be built of an area, in which the effect of wind is important, the boundaries should be located at deep water. At the boundaries, the water levels known from tidal prediction (without wind effect) should be used. The effect of the wind on the water levels is computed in the model (generated in the shallow areas).

The data needed for building the model

After the boundaries of the model have been chosen, a choice for the grid size $\Delta x = \Delta y$ should be made. The grid size can range from 10 m's - 50 km (Eastern Scheldt model - Sunda Shelf model; see Section 8.4 and Figure 8.4).

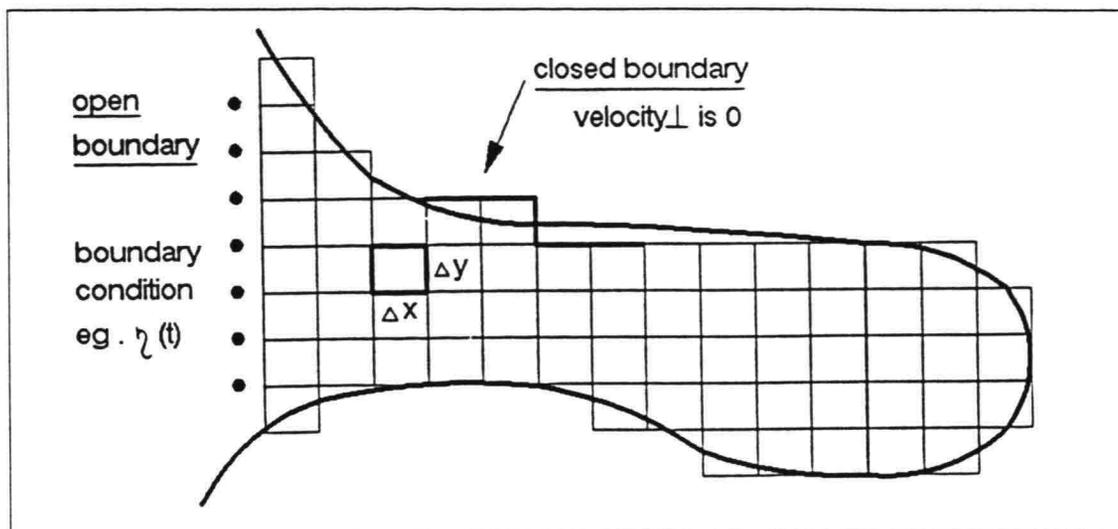


Figure 8.4 Example of a schematization of an estuary

The choice of the grid size depends on:

- required level of detail (small gullies in the model);
- size of the available computer (maximum number of grid points);
- efficiency (computing time is proportional to $(\Delta x)^3$).

Sometimes a small Δx is needed for the required level of detail, but the area should be large because civil engineering works have effect on the water levels at the boundaries. In such a case, the models should be nested (see Figure 8.5).

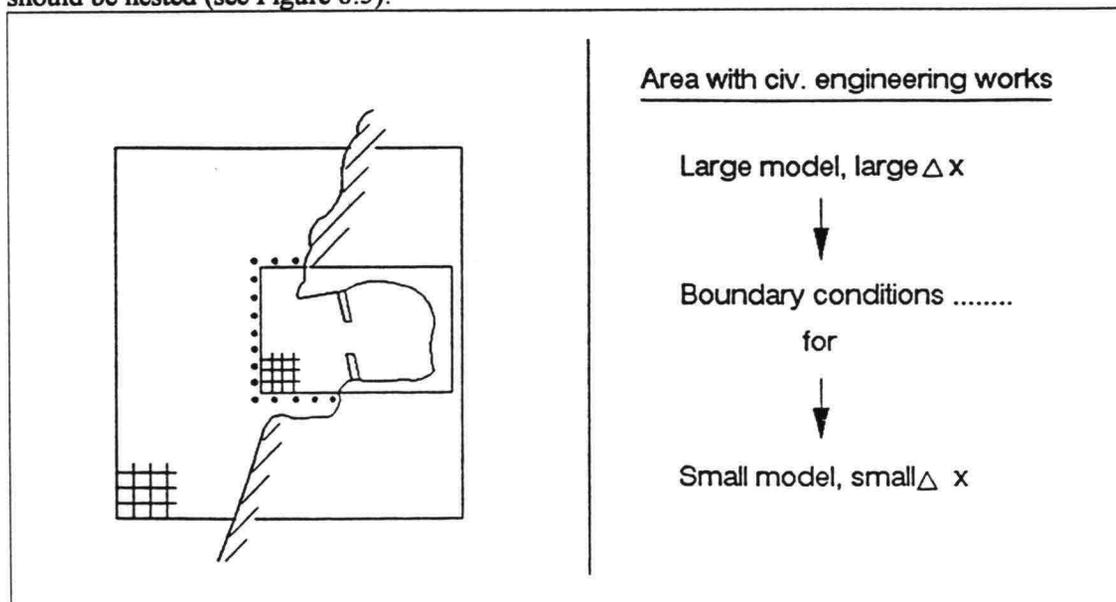


Figure 8.5 "Nested" models

Assume the area of Figure 8.4 and a grid size Δx . Then the area can be schematized to square boxes with grid size $\Delta x = \Delta y$. Along the shore the boundary is closed; the velocity perpendicular to the shore is zero. Along the open boundary, boundary conditions like water levels as function of time are needed.

Further information is needed on:

- depth per box. Depth values may originate from echo-soundings or sea maps. Data from sea maps should be interpreted with care, as they are prepared for ships. Often the depths taken from sea maps are too small;
- Chézy value per box. Often the model starts with a uniform Chézy value for the entire area, which is based on experience of that area.

Boundary conditions are needed at the open boundary. These may originate from:

- measurements (directly);
- measurements (analyzed): tidal analysis → harmonic components → tidal prediction;
- from other models (via “nesting”; see Figure 8.5);
- from literature (Schwiderski published tidal components for the entire world for every degree by degree area on the globe).

For running the model, time step Δt should be selected as follows:

- *explicit schemes*: Δt is restricted by the stability criterion

$$\frac{c\Delta t}{\Delta x} < 1 \rightarrow \Delta t < \frac{\Delta x}{c}$$
- *implicit schemes*: there is no restriction for stability. But the accuracy is less for large Δt (chosen by trial and error or experience).

8.3 Calibration of a model

To calibrate a model, the following data are needed:

1. Boundary conditions (like water levels at the open boundary);
2. Observed water levels/velocities in the model area.

The data should cover the same period. They can originate from:

- measurements;
- tidal prediction (from known components).

Figure 8.6 shows a schematic presentation of the Eastern Scheldt, The Netherlands. As an example, the calibration of this estuary will be presented.

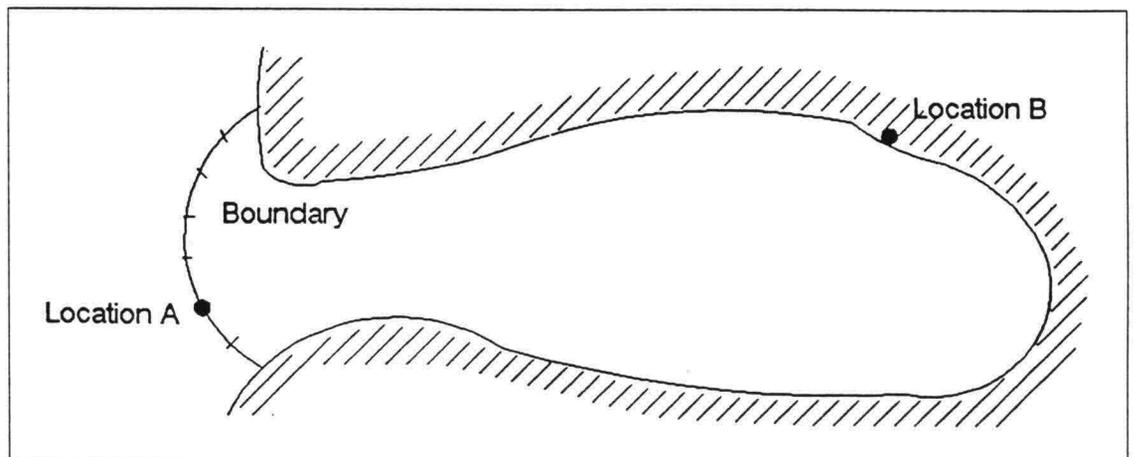


Figure 8.6 Schematic presentation of the Eastern Scheldt estuary, The Netherlands

Consider:

1. observed water levels as boundary condition in A;
2. observed water levels somewhere in the estuary in B.

With the first set of depths, Chézy values, and a chosen Δt , computations with the model are made. Such a computation is called the base case.

The observed water levels in A and B are decomposed into tidal components: diurnal, semi-diurnal, quarter diurnal, etc. Per component a plot is made of the amplification versus the time lag (see Figure 8.7).

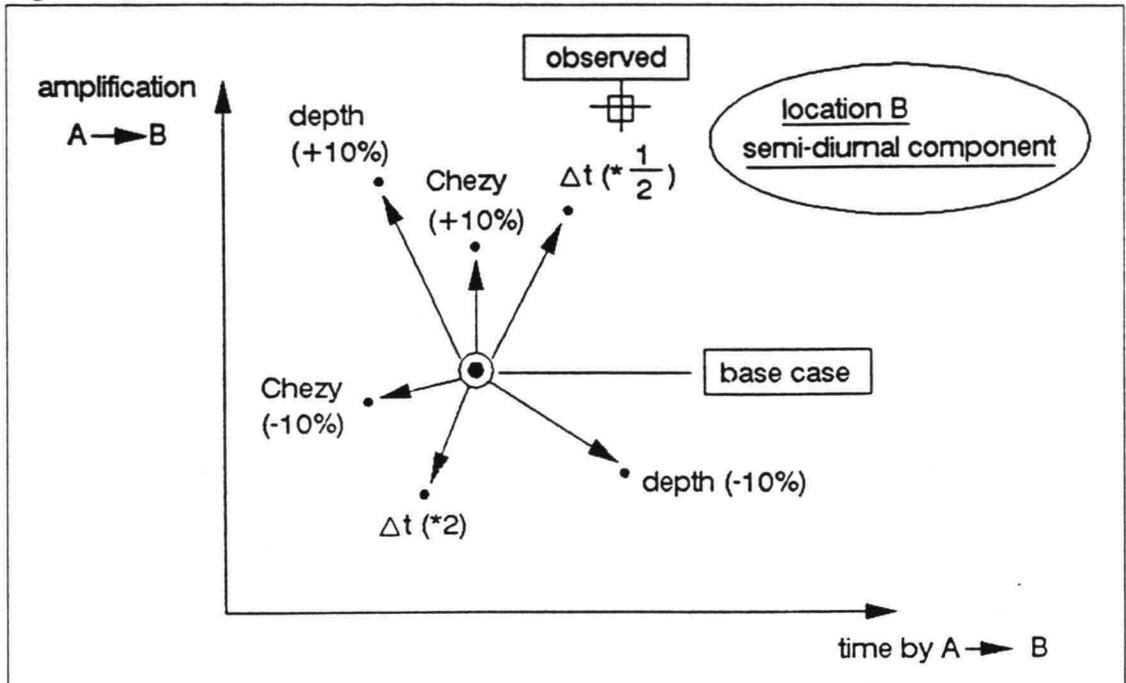


Figure 8.7 Plot of amplification versus time lag for points A and B of Figure 8.6

The next step is to vary the following parameters:

- depth: $\pm 10\%$;
- Chézy value $\pm 10\%$;
- time step Δt multiplied with $\frac{1}{2}$ and 2.

The decomposed results are also plotted in Figure 8.7. It can be concluded which parameter must be changed for a good simulation (in this case, Δt should be smaller).

In the calibration, the computed results are compared with:

- local water level measurements;
- local velocity measurements;
- known tidal pattern (from tidal atlases);
- known locations of Amphidromic points;

After the calibration of model, a validation computation should be made, for which boundary conditions and observed data from another period are used. In this way a check is made on computed and measured water levels and velocities.

8.4 Examples of mathematical models

This Section discusses two examples of models:

- *small scale models*, like the above model for the Eastern Scheldt (with emphasis on detailed flow computation);
- *large scale models*, which were for instance used for the Sunda Shelf (with emphasis on large scale tidal computation).

8.4.1 Mathematical models used for the Eastern Scheldt estuary

A storm surge barrier had to be constructed in the mouth of the Eastern Scheldt. The construction of such a civil engineering work has far reaching effects on the surrounding area, as it influences:

- tidal motion;
- exchange between the estuary and the sea;
- morphology.

The storm surge barrier is part of the so-called Delta Project, which was planned to offer a better protection of the southwestern region of The Netherlands (see Figure 8.8) against high storm surges. Under normal conditions, the barrier is open; it will be closed when high storm surges are expected.

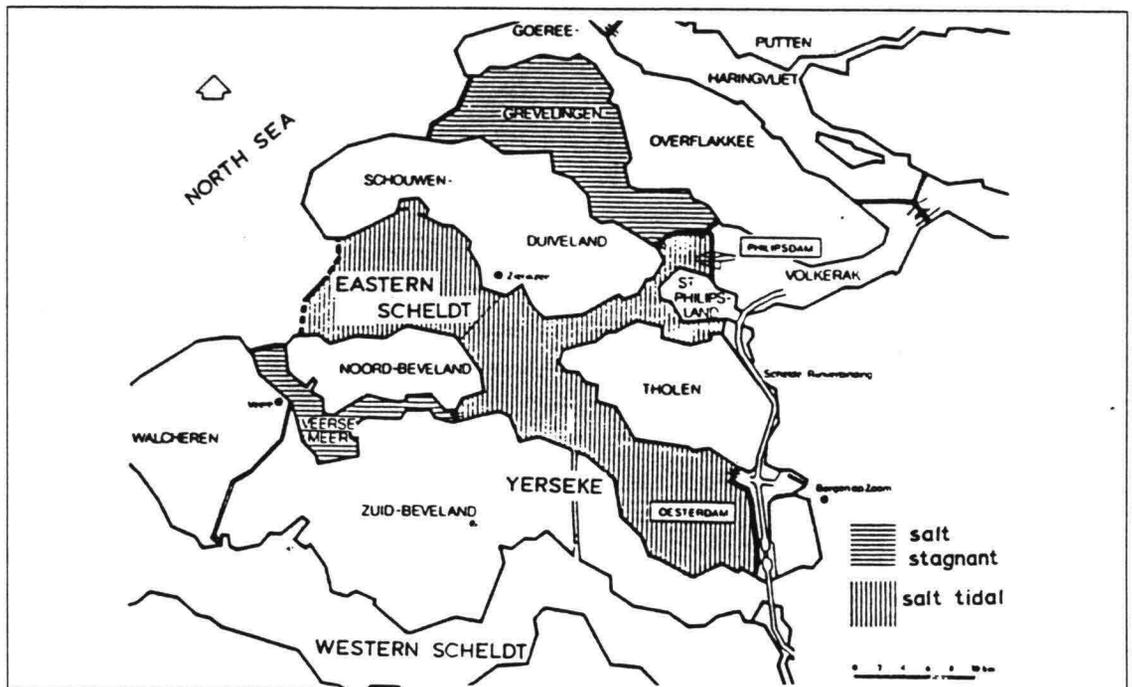


Figure 8.8 South-western part of The Netherlands (Delta region)

In the process of realization of the project, three stages were distinguished:

- *planning stage*, in which the lay out and general characteristics of the structure are determined
- *design stage*, in which the structure is designed;
- *execution stage*, in which the structure is constructed.

In each of the stages, models played an important role to provide answers on questions the designers face, like:

1. *Planning stage*
What is the effect of the aperture of the barrier on the tidal regime in the area?
What is the reduction of the tidal amplitude in the Eastern Scheldt?
2. *Design stage*
Which are the boundary conditions for the design of the barrier?
3. *Execution stage*
(What are) the detailed flow patterns at the barrier site during different construction phases?

Computations to answer those questions were carried out with a tidal computation model with an implicit scheme. That means that there was freedom in the choice of the time step, but careful interpretation was needed regarding the accuracy of the computed results.

Figure 8.9 shows a 2-dimensional overall model of the Eastern Scheldt, which has the following characteristics:

- grid size: 400 m;
- number of grid points: about 14,000;
- typical time step $\Delta t = 1.5$ minutes.

Figure 8.9 also shows the computed velocity field during maximum ebb for the situation without a barrier.

Boundary conditions were obtained from four tidal gauges at the boundary. For this, the following checks were made:

- for the selection of the boundary, first with a model of the larger area it was investigated if the water levels would be affected by the barrier;
- for the tide at the location of the tidal gauges, a tidal analysis of the measured water levels was made. From that analysis the components were obtained. With these components, a prediction could be made of the tide for any day in the future. This procedure was used, as the tidal motion during certain critical construction phases of the barrier were to be predicted in advance.

The model was calibrated extensively with observed water levels in the Eastern Scheldt and with measured discharges in the three tidal channels.

Calibration started with a base case computation with the first set of depths, Chézy values and time step Δt . The components of the decomposed water levels were plotted in a graph (Figure 8.10), which shows the amplification and phase lag for location 2 relative to location 1 (at the boundary of the model; see Figure 8.11).

Figure 8.11 shows the semi-diurnal component. The amplification and phase lag for the observed water level is plotted with symbol \square . The amplification and phase lag for the base case computation is plotted with symbol \circ .

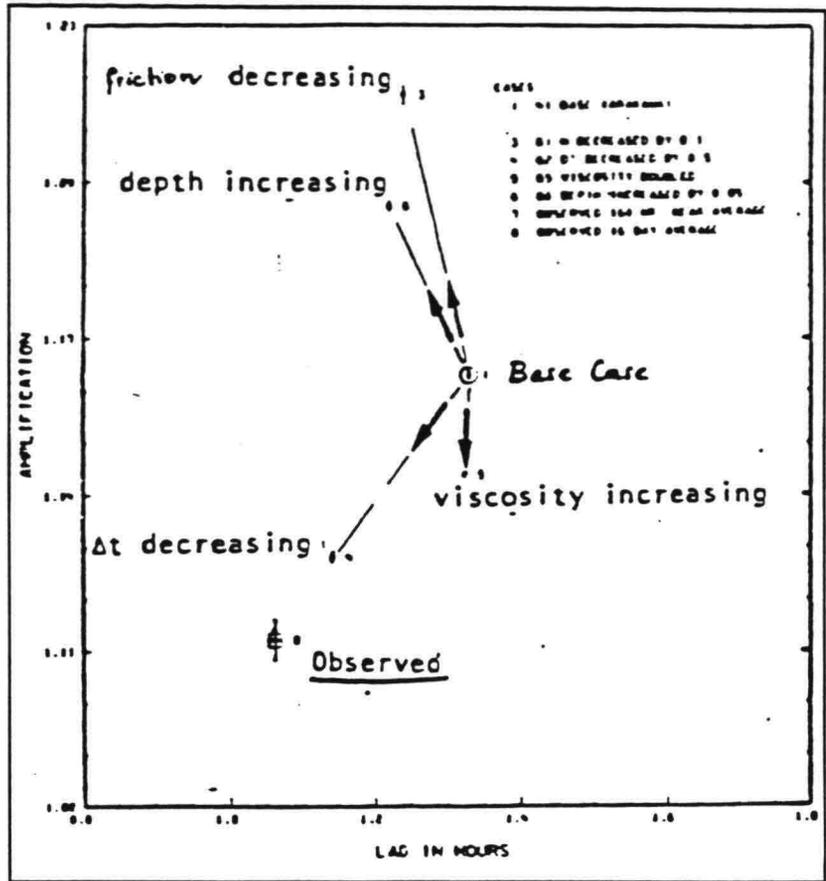


Figure 8.10 Decomposed water levels

The next step was to vary the depth, the friction and Δt . It appeared that decreasing the time step Δt was the most effective way to bring the base case closer to the observed data (in this example!).

The overall model (Figure 8.9) was used in the planning stage. For detailed flow computations, models with smaller grid sizes were utilized, which were nested in the overall model (see Figure 8.12). The overall model had a grid size of 400 m. For the flow computations during the successive building stages of the barriers, models with a grid size of 45 m's were needed, being the distance between the piers of the barrier.

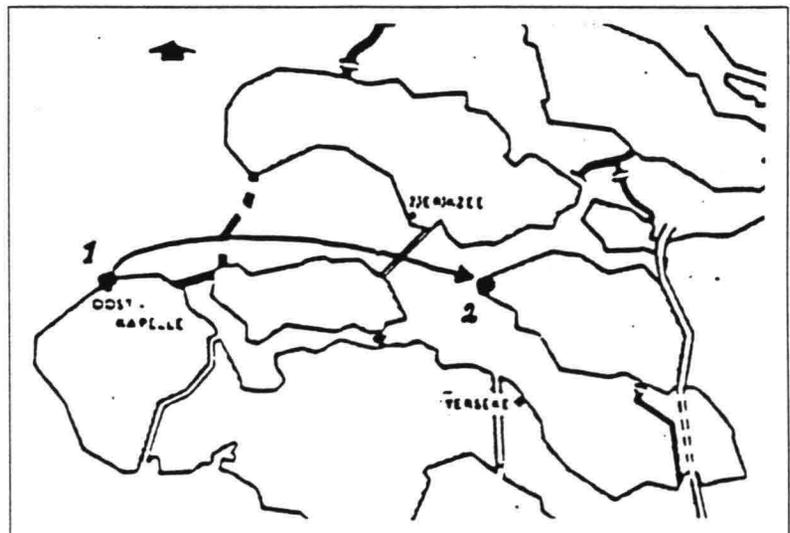


Figure 8.11 Locations 1 and 2 in the Eastern Scheldt estuary

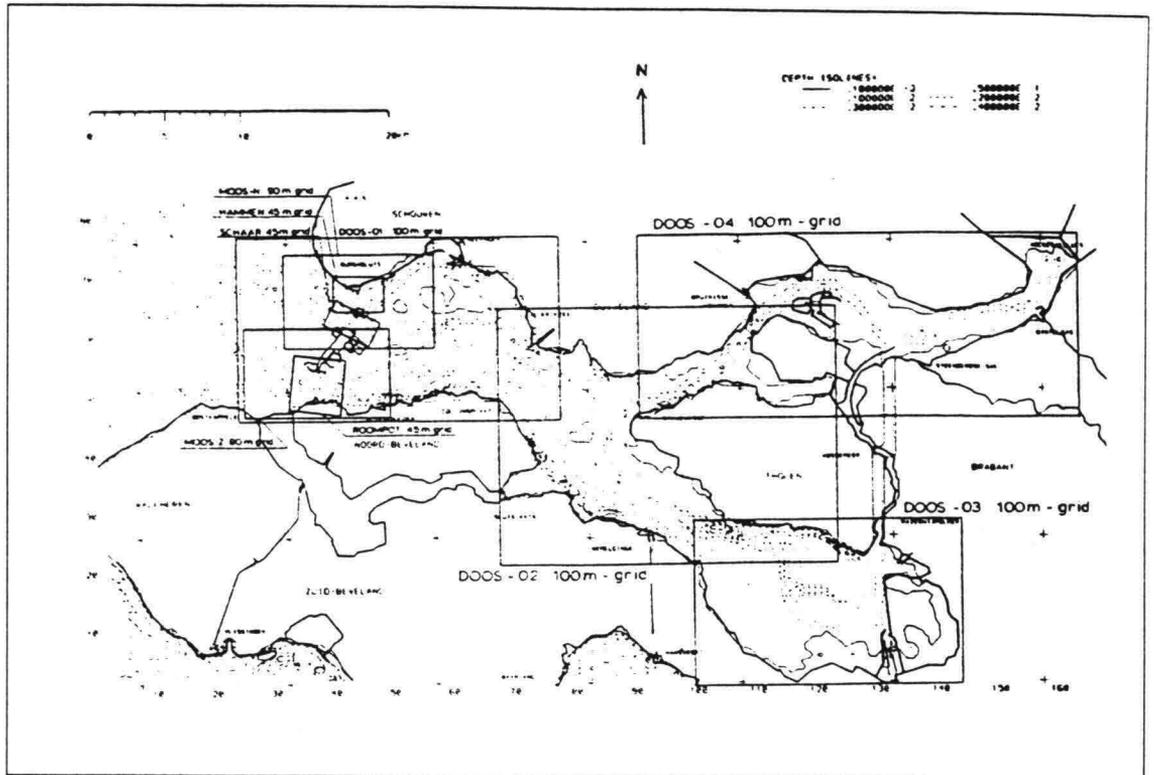


Figure 8.12 Models with smaller grid sizes nested in the overall model of the Eastern Scheldt estuary

For each tidal channel, a model with a grid size of 45 m was made. The step from 400 m to 45 m was too large for the transfer of boundary conditions in the area with complex geometry. Therefore, also models with 90 m grids were made.

Figure 8.13 presents the models in the mouth of the Eastern Scheldt. The boundary conditions for the 90 m grid models were obtained from the overall model, whereas the boundary conditions for the 45 m grid models were obtained from the 90 m grid models. A special technique was developed for the transfer of the boundary conditions to the small scale models in this area.

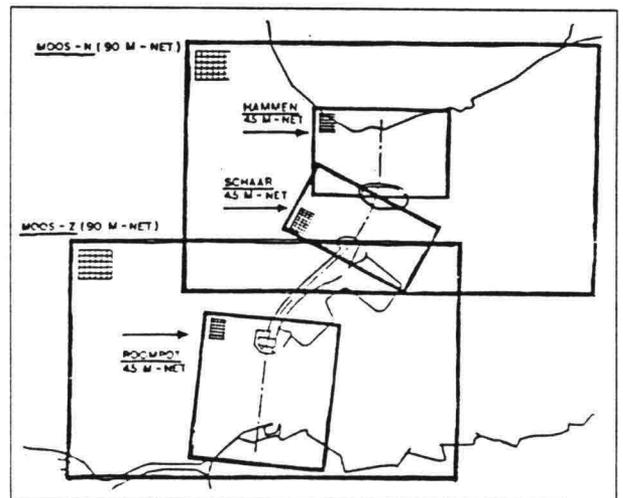


Figure 8.13 Models for the mouth of the Eastern Scheldt estuary

Figure 8.14 shows the results of a flow computation in the most northern tidal channel: the computed flow pattern during maximum ebb. The execution stage, with all piers placed, is investigated here.

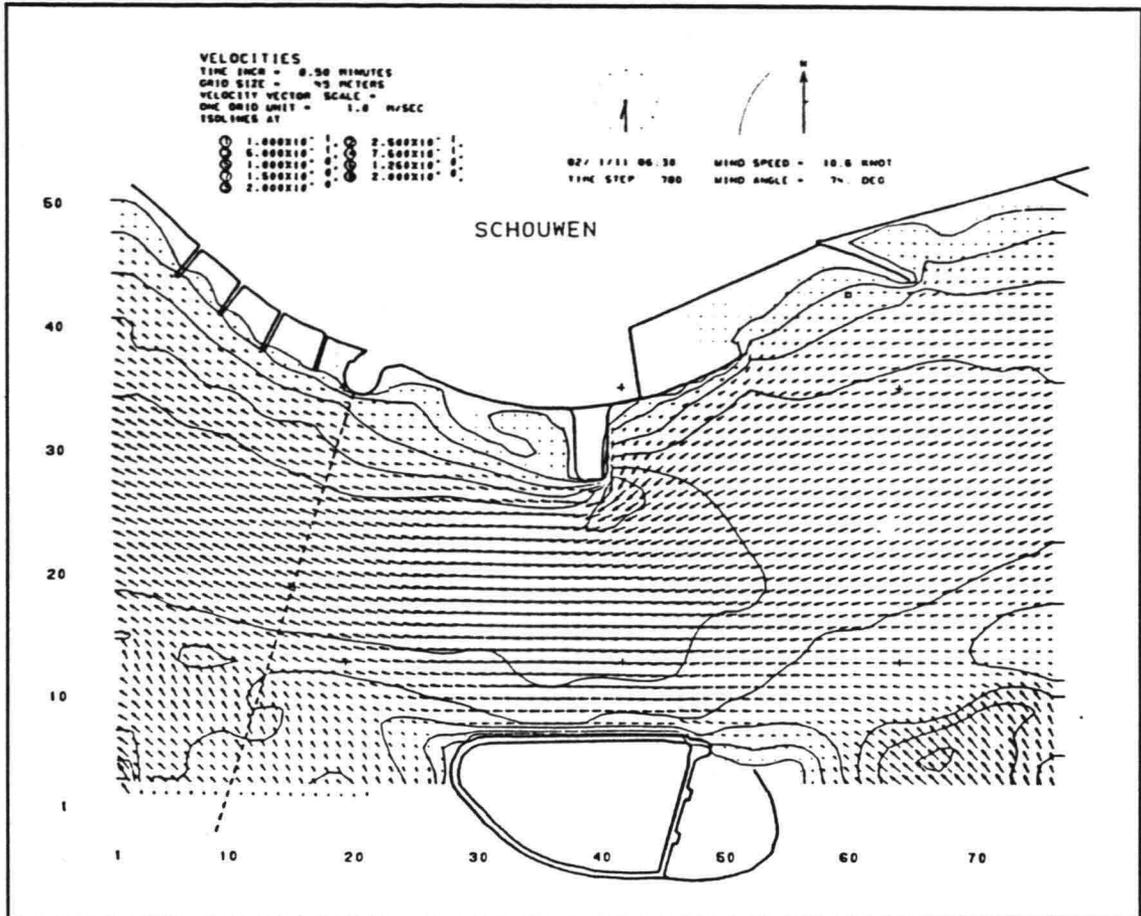


Figure 8.14 Results of flow computation in the northern tidal channel of the Eastern Scheldt estuary

8.4.2 Mathematical models used for the Sunda Shelf, South-East Asia

The second example concerns the model of the Sunda Shelf, South-East Asia. A model was needed of the coastal zone of the island of Java, Indonesia (see Figure 8.15). To obtain boundary conditions for that model, it had to be nested into a larger model of the Java Sea.

For the Java Sea model it was possible to compose boundary conditions from known components of the harmonic tide. However, monsoon winds play an important role in this sea, so the water levels at the shallow boundaries are affected by wind-set up.

To get proper boundary conditions for the wind situations of the Java Sea model, the boundaries had to be located in deep water. Therefore, first the Sunda Shelf Model had to be made, which is the large model of Figure 8.15.

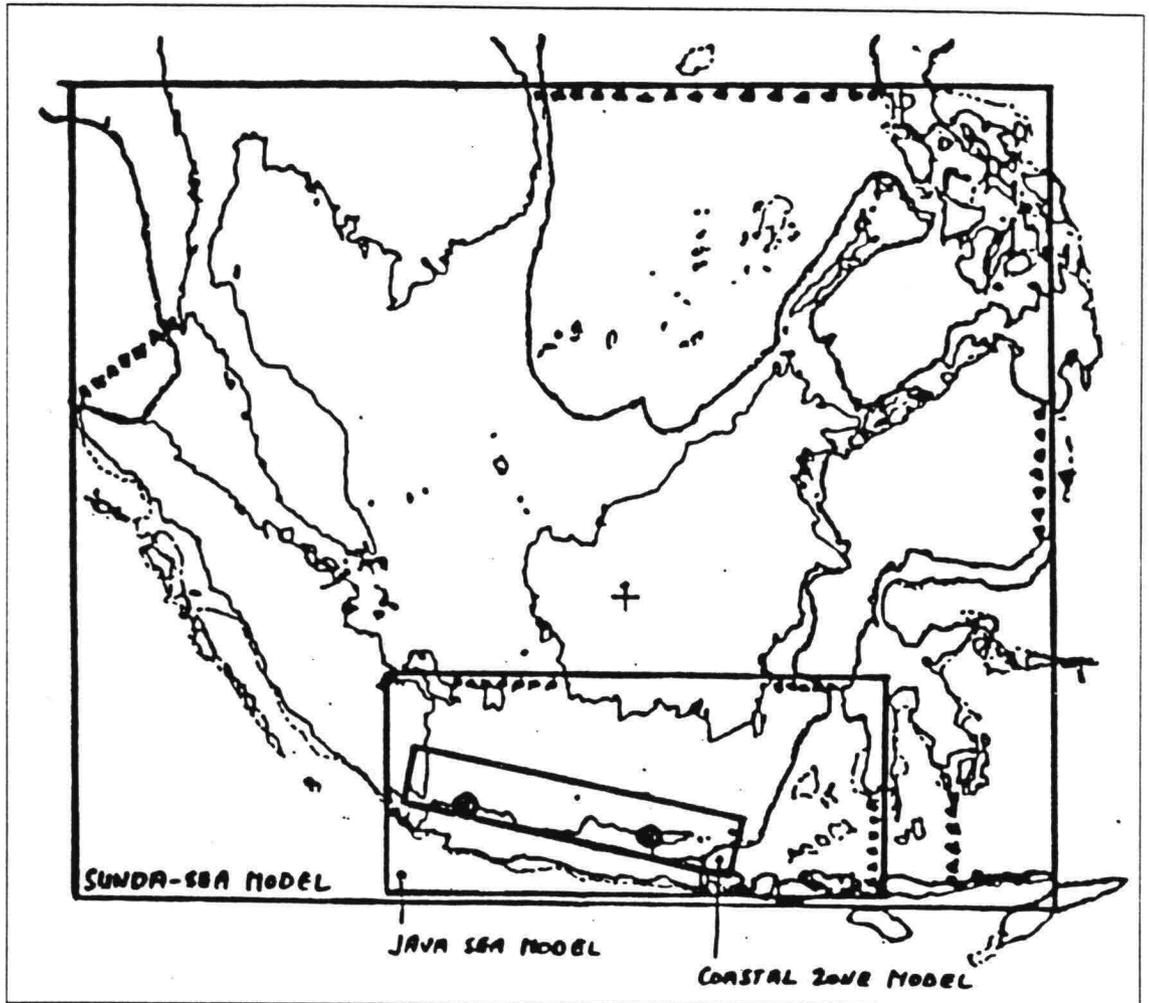


Figure 8.15 Coastal Zone model, Java Sea model and Sunda Sea model for the island of Java, Indonesia

Figure 8.16 shows the Sunda Shelf Model. Each dot represents the center of a grid cell. At the dots depths must be known, for which sea maps were used.

Some characteristics of the model are:

- grid size: 50 km²;
- number of grid points: 4000;
- time step 10 minutes.

Figure 8.17 shows the locations of the boundary conditions for the model. The black dots represent the composed water levels (●).

The water levels were composed (by a tidal prediction program) with the data of the constituents:

- diurnal O_1 , K_1 ;
- semi-diurnal M_2 , S_2 , K_2 , N_2 .

The data originate from Schwiderski, who published harmonic constituents for every degree by degree area on the globe.

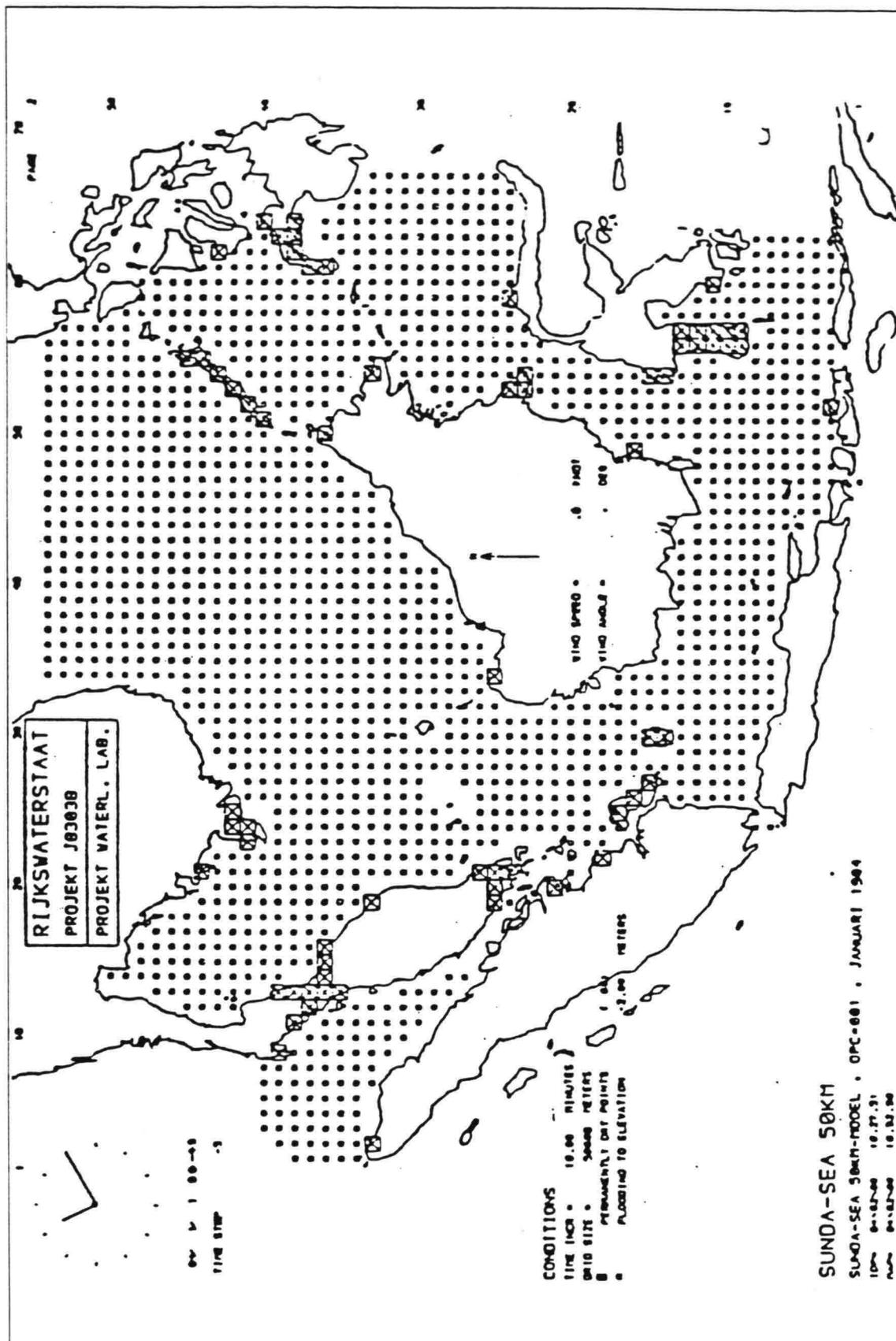


Figure 8.16 Sunda Shelf model

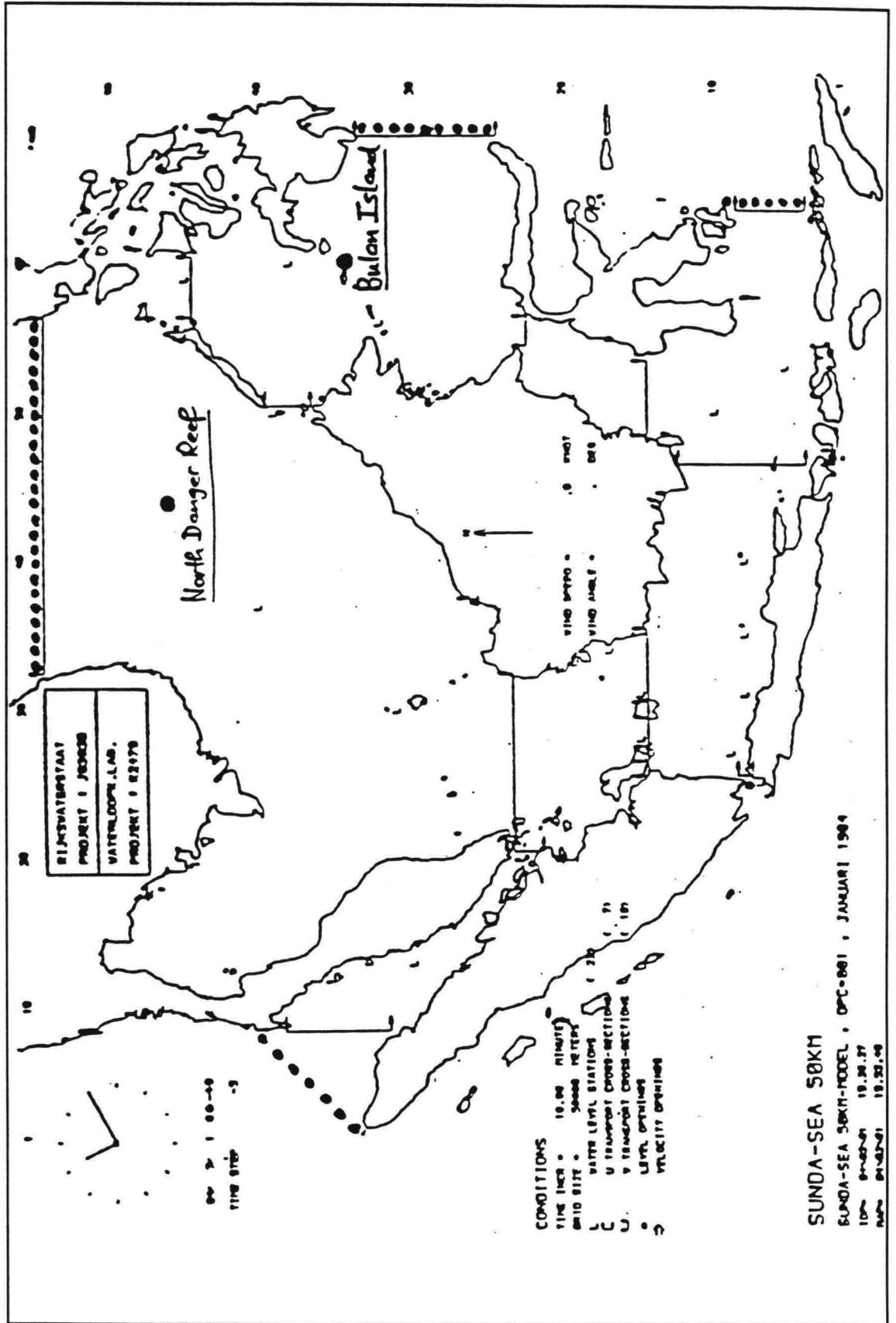


Figure 8.17 Location of boundaries in the Sunda Shelf model

Figure 8.18 shows a table from Schwiderski with amplitudes of the M_2 -component. The amplitudes are given per degree by degree area. This Table covers the area from Sri Lanka to the Philippines. There are also tables for:

- other constituents;
- other areas.

For a first calibration of the Sunda Shelf model, computed and "observed" water levels for the locations Bulan Island and North Danger Reef were used (see Figure 8.19; Bulan Island and North Danger Reef are shown in Figure 8.17; they are located in the North-Western part of the model). The solid line in Figure 8.19 is the outcome of the computations, whereas the dotted line represents the "observed" data. The agreement is reasonably, but not very good.

The term "observed" refers to the fact that no measured water levels were available in this area. Therefore, the observed water levels were composed of known harmonic components from the British Admiralty Tide Tables. These tables give amplitudes and phases for a series of components for many locations in the world. A tidal predication program was used for the composition of the observed water levels.

Numerical tidal computations

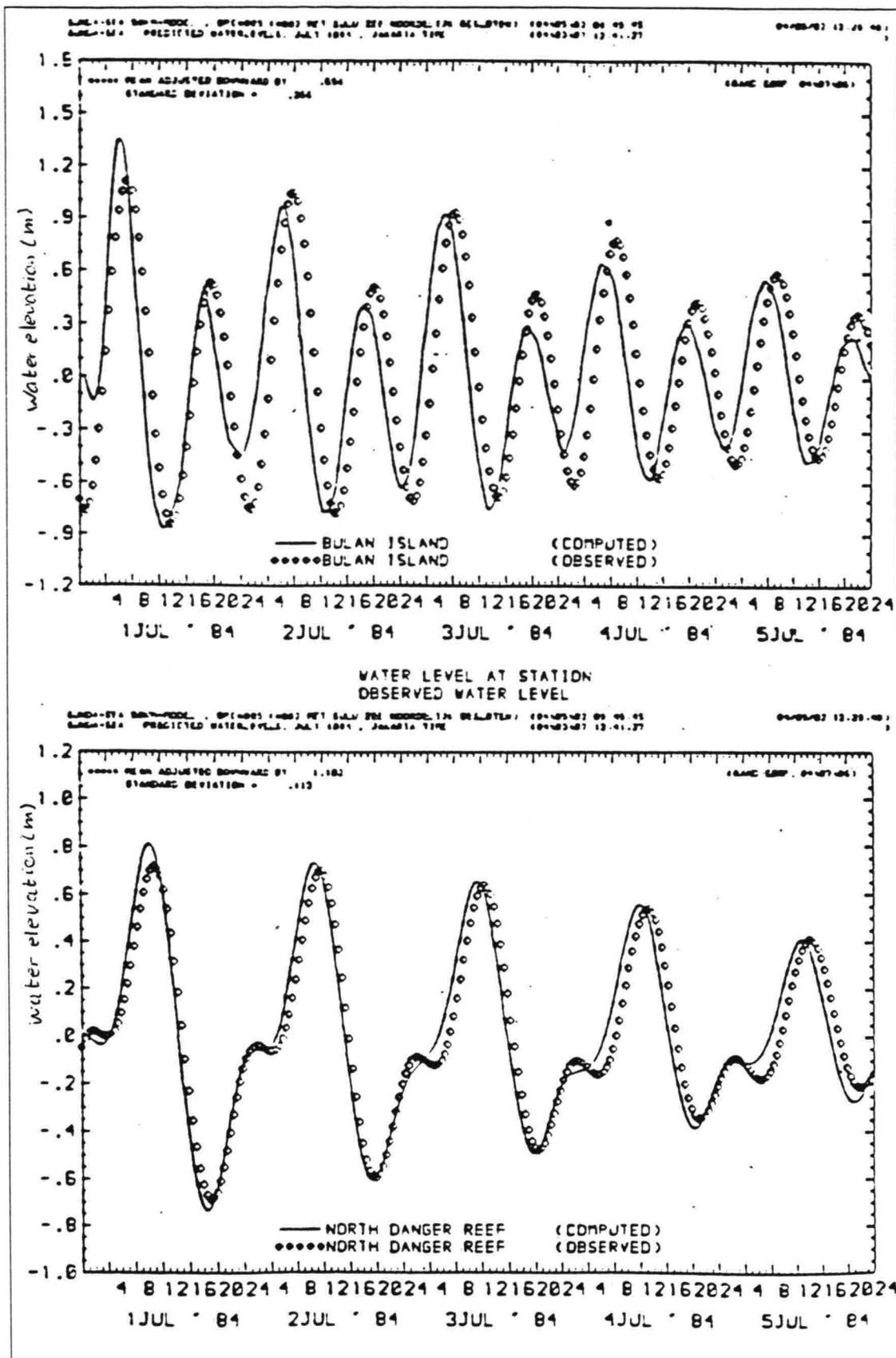


Figure 8.19 Computed and "observed" water levels for Bulan Island and North Danger Reef

References

- CHOW, VEN TE (1979). Open channel hydraulics. *McGraw-Hill, Tokyo*. IHE nr. H2.2/71
- DENBY, J. (1962). Fundamentals of celestial mechanisms. *New York*.
- DOODSON, A.T. (1921). The harmonic development of the tide generating potential. *Proceedings Royal Society, London*. Series A100, pp 306-328
- DOODSON, A.T. (19xx). The analysis of tidal observations. *Phil Trans. Royal Society, London*. Ser. A227, pp 223-279
- DOODSON, A.T. (1957). The analysis and prediction of tides in shallow water. *IHR 33*, pp 85-126
- DRONKERS, J.J. (1964). Tidal computations in rivers and coastal waters. *Elsevier, Amsterdam*. IHE nr H3.2/3
- GODIN, G. (1972). The analysis of tides. *Liverpool University Press, Liverpool*. 246pp. IHE nr H3.2/16
- HILL, M.N. (Ed)(1962). The Sea, Vol I and II. *New York*, IHE nr. H3.1/3 and H3.1/4
- IPPEN, A.T. (Ed)(1996). Estuary and coastline hydrodynamics. *New York*, IHE nr. H3.3/09 or IHE 33/47
- PILLSBURY G. (1956). Tidal hydraulics. *U.S. Army, Corps of Engineers, Waterway Experiment Station, Vicksburg, Miss*
- PUGH, D.T. (1987). Tides, surges and mean sea level. *Willey and Sons, Chichester, UK*. 472 pp TUD nr. 2009 5012
- SCHUREMAN, P. (1941). Manual of harmonic analysis and prediction of tides. *U.S. Coast and geodetic survey*. IHE nr. H3.2/42
- SCHWIDERSKI, E.W. (1979) Global ocean tides, Part 2. *Technical report NSWC-TR 79-414, U.S. Naval surface weapons centre*.

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