Analysis and Modelling of Networked Evolutionary Game Theory

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Nomenclature

Abbreviations

Abbreviation	Definition
EGT	Evolutionary Game Theory
PD	Prisoner's Dilemma
C	Cooperation
D	Defection
CGT	Cooperative game theory
NCGT	Non-cooperative game theory
NE	Nash equilibrium
ESS	Evolutionarily stable strategy
ORE	Optimal Replicator Equation
RE	Replicator Equation
BD	Bith-death
DB	Death-birth
IM	Imitation
AGOS	Averaged gradient of selection

Symbols

Symbol	Definition
N	Population size
β	Intensity of election
α	Heterogeneity
i	Number of iteration
f	Fitness
T	Temptation
S	Sucker

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Abstract

While many people believe in egoism and protectionism, the concept of ”win-win cooperation” is widely accepted worldwide. Investigating how cooperation behavior emerges and evolves can help us understand the interaction mechanism in human society.

In evolutionary game theory (EGT), the traditional methods are the **Replicator equation** and the **Moran Model**. However, these methods have limitations as they do not adequately consider the impact of network topology.

This thesis aims to investigate the cooperation behavior in different network topologies using analysis and modelling methods.

Networks in EGT are high-level representations of intricate systems, capturing the individuals in a real complex system and the relationships between them as nodes and connected edges. Each node can adopt one of two strategies: **Cooperation** or **Defection**. Nodes can also alter their strategies based on **max-payoff strategy updating rule**. Cooperation behavior is analyzed through the cooperation percentage during the generation.

We began by examining well-mixed populations. We then explored **homogeneous network** games. The results show that in well-mixed and circle networks, defection is the only stable state. However, in grid networks, there exists an infinite coexistence of cooperation and defection, which has a threshold on the size of the grid networks.

Lastly, we expanded our investigation to **heterogeneous network** games. Through numerical simulations, we demonstrated that both cooperation and defection can be stable states. We also discovered that heterogeneity does not directly promote cooperation, but rather indirectly influences it through the ”celebrity effect”.

Keywords: Evolutionary game theory, Prisoner’s Dilemma, Moran Model, Homogeneous network, Heterogeneous network

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Introduction

2.1. Game theory

Game theory (GT) was initially considered as ”the science of strategic thinking,” which emphasizes the impact of opponents’ decisions and their expectations of each other’s behavior. In the book ”The Strategy of Conflict,” [1] Thomas C. Schelling also described GT as ”the theory of interdependent decision.” Therefore, when we discuss a GT problem, we are considering the concepts of ”choice,” ”strategy,” and ”decision-making.”

The earliest research on GT can be traced back to the 19th century. In 1838, French mathematician Antoine Cournot studied the business behavior of two companies with the same product and different profits, inspired by the competition behavior in a spring water duopoly.

In the 20th century, GT was first completely and systematically analyzed. In 1944, the Hungarian mathematician John von Neumann and the economist Oskar Morgenstern published the seminal 1200-page work ”Theory of Games and Economic Behavior,” [2] which is the foundation for the theoretical framework of GT. In addition, some of the ideas in [2] originated from von Neumann’s previous paper ”On the Theory of Parlor Games,” published in 1928. Therefore, many scholars believe that 1928 is the true birth year of GT.

In the latter half of the 20th century, GT experienced rapid development. In 1951, American mathematician John Nash introduced the concept of the ”Nash equilibrium (NE)” (See subsection 2.1.3). It characterizes a state in which self-interested players reach a stable outcome through mutual interaction. This means that no player can enhance their profits by altering their strategies. The introduction of Nash equilibrium significantly advanced and refined the related theories of GT.

GT was further refined in the following years by other mathematicians such as Lloyd S. Shapley. He proposed the Shapley value, which describes how coalitional competitive forces influence the potential outcomes of a cooperative game.

By the end of the 20th century, GT evolved to a more structured theory. It became a crucial field of research in mathematics and economics. Remarkably, in 1994, economists Reinhard Selten and John Harsanyi received the Nobel Prize in Economics for their contributions to GT, which represents widespread academic recognition and acceptance of GT.

In the 21st century, GT has been used in interdisciplinary fields, including computer science. In 2009, Walid Saad [3] introduced GT concepts to communication networks. In 2013, Mohammad Hossein Manshaei [4] further used GT to analyze network security and privacy. During the same year, Xiannuan Liang [5] used GT to defend against cyber attacks.

In summary, even though strategic thinking in science is a prototype of GT, the formal discipline of GT was established and developed in the 1940s. It then became an integral part of mainstream economics by the end of the 20th century.

2.1.1. Classification

We use Olivier Chatain's categorization in his 2014 paper *Cooperative and Non-Cooperative Game Theory* [6], to classify GT into two branches: cooperative game theory (CGT) and non-cooperative game theory (NCGT).

- **Cooperative game theory (CGT)**The central concept of CGT is establishing a **binding contract** that guarantees an equitable division of the surplus value created by a cooperative group. Key issues in CGT focus on identifying stable groups and fairly distributing the resulting surplus value (For example, Sharply value mentioned in subsection 2.1).

- **Non-cooperative game theory (NCGT)** Players in a NCGT are not bound by a contract, so they **prioritize player gains**. It's important to note that cooperative and non-cooperative games are distinguished not by whether the outcome involves cooperation, but by the presence or absence of a contract to coordinate players' strategic behavior. Cooperation can still occur in non-cooperative games (which is the main argument of this dissertation), but it must be self-enforcing.

This thesis report focuses on non-cooperative games, specifically **two-person-non-zero-sum games** (NCGT can be subdivided into zero-sum and non-zero-sum games). The prisoner's Dilemma (PD) is one of the classical examples of two-person-non-zero-sum games (See subsection 2.1.4).

2.1.2. The strategic form

Game theory (GT) studies a mathematical model that encompasses the science of strategic thinking, consisting of three essential components: players, strategies, and payoffs. To represent a game incorporating these elements, a strategy form is used.

A strategy form is often presented as a payoff matrix, in which the rows and columns correspond to the strategies available to player 1 and player 2, respectively.

The entries a_{ij} and b_{ij} in the matrix represent the gain for player 1 and the cost for player 2 when player 1 selects strategy i and player 2 selects strategy j.

Suppose Player 1 has m strategies, and Player 2 has n strategies then strategy form is the payoff matrix $A \in \mathbb{R}^{m \times n}$ as shown in 2.1.

$$A = \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & \cdots & (a_{1n}, b_{1n}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & \cdots & (a_{2n}, b_{2n}) \\ \vdots & & \ddots & \vdots \\ (a_{m1}, b_{m1}) & (a_{m2}, b_{m2}) & \cdots & (a_{mn}, b_{mn}) \end{pmatrix} \tag{2.1}$$

2.1.3. Nash equilibrium

In GT, the central concept is the **Nash equilibrium (NE)**, which is defined as a set of strategies where no player can have a higher payoff by changing their strategy while others keep the same strategies. That is, each player selects the best response based on the choices of the other players.

Definition 1. *A strategy profile $s^* = (s_1^*, \ldots s_n^*)$ is a NE if and only if s_i^* is a best response to $s_{-i}^* = \left(s_1^*, \ldots s_{i-1}^*, s_{i+1}^*, \ldots s_N^*\right)$ for each i. That is, for all i, expected payoff u_i satisfies,*

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right) \quad \forall s_i \in S_i.$$

2.1.4. Prisoner's Dilemma

Next, we will use the **Prisoner's Dilemma (PD)** as an example to illustrate the GT. PD, also known as the Prisoner's Dilemma, it was first introduced by American mathematician Albert W. Tucker [7]. In this scenario, two players, noted as "prisoners," are apprehended under suspicion of a crime and have two options: to confess or not. Despite both aiming to minimize their time in jail (Maximum the profits), they paradoxically choose to confess. This highlights the conflict between player rationality and collective rationality.

The strategy form of PD can be written as follows.

	Confess	Not Confess
Confess	(5,5)	(1,20)
NotConfess	(20,1)	(2,2)

(2.2)

In the PD, the elements in 2.2, such as (1,20), represent the payoff for player 1 and player 2, respectively, in terms of years in jail. Regardless of player 2's decision, it's always in the best interest of player 1 to confess (because 20>5, 2>1). Player 2 faces a similar scenario. Consequently, we can prove the confession as the dominant strategy, and the strategy pair (confess, confess) as the NE in this game. However, (confess, confess) does not yield minimal jail time. It's important to note that the optimal scenario is (not confess, not confess) since 5+5>2+2.

2.1.5. Limitations of GT

Classical GT is based on the assumption that every player is perfectly rational. However, this assumption is no longer true in real-world scenarios. Human decision-making is influenced by environmental factors and irrationality. For instance, in biological evolution, equilibrium is not achieved instantly, but rather through a series of adjustments and gradual improvements. Thus, biologists and mathematicians further developed the biological evolution theories of Evolutionary Game Theory (EGT).

2.2. Evolutionary Game Theory

The formation and development of Evolutionary Game Theory (EGT) can be divided into three stages.

During stage 1 (Before 1961), biologists got inspiration from GT in economics. They applied GT to biology and constructed evolutionary models such as the predator-animal competition model. This stage is the application of GT in biology.

Stage 2 (1961 - 1990) biologists reformulating GT, redefining classical definitions such as NE to evolutionarily stable equilibrium (See subsection 2.2.1). This stage is the true birth of EGT. In 1961, Lewontin's paper "Evolution and the Theory of Games" [8] depicted genetic polymorphism as a stochastic strategy for adapting to changing environments, using the concept of fitness in biology to model changes in cooperation within populations. Lewontin is widely regarded as the forerunner of EGT.

In stage 3 (1990 - Now), economists and mathematicians got inspiration from biology and reintroduced evolutionary game theory (EGT) back to economics, contributing to its further development. An example of this progress is the shift from deterministic replicator dynamics [9] to frequency-based dynamics [10], which we will discuss in subsections 3.2.1 and 3.2.2.

In summary, EGT is an extension of GT. EGT provides a theoretical framework for studying the evolution of strategic behaviors in systems. The key idea behind EGT is how players make choices to adapt to their environment. In contrast to GT, rationality is not taken into account in EGT. Instead, the combination of genotype and network dynamics governs players' strategy choices.

2.2.1. Evolutionarily Stable Strategy

In 1972, Maynard Smith introduced the concept of an **evolutionarily stable strategy (ESS)** in the paper "The Logic of Animal Conflict" [11], which is an estimation of evolutionary stability. Unlike NE in GT, ESS is not a strategy profile but a certain (mixed) strategy.

Definition 2. *There are two conditions for a strategy x to be an ESS. For all $y \neq x$, either $u_i(x, x) > u_i(y, x)$ or $u_i(x, x) = u_i(y, x)$ and $u_i(x, y) > u_i(y, y)$.*

2.2.2. Hawk-Dove Game

Next, we use Hawk-Dove Game as an example to illustrate the EGT. This game, initially explored in the paper "The Logic of Animal Conflict" [11], is also commonly referred to as the Chicken game or the Snowdrift game. Within this game, two clearly defined genotypes, Hawks and Doves, engage in competition for resources, adhering to the following set of rules:

- When a Hawk competes with a Dove, the Hawk takes the whole of the resource. (The Dove is afraid to fight, leaving the whole of the resource to the Hawk.)

- When a Dove competes with a Dove, they share the resource equally.
- When a Hawk competes with a Hawk, they fight and have an equal chance of obtaining the resource or being injured.

Suppose the total resource is V and the cost of possible injury is C then the payoff matrix is shown as follows:

$$\begin{array}{lcc} & \text{Hawk} & \text{Dove} \\ \text{Hawk} & ((\text{V}-\text{C})/2, (\text{V}-\text{C})/2) & (V, 0) \\ \text{Dove} & (0, V) & (V/2, V/2) \end{array} \tag{2.3}$$

In the payoff matrix 2.3, if $V > C$, then $(V - C)/2 > 0$ and $V > V/2$, which means that Hawk is the dominant strategy for player 1. This applies to player 2 as well. Therefore, Hawk is dominant for both players, and the NE is (Hawk, Hawk). If $V < C$, then $(V - C)/2 < 0$ but $V > V/2$, indicating that Hawk is the best response for player 1 when player 2 chooses Dove, and Dove is the best response for player 1 when player 2 chooses Hawk. Consequently, the NE are (Dove, Hawk) and (Hawk, Dove).

So what about the evolutionary stability?

Consider the Hawk-Dove game when $V = 2, C = 6$ as follows:

$$\begin{array}{lcc} & \text{Hawk} & \text{Dove} \\ \text{Hawk} & (-2, -2) & (2, 0) \\ \text{Dove} & (0, 2) & (1, 1) \end{array} \tag{2.4}$$

Suppose we have a mixed strategy. For player 1, the probability of playing with Hawk is p and the probability of playing with Dove is $1-p$. Then, the expected payoffs of player 1 are $-2p + 2(1-p)$ and $1-p$ respectively. Therefore, we can construct an equation:

$$-2p + 2(1-p) = 1 - p \longrightarrow p = \frac{1}{3} \tag{2.5}$$

Therefore, player 1 choose Hawk with probability $\frac{1}{3}$ and Dove with probability $\frac{2}{3}$.

Use the definitions mentioned in subsection 2.2.1, and calculate the expected payoff of $u_i\left(\frac{1}{3}, q\right)$ and $u_i(q, q)$.

$$\begin{gathered} u_i\left(\frac{1}{3}, q\right) = \frac{1}{3}[q(-2) + (1-q)(2)] + \frac{2}{3}[q(0) + (1-q)(1)] = \frac{4}{3} - q \\ u_i(q, q) = q[q(-2) + (1-q)(2)] + (1-q)[q(0) + (1-q)(1)] = 1 - 4q - 3q^2 \end{gathered} \tag{2.6}$$

We need $u_i(\frac{1}{3}, q) > u_i(q, q)$ for all $q \neq p$.

$$
\begin{aligned}
u_i(\frac{1}{3}, q) &> u_i(q, q) \\
\frac{4}{3} - 2q &> 1 - 4q - 3q^2 \\
3q^2 - 2q - \frac{1}{3} &> 0 \\
9q^2 - 6q - 1 &> 0 \\
(3q - 1)^2 &> 0
\end{aligned}
$$

The inequality is always true as long as $q \neq \frac{1}{3}$. Thus, the population is evolutionarily stable when there are $\frac{1}{3}$ Hawks and $\frac{2}{3}$ Doves.

2.2.3. Generic payoff matrix

In subsections 2.1.4 and 2.2.2, we discussed two well-known games, the Prisoner's Dilemma (PD) and the Hawk-Dove Game. In addition to these games, there are other games such as the Stag-hunt game, introduced by Jean-Jacques Rousseau. In a scenario described by Rousseau, two hunters must choose between working together to chase a stag and going alone to hunt a hare. The conflict between mutual aid and player gain is exemplified by the game.

Can we create a generic payoff matrix that encompasses all these games? The answer is yes!

Suppose we have a generic payoff matrix as follows:

$$
\begin{array}{ccc}
 & C & D \\
C & (1,1) & (S,T) \\
D & (T,S) & (0,0)
\end{array}
\tag{2.7}
$$

In 2.7, Player 1 and Player 2 each have 2 strategies. C represents cooperation, and D represents defection. If both players cooperate, they each receive a **reward** (R) of 1. If both players defect, they each receive a **punishment** (P) of 0. When one player cooperates and the other defects, the defecting player receives **Temptation** (T), and the cooperating player receives **Sucker** (S).

The Temptation T can range from 0 to 2, and Sucker S can range from -1 to 1. The values of S and T vary depending on the specific game being played. For the Prisoner's Dilemma game, it is well-known that payoffs satisfy $T > 1 > 0 > S$. In the Dove-Hawk (Snowdrift) game, the conditions are $T > 1 > S > 0$. For the Harmony game and Stag-Hunt game, the conditions are $1 > T > S > 0$ and $1 > T > 0 > S$ respectively. Figure 2.1 provides an illustration of the generic payoff matrix, which is presented in a clockwise order starting with Stag Hunt: Stag Hunt - Harmony - Snowdrift - PD.

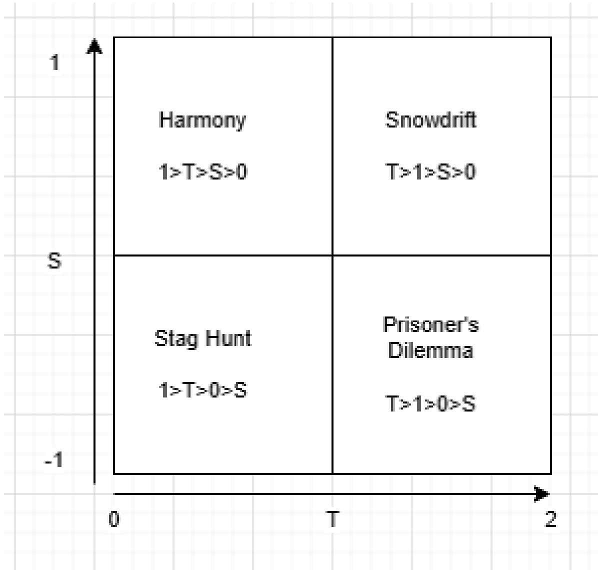


Figure 2.1: Generic payoff matrix for symmetric two-player games.

2.3. Research Questions

This thesis uses modelling and analytical techniques to investigate the evolutionary dynamics of EGT in complex network architecture. In order to represent players inside an actual complex system and their relationships as nodes and connected edges, respectively, complex networks are used as high-level abstractions of complex systems. Cooperation and Defection are the two techniques that define each node. Nodes can use the max-payoff updating rule to change their strategy.

In particular, we are interested in investigating cooperative behavior in the context of the Prisoner's Dilemma (PD) in complex networks. The PD, which is discussed in subsection 2.2, is an important tool for understanding human collaboration, a topic that crosses both science and philosophy. Moreover, characterizing evolutionary dynamics within populations requires an understanding of cooperation behavior. Notably, during the 125th anniversary of Science magazine in 2005, the topic "How did cooperative behavior evolve?" was selected as one of the most significant 25 questions.

As a result, the thesis question can be stated as follows:

● **Thesis Question:** *How does cooperative behavior depend on the topology of the network?*

In order to solve the thesis question, we can divide the thesis question into 2 sub-questions.

● **Sub Question 1:** *How did cooperative behavior evolve in the* ***homogeneous networks*** *under PD condition?*

First, we start with a simple homogeneous network. It is a symmetrical network in which all the nodes have the same environment. This network topology can help us simplify the questions.

Definition 3. *A network is called a homogeneous network if all its nodes have the same number of degrees.*

Definition 4. *The degree of a node can be written as*

$$d_i = \sum_{j=1}^{N} a_{ij}$$

If i and j are connected, $i \neq j$, then we have $a_{ij} = 1$, unless $a_{ij} = 0, N$ is the number of nodes in the network.

Examples shown in Figure 2.2 are the classical homogeneous network: Circle network (Left) and well-mixed network (Right). For the circle network, each node has degree 2. For the well-mixed network, each node has degree 4.

(a) Circle network **(b)** Well-mixed network

Figure 2.2: Examples of homogeneous network

● **Sub Question 2:** *How did cooperative behavior evolve in the* ***heterogeneous networks*** *under PD condition?*

Based on our analysis of homogeneous networks, we can continue with a more complicated heterogeneous network. It is an asymmetrical network in which all the nodes have different environments. Unlike the homogeneous network, a heterogeneous network is not a rare but a more general case. Most of the networks in reality are heterogeneous such as population networks.

Definition 5. *A network, that is not homogeneous, is called a heterogeneous network.*

Examples shown in Figure 2.3 are two classical heterogeneous networks: Linear network (Left) and Star-like network (Right). For the Linear network, node 1 and node 4 have degree 1. However, node 2 and node 3 have degree 2. For the star-like network, node 1 has degree 3 and node 2,3,4 have degree 1.

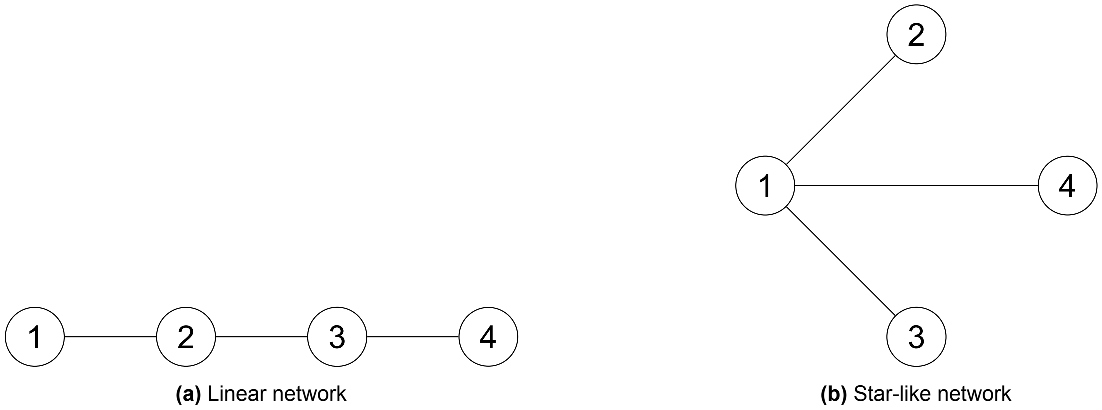


Figure 2.3: Examples of heterogeneous network

2.4. Research outline

To answer the research question of this thesis report, we will use a literature review, network analysis, and also build a simulation model. The outline of this thesis report is the following:

- In Chapter 1, we give a general overview.
- In Chapter 2, we introduce the backgrounds of GT and EGT.
- In Chapter 3, we conduct a literature review, to give a comprehensive idea of past research on networked EGT.
- In Chapter 4, we discuss the network analysis of heterogeneous networks.
- In Chapter 5, we present the analytic and simulation results of homogeneous and heterogeneous networks
- In Chapter 6, we make a conclusion.
- Reference and Appendix.

2.5. Summary

This chapter covered the fundamentals of homogeneous networks, heterogeneous networks, NE, ESS, and an introduction to GT and EGT. These ideas serve as the foundation for the analysis and modelling we will do in the upcoming chapters. In conclusion, we presented the primary thesis question along with two sub-questions.

3

Literature review

At the beginning, we make a literature review. This step allows us to gain a broad understanding of the topic and determine a specific research direction. In this thesis report, our literature review has three distinct aspects: the Replicator and Moran model, homogeneous network, and heterogeneous network. The list of literature is presented in Table 3.1 below.

Table 3.1: Overview of Papers and Perspectives

Name of The Paper	**Aspects**
Modeling, Analysis and Control of Networked Evolutionary Games [12] (See section 3.1)	Mathematical backgrounds
An Optimal Strategy to Solve the Prisoner's Dilemma [9] (See subsection 3.2.1)	Replicator Model
Coevolutionary Dynamics: From Finite to Infinite Populations [10] (See subsection 3.2.2)	Moran Model
Evolutionary games on cycles [13] (See section 3.3)	Homogeneous network
Evolutionary games and spatial chaos [14] (See section 3.3)	Homogeneous network
Spatial games and the maintenance of cooperation [15] (See section 3.3)	Homogeneous network
Evolutionary prisoner's dilemma game on a square lattice [16] (See section 3.3)	Homogeneous network
Evolutionary prisoner's dilemma game with dynamic preferential selection [17] (See section 3.3)	Homogeneous network
From Local to Global Dilemmas in Social Networks [18] (See section 3.4)	Heterogeneous network
Heterogeneous networks do not promote cooperation when humans play a Prisoner's Dilemma [19] (See section 3.4)	Heterogeneous network

3.1. Mathematical backgrounds

We use the paper ”Modeling, Analysis and Control of Networked Evolutionary Games” [12] for the mathematical foundations of networked EGT. In this paper, the authors thoroughly examined the rigorous definitions and theorems that we will utilize in our thesis project. Therefore, it serves as a ”Handbook” for us to begin with.

Before investigating networked EGT, it’s critical to comprehend the essence of a network. Our definition of networks is a combination of nodes and edges, as stated in subsection 2.3. In the paper [12], the authors described the network graph in the following way:

Given a set N and $E \subset N \times N, (N, E)$ is called a graph, where N is the set of nodes and E the set of edges. If $(i, j) \in E$ implies $(j, i) \in E$ the graph is undirected, otherwise, it is directed. Let $N' \subset N$, and $E' = (N' \times N') \cap E$. Then (N', E') is called a sub-graph of (N, E). Briefly, N' is a subgraph of N.

The main contents we will quote from this paper consist of 6 parts.

Part 1 is about the neighborhood node. For a node in networked EGT, only the neighborhood nodes can interact with it. The term neighborhood refers to the nodes that have edges to the target node.

Definition 6. *Let N be the set of nodes in a network, $E \subset N \times N$ the set of edges. $j \in N$ is called a neighborhood node of i, if either $(i, j) \in E$ or $(j, i) \in E$. The set of neighborhood nodes of i is called the neighborhood of i, denoted by $U(i)$. Throughout this paper it is assumed that $i \in U(i)$.*

Part 2 is about the fundamental network game (FNG). In subsection 2.1, we stated that GT has three core elements: players, strategies, and payoffs. Here is the same. FNG can be considered as a fundamental two-player game with the same three core elements as follows [12]:

1. n players $N = \{1, 2, \cdots, n\}$;
2. Player i has $S_i = \{1, \cdots, k_i\}$ strategies, $i = 1, \cdots, n, S := \prod_{i=1}^{n} S_i$ is the set of profiles;
3. Player i has its payoff function $c_i : S \to \mathbb{R}, i = 1, \cdots, n, c := (c_1, c_2, \cdots, c_n)$.

Definition 7. *(i) A normal game with two players is called a fundamental network game (FNG), if*

$$S_1 = S_2 := S_0 = \{1, 2, \cdots, k\}.$$

(ii) An FNG is symmetric, if

$$c_1(x, y) = c_2(y, x), \quad \forall x, y \in S_0.$$

Part 3 is about the overall payoff. In networked EGT, a node can have multiple neighborhood nodes (See Definition 6). Therefore, the overall payoff can be calculated by the sum of the payoff interacting with multiple neighborhood nodes.

Definition 8. *Let $c_{i,j}$ be the payoff of the FNG between i and j. $U(i)$ is the neighborhood of player i. Then the overall payoff of player i is*

$$c_i = \sum_{j \in U(i) \setminus i} c_{ij}, \quad i \in N.$$

Part 4 is about the strategy updating rule. For a node in networked EGT, it can change the strategy according to the strategy updating rule. It is vital to the cooperation evolution in networked EGT since two different strategy updating rules can lead to two completely opposite cooperation evolution. Examples are the cooperation updating rule (a node chooses cooperation strategy no matter what payoff or strategy it has) and the defection updating rule (a node chooses defection strategy no matter what payoff or strategy it has).

Definition 9. *A strategy updating rule for a Networked EGT, denoted as π, is a set of mappings:*

$$x_i(t+1) = f_i\left(\{x_j(t), c_j(t) \mid j \in U(i)\}\right), \quad t \geq 0, \quad i \in N.$$

That is, the strategy of each player at time $t+1$ depends on its neighborhood players' information at t, including their strategies and payoffs (See Definition 8).

Part 5 is about the networked EGT. Unlike FNG (See Definition 7), networked EGT has extra information strategy updating rules (See Definition 9) as follows:

Definition 10. *A networked evolutionary game, denoted by $((N, E), G, \Pi)$, consists of three parts*

1. *a network (graph) (N, E);*
2. *an FNG, G, such that if $(i, j) \in E$, then i and j play FNG repetitively with strategies $x_i(t)$ and $x_j(t)$ respectively. Particularly, if the FNG is not symmetric, then the corresponding network must be directed to show i is player one and j is player two;*
3. *a local information-based strategy updating rule, which is expressed in 9.*

Part 6 is about the evolutionary dynamics. Since we know the network (See Definition 10), overall payoff (See Definition 8), and the strategy updating rule (See Definition 9), we can determine how the cooperation behavior evolves in networked EGT.

Theorem 1. *The evolutionary dynamics can be expressed as*

$$x_i(t+1) = f_i\left(\{x_j(t) \mid j \in U_2(i)\}\right), \quad i \in N.$$

The proof of this theorem is provided in [12]. Evolutionary dynamics are employed throughout this paper to analyze the dynamic behaviors of the network. Additionally, various examples are also presented to illustrate the theoretical results.

Overall, the main contribution of this paper is the presentation of a mathematical model for networked EGT. These definitions and theorems remain consistent in the subsequent sections.

3.2. Evolutionary dynamics of well-mixed populations in PD

We use two papers to present a simple model for describing evolutionary dynamics in the context of the Prisoner's Dilemma. The paper ”An Optimal Strategy to Solve the Prisoner's Dilemma” [9] used an extension of the **Replicator Equation (RE)** known as the **Optimal Replicator Equation (ORE)**. The paper ”Coevolutionary Dynamics: From Finite to Infinite Populations” [10] used the **Moran process** in Evolutionary Game Theory (EGT) research. They both use deterministic and stochastic respectively to analyze the cooperation behavior. Even though the relationship between the frequency-dependent evolutionary dynamics and deterministic replicator dynamics is still unclear, the paper ”Coevolutionary Dynamics: From Finite to Infinite Populations” suggests that different microscopic stochastic processes lead to the standard or adjusted replicator dynamics.

3.2.1. An Optimal Strategy to Solve the Prisoner's Dilemma

The paper [9] aims to analyze the PD from a dynamic point of view. It compares two approaches **Replicator Equation (RE)** and **Optimal Replicator Equation (ORE)**, which is an extension of RE and motivated by the understanding that evolution occurs not only at the individual level within a population but also between competing populations.

The main contents we will quote from this paper consist of 2 parts.

Part 1 is about RE. The fundamental equation of evolutionary dynamics [9] is the RE

$$\dot{x}_a = x_a\left(f_a(\mathbf{x}) - \langle \mathbf{f} \rangle\right) \quad a = 1, 2, \tag{3.1}$$

where x_a is the relative abundance (frequency) of players of type a, $f_a(\mathbf{x})$ is the (frequency-dependent) fitness of type a, $\langle \mathbf{f} \rangle = x_1 f_1 + x_2 f_2$ is the average fitness of the population and the overdot denotes time derivative.

For the PD, there are only two possible strategies: C or D . Thus authors label the frequency of the population adopting each strategy as x_C and x_D respectively. Moreover, the fitness of each strategy is obtained from the payoff matrix (See subsection 2.1) by combining each payoff with the probability that the opponent chooses the corresponding strategy [9] as follows:

$$f_C(\mathbf{x}) = Rx_C + Sx_D, \quad f_D(\mathbf{x}) = Tx_C + Px_D. \tag{3.2}$$

Without loss of generality, authors assume $S = 0$. Using the fact that $x_C + x_D = 1$, after some algebra they rewrite the RE and the fitnesses in terms of x_C only, obtaining the following evolution equation [9] for the frequency of cooperators:

$$\dot{x}_C = -x_C\,(1 - x_C)\,[(T - R)x_C + P\,(1 - x_C)]\,, \tag{3.3}$$

Part 2 is about ORE. Optimal Control Theory (OCT) is applied to extend the RE in 3.1 to ORE and describes the dynamical equations for the coevolution of the frequencies x in the RE and the fitnesses f.

For $a = C = D$, as in the case of the PD, the dynamic system [9] is:

$$\begin{aligned} f_a &= \frac{1}{2}p_a, \\ \dot{x}_a &= \frac{x_a}{2}\left(p_a - \langle \mathbf{p} \rangle\right), \\ \dot{p}_a &= \frac{p_a}{2}\left(\langle \mathbf{p} \rangle - \frac{p_a}{2}\right). \end{aligned} \tag{3.4}$$

The additional variables p_C and p_D are usually called the co-states (See [9] for e Supplementary Information). The system in 3.4, together with the initial conditions $x_a(0) = x_a^0$ and the terminal conditions:

$$p_a(\tau) = \frac{\partial g(\mathbf{x}(\tau))}{\partial x_a}, \quad g(\mathbf{x}) := \langle \mathbf{f} \rangle \tag{3.5}$$

is the Optimal Replicator Equation (ORE).

A significant result in this paper is the introduction of the new ORE model for natural selection, which directly leads to a simple and natural guideline for the emergence of cooperation in the PD. Recall in subsection 2.1.4, we observed that both defections are the Nash Equilibrium (NE). However, paradox to conventional belief (all defection), this paper demonstrates that cooperative behavior can indeed emerge.

The RE and ORE in this paper are deterministic approaches. However, in reality, the evolution in biology or sociology is influenced by randomness, such as mutations. A beneficial mutation might quickly vanish due to random drift, but it also has the potential to persist and spread throughout the entire population. To account for this randomness, we introduce the Moran process in the next subsection 3.2.2.

3.2.2. Coevolutionary Dynamics: From Finite to Infinite Populations

Assuming a finite but constant population size the balance between selection and drift can be described by the **Moran process**. The paper [10] described the microscopic dynamics of the Moran process with three simple steps:

1. **Selection:** a player is randomly selected for reproduction with a probability proportional to its fitness;
2. **Reproduction:** the selected player produces one (identical) offspring;
3. **Replacement:** the offspring replaces a randomly selected player in the population.

The Moran process allows us to derive the fitness function, transition probability, and fixation probability of mutant genes. Based on these, the paper provided the stochastic approach to study evolutionary dynamics in finite populations.

The full population consists of both C and D. When each node interacts with a diverse group of others, the average payoff for a player categorized as C or D depends on the the rest of players. Excluding self-interactions, the average payoff of all i cooperators is calculated by the sum CC interaction and CD interaction [10]:

$$\pi^C(i) = \frac{(i-1) + S(N-i)}{N-1} \tag{3.6}$$

Similarly, the average payoff for all $(N-i)$ defectors is:

$$\pi^D(i) = \frac{i}{N-1}T \tag{3.7}$$

Suppose the payoff of the cooperator is π_i^C in equation 3.6, i is the number of cooperators, the fitness function of strategy C is defined to be a positive, convex combination of background fitness and expected payoff denoted as follows:

$$f_C^i(w) = 1 - w + w(\pi_i^C) \tag{3.8}$$

where $w \in [0,1]$ is called the strength of selection. In 3.8, w determines the relative contributions of the baseline fitness, which is associated with genetic predisposition, and the frequency-dependent contribution from interactions in the population.

- If $w = 0$ we call it neutral selection because the payoff matrix has no influence on fitness thus all strategies are deemed equal by selection.
- If $0 < w \ll 1$ we call it weak selection because the payoff matrix has a weak influence on selection.

For the Moran process, the transition probability that the number of C players increases from i to $i+1$ is:

$$T^{+}(i) = \frac{1 - w + w\pi_i^C}{1 - w + w\langle \pi_i \rangle} \frac{i}{N} \frac{N-i}{N}, \tag{3.9}$$

whereas it decreases from i to $i-1$ with probability:

$$T^{-}(i) = \frac{1 - w + w\pi_i^D}{1 - w + w\langle \pi_i \rangle} \frac{i}{N} \frac{N-i}{N}. \tag{3.10}$$

The derivation of equation 3.9 can be found in Appendix B.

To enhance computational efficiency, the paper [10] suggested a different approach for the microscopic process using local information. At each time step, a player, selected at random as b, compares its payoff to the payoff of another randomly selected player a. It then switches to the strategy of the other player with a certain probability:

$$p = \frac{1}{2} + \frac{w}{2} \frac{\pi_a - \pi_b}{\Delta\pi_{\max}} \tag{3.11}$$

where $\Delta\pi_{\max}$ is the maximum possible payoff difference and $0 < w \leq 1$ measures the strength of selection.

The transition matrix for the number of C players in this process i is then given by

$$\begin{aligned} T^{+}(i) &= \left(\frac{1}{2} + \frac{w}{2} \frac{\pi_i^A - \pi_i^B}{\Delta\pi_{\max}} \right) \frac{i}{N} \frac{N-i}{N} \\ T^{-}(i) &= \left(\frac{1}{2} + \frac{w}{2} \frac{\pi_i^B - \pi_i^A}{\Delta\pi_{\max}} \right) \frac{i}{N} \frac{N-i}{N} \end{aligned} \tag{3.12}$$

In both processes, the number of cooperators remains constant with probability $T^0(i) = 1 - T^+(i) - T^-(i)$. Furthermore, the states $i = 0$ and $i = N$ are absorbing states.

Fixation Probability in the Moran Process is the probability that a single player C in a population consisting of D's takes over the whole population. Finally, we can directly compute the fixation probability ϕ_1, as follows:

$$\phi_1 = \frac{1}{1 + \sum_{k=1}^{N-1} \prod_{i=1}^{k} \frac{T_i^-}{T_i^+}}. \tag{3.13}$$

The derivation of equation 3.13 can be found in Appendix B. It is the solution of the recursive equation $\phi_i = T_i^+ \phi_{i+1} + T_i^0 \phi_i + T_i^- \phi_{i-1}$.

The figure 3.1 below provides a more direct explanation of fixation probability.

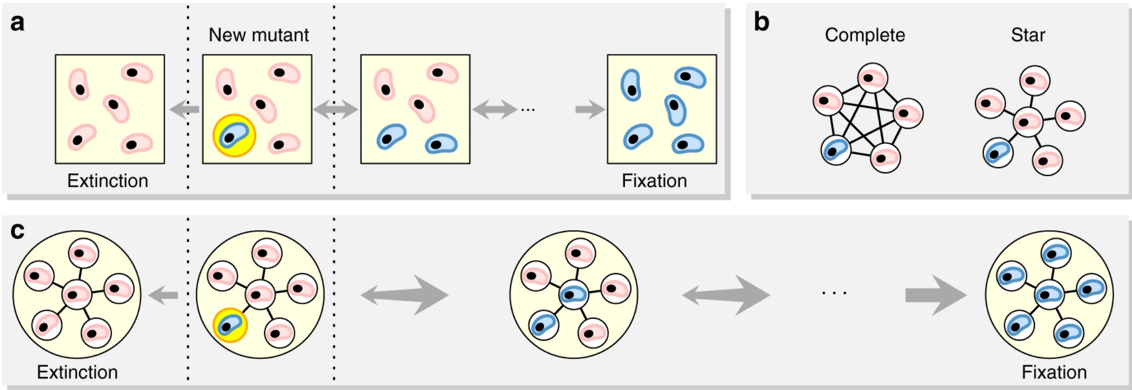


Figure 3.1: Illustration of Moran process [20]. (a) A new mutant (blue) appears in an unstructured population of finite size. The lineage of the new mutant can either become extinct or reach fixation. (b) A mutant (blue) appears in a networked population. (c) Moran process of a simple star-like network. The birth-death process is the same. However, network topology influences both the fixation probability and the fixation time.

This paper also introduces the Fokker-Planck equation and the Langevin equation to derive the Moran model from a finite to an infinite population. The relevant equations can be found in [10]. Although the approaches used in this paper [10] differ from the previous paper [9], they both lead to the emergence of cooperation. This presents a mathematically consistent transition from the description of the microscopic Moran process to a deterministic mean-field theory governed by ORE.

3.3. Evolutionary dynamics of homogeneous networks in PD

Compared to other networks, research on Networked EGT in homogeneous networks is the most well-documented and successful. The properties of homogeneous networks make our analysis and modeling easier.

The study of networked Evolutionary Game Theory (EGT) in homogeneous networks can be categorized from three different viewpoints: circle network, grid network, and adjusted grid network. The paper ”Evolutionary games on cycles” [13] is from the circle network perspective, while ”Evolutionary games and spatial chaos” [14] and ”Spatial games and the maintenance of cooperation” [15] are from the grid network perspective. Lastly, ”Evolutionary prisoner’s dilemma game on a square lattice” [16] and ”Evolutionary prisoner’s dilemma game with dynamic preferential selection” [17] are associated with the adjusted grid network perspective.

The paper ”Evolutionary games on cycles” [13] is a simple example of EGT in spatial settings. It demonstrates how the fixation probability, as discussed in the previous subsection 3.2.2, can be explicitly calculated to predict cooperation behavior. The paper explores three different updating rules: **’birth-death’ (BD), ’death–birth’ (DB), and ’imitation’ (IM)**. In [13], BD means that a player is selected for reproduction proportional to fitness and the offspring replaces a randomly chosen neighbor. DB means that a random player is eliminated, and the neighbors compete for the empty site proportional to their fitness. IM means that a random player is chosen to update its strategy. The study reveals that in a PD, BD updating consistently benefits defectors, while DB and IM updating can lead to a preference for cooperators over defectors.

The paper ”Evolutionary games and spatial chaos” [14] presents a purely deterministic, spatial version of the EGT problem in a grid network, with no memories among players and no strategical elaboration. It only considers two kinds of players: always cooperate and always defect. In each round of the game, players follow the strategy of their neighbor with the highest overall payoff (can be called as **max-payoff update**). The results generate chaotically changing spatial patterns, in which cooperators and defectors both persist indefinitely (See Appendix C). What is more, in the special initial condition, a gorgeous spatial pattern like a ”kaleidoscope” can occur (See Figure 5.4).

The paper ”Spatial games and the maintenance of cooperation” [15] published two years after the previous paper [14] can be viewed as an extension, which introduces the concept of **probabilistic winning** to make sure the spatial distribution is irregular. Other innovative extensions like three-dimensional grid networks, either symmetric or irregular and with deterministic or probabilistic winning have also been discussed. In all the extensions, the essential spatial results in [14] also remain valid in [15].

The paper ”Evolutionary prisoner’s dilemma game on a square lattice”[16] also presents an approach to study the cooperation behavior of the EGT in grid networks by Monte Carlo simulations and dynamical cluster techniques. Unlike the max-payoff update in [14] [15], in [16] nodes can adopt one of the neighboring strategies depending on the payoff difference (can be called as **Fermi update**, see equation 3.14). The simulation results provide direct evidence of two absorbing states and a significant threshold for Temptation T.

The paper ”Evolutionary prisoner’s dilemma game with dynamic preferential selection” [17] studied a modified PD on a disordered square lattice. Similar to [16], nodes can adopt one of the neighboring strategies based on the difference in payoff. However, the selection of the neighbor follows a dynamic **preferential rule**, meaning that the more frequently a neighbor’s strategy was adopted by the focal player in previous rounds, the higher the probability it will be chosen in subsequent rounds. The study found that cooperation is significantly enhanced due to this simple selection mechanism.

3.4. Evolutionary dynamics of heterogeneous networks in PD

Although the importance of heterogeneity was recognized long ago, many issues are still at the forefront of research. The challenges are great because the heterogeneity of player types and connectivity structures destroy symmetries. Most of the more traditional analytical methods are to a large extent no longer applicable, which means that numerical simulations are widely used.

The study of networked Evolutionary Game Theory (EGT) in heterogeneous networks mainly consists of two papers, ”From Local to Global Dilemmas in Social Networks” [18] and ”Heterogeneous networks do not promote cooperation when humans play a Prisoner’s Dilemma” [19]. They investigated the same types of heterogeneous networks with different methods.

3.4.1. From Local to Global Dilemmas in Social Networks

Inspired by paper [10] in subsection 3.2.2, the paper ”From Local to Global Dilemmas in Social Networks” [18] also uses the Stochastic Birth-Death Process, where in each time step a random player can choose birth or die. However, the paper introduces the new concept called **”averaged gradient of selection” (AGOS)**. Based on AGOS, the paper constructed a dynamical model and investigated both homogeneous and heterogeneous networks.

So how to compute AGOS?

In the structured networks, each node x adopts the strategy of a random neighbor y with probability given by the Fermi function (See previous paper [16] and paper [17]) as follows:

$$p \equiv \left[1 + e^{-\beta(f_y - f_x)}\right]^{-1} \tag{3.14}$$

where f_x and f_y is the accumulate payoff of x and y respectively. The parameter β is the intensity of selection.

Unlike the fitness function as shown in the previous section 3.2.2, we do not assume the nodes with the same strategy have the fitness. Thus fitness becomes context-dependent.

The same happens to transition probability, we need to consider the transitions that occur in every node in the network throughout the whole evolution.

For each node i in the structured population, we can compute the probability of changing behavior at time t,

$$T_i(t) = \frac{1}{k_i} \sum_{m=1}^{\bar{n}_i} \left[1 + e^{-\beta(f_m(t) - f_i(t))}\right]^{-1} \tag{3.15}$$

where k_i stands for the degree of node i and $\bar{n}_i$ for the number of neighbors of i having a strategy different from that of i.

Therefore, the time-dependent AGOS at a given time t of simulation p, where we have j cooperators in the population of size $\mathcal{N}$, is defined as the difference in probabilities to increase and decrease,

$$G_p(j,t) = T_A^+(j,t) - T_A^-(j,t) \tag{3.16}$$

where $T_4^{\pm}(j,t) = \frac{1}{N}\int_{i=1}^{AllDs} T_i(t)$.

The detailed methods can be found in [18].

The main results of this paper consist of 3 parts.

Part 1 is about the AGOS in homogeneous networks. The paper studied the dynamic behavior in homogeneous **random regular networks** via AGOS. A homogeneous random regular network is a network of size N and each node has a fixed degree. Grid network is one of the homogeneous random regular networks. Examples shown in Figure 3.2 are simple regular networks.

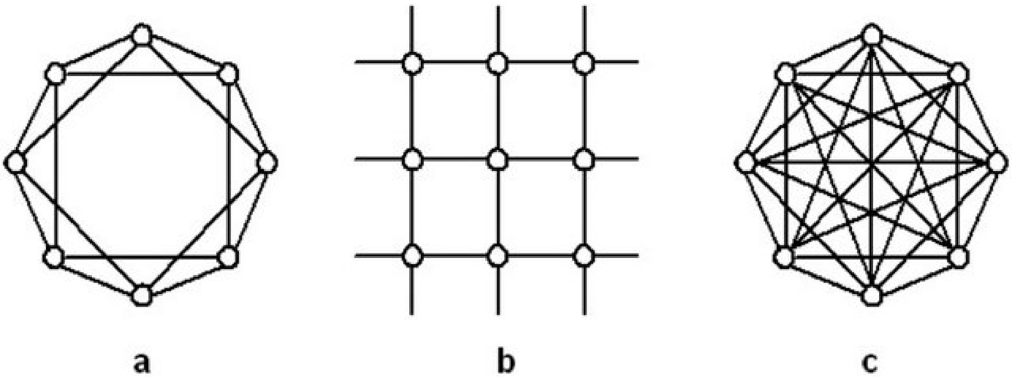


Figure 3.2: Examples of regular networks [21]. (a) periodic ring network, each node has degree 4. (b) grid network, each node has degree 4. (c) well-mixed network, each node has degree 7.

The study shows that the coexistence of C and D can occur from a global point of view even though every node engages in the PD games with its local neighbors. The coexistence points are associated with the internal root of AGOS. Moreover, the values of the coexistence point can be changed if the payoff matrix is changed.

We did a simulation similar to the simulation in paper [18]. Due to the computation efficiency, the parameters we used are smaller. For instance, we use random regular networks with size 100 instead of 1000. The results are shown in Figure 3.3 below.

In Figure 3.3, the x-axis is the initial cooperation percentage and the y-axis is the AGOS. The positive AGOS indicates that the cooperator behavior is more favorable. In other words, the cooperation percentage tends to increase. The negative AGOS is the opposite. Overall, we have a sin curve with a stable internal point around 0.6, which also implies the coexistence of 60% cooperation and 40% defection.

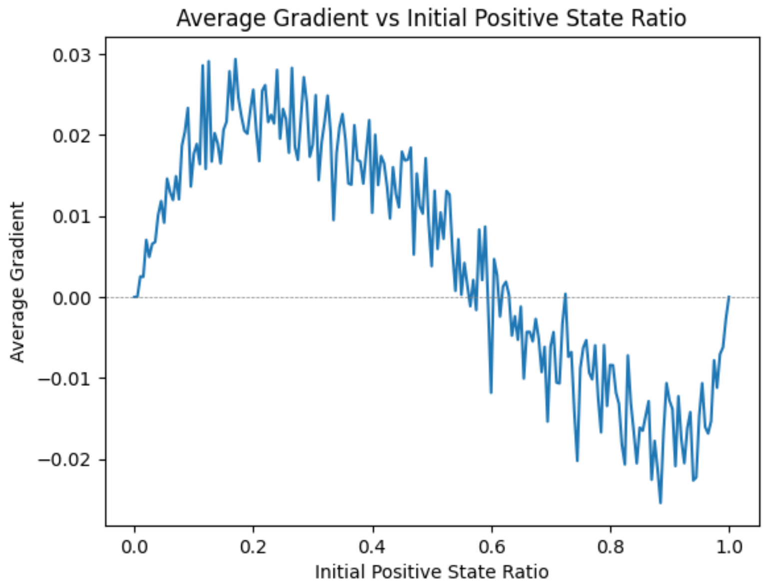


Figure 3.3: The simulation is done on a homogeneous random regular network with size $N = 100$. The payoff matrix has parameters $T = 1.25$ and $S = -0.25$. It uses the Fermi updating rule with $\beta = 1$. The experiment was repeated 10 times.

Part 2 is about the time dependence of AGOS. Recall the formula of AGOS in 3.16, the term ”averaged” means that we need to average over the entire evolution. Thus, AGOS results can vary significantly depending on the number of generations.

The paper explored how AGOS behaves in a homogeneous regular network with different generations. It finds that there exists the time evolution of the internal roots towards the coexistence root. As the population evolves, C and D nodes interact with each other, and finally cooperation percentage stabilizes. Different generations of AGOS internal points correspond to various stages in the evolution of cooperation.

Part 3 is about the AGOS in the heterogeneous network. Specifically, the paper computes AGOS in **scale-free (SF) networks** by **Barabasi and Albert (BA) methods**. SF is a network that the degree follows a power law distribution. Figure 3.4 shows an example of an SF network.

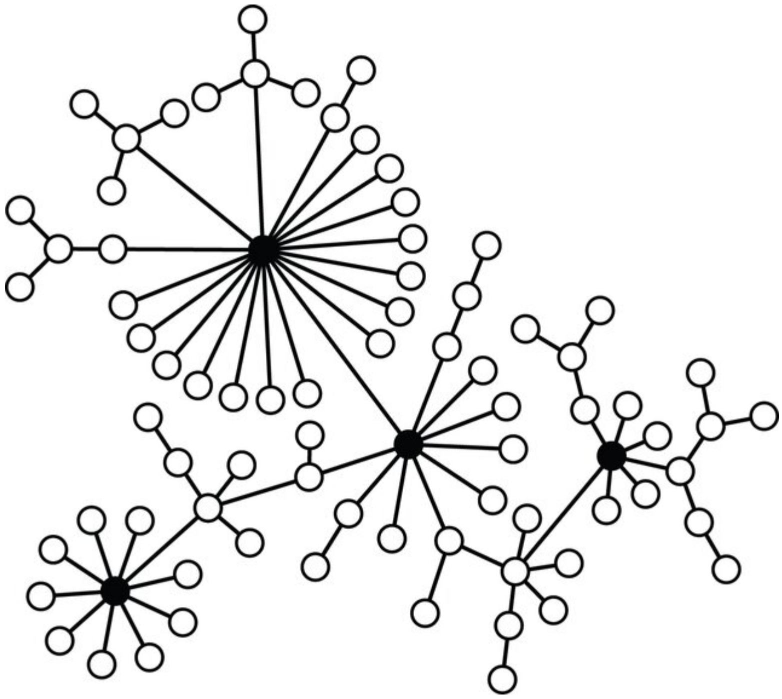


Figure 3.4: An example of a scale-free (SF) network [22]. The "hubs", nodes with high degree are painted black, while the low degree nodes are represented in white. Note that many nodes are related to just one hub.

The study shows that the coexistence points in parts 1 and 2 become unstable in part 3, which represents the non-coexistence for cooperation percentage. In other words, above a certain initial cooperation percentage, cooperation is advantageous and below this value defection is advantageous. This is due to the special configuration of the SF network. As shown in Figure 3.4, there are many clusters. On SF networks, a certain initial cooperation percentage is required to ensure the formation of a cooperation star-like cluster that persists in the invasion of defection.

To summarize, this paper uses AGOS to understand the coexistence behavior in homogeneous networks and non-coexistence behavior in heterogeneous networks. The AGoS method is universal. It can be used for arbitrary intensity of selection, arbitrary population structure, and arbitrary game. However, there is still a gap in the research on how heterogeneity in SF networks influences cooperation evolution specifically. In the following subsection, we will focus on that.

3.4.2. Heterogeneous networks do not promote cooperation when humans play a Prisoner's Dilemma

Since scale-free (SF) networks exhibit obvious heterogeneity, there has been extensive research on the role of this heterogeneity.

Let us start with two early papers. In 2005, F. C. Santos [23] first investigated the cooperation behavior in SF networks. Unlike the Fermi update in [18], the paper [23] uses the **accumulated payoff** and the probability of changing the state is directly proportional to the accumulated payoff difference. It finds that SF networks have an advantage in promoting and sustaining cooperation behavior compared to regular networks. In 2007, ZhiXi Wu [24] used Monte Carlo (MC) simulations and implemented both **synchronous** (update all the nodes in one round) and **asynchronous** Fermi update (update one node in one round) dynamics to calculate the average density of cooperators as a function of the temptation T in the equilibrium state. Unlike Santos's paper [23], Wu [24] finds the result that cooperation is sometimes enhanced and sometimes even lower than the regular network using the same methods.

Research in the SF networks has different results. So, the question arises, what is the impact of heterogeneity on cooperation behavior in reality?

To answer this question, we will introduce another paper "Heterogeneous networks do not promote cooperation when humans play a Prisoner's Dilemma" [19]. It does not use computer simulation but a performs real experiment by playing a spatial PD game among 1200 high school students on a grid and an SF network. It carried out two treatments, either players remained at the same positions in the network with the same neighbors or shuffled the neighbors to remove the effect of the networks. For both treatments, the levels of cooperation attained in a grid and an SF network are indistinguishable, suggesting that population structure has little relevance as a cooperation promoter in population evolution.

3.5. Discussion and Summary

In this chapter, we reviewed a list of papers discussing unstructured populations and structured populations, e.g. homogeneous or heterogeneous networks.

The mathematical analysis and Modelling results from those papers show that they all can lead to the emergence of cooperation in PD. Moreover, the evolution of cooperation is indeed different in different network typologies. For example, the paper [18] shows that the homogeneous network has a co-existence point but the heterogeneous network does not. This result can also be found in papers [14] and [23]. It provides us an insight into the thesis question. However, there are still many questions regarding the evolution of networked EGT. For example, two papers of the same network topology such as [23] and [24] can even lead to opposite results.

So apart from network topology, what is the key? It is a strategy updating rule. The papers choose the same network topology but different updating rules can have different results.

Generally speaking, we can classify the updating rule as deterministic (max-payoff update) or probabilistic (Fermi update); synchronous or asynchronous [24]; with preference or not [17]; etc. Much research (Moran process [10], AGOS [18]) is done using the probabilistic updating rule. However, for the deterministic updating rule, there remains a gap. In the following chapters, we will focus on the **synchronous max-payoff update without preference**.

4 Network Analysis

At the beginning of the research in this thesis report, we start with the analysis of network topology. Recall that the definition of homogeneous and heterogeneous networks has already been introduced in the introduction chapter 2 subsection 2.3. Due to the symmetry of homogeneous networks, We will mainly discuss the heterogeneous networks in this chapter.

4.1. An Analysis of Special Heterogeneous networks

Much research has been done on famous heterogeneous networks such as SF networks [18] [23] [24] [19]. However, for more general heterogeneous networks, there remains partly unknown. In this thesis report, we adopt the special algorithm proposed in the paper ”Intermediate Levels of Network Heterogeneity Provide the Best Evolutionary Outcomes” [25]. Based on this algorithm, we can build up a heterogeneous network parameterized by heterogeneity denoted as α.

The steps to generate a random heterogeneous network with N nodes are the following:

1. Construct a simple circle network with three connected nodes.
2. Introduce a new node.
3. Make two connections between the new node and existing nodes.
4. The probability of making connection with node i is $t^{\alpha}(\alpha \geq 0.0)$, where t is the age of node i.
5. Repeat steps 2, 3, and 4.
6. Reach the end when we successfully add remaining $N-3$ nodes.

In this algorithm, α takes the role of the degree heterogeneity parameter, so that a low (large) α is associated with a low (large) level of degree variance or heterogeneity.

Figure 4.1 below is a more direct illustration of what we did. Step 1 is all the same. But steps 2, 3, and so on are probabilistic.

The corresponding Python code can be found in Appendix A.

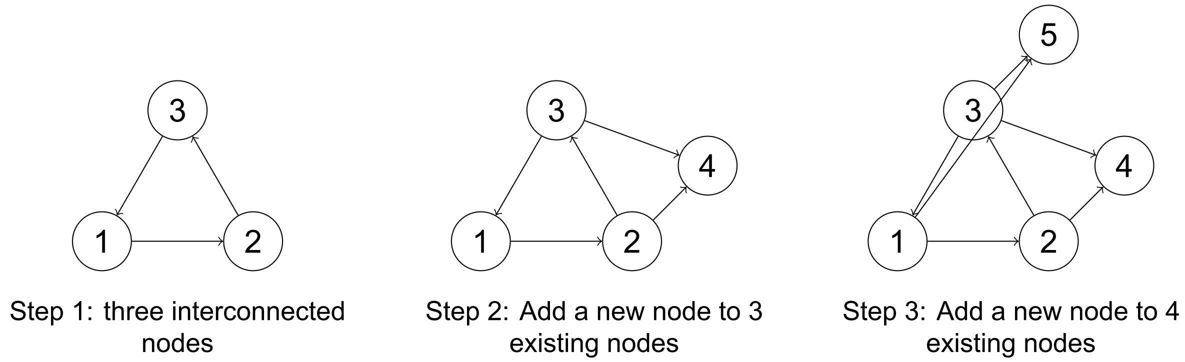


Figure 4.1: First three steps of formation a heterogeneous network

We randomly generated 10000 heterogeneous networks with a population size of 100. They had different heterogeneity α ranging from 0 to 100 with step size 0.01. The histogram of degree distribution was also plotted. By inspecting the network and histogram results, we found out that the network gradually became ”star-like” as heterogeneity α increases. Figure 2.3 shows three examples of heterogeneous networks we generated with α = 1, 10, 100.

To further investigate the properties of networks in terms of degree, we selected and several values of α (0, 1, 10, 100, 1000). For each value of α, we generated 100 random networks and calculated the mean average degree and the variation. The results are shown in Table 4.1.

Table 4.1: Mean Average Degree and Variation for Different α Values

α	Mean Average Degree	Variation
0	3.8870	0.00129900
1	3.8406	0.00203564
10	3.5782	0.00404876
100	2.4570	0.00645900
1000	2.0000	0.00000000

Why we have a mean average degree of 3.8870 in Table 4.1?

When $\alpha = 0$, the probabilities of making connection with new node and all existing nodes are the same. Consider the nodes 1, 2, and 3, the expected (estimated) degree is the same:

$$2 + \frac{2}{3} + \frac{2}{4} + \ldots + \frac{2}{N-1} = 2\log(N-1) - 1 \tag{4.1}$$

The expected degree for node 4 is:

$$2 + \frac{2}{4} + \frac{2}{5} + \ldots + \frac{2}{N-1} = 2\log(N-1) - 2 - \frac{2}{3} \tag{4.2}$$

The expected degree for node i is:

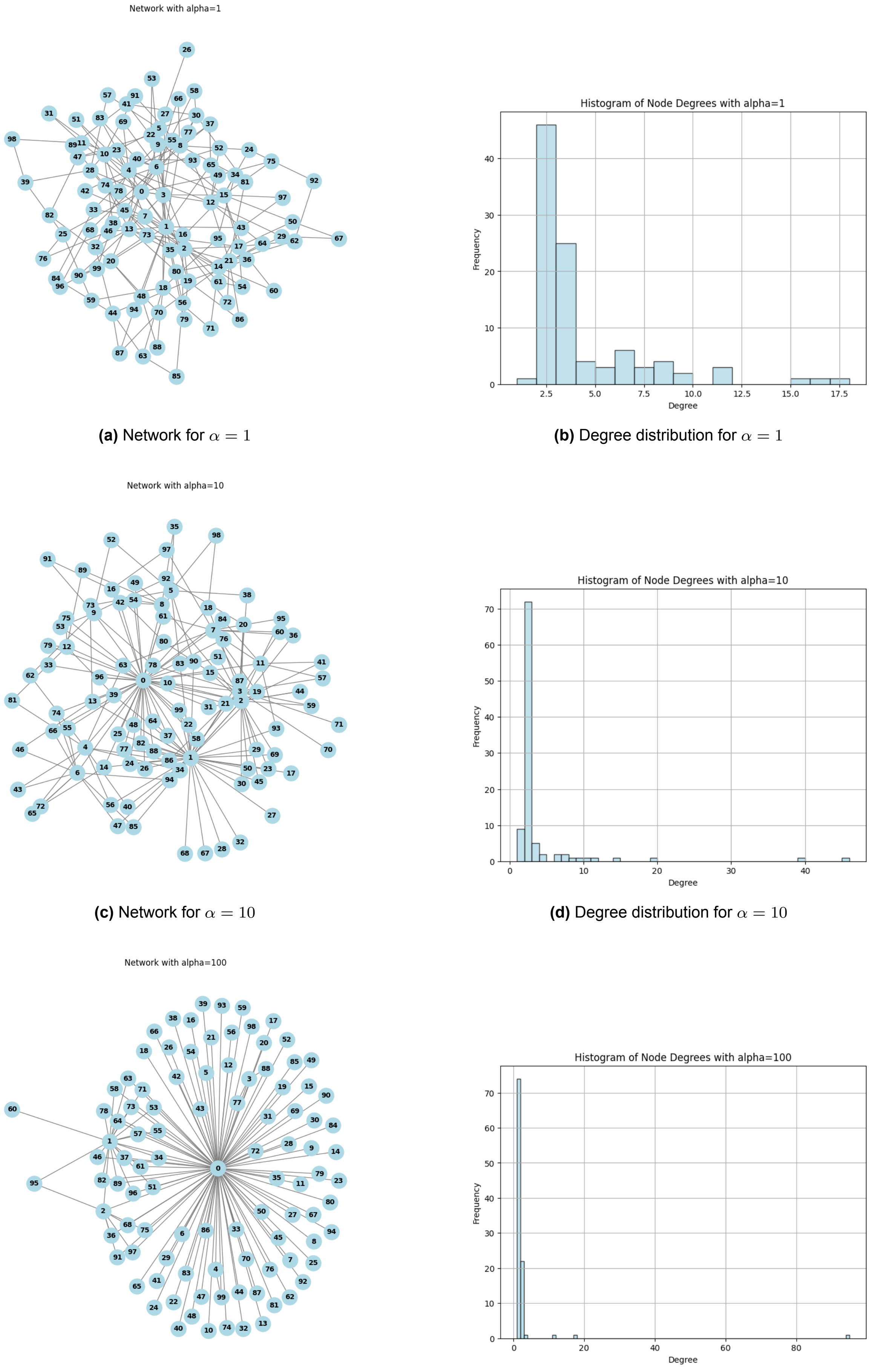
$$\begin{aligned} 2 + \frac{2}{i} + \frac{2}{i+1} + \ldots + \frac{2}{N-1} &= 2\log(N-1) - 1 - \frac{2}{3} - \frac{2}{4} - \ldots - \frac{2}{i-1} \\ 2 + \frac{2}{i} + \frac{2}{i+1} + \ldots + \frac{2}{N-1} &= 2 + 2\log(N-1) - 2\log(i-1) \end{aligned} \tag{4.3}$$

Therefore, the average degree is:

$$\frac{3(2\log(N-1)-1) + \sum_{i=4}^{N}(2 + 2\log(N-1) - 2\log(i-1))}{N} \tag{4.4}$$

We implemented the equation 4.4 into Mathematica and we found that the average degree is approximately 3.9.

The mean average degree 2 (α =1000) also coincides with our prediction. For a star-like network, we have 1 node with degree $N-1$ and $N-1$ nodes with degree 1 (Moreover, we have at least 1 circle with three nodes). Thus, the mean average degree is 2.



(a) Network for $\alpha = 1$ **(b)** Degree distribution for $\alpha = 1$

(c) Network for $\alpha = 10$ **(d)** Degree distribution for $\alpha = 10$

(e) Network for $\alpha = 100$ **(f)** Degree distribution for $\alpha = 100$

Figure 4.2: Example networks and corresponding degree distributions for different alpha values. When the alpha value is small, the network is more random, and the degree distribution is more spread out. As the alpha value increases, the network tends to radialize, and the degree distribution becomes more centralized.

To classify the heterogeneous networks, we can roughly divide them into three partitions. The first is the ”random” partition. The second is the ”transition” partition. The third is the ”star-like” partition.

So what are the classification criteria?

In order to have the detailed range for each partition, we again generated 10000 networks with size 100 and random α (range from 0 to 100). Then, we used the ”**K-means clustering**” method in Machine Learning to classify those networks. Since the inflection point using the Elbow method is unclear, here we used **Silhouette Method** to find the optimal cluster number.

The Silhouette Method calculates the silhouette score, which measures how similar an object is to its cluster (cohesion) compared to other clusters (separation). The value of the Silhouette score ranges from -1 to 1. The interpretation of the Silhouette score is as follows:

- 1: Points are perfectly assigned in a cluster and clusters are easily distinguishable.
- 0: Clusters are overlapping.
- -1: Points are wrongly assigned in a cluster.

The Silhouette score results are shown in Figure 4.3 below.

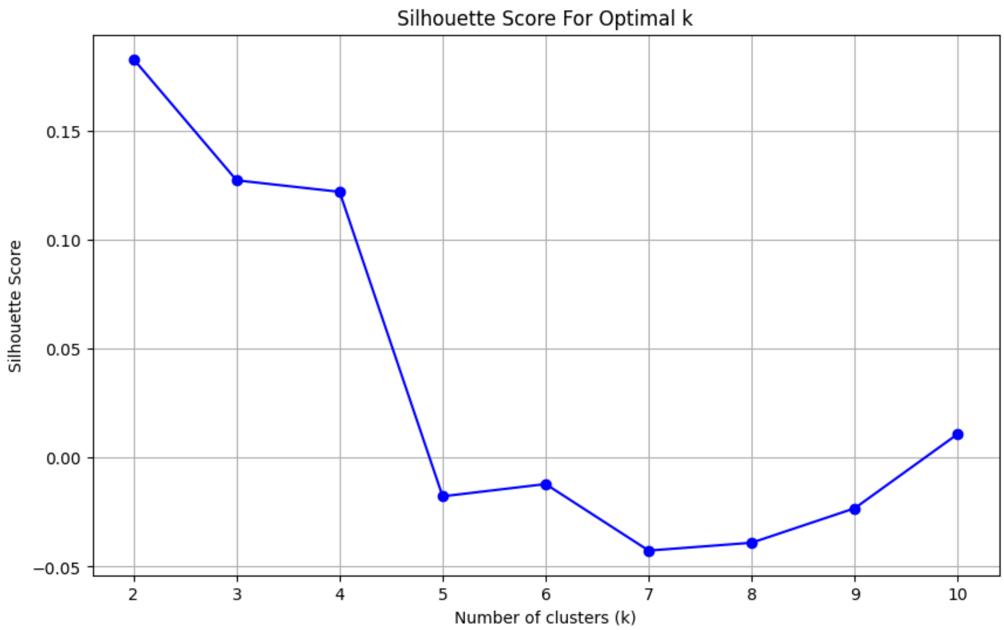


Figure 4.3: Silhouette Scores for an optimal number of clusters.

In Figure 4.3, we find that the highest silhouette score occurs when the number of clusters is 2, which means the optimal cluster number is 2.

Then, we divided 10000 networks into 2 clusters. The results are shown in Table 4.2. Cluster 0 has α values ranging from 0 to 27. Cluster 1 has α values ranging from 14 to 100.

Cluster	Alpha Range
0	(0.001, 27.410)
1	(14.089, 99.991)

Table 4.2: Alpha Ranges for Each Cluster

Eventually, we can divide the heterogeneous networks into three partitions. The first partition has α values ranging from 0 to 14. These networks are more ”random” and have a higher average degree. The second partition has α values ranging from 14 to 27. It is the short transition partition. The third partition has α values over 27. These networks are more ”star-like” and have a lower average degree.

4.2. Summary

In this chapter, we made the network analysis about special heterogeneous networks from paper [25]. The study clearly shows that this heterogeneous network can be divided into three partitions. The network topology and degree distribution in the three partitions show obvious differences. Thus, for simplicity, in the next chapter 5, we will pick the representative networks in each partition to simulate the evolution of cooperation.

5

Analysis and Modelling of Networked EGT

In Chapter 3, we reviewed the literature research on homogeneous and heterogeneous networks. In this chapter, we follow the same structure: start with unstructured population to homogeneous networks such as circle and grid networks, and eventually, we discuss the networks mentioned in Chapter 4. This chapter introduces a new algorithm ”**uni-network**” to measure the cooperation behavior in different networks. All the analysis and simulations have been done using **synchronous max-payoff update without preference** (In short **Max-payoff updating rule**). It is a deterministic updating rule that all the nodes copy the strategy of their neighbor with the highest overall payoff in one round. Also, all the payoff matrix satisfies the PD conditions $T > 1 > 0 > S$ (See subsection 2.1).

5.1. Uni-networks

Inspired by the AGOS in subsection 5.5, the fitness function is context-dependent, we design a new ”uni-networks” algorithm (This algorithm is not from any paper and it is totally new.) for deterministic updating rule such as max-payoff updating rule. This algorithm counts the number of times a node D (C) changes to node C (D) in a uni network within all the spatial configurations.

The term ”uni network” refers to the smallest unit in the networks.

> **Definition 11.** *A uni network for node i is a minimum sub-network that contains all the elements needed for node i to use the max-payoff updating rule. Node i is the uni node (center node).*

According to the definition, the size of the uni network depends on the network topology and updating rule. For example, for a circle network and max-payoff updating rule, the uni networks contain only 5 nodes. However, for a grid network and max-payoff updating rule, the uni networks contain 13 nodes. Figure 5.1 is a more direct illustration.

Figure 5.1: Example of uni networks. Left: ni network of circle networks. Right: Uni network of grid networks.

Suppose we have a uni network with size N, the uni node is C (D) and other nodes can have 2 strategies C or D, then the total possible spatial configurations are 2^{N-1}. For each spatial configuration, we can determine whether the uni node will change from C (D) to D (C). If the initial condition is randomly distributed C and D. Uni node C (D) has a probability to be in one of these spatial configurations. Then, the probability of a random C (D) node changing strategy can be estimated by the number of spatial configurations uni node C (D) changes the state to D (C) divided by the total number of spatial configurations. This can be considered as the analogy of transition probability $T+$ ($T-$). Therefore, the difference in transition probability can be calculated. It tells us how the cooperation percentage tends to evolve.

The complete procedures of the uni network algorithm are the following:

1. Find the uni network of a specific network topology.
2. Randomly distributed C and D in the uni networks.
3. Calculate the overall payoff of the uni node and neighboring nodes.
4. Determine whether the uni node will change the state using the max-payoff updating rule.
5. The process is repeated until all the possible spatial configuration is considered.
6. The transition probability is determined by the number of times the uni node can change the strategy.

The code for the uni network algorithm can be found in appendix A.

Compared to the classical method (generate the whole network assign initial conditions and update nodes one by one in each round), the uni network algorithm dramatically saves much computing time. For example, running a uni network in a 100×100 grid network can be done within several seconds while the classical method can take several minutes. As the size of networks increases, the algorithm becomes even more efficient. This is because the uni network algorithm only considers the smallest sub-network.

However, there are three key limitations of the uni network algorithm.

First, this algorithm works well for homogeneous networks, but for heterogeneous networks, it is not the case. For instance, the unit network of a star-like network is the whole network. The uni network algorithm is just wasting time. Second, this algorithm cannot be used for some spatial initial conditions—for instance, a grid network with half C on the left and half D on the right. Last but not least, the uni network method only works when the size of the network is large enough. If the size of the network is too small, we need to consider the case independently.

5.2. Evolutionary dynamics of well-mixed populations in PD

Suppose we have a well-mixed network with a total of N players. Each node has connections with all the other nodes. k nodes choose strategy C and $N-k$ nodes choose strategy D. The payoff matrix satisfies the PD conditions $T > 1 > 0 > S$.

For a random node C, it is connected to $k-1$ other C nodes and $N-k$ D nodes. A random D node is connected to k C nodes and $N-k-1$ other D nodes. Therefore, the payoff for this C node is $(k-1)+S(N-k)$. Similarly, the payoff for a random D node is kT.

It is obvious that the payoff for D nodes is always greater than that for C nodes. By the max-payoff updating rule, all C nodes will eventually switch to the D nodes as long as the initial condition is not all C. In conclusion, all D is the only stable state for a well-mixed population using the max-payoff updating rule.

5.3. Evolutionary dynamics of circle networks in PD

In this section, we use the ”uni network” algorithm to analyze the circle networks. First, we analyze the simplest three-node network. Since any circle network larger than 5 can be considered as several linear networks gluing together and the max-payoff updating rule only needs the local information, we consider a 5-node uni network with different spatial configurations. Then, by analyzing those configurations, we can derive the conclusion from the base case to a more general case.

Suppose we have a three-node circle network as shown in Figure 5.2.

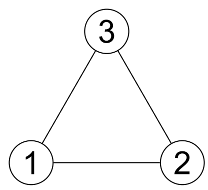


Figure 5.2: three-node circle network

Each node can randomly choose C or D. Depending on the strategies chosen, we can categorize our problem into the following $2^3 = 8$ cases.

Case	1	2	3	4	5	6	7	8
Node1	C	D	C	C	C	D	D	D
Node2	C	C	D	C	D	C	D	D
Node3	C	C	C	D	D	D	C	D

(5.1)

By inspecting Table 5.1, we found out that the cases 2, 3, and 4 are the same due to the symmetry (1 D and 2 C). Similarly, the cases 5, 6, and 7 are the same due to symmetry (1 C and 2 D). Thus, we can simplify the problem into 4 cases:

- Case 1: 3 C
- Case 2: 1 D and 2 C
- Case 3: 1 C and 2 D
- Case 4: 3 D

Calculate the overall payoff of each node in the 4 cases.

- Case 1: All nodes have payoff 2
- Case 2: C nodes have payoff $1 + S$ and D node has payoff $2T$
- Case 3: C node has payoff $2S$ and D nodes have payoff T
- Case 4: All nodes have payoff 0

We choose the max-payoff updating rule and determine the strategy in the next turn. Cases 2, 3, and 4 become all D since $(1 + S) < 2T, T > 2S$ in PD. Case 1 remains the same since it is a stable case. We show that defection is the only stable state in a three-node circle network using the max-payoff updating rule. The same can be done for 4-node and 5-node circle networks.

Next, we will extend our conclusion to a more general case. For a random circle network (N > 5), the problem becomes more complex, but the key is to find the uni network. Because each node will choose the neighboring node with the largest total payoff, and the neighboring nodes also need the left and right neighboring nodes to calculate the overall payoff. So, the strategy choice of each node for the next round depends on the four nodes to the left and right (See figure 5.1). The size of the uni network is 2^4 if the center node is C and 2^4 if the center node is D. We can categorize our problem into the following $2^5 = 32$ spatial configurations. Similar to a 3-node circle network, we can further simplify the 32 spatial configurations into 6 cases as follows:

- Case 1: 5 C
- Case 2: 1 D and 4 C
- Case 3: 2 D and 3 C
- Case 4: 3 D and 2 C

- Case 5: 4 D and 1 C
- Case 6: 5 D

The detailed discussion of each case can be found in Appendix D. The results show that unless the initial state is all C, no matter what the initial state and the S and T are, eventually the result must become all D. In conclusion, all D is the only stable state for circle networks using the max-payoff updating rule.

5.4. Evolutionary dynamics of grid networks in PD

Recall that in [14] (See subsection 3.3), the paper investigated cooperation behavior in grid networks using max-payoff update. Two main results are obtained, one is infinitely chaotic cooperation behavior in the random initial condition (See Appendix C), one is the symmetrical beautiful spatial patterns in the initial condition that 1 D in the middle node and all the other nodes are C (See Figure 5.4). However, the mathematical explanation remains a gap. In this section, we would like to further investigate grid networks by uni network algorithm.

Suppose we have a 3×3 grid network as shown in 5.3.

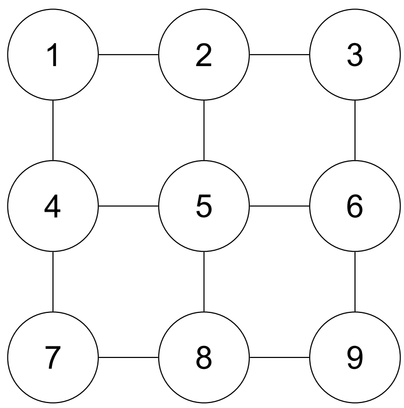


Figure 5.3: 3×3 Grid network

Since there are a total of $2^9 = 512$ different spatial configurations. Even though we can simplify the 512 spatial configurations by symmetry (Nodes 2, 4, 6, and 8 are symmetrical and Nodes 1, 3, 7, and 9 are symmetrical), it is still too tedious to calculate them by hand. We will do it using computer simulation. We generated 512 networks, each network has a different spatial configuration. In each generation, all the nodes use the max-payoff updating rule to change their strategy. Eventually, we plot the cooperation percentage with respect to the generation. The results show that unless the initial state is all C, no matter what the initial state and the S and T are, eventually it must become all D. In conclusion, all D is the only stable state for 3x3 grid networks using the max-payoff updating rule.

Next, we will extend our conclusion to a more general case. For a random grid network (size > 3×3), the approach is similar to circle networks discussed in the previous subsection 5.3. Because each node will choose the neighboring node with the largest total payoff, and the neighboring nodes also need the left, right, up, and down neighboring nodes to calculate the overall payoff. So, the strategy choice of each node for the next round depends on the 12 nodes around it (See Figure 5.1). Thus, the size of the uni network is 13. We can categorize our problem into the following $2^{13} = 8192$ spatial configurations ($2^{12} = 4096$ spatial configurations if the center node is C and $2^{12} = 4096$ spatial configurations if the center node is D). The simulation result can be found in Table 5.1.

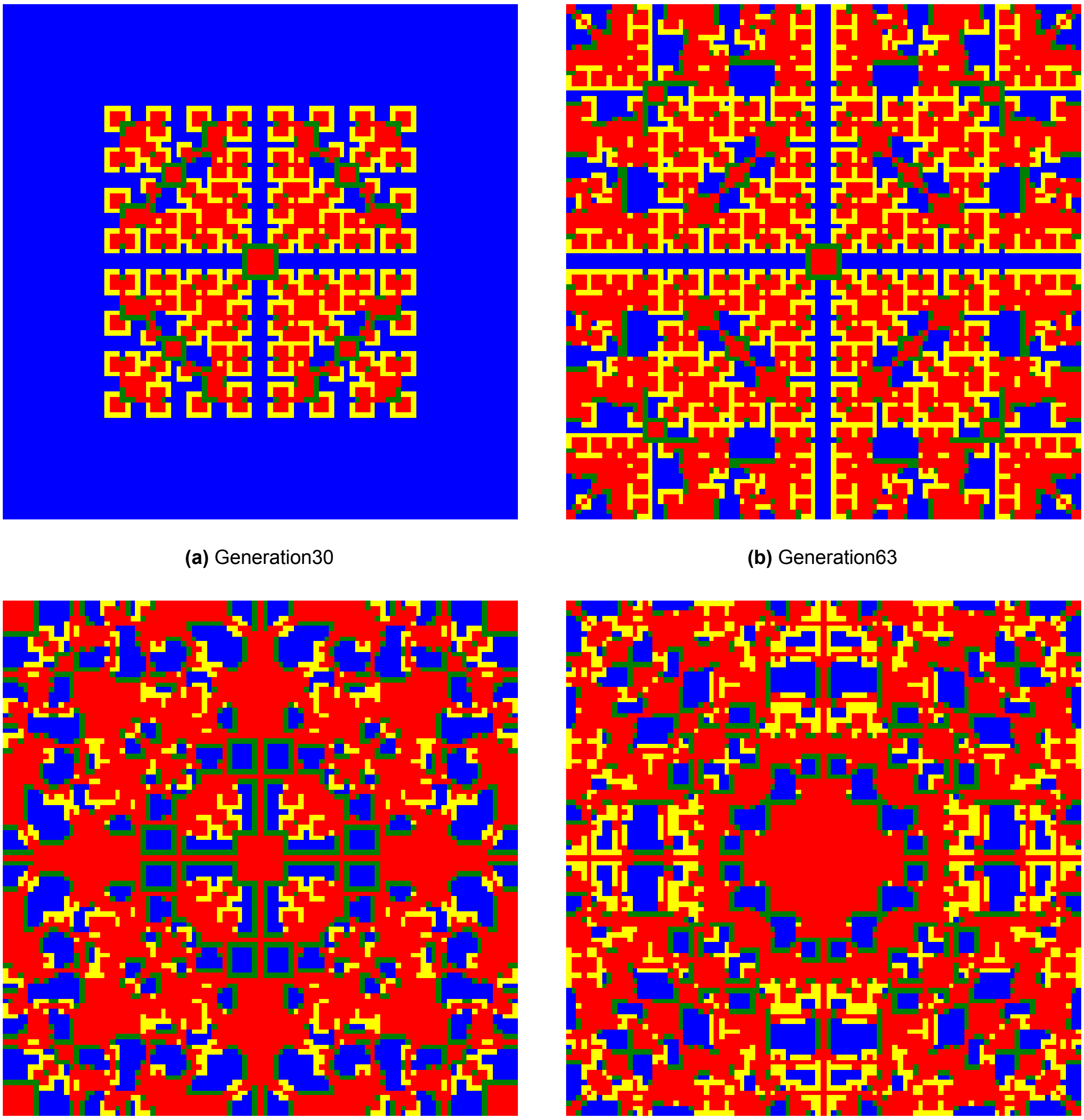
The results in Table 5.1 show that not all the spatial configurations lead to defection. For instance, there are 3288 out of a total 4096 spatial configurations that are more favorable to defection if the center node is C and T values range from 1 to 1.45. Similarly, not all the spatial configurations lead to cooperation. For instance, there are 856 out of a total 4096 spatial configurations that are more favorable to cooperation if the center node is D and T values range from 1 to 1.45. In conclusion, all C and all D are not stable states for grid networks using the uni network algorithm.

Table 5.1: Simulation results for uni network with size 13 (Sucker S = - 0.05)

Temptation T Value	C to D Changes	D to C Changes
1.00	3288	856
1.05	3288	856
1.10	3288	856
1.15	3288	856
1.20	3288	856
1.25	3288	856
1.30	3288	856
1.35	3288	856
1.40	3288	856
1.45	3288	856
1.50	3672	512
1.55	3672	512
1.60	3672	512
1.65	3672	512
1.70	3672	512
1.75	3672	512
1.80	3672	512
1.85	3672	512
1.90	3712	128
1.95	3712	128
2.00	3712	128

5.4.1. Grid networks with special initial conditions

In this subsection, we will focus on the special initial conditions in [14], i.e., the center node is D and all other nodes are C. Recall in [14], the beautiful ”kaleidoscope” pattern can occur under special initial conditions. Figure 5.4 below is an example of a grid network with size 101×101.



(a) Generation30

(b) Generation63

(c) Generation100

(d) Generation208

Figure 5.4: The simulation begins with a single D at the center of a 101×101 square-lattice of C with fixed boundary conditions, when S = - 0.01, T = 1.95 (ensuring $1.8 < T < 2$) [14]. Blue for a cooperator (C) that was already a C in the previous generation, red for a defector (D) following another D, yellow for a D following a C, and green for a C following a D. By max-payoff updating rule, we have an endless variety of different fractal patterns. It is symmetrical since the initial condition is symmetrical.

By calculating the cooperation percentage in different generations, we found out that the cooperation percentage decreases and eventually reaches an asymptotic value around 0.35 (This asymptotic value is independent of the payoff matrix [14]). The results are shown in Figure 5.5.

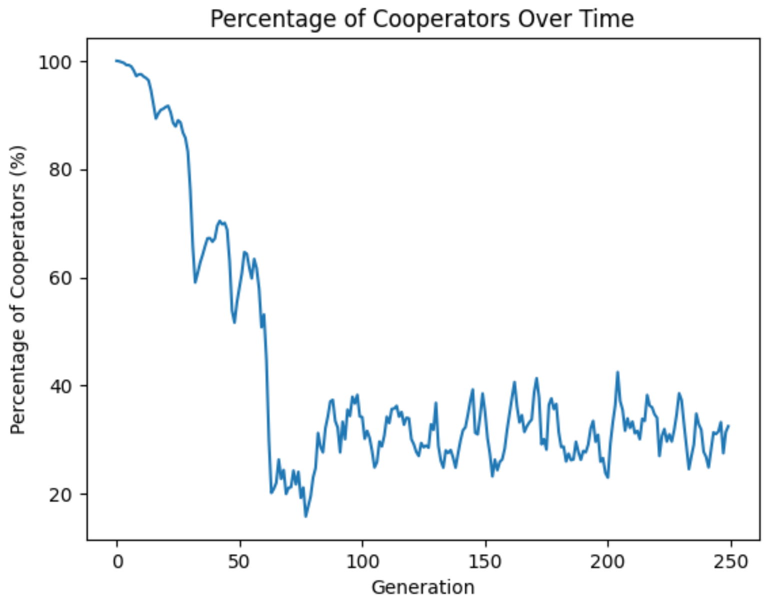


Figure 5.5: Cooperation evolution of a 101×101 square-lattice in Figure 5.4

This asymptotic value is clear evidence of the co-existence of cooperation and defection. Even though we don't know why is 0.35, we can still verify this value using AGOS internal point.

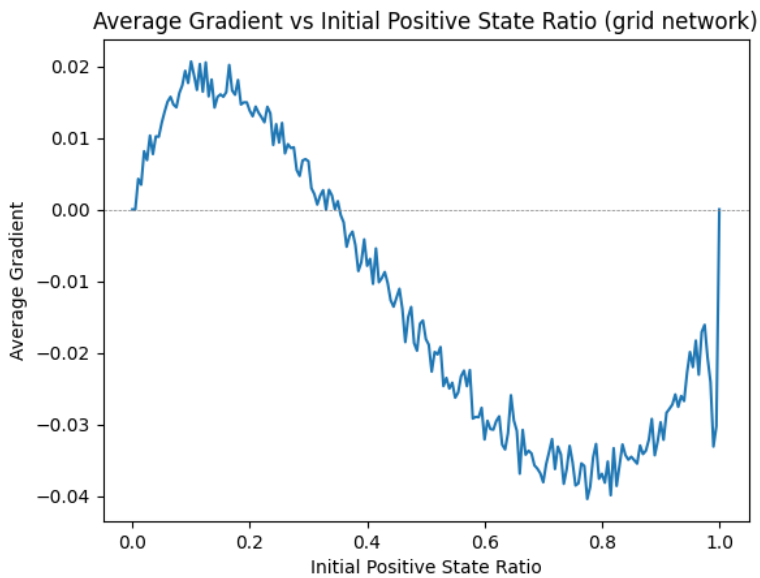


Figure 5.6: The simulation is done on a grid network with size 10×10. The payoff matrix has parameters $T = 1.95$ and $S = 0.01$. It uses the Fermi updating rule with $\beta = 1$. The experiment was repeated 10 times. The internal point is between 0.3 and 0.4.

Next, we start with the simple 3×3 grid network, calculate the overall payoffs of each node, and determine the strategy in the next turn. The results can be found in Table 5.2.

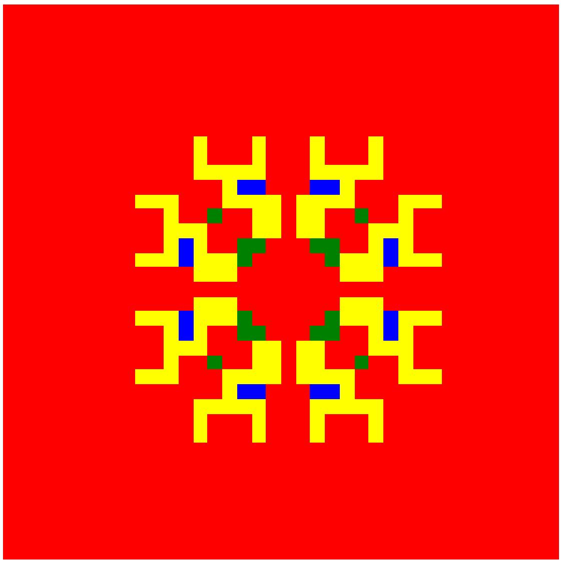
Node	1	2	3	4	5	6	7	8	9
Initial Condition	C	C	C	C	D	C	C	C	C
Payoff 1	4	3 + S	4	3 + S	4T	3 + S	4	3 + S	4
Generation 1	C	D	C	D	D	D	C	D	C
Payoff 2	2	2T	2	2T	0	2T	2	2T	2
Generation 2	D	D	D	D	D	D	D	D	D

Table 5.2: Strategies and Payoffs in a 3×3 grid network

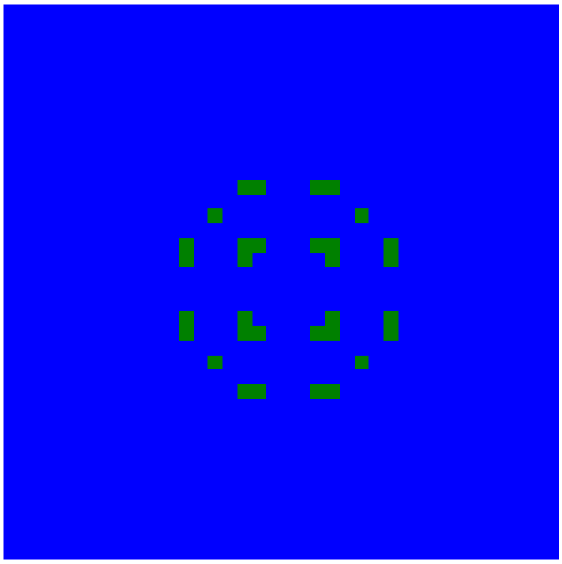
In Table 5.2, we observe that eventually, all nodes change to D. This is not at all the same as the kaleidoscope-like result we got in Figure 5.4. So, how large is N can we have the kaleidoscope patterns?

We conducted the simulations as follows. We generated in total 100 grid networks with different sizes ranging from 1×1 to 100×100. For each grid network, the middle point was D and all the other points were C. Set S = - 0.01 and T = 1.95. In each generation, all the players use the max-payoff updating rule to change their strategy. Eventually, we plot the cooperation percentage with respect to the generation. Related Python code can be found in Appendix A

The results are shown in Figure 5.7 and Figure 5.8 below.

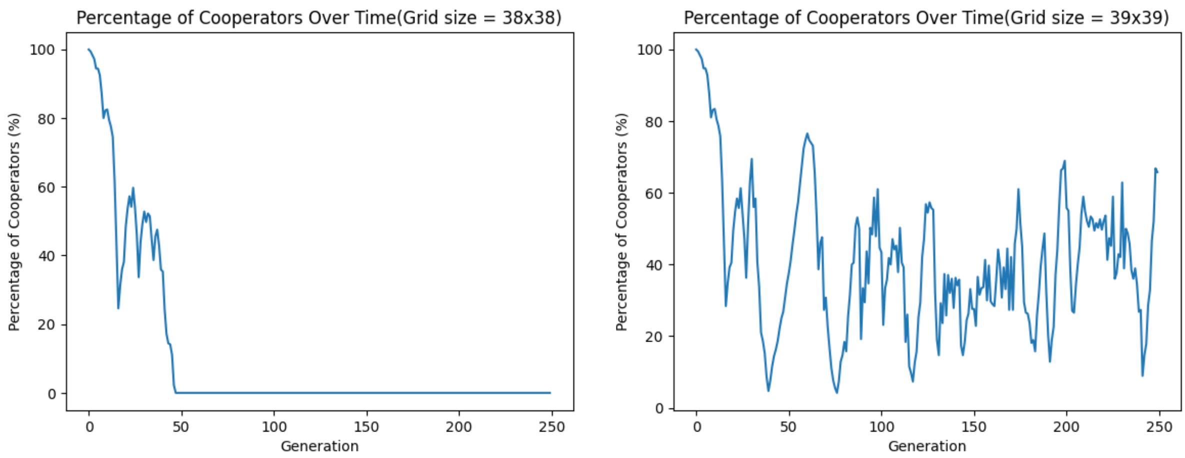


(a) Generation 45



(b) Generation 46

Figure 5.7: 38x38 Grid network. Spatial game patterns suddenly disappear at generation 47.



(a) Grid size = 38x38

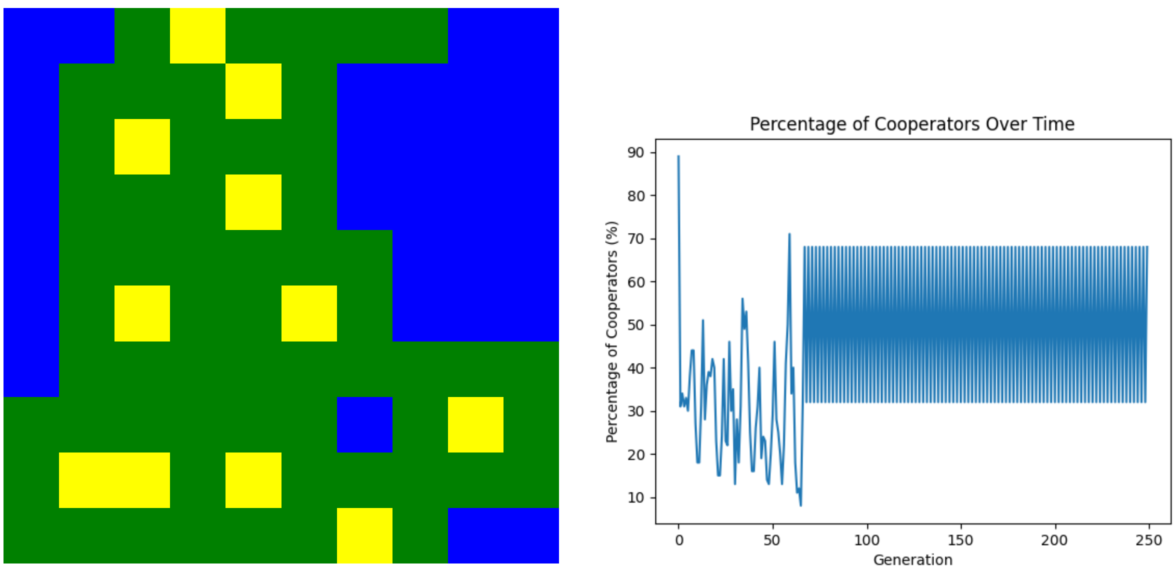
(b) Grid size = 39x39

Figure 5.8: Evolution of cooperation with different grid sizes. Grid size 38x38 has the 0 cooperation percentage at the end but grid size 39x39 is not the case.

Through numerical simulations, we find out that there exists a threshold for N. When $N < 38$, the ”kaleidoscope” pattern does not appear and the cooperation ratio drops to 0. When $N = 39$, the ”kaleidoscope” pattern remains infinitely.

So is this threshold the same for all or only for this special initial condition?

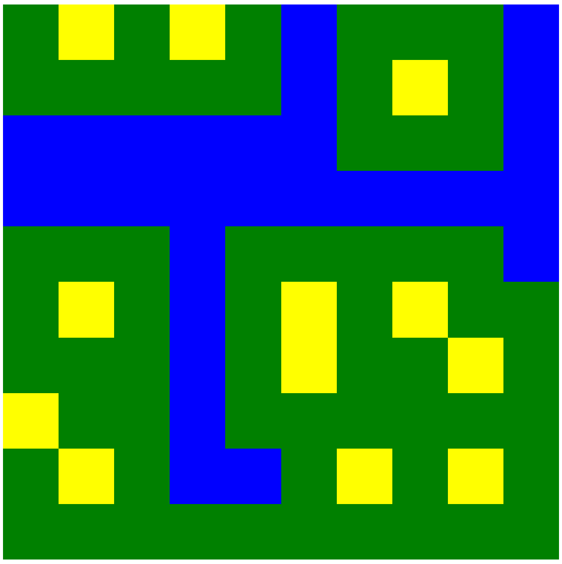
We generated in total 100 grid networks with different sizes ranging from 1×1 to 100×100. For each network, we settled 10 **random initial conditions** (S = - 0.01, T = 1.95, 90% C and 10% D). Nodes can change their strategy using the max-payoff updating rule. The results of cooperation evolution sometimes are all D but sometimes are finitely oscillations. Figures 5.9 and 5.10 below are examples.



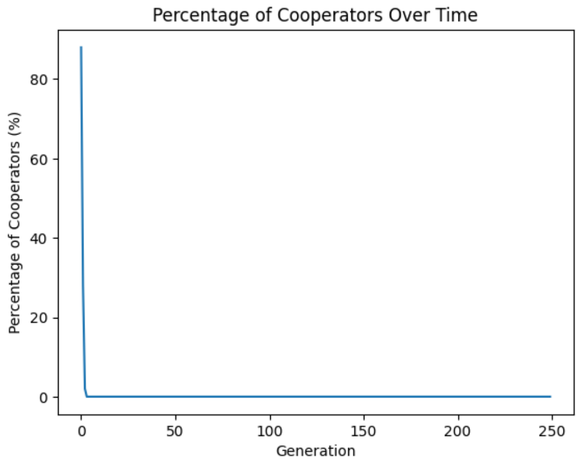
(a) Initial condition

(b) Cooperation evolution

Figure 5.9: 10x10 Grid network. Spatial games oscillate infinitely.



(a) Initial condition



(b) Cooperation evolution

Figure 5.10: 10x10 Grid network. Spatial games suddenly disappear.

The plot suggests that there exists a complicated relationship between cooperation evolution in grid networks with initial spatial configurations and the size of the networks. Even though we do not have a solid explanation for why in Figure 5.9 and 5.10 we have completely different cooperation evolution, the possible reason could be the significant spatial structure effect. In Figure 5.9, the cooperators are closer to each other to resist the invasion of defectors.

In conclusion, cooperation evolution in grid networks with the special initial condition can only produce kaleidoscopic patterns if the network size satisfies a certain threshold. For other random initial conditions, more research needs to be done.

5.5. Evolutionary dynamics of heterogeneous networks in PD

Recall in Chapter 4, the heterogeneous networks can be categorized into 2 clusters. Two clusters have different network topology, one has lower heterogeneity and one has higher heterogeneity. So, what is the difference in the cooperation behaviors of these two clusters? In other words, will the heterogeneous cluster (cluster 1) promote cooperation behavior? In this section, we will focus on this question.

We generated 1000 networks with size 50 and randomly selected α ranging from 0 to 14 (cluster 0). Each network starts with a random initial condition with 50% C and 50% D. For the payoff matrix, S = - 0.05, T = 1.6. In each generation, nodes can change the strategy using the max-payoff updating rule. If the cooperation percentage reaches 0% or 100%, stop the experiment. The same process is been done for α ranging from 27 to 100 (cluster 1). Relevant code can be found in Appendix A.

The results show that there is no coexistence of C and D. The possible final state after cooperation evolution will be either all C or all D. The frequencies of all C and all D for two clusters in 1000 networks are shown in Table 5.3.

Alpha Range	All Cooperation Ratio	All Defection Ratio
0-14	0.103	0.897
27-100	0.164	0.836

Table 5.3: Cooperation and Defection Ratios for Cluster 0 and 1 (50% C, 50% D).

In Table 5.3, it is clear that the cooperation ratio of cluster 1 is larger than the ratio of cluster 0, which indicates that heterogeneity indeed promotes the emergence of cooperation.

But why is that? One possible explanation is due to the network topology. The heterogeneity of the network topology creates an imbalance. For example, in cluster 1, center nodes in a star-like network play an essential role in cooperation evolution.

Suppose we have a star-like network with size N. The Numbers of C and D in the network are k and $N-k$ respectively. The center node 0 chooses C. So for the peripheral nodes, the Numbers of C and D in the network are $k-1$ and $N-k$. Overall, the payoff for the center node is $(k-1)+S(N-k)$ the payoff for peripheral nodes are 1 for C nodes T for D nodes. If we want to have all C, we need:

$$(k-1)+S(N-k) > T \tag{5.2}$$

Substitute N = 50, k =25, S = - 0.05, T = 1.6 into the expression 5.2.

$$24 - 0.05 \times 25 > 1.6 \tag{5.3}$$

The inequality is always true. It implies that if the center node is C, then all nodes will choose C after 1 generation.

The same calculation can be done when the center node is D. The Numbers of C and D in the network are k and $N-k$ respectively. The center node 0 chooses D. So for the peripheral nodes, the Numbers of C and D in the network are k and $N-k-1$. Overall, the payoff for the center node is kT the payoff for peripheral nodes are S for C nodes 0 for D nodes. If we want to have all C, we need:

$$kT < S \tag{5.4}$$

Substitute N = 50, k =25, S = - 0.05, T = 1.6 into the expression 5.4.

$$25 \times 1.6 < 0 \tag{5.5}$$

The inequality cannot be true. It implies that if the center node is D, then all nodes will choose D after 1 generation.

We can design an experiment to verify this. We generated 1000 networks with size 50 and α = 1000. 50% C and 50% D nodes are randomly distributed in the networks but node 0 is constrained to choose C or D. Count the frequencies of all C and all D. The results validate our expectations (See Table 5.4).

Table 5.4: Cooperation and Defection Ratios for $\alpha = 1000$

Node 0	All Cooperation Ratio	All Defection Ratio
C	1	0
D	0	1

In Table 5.3, an experiment is designed to have α in a range. It means we averaged the spatial effect of the networks in a cluster. Therefore, our results are more of an summary of cooperation behavior in a cluster. So will the cooperation strategy of the center node promote the cooperation behavior in the cluster?

We modified the previous experiment. We generated 1000 networks with size 50 and α within the range 0-14 or 27-100. 50% C and 50% D are randomly distributed in the networks but node 0 is constrained to choose C. The results are shown in Table 5.5.

Alpha Range	All Cooperation Ratio	All Defection Ratio
0-14	0.214	0.786
27-100	0.329	0.671

Table 5.5: Cooperation and Defection Ratios for Cluster 0 and 1 (50% C, 50% D, Node 0 is C).

The cooperation ratios for two clusters in Table 5.5 are over two times larger than the results in in Table 5.3. It implies that the center node also has a significant impact on cooperation behavior in terms of the cluster. The center node is more like a ”celebrity” in the network, the choice it made has the celebrity effect, which inspires and motivates other nodes to take the same strategy. When heterogeneity increases, the celebrity effect becomes more significant.

So will other older nodes such as node 1 or node 2 have this ”celebrity effect”? In order to investigate this problem, we conducted a new experiment and constrained both node 0 and node 1 to be C. The results are shown in Table 5.6.

Alpha Range	All Cooperation Ratio	All Defection Ratio
0-14	0.39	0.61
27-100	0.678	0.322

Table 5.6: Cooperation and Defection Ratios for Cluster 0 and 1 (50% C, 50% D, Nodes 0 and 1 are C).

Apparently, the cooperation ratios in Table 5.6 are again two times larger than the results in Table 5.5. It implies that the older node can also promote cooperation behavior.

In Table 5.7, we further constrained all the nodes 0, 1, and 2 to be C.

Alpha Range	All Cooperation Ratio	All Defection Ratio
0-14	0.65	0.35
27-100	0.865	0.135

Table 5.7: Cooperation and Defection Ratios for Cluster 0 and 1 (50% C, 50% D, Nodes 0, 1, and 2 are C).

In summary, our research yields three key findings. Firstly, under the max-payoff updating rule, both cooperation and defection remain stable states. Secondly, the ratio of cooperation in cluster 1 consistently exceeds that of cluster 0, suggesting that a more heterogeneous network topology can effectively encourage cooperative behavior. Lastly, we observed that older nodes, particularly node 0, can also foster cooperative behavior by ”celebrity effect”.

5.6. Discussion and Summary

In this chapter, we studied the EGT using the max-payoff updating rule in different network topologies. Most importantly, we introduced a new algorithm called ”uni network”. It is used for analyzing the cooperation behavior of homogeneous networks. We found that in unstructured and circle networks, defection is the only stable state. Cooperative behavior is pretty simple. Initially, the defectors increase and cooperators decrease since defectors are always more advantageous. Then, all nodes become defectors and it is a stable state.

In a grid network, we observed the coexistence of both cooperation and defection. Specifically, the cooperation percentage finally reaches an asymptotic value of around 0.35, which can be verified via AGOS internal points. The asymptotic value is independent of the payoff matrix. However, we found that an asymptotic value can only occur when the grid size is larger than 38×38. Overall, the cooperative behavior that evolves in the grid network is an analogy of the stationary wave. Initially, the cooperators and defectors expend the whole space. Then, they reflect just like how light bounces back and forth between two walls. Finally, a complicated spatial pattern is formed with a final cooperation percentage around 0.35.

In a heterogeneous network, both cooperation and defection are stable states. Unlike a homogeneous network, the cooperative behavior in a heterogeneous network is no longer predictable. The heterogeneity in the network creates an imbalance. The larger the heterogeneity, the larger the imbalance. Some nodes are always advantageous to other nodes no matter what strategy they choose. For example, the center point in a star-like network is always the most beneficial. If the center point is a cooperator, other nodes will more likely change to cooperators, and eventually all cooperation. If the center point is a defector, other nodes will more likely change to defectors, and eventually all defection.

The findings demonstrate that cooperative behavior is influenced by the network topology. Generally speaking, a homogeneous network fosters coexistence, while a heterogeneous network results in either all cooperation or all defection. These results provide new insight into our research question: ”How does cooperative behavior vary based on network topology?” However, our study has limitations as we only focused on the deterministic updating rule (e.g. Max payoff update). Further research is needed to investigate the probabilistic updating rule.

6

Conclusion

In the report, we discussed the Evolutionary Game Theory (EGT) in complex networks. Biologists first proposed EGT as a way to explain population evolution, and it is currently widely applied in a number of disciplines, including social studies and economics. The use of EGT concepts on networks in networked EGT involves players interacting with their immediate neighbors, each of whom has two strategies: C or D. By meeting the PD condition ($T > 1 > 0 > S$), a payoff matrix is used to determine the overall payout for nodes. Players can use the max-payoff updating rule to change their strategy in every generation.

In Chapter 4, we introduced a network analysis of heterogeneous networks and discovered that they can be divided into two clusters. Cluster 0 is a more ”random” nature and has a higher average degree, while Cluster 1 is more ”star-like” and has a lower average degree.

In Chapter 5, we used analysis and modeling methods to discuss different network topologies. In subsection 5.2, we analyzed well-mixed populations by simple calculations and found that defection is the only stable state. In subsection 5.3, we used the method of induction to analyze circle networks, yielding similar results where defection is the only stable state. In subsection 5.4, we investigated the spatial game on grid networks as a continuation of paper [14]. We discovered the existence of a threshold for the grid size. Specifically, the infinite oscillation can only occur when the grid size is larger than 38×38. In subsection 5.5, we conducted several simulations on heterogeneous networks. The results suggest that heterogeneity has a positive effect on cooperation behavior. However, this positive effect also depends on other factors such as the initial condition (center node such as node 0 exhibits a ”celebrity effect”).

Overall, the study demonstrates that the spatial effect in networked EGT actually promotes the emergence of cooperation. Despite each node engaging in a PD game with a deterministic updating rule (Defection is the NE), we observed a global tendency toward cooperation, resulting in counter-intuitive outcomes. This provides us with a deeper understanding of cooperative behaviors in real-world scenarios.

Contribution: Feng developed the uni network algorithm in subsection 5.1 and performed the analytic calculations for well-mixed and circle networks. For the grid network, the numerical simulations are based on the paper [14]. Feng made an extension of the grid network, such as AGOS in Figure 5.6. Feng also carried out experiments for heterogeneous networks developed in paper [25]. Both supervisors Johan and Eric were involved during the research. They provided critical feedback and helped shape the analysis, modelling, and manuscript. In Chapters 2 and 3, many paragraphs are polished by Grammarly (with implementation of AI). However, those paragraphs have been changed again by Feng to reduce the AI similarity.

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A

Source Code Example

A.1. Sihouette Scores

```
import networkx as nx
import matplotlib.pyplot as plt
import random
import numpy as np
from sklearn.cluster import KMeans
from sklearn.preprocessing import StandardScaler
from sklearn.metrics import silhouette_score

def generate_network(Z, alpha):
    G = nx.complete_graph(3)
    for i in range(3, Z):
        attachment_probs = [(i - j) ** alpha for j in G.nodes]
        total_prob = sum(attachment_probs)
        attachment_probs = [p / total_prob for p in attachment_probs]
        selected_nodes = random.choices(list(G.nodes), weights=
            attachment_probs, k=2)
        G.add_node(i)
        G.add_edges_from([(i, selected_nodes[0]), (i, selected_nodes[1])])
    return G

Z = 100
alpha_values = np.arange(0, 100.01, 0.01)
degree_distributions = []
max_length = 0
for alpha in alpha_values:
    G = generate_network(Z, alpha)
    degree_sequence = [d for n, d in G.degree()]
    hist, _ = np.histogram(degree_sequence, bins=range(max(degree_sequence
        ) + 2), density=True)
    degree_distributions.append(hist)
    if len(hist) > max_length:
        max_length = len(hist)
padded_distributions = [np.pad(hist, (0, max_length - len(hist)), '
    constant') for hist in degree_distributions]
padded_distributions = np.array(padded_distributions)
scaler = StandardScaler()
```

```
degree_distributions_scaled = scaler.fit_transform(padded_distributions)
inertia = []
silhouette_scores = []
range_n_clusters = range(2, 11)

for n_clusters in range_n_clusters:
    kmeans = KMeans(n_clusters=n_clusters, random_state=0)
    cluster_labels = kmeans.fit_predict(degree_distributions_scaled)

    inertia.append(kmeans.inertia_)
    silhouette_avg = silhouette_score(degree_distributions_scaled,
        cluster_labels)
    silhouette_scores.append(silhouette_avg)


plt.figure(figsize=(10, 6))
plt.plot(range_n_clusters, inertia, 'bx-')
plt.xlabel('Number of clusters')
plt.ylabel('Inertia')
plt.title('Elbow Method for Optimal Number of Clusters')
plt.grid(True)
plt.show()


plt.figure(figsize=(10, 6))
plt.plot(range_n_clusters, silhouette_scores, 'bx-')
plt.xlabel('Number of clusters')
plt.ylabel('Silhouette Score')
plt.title('Silhouette Scores for Optimal Number of Clusters')
plt.grid(True)
plt.show()
```

A.2. Grid Network

```
import numpy as np
import random
import matplotlib.pyplot as plt
import matplotlib.colors as mcolors

def initialize_grid(N, p_defector):
    grid = np.zeros((N, N), dtype=int)
    # Initialize defector at the center
    grid[N // 2, N // 2] = 1
    # Randomly initialize other players
    for i in range(N):
        for j in range(N):
            if (i, j) != (N // 2, N // 2):
                if np.random.random() < p_defector:
                    grid[i, j] = 1
    return grid

def calculate_payoff(grid, i, j, payoff_matrix):
    N = grid.shape[0]
    payoff = 0
    for dx in range(-1, 2):
        for dy in range(-1, 2):
```

```
            x, y = (i + dx) % N, (j + dy) % N  # toroidal boundary conditions
            payoff += payoff_matrix[grid[i, j], grid[x, y]]
    return payoff

def femi_function(x, y, beta):
    return 1 / (1 + np.exp(beta * (x - y)))

def evolve(grid, payoff_matrix):
    N = grid.shape[0]
    new_grid = np.zeros((N, N), dtype=int)
    transition_grid = np.zeros((N, N), dtype=int)  # Grid to keep track of state transitions
    for i in range(N):
        for j in range(N):
            max_payoff = -1
            max_strategy = -1
            for dx in range(-1, 2):
                for dy in range(-1, 2):
                    x, y = (i + dx) % N, (j + dy) % N  # toroidal boundary conditions
                    payoff = calculate_payoff(grid, x, y, payoff_matrix)
                    if payoff > max_payoff:
                        max_payoff = payoff
                        max_strategy = grid[x, y]
            new_grid[i, j] = max_strategy
            if grid[i, j] == 0 and max_strategy == 0:
                transition_grid[i, j] = 0  # blue: C -> C
            elif grid[i, j] == 1 and max_strategy == 1:
                transition_grid[i, j] = 1  # red: D -> D
            elif grid[i, j] == 0 and max_strategy == 1:
                transition_grid[i, j] = 2  # yellow: C -> D
            elif grid[i, j] == 1 and max_strategy == 0:
                transition_grid[i, j] = 3  # green: D -> C
    return new_grid, transition_grid

def plot_grid(transition_grid):
    cmap = mcolors.ListedColormap(['blue', 'red', 'yellow', 'green'])
    plt.imshow(transition_grid, cmap=cmap)
    plt.axis('off')
    plt.show()

def plot_cooperator_percentage(percentages):
    plt.plot(percentages)
    plt.xlabel('Generation')
    plt.ylabel('Percentage of Cooperators (%)')
    plt.title('Percentage of Cooperators Over Time')
    plt.show()

N = 101  # Grid size
p_defector = 0.1  # Initial proportion of defectors
num_generations = 250  # Number of generations
payoff_matrix = np.array([[1, 0],
                          [1.95, 0]])
grid = initialize_grid(N, p_defector)
cooperator_percentages = []
```

```
for generation in range(num_generations):
    print(f"Generation: {generation}")
    cooperator_count = np.sum(grid == 0)
    cooperator_percentage = (cooperator_count / (N * N)) * 100
    cooperator_percentages.append(cooperator_percentage)
    grid, transition_grid = evolve(grid, payoff_matrix)
    plot_grid(transition_grid)


plot_cooperator_percentage(cooperator_percentages)
```

A.3. Heterogeneous Networks

```
import numpy as np
import itertools

def calculate_overall_payoff(grid, i, j, payoff_matrix):
    overall_payoff = 0
    neighbors = [(-1, 0), (1, 0), (0, -1), (0, 1)]  # Up, Down, Left, Right
    for dx, dy in neighbors:
        ni, nj = i + dx, j + dy
        if 0 <= ni < grid.shape[0] and 0 <= nj < grid.shape[1]:
            overall_payoff += payoff_matrix[grid[i, j], grid[ni, nj]]
    return overall_payoff

def main(T):
    # Parameters
    num_positions = 13  # Total number of positions (3x3 grid + 4 surrounding points)
    payoff_matrix = np.array([[1, 0],
                              [T, 0]])

    # All possible configurations
    configurations = itertools.product([0, 1], repeat=num_positions)

    # Statistics
    C_to_D_changes = 0
    D_to_C_changes = 0

    for config in configurations:
        # Create the 5x5 grid
        grid = np.zeros((5, 5), dtype=int)
        grid_positions = [(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (0, 2), (4, 2), (2, 0), (2, 4)]

        # Assign the configuration to the grid
        for pos, val in zip(grid_positions, config):
            grid[pos] = val

        center_point = (2, 2)
        surrounding_points = [(1, 2), (3, 2), (2, 1), (2, 3)]

        center_payoff = calculate_overall_payoff(grid, *center_point,
```

```
            payoff_matrix)
        surrounding_payoffs = [calculate_overall_payoff(grid, x, y, payoff_matrix) for x, y in surrounding_points]

        # Check the center point strategy
        center_strategy = grid[center_point]
        max_neighbor_payoff = max(surrounding_payoffs)

        if center_strategy == 0 and any(grid[x, y] == 1 and surrounding_payoffs[i] > center_payoff for i, (x, y) in enumerate(surrounding_points)):
            C_to_D_changes += 1
        elif center_strategy == 1 and any(grid[x, y] == 0 and surrounding_payoffs[i] > center_payoff for i, (x, y) in enumerate(surrounding_points)):
            D_to_C_changes += 1

    return C_to_D_changes, D_to_C_changes

if __name__ == "__main__":
    T_values = np.arange(1, 2.05, 0.05)
    results = []

    for T in T_values:
        C_to_D_changes, D_to_C_changes = main(T)
        results.append((T, C_to_D_changes, D_to_C_changes))

    for T, C_to_D_changes, D_to_C_changes in results:
        print(f"T = {T:.2f}")
        print(f"  Center point changes from C to D: {C_to_D_changes} times")
        print(f"  Center point changes from D to C: {D_to_C_changes} times")
```

B

Proofs

B.1. Transition Probability

Proof. The probability proportional to its fitness is $\frac{if_C^i(w)}{jf_C^i(w)+(N-i)f_D^i(w)}$ and the probability of selecting a D node is $\frac{N-i}{N}$, thus,

$$T^+(i) = \frac{if_C^i(w)}{if_C^i(w) + (N-i)f_D^i(w)} \frac{N-i}{N}$$

Plug in $f_C^i(w) = 1 - w + w(\pi_i^C)$

$$T^+(i) = \frac{i(1 - w + w(\pi_i^C))}{i(1 - w + w(\pi_i^C)) + (N-i)(1 - w + w(\pi_i^D))} \frac{N-i}{N}$$

Rearrange the formula

$$T^+(i) = \frac{i(1 - w + w(\pi_i^C))}{i - iw + iw(\pi_i^C) + N - Nw + Nw(\pi_i^D) - i + iw - iw(\pi_i^D)} \frac{N-i}{N}$$

$$T^+(i) = \frac{1 - w + w(\pi_i^C)}{iw(\pi_i^C - \pi_i^D) + N - Nw + Nw(\pi_i^D)} i \frac{N-i}{N}$$

$$T^+(i) = \frac{1 - w + w(\pi_i^C)}{iw(\pi_i^C - \pi_i^D) + N(1 - w + w(\pi_i^D))} i \frac{N-i}{N}$$

$$T^+(i) = \frac{1 - w + w(\pi_i^C)}{\frac{i}{N}w(\pi_i^C - \pi_i^D) + 1 - w + w(\pi_i^D)} \frac{i}{N} \frac{N-i}{N}$$

$$T^+(i) = \frac{1 - w + w(\pi_i^C)}{w(\frac{i}{N}\pi_i^C + \frac{N-i}{N}\pi_i^D) + 1 - w} \frac{i}{N} \frac{N-i}{N}$$

$$T^+(i) = \frac{1 - w + w\pi_i^C}{1 - w + w\langle \pi_i \rangle} \frac{i}{N} \frac{N-i}{N}$$

□

B.2. Fixation Probability

Proof. To compute probability ϕ_1, we need to solve the recursive equation.

$$\begin{cases} \phi_i = T_i^+ \phi_{i+1} + T_i^- \phi_{i-1} + \left(1 - T_i^+ - T_i^-\right) \phi_i \\ \phi_0 = 0 \\ \phi_N = 1 \end{cases}$$

Use substitution, suppose $V_i = \phi_i - \phi_{i-1}$ then $V_{i+1} = \phi_{i+1} - \phi_i$

$$\begin{aligned} &\phi_i = T_i^+ \phi_{i+1} + T_i^- \phi_{i-1} + \phi_i - T_i^+ \phi_i - T_i^- \phi_i \\ &\phi_i = T_i^+ \left(\phi_{i+1} - \phi_i\right) + T_i^- \left(\phi_{i-1} - \phi_i\right) + \phi_i \\ &0 = T_i^+ \left(\phi_{i+1} - \phi_i\right) + T_i^- \left(\phi_{i-1} - \phi_i\right) \\ &T_i^+ \left(\phi_{i+1} - \phi_i\right) = T_i^- \left(\phi_i - \phi_{i-1}\right) \\ &T_i^+ V_{i+1} = T_i^- V_i \\ &V_{i+1} = \frac{T_i^-}{T_i^+} V_i \end{aligned}$$

or rewrite it us $V_i = \frac{T_i^-}{T_i^+} V_{i-1}$ and use the boundary condition.

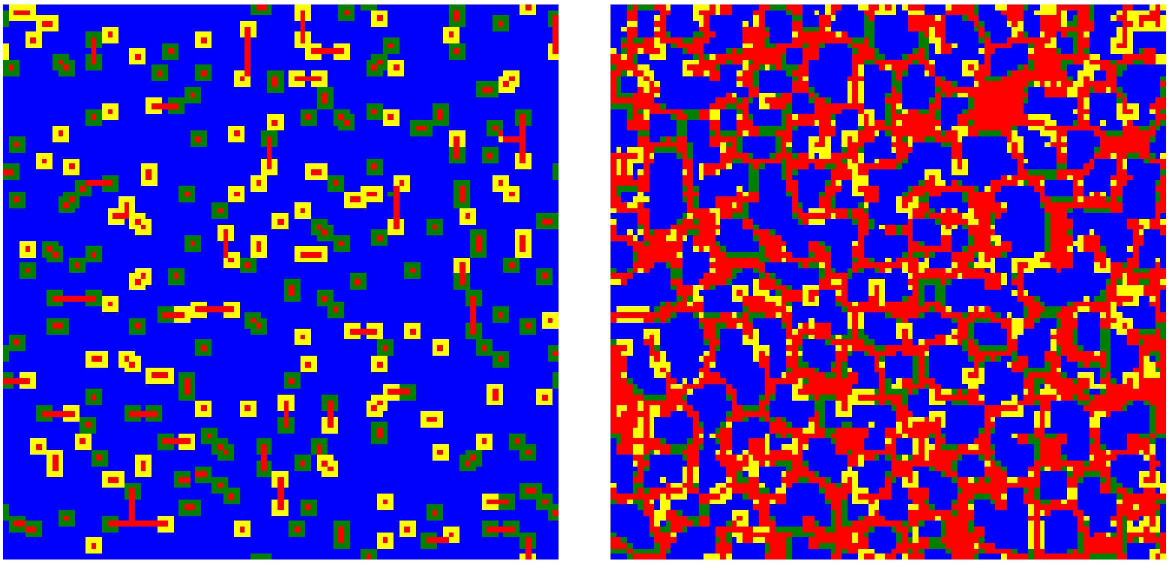
$$\begin{aligned} &V_i = \frac{T_i}{T_i^\dagger} \cdot \frac{T_{i-1}}{T_{i-1}^\dagger} \cdots \phi_1 = \left(\prod_{j=1}^{i-1} \frac{T_j}{T_j} \right) \phi_1 \\ &\sum_{k=1}^{N} V_k = \phi_N = 1 \\ &1 = \sum_{k=1}^{N} V_k = \phi_1 \sum_{k=1}^{N} \prod_{j=1}^{N-1} \frac{T_j}{T_j} = \phi_1 \left(1 + \sum_{k=1}^{N-1} \prod_{j=1}^{k} \frac{T_j}{T_j} \right) \end{aligned}$$

rearrange $\phi_1 = \frac{1}{1+N_{kj}^{N-1} \pi_{j=1}^{k} y_j}$ where $\gamma_j = \frac{T_j^-}{T_j^+}$

□

C

Figures



(a) T = 1.3

(b) T = 1.78

Figure C.1: The simulation begins with the random distribution of C and D on a 101×101 square grid network with fixed boundary conditions. It employs the max-payoff updating rule (refer to [14]) and the PD condition. When T is less than 1, all the nodes cooperate. When T ranges from 1 to 1.8, the spatial pattern changes from isolated points to a spider web.

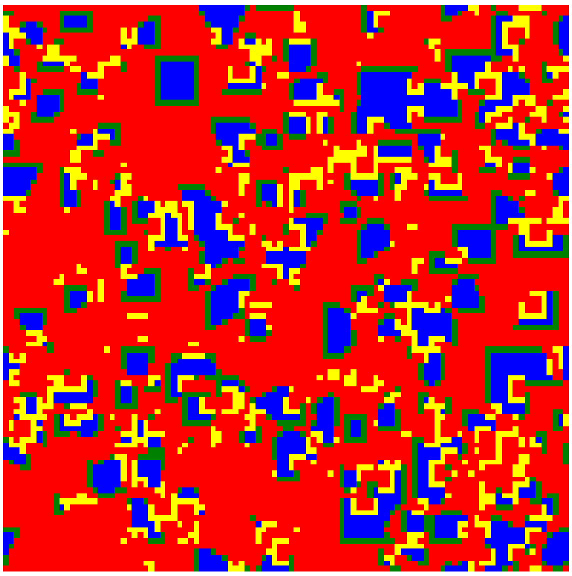


Figure C.2: Spatial chaos characterizes the region $1.8 < T < 2$. C and D coexist indefinitely in a chaotically shifting balance, with the frequency of C being almost completely independent of the initial conditions at approximately 0.318.

D

Tables

D.1. Circle Networks

Based on the symmetry of the circle network, we can categorize similar situations into one class.

Class 1

First, there are 8 special cases. In all eight cases, the center node and the left and right nodes have the same strategy, and the center point does not change its strategy regardless of the total return of the left and right nodes.

Node1	Node 2	Node3	Node 4	Node 5
C	C	C	C	C
C	C	C	C	D
D	C	C	C	C
D	C	C	C	D
C	D	D	D	D
D	D	D	D	C
C	D	D	D	C
D	D	D	D	D

(D.1)

Class 2

The four cases can be summarized as having a central node C, which has two D's connected to one side, and C and a random point on the other side.

Node1	Node 2	Node3	Node 4	Node 5
D	D	C	C	C
D	D	C	C	D
C	C	C	D	D
D	C	C	D	D

(D.2)

Calculate the overall payoff of each node under prisoner's dilemma.

Node 2	Node3	Node 4
T	1 + S	2
T	1 + S	1 + S
2	1 + S	T
1 + S	1 + S	T

(D.3)

Then, change the middle node to D, we have extra 4 cases.

Node1	Node 2	Node3	Node 4	Node 5
D	D	D	C	C
D	D	D	C	D
C	C	D	D	D
D	C	D	D	D

(D.4)

Calculate the overall payoff of each node under prisoner's dilemma.

Node 2	Node3	Node 4
0	T	1 + S
0	T	2S
1 + S	T	0
2S	T	0

(D.5)

By analyzing D.3, we find that if the center node is C, C becomes D unless there is a side linking two C points. And analyzing D.5, we find that point D does not change in this case.

Class 3

The four cases can be summarized as having a central node C, which has two D's connected to two sides.

Node1	Node 2	Node3	Node 4	Node 5
D	D	C	D	C
D	D	C	D	D
C	D	C	D	D
D	D	C	D	D
C	D	C	D	C

(D.6)

Calculate the overall payoff of each node under the prisoner's dilemma.

Node 2	Node3	Node 4
T	2S	2T
T	2S	T
2T	2S	T
T	2S	T
2T	2S	2T

(D.7)

Then, change the middle node to D, which has two C's connected to two sides.

Node1	Node 2	Node3	Node 4	Node 5
D	C	D	C	C
D	C	D	C	D
C	C	D	C	D
D	C	D	C	C
C	C	D	C	C

(D.8)

Calculate the overall payoff of each node under the prisoner's dilemma.

Node 2	Node3	Node 4
2S	2T	1 + S
2S	2T	2S
1 + S	2T	2S
2S	2T	1 + S
1 + S	2T	1 + S

(D.9)

By analyzing D.7, we find that C becomes D. And by analyzing D.9, we find that the central node D does not change in this case.

Class 4

The fourth case can be summarized as a C at the center and a C and a D to the left and right, respectively.

Node1	Node 2	Node3	Node 4	Node 5
C	D	C	C	D
D	C	C	D	C
C	D	C	C	C
C	C	C	D	C

(D.10)

Calculate the overall payoff of each node under the prisoner's dilemma.

Node 2	Node3	Node 4
2T	1 + S	1 + S
1 + S	1 + S	2T
2T	1 + S	2
2	1 + S	2T

(D.11)

Then, change the middle node to D, we have an extra 3 cases.

Node1	Node 2	Node3	Node 4	Node 5
C	D	D	C	D
D	C	D	D	C

(D.12)

Calculate the overall payoff of each node under the prisoner's dilemma.

Node 2	Node3	Node 4
T	1 + S	2S
2S	1 + S	T

(D.13)

By analyzing D.11, we find that the central node C becomes D. And by analyzing D.13, we find that the central node D does not change in this case.

After summarizing all the cases, we find that there are only two results, one in which the central node changes from C to D, and one in which it stays the same. Therefore, we can infer that as the number of iterations increases, more and more C's will turn into D's, eventually leading to all nodes being D's. The evolution of cooperation of the circle network tends to decrease to 0.