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SAHLQVIST VIA TRANSLATION

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Abstract. In recent years, *unified correspondence* has been developed as a generalized Sahlqvist theory which applies uniformly to all signatures of normal and regular (distributive) lattice expansions. A fundamental tool for attaining this level of generality and uniformity is a principled way, based on order theory, to define the Sahlqvist and inductive formulas and inequalities in every such signature. This definition covers in particular all (bi-)intuitionistic modal logics. The theory of these logics has been intensively studied over the past seventy years in connection with classical polyadic modal logics, using versions of Gödel-McKinsey-Tarski translations, suitably defined in each signature, as main tools. In view of this state-of-the-art, it is natural to ask (1) whether a general perspective on Gödel-McKinsey-Tarski translations can be attained, also based on order-theoretic principles like those underlying the general definition of Sahlqvist and inductive formulas and inequalities, which accounts for the known Gödel-McKinsey-Tarski translations and applies uniformly to all signatures of normal (distributive) lattice expansions; (2) whether this general perspective can be used to transfer correspondence and canonicity theorems for Sahlqvist and inductive formulas and inequalities in all signatures described above under Gödel-McKinsey-Tarski translations.

In the present paper, we set out to answer these questions. We answer (1) in the affirmative; as to (2), we prove the transfer of the correspondence theorem for inductive inequalities of arbitrary signatures of normal distributive lattice expansions. We also prove the transfer of canonicity for inductive inequalities, but only restricted to arbitrary normal modal expansions of *bi-intuitionistic logic*. We also analyze the difficulties involved in obtaining the transfer of canonicity outside this setting, and indicate a route to extend the transfer of canonicity to all signatures of normal distributive lattice expansions.

Key words and phrases: Sahlqvist theory, Gödel-McKinsey-Tarski translation, algorithmic correspondence, canonicity, normal distributive lattice expansions, Heyting algebras, co-Heyting algebras, bi-Heyting algebras.

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1. Introduction

Sahlqvist theory has a long history in normal modal logic, going back to [58]. The Sahlqvist theorem in [58] gives a syntactic definition of a class of modal formulas, the *Sahlqvist class*, each member of which defines an elementary (i.e. first-order definable) class of frames and is canonical.

Over the years, many extensions, variations and analogues of this result have appeared, including alternative proofs in e.g. [59], generalizations to arbitrary modal signatures [22], variations of the correspondence language [51, 63], Sahlqvist-type results for hybrid logics [61], various substructural logics [41, 25, 30], mu-calculus [64], and enlargements of the Sahlqvist class to e.g. the *inductive* formulas of [36], to mention but a few.

Recently, a uniform and modular theory has emerged, called *unified correspondence* [11], which subsumes the above results and extends them to logics with a *non-classical* propositional base. It is built on duality-theoretic insights [18] and uniformly exports the state-of-the-art in Sahlqvist theory from normal modal logic to a wide range of logics which include, among others, intuitionistic and distributive and general (non-distributive) lattice-based (modal) logics [15, 17], non-normal (regular) modal logics based on distributive lattices of arbitrary modal signature [53], hybrid logics [20], many valued logics, [43] and bi-intuitionistic and lattice-based modal mu-calculus [5, 7, 6].

The breadth of this work has stimulated many and varied applications. Some are closely related to the core concerns of the theory itself, such as understanding the relationship between different methodologies for obtaining canonicity results [52, 16], or the exploration of the limits of applicability of the theory [71, 70] or of the phenomenon of pseudocorrespondence [19]. Other, possibly surprising applications include the dual characterizations of classes of finite lattices [29], the identification of the syntactic shape of axioms which can be translated into structural rules of a proper display calculus [38] and of internal Gentzen calculi for the logics of strict implication [45], and the epistemic interpretation of lattice-based modal logic in terms of categorization theory in management science [9, 8]. The approach underlying these results relies only on the order-theoretic properties of the algebraic interpretations of logical connectives, abstracting away from specific logical signatures.

Featuring prominently among the logics targeted by these developments are (bi-)intuitionistic logic and their all modal expansions. The theory of these logics has been intensively studied over the past seventy years using the Gödel-McKinsey-Tarski translation [34, 50], henceforth simply the GMT translation, as a key tool. Specifically, since the 1940s and up to the present day, versions of the GMT translation have been used for transferring and reflecting results between classical and intuitionistic logics and their extensions and expansions (see e.g. [48, 2, 27, 28, 4, 67, 68, 69, 65]. More on this in Section 3.1).

In view of this state-of-the-art, it is natural to ask (1) whether a general perspective on Gödel-McKinsey-Tarski translations can be attained, also based on order-theoretic principles like those underlying the general definition of Sahlqvist and inductive formulas and inequalities, which accounts for the known Gödel-McKinsey-Tarski translations and applies uniformly to all signatures of normal (distributive) lattice expansions; (2) whether this general perspective can be used to transfer correspondence and canonicity theorems for Sahlqvist and inductive formulas and inequalities in the signatures described above under Gödel-McKinsey-Tarski translations. In the present paper, we set out to answer these questions.

Notice that in general, GMT translations do not preserve the Sahlqvist shape. For instance, the original GMT translation transforms the Sahlqvist inequality $\Box\Diamond p \leq \Diamond p$ into $\Box\Diamond\Box_G p \leq \Diamond\Box_G p$, which is not Sahlqvist, and in fact does not even have a first-order correspondent [62]. Any translation which 'boxes' propositional variables would suffer from this problem (for further discussion see [11, Section 36.9]). However, *some* GMT translations preserve the shape of *some* Sahlqvist and inductive formulas or inequalities. This has been exploited by Gehrke, Nagahashi and Venema in [32] to obtain the correspondence part of their Sahlqvist theorem for Distributive Modal Logic. The need to establish a suitable match between GMT translations and Sahlqvist and inductive formulas or inequalities in each signature gives us a concrete reason to investigate GMT translations as a class.

The starting point of our analysis, and first contribution of the present paper, is an order-theoretic generalization of the main semantic property of the original GMT translation. We show that this generalization provides a unifying pattern, instantiated in the concrete GMT translations in each setting of interest to the present paper. As an application of this generalization, we prove two transfer results, which are the main contributions of the present paper: the transfer of *generalized Sahlqvist correspondence* from multi-modal (classical) modal logic to logics of arbitrary normal distributive lattice expansions and, in a more restricted setting, the transfer of *generalized Sahlqvist canonicity* from multi-modal (classical) modal logic to logics of arbitrary normal *bi-Heyting algebra* expansions. The transfer of correspondence extends [32, Theorem 3.7] both as regards the setting (from Distributive Modal Logic to arbitrary normal DLE-logics) and the scope (from Sahlqvist to inductive inequalities). The transfer of canonicity is entirely novel also in its formulation, in the sense that it targets specific, syntactically defined classes. Finally, we analyze the difficulties in extending the transfer of canonicity to normal DLE-logics. Thanks to this analysis, we identify a route towards this result, which – however – calls for a much higher level of technical sophistication than required by the usual route.

The paper is structured as follows. Section 2 collects preliminaries on logics of normal DLEs and their algebraic and relational semantics. In Section 3 we discuss GMT translations in various DLE settings in the literature and the semantic underpinnings of the GMT translation for intuitionistic logic. In Section 4 we introduce a general template which accounts for the main semantic property of GMT translations—their being full and faithful—in the setting of ordered algebras of arbitrary signatures. In Section 5, we show that the GMT translations of interest instantiate this template. This sets the stage for Sections 6 and 7 where we present our transfer results. We conclude in Section 8.

2. Preliminaries on normal DLEs and their logics

In this section we collect some basic background on logics of normal distributive lattice expansions (DLEs). All the logics which we will consider in this paper are particular instances of DLE-logics.

2.1. **Language and axiomatization of basic DLE-logics.**

Our base language is an unspecified but fixed language $\mathcal{L}_{\mathrm{DLE}}$, to be interpreted over distributive lattice expansions of compatible similarity type. We will make heavy use of the following auxiliary definition:

an *order-type* over $n \in \mathbb{N}$[1] is an n-tuple $\varepsilon \in \{1, \partial\}^n$. For every order type ε, we denote its *opposite* order type by ε^∂, that is, $\varepsilon^\partial_i = 1$ iff $\varepsilon_i = \partial$ for every $1 \leq i \leq n$. For any lattice $\mathbb{A}$, we let $\mathbb{A}^1 := \mathbb{A}$ and $\mathbb{A}^\partial$ be the dual lattice, that is, the lattice associated with the converse partial order of $\mathbb{A}$. For any order type ε, we let $\mathbb{A}^\varepsilon := \Pi_{i=1}^n \mathbb{A}^{\varepsilon_i}$.

The language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$ (from now on abbreviated as $\mathcal{L}_{\mathrm{DLE}}$) takes as parameters: 1) a denumerable set PROP of proposition letters, elements of which are denoted p, q, r, possibly with indexes; 2) disjoint sets of connectives $\mathcal{F}$ and $\mathcal{G}$. Each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ has arity $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) and is associated with some order-type ε_f over n_f (resp. ε_g over n_g).[2] The terms (formulas) of $\mathcal{L}_{\mathrm{DLE}}$ are defined recursively as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

where $p \in \mathsf{PROP}$, $f \in \mathcal{F}$, $g \in \mathcal{G}$. Terms in $\mathcal{L}_{\mathrm{DLE}}$ will be denoted either by s, t, or by lowercase Greek letters such as φ, ψ, γ etc.

Definition 2.1. For any language $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, the *basic*, or *minimal* $\mathcal{L}_{\mathrm{DLE}}$-*logic* is a set of sequents $\varphi \vdash \psi$, with $\varphi, \psi \in \mathcal{L}_{\mathrm{LE}}$, which contains the following axioms:

- Sequents for lattice operations:

$$p \vdash p, \quad \bot \vdash p, \quad p \vdash \top, \quad p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r),$$
$$p \vdash p \vee q, \quad q \vdash p \vee q, \quad p \wedge q \vdash p, \quad p \wedge q \vdash q,$$

- Sequents for $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$f(p_1, \ldots, \bot, \ldots, p_{n_f}) \vdash \bot, \text{ for } \varepsilon_f(i) = 1,$$
$$f(p_1, \ldots, \top, \ldots, p_{n_f}) \vdash \bot, \text{ for } \varepsilon_f(i) = \partial,$$
$$\top \vdash g(p_1, \ldots, \top, \ldots, p_{n_g}), \text{ for } \varepsilon_g(i) = 1,$$
$$\top \vdash g(p_1, \ldots, \bot, \ldots, p_{n_g}), \text{ for } \varepsilon_g(i) = \partial,$$
$$f(p_1, \ldots, p \vee q, \ldots, p_{n_f}) \vdash f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f}), \text{ for } \varepsilon_f(i) = 1,$$
$$f(p_1, \ldots, p \wedge q, \ldots, p_{n_f}) \vdash f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f}), \text{ for } \varepsilon_f(i) = \partial,$$
$$g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g}) \vdash g(p_1, \ldots, p \wedge q, \ldots, p_{n_g}), \text{ for } \varepsilon_g(i) = 1,$$
$$g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g}) \vdash g(p_1, \ldots, p \vee q, \ldots, p_{n_g}), \text{ for } \varepsilon_g(i) = \partial,$$

and is closed under the following inference rules:

$$\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \quad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \quad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \quad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$$

$$\frac{\varphi \vdash \psi}{f(p_1, \ldots, \varphi, \ldots, p_n) \vdash f(p_1, \ldots, \psi, \ldots, p_n)} \ (\varepsilon_f(i) = 1)$$

$$\frac{\varphi \vdash \psi}{f(p_1, \ldots, \psi, \ldots, p_n) \vdash f(p_1, \ldots, \varphi, \ldots, p_n)} \ (\varepsilon_f(i) = \partial)$$

$$\frac{\varphi \vdash \psi}{g(p_1, \ldots, \varphi, \ldots, p_n) \vdash g(p_1, \ldots, \psi, \ldots, p_n)} \ (\varepsilon_g(i) = 1)$$

[1] Throughout the paper, order-types will be typically associated with arrays of variables $\vec{p} := (p_1, \ldots, p_n)$. When the order of the variables in $\vec{p}$ is not specified, we will sometimes abuse notation and write $\varepsilon(p) = 1$ or $\varepsilon(p) = \partial$.

[2] Unary f (resp. g) will be sometimes denoted as $\Diamond$ (resp. $\Box$) if the order-type is 1, and $\lhd$ (resp. $\rhd$) if the order-type is ∂.

$$\frac{\varphi \vdash \psi}{g(p_1, \ldots, \psi, \ldots, p_n) \vdash g(p_1, \ldots, \varphi, \ldots, p_n)}\ (\varepsilon_g(i) = \partial).$$

The minimal DLE-logic is denoted by $\mathbf{L}_{\mathrm{DLE}}$. For any DLE-language $\mathcal{L}_{\mathrm{DLE}}$, by an DLE-*logic* we understand any axiomatic extension of the basic $\mathcal{L}_{\mathrm{DLE}}$-logic in $\mathcal{L}_{\mathrm{DLE}}$.

Example 2.1. *The DLE setting is extremely general, covering a wide spectrum of non-classical logics. Here we give a few examples, showing how various languages are obtained as specific instantiations of $\mathcal{F}$ and $\mathcal{G}$. The associated logics extended the basic DLE logics corresponding to these instantiations.*

The formulas of intuitionistic logic are obtained by instantiating $\mathcal{F} := \varnothing$ and $\mathcal{G} := \{\rightarrow\}$ with $n_\rightarrow = 2$, and $\varepsilon_\rightarrow = (\partial, 1)$. The formulas of bi-intuitionistic logic (cf. [56]) are obtained by instantiating $\mathcal{F} := \{\succ\!\!-\}$ with $n_{\succ\!-} = 2$ and $\varepsilon_{\succ\!-} = (\partial, 1)$, and $\mathcal{G} := \{\rightarrow\}$ with $n_\rightarrow = 2$ and $\varepsilon_\rightarrow = (\partial, 1)$. The formulas of Fischer Servi's intuitionistic modal logic (cf. [28]), Prior's MIPC (cf. [54]), and G. Bezhanishvili's intuitionistic modal logic with universal modalities (cf. [1]) are obtained by instantiating $\mathcal{F} := \{\Diamond\}$ with $n_\Diamond = 1$ and $\varepsilon_\Diamond = 1$ and $\mathcal{G} := \{\rightarrow, \Box\}$ with $n_\rightarrow = 2$, and $\varepsilon_\rightarrow = (\partial, 1)$, and $n_\Box = 1$, and $\varepsilon_\Box = 1$. The formulas of Wolter's bi-intuitionistic modal logic (cf. [66]) are obtained by instantiating $\mathcal{F} := \{\succ\!\!-, \Diamond\}$ with $n_\Diamond = 1$ and $\varepsilon_\Diamond = 1$ and $n_{\succ\!-} = 2$ and $\varepsilon_{\succ\!-} = (\partial, 1)$, and $\mathcal{G} := \{\rightarrow, \Box\}$ with $n_\rightarrow = 2$, and $\varepsilon_\rightarrow = (\partial, 1)$, and $n_\Box = 1$, and $\varepsilon_\Box = 1$. The formulas of Dunn's positive modal logic *(cf. [24]) are obtained by instantiating $\mathcal{F} := \{\Diamond\}$ with $n_\Diamond = 1$, $\varepsilon_\Diamond = 1$ and $\mathcal{G} := \{\Box\}$ with $n_\Box = 1$ and $\varepsilon_\Box = 1$. The language of Gehrke, Nagahashi and Venema's* distributive modal logic *(cf. [32]) is an expansion of positive modal logic and is obtained by adding the connectives $\lhd$ and $\rhd$ to $\mathcal{F}$ and $\mathcal{G}$, respectively, with $n_\lhd = n_\rhd = 1$ and $\varepsilon_\lhd = \varepsilon_\rhd = \partial$.*

2.2. **Algebraic and relational semantics for basic DLE-logics.** The following definition captures the algebraic setting of the present paper:

Definition 2.2. For any tuple $(\mathcal{F}, \mathcal{G})$ of disjoint sets of function symbols as above, a *distributive lattice expansion* (abbreviated as DLE) is a tuple $\mathbb{A} = (L, \mathcal{F}^\mathbb{A}, \mathcal{G}^\mathbb{A})$ such that L is a bounded distributive lattice, $\mathcal{F}^\mathbb{A} = \{f^\mathbb{A} \mid f \in \mathcal{F}\}$ and $\mathcal{G}^\mathbb{A} = \{g^\mathbb{A} \mid g \in \mathcal{G}\}$, such that every $f^\mathbb{A} \in \mathcal{F}^\mathbb{A}$ (resp. $g^\mathbb{A} \in \mathcal{G}^\mathbb{A}$) is an n_f-ary (resp. n_g-ary) operation on $\mathbb{A}$. A DLE is *normal* if every $f^\mathbb{A} \in \mathcal{F}^\mathbb{A}$ (resp. $g^\mathbb{A} \in \mathcal{G}^\mathbb{A}$) preserves finite (hence also empty) joins (resp. meets) in each coordinate with $\varepsilon_f(i) = 1$ (resp. $\varepsilon_g(i) = 1$) and reverses finite (hence also empty) meets (resp. joins) in each coordinate with $\varepsilon_f(i) = \partial$ (resp. $\varepsilon_g(i) = \partial$).[3] Let $\mathbb{DLE}$ be the class of normal DLEs. Sometimes we will refer to certain DLEs as $\mathcal{L}_{\mathrm{DLE}}$-algebras when we wish to emphasize that these algebras have a compatible signature with the logical language we have fixed. A distributive lattice is *perfect* if it is complete, completely distributive and completely join-generated by its completely join-prime elements. Equivalently, a distributive

[3] Normal DLEs are sometimes referred to as *distributive lattices with operators* (DLOs). This terminology derives from the setting of Boolean algebras with operators, in which operators are understood as operations which preserve finite (hence also empty) joins in each coordinate. Thanks to the Boolean negation, operators are typically taken as primitive connectives, and all the other modal operations are reduced to these. However, this terminology is somewhat ambiguous in the lattice setting, in which primitive operations are typically maps which are operators if seen as $\mathbb{A}^\varepsilon \to \mathbb{A}^\eta$ for some order-type ε on n and some order-type $\eta \in \{1, \partial\}$. Rather than speaking of lattices with (ε, η)-operators, we then speak of normal DLEs. This terminology is also used in other papers developing Sahlqvist-type results at a level of generality comparable to that of the present paper, e.g. [38, 6].

lattice is perfect iff it is isomorphic to the lattice of up-sets of some poset. A normal DLE is *perfect* if its lattice-reduct is a perfect distributive lattice, and each f-operation (resp. g-operation) is completely join-preserving (resp. meet-preserving) in the coordinates i such that $\varepsilon_f(i) = 1$ (resp. $\varepsilon_g(i) = 1$) and completely meet-reversing (resp. join-reversing) in the coordinates i such that $\varepsilon_f(i) = \partial$ (resp. $\varepsilon_g(i) = \partial$). The *canonical extension* of a normal DLE $\mathbb{A} = (L, \mathcal{F}, \mathcal{G})$ is the perfect normal DLE $\mathbb{A}^\delta := (L^\delta, \mathcal{F}^\sigma, \mathcal{G}^\pi)$, where L^δ is the canonical extension of L,[4] and $\mathcal{F}^\sigma := \{f^\sigma \mid f \in \mathcal{F}\}$ and $\mathcal{G}^\pi := \{g^\pi \mid g \in \mathcal{G}\}$.[5] Canonical extensions of Heyting algebras, Brouwerian algebras and bi-Heyting algebras are defined by instantiating the definition above in the corresponding signatures. The canonical extension of any Heyting (resp. Brouwerian, bi Heyting) algebra is a (perfect) Heyting (resp. Brouwerian, bi-Heyting) algebra.

In the present paper we also find it convenient to talk of normal Boolean algebra expansions (BAEs) (respectively, normal Heyting algebra expansions (HAEs), normal bi-Heyting algebra expansions (bHAEs)) which are structures defined as in Definition 2.2, but replacing the distributive lattice L with a Boolean algebra (respectively, Heyting algebra, bi-Heyting algebra). The logics corresponding to these classes will be collectively referred to as normal BAE-logics, normal HAE-logics, normal bHAE-logics. In what follows we will typically drop the adjective 'normal'.

[4]The *canonical extension* of a bounded lattice L is a complete lattice L^δ containing L as a sublattice, such that:

1. *(denseness)* every element of L^δ is both the join of meets and the meet of joins of elements from L;
2. *(compactness)* for all $S, T \subseteq L$, if $\bigwedge S \leq \bigvee T$ in L^δ, then $\bigwedge F \leq \bigvee G$ for some finite sets $F \subseteq S$ and $G \subseteq T$.

[5]An element $k \in L^\delta$ (resp. $o \in L^\delta$) is *closed* (resp. *open*) if is the meet (resp. join) of some subset of L. We let $K(L^\delta)$ (resp. $O(L^\delta)$) denote the set of the closed (resp. open) elements of L^δ. For every unary, order-preserving map $h : L \to M$ between bounded lattices, the σ-*extension* of h is defined firstly by declaring, for every $k \in K(L^\delta)$,

$$h^\sigma(k) := \bigwedge\{h(a) \mid a \in L \text{ and } k \leq a\},$$

and then, for every $u \in L^\delta$,

$$h^\sigma(u) := \bigvee\{h^\sigma(k) \mid k \in K(L^\delta) \text{ and } k \leq u\}.$$

The π-*extension* of f is defined firstly by declaring, for every $o \in O(L^\delta)$,

$$h^\pi(o) := \bigvee\{h(a) \mid a \in L \text{ and } a \leq o\},$$

and then, for every $u \in L^\delta$,

$$h^\pi(u) := \bigwedge\{h^\pi(o) \mid o \in O(L^\delta) \text{ and } u \leq o\}.$$

The definitions above apply also to operations of any finite arity and order-type. Indeed, taking order-duals interchanges closed and open elements: $K((L^\delta)^\partial) = O(L^\delta)$ and $O((L^\delta)^\partial) = K(L^\delta)$; similarly, $K((L^n)^\delta) = K(L^\delta)^n$, and $O((L^n)^\delta) = O(L^\delta)^n$. Hence, $K((L^\delta)^\varepsilon) = \prod_i K(L^\delta)^{\varepsilon(i)}$ and $O((L^\delta)^\varepsilon) = \prod_i O(L^\delta)^{\varepsilon(i)}$ for every lattice L and every order-type ε over any $n \in \mathbb{N}$, where

$$K(L^\delta)^{\varepsilon(i)} := \begin{cases} K(L^\delta) & \text{if } \varepsilon(i) = 1 \\ O(L^\delta) & \text{if } \varepsilon(i) = \partial \end{cases} \quad O(L^\delta)^{\varepsilon(i)} := \begin{cases} O(L^\delta) & \text{if } \varepsilon(i) = 1 \\ K(L^\delta) & \text{if } \varepsilon(i) = \partial. \end{cases}$$

From this it follows that $(L^\partial)^\delta$ can be identified with $(L^\delta)^\partial$, $(L^n)^\delta$ with $(L^\delta)^n$, and $(L^\varepsilon)^\delta$ with $(L^\delta)^\varepsilon$ for any order type ε over n, where $L^\varepsilon := \prod_{i=1}^n L^{\varepsilon(i)}$. These identifications make it possible to obtain the definition of σ-and π-extensions of ε-monotone operations of any arity n and order-type ε over n by instantiating the corresponding definitions given above for monotone and unary functions.

In the remainder of the paper, we will abuse notation and write e.g. f for $f^{\mathbb{A}}$ when this causes no confusion. Normal DLEs constitute the main semantic environment of the present paper. Henceforth, since every DLE is assumed to be normal, the adjective will be typically dropped. The class of all DLEs is equational, and can be axiomatized by the usual distributive lattice identities and the following equations for any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) and $1 \leq i \leq n_f$ (resp. for each $1 \leq j \leq n_g$):

- if $\varepsilon_f(i) = 1$, then $f(p_1, \ldots, p \vee q, \ldots, p_{n_f}) = f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f})$; moreover if $f \in \mathcal{F}_n$, then $f(p_1, \ldots, \bot, \ldots, p_{n_f}) = \bot$,
- if $\varepsilon_f(i) = \partial$, then $f(p_1, \ldots, p \wedge q, \ldots, p_{n_f}) = f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f})$; moreover if $f \in \mathcal{F}_n$, then $f(p_1, \ldots, \top, \ldots, p_{n_f}) = \bot$,
- if $\varepsilon_g(j) = 1$, then $g(p_1, \ldots, p \wedge q, \ldots, p_{n_g}) = g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g})$; moreover if $g \in \mathcal{G}_n$, then $g(p_1, \ldots, \top, \ldots, p_{n_g}) = \top$,
- if $\varepsilon_g(j) = \partial$, then $g(p_1, \ldots, p \vee q, \ldots, p_{n_g}) = g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g})$; moreover if $g \in \mathcal{G}_n$, then $g(p_1, \ldots, \bot, \ldots, p_{n_g}) = \top$.

Each language $\mathcal{L}_{\mathrm{DLE}}$ is interpreted in the appropriate class of DLEs. In particular, for every DLE $\mathbb{A}$, each operation $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) is finitely join-preserving (resp. meet-preserving) in each coordinate when regarded as a map $f^{\mathbb{A}} : \mathbb{A}^{\varepsilon_f} \to \mathbb{A}$ (resp. $g^{\mathbb{A}} : \mathbb{A}^{\varepsilon_g} \to \mathbb{A}$).

For every DLE $\mathbb{A}$, the symbol $\vdash$ is interpreted as the lattice order $\leq$. A sequent $\varphi \vdash \psi$ is valid in $\mathbb{A}$ if $h(\varphi) \leq h(\psi)$ for every homomorphism h from the $\mathcal{L}_{\mathrm{DLE}}$-algebra of formulas over PROP to $\mathbb{A}$. The notation $\mathbb{DLE} \models \varphi \vdash \psi$ indicates that $\varphi \vdash \psi$ is valid in every DLE. Then, by means of a routine Lindenbaum-Tarski construction, it can be shown that the minimal DLE-logic $\mathbf{L}_{\mathrm{DLE}}$ is sound and complete with respect to its corresponding class of algebras $\mathbb{DLE}$, i.e. that any sequent $\varphi \vdash \psi$ is provable in $\mathbf{L}_{\mathrm{DLE}}$ iff $\mathbb{DLE} \models \varphi \vdash \psi$.

Definition 2.3. An $\mathcal{L}_{\mathrm{DLE}}$-*frame* is a tuple $\mathbb{F} = (\mathbb{X}, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$ such that $\mathbb{X} = (W, \leq)$ is a (nonempty) poset, $\mathcal{R}_{\mathcal{F}} = \{R_f \mid f \in \mathcal{F}\}$, and $\mathcal{R}_{\mathcal{G}} = \{R_g \mid g \in \mathcal{G}\}$ such that for each $f \in \mathcal{F}$, the symbol R_f denotes an $(n_f + 1)$-ary relation on W such that for all $\overline{w}, \overline{v} \in \mathbb{X}^{\eta_f}$,

$$\text{if } R_f(\overline{w}) \text{ and } \overline{w} \leq^{\eta_f} \overline{v}, \text{ then } R_f(\overline{v}), \tag{2.1}$$

where η_f is the order-type on $n_f + 1$ defined as follows: $\eta_f(1) = 1$ and $\eta_f(i+1) = \varepsilon_f^{\partial}(i)$ for each $1 \leq i \leq n_f$.

Likewise, for each $g \in \mathcal{G}$, the symbol R_g denotes an $(n_g + 1)$-ary relation on W such that for all $\overline{w}, \overline{v} \in \mathbb{X}^{\eta_g}$,

$$\text{if } R_g(\overline{w}) \text{ and } \overline{w} \geq^{\eta_g} \overline{v}, \text{ then } R_g(\overline{v}), \tag{2.2}$$

where η_g is the order-type on $n_g + 1$ defined as follows: $\eta_g(1) = 1$ and $\eta_g(i+1) = \varepsilon_g^{\partial}(i)$ for each $1 \leq i \leq n_g$.

An $\mathcal{L}_{\mathrm{DLE}}$-*model* is a tuple $\mathbb{M} = (\mathbb{F}, V)$ such that $\mathbb{F}$ is an $\mathcal{L}_{\mathrm{DLE}}$-frame, and $V : \mathsf{Prop} \to \mathcal{P}^{\uparrow}(W)$ is a persistent valuation.

The defining clauses for the interpretation of each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ on $\mathcal{L}_{\mathrm{DLE}}$-models are given as follows:

$$
\begin{array}{lll}
\mathbb{M}, w \Vdash f(\overline{\varphi}) & \text{iff} & \text{there exists some } \overline{v} \in W^{n_f} \text{ s.t. } R_f(w, \overline{v}) \\
& & \text{and } \mathbb{M}, v_i \Vdash^{\varepsilon_f(i)} \varphi_i \text{ for each } 1 \leq i \leq n_f, \\
\mathbb{M}, w \Vdash g(\overline{\varphi}) & \text{iff} & \text{for any } \overline{v} \in W^{n_g}, \text{ if } R_g(w, \overline{v}) \text{ then } \mathbb{M}, v_i \Vdash^{\varepsilon_g(i)} \varphi_i \\
& & \text{for some } 1 \leq i \leq n_g,
\end{array}
$$

where $\Vdash^1$ is $\Vdash$ and $\Vdash^{\partial}$ is $\nVdash$.

2.3. **Sahlqvist and Inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities.** In the present subsection, we recall the definitions of *Sahlqvist* and *inductive* $\mathcal{L}_{\mathrm{DLE}}$-inequalities (cf. [17, Definition 3.4]), which we will show to be preserved and reflected under suitable GMT translations in Section 6. These definitions capture the non-classical counterparts, in each normal modal signature, of the classes of Sahlqvist ([58]) and inductive ([36]) formulas. The definition is given in terms of the order-theoretic properties of the interpretation of the logical connectives (cf. [11, 15, 17] for expanded discussions on the design principles of this definition). The fact that this notion applies uniformly across arbitrary normal modal signatures makes it possible to give a very general yet mathematically precise formulation to the question about the transfer of Sahlqvist-type results under GMT translations.

Technically speaking, these definitions are given parametrically in an order type. This contrasts to with the classical case, where the constantly-1 order-type is sufficient to encompass all Sahlqvist formulas. As a result, the preservation of the syntactic shape of each of these inequalities requires a GMT translation parametrized by the same order-type. These parametric translations will be introduced in Section 5.2.

Definition 2.4 (**Signed Generation Tree**)**.** The *positive* (resp. *negative*) *generation tree* of any $\mathcal{L}_{\mathrm{DLE}}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows:

- For any node labelled with $\vee$ or $\wedge$, assign the same sign to its children nodes.
- For any node labelled with $h \in \mathcal{F} \cup \mathcal{G}$ of arity $n_h \geq 1$, and for any $1 \leq i \leq n_h$, assign the same (resp. the opposite) sign to its ith child node if $\varepsilon_h(i) = 1$ (resp. if $\varepsilon_h(i) = \partial$).

Nodes in signed generation trees are *positive* (resp. *negative*) if are signed $+$ (resp. $-$).

Signed generation trees will be mostly used in the context of term inequalities $s \leq t$. In this context we will typically consider the positive generation tree $+s$ for the left-hand side and the negative one $-t$ for the right-hand side.

For any term $s(p_1, \ldots p_n)$, any order type ε over n, and any $1 \leq i \leq n$, an *ε-critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\varepsilon_i = 1$ or $-p_i$ with $\varepsilon_i = \partial$. An *ε-critical branch* in the tree is a branch from an ε-critical node. The intuition is that variable occurrences corresponding to ε-critical nodes are those that the algorithm ALBA will solve for in the process of eliminating them.

For every term $s(p_1, \ldots p_n)$ and every order type ε, we say that $+s$ (resp. $-s$) *agrees with* ε, and write $\varepsilon(+s)$ (resp. $\varepsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ε-critical. In other words, $\varepsilon(+s)$ (resp. $\varepsilon(-s)$) means that all variable occurrences corresponding to leaves of $+s$ (resp. $-s$) are to be solved for according to ε. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\varepsilon(\gamma) \prec *s$ (resp. $\varepsilon^\partial(\gamma) \prec *s$) to indicate that the signed subtree γ, with the sign inherited from $*s$, agrees with ε (resp. with ε^∂).

Definition 2.5. Nodes in signed generation trees will be called Δ-*adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, and *syntactically right adjoint (SRA)*, according to the specification given in Table 1. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and P_2 consists (apart from variable nodes) only of Skeleton-nodes. A good branch in which the nodes in P_1 are all SRA is called *excellent*.

Skeleton	PIA
Δ-adjoints	SRA
$+\ \vee\ \wedge$	$+\ \wedge\ g$ with $n_g = 1$
$-\ \wedge\ \vee$	$-\ \vee\ f$ with $n_f = 1$
SLR	SRR
$+\ \wedge\ f$ with $n_f \geq 1$	$+\ \vee\ g$ with $n_g \geq 2$
$-\ \vee\ g$ with $n_g \geq 1$	$-\ \wedge\ f$ with $n_f \geq 2$

TABLE 1. Skeleton and PIA nodes for DLE.

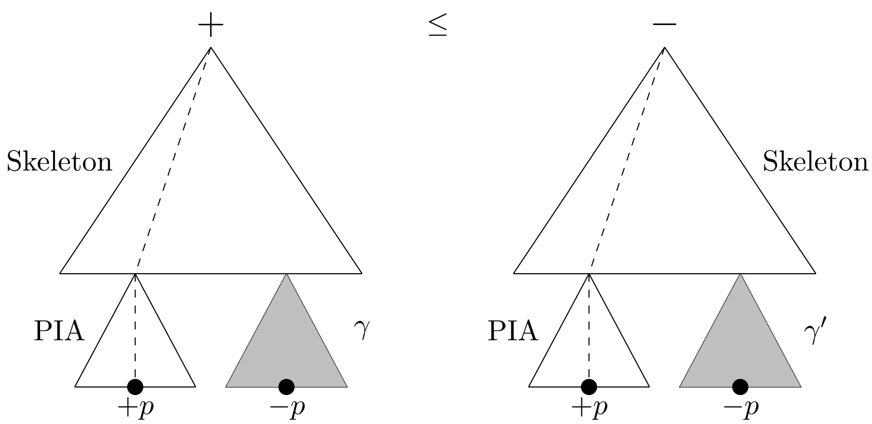


FIGURE 1. A schematic representation of inductive inequalities.

Definition 2.6 (Sahlqvist and Inductive inequalities)**.** For any order type ε, the signed generation tree $*s$ ($* \in \{-,+\}$) of a term $s(p_1,\dots p_n)$ is *ε-Sahlqvist* if for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is excellent (cf. Definition 2.5). An inequality $s \leq t$ is *ε-Sahlqvist* if the signed generation trees $+s$ and $-t$ are ε-Sahlqvist. An inequality $s \leq t$ is *Sahlqvist* if it is ε-Sahlqvist for some ε.

For any order type ε and any irreflexive and transitive relation $<_\Omega$ on $p_1,\dots p_n$, the signed generation tree $*s$ ($* \in \{-,+\}$) of a term $s(p_1,\dots p_n)$ is *(Ω,ε)-inductive* if

1. for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is good (cf. Definition 2.5);
2. every m-ary SRR-node occurring in the critical branch is of the form $\circledast(\gamma_1,\dots,\gamma_{j-1},\beta,\gamma_{j+1}\dots,\gamma_m)$, where for any $h \in \{1,\dots,m\} \setminus j$:
   (a) $\varepsilon^\partial(\gamma_h) \prec *s$ (cf. discussion before Definition 2.5), and
   (b) $p_k <_\Omega p_i$ for every p_k occurring in γ_h and for every $1 \leq k \leq n$.

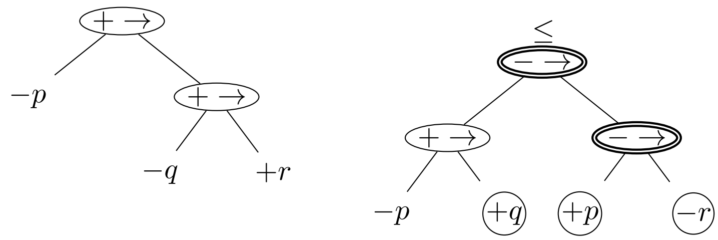
We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is *(Ω,ε)-inductive* if the signed generation trees $+s$ and $-t$ are (Ω,ε)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω,ε)-inductive for some $<_\Omega$ and ε.

The definition above specializes so as to account for all the settings in which GMT translations have been defined. Below we expand on a selection of these.

Example 2.2 (Intuitionistic language)**.** *As observed in* [15]*, the Frege inequality*

$$p \to (q \to r) \leq (p \to q) \to (p \to r)$$

is not Sahlqvist for any order type, but is (Ω, ε)*-inductive, e.g. for* $r <_\Omega p <_\Omega q$ *and* $\varepsilon(p, q, r) = (1, 1, \partial)$*, as can be seen from the signed generation trees below:*



In the picture above, the circled variable occurrences are the ε*-critical ones, the nodes in double ellipses are Skeleton, and those in single ellipses are PIA.*

Example 2.3 (Bi-intuitionistic language)**.** *In* [56, Section 4]*, Rauszer axiomatizes bi-intuitionistic logic considering the following axioms among others, which we present in the form of inequalities:*

$$r \succ\!\!- (q \succ\!\!- p) \leq (p \vee q) \succ\!\!- p \qquad (q \succ\!\!- p) \to \bot \leq p \to q.$$

The first inequality is not (Ω, ε)*-inductive for any* Ω *and* ε*; indeed, in the negative generation tree of* $(p \vee q) \succ\!\!- p$*, the variable* p *occurs in both subtrees rooted at the children of the root, which is a binary SRR node, making it impossible to satisfy condition 2(b) of Definition 2.6 for any order-type* ε *and strict ordering* Ω*.*

The second inequality is ε*-Sahlqvist for* $\varepsilon(p) = 1$ *and* $\varepsilon(q) = \partial$*, and is also* (Ω, ε)*-inductive but not Sahlqvist for* $q <_\Omega p$ *and* $\varepsilon(p) = \varepsilon(q) = \partial$*. It is also* (Ω, ε)*-inductive but not Sahlqvist for* $p <_\Omega q$ *and* $\varepsilon(p) = \varepsilon(q) = 1$*.*

Example 2.4 (Intuitionistic bi-modal language)**.** *The following Fischer Servi inequalities (cf.* [28]*)*

$$\Diamond(q \to p) \leq \Box q \to \Diamond p \qquad \Diamond q \to \Box p \leq \Box(q \to p),$$

are both ε*-Sahlqvist for* $\varepsilon(p) = \partial$ *and* $\varepsilon(q) = 1$*, and are also both* (Ω, ε)*-inductive but not Sahlqvist for* $p <_\Omega q$ *and* $\varepsilon(p) = \partial$ *and* $\varepsilon(q) = \partial$*.*

Example 2.5 (Distributive modal language)**.** *The following inequalities are key to Dunn's positive modal logic* [24]*, the language of which is the* $\{\lhd, \rhd\}$*-free fragment of the language of Distributive Modal Logic* [32]*:*

$$\Box q \wedge \Diamond p \leq \Diamond(q \wedge p) \qquad \Box(q \vee p) \leq \Diamond q \vee \Box p.$$

The inequality on the left (resp. right) is ε*-Sahlqvist for* $\varepsilon(p) = \varepsilon(q) = 1$ *(resp.* $\varepsilon(p) = \varepsilon(q) = \partial$*), and is* (Ω, ε)*-inductive but not Sahlqvist for* $p <_\Omega q$ *and* $\varepsilon(p) = 1$ *and* $\varepsilon(q) = \partial$ *(resp.* $p <_\Omega q$ *and* $\varepsilon(p) = \partial$ *and* $\varepsilon(q) = 1$*).*

3. The GMT translations

In this section we give a brief overview of some highlights in the history of the Gödel-McKinsey-Tarski translation, its extensions and variations and the uses to which they have been put. We then recall the technical details of the translation and how its is founded upon the interplay of persistent and arbitrary valuations in intuitionistic Kripke frames.

3.1. **Brief history.** The GMT translation originates in the work of Gödel [34] and its algebraic underpinnings were analysed by McKinsey and Tarski in [49, 47]. In particular, in [49] develops the theory of closure algebras, i.e. S4-algebras, while [47] shows that every Brouwerian algebra embeds as the subalgebra of closed subsets of some closure algebra. In [50] this analysis is used to show that the GMT translation and a number of variations are full and faithful. This result is extended by Dummett and Lemmon [23] in 1959 to all intermediate logics. In 1967 Grzegorczyk [39] showed that Dummett and Lemmon's result also holds if one replaces normal extensions of S4 with normal extension of what is now known as Grzegorczyk's logic. In light these developments, Maksimova and Rybakov [46] launched a systematic study of the relationship between the lattice of intermediate logics and that of normal extensions of S4.

In 1976 Blok [2] and Esakia [27] independently built on this work to establish an isomorphism, based the GMT translation, between the lattice of intermediate logics and the lattice of normal extensions of Grzegorczyk's logic.

Between 1989 and 1992, several theorems are proven by Zakharyaschev and Chagrov about the preservation under GMT-based translations of properties such as decidability, finite model property, Kripke and Hallden completeness, disjunction property, and compactness. For detailed surveys on this line of research, the reader is referred to [3] and [69].

The first extension of the GMT translation to modal expansions of intuitionistic logic is introduced by Fischer Servi [28] in 1977. Between 1979 and 1984 Shehtman and Sotirov also published on extensions (cf. discussions in [3, 69]).

In the mid 90s, building on the work of Fischer Servi and Shehtman, Wolter and Zakharyaschev [68] use an extension of the GMT translation to prove the transfer of a number of results, including finite model property, canonicity, decidability, tabularity and Kripke completeness, for intuitionistic modal logic with box only. In Remark 7.2 we will expand on the relationship of their canonicity results (cf. [68, Theorem 12]) with those of the present paper. In [67] this line of work is extended to intuitionistic modal logics with box and diamond. Duality theory is developed and the transfer of decidability, finite model property and tabularity under a suitably extended GMT translation is established. Moreover, a Blok-Esakia theorem is proved in this setting.

This research programme is further pursued by Wolter [66] in the setting bi-intuitionistic modal logic, obtained by adding the left residual of the disjunction (also known as co-implication) to the language of intuitionistic modal logic. He develops duality theory for these logics, extends the GMT translation and established a Blok-Esakia theorem. Recent developments include work by G. Bezhanishvili who considers expansions of Prior's MIPC with universal modalities, extends the GMT translation and establishes a Blok-Esakia theorem.

As the outline above shows, the research program on the transfer of results via GMT translations includes many transfers of completeness and canonicity (the latter in the form of d-persistence) which are closely related to the focus of the present paper. However, there are very few transfer results which specifically concern Sahlqvist theory, and this for the obvious reason that the formulation of such results depends on the availability of an independent definition of Sahlqvist formulas for each setting of (modal expansions of) non-classical logics, and these definitions have not been introduced in any of these settings, neither together nor in isolation, until 2005, when Gehrke, Nagahashi and Venema [32] introduced the notion of

Sahlqvist inequalities in the language of Distributive Modal Logic (DML).[6] This definition made it possible to formulate the Sahlqvist correspondence theorem for DML-inequalities (cf. [32, Theorem 3.7]) and prove it via reduction to a suitable classical poly-modal logic using GMT translations.

We conclude this brief survey by mentioning a recent paper by van Benthem, N. Bezhanishvili and Holliday [65] which studies a GMT-like translation from modal logic with possibility semantics into bi-modal logic. They prove that this translation transfers and reflects first-order correspondence, but point out that it destroys the Sahlqvist shape of formulas in all but a few special cases.

3.2. **Semantic analysis.** In the present section we recall the definition of the GMT translation in its original setting, highlight its basic semantic underpinning as a toggle between persistent and non-persistent valuations on S4-frames. This analysis will be extended to a uniform account of the GMT translation for arbitrary normal DLE-logics in the next section.

In what follows, for any partial order $(W, \leq)$, we let $w{\uparrow} := \{v \in W \mid w \leq v\}$, $w{\downarrow} := \{v \in W \mid w \geq v\}$ for every $w \in W$, and for every $X \subseteq W$, we let $X{\uparrow} := \bigcup_{x\in X} x{\uparrow}$ and $X{\downarrow} := \bigcup_{x\in X} x{\downarrow}$. *Up-sets* (resp. *down-sets*) of $(W, \leq)$ are subsets $X \subseteq W$ such that $X = X{\uparrow}$ (resp. $X = X{\downarrow}$). We denote by $\mathcal{P}(W)$ the Boolean algebra of subsets of W, and by $\mathcal{P}^{\uparrow}(W)$ (resp. $\mathcal{P}^{\downarrow}(W)$) the (bi-)Heyting algebra of up-sets (resp. down-sets) of $(W, \leq)$. Finally we let X^c denote the relative complement $W \setminus X$ of every subset $X \subseteq W$.

Fix a denumerable set Prop of propositional variables. The language of intuitionistic logic over Prop is given by

$$\mathcal{L}_I \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

The language of the normal modal logic S4 over Prop is given by

$$\mathcal{L}_{S4\Box} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \Box_{\leq}\alpha.$$

The GMT translation is the map $\tau\colon \mathcal{L}_I \to \mathcal{L}_{S4\Box}$ defined by the following recursion:

$$\begin{array}{rcl}
\tau(p) & = & \Box_{\leq} p\\
\tau(\bot) & = & \bot\\
\tau(\top) & = & \top\\
\tau(\varphi \wedge \psi) & = & \tau(\varphi) \wedge \tau(\psi)\\
\tau(\varphi \vee \psi) & = & \tau(\varphi) \vee \tau(\psi)\\
\tau(\varphi \to \psi) & = & \Box_{\leq}(\neg\tau(\varphi) \vee \tau(\psi)).
\end{array}$$

Both intuitionistic and S4-formulas can be interpreted on partial orders $\mathbb{F} = (W, \leq)$, as follows: an S4-model is a tuple $(\mathbb{F}, U)$ where $U : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation. The interpretation $\Vdash^*$ of S4-formulas on S4-models is defined recursively as follows: for an $w \in W$,

$$\begin{array}{ll}
\mathbb{F}, w, U \Vdash^* p & \text{iff } p \in U(p)\\
\mathbb{F}, w, U \Vdash^* \bot & \text{never}\\
\mathbb{F}, w, U \Vdash^* \top & \text{always}\\
\mathbb{F}, w, U \Vdash^* \alpha \wedge \beta & \text{iff } \mathbb{F}, w, U \Vdash^* \alpha \text{ and } \mathbb{F}, w, U \Vdash^* \beta\\
\mathbb{F}, w, U \Vdash^* \alpha \vee \beta & \text{iff } \mathbb{F}, w, U \Vdash^* \alpha \text{ or } \mathbb{F}, w, U \Vdash^* \beta\\
\mathbb{F}, w, U \Vdash^* \neg\alpha & \text{iff } \mathbb{F}, w, U \nVdash^* \alpha\\
\mathbb{F}, w, U \Vdash^* \Box_{\leq}\alpha & \text{iff } \mathbb{F}, v, U \Vdash^* \alpha \text{ for any } v \in w{\uparrow}.
\end{array}$$

[6] We refer to [11, 17] for a systematic comparison between [32, Definition 3.4] and the definition of Sahlqvist and inductive inequalities for normal DLE-languages (cf. Definition 2.6).

For any S4-formula α we let $(\![\alpha]\!)_U := \{w \mid \mathbb{F}, w, U \Vdash^* \alpha\}$. It is not difficult to verify that for every $\alpha \in \mathcal{L}_{S4}$ and any valuation U,

$$(\![\Box_{\leq}\alpha]\!)_U = (\![\alpha]\!)_U^c{\downarrow}^c. \tag{3.1}$$

An intuitionistic model is a tuple $(\mathbb{F}, V)$ where $V : \mathsf{Prop} \to \mathcal{P}^{\uparrow}(W)$ is a *persistent* valuation. The interpretation $\Vdash^*$ of S4-formulas on S4-models is defined recursively as follows: for an $w \in W$,

$\mathbb{F}, w, V \Vdash p$	iff $p \in V(p)$
$\mathbb{F}, w, V \Vdash \bot$	never
$\mathbb{F}, w, V \Vdash \top$	always
$\mathbb{F}, w, V \Vdash \varphi \wedge \psi$	iff $\mathbb{F}, w, V \Vdash \varphi$ and $\mathbb{F}, w, V \Vdash \psi$
$\mathbb{F}, w, V \Vdash \varphi \vee \psi$	iff $\mathbb{F}, w, V \Vdash \varphi$ or $\mathbb{F}, w, V \Vdash \psi$
$\mathbb{F}, w, V \Vdash \varphi \to \psi$	iff either $\mathbb{F}, v, V \nVdash \varphi$ or $\mathbb{F}, v, V \Vdash \psi$ for any $v \in w{\uparrow}$.

For any intuitionistic formula φ we let $[\![\varphi]\!]_V := \{w \mid \mathbb{F}, w, V \Vdash \varphi\}$. It is not difficult to verify that for all $\varphi, \psi \in \mathcal{L}_I$ and any persistent valuation V,

$$[\![\varphi \to \psi]\!]_V = ([\![\varphi]\!]_V^c \cup [\![\psi]\!]_V)^c{\downarrow}^c. \tag{3.2}$$

Clearly, every persistent valuation V on $\mathbb{F}$ is also a valuation on $\mathbb{F}$. Moreover, for every valuation U on $\mathcal{F}$, the assignment mapping every $p \in \mathsf{Prop}$ to $U(p)^c{\downarrow}^c$ defines a persistent valuation $U^{\uparrow}$ on $\mathbb{F}$. The main semantic property of the GMT translation is stated in the following well-known proposition:

Proposition 3.1. *For every intuitionistic formula φ and every partial order $\mathbb{F} = (W, \leq)$,*

$$\mathbb{F} \Vdash \varphi \quad \textit{iff} \quad \mathbb{F} \Vdash^* \tau(\varphi).$$

Proof. If $\mathbb{F} \nVdash \varphi$, then $\mathbb{F}, w, V \nVdash \varphi$ for some persistent valuation V and $w \in W$. That is, $w \notin [\![\varphi]\!]_V = (\![\tau(\varphi)]\!)_V$, the last identity holding by item 1 of Lemma 3.2. Hence, $\mathbb{F}, w, V \nVdash^* \tau(\varphi)$, i.e. $\mathbb{F} \nVdash^* \tau(p)$. Conversely, if $\mathbb{F} \nVdash^* \tau(\varphi)$, then $\mathbb{F}, w, U \nVdash \tau(\varphi)$ for some valuation U and $w \in W$. That is, $w \notin (\![\tau(\varphi)]\!)_U = [\![\varphi]\!]_{U^{\uparrow}}$, the last identity holding by item 2 of Lemma 3.2. Hence, $\mathbb{F}, w, U^{\uparrow} \nVdash \varphi$, yielding $\mathbb{F} \nVdash \varphi$. ∎

Lemma 3.2. *For every intuitionistic formula φ and every partial order $\mathbb{F} = (W, \leq)$,*
1. *$[\![\varphi]\!]_V = (\![\tau(\varphi)]\!)_V$ for every persistent valuation V on $\mathbb{F}$;*
2. *$(\![\tau(\varphi)]\!)_U = [\![\varphi]\!]_{U^{\uparrow}}$ for every valuation U on $\mathbb{F}$.*

Proof. 1. By induction on φ. As for the base case, let $\varphi := p \in \mathsf{Prop}$. Then, for any persistent valuation V,

$$\begin{array}{rcll} [\![p]\!]_V & = & V(p) & \text{(def. of } [\![\cdot]\!]_V) \\ & = & V(p)^c{\downarrow}^c & (V \text{ persistent}) \\ & = & (\![\Box_{\leq} p]\!)_V & \text{(equation (3.1))} \\ & = & (\![\tau(p)]\!)_V, & \text{(def. of } \tau) \end{array}$$

as required. As for the inductive step, let $\varphi := \psi \to \chi$. Then, for any persistent valuation V,

$$\begin{array}{rcll} [\![\psi \to \chi]\!]_V & = & ([\![\psi]\!]_V^c \cup [\![\chi]\!]_V)^c{\downarrow}^c & \text{(equation (3.1))} \\ & = & ((\![\tau(\psi)]\!)_V^c \cup (\![\tau(\chi)]\!)_V)^c{\downarrow}^c & \text{(induction hypothesis)} \\ & = & (\![\Box_{\leq}(\neg\tau(\psi) \vee \tau(\chi))]\!)_V & \text{(equation (3.1), def. of } (\![\cdot]\!)_V) \\ & = & (\![\tau(\psi \to \chi)]\!)_V, & \text{(def. of } \tau) \end{array}$$

as required. The remaining cases are omitted.

2. By induction on φ. As for the base case, let $\varphi := p \in \mathsf{Prop}$. Then, for any valuation U,

$$\begin{array}{rcll} (\![\tau(p)]\!)_U & = & (\![\Box_{\leq} p]\!)_U & \text{(def. of } \tau) \\ & = & (\![p]\!)_U^c {\downarrow}^c & \text{(equation (3.1))} \\ & = & U(p)^c {\downarrow}^c & \text{(def. of } (\![\cdot]\!)_U) \\ & = & [\![p]\!]_{U^{\uparrow}}, & \text{(def. of } U^{\uparrow}) \end{array}$$

as required. As for the inductive step, let $\varphi := \psi \to \chi$. Then, for any valuation U,

$$\begin{array}{rcll} (\![\tau(\psi \to \chi)]\!)_U & = & (\![\Box_{\leq}(\neg\tau(\psi) \vee \tau(\chi))]\!)_U & \text{(def. of } \tau) \\ & = & (\![\neg\tau(\psi) \vee \tau(\chi)]\!)_U^c {\downarrow}^c & \text{(equation (3.1))} \\ & = & ((\![\tau(\psi)]\!)_U^c \cup (\![\tau(\chi)]\!)_U)^c {\downarrow}^c & \text{(def. of } (\![\cdot]\!)_U) \\ & = & ([\![\psi]\!]_{U^{\uparrow}}^c \cup [\![\chi]\!]_{U^{\uparrow}})^c {\downarrow}^c & \text{(induction hypothesis)} \\ & = & [\![\psi \to \chi]\!]_{U^{\uparrow}}, & \text{(equation (3.2), } U^{\uparrow} \text{ persistent)} \end{array}$$

as required. The remaining cases are omitted. □

Hence, the main semantic property of GMT translation, stated in Proposition 3.1, can be understood in terms of the interplay between persistent and nonpersistent valuations, as captured in the above lemma. In the next section, we are going to establish a general template for this interplay and then apply it to the algebraic analysis of GMT translations in various settings.

4. Unifying analysis of GMT translations

In the present section, we generalize the key mechanism captured in Section 3.2 and guaranteeing the preservation and reflection of validity under the GMT translation. Being able to identify this pattern in generality will make it possible to recognise this mechanism in several logical settings, as we do in the following section.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages over a given set X, and let $\mathbb{A}$ and $\mathbb{B}$ be ordered $\mathcal{L}_1$- and $\mathcal{L}_2$-algebras respectively, such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists. For each $V \in \mathbb{A}^X$ and $U \in \mathbb{B}^X$, let $[\![\cdot]\!]_V$ and $(\![\cdot]\!)_U$ denote their unique homomorphic extensions to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Clearly, $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ lifts to a map $\overline{e} : \mathbb{A}^X \to \mathbb{B}^X$ by the assignment $V \mapsto e \circ V$.

Proposition 4.1. *Let $\tau\colon \mathcal{L}_1 \to \mathcal{L}_2$ and $r\colon \mathbb{B}^X \to \mathbb{A}^X$ be such that the following conditions hold for every $\varphi \in \mathcal{L}_1$:*

(a) *$e([\![\varphi]\!]_V) = (\![\tau(\varphi)]\!)_{\overline{e}(V)}$ for every $V \in \mathbb{A}^X$;*

(b) *$(\![\tau(\varphi)]\!)_U = e([\![\varphi]\!]_{r(U)})$ for every $U \in \mathbb{B}^X$.*

Then, for all $\varphi, \psi \in \mathcal{L}_1$,

$$\mathbb{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B} \models \tau(\varphi) \leq \tau(\psi).$$

Proof. From left to right, suppose contrapositively that $(\mathbb{B}, U) \not\models \tau(\varphi) \leq \tau(\psi)$ for some $U \in \mathbb{B}^X$, that is, $(\![\tau(\varphi)]\!)_U \not\leq (\![\tau(\psi)]\!)_U$. By item (b) above, this non-inequality is equivalent to $e([\![\varphi]\!]_{r(U)}) \not\leq e([\![\psi]\!]_{r(U)})$, which, by the monotonicity of e, implies that $[\![\varphi]\!]_{r(U)} \not\leq [\![\psi]\!]_{r(U)}$, that is, $(\mathbb{A}, r(U)) \not\models \varphi \leq \psi$, as required. Conversely, if $(\mathbb{A}, V) \not\models \varphi \leq \psi$ for some $V \in \mathbb{A}^X$, then $[\![\varphi]\!]_V \not\leq [\![\psi]\!]_V$, and hence, since e is an order-embedding and by item (a) above,

$(\!|\tau(\varphi)|\!)_{\overline{e}(V)} = e([\![\varphi]\!]_V) \not\leq e([\![\psi]\!]_V) = (\!|\tau(\psi)|\!)_{\overline{e}(V)}$, that is $(\mathbb{B}, \overline{e}(V)) \not\models \tau(\varphi) \leq \tau(\psi)$, as required. □

In the proof above we have only made use of the assumption that e is an order-embedding, but have not needed to assume any property of r. Moreover, the proposition above is independent of the logical/algebraic signature of choice, and hence can be used as a general template accounting for the main property of GMT translations. Finally, the proposition holds for *arbitrary* ordered algebras. This latter point is key to the treatment of Sahlqvist canonicity via translation.

5. Instantiations of the general template for GMT translations

In the present section, we look into a family of GMT-type translations, defined for different logics, to which we apply the template of Section 4. We organize these considerations into two subsections, in the first of which we treat the GMT translations for intuitionistic and co-intuitionistic logic and discuss how these extend to bi-intuitionistic logic. In the second subsection we consider parametrized versions of the GMT translation in the style of Gehrke, Nagahashi and Venema [32] for normal modal expansions of bi-intuitionistic and DLE-logics.

5.1. Non-parametric GMT translations.

As is well known, the semantic underpinnings of the GMT translations for intuitionistic and co-intuitionistic logic are the embeddings of Heyting algebras as the algebras of open elements of interior algebras and of Brouwerian algebras (aka co-Heyting algebras) as algebras of closed elements of closure algebras [47]. In the following subsections, the existence of these embeddings will be used to satisfy the key assumptions of Section 4.

5.1.1. *Interior operator analysis of the GMT translation for intuitionistic logic.*

As observed above, Proposition 4.1 generalizes Proposition 3.1 in more than one way. In the present subsection, we show that the GMT translation for intuitionistic logic verifies the conditions of Proposition 4.1. This is an alternative proof of the well-known fact that the GMT-translation is full and faithful, not only with respect to perfect algebras (dual to frames), but also with respect to general algebras. This fact is necessary for the proof of the transfer of Sahlqvist canonicity (cf. Section 7). Towards this goal, we let $X := \mathsf{Prop}$, $\mathcal{L}_1 := \mathcal{L}_I$, and $\mathcal{L}_2 := \mathcal{L}_{S4}$. Moreover, we let $\mathbb{A}$ be a Heyting algebra, and $\mathbb{B}$ a Boolean algebra such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and has a right adjoint[7] $\iota\colon \mathbb{B} \to \mathbb{A}$ such that for all $a, b \in \mathbb{A}$,

$$a \to^{\mathbb{A}} b = \iota(\neg^{\mathbb{B}} e(a) \vee^{\mathbb{B}} e(b)). \tag{5.1}$$

Then $\mathbb{B}$ can be endowed with a natural structure of Boolean algebra expansion (BAE) by defining $\Box^{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ \iota)(b)$. The following is a well known fact in algebraic modal logic:

[7] That is, $e(a) \leq b$ iff $a \leq \iota(b)$ for every $a \in \mathbb{A}$ and $b \in \mathbb{B}$. By well known order-theoretic facts (cf. [21]), $e \circ \iota$ is an *interior operator*, that is, for every $b, b' \in \mathbb{B}$,

i1. $(e \circ \iota)(b) \leq b$;

i2. if $b \leq b'$ then $(e \circ \iota)(b) \leq (e \circ \iota)(b')$;

i3. $(e \circ \iota)(b) \leq (e \circ \iota)((e \circ \iota)(b))$.

Moreover, $e \circ \iota \circ e = e$ and $\iota = \iota \circ e \circ \iota$ (cf. [21, Lemma 7.26]).

Proposition 5.1. *The BAE* $(\mathbb{B}, \Box^{\mathbb{B}})$*, with* $\Box^{\mathbb{B}}$ *defined above, is normal and is also an S4-modal algebra.*

Proof. The fact that $\Box^{\mathbb{B}}$ preserves finite (hence empty) meets readily follows from the fact that ι is a right adjoint, and hence preserves existing (thus all finite) meets of $\mathbb{B}$, and e is a lattice homomorphism. For every $b \in \mathbb{B}$, $\iota(b) \leq \iota(b)$ implies that $\Box^{\mathbb{B}} b = e(\iota(b)) \leq b$, which proves (T). For every $b \in \mathbb{B}$, $e(\iota(b)) \leq e(\iota(b))$ implies that $\iota(b) \leq \iota(e(\iota(b)))$ and hence $\Box^{\mathbb{B}} b = e(\iota(b)) \leq e(\iota(e(\iota(b)))) = (e \circ \iota)((e \circ \iota)(b)) = \Box^{\mathbb{B}}\Box^{\mathbb{B}} b$, which proves K4. □

Finally, we let $r : \mathbb{B}^X \to \mathbb{A}^X$ be defined by the assignment $U \mapsto (\iota \circ U)$.

Proposition 5.2. *Let* $\mathbb{A}$*,* $\mathbb{B}$*,* $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ *and* $r\colon \mathbb{B}^X \to \mathbb{A}^X$ *be as above.*[8] *Then the GMT translation* τ *satisfies conditions (a) and (b) of Proposition 4.1 for any formula* $\varphi \in \mathcal{L}_I$*.*

Proof. By induction on φ. As for the base case, let $\varphi := p \in \mathsf{Prop}$. Then, for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

$$\begin{array}{rcll}
e([\![p]\!]_{r(U)}) & = & e((\iota \circ U)(p)) & \\
& = & (e \circ \iota)((\![p]\!)_U) & \text{assoc. of } \circ \\
& = & \Box^{\mathbb{B}}(\![p]\!)_U & \\
& = & (\![\Box_{\leq} p]\!)_U & \\
& = & (\![\tau(p)]\!)_U. &
\end{array}
\qquad
\begin{array}{rcll}
(\![\tau(p)]\!)_{\overline{e}(V)} & = & (\![\Box_{\leq} p]\!)_{\overline{e}(V)} & \\
& = & \Box^{\mathbb{B}}(\![p]\!)_{\overline{e}(V)} & \\
& = & \Box^{\mathbb{B}}((e \circ V)(p)) & \\
& = & (e \circ \iota)((e \circ V)(p)) & \\
& = & e((\iota \circ e)(V(p))) & \text{assoc. of } \circ \\
& = & e(V(p)) & e \circ (\iota \circ e) = e \\
& = & e([\![p]\!]_V), &
\end{array}$$

which proves the base cases of (b) and (a) respectively. As for the inductive step, let $\varphi := \psi \to \chi$. Then, for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

$$\begin{array}{rcll}
e([\![\psi \to \chi]\!]_{r(U)}) & = & e([\![\psi]\!]_{r(U)} \to^{\mathbb{A}} [\![\chi]\!]_{r(U)}) & \\
& = & e(\iota(\neg^{\mathbb{B}} e([\![\psi]\!]_{r(U)}) \vee^{\mathbb{B}} e([\![\chi]\!]_{r(U)}))) & \text{assumption (5.1)} \\
& = & e(\iota(\neg^{\mathbb{B}}(\![\tau(\psi)]\!)_U \vee^{\mathbb{B}} (\![\tau(\chi)]\!)_U)) & \text{(induction hypothesis)} \\
& = & (e \circ \iota)(\neg^{\mathbb{B}}(\![\tau(\psi)]\!)_U \vee^{\mathbb{B}} (\![\tau(\chi)]\!)_U) & \\
& = & \Box^{\mathbb{B}}(\neg^{\mathbb{B}}(\![\tau(\psi)]\!)_U \vee^{\mathbb{B}} (\![\tau(\chi)]\!)_U) & \\
& = & (\![\Box_{\leq}(\neg\tau(\psi) \vee \tau(\chi))]\!)_U & \\
& = & (\![\tau(\psi \to \chi)]\!)_U. &
\end{array}$$

$$\begin{array}{rcll}
e([\![\psi \to \chi]\!]_V) & = & e([\![\psi]\!]_V \to^{\mathbb{A}} [\![\chi]\!]_V) & \\
& = & e(\iota(\neg^{\mathbb{B}} e([\![\psi]\!]_V) \vee^{\mathbb{B}} e([\![\chi]\!]_V))) & \text{assumption (5.1)} \\
& = & e(\iota(\neg^{\mathbb{B}}(\![\tau(\psi)]\!)_{\overline{e}(V)} \vee^{\mathbb{B}} (\![\tau(\chi)]\!)_{\overline{e}(V)})) & \text{(induction hypothesis)} \\
& = & (e \circ \iota)(\neg^{\mathbb{B}}(\![\tau(\psi)]\!)_{\overline{e}(V)} \vee^{\mathbb{B}} (\![\tau(\chi)]\!)_{\overline{e}(V)}) & \\
& = & \Box^{\mathbb{B}}(\neg^{\mathbb{B}}(\![\tau(\psi)]\!)_{\overline{e}(V)} \vee^{\mathbb{B}} (\![\tau(\chi)]\!)_{\overline{e}(V)}) & \\
& = & (\![\Box_{\leq}(\tau(\psi) \vee \tau(\chi))]\!)_{\overline{e}(V)} & \\
& = & (\![\tau(\psi \to \chi)]\!)_{\overline{e}(V)}. &
\end{array}$$

The remaining cases are straightforward, and are left to the reader. □

The following strengthening of Proposition 3.1 immediately follows from Propositions 4.1 and 5.2:

[8]The assumption that e is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ is needed for the inductive steps relative to $\bot, \top, \wedge, \vee$ in the proof this proposition, while condition (5.1) is needed for the step relative to $\to$.

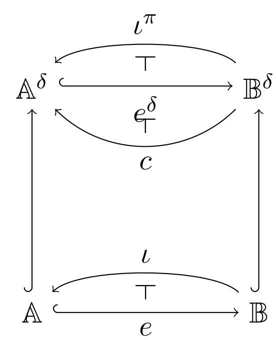
Corollary 5.3. *Let $\mathbb{A}$ be a Heyting algebra and $\mathbb{B}$ a Boolean algebra such that $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ and $\iota\colon \mathbb{B} \to \mathbb{A}$ exist as above. Then for all intuitionistic formulas φ and ψ,*

$$\mathbb{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B} \models \tau(\varphi) \leq \tau(\psi),$$

where τ is the GMT translation.

We conclude this subsection by showing that the required embedding of Heyting algebras into Boolean algebras exists. This well known fact appears in its order-dual version already in [47, Theorem 1.15], where is shown in a purely algebraic way. We give an alternative proof, based on very well known duality-theoretic facts. The format of the statement is different from the dual of [47, Theorem 1.15] both because it is in the form required by Proposition 4.1, and also because it highlights that the embedding lifts to the canonical extensions of the algebras involved, where it acquires additional properties. This is relevant to the analysis of canonicity via translation in Section 7.

Proposition 5.4. *For every Heyting algebra $\mathbb{A}$, there exists a Boolean algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ verifying condition* (5.1). *Finally, these facts lift to the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ as in the following diagram:*



Proof. Via Esakia duality [26], the Heyting algebra $\mathbb{A}$ can be identified with the algebra of clopen up-sets of its associated Esakia space $\mathbb{X}_{\mathbb{A}}$, which is a Priestley space, hence a Stone space. Let $\mathbb{B}$ be the Boolean algebra of the clopen sub*sets* of $\mathbb{X}_{\mathbb{A}}$. Since any clopen up-set is in particular a clopen subset, a natural order embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a lattice homomorphism between $\mathbb{A}$ and $\mathbb{B}$. This shows the first part of the claim.

As to the second part, notice that Esakia spaces are Priestley spaces in which the downward-closure of a clopen set is a clopen set.

Therefore, we can define the map $\iota\colon \mathbb{B} \to \mathbb{A}$ by the assignment $b \mapsto \neg((\neg b){\downarrow})$ where b is identified with its corresponding clopen set in $\mathbb{X}_{\mathbb{A}}$, $\neg b$ is identified with the relative complement of the clopen set b, and $(\neg b){\downarrow}$ is defined using the order in $\mathbb{X}_{\mathbb{A}}$. It can be readily verified that ι is the right adjoint of e and that moreover condition (5.1) holds.

Finally, $e : \mathbb{A} \to \mathbb{B}$ being also a homomorphism between the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ implies that e is smooth and its canonical extension $e^{\delta}\colon \mathbb{A}^{\delta} \to \mathbb{B}^{\delta}$, besides being an order-embedding, is a complete homomorphism between the lattice reducts of $\mathbb{A}^{\delta}$ and $\mathbb{B}^{\delta}$ (cf. [31, Corollary 4.8]), and hence is endowed with both a left and a right adjoint. Furthermore, the right adjoint of e^{δ} coincides with ι^{π} (cf. [33, Proposition 4.2]). Hence, $\mathbb{B}^{\delta}$ can be endowed with a natural structure of S4 bi-modal algebra by defining $\Box_{\leq}^{\mathbb{B}^{\delta}} : \mathbb{B}^{\delta} \to \mathbb{B}^{\delta}$ by the assignment $u \mapsto (e^{\delta} \circ \iota^{\pi})(u)$, and $\Diamond_{\geq}^{\mathbb{B}^{\delta}} : \mathbb{B}^{\delta} \to \mathbb{B}^{\delta}$ by the assignment $u \mapsto (e^{\delta} \circ c)(u)$. □

5.1.2. *The GMT translation for co-intuitionistic logic.* In the present subsection, we show that the GMT translation for co-intuitionistic logic (which we will sometimes refer to as the co-GMT translation) verifies the conditions of Proposition 4.1. Our presentation is a straightforward dualization of the previous subsection. We include it for the sake of completeness and for introducing some notation. Semantically, this dualization involves replacing Heyting algebras with co-Heyting algebras (aka Brouwerian algebras cf. [47]), and interior algebras (aka S4 algebras with box) with closure algebras (aka S4 algebras with diamond).

Fix a denumerable set Prop of propositional variables. The language of co-intuitionistic logic over Prop is given by

$$\mathcal{L}_C \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \succ\!\!- \varphi.$$

The target language for translating co-intuitionistic logic is that of the normal modal logic S4$\Diamond$ over Prop, given by

$$\mathcal{L}_{S4\Diamond} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \Diamond_{\geq}\alpha.$$

Just like intuitionistic logic, formulas of co-intuitionistic logic can be interpreted on partial orders $\mathbb{F} = (W, \leq)$ with persistent valuations. Here we only report on the interpretation of $\Diamond_{\geq}$-formulas in $\mathcal{L}_{S4\Diamond}$ and $\succ\!\!-$-formulas in $\mathcal{L}_C$:

$$\begin{array}{lll} \mathbb{F}, w, U \Vdash^* \Diamond_{\geq}\varphi & \text{iff} & \mathbb{F}, v, U \Vdash^* \varphi \ \text{ for some } v \in w{\downarrow}. \\ \mathbb{F}, w, V \Vdash \varphi \succ\!\!- \psi & \text{iff} & \mathbb{F}, v, V \not\Vdash \varphi \ \text{ and } \ \mathbb{F}, v, V \Vdash \psi \text{ for some } v \in w{\downarrow}. \end{array}$$

The language $\mathcal{L}_C$ is naturally interpreted in co-Heyting algebras. The connective $\succ\!\!-$ is interpreted as the left residual of $\vee$. The co-GMT translation is the map $\sigma\colon \mathcal{L}_C \to \mathcal{L}_{S4\Diamond}$ defined by the following recursion:

$$\begin{array}{rcl} \sigma(p) & = & \Diamond_{\geq} p \\ \sigma(\bot) & = & \bot \\ \sigma(\top) & = & \top \\ \sigma(\varphi \wedge \psi) & = & \sigma(\varphi) \wedge \sigma(\psi) \\ \sigma(\varphi \vee \psi) & = & \sigma(\varphi) \vee \sigma(\psi) \\ \sigma(\varphi \succ\!\!- \psi) & = & \Diamond_{\geq}(\neg\sigma(\varphi) \wedge \sigma(\psi)) \end{array}$$

Next, we show that Proposition 4.1 applies to the co-GMT translation. We let $X := \mathsf{Prop}$, $\mathcal{L}_1 := \mathcal{L}_C$, and $\mathcal{L}_2 := \mathcal{L}_{S4\Diamond}$. Moreover, we let $\mathbb{A}$ be a co-Heyting algebra, and $\mathbb{B}$ a Boolean algebra such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and has a left adjoint[9] $c\colon \mathbb{B} \to \mathbb{A}$ such that for all $a, b \in \mathbb{A}$,

$$a \succ\!\!-^{\mathbb{A}} b = c(\neg^{\mathbb{B}} e(a) \wedge^{\mathbb{B}} e(b)). \tag{5.2}$$

Then $\mathbb{B}$ can be endowed with a natural structure of Boolean algebra expansion (BAE) by defining $\Diamond^{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ c)(b)$. The following is the dual of Proposition 5.1 and its proof is omitted.

Proposition 5.5. *The BAE $(\mathbb{B}, \Diamond^{\mathbb{B}})$, with $\Diamond^{\mathbb{B}}$ defined above, is normal and is also an S4$\Diamond$-modal algebra.*

[9]That is, $c(b) \leq a$ iff $b \leq e(a)$ for every $a \in \mathbb{A}$ and $b \in \mathbb{B}$. By well known order-theoretic facts (cf. [21]), $e \circ c$ is an *interior operator*, that is, for every $b, b' \in \mathbb{B}$,

c1. $b \leq (e \circ c)(b)$;
c2. if $b \leq b'$ then $(e \circ c)(b) \leq (e \circ c)(b')$;
c3. $(e \circ c)((e \circ c)(b)) \leq (e \circ c)(b)$.

Moreover, $e \circ c \circ e = e$ and $c = c \circ e \circ c$ (cf. [21, Lemma 7.26]).

Finally, we let $r : \mathbb{B}^X \to \mathbb{A}^X$ be defined by the assignment $U \mapsto (c \circ U)$. The proof of the following proposition is similar to that of Proposition 5.2, and its proof is omitted.

Proposition 5.6. *Let $\mathbb{A}$, $\mathbb{B}$, $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ and $r\colon \mathbb{B}^X \to \mathbb{A}^X$ be as above.*[10] *Then the co-GMT translation σ satisfies conditions (a) and (b) of Proposition 4.1 for any formula $\varphi \in \mathcal{L}_C$.*

The following corollary immediately follows from Propositions 4.1 and 5.6:

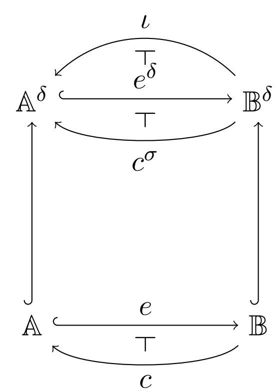
Corollary 5.7. *Let $\mathbb{A}$ be a co-Heyting algebra and $\mathbb{B}$ a Boolean algebra such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and has a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ such that condition (5.2) holds for all $a, b \in \mathbb{A}$. Then for all $\varphi, \psi \in \mathcal{L}_C$,*

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \sigma(\varphi) \leq \sigma(\psi),$$

where σ is the co-GMT translation.

As in the previous subsection, we conclude by giving a version of [47, Theorem 1.15] which we need to instantiate Proposition 4.1 and for the analysis of canonicity via translation in Section 7. The proof is dual to that of Proposition 5.4.

Proposition 5.8. *For every co-Heyting algebra $\mathbb{A}$, there exists a Boolean algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and has a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ verifying condition (5.2). Finally, these facts lift to the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ as in the following diagram:*



5.1.3. *Extending the GMT and co-GMT translations to bi-intuitionistic logic.* In the present subsection we consider the extensions of the GMT and co-GMT translations to bi-intuitionistic logic (aka Heyting-Brouwer logic according to the terminology of Rauszer [56] who introduced this logic in the same paper, see also [55, 57]). The extension τ' of the GMT translation considered below coincides with the one introduced by Wolter in [66] restricted to the language of bi-intuitionistic logic. The paper [66] considers a modal expansion of bi-intuitionistic logic with box and diamond operators where the extended GMT translation is used to establish a Blok-Esakia result and to transfer properties such as completeness, finite model property and decidability.

[10] The assumption that e is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ is needed for the inductive steps relative to $\bot, \top, \wedge, \vee$ in the proof this proposition, while condition (5.2) is needed for the step relative to $\succ\!\!-$.

The language of bi-intuitionistic logic is given by

$$\mathcal{L}_B \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \succ\!\!- \varphi$$

The language of the normal bi-modal logic S4 is given by

$$\mathcal{L}_{S4B} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \neg\alpha \mid \Box_{\leq}\alpha \mid \Diamond_{\geq}\alpha$$

The GMT and the co-GMT translations τ and σ can be extended to the bi-intuitionistic language as the maps $\tau', \sigma' \colon \mathcal{L}_B \to \mathcal{L}_{S4B}$ defined by the following recursions:

$$\begin{array}{rclcrcl}
\tau'(p) & = & \Box_{\leq} p & & \sigma'(p) & = & \Diamond_{\geq} p \\
\tau'(\bot) & = & \bot & & \sigma'(\bot) & = & \bot \\
\tau'(\top) & = & \top & & \sigma'(\top) & = & \top \\
\tau'(\varphi \wedge \psi) & = & \tau'(\varphi) \wedge \tau'(\psi) & & \sigma(\varphi \wedge \psi) & = & \sigma'(\varphi) \wedge \sigma'(\psi) \\
\tau'(\varphi \vee \psi) & = & \tau'(\varphi) \vee \tau'(\psi) & & \sigma'(\varphi \vee \psi) & = & \sigma'(\varphi) \vee \sigma'(\psi) \\
\tau'(\varphi \to \psi) & = & \Box_{\leq}(\neg\tau'(\varphi) \vee \tau'(\psi)) & & \sigma'(\varphi \to \psi) & = & \Box_{\leq}(\neg\sigma'(\varphi) \vee \sigma'(\psi)). \\
\tau'(\varphi \succ\!\!- \psi) & = & \Diamond_{\geq}(\neg\tau'(\varphi) \wedge \tau'(\psi)) & & \sigma'(\varphi \succ\!\!- \psi) & = & \Diamond_{\geq}(\neg\sigma'(\varphi) \wedge \sigma'(\psi)).
\end{array}$$

Notice that τ' and σ' agree on each defining clause but those relative to the proposition variables. Let $\mathbb{A}$ be a bi-Heyting algebra and $\mathbb{B}$ a Boolean algebra such that $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ is an order-embedding and a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$. Suppose that e has both a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ and a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ such that identities (5.1) and (5.2) hold for every $a, b \in \mathbb{A}$. Then $\mathbb{B}$ can be endowed with a natural structure of bi-modal S4-algebra by defining $\Box^{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ \iota)(b)$ and $\Diamond^{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ c)(b)$.

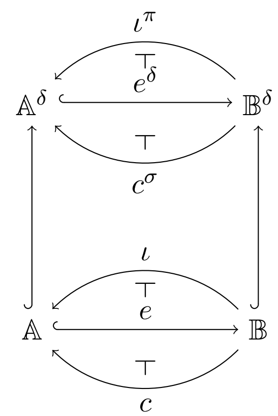
Proposition 5.9. *The BAE $(\mathbb{B}, \Box^{\mathbb{B}}, \Diamond^{\mathbb{B}})$, with $\Box^{\mathbb{B}}, \Diamond^{\mathbb{B}}$ defined as above, is normal and an S4-bimodal algebra.*

The following proposition show that Proposition 4.1 applies to τ' and σ'. We let $X :=$ Prop. The proof is similar to those of Propositions 5.2 and 5.6, and is omitted.

Proposition 5.10. *The translation τ' (resp. σ') defined above satisfies conditions (a) and (b) of Proposition 4.1 relative to $r : \mathbb{B}^X \to \mathbb{A}^X$ defined by $U \mapsto (\iota \circ U)$ (resp. defined by $U \mapsto (c \circ U)$).*

Thanks to the proposition above, Proposition 4.1 applies to both τ' and σ' provided suitable embeddings of bi-Heyting algebras into Boolean algebras exist. The existence of such embeddings is proven by Rauszer in [56, Section 4]. We now give a reformulation of this result in the format required to instantiate Proposition 4.1 and highlighting the compatibility of this embedding with canonical extensions — as well as a duality-based proof.

Proposition 5.11. *For every bi-Heyting algebra $\mathbb{A}$, there exists a Boolean algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has both a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ and a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ verifying conditions* (5.1) *and* (5.2)*. Finally, all these facts lift to the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ as in the following diagram:*



Proof. Via Esakia-type duality [26, 66], the bi-Heyting algebra $\mathbb{A}$ can be identified with the algebra of clopen up-sets of its associated dual space $\mathbb{X}_\mathbb{A}$ (referred to here as a a bi-Esakia space), which is a Priestley space, hence a Stone space. Let $\mathbb{B}$ be the Boolean algebra of the clopen sub*sets* of $\mathbb{X}_\mathbb{A}$. Since any clopen up-set is in particular a clopen subset, a natural order embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a lattice homomorphism between $\mathbb{A}$ and $\mathbb{B}$. This shows the first part of the claim. As to the second part, bi-Esakia spaces are Priestley spaces such that both the upward-closure and the downward-closure of a clopen set is a clopen set.

Therefore, we can define the maps $c\colon \mathbb{B} \to \mathbb{A}$ and $\iota\colon \mathbb{B} \to \mathbb{A}$ by the assignments $b \mapsto b{\uparrow}$ and $b \mapsto \neg((\neg b){\downarrow})$ respectively, where b is identified with its corresponding clopen set in $\mathbb{X}_\mathbb{A}$, $\neg b$ is defined as the relative complement of b in $\mathbb{X}_\mathbb{A}$, and $b{\uparrow}$ and $(\neg b){\downarrow}$ are defined using the order in $\mathbb{X}_\mathbb{A}$. It can be readily verified that c and ι are the left and right adjoints of e respectively, and that moreover conditions (5.1) and (5.2) hold.

Finally, $e : \mathbb{A} \to \mathbb{B}$ being also a homomorphism between the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ implies that e is smooth and its canonical extension $e^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$, besides being an order-embedding, is a complete homomorphism between the lattice reducts of $\mathbb{A}^\delta$ and $\mathbb{B}^\delta$ (cf. [31, Corollary 4.8]), and hence is endowed with both a left and a right adjoint. Furthermore, the left (resp. right) adjoint of e^δ coincides with c^σ (resp. with ι^π) (cf. [33, Proposition 4.2]). Hence, $\mathbb{B}^\delta$ can be naturally endowed with the structure of an S4 bi-modal algebra by defining $\Box_\leq^{\mathbb{B}^\delta}\colon \mathbb{B}^\delta \to \mathbb{B}^\delta$ by the assignment $u \mapsto (e^\delta \circ \iota^\pi)(u)$, and $\Diamond_\geq^{\mathbb{B}^\delta}\colon \mathbb{B}^\delta \to \mathbb{B}^\delta$ by the assignment $u \mapsto (e^\delta \circ c^\sigma)(u)$. □

5.2. **Parametric GMT translations.** In this section we extend the previous translations to a parametric set of GMT translations for each normal DLE- and bHAE-logic (cf. Section 2). Parametric GMT translations were already considered in [32] in the context of one specific DLE signature, namely that of Distributive Modal Logic, where they are used to prove the transfer of correspondence results for ε-Sahlqvist inequalities in every order-type ε (cf. Definition 2.6). We generalize this idea to arbitrary DLE-logics, and explore the additional properties of GMT translations in the setting of bHAE-logics.

5.2.1. *Parametric GMT translations for normal DLE-logics.* In the present section we consider parametric GMT translations for the general DLE-setting. We will use the following notation: for every Boolean algebra $\mathbb{B}$, n-tuple $\overline{b}$ of elements of $\mathbb{B}$ and every order-type η on n, we let $\overline{b}^\eta := (b'_i)_{i=1}^n$ where $b'_i = b_i$ if $\eta(i) = 1$ and $b'_i = \neg b_i$ if $\eta(i) = \partial$. Let us fix a normal DLE-signature $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$. We first identify the target language for these translations. This is the normal BAE-signature $\mathcal{L}^\circ_{\mathrm{BAE}} = \mathcal{L}_{\mathrm{BAE}}(\mathcal{F}^\circ, \mathcal{G}^\circ)$ associated with $\mathcal{L}_{\mathrm{DLE}}$, where $\mathcal{F}^\circ := \{\Diamond_\geq\} \cup \{f^\circ \mid f \in \mathcal{F}\}$, and $\mathcal{G}^\circ := \{\Box_\leq\} \cup \{g^\circ \mid g \in \mathcal{G}\}$, and for every $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$), the connective f° (resp. g°) is such that $n_{f^\circ} = n_f$ (resp. $n_{g^\circ} = n_g$) and $\varepsilon_{f^\circ}(i) = 1$ for each $1 \leq i \leq n_f$ (resp. $\varepsilon_{g^\circ}(i) = 1$ for each $1 \leq i \leq n_g$).

We assume that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, such that both the left and right adjoint $c\colon \mathbb{B} \to \mathbb{A}$ and $\iota\colon \mathbb{B} \to \mathbb{A}$ exist and moreover the following diagrams commute for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$:[11]

$$\begin{array}{ccc} \mathbb{A}^{n_f} & \xrightarrow{e^{\varepsilon_f}} & \mathbb{B}^{n_f} \\ \downarrow f^{\mathbb{A}} & & \downarrow f^{\circ\mathbb{B}} \\ \mathbb{A} & \xleftarrow{c} & \mathbb{B} \end{array} \qquad \begin{array}{ccc} \mathbb{A}^{n_g} & \xrightarrow{e^{\varepsilon_g}} & \mathbb{B}^{n_g} \\ \downarrow g^{\mathbb{A}} & & \downarrow g^{\circ\mathbb{B}} \\ \mathbb{A} & \xleftarrow{\iota} & \mathbb{B} \end{array} \tag{5.3}$$

where $e^{\varepsilon_f}(\overline{a}) := \overline{e(a)}^{\varepsilon_f}$ and $e^{\varepsilon_g}(\overline{a}) := \overline{e(a)}^{\varepsilon_g}$. Then, as discussed early on, the Boolean reduct of $\mathbb{B}$ can be endowed with a natural structure of bi-modal S4-algebra by defining $\Box^{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ \iota)(b)$ and $\Diamond^{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ c)(b)$.

The target language for the parametrized GMT translations over Prop is given by

$$\mathcal{L}^\circ_{BAE} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid f^\circ(\overline{\alpha}) \mid g^\circ(\overline{\alpha}) \mid \Diamond_\geq \alpha \mid \Box_\leq \alpha.$$

Let $X := \mathsf{Prop}$. For any order-type ε on X, define the translation $\tau_\varepsilon : \mathcal{L}_{\mathrm{DLE}} \to \mathcal{L}^\circ_{\mathrm{BAE}}$ by the following recursion:

$$\tau_\varepsilon(p) = \begin{cases} \Box_\leq p & \text{if } \varepsilon(p) = 1 \\ \Diamond_\geq p & \text{if } \varepsilon(p) = \partial, \end{cases} \qquad \begin{array}{rcl} \tau_\varepsilon(\bot) & = & \bot \\ \tau_\varepsilon(\top) & = & \top \\ \tau_\varepsilon(\varphi \wedge \psi) & = & \tau_\varepsilon(\varphi) \wedge \tau_\varepsilon(\psi) \\ \tau_\varepsilon(\varphi \vee \psi) & = & \tau_\varepsilon(\varphi) \vee \tau_\varepsilon(\psi) \\ \tau_\varepsilon(f(\overline{\varphi})) & = & \Diamond_\geq f^\circ(\overline{\tau_\varepsilon(\varphi)}^{\varepsilon_f}) \\ \tau_\varepsilon(g(\overline{\varphi})) & = & \Box_\leq g^\circ(\overline{\tau_\varepsilon(\varphi)}^{\varepsilon_g}) \end{array}$$

where for each order-type η on n and any n-tuple $\overline{\psi}$ of $\mathcal{L}^\circ_{\mathrm{BAE}}$-formulas, $\overline{\psi}^\eta$ denotes the n-tuple $(\psi'_i)_{i=1}^n$, where $\psi'_i = \psi_i$ if $\eta(i) = 1$ and $\psi'_i = \neg\psi_i$ if $\eta(i) = \partial$.

Let $\mathbb{A}$ be a $\mathcal{L}_{\mathrm{DLE}}$-algebra and $\mathbb{B}$ be a $\mathcal{L}^\circ_{\mathrm{BAE}}$-algebra such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is a homomorphism of the lattice-reducts of $\mathbb{A}$ and $\mathbb{B}$, is endowed with both right and left adjoints, and satisfies the commutativity of the diagrams (5.3) for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$. For every order-type ε on X, consider the map $r_\varepsilon : \mathbb{B}^X \to \mathbb{A}^X$ defined, for any $U \in \mathbb{B}^X$ and $p \in X$, by:

$$r_\varepsilon(U)(p) = \begin{cases} (\iota \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

Proposition 5.12. *For every order-type ε on X, the translation τ_ε defined above satisfies conditions (a) and (b) of Proposition 4.1 relative to r_ε.*

[11] Notice that equations (5.1) and (5.2) encode the special cases of the commutativity of the diagrams (5.3) for $f(\varphi, \psi) := \varphi \succ\!\!-\psi$ (in which case, $f^\circ(\neg\alpha, \beta) := \neg\alpha \wedge \beta$) and $g(\varphi, \psi) := \varphi \to \psi$ (in which case, $g^\circ(\neg\alpha, \beta) := \neg\alpha \vee \beta$).

Proof. By induction on φ. As for the base case, let $\varphi := p \in \mathsf{Prop}$. If $\varepsilon(p) = \partial$, then for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

$$\begin{array}{rcll} e(\llbracket p \rrbracket_{r_\varepsilon(U)}) & = & e((c \circ U)(p)) & \text{(def. of } r_\varepsilon) \\ & = & (e \circ c)(\!\llbracket p \rrbracket\!)_U & \text{(assoc. of } \circ) \\ & = & \Diamond^{\mathbb{B}} (\!\llbracket p \rrbracket\!)_U & \text{(def. of } \Diamond^{\mathbb{B}}) \\ & = & (\!\llbracket \Diamond_{\geq} p \rrbracket\!)_U & \text{(def. of } (\!\llbracket \cdot \rrbracket\!)_U) \\ & = & (\!\llbracket \tau_\varepsilon(p) \rrbracket\!)_U. & \text{(def. of } \tau_\varepsilon) \end{array}$$

$$\begin{array}{rcll} (\!\llbracket \tau_\varepsilon(p) \rrbracket\!)_{\overline{e}(V)} & = & (\!\llbracket \Diamond_{\geq} p \rrbracket\!)_{\overline{e}(V)} & \text{(def. of } \tau_\varepsilon) \\ & = & \Diamond^{\mathbb{B}} (\!\llbracket p \rrbracket\!)_{\overline{e}(V)} & \text{(def. of } (\!\llbracket \cdot \rrbracket\!)_U) \\ & = & \Diamond^{\mathbb{B}}((e \circ V)(p)) & \text{(def. of } \overline{e}(V)) \\ & = & (e \circ c)((e \circ V)(p)) & \text{(def. of } \Diamond^{\mathbb{B}}) \\ & = & e((c \circ e)(V(p))) & \text{(assoc. of } \circ) \\ & = & e(V(p)) & (e \circ (c \circ e) = e) \\ & = & e(\llbracket p \rrbracket_V). & \text{(def. of } \llbracket \cdot \rrbracket_V) \end{array}$$

If $\varepsilon(p) = 1$, then for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

$$\begin{array}{rcll} e(\llbracket p \rrbracket_{r_\varepsilon(U)}) & = & e((\iota \circ U)(p)) & \text{(def. of } r_\varepsilon) \\ & = & (e \circ \iota)(\!\llbracket p \rrbracket\!)_U & \text{(assoc. of } \circ) \\ & = & \Box^{\mathbb{B}} (\!\llbracket p \rrbracket\!)_U & \text{(def. of } \Box^{\mathbb{B}}) \\ & = & (\!\llbracket \Box_{\leq} p \rrbracket\!)_U & \text{(def. of } (\!\llbracket \cdot \rrbracket\!)_U) \\ & = & (\!\llbracket \tau_\varepsilon(p) \rrbracket\!)_U. & \text{(def. of } \tau_\varepsilon) \end{array}$$

$$\begin{array}{rcll} (\!\llbracket \tau_\varepsilon(p) \rrbracket\!)_{\overline{e}(V)} & = & (\!\llbracket \Box_{\leq} p \rrbracket\!)_{\overline{e}(V)} & \text{(def. of } \tau_\varepsilon) \\ & = & \Box^{\mathbb{B}} (\!\llbracket p \rrbracket\!)_{\overline{e}(V)} & \text{(def. of } (\!\llbracket \cdot \rrbracket\!)_U) \\ & = & \Box^{\mathbb{B}}((e \circ V)(p)) & \text{(def. of } \overline{e}(V)) \\ & = & (e \circ \iota)((e \circ V)(p)) & \text{(def. of } \Box^{\mathbb{B}}) \\ & = & e((\iota \circ e)(V(p))) & \text{(assoc. of } \circ) \\ & = & e(V(p)) & (e \circ (\iota \circ e) = e) \\ & = & e(\llbracket p \rrbracket_V). & \text{(def. of } \llbracket \cdot \rrbracket_V) \end{array}$$

Let $\varphi := f(\overline{\varphi})$. Then for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

$$\begin{array}{cll} & e(\llbracket f(\overline{\varphi}) \rrbracket_{r_\varepsilon(U)}) & \\ = & e(f(\overline{\llbracket \varphi \rrbracket_{r_\varepsilon(U)}})) & \text{(def. of } \llbracket \cdot \rrbracket_{r_\varepsilon(U)}) \\ = & e(c \circ f^\circ(\overline{e(\llbracket \varphi \rrbracket_{r_\varepsilon(U)})}^{\varepsilon_f})) & \text{(assump. (5.3))} \\ = & \Diamond^{\mathbb{B}} f^\circ(\overline{(\!\llbracket \tau_\varepsilon(\varphi) \rrbracket\!)_U}^{\varepsilon_f}) & \text{(IH \& def. of } \Diamond^{\mathbb{B}}) \\ = & (\!\llbracket \Diamond_{\geq} f^\circ(\overline{\tau_\varepsilon(\varphi)}^{\varepsilon_f}) \rrbracket\!)_U & \text{(def. of } (\!\llbracket \cdot \rrbracket\!)_U) \\ = & (\!\llbracket \tau_\varepsilon(f(\overline{\varphi})) \rrbracket\!)_U. & \text{(def. of } \tau_\varepsilon) \end{array}$$

$$\begin{array}{cll} & (\!\llbracket \tau_\varepsilon(f(\overline{\varphi})) \rrbracket\!)_{\overline{e}(V)} & \\ = & (\!\llbracket \Diamond_{\geq} f^\circ(\overline{\tau_\varepsilon(\varphi)}^{\varepsilon_f}) \rrbracket\!)_{\overline{e}(V)} & \text{(def. of } \tau_\varepsilon) \\ = & \Diamond^{\mathbb{B}} f^\circ(\overline{(\!\llbracket \tau_\varepsilon(\varphi) \rrbracket\!)_{\overline{e}(V)}}^{\varepsilon_f}) & \text{(def. of } (\!\llbracket \cdot \rrbracket\!)_{\overline{e}(V)}) \\ = & \Diamond^{\mathbb{B}} f^\circ(\overline{e(\llbracket \varphi \rrbracket_V)}^{\varepsilon_f}) & \text{(IH)} \\ = & e(c \circ f^\circ(\overline{e(\llbracket \varphi \rrbracket_V)}^{\varepsilon_f})) & \text{(def. of } \Diamond^{\mathbb{B}}) \\ = & e(f(\overline{\llbracket \varphi \rrbracket_V})) & \text{(assump. (5.3))} \\ = & e(\llbracket f(\overline{\varphi}) \rrbracket_V) & \text{(def. of } \llbracket \cdot \rrbracket_V) \end{array}$$

The remaining cases are analogous and are omitted. □

As a consequence of the proposition above, Proposition 4.1 applies to τ_ε for any order-type ε on X. Hence:

Corollary 5.13. *Let $\mathbb{A}$ be a $\mathcal{L}_{\mathrm{DLE}}$-algebra. If an embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exists into a $\mathcal{L}^\circ_{\mathrm{BAE}}$-algebra $\mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and e has both a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ and a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ satisfying the commutativity of the diagrams* (5.3) *for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, then for any $\mathcal{L}_{\mathrm{DLE}}$-inequality $\varphi \leq \psi$,*

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

We finish this subsection by showing that every *perfect* $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$ (cf. Definition 2.2) embeds into a *perfect* $\mathcal{L}^\circ_{\mathrm{BAE}}$-algebra $\mathbb{B}$ in the way described in Corollary 5.13:

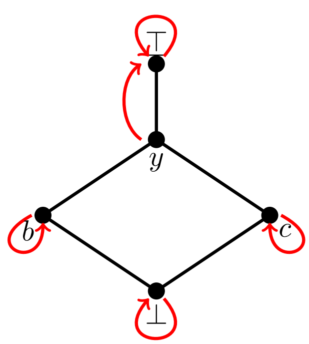
Proposition 5.14. *For every perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$, there exists a perfect $\mathcal{L}^\circ_{\mathrm{BAE}}$–algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is also a homomorphism* *of the lattice reducts of* $\mathbb{A}$ *and* $\mathbb{B}$ *and has both a left adjoint* $c\colon \mathbb{B} \to \mathbb{A}$ *and a right adjoint* $\iota\colon \mathbb{B} \to \mathbb{A}$ *satisfying the commutativity of the diagrams* (5.3)*.*

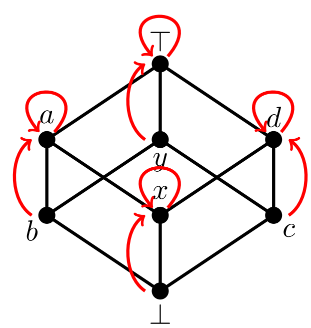
Proof. Via expanded Birkhoff's duality (cf. e.g. [21, 60]) the perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$ can be identified with the algebra of up-sets of its associated prime element $\mathcal{L}_{\mathrm{DLE}}$-frame $\mathbb{X}_{\mathbb{A}}$, which is based on a poset. Let $\mathbb{B}$ be the powerset algebra of the universe of $\mathbb{X}_{\mathbb{A}}$. Since any up-set is in particular a subset, a natural order embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a complete lattice homomorphism between $\mathbb{A}$ and $\mathbb{B}$. This shows the first part of the claim.

As to the second part, because e is a complete homomorphism between complete lattices, it has both a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ and a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$, respectively defined by the assignments $b \mapsto b{\uparrow}$ and $b \mapsto \neg((\neg b){\downarrow})$, where b is identified with its corresponding subset in $\mathbb{X}_{\mathbb{A}}$, $\neg b$ is defined as the relative complement of b in $\mathbb{X}_{\mathbb{A}}$, and $b{\uparrow}$ and $(\neg b){\downarrow}$ are defined using the order in $\mathbb{X}_{\mathbb{A}}$.

Finally, notice that any $\mathcal{L}_{\mathrm{DLE}}$-frame $\mathbb{F}$ is also an $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-frame by interpreting the f-type connective $\Diamond_{\geq}$ by means of the binary relation $\geq$, the g-type connective $\Box_{\leq}$ by means of the binary relation $\leq$, each $f^{\circ} \in \mathcal{F}^{\circ}$ by means of R_f and each $g^{\circ} \in \mathcal{G}^{\circ}$ by means of R_g. Moreover, the additional properties (2.1) and (2.2) of the relations R_f and R_g guarantee that the diagrams (5.3) commute for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$. □

Remark 5.15. The parametric GMT translations defined in this section do not just generalize those of [32] w.r.t. the signature, but also differ from them in terms of their definition, and the assumptions each of which requires. Specifically, the parametric translations of f-formulas (resp. g-formulas) add an extra $\Diamond_{\geq}$ (resp. a $\Box_{\leq}$) on top of the corresponding f° (resp. g°) connective, while in [32], the extra $\Diamond_{\geq}$ and $\Box_{\leq}$ are not used in the definition of the translation of formulas with a modal operator as main connective. This simpler definition is sound only w.r.t. semantic settings, such as that of [32], in which the relations interpreting the modal connectives satisfy the additional properties (2.1) and (2.2). This corresponds algebraically to the operations interpreting the classical modal connectives restricting nicely to the algebra of targets of persistent valuations, and corresponds syntactically to the *mix axioms* (e.g. $\Box_{\leq}\Box^{\circ}\Box_{\leq}p \leftrightarrow \Box^{\circ}p$ and $\Diamond_{\geq}\Diamond^{\circ}\Diamond_{\geq}p \leftrightarrow \Diamond^{\circ}p$, cf. [66, Section 6]) being valid. The particular algebras and maps picked in the proof of Proposition 5.14 happen to validate also the mix axioms. However, the mix axioms are not a necessary condition for satisfying Proposition 5.14, as the following example shows.





The (finite, hence perfect) distributive lattice with $\Box$ on the left embeds as a complete lattice into the (finite, hence perfect) Boolean algebra with $\Box^{\circ}$ on the right. Hence, the corresponding embedding e has both a right adjoint ι and a left adjoint c. The composition $e \circ \iota$ gives rise to the S4-operation $\Box_{\leq} := e \circ \iota$ which by construction maps a to b, d to

c, x to $\bot$, and every other element to itself. It it easy to check that $\Box$ and $\Box^\circ$ verify the commutativity of diagram (5.3). However, $\Box_\leq \Box^\circ \Box_\leq b = b \neq a = \Box^\circ b$, which shows that the mix axiom is not valid.

Notice that Proposition 5.14 has a more restricted scope than analogous propositions such as Propositions 5.11 or 5.4. Indeed, via expanded Priestley duality (cf. e.g. [60]), any DLE $\mathbb{A}$ is isomorphic to the DLE of clopen up-sets of its dual (relational) Priestley space $\mathbb{X}_\mathbb{A}$, which is a Stone space in particular, and this yields a natural embedding of $\mathbb{A}$ into the BAE of the clopen subsets of $\mathbb{X}_\mathbb{A}$. However, this embedding has in general neither a right nor a left adjoint. In Section 6, we will see that Proposition 5.14 is enough to obtain the correspondence theorem for inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities via translation from the correspondence theorem for inductive $\mathcal{L}_{\mathrm{BAE}}$-inequalities. However, we will see in Section 7 that canonicity cannot be straightforwardly obtained in the same way, precisely due to the restriction on Proposition 5.14. As we show next, this restriction can be removed if we confine ourselves setting to arbitrary normal bHAEs. In this setting, we are going to show a strengthened version of Proposition 5.14 which will be key for the transfer of canonicity of Section 7.1.

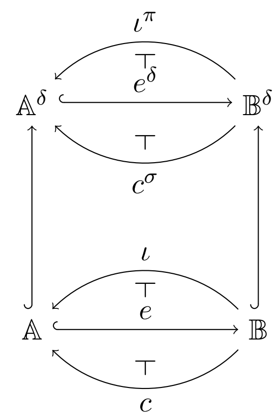
5.2.2. *Parametric GMT translations for bHAE-logics.* The considerations collected in Section 5.2.1 apply to the more restricted setting of bHAEs (cf. Section 2.2). Let us fix a bHAE-signature $\mathcal{L}_{\mathrm{bHAE}} = \mathcal{L}_{\mathrm{bHAE}}(\mathcal{F}, \mathcal{G})$ and let $\mathcal{L}^\circ_{\mathrm{BAE}}$ denote its corresponding target signature (cf. Section 5.2.1). Then, Corollary 5.13 specializes as follows:

Corollary 5.16. *Let $\mathbb{A}$ be an $\mathcal{L}_{\mathrm{bHAE}}$-algebra. If an embedding $e : \mathbb{A} \to \mathbb{B}$ exists into an $\mathcal{L}^\circ_{\mathrm{BAE}}$-algebra $\mathbb{B}$ which is a homomorphism of their lattice reducts and e has both a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ and a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ satisfying* (5.1)*,* (5.2) *and* (5.3)*, then for any $\mathcal{L}_{\mathrm{bHAE}}$-inequality $\varphi \leq \psi$,*

$$\mathbb{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

As discussed above, the present setting is characterized by the fact that, for any $\mathcal{L}_{\mathrm{bHAE}}$-algebras $\mathbb{A}$, the right and left adjoints of the embedding map $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exist, as shown by the following proposition.

Proposition 5.17. *For every $\mathcal{L}_{\mathrm{bHAE}}$-algebra $\mathbb{A}$, there exists an $\mathcal{L}^\circ_{\mathrm{BAE}}$-algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has both a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ and a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ satisfying* (5.1)*,* (5.2) *and* (5.3)*. Finally, all these facts lift to the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ as in the following diagram:*



Proof. By Proposition 5.11, to complete the proof of the first part of the statement, we need to address the claims regarding the expansions. This is done by using a version of the duality in [60] restricted to those Priestley spaces which are also bi-Esakia spaces. As to the second part, notice that the commutativity of the diagrams (5.3) can be written in the form of pairs of inequalities (i.e. $f = cf^\circ e^{\varepsilon_f}$ and $g = \iota g^\circ e^{\varepsilon_g}$) which are Sahlqvist and hence lift to the upper part of the diagram above. □

6. Correspondence via translation

The theory developed so far puts us in a position to meaningfully formulate and prove the transfer of first-order correspondence to Sahlqvist and inductive DLE-inequalities from suitable classical poly-modal cases. These general results specialize to the logics mentioned above, e.g. those mentioned in Example 2.1.

In what follows, we let $\mathcal{L}$ denote an arbitrary but fixed DLE-language and $\mathcal{L}^\circ$ its associated target language (cf. Section 5.2.1). The general definition of inductive inequalities (cf. Definition 2.6) applies both $\mathcal{L}$ and $\mathcal{L}^\circ$. In particular, the Boolean negation in $\mathcal{L}^\circ$ enjoys both the order-theoretic properties of a unary f-type connective and of a unary g-type connective. Hence, Boolean negation occurs unrestricted in inductive $\mathcal{L}^\circ$-inequalities. Moreover, the algebraic interpretations of the S4-connectives $\Box_\leq$ and $\Diamond_\geq$ enjoy the order-theoretic properties of normal unary f-type and g-type connectives respectively. Hence, the occurrence of $\Box_\leq$ and $\Diamond_\geq$ in inductive $\mathcal{L}^\circ$-inequalities is subject to the same restrictions applied to any connective pertaining to the same class to which they belong.

The following correspondence theorem is a straightforward extension to the $\mathcal{L}^\circ$-setting of the correspondence result for classical normal modal logic in [13]:

Proposition 6.1. *Every inductive $\mathcal{L}^\circ$-inequality has a first-order correspondent over its class of $\mathcal{L}^\circ$-frames.*

In what follows, we aim to transfer the correspondence theorem for inductive $\mathcal{L}^\circ$-inequalities as stated in the proposition above to inductive $\mathcal{L}$-inequalities. The next proposition is the first step towards this goal. As before, let $X := \mathsf{Prop}$.

Proposition 6.2. *The following are equivalent for any order-type ε on X, and any $\mathcal{L}$-inequality $\varphi \leq \psi$:*

1. *$\varphi \leq \psi$ is an (Ω, ε)-inductive $\mathcal{L}$-inequality;*

2. *$\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ is an (Ω, ε)-inductive $\mathcal{L}^\circ$-inequality.*

Proof. By induction on the shape of $\varphi \leq \psi$. In a nutshell: the definitions involved guarantee that: (1) PIA nodes are introduced immediately above ε-critical occurrences of proposition variables; (2) Skeleton nodes are translated as (one or more) Skeleton nodes; (3) PIA nodes are translated as (one or more) PIA nodes. Moreover, this translation does not disturb the dependency order Ω. Hence, from item 1 to item 2, the translation does not introduce any violation on ε-critical branches, and, from item 2 to item 1, the translation does not amend any violation. □

Theorem 6.1 (Correspondence via translation)**.** *The correspondence theorem for inductive $\mathcal{L}^\circ$-inequalities transfers to inductive $\mathcal{L}$-inequalities.*

Proof. Let $\varphi \leq \psi$ be an (Ω, ε)-inductive $\mathcal{L}$-inequality, and $\mathbb{F}$ be an $\mathcal{L}$-frame such that $\mathbb{F} \Vdash \varphi \leq \psi$. By the discrete duality between perfect $\mathcal{L}$-algebras and $\mathcal{L}$-frames, this assumption is equivalent to $\mathbb{A} \models \varphi \leq \psi$, where $\mathbb{A}$ denotes the complex $\mathcal{L}$-algebra of $\mathbb{F}$. Since $\mathbb{A}$ is a perfect $\mathcal{L}$-algebra, by Proposition 5.14, a perfect $\mathcal{L}^\circ$-algebra $\mathbb{B}$ exists with a natural embedding $e : \mathbb{A} \to \mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has both a right adjoint $\iota \colon \mathbb{B} \to \mathbb{A}$ and a left adjoint $c \colon \mathbb{B} \to \mathbb{A}$ such that diagrams (5.3) commute. Hence, Corollary 5.13 is applicable and yields $\mathbb{A} \models \varphi \leq \psi$ iff $\mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$, which is equivalent to $\mathbb{F} \Vdash^* \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ by the discrete duality between perfect $\mathcal{L}^\circ$-algebras and $\mathcal{L}^\circ$-frames.

By Proposition 6.2, $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ is an (Ω, ε)-inductive $\mathcal{L}^\circ$-inequality, and hence, by Proposition 6.1, $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ has a first-order correspondent $\mathsf{FO}(\varphi)$ on $\mathcal{L}^\circ$-frames. Therefore $\mathbb{F} \Vdash^* \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ iff $\mathbb{F} \models \mathsf{FO}(\varphi)$. Since the first-order frame correspondence languages of $\mathcal{L}$ and $\mathcal{L}^\circ$ are the same, it follows that $\mathsf{FO}(\varphi)$ is also the first-order correspondent of $\varphi \leq \psi$. The steps of this argument are summarized in the following chain of equivalences:

$$\begin{array}{lll} & \mathbb{F} \Vdash \varphi \leq \psi & \\ \text{iff} & \mathbb{A} \models \varphi \leq \psi & \text{(discrete duality for } \mathcal{L}\text{-frames)} \\ \text{iff} & \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \text{(Proposition 5.14, Corollary 5.13)} \\ \text{iff} & \mathbb{F} \Vdash^* \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \text{(discrete duality for } \mathcal{L}^\circ\text{-frames)} \\ \text{iff} & \mathbb{F} \models \mathsf{FO}(\varphi) & \text{(Proposition 6.1)} \end{array}$$

□

Remark 6.3. In Example 2.1, we showed that the languages of Rauszer's bi-intuitionistic logic, Fischer Servi's intuitionistic modal logic, Wolter's bi-intuitionistic modal logic, Bezhanishvili's MIPC with universal modalities, Dunn's positive modal logic and Gehrke Nagahashi and Venema's Distributive Modal Logic are specific instances of DLE-logics. Hence, Theorem 6.1, applied to each of these settings, enables one to transfer generalized Sahlqvist correspondence theorems to each of these logics. In all these settings but the latter two, this transferability is a new result which can be added to the list of known transfer results for these logics (cf. e.g. [3, Section 4.1]). In positive modal logic and distributive modal logic, [32, Theorem 3.7] proves the transfer of Sahlqvist correspondence. Our result strengthens this to the transfer of correspondence for the larger class of inductive formulas.[12]

[12]In [36] it is shown that inductive formulas exist which are not semantically equivalent to any Sahlqvist formula.

7. Canonicity via translation

In this section we apply the results of Section 5, and in particular those of Section 5.2.2, to show that the canonicity of the inductive $\mathcal{L}_{\mathrm{bHAE}}$-inequalities transfers from classical multi-modal logic via parametrized GMT translations. The proof strategy of this result does not generalize successfully to DLE-logics or, indeed, to intuitionistic or co-intuitionistic modal logics. We discuss the reasons for this and propose possible alternative strategies.

7.1. **Canonicity transfer to inductive bHAE-inequalities.** Throughout the present section, let us fix a bHAE-signature $\mathcal{L}_{\mathrm{bHAE}} = \mathcal{L}_{\mathrm{bHAE}}(\mathcal{F}, \mathcal{G})$, and let $\mathcal{L}^{\circ}_{\mathrm{BAE}} = \mathcal{L}_{\mathrm{BAE}}(\mathcal{F}^{\circ}, \mathcal{G}^{\circ})$ be the target language for the parametric GMT translations for $\mathcal{L}_{\mathrm{bHAE}}$ (cf. Section 5.2.1). The following canonicity theorem is a straightforward algebraic reformulation of the canonicity result for classical normal polyadic modal logic in [35] and [14]:

Proposition 7.1. *For every inductive $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-inequality $\alpha \leq \beta$ and every $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-algebra $\mathbb{B}$,*

$$\text{if } \mathbb{B} \models \alpha \leq \beta \quad \text{then } \mathbb{B}^{\delta} \models \alpha \leq \beta.$$

In what follows, we show that the canonicity of inductive $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-inequalities, given by the proposition above, transfers to inductive $\mathcal{L}_{\mathrm{bHAE}}$-inequalities via suitable parametrized GMT translations.

Theorem 7.1 (Canonicity via translation)**.** *The canonicity theorem for inductive $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-inequalities transfers to inductive $\mathcal{L}_{\mathrm{bHAE}}$-inequalities.*

Proof. Fix an $\mathcal{L}_{\mathrm{bHAE}}$-algebra $\mathbb{A}$ and let $\varphi \leq \psi$ be an inductive $\mathcal{L}_{\mathrm{bHAE}}$-inequality such that $\mathbb{A} \models \varphi \leq \psi$. We show $\mathbb{A}^{\delta} \models \varphi \leq \psi$. By Proposition 5.17, an $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-algebra $\mathbb{B}$ exists with a natural embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has both a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ and a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ such that conditions (5.1), (5.2), and (5.3) hold. Hence, Corollary 5.16 is applicable, which yields $\mathbb{A} \models \varphi \leq \psi$ iff $\mathbb{B} \models \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$.

By Proposition 6.2, $\tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$ is an (Ω, ε)-inductive $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-inequality, and hence, by Proposition 7.1, $\mathbb{B}^{\delta} \models \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)$. By the last part of the statement of Proposition 5.17, Corollary 5.16 applies also to $\mathbb{A}^{\delta}$ and $\mathbb{B}^{\delta}$, and thus $\mathbb{A}^{\delta} \models \varphi \leq \psi$, as required. The steps of this argument are summarized in the following U-shaped diagram:

$$\begin{array}{lcl}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^{\delta} \models \varphi \leq \psi \\
\quad \Updownarrow \text{(Prop 5.17, Cor 5.16)} & & \quad \Updownarrow \text{(Prop 5.17, Cor 5.16)} \\
\mathbb{B} \models \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi) & \Leftrightarrow & \mathbb{B}^{\delta} \models \tau_{\varepsilon}(\varphi) \leq \tau_{\varepsilon}(\psi)
\end{array}$$

□

Remark 7.2. Theorem 7.1 applies to both Rauszer's bi-intuitionistic logic[13] [56] and Wolter's bi-intuitionistic modal logic [66]. Hence, transfer of generalized Sahlqvist canonicity theorems is available for each of these logics. This transferability is a new result for these settings, and is different from the transfer of d-persistence as was shown e.g. in [68, Theorem 12] in the context of intuitionistic modal logic, in at least two respects; first, it hinges specifically on the preservation and reflection of the shape of inductive formulas; second, it

[13]As discussed in Example 2.3, not all axioms in Rauszer's axiomatization of bi-intuitionistic logic are inductive. However, in [37], Goré introduces a proper display calculus for bi-intuitionistic logic, which implies, by the characterization given in [38], that an axiomatization for bi-intuitionistic logic exists which consists only of inductive formulas.

does not rely on any assumptions about the interaction between the S4 and other modalities in the target logic such as those captured by the mix axiom (cf. Remark 5.15).

7.2. **Generalizing the canonicity-via-translation argument.** In the present subsection, we discuss the extent to which the proof pattern described in the previous subsection can be applied to the settings of normal Heyting and co-Heyting algebra expansions (HAEs, cHAEs), and to normal DLEs. In the setting of bHAEs, the order embedding e has both a left and a right adjoint, the existence of which is shown in Proposition 5.17, while for HAEs, cHAEs and DLEs at most one of the two adjoints was shown to exist in general (cf. Propositions 5.4 and 5.8), while both adjoints exist if the algebra is perfect (cf. Proposition 5.14).

This implies that the the U-shaped argument discussed in the proof of Theorem 7.1 is not straightforwardly applicable to HAEs, cHAEs and DLEs. Indeed, in each of these settings, the equivalence on the side of the perfect algebras can still be argued using Proposition 5.14 and Corollary 5.13, but the one on the side of general algebras (left-hand side of the diagram) cannot, precisely because Proposition 5.14 does not generalize to arbitrary DLEs (resp. HAEs, cHAEs).

$$\begin{array}{lcl} \mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\ \quad \Updownarrow ? & & \quad \Updownarrow \text{(Prop 5.14, Cor 5.13)} \\ \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \Leftrightarrow & \mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) \end{array}$$

In what follows, we employ a more refined argument to show that the left-hand side equivalence holds. That is, the question mark in the U-shaped diagram above can be replaced by Proposition 7.4 below. We work in the setting of $\mathcal{L}$-algebras for an arbitrarily fixed DLE-signature $\mathcal{L}$, with $\mathcal{L}^\circ$ its associated target signature. Recall that the canonical extension $e^\delta : \mathbb{A}^\delta \to \mathbb{B}^\delta$ of the embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ is a complete lattice homomorphism, and hence both its left and right adjoints exist, which we respectively denote $c : \mathbb{B}^\delta \to \mathbb{A}^\delta$ and $\iota : \mathbb{B}^\delta \to \mathbb{A}^\delta$. It is well known from the theory of canonical extensions that $c(b) \in K(\mathbb{A}^\delta)$ and $\iota(b) \in O(\mathbb{A}^\delta)$ for every $b \in \mathbb{B}$ (cf. [17, Lemma 10.3]). Hence, if $r_\varepsilon : (\mathbb{B}^\delta)^X \to (\mathbb{A}^\delta)^X$ is the map defined for any $U \in (\mathbb{B}^\delta)^X$ and $p \in X$ by:

$$r_\varepsilon(U)(p) = \begin{cases} (\iota \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

then $(r_\varepsilon(U))(p) \in K(\mathbb{A}^\delta)$ if $\varepsilon(p) = \partial$ and $(r_\varepsilon(U))(p) \in O(\mathbb{A}^\delta)$ if $\varepsilon(p) = 1$ for any 'admissible valuation' $U \in \mathbb{B}^X$ and $p \in X$.

Lemma 7.3. *Let $\mathbb{A}$ be an $\mathcal{L}$-algebra, and $e : \mathbb{A} \hookrightarrow \mathbb{B}$ be an embedding of $\mathbb{A}$ into an $\mathcal{L}^\circ$-algebra $\mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ such that the left and right adjoints of $e^\delta : \mathbb{A}^\delta \to \mathbb{B}^\delta$ make the diagrams* (5.3) *commute. Then, for every order-type ε on X, the following conditions hold for every $\varphi \in \mathcal{L}$:*

(a) *$e(\llbracket\varphi\rrbracket_V) = (\!|\tau_\varepsilon(\varphi)|\!)_{\overline{e}(V)}$ for every $V \in \mathbb{A}^X$;*
(b) *$(\!|\tau_\varepsilon(\varphi)|\!)_U = e^\delta(\llbracket\varphi\rrbracket_{r_\varepsilon(U)})$ for every $U \in \mathbb{B}^X$.*

Proof. The statement immediately follows from Proposition 5.12 applied to $e^\delta : \mathbb{A}^\delta \to \mathbb{B}^\delta$. □

Proposition 7.4. *Let $\mathbb{A}$ be an $\mathcal{L}$-algebra, and $e : \mathbb{A} \hookrightarrow \mathbb{B}$ be an embedding of $\mathbb{A}$ into an $\mathcal{L}^\circ$-algebra $\mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ such that the left* *and right adjoints of* $e^\delta : \mathbb{A}^\delta \to \mathbb{B}^\delta$ *make the diagrams* (5.3) *commute. Then, for every* (Ω, ε)*-inductive* $\mathcal{L}$*-inequality* $\varphi \leq \psi$*,*

$$\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B}^\delta \models_{\mathbb{B}} \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

Sketch of proof. From right to left, if $(\mathbb{A}^\delta, V) \not\models \varphi \leq \psi$ for some $V \in \mathbb{A}^X$, then $[\![\varphi]\!]_V \not\leq [\![\psi]\!]_V$. By Lemma 7.3 (a), this implies that $(\![\tau_\varepsilon(\varphi)]\!)_{\overline{e}(V)} = e([\![\varphi]\!]_V) \not\leq e([\![\psi]\!]_V) = (\![\tau_\varepsilon(\psi)]\!)_{\overline{e}(V)}$, that is $(\mathbb{B}^\delta, \overline{e}(V)) \not\models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$, as required.

Conversely, assume contrapositively that $(\mathbb{B}^\delta, U) \not\models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ for some $U \in \mathbb{B}^X$, that is, $(\![\tau_\varepsilon(\varphi)]\!)_U \not\leq (\![\tau_\varepsilon(\psi)]\!)_U$. By Lemma 7.3 (b), this is equivalent to $e^\delta([\![\varphi]\!]_{r_\varepsilon(U)}) \not\leq e^\delta([\![\psi]\!]_{r_\varepsilon(U)})$, which, by the monotonicity of e^δ, implies that $[\![\varphi]\!]_{r_\varepsilon(U)} \not\leq [\![\psi]\!]_{r_\varepsilon(U)}$, that is, $(\mathbb{A}, r_\varepsilon(U)) \not\models \varphi \leq \psi$. This is not enough to finish the proof, since $r_\varepsilon(U)$ is not guaranteed to belong in $\mathbb{A}^X$; however, as observed above, $(r_\varepsilon(U))(p) \in K(\mathbb{A}^\delta)$ if $\varepsilon(p) = \partial$ and $(r_\varepsilon(U))(p) \in O(\mathbb{A}^\delta)$ if $\varepsilon(p) = 1$ for each proposition variable p. To finish the proof, we need to show that an admissible valuation $V' \in \mathbb{A}^X$ can be manufactured from $r_\varepsilon(U)$ and $\varphi \leq \psi$ in such a way that $(\mathbb{A}^\delta, V') \not\models \varphi \leq \psi$. In what follows, we provide a sketch of the proof of the existence of the required V'. Assume that $\varepsilon(q) = \partial$ for some proposition variable q occurring in $\varphi \leq \psi$ (the case of $\varepsilon(q) = 1$ is analogous and is omitted). Then we define $V'(q) \in \mathbb{A}$ as follows. We run ALBA on $\varphi \leq \psi$ according to the dependency order $<_\Omega$, up to the point when we solve for the negative occurrences of q, which by assumption are ε-critical. Notice that ALBA preserves truth under assignments.[14] Then the inequality providing the minimal valuation of q is of the form $q \leq \alpha$, where α is *pure* (i.e. no proposition variables occur in α). By [15, Lemma 9.5], every inequality in the antecedent of the quasi-inequality obtained by applying first approximation to an inductive inequality is of the form $\gamma \leq \delta$ with γ syntactically closed and δ syntactically open. Hence, α is pure and syntactically open, which means that the interpretation of α is an element in $O(\mathbb{A}^\delta)$. Therefore, by compactness, there exists some $a \in \mathbb{A}$ such that $r_\varepsilon(U)(q) \leq a \leq \alpha$. Then we define $V'(q) = a$. Finally, it remains to be shown that $(\mathbb{A}^\delta, V') \not\models \varphi \leq \psi$. This immediately follows from the fact that ALBA steps preserves truth under assignments, and that all the inequalities in the system are preserved in the change from $r_\varepsilon(U)$ to V'. □

However, Proposition 7.4 is still not enough for the U-shaped argument above to go through. Indeed, notice that, whenever $e : \mathbb{A} \to \mathbb{B}$ misses one of the two adjoints (e.g. the left adjoint), for any (Ω, ε)-inductive $\mathcal{L}$-inequality $\varphi \leq \psi$ containing some q with $\varepsilon(q) = \partial$, its translation $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ contains occurrences of the connective $\Diamond_\geq$, the algebraic interpretation of which in $\mathbb{B}^\delta$ is based on the left adjoint c of e^δ, which, as discussed above, maps elements in $\mathbb{B}$ to elements in $K(\mathbb{B}^\delta)$. Hence, the canonicity of $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$, understood as the preservation of its validity from $\mathbb{B}$ to $\mathbb{B}^\delta$, cannot be argued by appealing to Proposition 7.1: indeed, Proposition 7.1 holds under the assumption that $\mathbb{B}$ is an $\mathcal{L}^\circ$-subalgebra of $\mathbb{B}^\delta$, while, as discussed above, $\mathbb{B}$ is not in general closed under $\Diamond_\geq$.

In order to be able to adapt the canonicity-via-translation argument to the case of HAEs, cHAEs and DLEs, we would need to strengthen Proposition 7.1 so as to obtain the

[14]In [15] it is proved that ALBA steps preserve validity of quasi-inequalities. In fact, something stronger is ensured, namely that truth under assignments is preserved, modulo the values of introduced and eliminated variables. This notion of equivalence is studied in e.g. [12]. We are therefore justified in our assumption that the value of q is held constant as are the values of all variables occurring in $\varphi \leq \psi$ which have not yet been eliminated up to the point where q is solved for.

following equivalence for any inductive $\mathcal{L}^\circ$-inequality $\alpha \leq \beta$:

$$\mathbb{B}^\delta \models_{\mathbb{B}} \alpha \leq \beta \text{ iff } \mathbb{B}^\delta \models \alpha \leq \beta \tag{7.1}$$

in a setting in which the interpretations of $\Diamond_{\geq}$ and $\Box_{\leq}$ exist only in $\mathbb{B}^\delta$ and all we have in general for $b \in \mathbb{B}$ is that $\Diamond^{\mathbb{B}^\delta}(b) \in K(\mathbb{B}^\delta)$ and $\Box^{\mathbb{B}^\delta}(b) \in O(\mathbb{B}^\delta)$.

Such a strengthening cannot be straightforwardly obtained with the tools provided by the present state-of-the-art in canonicity theory. To see where the problem lies, let us try and apply ALBA/SQEMA in an attempt to prove the left-to-right direction of (7.1) for the 'Sahlqvist' inequality $\Box_{\leq} p \leq \Diamond_{\geq}\Box_{\leq} p$, assuming that $\Diamond_{\geq}$ is left adjoint to $\Box_{\leq}$, and $(\![\Box_{\leq} p]\!)_U \in O(\mathbb{B}^\delta)$ and $(\![\Diamond_{\geq} p]\!)_U \in K(\mathbb{B}^\delta)$ for any admissible valuation $U \in \mathbb{B}^X$:

$$\begin{array}{ll} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p[\Box_{\leq} p \leq \Diamond_{\geq}\Box_{\leq} p] \\ \text{iff} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box_{\leq} p \ \&\ \Diamond_{\geq}\Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\Diamond_{\geq}\mathbf{i} \leq p \ \&\ \Diamond_{\geq}\Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \end{array}$$

The minimal valuation term $\Diamond_{\geq}\mathbf{j}$, computed by ALBA/SQEMA when solving for the negative occurrence of p, is closed. However, substituting this minimal valuation into $\Diamond_{\geq}\Box_{\leq} p \leq \mathbf{m}$ would get us $\Diamond_{\geq}\Box_{\leq}\Diamond_{\geq}\mathbf{j} \leq \mathbf{m}$ with $\Diamond_{\geq}\Box_{\leq}\Diamond_{\geq}\mathbf{j}$ neither closed nor open. Hence, we cannot anymore appeal to the Esakia lemma in order to prove the following equivalence:[15]

$$\begin{array}{ll} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\Diamond_{\geq}\mathbf{i} \leq p \ \&\ \Diamond_{\geq}\Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall \mathbf{i} \forall \mathbf{m}[\Diamond_{\geq}\Box_{\leq}\Diamond_{\geq}\mathbf{i} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \end{array}$$

An analogous situation arises when solving for the positive occurrence of p. Other techniques for proving canonicity, such as Jónsson-style canonicity [40, 52], display the same problem, since they also rely on an Esakia lemma which is not available if $\mathbb{B}$ is not closed under $\Box_{\leq}$ and $\Diamond_{\geq}$.

8. Conclusions and further directions

Contributions. In the present paper, we have laid the groundwork for a general and uniform theory of transfer of generalized Sahlqvist correspondence and canonicity from normal BAE-logics to normal DLE-logics. Towards this goal, we have introduced a unifying template for GMT translations, of which the GMT translations in the literature can be recognized as instances. We have proved that generalized Sahlqvist correspondence transfers for all DLE-logics, while generalized Sahlqvist canonicity transfers for the more restricted setting of bHAE-logics. The formulation of these results has been made possible by the recent introduction of a general mechanism for identifying Sahlqvist and inductive classes for any normal DLE-signature [17]. Consequently, there are not many transfer results in the literature to which these results can be compared, the only exceptions being the transfer of Sahlqvist correspondence for DML-inequalities of [32, Theorem 3.7], and the transfer of canonicity in the form of d-persistence for intuitionistic modal formulas of [68, Theorem 12]. The transfer of correspondence shown in this paper generalizes [32, Theorem 3.7] both as regards the setting (from DML to general normal DLE-logics) and the scope (from Sahlqvist to inductive inequalities). As discussed in Remark 7.2, the transfer of canonicity shown in the present paper is neither subsumed by, nor does it subsume [68, Theorem 12].

[15] In other words, if $\mathbb{B}$ is not closed under $\Diamond_{\geq}$ or $\Box_{\leq}$, the soundness of the application of the Ackermann rule under admissible assignments cannot be argued anymore by appealing to the Esakia lemma, and hence, to the topological Ackermann lemma.

Regarding insights, we have also gained a better understanding of the nature of the difficulties in the transfer of (generalized) Sahlqvist canonicity via GMT translations. These difficulties were discussed as follows in the conclusions of [32]:

> [...] a reduction to the classical result for canonicity seems to be much harder than for correspondence, due to the following reason. In the correspondence case, where we are working with perfect DMAs, there is an obvious way to connect with Boolean algebras with operators, namely by taking the (Boolean) complex algebra of the dual frame. In the canonicity case however, we would need to embed arbitrary DMAs into BAOs in a way that would interact nicely with taking canonical extensions, and we do not see a natural, general way for doing so.

Our analysis shows that, actually, the problem does not lie in the interaction between the embedding and the canonical extensions, but rather in the fact that the embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ of an arbitrary DLE into a suitable BAE lacks the required adjoint maps, and that while the role of these adjoints can be played to a certain extent by the adjoints of $e^\delta : \mathbb{A}^\delta \hookrightarrow \mathbb{B}^\delta$ (cf. Proposition 7.4), we would need to develop a much stronger theory of algebraic (generalized) Sahlqvist canonicity in the BAE setting to be able to reduce (generalized) Sahlqvist canonicity for DLEs to the Boolean setting.

Further directions. As mentioned above, our analysis suggests a way to obtain the transfer of generalized Sahlqvist canonicity for arbitrary DLE-logics, namely to develop a generalized canonicity theory in the setting of BAEs which relies on the order-theoretic properties of maps $\mathbb{A} \to \mathbb{B}^\delta$. The way to this theory has already been paved in [52], where a generalization of the standard theory of canonical extensions of maps is developed, accounting for maps $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^\delta$ such that the value of $f^{\mathbb{A}}$ is not restricted to clopen elements in $\mathbb{B}$.

Blok-Esakia theorem for DLE-logics. The uniform perspective on GMT translations developed in this paper can perhaps be useful to systematically explore the possible variants of the notion of 'modal companion' of a given intuitionistic modal logic, and extend the Blok-Esakia theorem uniformly to DLE-logics.

Methodology: generalization through algebras via duality. The generalized canonicity-via-translation result for the bi-intuitionistic setting comes from embracing the full extent of the algebraic analysis. Specifically, canonicity-via-translation hinges upon the fact that the interplay of persistent and non-persistent valuations on frames can be understood and reformulated in terms of an adjunction situation between two complex algebras of the same frame. In its turn, this adjunction situation generalizes to arbitrary algebras. The same modus operandi, which achieves generalization through algebras via duality, has been fruitfully employed by some of the authors also for very different purposes, such as the definition of the non-classical counterpart of a given logical framework (cf. [42, 44, 10]).

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