Report LR-657

# Programming of the complex logarithm function in the solution of the cracked anisotropic plate loaded by a point force

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## Abstract

In programming solutions of complex function theory, the complex logarithm function is replaced by the complex logarithmic function, introducing a discontinuity along the branch cut into the programmed solution which was not present in the mathematical solution. Recently, Liaw and Kamel presented their solution of the infinite anisotropic centrally cracked plate loaded by an arbitrary point force, which they used as Green's function in a boundary element method intended to evaluate the stress intensity factor at the tip of a crack originating from an elliptical hole. Their solution may be used as Green's function for many more numerical methods involving anisotropic elasticity. In programming applications of Liaw and Kamel's solution, the standard definition of the logarithmic function with the branch cut at the non-positive real axis cannot provide a reliable computation of the displacement field for Liaw and Kamel's solution. Either the branch cut should be redefined outside the domain of the logarithmic function, after proving that the domain is limited to a part of the plane, or the logarithm function should be defined on its Riemann surface. A two dimensional line fractal can provide the link between all mesh points on the plane, essential to evaluate the logarithm function on its Riemann surface. As an example, a two dimensional line fractal is defined for a mesh once used by Erdogan and Arin.

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# 1 Introduction

The theory of complex functions is used in engineering mathematics for many different applications. Complex functions are used e.g. to describe the flow of a two dimensional incompressible frictionless fluid, the deformations in an isotropic plate and - the application discussed in this paper - the deformations in an anisotropic plate. In all these applications, the analytical nature of complex functions is essential: being an analytical function, the complex function forms a solution of the governing differential equations.

The limited number of elementary complex functions necessitates the use of power series of complex functions in most applications. Especially boundary conditions can usually not be satisfied with single elementary complex functions.

Among the complex functions, the complex logarithm function is special: the singularity it contains describes some special events in its application to two dimensional aerodynamics and elasticity. In two dimensional aerodynamics, the complex logarithm function describes the flow around a source or a vortex, situated at the location of the singularity in the flow field; in two dimensional elasticity, the complex logarithm function describes the displacements around a point force, applied at the location of the singularity.

Programming the complex logarithm function, e.g. in applications to derive a flow field in aerodynamics or a displacement field in elasticity, can lead to difficulties. In the mathematical theory from which the governing equations are derived, the complex *logarithm* function is used. The logarithm function is a multi valued function:

$$log(z) = ln|z| + i(Arg(z) + 2k\pi), k = integer \qquad (1.1)$$

The mathematical solution of the differential equations in practical applications are always described in such a way that the multi valued logarithm function never leads to multi valued solutions (e.g. multi valued velocities or displacements). In programming however, the logarithm function has to be programmed as the *logarithmic* function: the logarithmic function is defined as one single valued branch of the logarithm function. Being a single valued function, the logarithmic function can be programmed. Usually, the logarithmic function is defined as the principal value of the logarithm function:

$$Log(z) = ln|z| + iArg(z), -\pi < Arg(z) \leq \pi \qquad (1.2)$$

By choosing one branch of the logarithm function to define the logarithmic function, a line of discontinuity is introduced into the programmed solution. For the logarithmic function above this line is the non-positive real axis. In the mathematical solution the logarithm function is continuous on the entire plane, while in the programmed solution the logarithmic function, which replaces the logarithm function, always has a line of discontinuity. This inevitable defect in the programmed solution can lead to discontinuities in the results derived with the programmed solution (e.g. discontinuities in the flow field or displacement field). These discontinuities are only caused by programming errors.

In this paper, an attempt is made to describe methods to solve the difficulties in programming solutions containing the complex logarithm function. Two methods are described. A simple one can be used when the domain of the complex logarithm function in the solution is known; then the branch cut (the line of discontinuity in the logarithmic function) can simply be placed outside this domain. A more intricate method, using fractals to link the locations in the mesh where the complex logarithm function is to be evaluated, does not require knowledge about the domain and can function fully automated for any domain.

Both methods will be demonstrated using Liaw and Kamel's solution [1] describing the deformations in an infinite centrally cracked anisotropic plate loaded by an arbitrary in-plane point force, as an example. Liaw and Kamel used their solution as Green's function in a boundary element method intended to evaluate the stress intensity factor at the tip of a crack originating from an elliptical hole. In general, any numerical method in isotropic elasticity based on the use of the solution of the cracked plate loaded by a point force as Green's function, may be extended to anisotropic elasticity.

Liaw and Kamel's solution is presented within the framework of Lekhnitskii's theory of anisotropic plates. The complex logarithm function in Liaw and Kamel's solution describes the displacement field around the point force and the displacement field caused by the correction terms eliminating the stresses at the crack surface. Defects in programming the complex logarithm function in their solution may result in a discontinuous displacement field. The evaluation of the Green's function from a discontinuous displacement field would not yield reliable results.

Explaining both methods requires some basic knowledge on the concepts of the complex logarithm function, its Riemann surface, branch cuts and their relation to the definition of the complex logarithmic function. These subjects will be discussed in the following chapter. Many books on basic complex function theory provide a treatise on the complex logarithm function. The discussion presented in the following chapter is extracted from the book of Paliouras [2].

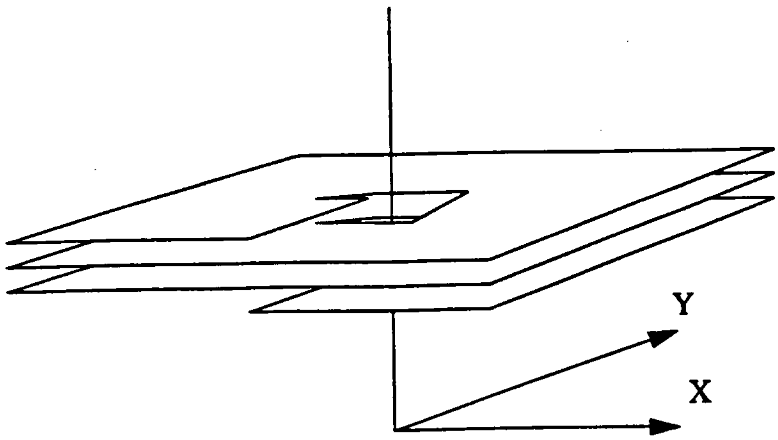


*Figure 2.1: Riemann surface of the logarithm function*

## 2 The complex logarithm function

The ordinary logarithm function projects all non-negative real numbers onto the entire set of real numbers:

$$y = ln(x), x \in (0, \infty), y \in (-\infty, \infty) \quad (2.1)$$

The concept of the ordinary logarithm function has been extended to the complex logarithm function:

$$log(z) = ln|z| + i \cdot arg(z), -\infty < arg(z) < \infty \quad (2.2)$$

On the non-negative real axis, the complex logarithm function coincides with the ordinary logarithm function if the argument of the complex variable $z$ is chosen to be equal to zero on this line. This choice - the choice that the argument of the complex variable $z$ is zero on the non-negative real axis - addresses the key cause of problems in programming the complex logarithm function. The logarithm function is a multi valued function:

$$log(z) = ln|z| + i(Arg(z) + 2k\pi), -\pi < Arg(z) \le \pi, k \in Z \quad (2.3)$$

The argument function *Arg(z)* is by definition equal to zero on the non-negative real axis. If the variable $k$ is set equal to zero, the logarithmic function from equation (1.2) is obtained again. Any other choice of the value of the variable $k$ would yield an equally valid definition of the complex logarithmic function. Fixing the value of the variable $k$ is sufficient to obtain a valid definition of the logarithmic function from the logarithm function.

The Riemann surface of the logarithm function can be used to visualize the definition of the logarithmic function. The Riemann surface for the logarithm function resembles an infinite circular staircase (figure 2.1).

Each level - or branch - is composed of a full plane with the origin - the branch point - failing. Travelling around the origin, one descends or ascends to the neighbouring branches. By assigning one branch to each distinct value of the variable $k$ in equation (2.3), the logarithm function is defined as a single valued function on its Riemann surface.

To define the logarithmic function from the logarithm function on the Riemann surface, one branch must be cut from the Riemann surface. Any line from the origin to infinity that does not intersect itself, will do to cut the Riemann surface into an infinite number of branches. Each branch covers the full plane and can be chosen to define the logarithmic function. In this light, the definition of the logarithmic function given in equation (1.2) is obtained by using the non-positive real axis as branch cut and choosing that level that yields an argument equal to zero on the non-negative real axis. Clearly this is only one of many possible options to define the logarithmic function.

When the branch cut is determined, the choice which branch is used will be of minor importance in practical applications: choosing a higher or lower branch only leads to a constant shift in the potential in two dimensional aerodynamics and to a constant rigid body motion in elasticity.

In programming practical applications, the branch cut should be placed outside any region where the complex logarithmic function is to be evaluated. If the branch cut intersects the domain where the complex logarithmic function is evaluated, the logarithmic function would show two separated areas on the plane as a result. The separation of the domain into two parts would lead to discontinuous displacement or flow fields.

Where to place the branch cut in defining the logarithmic function must be determined for each application separately. Liaw and Kamel's solution describing the deformations in an infinite centrally cracked plate loaded by a point force will be used to demonstrate two methods to program the complex logarithm function. Liaw and Kamel gave their solution within the framework of Lekhnitskii's theory of anisotropic plates [3]. After some explanations on Lekhnitskii's theory and Liaw and Kamel's solution in the following chapter, both methods to program the logarithmic function will be described in chapter 4.

# 3 Lekhnitskii's theory of anisotropic plates

Lekhnitskii's theory of anisotropic elasticity [3] describes the deformations in anisotropic plates. His theory is related to Mushkelishvili's theory of isotropic elasticity, but is simpler in most aspects.

Lekhnitskii's theory provides the solution of the differential equations governing the deformation of an anisotropic plate. The solution is presented in terms of two functions $\phi_1$ and $\phi_2$ expressed in the variables $z_1$ and $z_2$:

$$u(x, y) = 2Re[p_1\phi_1(z_1) + p_2\phi_2(z_2)] + C_1y + C_2 \quad (3.1.a)$$

$$v(x, y) = 2Re[q_1\phi_1(z_1) + q_2\phi_2(z_2)] - C_1x + C_4 \quad (3.1.b)$$

where:

$$z_1 = x + s_1y \quad (3.2.a)$$

$$z_2 = x + s_2y \quad (3.2.b)$$

$$p_1 = a_{11}s_1^2 + a_{12} - a_{16}s_1 \quad (3.3.a)$$

$$p_2 = a_{11}s_2^2 + a_{12} - a_{16}s_2 \quad (3.3.b)$$

$$q_1 = a_{12}s_1 + \frac{a_{22}}{s_1} - a_{26} \quad (3.3.c)$$

$$q_2 = a_{12}s_2 + \frac{a_{22}}{s_2} - a_{26} \quad (3.3.d)$$

$$\varepsilon_i = a_{ij}\sigma_j \quad (3.4)$$

and $s_1$ and $s_2$ are the roots of the equation:

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0 \quad (3.5)$$

The functions $\phi_1$ and $\phi_2$ are unique to each solution. From the complex functions $\phi_1(z_1)$ and $\phi_2(z_2)$ the stresses in the anisotropic plate can be derived:

$$\sigma_x = 2Re[s_1^2\phi'_1(z_1) + s_2^2\phi'_2(z_2)] \quad (3.6.a)$$

$$\sigma_y = 2Re[\phi'_1(z_1) + \phi'_2(z_2)] \tag{3.6.b}$$

$$\sigma_x = 2Re[s_1\phi'_1(z_1) + s_2\phi'_2(z_2)] \tag{3.6.c}$$

The number of closed form solutions known for Lekhnitskii's theory is limited. Only recently, Liaw and Kamel [1] published their solution describing the deformations in an infinite centrally cracked anisotropic plate loaded by an arbitrary point force. The derivation of this solution has not yet been published [4]. Liaw and Kamel described their solution as:

$$\phi_j(z_j) = B_j \log(z_j - z_j^0) - B_j \log(\zeta_j - t_{j1}^0) - \bar{B}_j\left(\frac{s_k - \bar{s}_j}{s_k - sj}\right)\log(\zeta_j - t_{j2}^0)$$

$$-\bar{B}_k\left(\frac{s_k - \bar{s}_k}{s_k - s_j}\right)\log(\zeta_j - t_{k2}^0),\ (j = 1, 2, k = 2, 1) \tag{3.7}$$

where the loaded point is designated by $z^0{}_j$.

The first terms of the equations - the terms $B_1 \log(z_1 - z^0{}_1)$ and $B_2 \log(z_2 - z^0{}_2)$ - describe the solution of the infinite plate without a crack loaded by a point force. This solution is well known and can be found e.g. in Lekhnitskii's book [3, page 130]. Both terms contain a multi valued logarithm function. The restriction that the displacement field around the point force is single valued can be satisfied by appropriately choosing the amplitude coefficient $B_1$ and $B_2$. For these coefficients, Spies [5] gives simple expressions, which are valid when both $a_{16}$ and $a_{26}$ are equal to zero:

$$B_1 = \frac{iX}{4\pi\left(s_1 - \frac{q_1}{q_2}s_2\right)} - \frac{iY}{4\pi\left(1 - \frac{p_1}{p_2}\right)} \tag{3.8.a}$$

$$B_2 = \frac{iX}{4\pi\left(s_2 - \frac{q_2}{q_1}s_1\right)} - \frac{iY}{4\pi\left(1 - \frac{p_2}{p_1}\right)} \tag{3.8.b}$$

where X and Y denote the magnitude of the force components in x and y-direction. To visualize the transition from two multi valued logarithm functions to a single valued displacement field, the Riemann surface may be used: the amplitude coefficients ensure that while the Riemann surface of one of the terms containing a logarithm function is rotating counter clockwise, the Riemann surface of the other term is rotating clockwise. Travelling over a full circle around the point force yields a nett single valued displacement field when both terms are summed.

Usually, the logarithmic function used to evaluate the terms described above is defined as the principal value of the logarithm function (equation 1.2). In this definition, the

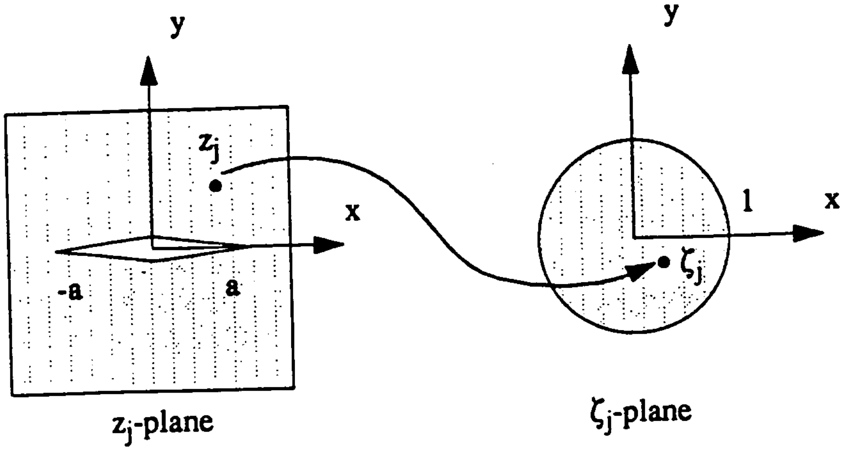


*Figure 3.1 Conformal mapping from the $z_j$-plane to the $\zeta_j$-plane in Liaw's solution of the infinite, centrally cracked plate loaded by a point force*

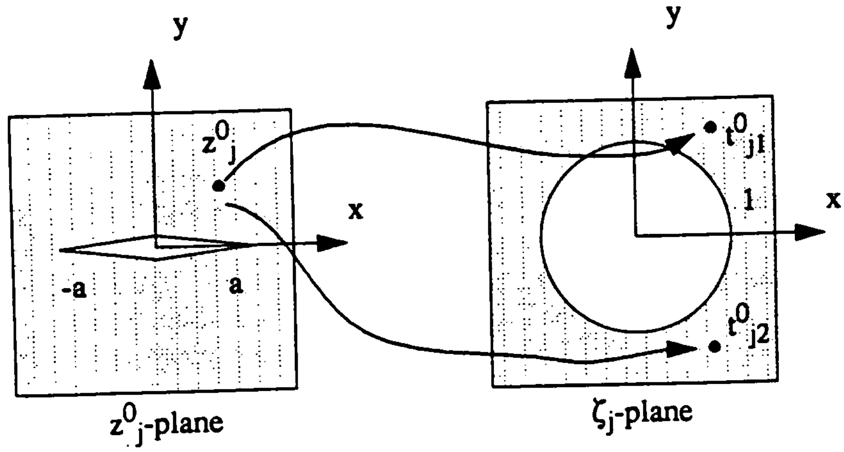


*Figure 3.2 Conformal mapping from the $z^0_j$-plane to the $t^0_{j1}$ and $t^0_{j2}$-plane in Liaw's solution of the infinite, centrally cracked plate loaded by a point force*

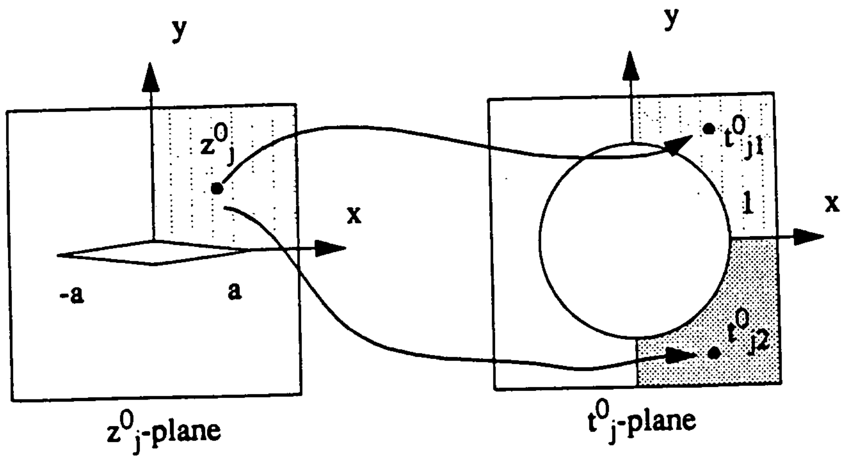


*Figure 3.3 Conformal mapping from the $z^0_j$-plane to the $t^0_{j1}$ and $t^0_{j2}$-plane, with the loaded point $z^0_j$ limited to the first quadrant of the plane*

branch cut lies at the non-positive real axis. At this line, the variables $z_1$ and $z_2$ coincide. This ensures that the discontinuities in both terms arising from the logarithmic function, cancel out pair wise on this line. Theoretically, they cancel out exactly; practically, they can be made to cancel out with any desired accuracy. Only the non-positive and the non-negative real axis provide such a convenient location for a branch cut. On any other radial line from the origin to infinity, it is not at all obvious that the discontinuities from the logarithmic function will cancel out pair wise for these terms.

The remaining terms in Liaw and Kamel's solution - the ones expressed in $\zeta_j$, $t^0_{j1}$ and $t^0_{j2}$ - eliminate the stresses at the crack surface. They correct the solution of the plate without a crack for the presence of the crack on the real axis.

The variables $\zeta_j$ are derived through conformal mapping from the variables $z_j$: the $z_j$-plane - containing all points in the plate where the displacements and stresses are sought - is mapped into the unit circle on the $\zeta_j$-plane (figure 3.1).

The variables $t^0_{j1}$ and $t^0_{j2}$ are derived through an almost identical conformal mapping from the variables $z^0_j$: now the entire $z^0_j$-plane, containing all the point where the point force may be applied, is mapped outside the unit circle on the $t^0_{j1}$ and $t^0_{j2}$-plane (figure 3.2).

The loaded point $z^0_j$ is mapped outside the unit circle on the $t^0_{j1}$ and $t^0_{j2}$-plane. The point $z_j$ where the displacements are sought, is mapped inside the unit circle on the $\zeta_j$-plane. Both groups of points are therefore geometrically separated: one group is inside the unit circle, the other is outside the unit circle. Only when loads are applied at the crack surface, both groups overlap on the unit circle. This will be avoided throughout this paper. Since points from both groups can never coincide, the logarithm function expressed in the variables $\zeta_j$, $t^0_{j1}$ and $t^0_{j2}$ can never have an argument which is equal to zero. Therefore these logarithm terms do not cause singularities in the solution.

The problems encountered in defining the logarithmic function for the correction terms are of a different nature compared to the terms from the basic solution. Whereas for the basic solution the problems arise from travelling around the singularity, the correction terms do not contain such a singularity. For the correction terms, the problem is that there is no obvious branch cut available, along which the discontinuities in the logarithmic function cancel out pair wise. Consequently, evaluation of these terms near the branch cut in the logarithmic function should be avoided.

One way to do this, is carefully placing the branch cut outside any region where these terms are evaluated. For all points in the plate, the variables $z_j$ are contained within or on the unit circle on the $\zeta_j$-plane. The variables $z^0_j$ are mapped outside or on the unit circle on the $t^0_{j1}$ and $t^0_{j2}$-plane. The arguments of the logarithm terms expressed in $\zeta_j$, $t^0_{j1}$ and $t^0_{j2}$ can take any value if both $z_j$ and $z^0_j$ are free to be chosen. Limiting the range of the variables $z_j$ on the $\zeta_j$-plane does not change this behaviour. Limiting the range of the variables $z^0_j$ can limit the domain of the terms containing logarithm functions: by e.g. choosing the first quadrant of the plane as the only quadrant where $z^0_j$ may be chosen - that is the only quadrant where the point force may be applied - $z^0_j$ is mapped only to the first quadrant on the $t^0_{j1}$-plane and to the fourth quadrant on the $t^0_{j2}$-plane, in both cases outside the unit circle (figure 3.3).

The argument of the logarithm functions is expressed as the difference of $\zeta_j$and $t^0_{j1}$ or $t^0_{j2}$. These expressions form the domain of the logarithm functions. If the loaded point

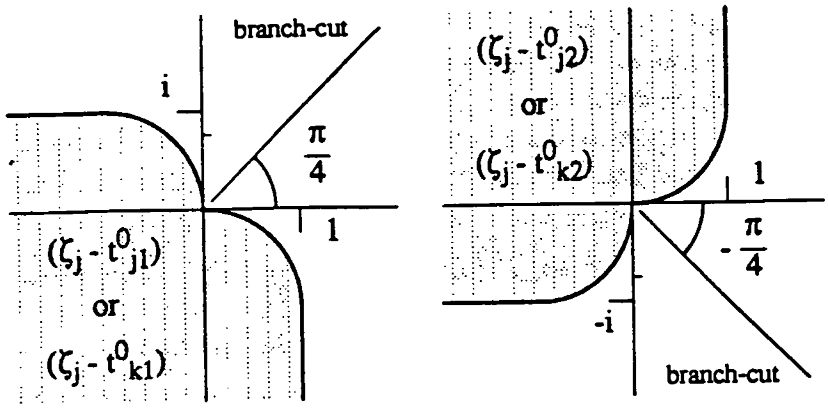


*Figure 3.4 Domain of the logarithm-functions in the terms correcting for the presence of the crack, with the loaded point $z^0 j$ limited to the first quadrant of the plane*

$z^0{}_j$ is limited to the first quadrant of the plane, the domain of the logarithm functions only covers three quadrants (figure 3.4).

The branch cut in the definition of the logarithmic function for these terms can appropriately and conveniently be chosen on radial lines from the origin in the direction of $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. With the loaded point limited to the first quadrant, the domain of the logarithm functions in the correction terms will never cross these lines.

When the deformations in the plate are to be known for point forces in any of the other quadrants, either symmetry conditions or an appropriate redefinition of the branch cut will suffice. In the following chapter, a method to program the branch cut at an arbitrary radial line from the origin will be explained.

The method of positioning the branch cut in the definition of the logarithmic function outside the domain of the logarithm function, requires prove that this domain is restricted to part of the plane. For applications where this prove is not available or where restricting the domain is not convenient, a method is described in the next chapter which allows under certain restrictions programming of the logarithm function on the entire Riemann surface.

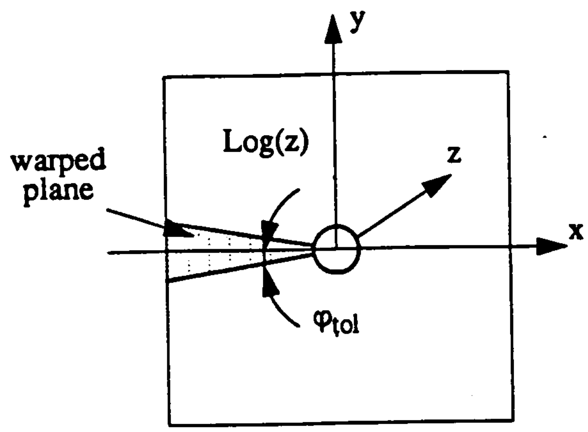


*Figure 4.1 Connection of the edges of a branch of the Riemann surface by a warped plane*

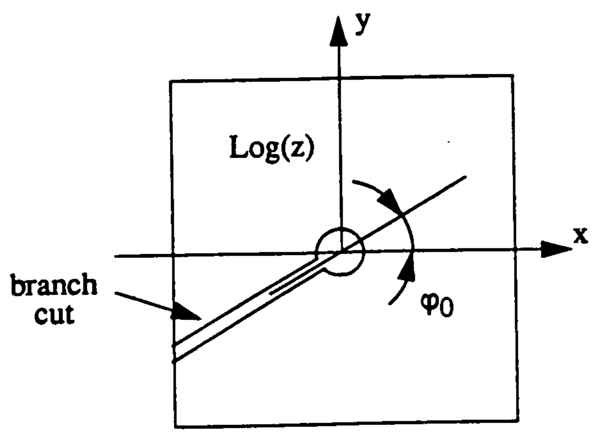


*Figure 4.2 Definition of the logarithmic function on a plane with the branch cut on a radial line from the origin*

# 4 Programming of the complex logarithm function

In Fortran, the standard complex logarithmic function is defined as the principal value of the logarithm function:

$$Log(z) = ln|z| + iArg(z), -\pi < Arg(z) \le \pi \qquad (4.1)$$

The branch cut lies at the non-positive real axis. Points on the branch cut are assumed to have a phase angle of $\pi$ radians, as if the Riemann surface from above the non-positive real axis extends over the non-positive real axis. This choice is arbitrary: with equal validity a phase angle of $-\pi$ could have been chosen on the branch cut.

To avoid round off errors near the branch cut in programming this definition of the complex logarithmic function, the edges of the Riemann surface near the branch cut may be reconnected by a warped plane over a small angle above and below the non-positive real axis (figure 4.1). A mathematical definition of the warped plane would look like:

$$Log(z) = ln|z| + i\left(1 - \frac{\pi}{\varphi_{tol}}\right)\frac{y}{x},\ |y| < -\varphi_{tol}x \qquad (4.2)$$

With this definition, the zero order discontinuity along the branch cut is replaced by two first order discontinuities along the lines connecting the warped plane and the Riemann surface. In practice, the logarithm function may more often be evaluated right on the branch cut than just beside it.

The definition of the warped plane may prevent round off errors in evaluating the logarithmic function on the branch cut. In Liaw and Kamel's solution e.g., the logarithmic function in the first two terms $B_j \log(z_j - z^0_j)$ is evaluated right on the non-positive real axis when the point $z_j$ where the displacement is sought, lies exactly left of the point $z^0_j$ where the point force is applied. To obtain the right displacement, the phase angle of both arguments should be identical in the summation of both terms: either both $\pi$, zero or $-\pi$ radians. Round off errors could move one argument just above the real axis and the other one just below. If the warped function were not used, an erroneous displacement could be obtained.

As discussed in chapter 2, the definition of the logarithmic function from the logarithm function with the branch cut along the non-positive real axis is only one of the many valid definitions. The branch cut can be rotated to any radial position from the origin by multiplying the argument of the standard logarithmic function by a factor $e^{-i\varphi_0}$ and adding an amount of $i\varphi_0$ to the imaginary part off the result:

$$Log(z) = Log(ze^{-i\varphi_0}) + i\varphi_0 \qquad (4.3)$$

In this definition, the angle $\varphi_0$ indicates the direction opposite the branch cut (figure 4.2). Setting $\varphi_0$ equal to zero, one obtains again the standard branch cut at the non-positive real axis. Effectively, the multiplication of the argument of the logarithmic function by a factor $e^{-i\varphi_0}$ causes a rotation of the xy-axis system over an angle $\varphi_0$. After evaluating the logarithmic function with the standard Fortran function, an amount of $i\varphi_0$ must be added to the imaginary part to compensate for the rotation of the xy-axis system.

One may consider introducing the warped plane at the branch cut also for this definition of the logarithmic function. Applying this definition however requires an intentional rotation of the branch cut. As discussed in chapter 2, the position where the branch cut is placed, should be outside any region where the logarithmic function is expected to be evaluated. Instead of the warped plane an error message to warn for an unexpected evaluation near the branch cut, may be more appropriate.

Not always it may be easy to find an appropriate location to situate the branch cut. In Liaw and Kamel's solution e.g. limiting the area where the point force may be applied, reduced the domain of the logarithmic functions to less than three quadrants of the plane. The quadrant unoccupied by the domain provided an appropriate location for the branch cut. In other cases, either the domain of the logarithmic functions may be unknown or limiting the domain may be undesirable. For these cases, the logarithmic function may be programmed on its entire Riemann surface.

The Riemann surface of the logarithmic function consists of an infinite number of planes formed into an infinite circular staircase around the origin. On successive levels of the Riemann surface, the argument of a complex variable z differs by a magnitude of $2\pi$ radians. As a result, the logarithm function of the variable z differs by a magnitude $2\pi i$.

In solutions containing the complex logarithmic function, any continuous line in the domain - e.g. a continuous line on the plate in Liaw and Kamel's solution - should remain a continuous line on the Riemann surface, even after mapping procedures. The line can not jump from one level of the Riemann surface to the next level; it must travel around the origin to move up and down on the staircase. Splitting the continuous line in a series of neighbouring points, one can state that neighbouring points in the domain lie close to each other on the Riemann surface. How close they lie on the Riemann surface mainly depends on how close they lie in the domain. When it is assumed that in the domain the spacing between neighbouring point is close enough, neighbouring points on the Riemann surface may be assumed to lie on the same level of the Riemann surface. That is, they differ in phase angle less than $\pi$ radians.

To compute the logarithm function in one point, the phase angle of its close neighbour may then be used to define the branch cut in the logarithmic function. Mathematically, their relation looks like:

$$Log(z_{n+1}) = Log(z_{n+1}e^{-i\varphi_n}) + i\varphi_n \tag{4.4.a}$$

$$\varphi_{n+1} = Im[Log(z_{n+1})] \tag{4.4.b}$$

When applied to a consecutive single file of points forming points and neighbours, an iteration process would develop. The iteration process does not accumulate errors. The starting value of the phase angle used to evaluate the first point in the row, would only cause a constant shift in phase angle for all the points in the row. As argued in chapter 2, a constant shift in phase angle is of minor importance in most applications. In Liaw and Kamel's solution e.g. it only causes a rigid body motion of the plate.

Proving that the spacing between neighbouring points in the domain of the logarithmic function is small enough to ensure that they differ in phase angle less than $\pi$ radians, is quite difficult. Around the point force in Liaw and Kamel's solution e.g., the phase angle changes over a complete $2\pi$ radians in the domain of the terms $B_j \log (z_j - z^0{}_j)$. When two neighbouring point would have the loaded point right in between them, on the line connecting them, they would have a difference in phase angle of exactly $\pi$ radians. In this example, avoiding the problem is possible and prove can be given that the problem will not occur if the right measures are taken. In general, a mathematical prove that the points in the domain lie close enough may be difficult to give.

In the method presently discussed, it is essential that a neighbour is assigned to every point where the logarithmic function is evaluated. More over, a unique starting point must be used and all point must be reached along a path starting from this point, to ensure that all points are connected in one group on the Riemann surface. In finite element meshes, the wavefront optimization routines may serve to indicate neighbours to each point. Consecutive wavefronts start from one point or a line of points and progress through the mesh without leaving spaces. Other types of meshes, such as the mesh Erdogan and Arin used [6], do not have connectivity conditions imposed on them. Their mesh - which will be referred to as Erdogan's mesh only for the sake of brevity - was intended for a numerical surface integration procedure. Not the connectivity of his elements but the fact that they covered the entire plane was of importance. For Erdogan's mesh, line fractals may be suited to order all elements in a single file. Wavefront methods may require more programming effort, since they do not already form a part of the numerical procedure as in finite element methods.

In the next chapter, after some explanations on fractal theory, it will be shown that Erdogan's mesh can be generated with the assistance of a line fractal, obtaining all elements ordered in a single file at the same time.

Finally, in chapter 6 all methods to program the complex logarithmic function discussed above will be demonstrated using Liaw and Kamel's solution of the anisotropic centrally cracked plate loaded by a point force, and Erdogan's mesh as an example.

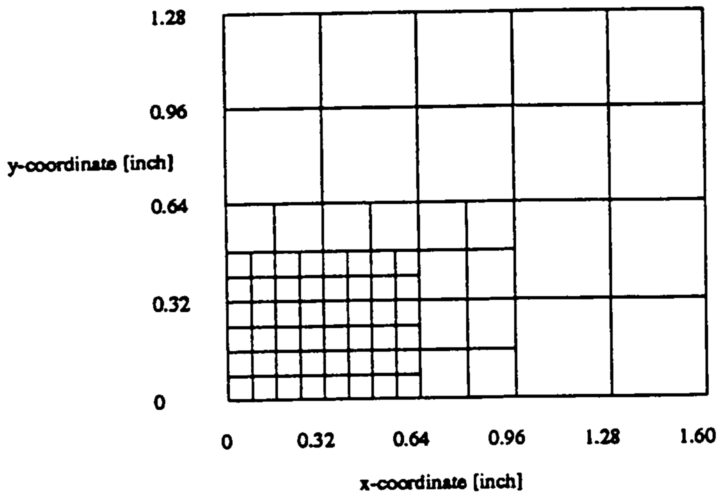


*Figure 5.1 Erdogan's mesh for a surface integration procedure*

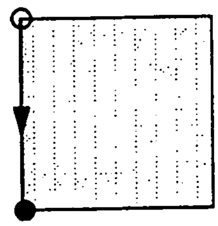

*Figure 5.2 Initial shape of the fractal used for Erdogan's mesh*

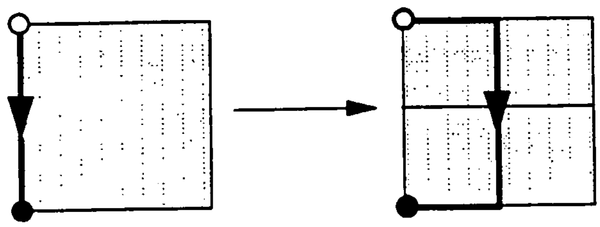

*Figure 5.3 Refinement rule for the fractal used for Erdogan's mesh*

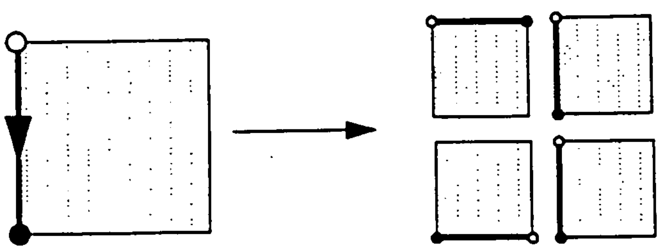

*Figure 5.4 Self-similarity in the fractal used for Erdogan's mesh*

## 5 Fractal assisted mesh generation

The past years, many books have been published on fractal theory, some scientific and some more popular in their approach. For the present study, the more popular book of Lauwerier [7] has been used as guidance.

Fractals don't behave according to the rules known for lines and points. Where lines and points have dimensions of one and zero, fractals can have any dimension in between whole numbers. The dimension of a fractal on a plane can be any fraction - hence the name - in between zero and two. The fractional dimension of a fractal on a plane describes the amount of area covered by the fractal: a fractal with a dimension in between zero and one is an infinite collection of points and forms a 'dust' pattern on the plane; a fractal with a dimension in between one and two is an infinite collection of infinitely small line sections, joined in a continuous coiled line.

Being a continuous line on the plane, the line type fractal supplies the single file of consecutive points and neighbours required to evaluate the complex logarithm function on the Riemann surface, as described in the previous chapter. To ensure that at each point of the plane the logarithm function can be evaluated, the fractal should be able to reach all points of the plane. A two dimensional line fractal is required for this task: being of the same dimension as the plane, it can only reach its two dimensional form by covering the entire plane, provided it doesn't intersect itself at any point.

Fractals are best known for their self similarity on increasing levels of refinement: zooming in on a fractal will show an identical fractal. The feature of self similarity enables an effective procedure to generate meshes with the assistance of the fractal. Refining the fractal requires few programming rules and varying the level of refinement over the surface of the plane provides a varying level of mesh refinement.

One fractal can't perform the miracle to provide a mesh generation routine for any mesh: each type of mesh requires its own fractal. In this chapter, a fractal suited for Erdogan's mesh [6] will be described. Erdogan presented his mesh as shown in figure 5.1.

Erdogan used his mesh in a method to model cracks in metal composite laminates. The method involved the solution of the infinite, centrally cracked, isotropic plate loaded by a point force, presented within the framework of Mushkelishvili's theory of isotropic elasticity. A numerical surface integration procedure formed the core of Erdogan's computations. Being intended for a surface integration, the mesh didn't have to obey the strict connectivity rules as e.g. have to be observed for finite element meshes. Basically, the only rule was that the entire surface had to be covered. Around the crack tip, where large gradients in the solution where expected, the mesh was refined.

Erdogan's mesh is composed of square elements of different sizes. To define the fractal for Erdogan's mesh, two items must be specified: first the initial shape of the fractal on one element, next the rule to obtain successive refinement levels from this basic element. Starting from one of the largest elements, the initial shape of the fractal is defined as a line along one side of the element (figure 5.2).

This line is the start of the fractal. The open dot indicates the starting point, the closed dot the end point. Successive levels of refinement are obtained by applying the base refinement rule, unique to each fractal. For Erdogan's mesh, the refinement rule implies

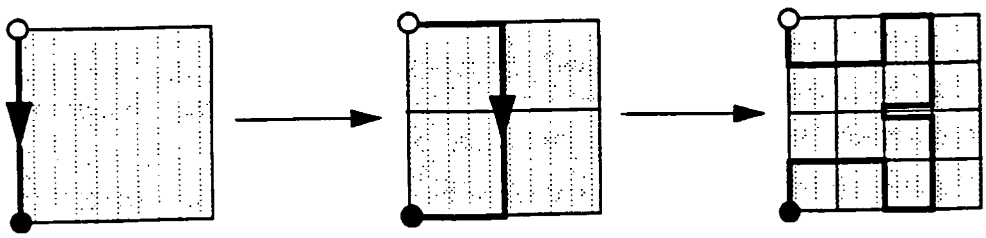

*Figure 5.5 Fractal used for Erdogan's mesh at the second level of refinement*

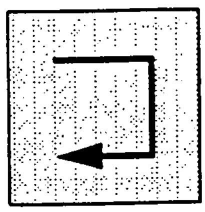

*Figure 5.6 Notation for the fractal refinement rule*

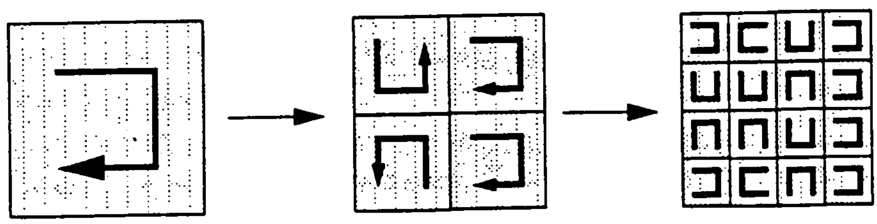

*Figure 5.7 Successive refinement of the fractal in symbol notation*

replacing one line by four lines with each half the original length in the way presented in figure 5.3.

Each smaller square can be identified with one of the smaller lines with half the length of the original lines. Observing each smaller square separately, each is seen to be a smaller copy of the original square (figure 5.4). Applying the refinement rule twice, already yields a slightly coiled line (figure 5.5).

The fractal is seen to return on its path on parts of the fractal. To avoid confusion when drawing the fractal, the notation is changed. Each element is identified by a symbol indicating in which order of four smaller elements it is to be divided. The first, largest, element is indicated as shown in figure 5.6.

Successive refinements of the fractal in this notation show as a collection of squares with curved arrows (figure 5.7). Smaller squares are not large enough to show the full arrow.

Four types of elements can be identified, when the direction and starting point of the arrow is taken as discerning feature. For each type of element, the fractal refinement rule can be described as a substitution of one element by four smaller elements of fixed type and position (figure 5.8).

In fact, all four types of elements and their refinement rules are identical: their only distinction can be solved by rotation and mirroring. In programming, it is convenient to distinguish between the four types of elements.

Starting from one element, the fractal refinement will yield a single file of smaller elements inside this first element. Programming the refinement procedure amounts to replacing one element in the file by four new elements, indeed a simple task.

For Erdogan's mesh, the most convenient collection of elements to start from is not a single element but a series of elements. In symbol notation, the starting mesh looks as shown in figure 5.9.

Refining this starting collection of elements, yields a continuous fractal line, travelling across Erdogan's mesh (figure 5.10).

Observing the fractal in a refined stage, it seems obvious that the fractal can at least approach any point on the plane. Mandelbrot's definition of fractal dimension even yields a dimension equal to two for the present fractal. Mandelbrot's definition of fractal dimension can be found in most books on fractal theory. In the book of Lauwerier [7] it was quoted as:

$$D = \frac{\log(N)}{\log\left(\frac{1}{a}\right)} \tag{5.1}$$

where D is the fractal dimension, N the number of line segments on one refinement level of the fractal and a the length of these line segments. For the present fractal, the number of line segments and the length of each line segment can be expressed in two formulas (table 5.1), as can be verified observing figure 4.4. Substituting in Mandelbrot's equation yields a dimension equal to two for the fractal:

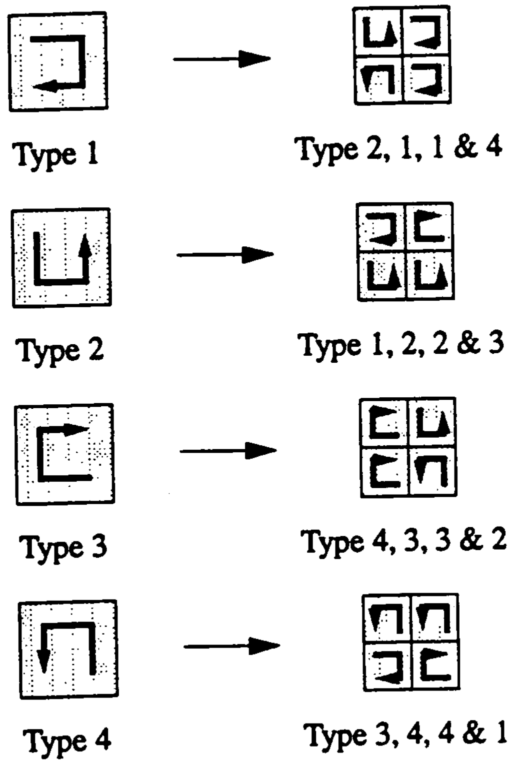


*Figure 5.8 Refinement rules in symbol notation for the four types of mesh elements*

**Table 5.1 Mandelbrot dimension of the fractal used for Erdogan's mesh**

| level of refinement | 0 | 1 | 2 | 3 | ... | n |
|---|---|---|---|---|---|---|
| number of line segments N | 1 | 4 | 16 | 64 | ... | $2^{2n}$ |
| length of line segments a | 1 | 1/2 | 1/4 | 1/8 | ... | $1/2^{n}$ |

$$D = \frac{log\,(N)}{log\,(\frac{1}{a})} = \frac{log\,(2^{2n})}{log\,(2^{n})} = \frac{2n}{n} = 2 \qquad (5.2)$$

Only the small parts where the fractal returns on its path prevent a mathematical prove that the fractal covers the entire plane and reaches any point. The fractal does approach any point inside the starting element and on its boundaries at a sufficient level of refinement. For an engineering application this seems to be sufficient.

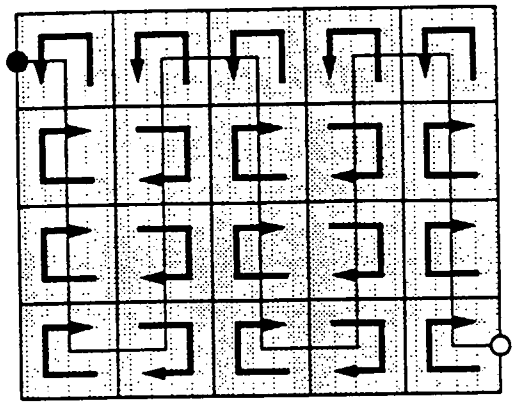

*Figure 5.9 Collection of elements to start the generation of Erdogan's mesh*

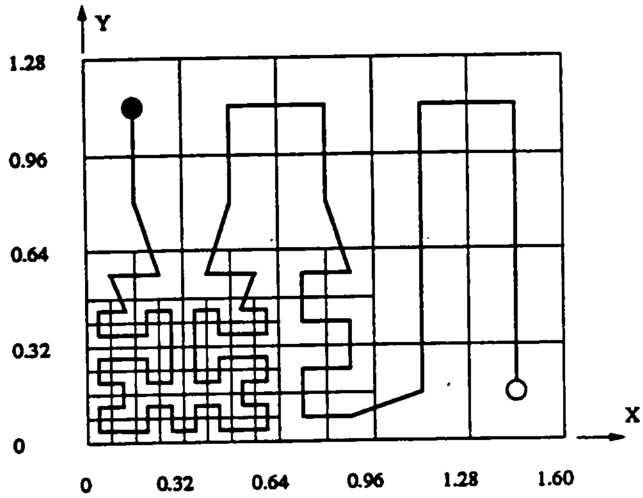


*Figure 5.10 Erdogan's mesh generated with the assistance of the line fractal*

**Table 6.1 Geometrical and material properties of the plate used to demonstrate Liaw and Kamel's solution**

| | | |
|---|---|---|
| Young's modulus | $E_x$ | 3.5 MPa |
| | $E_y$ | 3.5 MPa |
| Shear modulus | $G_{xy}$ | 0.7114 MPa |
| Poisson's ratio | $\nu_{xy}$ | 0.3 |
| crack length | a | 0.4 inch |
| position of the point force Y | $x^0$ | 0.48 inch |
| | $y^0$ | 0.16 inch |

# 6 The complex logarithm function in Liaw and Kamel's solution of the cracked anisotropic plate

Liaw and Kamel's solution of the infinite centrally cracked anisotropic plate loaded by a point force, can serve as an illuminating example of an engineering application of the complex logarithm function. Liaw and Kamel's solution contains eight terms with a logarithm function. As discussed in chapter 3, six of these terms are sensitive to the position of the branch cut in the definition of the logarithmic function. The standard definition of the logarithmic function with the branch cut at the non-positive real axis, as described in chapter 2, leads to discontinuities in the computed displacement field at parts of the plane. In this chapter, these discontinuities will be shown and the branch cut in the definition of the logarithmic function will be shown to be the cause of the errors.

As discussed in chapter 3, a redefinition of the position of the branch cut in the logarithmic function or the definition of the logarithm function on its entire Riemann surface with the assistance of fractals, can solve these problems. Both will be demonstrated. The programming methods described in chapter 4 will be used.

The displacement field will be derived on all four quadrants of the plane, using Erdogan's mesh on each quadrant as shown in figure 6.1. The displacements are derived at the centre points of the elements. In the definition of the logarithm function on the Riemann surface, the fractal presented in figure 5.10 will be used. Liaw and Kamel's solution is computed for a cracked plate with the - somewhat symbolic - geometrical and material properties presented in table 6.1.

To avoid the singularity in the logarithm functions, the point force was chosen not to coincide with any point where the displacements are derived.

The discontinuities in the displacement field can sprout from all six terms which contain a logarithm function expressed in the variables $\zeta_j$, $t^0{}_{j1}$ and $t^0{}_{j2}$. The variables $\zeta_j$ describe the point where the displacement is sought. They are mapped from the variables $z_j$ into the unit circle on the $\zeta_j$-plane. For the present geometrical and material properties, the fractal describing Erdogan's mesh on the $\zeta_j$-plane is presented in figure 6.2. The variables $t^0{}_{j1}$ and $t^0{}_{j2}$ contain all the points where the point force may be applied. Presently, this amounts to only one point. The variables $t^0{}_{j1}$ and $t^0{}_{j2}$ are mapped from the variables $z^0{}_j$ outside the unit circle on the $t^0{}_{j1}$ and $t^0{}_{j2}$-plane, as shown in figure 6.2.

The domain of the six logarithm functions in Liaw and Kamel's solution is expressed as the difference of a $\zeta$-variable and a $t^0$-variable. Plotting e.g the domain of the two logarithm terms $\log(\zeta_1 - t^0_{11})$ and $\log(\zeta_2 - t^0_{21})$ as shown in figure 6.3, explicitly shows the error in applying the standard definition of the logarithmic function: the domain of the term $\log(\zeta_2 - t^0_{21})$ crosses the non-positive real axis. Computing the logarithm function with this standard definition of the logarithmic function splits the domain of the term $\log(\zeta_2 - t^0_{21})$ into two parts, as can be seen in figure 6.4. The domain of the term $\log(\zeta_1 - t^0_{11})$ remains intact. The part of the displacement field where the separated portion of the domain of the term $\log(\zeta_2 - t^0_{21})$ contributes to, is not compatible with the surrounding displacement field (figure 6.5). For the term $\log(\zeta_2 - t^0_{21})$ the separated portion of the domain originates from an area below the crack. The area is separated from the main part of the plate by a curved, parabolic like line. The discontinuity in displacement across this line is not too obvious. Above the crack, two similar areas with discontinuous displacements can be found. The larger and smaller area are respectively related to the terms $\log(\zeta_2 - t^0_{22})$ and $\log(\zeta_1 - t^0_{22})$. Especially for the smaller area related to the term $\log(\zeta_1 - t^0_{22})$ the discontinuity in the displacements across the edge is obvious. For the present example, the number of areas with discontinuous displacement fields is limited to three. Depending on the position of the applied point force, both the number of these areas and the shape of their edges could vary.

Observing the domain of the term $\log(\zeta_2 - t^0_{21})$ in figure 6.3, the choice of the position of the branch cut at an angle of $\frac{\pi}{4}$, as discussed in chapter 3, seems justified. Computing the logarithmic with the branch cut at this angle leaves the domains of both terms $\log(\zeta_1 - t^0_{11})$ and $\log(\zeta_2 - t^0_{21})$ intact (figure 6.6).The displacement field derived with this definition of the logarithmic function is fully continuous (figure 6.7).

A redefinition of the branch cut in the logarithmic function suffices to solve the discontinuities in the displacement field. The fractal method, using the definition of the logarithmic function on the entire Riemann surface, can't add anything to these results in the present example. Computing the logarithm function in the terms $\log(\zeta_1 - t^0_{11})$ and $\log(\zeta_2 - t^0_{21})$ with the fractal method only results in a constant shift in the imaginary part of both terms on the entire domain, when compared to the method with the branch cut at an angle of $\frac{\pi}{4}$ (figure 6.8). The logarithm function is computed on one level higher on the Riemann surface, resulting in a shift in the imaginary part equal to $2\pi$ on the entire domain. The constant shift in both terms $\log(\zeta_1 - t^0_{11})$ and $\log(\zeta_2 - t^0_{21})$ doesn't even cause a rigid body displacement (figure 6.9).

The standard definition of the logarithmic function with the branch cut at the non-positive real axis has been shown to cause discontinuities in the displacement field: the domain of the term $\log(\zeta_2 - t^0_{21})$ is intersected by the non-positive real axis (figure 6.3), causing the domain to be split into two parts after computation of the logarithmic function (figure 6.4) and causing a discontinuous displacement field (figure 6.5). Both the definition of the logarithmic function with the branch cut at $\frac{\pi}{4}$ and the definition of the logarithm function on its entire Riemann surface are seen to solve these problems (figure 6.7 and 6.9).

## 7 Conclusions

Lekhnitskii's theory of anisotropic plates is one of the applications of complex function theory to engineering mathematics. Within Lekhnitskii's theory, the complex logarithm function describes the displacements around a concentrated point force. Programming solutions of complex function theories containing the complex logarithm function, can lead to problems. In the mathematical solution, the logarithm function is a multi valued function. Within Lekhnitskii's theory, the solutions are described in such a way that the multi valued logarithm function does not lead to multi valued displacement fields. In programs, the multi valued logarithm function has to be replaced by the single valued logarithmic function. The logarithmic function is defined as one single valued branch of the logarithm function. Across the branch cut - the line along which the one single valued branch is cut from the logarithm function - the logarithmic function is discontinuous. The discontinuity in the logarithmic function across the branch cut may introduce a line of discontinuity in the programmed solution, which was not present in the mathematical solution.

In the solution of the infinite anisotropic plate loaded by a point force, known form Lekhnitskii's book [3], the branch cut in the logarithmic function never led to problems. Most commonly, the standard definition of the logarithmic function with the branch cut at the non-positive real axis is used. For this definition of the branch cut, the discontinuities cancel out internally in the solution.

In Liaw and Kamel's solution of the infinite centrally cracked anisotropic plate loaded by a point force, the branch cut leads to problems. No definition of the branch cut, also the standard definition at the non-positive real axis, can provide a reliable computation of the displacement field, if the position of the point force is free to be chosen anywhere on the plane. Depending on the position of the point force, the position of the branch cut should be determined. Limiting the position of the point force to the first quadrant of the plane, enables a reliable computation of the displacement field by situating the branch cut outside the domain where the complex logarithm functions are evaluated. For terms in Liaw and Kamel's solution containing the variable $t^0_{j1}$, the branch cut should be defined on a radial line from the origin at an angle of $\frac{\pi}{4}$ radians relative to the positive real axis; for the terms containing the variable $t^0_{j2}$, the branch cut should be defined at a radial line at an angle of $-\frac{\pi}{4}$ radians relative to the positive real axis.

Besides the solution of redefining the branch cut, the logarithm function may also be programmed on its entire Riemann surface. For this second solution, the points on the plane where the logarithm function is to be evaluated, should be ordered in a single file of consecutive points and neighbours, forming a continuous line on the plane. Assuming that it has been proven that the phase angle between neighbouring points in the single file of points is less than $\pi$ radians, than the logarithmic function for each point can be evaluated with the branch cut opposite the position of its neighbour on the Riemann surface.

For Erdogan's mesh, a two dimensional line fractal can supply the single file of points across the mesh. Erdogan used his mesh in a method to model cracks in metal composite laminates, involving Erdogan's own solution of the cracked isotropic plate loaded by a point force. With the assistance of a fractal, Erdogan's mesh can be generated and simultaneously the elements ordered in a single file. Programming the logarithm function on its Riemann surface provides an automated evaluation procedure for any domain of the logarithm function. Especially when the domain of the logarithm function is unknown or can't be contained to a part of the plane to allow a reliable definition of the branch cut, the fractal method may solve the problems.

## References

[1] B.M. Liaw and M. Kamel, Stress intensity factors for cracks emanating from a curvilinear hole in an anisotropic plane under arbitrary loadings. *Engng Fracture Mech.* **36**, 669-681 (1990).

[2] J.D. Paliouras, *Complex variables for scientists and engineers.* Macmillan Publishing Co., New York (1975).

[3] S.G. Lekhnitskii, *Anisotropic plates.* translated from the 2nd Russian edition by S.W. Tsai and T. Cheron, Gordon and Breach Science Publishers, New York (1968).

[4] M. Kamel and B.M. Liaw, Fundamental solutions for an anisotropic plane containing an elliptical hole or a crack. Submitted for publication in *J. Appl. Mech.*

[5] G.J. Spies, *Versterkte materialen.* (Lecture notes in Dutch), Delft University of Technology, Delft, Netherlands (1978).

[6] F. Erdogan and K. Arin, A sandwich plate with a part-through and a debonding crack. *Engng Fracture Mech.* **4**, 449-458 (1972).

[7] H. Lauwerier, *Fractals* (In Dutch). Aramith Uitgevers, Bloemendaal, Netherlands (1987).

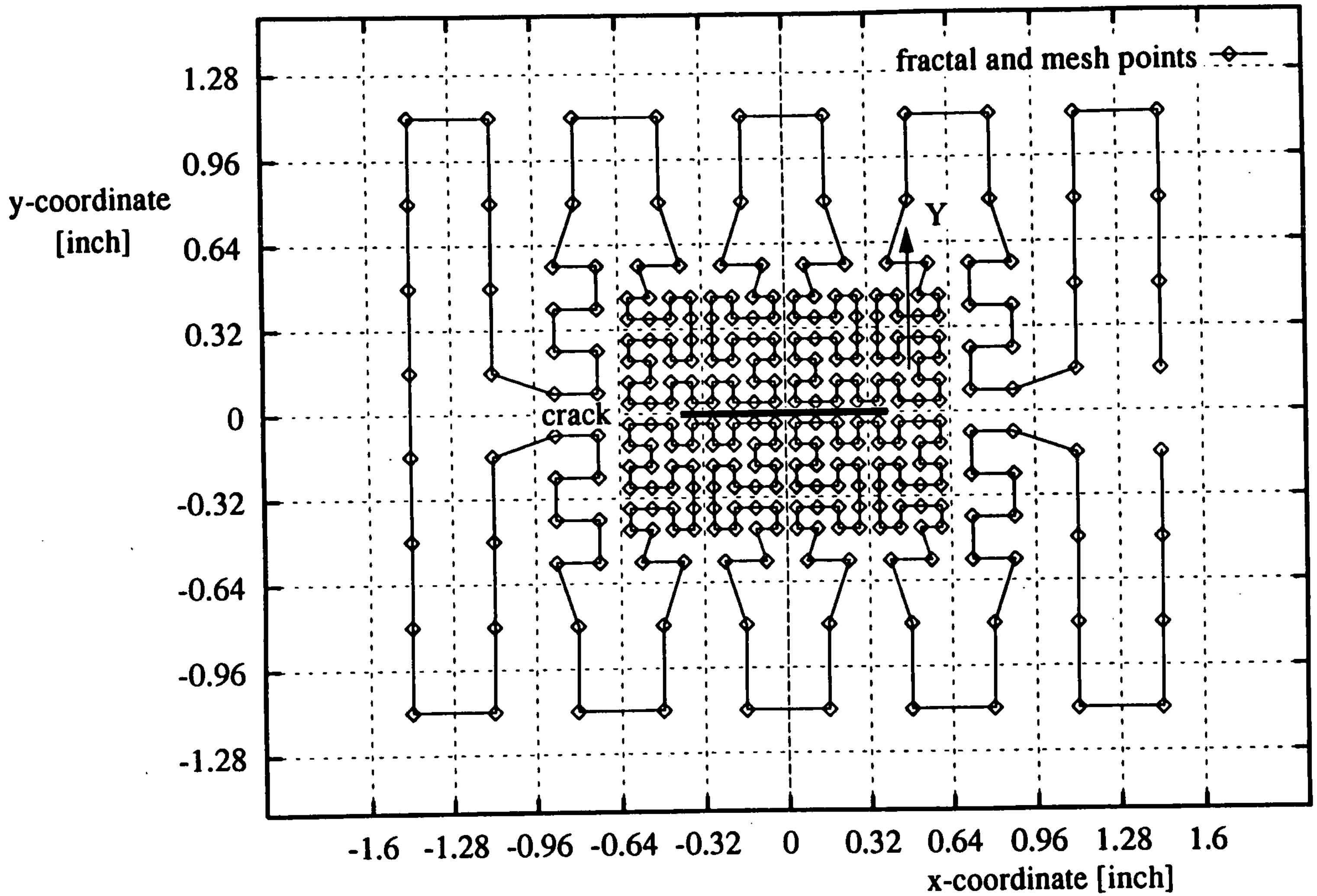


Figure 6.1 Erdogan's mesh and the fractal on all four quadrants of the plane

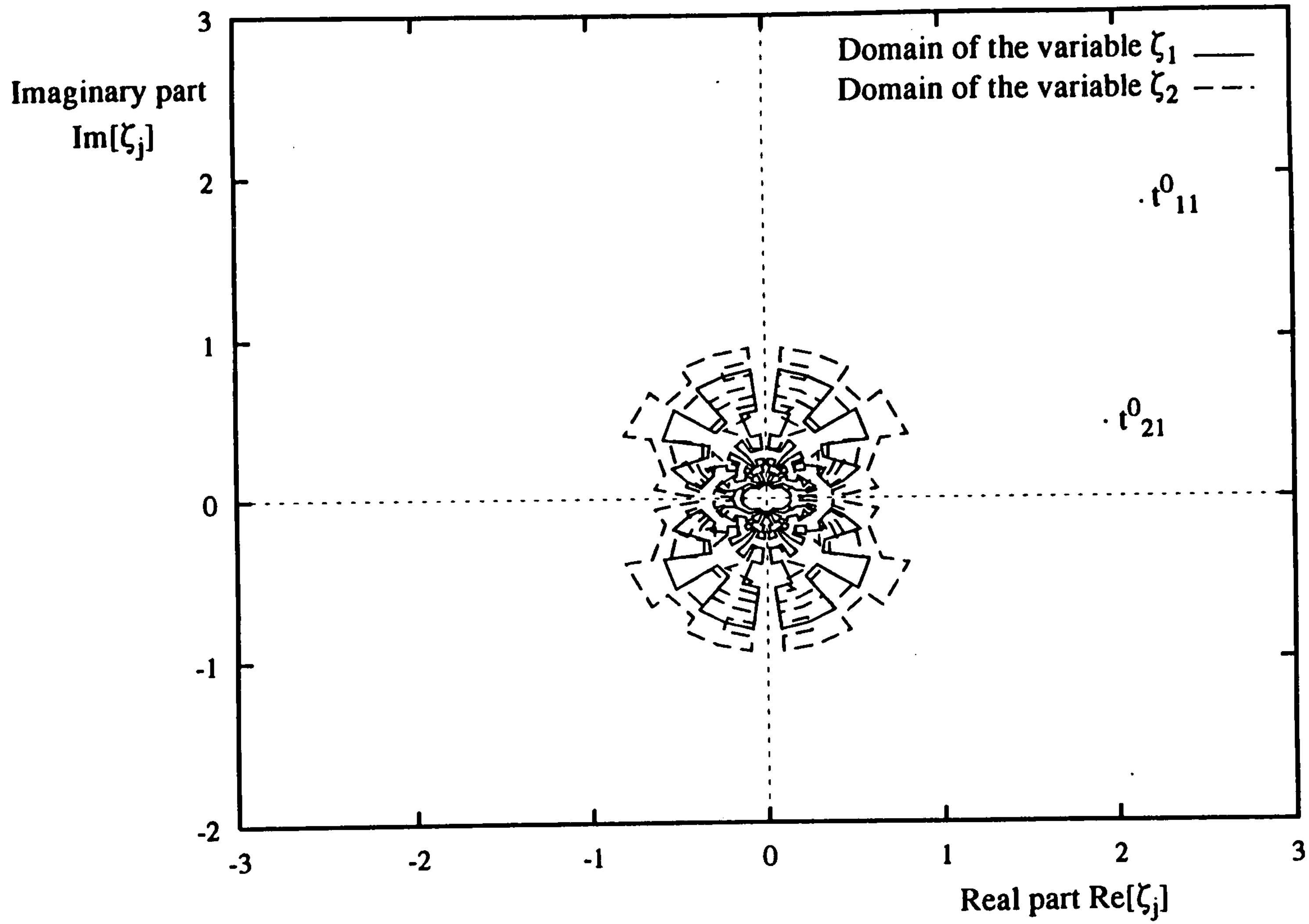


Figure 6.2 Domain of the complex variables $\zeta_1$ and $\zeta_2$ after conformal mapping of the fractal and the position of the complex variables $t^0{}_{11}$ and $t^0{}_{21}$ in the example of Liaw and Kamel's solution

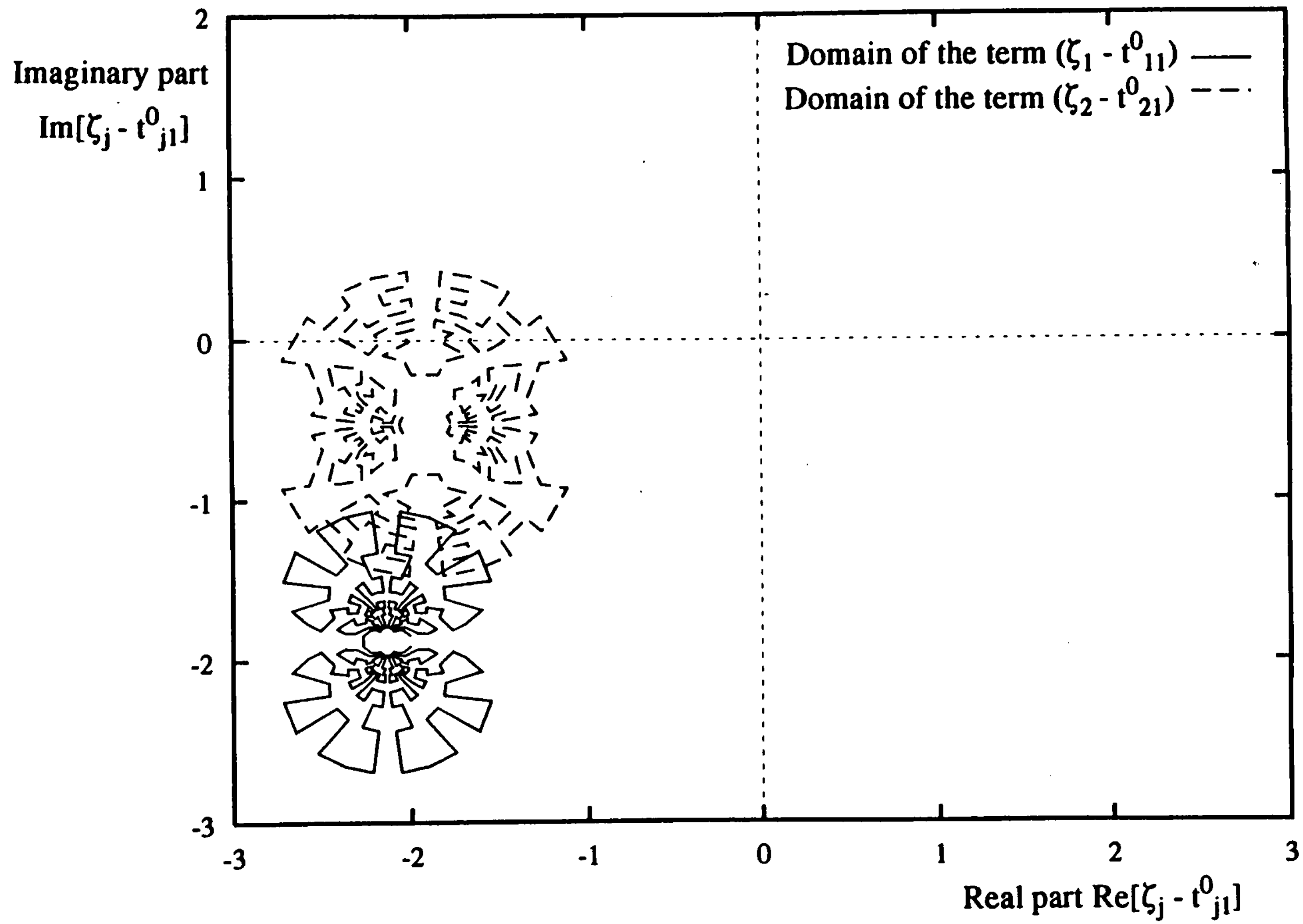


Figure 6.3 Domain of the terms $(\zeta_1 - t^0_{11})$ and $(\zeta_2 - t^0_{21})$ in Liaw and Kamel's solution

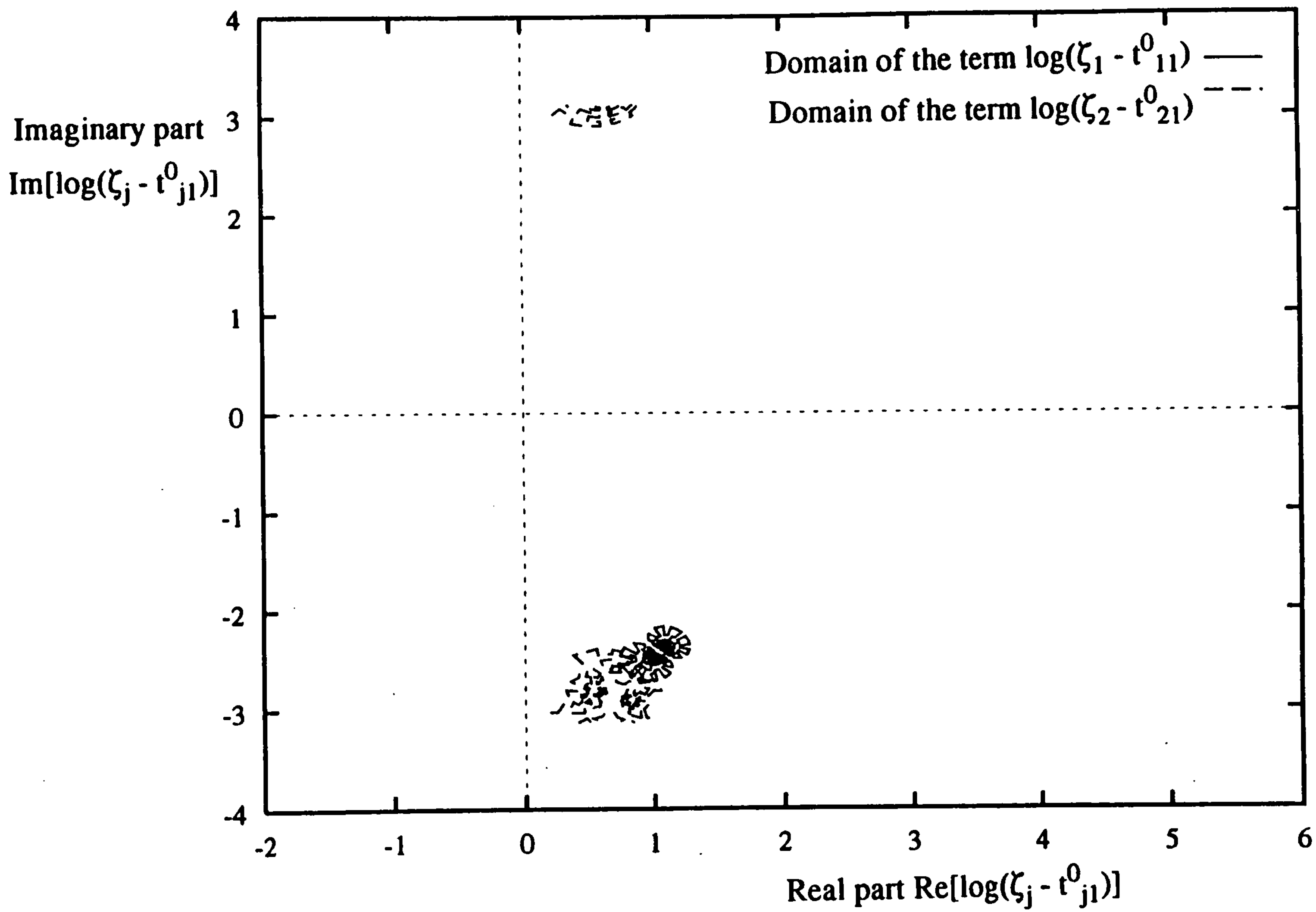


Figure 6.4 Domain of the terms $\log(\zeta_1 - t^0_{11})$ and $\log(\zeta_2 - t^0_{21})$ in Liaw and Kamel's solution, with the branch cut in the logarithmic function at the non-positive real axis

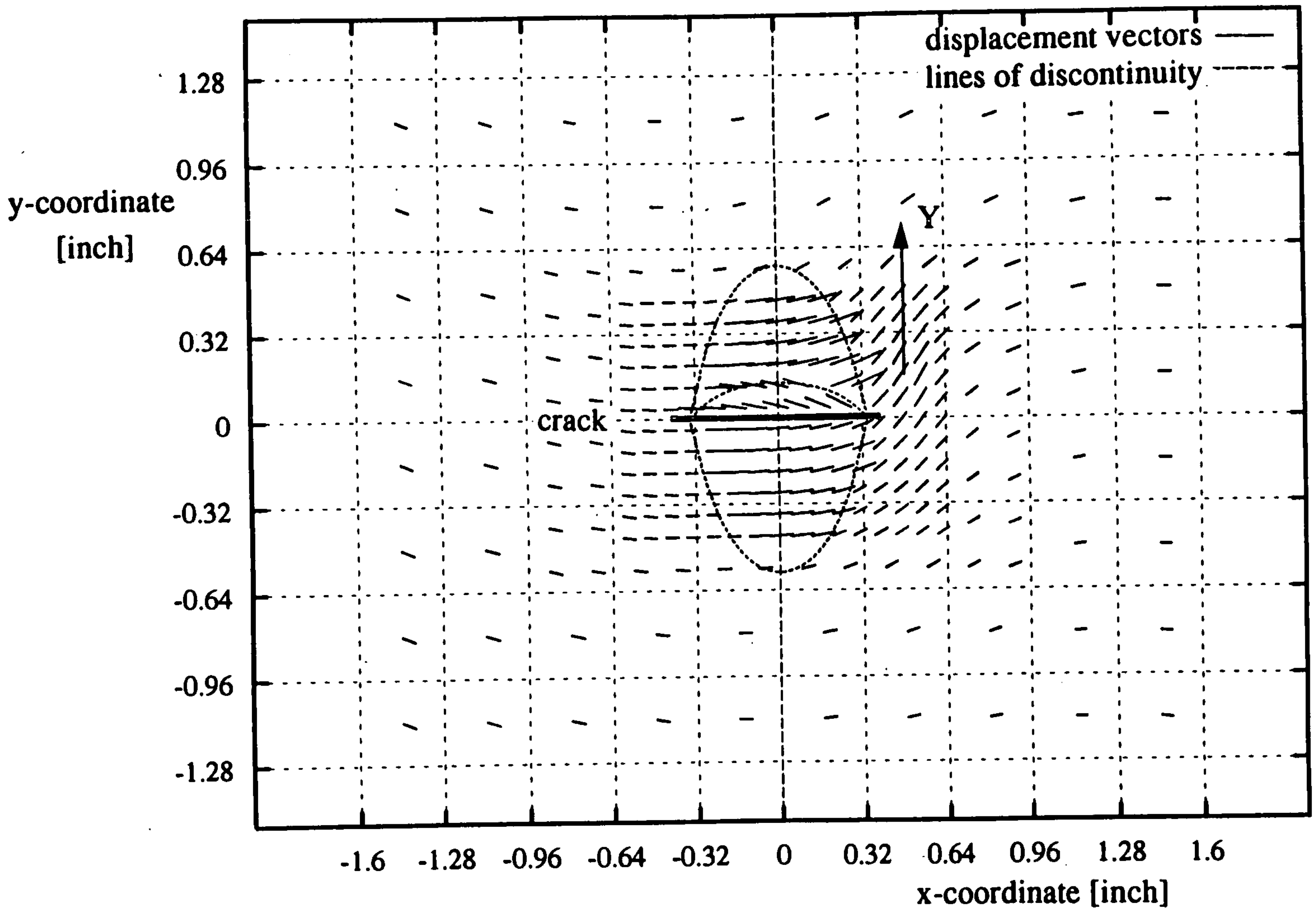


Figure 6.5 Displacement field in Liaw and Kamel's solution, computed with the branch cut in the logarithmic function at the non-positive real axis

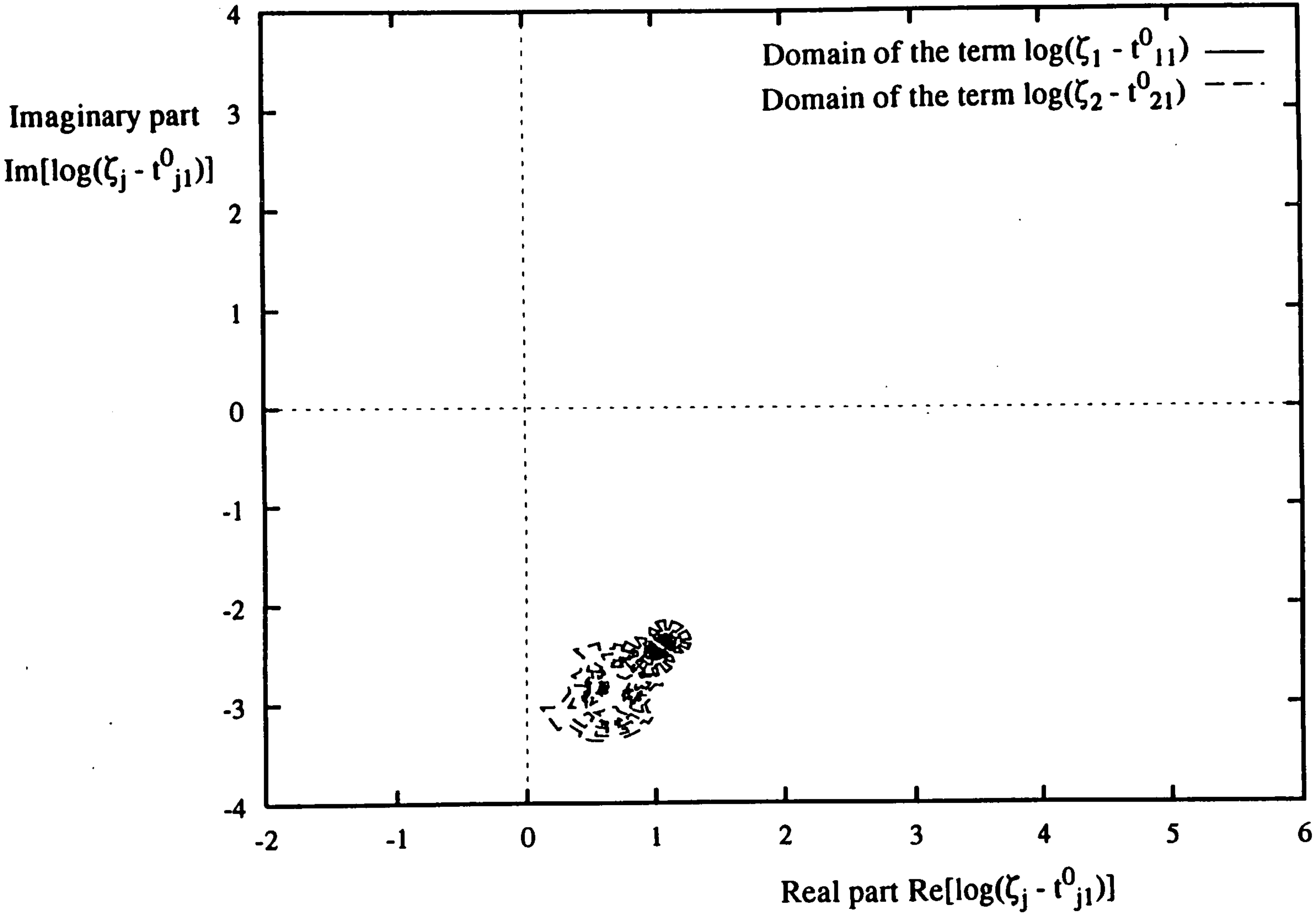


Figure 6.6 Domain of the terms $\log(\zeta_1 - t^0_{11})$ and $\log(\zeta_2 - t^0_{21})$ in Liaw and Kamel's solution, with the branch cut in the logarithmic function situated outside the domain of the logarithm function

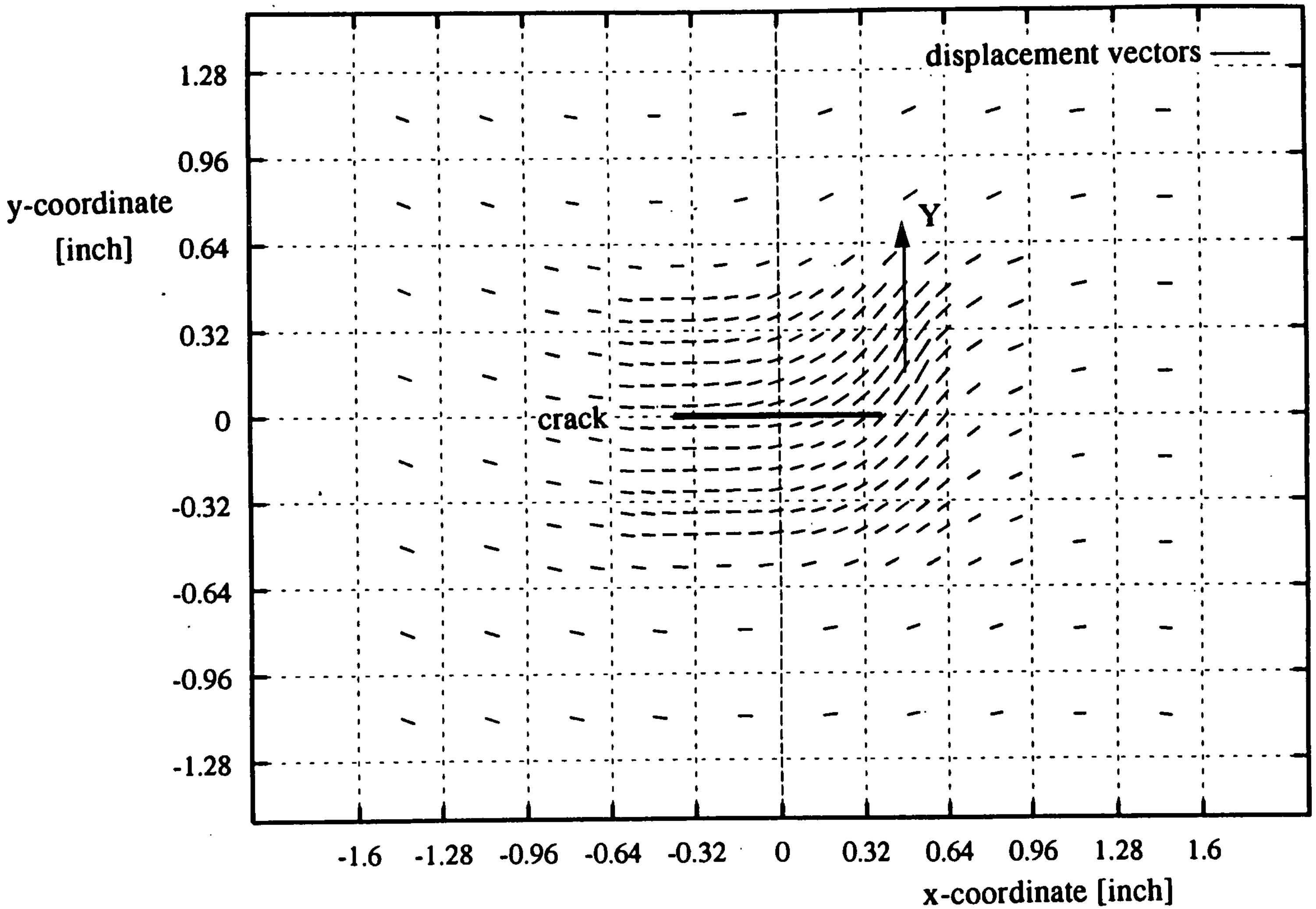


Figure 6.7 Displacement field in Liaw and Kamel's solution, computed with the branch cut in the logarithmic function situated outside the domain of the logarithm function

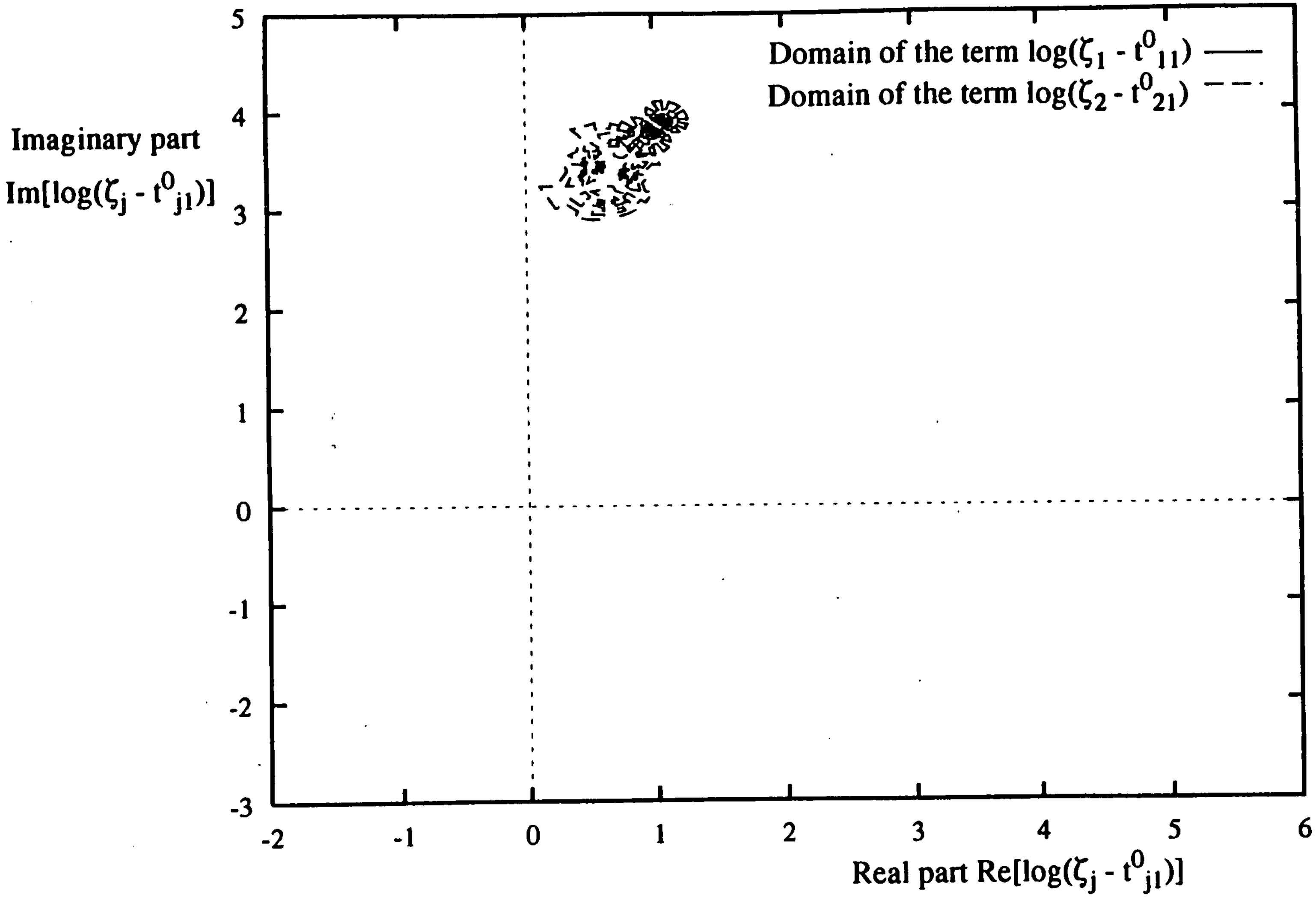


Figure 6.8 Domain of the terms $\log(\zeta_1 - t^0_{11})$ and $\log(\zeta_2 - t^0_{21})$ in Liaw and Kamel's solution, with the logarithm function programmed on the Riemann surface

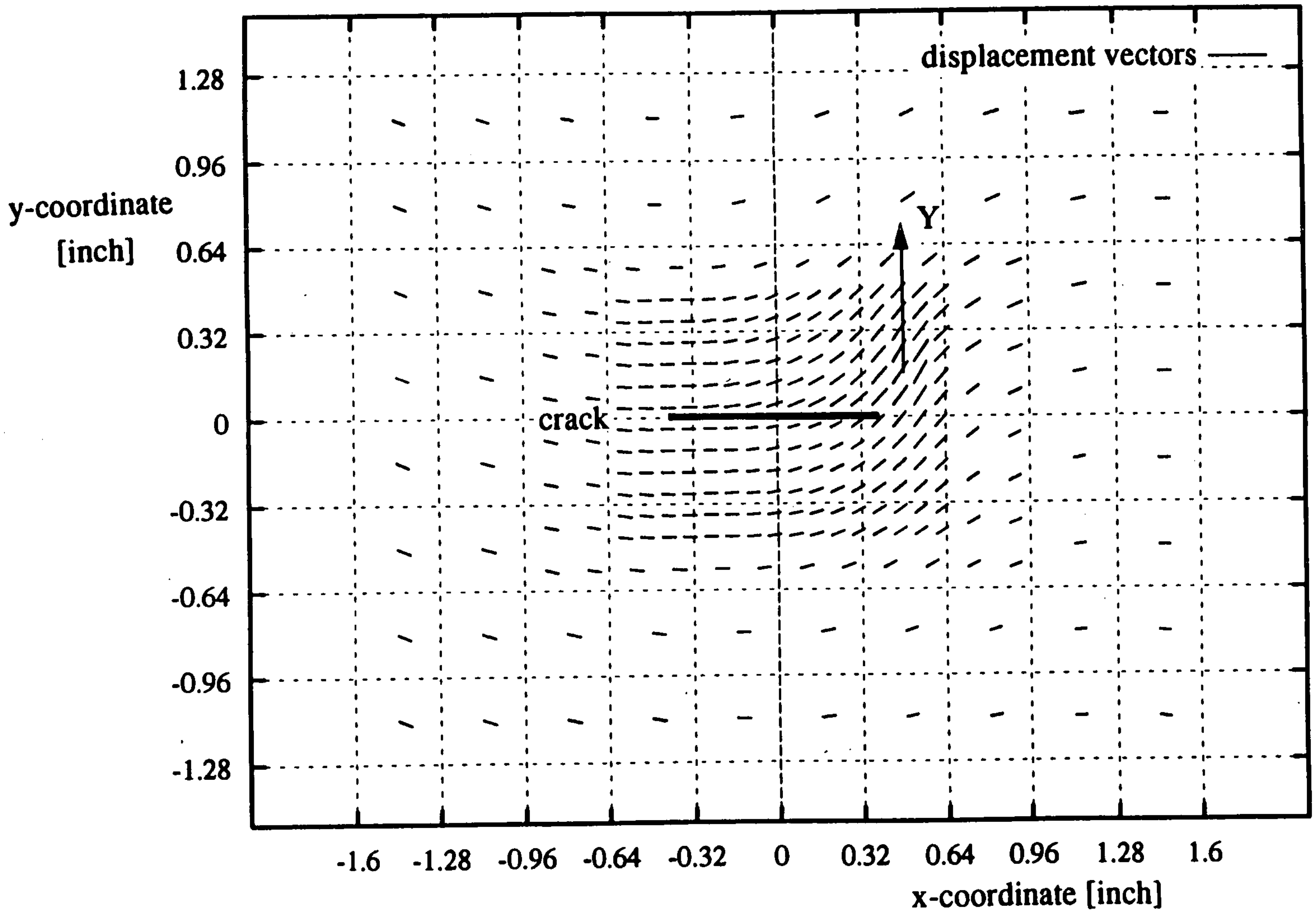


Figure 6.9 Displacement field in Liaw and Kamel's solution, computed with the logarithm function programmed on the Riemann surface

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