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Unified matrix–vector wave equation, reciprocity and representations

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SUMMARY

The matrix–vector wave equation is a compact first-order differential equation. It was originally used for the analysis of elastodynamic plane waves in laterally invariant media. It has been extended by various authors for laterally varying media. Other authors derived a similar formalism for other wave phenomena. This paper starts with a unified formulation of the matrix–vector wave equation for 3-D inhomogeneous, dissipative media. The wave vector, source vector and operator matrix are specified in the appendices for acoustic, quantum mechanical, electromagnetic, elastodynamic, poroelastodynamic, piezoelectric and seismoelectric waves. It is shown that the operator matrix obeys unified symmetry relations for all these wave phenomena. Next, unified matrix–vector reciprocity theorems of the convolution and correlation type are derived, utilizing the symmetry properties of the operator matrix. These theorems formulate mathematical relations between two wave states in the same spatial domain. A unified wavefield representation is obtained by replacing one of the states in the convolution-type reciprocity theorem by a Green's state. By replacing both states in the correlation-type reciprocity theorem by Green's states, a unified representation of the homogeneous Green's matrix is obtained. Applications of the unified reciprocity theorems and representations for forward and inverse wave problems are briefly indicated.

Key words: Electromagnetic theory; Theoretical seismology; Wave propagation.

1 INTRODUCTION

The basic equations for wave propagation in an inhomogeneous medium can be organized in a compact matrix–vector wave equation. This equation expresses the vertical derivative of a wave vector in terms of an operator matrix acting on this wave vector. This specific form of the wave equation is useful, for example, to evaluate wave problems in media of which the medium parameters vary more rapidly in the vertical direction than in the lateral directions. It is also particularly useful for situations in which the vertical direction is the preferred direction of wave propagation. However, the theoretical treatment of the matrix–vector wave equation in this paper is not limited to these special situations.

The matrix–vector wave equation finds its roots in early work on the analysis of plane waves in laterally invariant media. Thomson (1950) introduced a matrix formalism for the analysis of elastodynamic plane waves propagating through a stratified solid medium. Haskell (1953) used the same formalism to analyse the dispersion of surface waves in layered media. Backus (1962) used similar concepts to derive long-wave effective anisotropic parameters for stratified media. This approach has become known as Backus averaging (Mavko *et al.* 2009). Gilbert & Backus (1966) used the matrix–vector wave equation to derive so-called propagator matrices for elastodynamic wave problems in stratified media. Woodhouse (1974) extended the formalism for arbitrary anisotropic inhomogeneous media and used it for the study of surface waves in laterally varying layered media. Frasier (1970), Kennett *et al.* (1978), Frazer & Fryer (1989) and Chapman (1994) used the matrix–vector wave equation to derive symmetry properties of reflection and transmission responses of laterally invariant media. Haines (1988), Kennett *et al.* (1990), Koketsu *et al.* (1991) and Takenaka *et al.* (1993) exploited the symmetry properties of the matrix–vector wave equation to derive so-called propagation invariants for laterally varying layered media and used this for modelling of reflection and transmission responses of such media. Using the same symmetry properties, Haines & de Hoop (1996) and Wapenaar (1996b) derived reciprocity theorems and representations for the acoustic wave vector.

The matrix–vector wave equation has been used by many authors as the starting point for decomposition into coupled wave equations for downgoing and upgoing waves, for example for modelling in horizontally layered media (Kennett & Kerry 1979; Kennett & Illingworth 1981), for wide-angle propagation in laterally variant media (Fishman & McCoy 1984; Weston 1989; Fishman 1992), and for deriving reciprocity theorems for coupled downward and upward propagating waves (Wapenaar & Grimbergen 1996; Thomson 2015a,b), generalized Bremmer

Table 1. Wavefield vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ for the different wave phenomena considered in this paper. Labels A to G refer to the appendices in which these wave phenomena are discussed in more detail.

	$\mathbf{q}_1$	$\mathbf{q}_2$
A: Acoustic	p	v_3
B: Quantum mechanic	ψ	$\frac{2\hbar}{mi}\partial_3\psi$
C: Electromagnetic	$\mathbf{E}_0 = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$	$\mathbf{H}_0 = \begin{pmatrix} H_2 \\ -H_1 \end{pmatrix}$
D: Elastodynamic	$-\boldsymbol{\tau}_3 = -\begin{pmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{pmatrix}$	$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
E: Poroelastodynamic	$\begin{pmatrix} -\boldsymbol{\tau}_3^b \\ p^f \end{pmatrix}$	$\begin{pmatrix} \mathbf{v}^s \\ \phi(v_3^f - v_3^s) \end{pmatrix}$
F: Piezoelectric	$\begin{pmatrix} -\boldsymbol{\tau}_3 \\ \mathbf{E}_0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{v} \\ \mathbf{H}_0 \end{pmatrix}$
G: Seismoelectric	$\begin{pmatrix} -\boldsymbol{\tau}_3^b \\ p^f \\ \mathbf{E}_0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{v}^s \\ \phi(v_3^f - v_3^s) \\ \mathbf{H}_0 \end{pmatrix}$

series representations for reflection data (Corones 1975; Haines & de Hoop 1996; Wapenaar 1996a; de Hoop 1996b) and representations for seismic interferometry (Wapenaar 2003).

This paper discusses the matrix–vector wave equation and its symmetry properties for a range of wave phenomena in a unified way (Section 2 and Appendices A to G). The treatment builds on earlier systematic treatments of different wave phenomena by Auld (1973), Ursin (1983), Kennett (1983), Müller (1985), Wapenaar & Berkhout (1989), de Hoop (1995, 1996a), Gangi (2000), Carcione (2007) and Mittet (2015). The matrix–vector wave equation forms the basis for the derivation of unified matrix–vector reciprocity theorems (Section 3) and representations (Section 4), analogous to those for the acoustic wave vector (Haines & de Hoop 1996; Wapenaar 1996b).

2 THE UNIFIED MATRIX–VECTOR WAVE EQUATION AND ITS SYMMETRY PROPERTIES

2.1 The matrix–vector wave equation

The unified matrix–vector wave equation has the form

$$\partial_3 \mathbf{q} = \boldsymbol{\mathcal{A}}\mathbf{q} + \mathbf{d}. \tag{1}$$

Here $\mathbf{q}$ is the wavefield vector, $\mathbf{d}$ the source vector and $\boldsymbol{\mathcal{A}}$ the operator matrix. All quantities are defined in the space–frequency domain, hence $\mathbf{q} = \mathbf{q}(\mathbf{x}, \omega)$, etc., where $\mathbf{x}$ denotes the Cartesian coordinate vector (x_1, x_2, x_3) and ω the angular frequency. The positive x_3-axis is pointing downward. Operator ∂_3 stands for the spatial differential operator $\partial/\partial x_3$. The vectors and matrix in eq. (1) are partitioned as follows:

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix}, \quad \boldsymbol{\mathcal{A}} = \begin{pmatrix} \boldsymbol{\mathcal{A}}_{11} & \boldsymbol{\mathcal{A}}_{12} \\ \boldsymbol{\mathcal{A}}_{21} & \boldsymbol{\mathcal{A}}_{22} \end{pmatrix}, \tag{2}$$

hence,

$$\partial_3 \mathbf{q}_1 = \boldsymbol{\mathcal{A}}_{11}\mathbf{q}_1 + \boldsymbol{\mathcal{A}}_{12}\mathbf{q}_2 + \mathbf{d}_1, \tag{3}$$

$$\partial_3 \mathbf{q}_2 = \boldsymbol{\mathcal{A}}_{21}\mathbf{q}_1 + \boldsymbol{\mathcal{A}}_{22}\mathbf{q}_2 + \mathbf{d}_2. \tag{4}$$

The vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are specified in rows A to G of Table 1 for the different wave phenomena considered in this paper. The wavefield quantities contained in these vectors are defined in Appendices A to G. For acoustic and quantum mechanical waves (rows A and B), $\mathbf{q}_1$ and $\mathbf{q}_2$ are scalars. For electromagnetic, elastodynamic and poroelastodynamic waves (rows C to E) they are 2×1, 3×1 and 4×1 vectors, respectively (superscripts b, f and s in row E stand for bulk, fluid and solid, respectively). Rows F and G represent coupled electromagnetic and (poro)elastodynamic waves. For piezoelectric waves (row F), constitutive eqs (F1) and (F2) account for the coupling. For this situation the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are combinations of those for electromagnetic and elastodynamic waves (rows C and D). For seismoelectric waves (row G), constitutive eqs (G1) and (G2) account for the coupling. In this case the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are combinations of those for electromagnetic and poroelastodynamic waves (rows C and E).

In all cases, except for quantum mechanical waves, the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are defined such that they constitute the power-flux density j in the x_3-direction via

$$j = \frac{1}{4}(\mathbf{q}_1^\dagger \mathbf{q}_2 + \mathbf{q}_2^\dagger \mathbf{q}_1), \tag{5}$$

where the superscript † denotes transposition and complex conjugation. For quantum mechanical waves, j represents the probability current density in the x_3-direction. Vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ and operator matrices $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ are defined in Appendices A to G for the different wave phenomena. The operator matrices contain specific combinations of space-dependent medium parameters (or, for quantum mechanics, the potential V and mass m) and spatial differential operators ∂_1 and ∂_2 (standing for $\partial/\partial x_1$ and $\partial/\partial x_2$, respectively).

Eq. (1), with the operator matrix specified in the appendices, may be used as a starting point for generalizing many of the applications mentioned in the introduction (analysis of surface waves, derivation of long-wave effective medium parameters, derivation of propagator matrices, decomposition into downgoing and upgoing waves, modelling wide-angle propagation in laterally variant media, etc.). A discussion of these applications is beyond the scope of this paper. Here we focus on the symmetry of the operator matrix and its use in unified reciprocity theorems and representations.

2.2 Symmetry properties of the operator matrix

We discuss the symmetry properties of the operator matrix. First, consider a scalar operator $\mathcal{U}$, containing space-dependent parameters and differential operators ∂_1 and ∂_2. We introduce its transpose $\mathcal{U}^t$ and adjoint $\mathcal{U}^\dagger$ via their integral properties

$$\int_{\mathbb{A}} (\mathcal{U} f) g \,\mathrm{d}^2\mathbf{x}_\mathrm{H} = \int_{\mathbb{A}} f (\mathcal{U}^t g) \,\mathrm{d}^2\mathbf{x}_\mathrm{H} \tag{6}$$

and

$$\int_{\mathbb{A}} (\mathcal{U} f)^* g \,\mathrm{d}^2\mathbf{x}_\mathrm{H} = \int_{\mathbb{A}} f^* (\mathcal{U}^\dagger g) \,\mathrm{d}^2\mathbf{x}_\mathrm{H}. \tag{7}$$

Here $\mathbf{x}_\mathrm{H}$ is the horizontal coordinate vector (x_1, x_2), superscript $*$ denotes complex conjugation, $\mathbb{A}$ denotes an infinite horizontal integration boundary at arbitrary depth x_3, and $f = f(\mathbf{x})$ and $g = g(\mathbf{x})$ are space-dependent functions with sufficient decay along $\mathbb{A}$ towards infinity. Eq. (6) implies

$$(\mathcal{U}\mathcal{V}\mathcal{W})^t = \mathcal{W}^t\mathcal{V}^t\mathcal{U}^t, \tag{8}$$

where also $\mathcal{V}$ and $\mathcal{W}$ are scalar operators. Eqs (6) and (7) imply

$$\mathcal{U}^\dagger = (\mathcal{U}^t)^*. \tag{9}$$

For the special case that $\mathcal{U} = \partial_1$, eq. (6) implies (via integration by parts) $\partial_1^t = -\partial_1$. Similarly, $\partial_2^t = -\partial_2$. Hence,

$$\partial_\alpha^t = -\partial_\alpha, \tag{10}$$

where Greek subscripts take on the values 1 and 2. Using this property and eq. (8), we find for example for the operator in eq. (A22), $(\partial_\alpha b_{\alpha\beta}\partial_\beta)^t = \partial_\beta b_{\alpha\beta}\partial_\alpha$ (Einstein's summation convention applies to repeated subscripts). Since $b_{\alpha\beta} = b_{\beta\alpha}$, this implies $(\partial_\alpha b_{\alpha\beta}\partial_\beta)^t = \partial_\beta b_{\beta\alpha}\partial_\alpha = \partial_\alpha b_{\alpha\beta}\partial_\beta$ and, using eq. (9), $(\partial_\alpha b_{\alpha\beta}\partial_\beta)^\dagger = (\partial_\alpha b_{\alpha\beta}\partial_\beta)^* = \partial_\alpha b^*_{\alpha\beta}\partial_\beta$.

Next, we consider an operator matrix $\boldsymbol{\mathcal{U}}$, of which the entries are operators containing space-dependent parameters and differential operators ∂_1 and ∂_2. Analogous to eqs (6) and (7), we introduce its transpose $\boldsymbol{\mathcal{U}}^t$ and its adjoint $\boldsymbol{\mathcal{U}}^\dagger$ via

$$\int_{\mathbb{A}} (\boldsymbol{\mathcal{U}}\mathbf{f})^t \mathbf{g} \,\mathrm{d}^2\mathbf{x}_\mathrm{H} = \int_{\mathbb{A}} \mathbf{f}^t (\boldsymbol{\mathcal{U}}^t \mathbf{g}) \,\mathrm{d}^2\mathbf{x}_\mathrm{H} \tag{11}$$

and

$$\int_{\mathbb{A}} (\boldsymbol{\mathcal{U}}\mathbf{f})^\dagger \mathbf{g} \,\mathrm{d}^2\mathbf{x}_\mathrm{H} = \int_{\mathbb{A}} \mathbf{f}^\dagger (\boldsymbol{\mathcal{U}}^\dagger \mathbf{g}) \,\mathrm{d}^2\mathbf{x}_\mathrm{H}, \tag{12}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{x})$ and $\mathbf{g} = \mathbf{g}(\mathbf{x})$ are space-dependent vector functions with sufficient decay along $\mathbb{A}$ towards infinity. Eq. (11) implies that $\boldsymbol{\mathcal{U}}^t$ involves transposition of the matrix and transposition of the operators contained in the matrix. For example, for a 2×2 operator matrix $\boldsymbol{\mathcal{U}}$, we have

$$\begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ \mathcal{U}_{21} & \mathcal{U}_{22} \end{pmatrix}^t = \begin{pmatrix} \mathcal{U}_{11}^t & \mathcal{U}_{21}^t \\ \mathcal{U}_{12}^t & \mathcal{U}_{22}^t \end{pmatrix}. \tag{13}$$

Eq. (11) implies

$$(\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{W}})^t = \boldsymbol{\mathcal{W}}^t\boldsymbol{\mathcal{V}}^t\boldsymbol{\mathcal{U}}^t, \tag{14}$$

where also $\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\mathcal{W}}$ are operator matrices. Eqs (11) and (12) imply

$$\boldsymbol{\mathcal{U}}^\dagger = (\boldsymbol{\mathcal{U}}^t)^*. \tag{15}$$

Using eqs (8), (10) and (14), it follows that operator matrices $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$, defined in Appendices A to G, obey the following symmetry relations

$$\boldsymbol{\mathcal{A}}_{11}^t = -\boldsymbol{\mathcal{A}}_{22}, \tag{16}$$

$$\mathcal{A}^t_{12} = \mathcal{A}_{12}, \tag{17}$$

$$\mathcal{A}^t_{21} = \mathcal{A}_{21}. \tag{18}$$

In Appendices A to G we define adjoint medium parameters (or, for quantum mechanics, an adjoint potential). When a medium is dissipative, its adjoint is effectual, and vice versa (de Hoop 1987, 1988; Wapenaar *et al.* 2001). A wave propagating through an effectual medium gains energy. Effectual media play a role in the reciprocity theorems and representations, discussed in Sections 3 and 4. An adjoint medium parameter is denoted by an overbar. An operator with an overbar means that the medium parameters contained in that operator are replaced by their adjoints. For example, for the operator in eq. (A22) we have $\overline{\frac{1}{i\omega}\partial_\alpha b_{\alpha\beta}\partial_\beta} = \frac{1}{i\omega}\partial_\alpha \bar{b}_{\alpha\beta}\partial_\beta$. Since $\bar{b}_{\alpha\beta} = b^*_{\alpha\beta}$ this becomes $\overline{\frac{1}{i\omega}\partial_\alpha b_{\alpha\beta}\partial_\beta} = \frac{1}{i\omega}\partial_\alpha b^*_{\alpha\beta}\partial_\beta = -(\frac{1}{i\omega}\partial_\alpha b_{\alpha\beta}\partial_\beta)^*$.

For the operator matrices in Appendices A to G in an adjoint medium we have

$$\bar{\mathcal{A}}_{11} = \mathcal{A}^*_{11}, \tag{19}$$

$$\bar{\mathcal{A}}_{12} = -\mathcal{A}^*_{12}, \tag{20}$$

$$\bar{\mathcal{A}}_{21} = -\mathcal{A}^*_{21}, \tag{21}$$

$$\bar{\mathcal{A}}_{22} = \mathcal{A}^*_{22}. \tag{22}$$

Using eq. (15), we find from eqs (16) to (22)

$$\mathcal{A}^\dagger_{11} = -\bar{\mathcal{A}}_{22}, \tag{23}$$

$$\mathcal{A}^\dagger_{12} = -\bar{\mathcal{A}}_{12}, \tag{24}$$

$$\mathcal{A}^\dagger_{21} = -\bar{\mathcal{A}}_{21}, \tag{25}$$

$$\mathcal{A}^\dagger_{22} = -\bar{\mathcal{A}}_{11}. \tag{26}$$

From eqs (16) to (26), we find for the operator matrix $\mathcal{A}$ defined in eq. (2)

$$\mathcal{A}^t\mathbf{N} = -\mathbf{N}\mathcal{A}, \tag{27}$$

$$\mathcal{A}^*\mathbf{J} = \mathbf{J}\bar{\mathcal{A}}, \tag{28}$$

$$\mathcal{A}^\dagger\mathbf{K} = -\mathbf{K}\bar{\mathcal{A}}, \tag{29}$$

with

$$\mathbf{N} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}, \tag{30}$$

where $\mathbf{O}$ and $\mathbf{I}$ are zero and identity matrices of appropriate size. Symmetry relations in the wavenumber-frequency domain for the special case of a laterally invariant medium (or potential) are given in Appendix H.

3 MATRIX–VECTOR WAVEFIELD RECIPROCITY THEOREMS

In wave theory, a reciprocity theorem formulates a mathematical relation between two states (wavefields, sources and medium parameters) in the same spatial domain. An early reference for the acoustic reciprocity theorem is Rayleigh (1878), who referred to it as Helmholtz's theorem. Lorentz (1895) formulated a reciprocity theorem for electromagnetic fields. Early references for elastodynamic reciprocity theorems

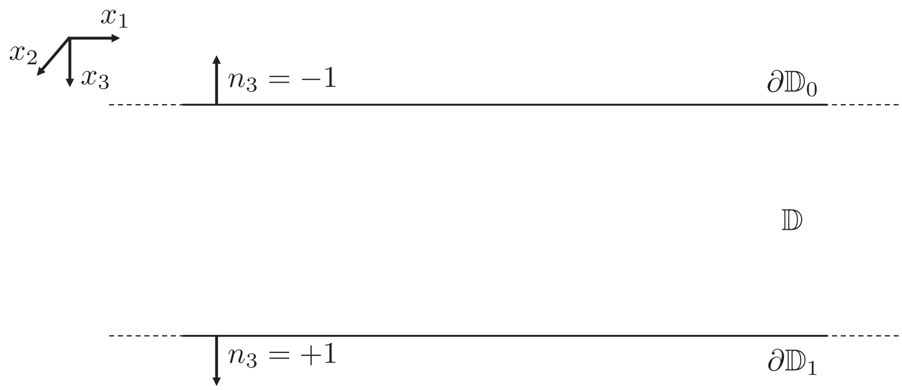

Figure 1. Configuration for the matrix–vector reciprocity theorems, eqs (36) and (37). The combination of boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_1$ is called $\partial\mathbb{D}$ in these equations.

are Knopoff & Gangi (1959) and de Hoop (1966). Auld (1979) and Pride & Haartsen (1996) formulated reciprocity theorems for piezoelectric and seismoelectric waves, respectively. Comprehensive overviews of the history of reciprocity theorems and their applications are given by Fokkema & van den Berg (1993), de Hoop (1995) and Achenbach (2003).

Matrix–vector wave eq. (1) and symmetry relations (27) and (29) underly unified matrix–vector reciprocity theorems in an inhomogeneous medium (or potential). We consider two states A and B, characterized by independent wave vectors $\mathbf{q}_A(\mathbf{x},\omega)$ and $\mathbf{q}_B(\mathbf{x},\omega)$, obeying matrix–vector wave eq. (1), with source vectors $\mathbf{d}_A(\mathbf{x},\omega)$ and $\mathbf{d}_B(\mathbf{x},\omega)$, and operator matrices $\boldsymbol{\mathcal{A}}_A(\mathbf{x},\omega)$ and $\boldsymbol{\mathcal{A}}_B(\mathbf{x},\omega)$. The subscripts A and B of these operator matrices refer to the, possibly different, medium parameters in states A and B. We assume that outside a finite domain, the medium (or potential) and its adjoint are lossless in both states. We consider a spatial domain $\mathbb{D}$ enclosed by two infinite horizontal boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_1$ (with $\partial\mathbb{D}_1$ below $\partial\mathbb{D}_0$), together denoted by $\partial\mathbb{D}$, see Fig. 1. The starting point for deriving reciprocity theorems for the wavefields in states A and B is formed by the quantities $\partial_3\{\mathbf{q}_A^t\mathbf{N}\mathbf{q}_B\}$ and $\partial_3\{\mathbf{q}_A^\dagger\mathbf{K}\mathbf{q}_B\}$ in domain $\mathbb{D}$. Applying the product rule for differentiation gives

$$\partial_3\{\mathbf{q}_A^t\mathbf{N}\mathbf{q}_B\} = (\partial_3\mathbf{q}_A^t)\mathbf{N}\mathbf{q}_B + \mathbf{q}_A^t\mathbf{N}(\partial_3\mathbf{q}_B), \tag{31}$$

$$\partial_3\{\mathbf{q}_A^\dagger\mathbf{K}\mathbf{q}_B\} = (\partial_3\mathbf{q}_A^\dagger)\mathbf{K}\mathbf{q}_B + \mathbf{q}_A^\dagger\mathbf{K}(\partial_3\mathbf{q}_B). \tag{32}$$

Note that eq. (10), which defines the transpose of the horizontal differential operator ∂_α, does not apply to the vertical differential operator ∂_3. Hence, we may replace $(\partial_3\mathbf{q}_A^t)$ by $(\partial_3\mathbf{q}_A)^t$ in eq. (31), and $(\partial_3\mathbf{q}_A^\dagger)$ by $(\partial_3\mathbf{q}_A)^\dagger$ in eq. (32). Using wave eq. (1) for both states in the right-hand sides of eqs (31) and (32), integrating both sides of these equations over domain $\mathbb{D}$ and applying the theorem of Gauss to the left-hand sides, we obtain

$$\int_{\partial\mathbb{D}} \mathbf{q}_A^t\mathbf{N}\mathbf{q}_B n_3 \mathrm{d}^2\mathbf{x}_\mathrm{H} = \int_{\mathbb{D}} \Big[\big((\boldsymbol{\mathcal{A}}_A\mathbf{q}_A)^t + \mathbf{d}_A^t\big)\mathbf{N}\mathbf{q}_B + \mathbf{q}_A^t\mathbf{N}\big(\boldsymbol{\mathcal{A}}_B\mathbf{q}_B + \mathbf{d}_B\big)\Big]\mathrm{d}^3\mathbf{x} \tag{33}$$

and

$$\int_{\partial\mathbb{D}} \mathbf{q}_A^\dagger\mathbf{K}\mathbf{q}_B n_3 \mathrm{d}^2\mathbf{x}_\mathrm{H} = \int_{\mathbb{D}} \Big[\big((\boldsymbol{\mathcal{A}}_A\mathbf{q}_A)^\dagger + \mathbf{d}_A^\dagger\big)\mathbf{K}\mathbf{q}_B + \mathbf{q}_A^\dagger\mathbf{K}\big(\boldsymbol{\mathcal{A}}_B\mathbf{q}_B + \mathbf{d}_B\big)\Big]\mathrm{d}^3\mathbf{x}. \tag{34}$$

Here n_3 is the vertical component of the outward pointing normal vector on $\partial\mathbb{D}$, with $n_3 = -1$ at the upper boundary $\partial\mathbb{D}_0$ and $n_3 = +1$ at the lower boundary $\partial\mathbb{D}_1$. The integrals on the right-hand sides can be written as

$$\int_{\mathbb{D}} \{\cdots\}\mathrm{d}^3\mathbf{x} = \int_{x_{3,0}}^{x_{3,1}} \mathrm{d}x_3 \int_{\mathbb{A}} \{\cdots\}\mathrm{d}^2\mathbf{x}_\mathrm{H}, \tag{35}$$

where $x_{3,0}$ and $x_{3,1}$ denote the depths of $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_1$, respectively. Hence, at each depth level between $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_1$ we can use the integral properties of transpose and adjoint operators, as formulated by eqs (11) and (12). Together with the symmetry relations (27) and (29) for operator $\boldsymbol{\mathcal{A}}_A$, we thus obtain the following matrix–vector reciprocity theorems

$$\int_{\mathbb{D}} (\mathbf{d}_A^t\mathbf{N}\mathbf{q}_B + \mathbf{q}_A^t\mathbf{N}\mathbf{d}_B)\mathrm{d}^3\mathbf{x} = \int_{\partial\mathbb{D}} \mathbf{q}_A^t\mathbf{N}\mathbf{q}_B n_3 \mathrm{d}^2\mathbf{x}_\mathrm{H} + \int_{\mathbb{D}} \mathbf{q}_A^t\mathbf{N}(\boldsymbol{\mathcal{A}}_A - \boldsymbol{\mathcal{A}}_B)\mathbf{q}_B \mathrm{d}^3\mathbf{x} \tag{36}$$

and

$$\int_{\mathbb{D}} (\mathbf{d}_A^\dagger\mathbf{K}\mathbf{q}_B + \mathbf{q}_A^\dagger\mathbf{K}\mathbf{d}_B)\mathrm{d}^3\mathbf{x} = \int_{\partial\mathbb{D}} \mathbf{q}_A^\dagger\mathbf{K}\mathbf{q}_B n_3 \mathrm{d}^2\mathbf{x}_\mathrm{H} + \int_{\mathbb{D}} \mathbf{q}_A^\dagger\mathbf{K}(\bar{\boldsymbol{\mathcal{A}}}_A - \boldsymbol{\mathcal{A}}_B)\mathbf{q}_B \mathrm{d}^3\mathbf{x}. \tag{37}$$

Eq. (36) is a convolution-type reciprocity theorem (Fokkema & van den Berg 1993; de Hoop 1995) because products like $\mathbf{q}_A^t\mathbf{N}\mathbf{q}_B$ in the frequency domain correspond to convolutions in the time domain. Eq. (37) is a correlation-type reciprocity theorem (Bojarski 1983) because products like $\mathbf{q}_A^\dagger\mathbf{K}\mathbf{q}_B$ in the frequency domain correspond to correlations in the time domain. These matrix–vector reciprocity theorems have been previously derived for acoustic waves (Haines & de Hoop 1996; Wapenaar 1996b). Because these theorems follow from the unified matrix–vector wave eq. (1), with unified symmetry relations (27) and (29), they hold for all wave phenomena listed in Table 1. In the next section we use these theorems as the basis for matrix–vector wavefield representations. Here we consider some special cases of these theorems.

Power balance

When the sources, medium parameters and wavefields are identical in both states, we may drop the subscripts A and B. In this case eq. (37) simplifies to

$$\int_{\mathbb{D}} \frac{1}{4}(\mathbf{d}^{\dagger}\mathbf{K}\mathbf{q} + \mathbf{q}^{\dagger}\mathbf{K}\mathbf{d})\mathrm{d}^3\mathbf{x} = \int_{\partial\mathbb{D}} \frac{1}{4}\mathbf{q}^{\dagger}\mathbf{K}\mathbf{q}n_3\mathrm{d}^2\mathbf{x}_{\mathrm{H}} + \int_{\mathbb{D}} \frac{1}{4}\mathbf{q}^{\dagger}\mathbf{K}(\bar{\boldsymbol{\mathcal{A}}} - \boldsymbol{\mathcal{A}})\mathbf{q}\mathrm{d}^3\mathbf{x}. \tag{38}$$

Because $\frac{1}{4}\mathbf{q}^{\dagger}\mathbf{K}\mathbf{q} = \frac{1}{4}(\mathbf{q}_1^{\dagger}\mathbf{q}_2 + \mathbf{q}_2^{\dagger}\mathbf{q}_1) = j$, the first term on the right-hand side is the power flux (or probability current) through the boundary $\partial\mathbb{D} = \partial\mathbb{D}_0 \cup \partial\mathbb{D}_1$ (i.e. the power leaving the domain $\mathbb{D}$). Hence, eq. (38) formulates the unified power balance. The term on the left-hand side is the power generated by the sources in $\mathbb{D}$ and the second term on the right-hand side is the dissipated power in $\mathbb{D}$.

Propagation invariants

When there are no sources in $\mathbb{D}$ and the medium parameters in $\mathbb{D}$ are equal in the two states, the domain integrals in eq. (36) vanish, hence

$$\int_{\partial\mathbb{D}_0\cup\partial\mathbb{D}_1} \mathbf{q}_A^t\mathbf{N}\mathbf{q}_B n_3\mathrm{d}^2\mathbf{x}_{\mathrm{H}} = 0, \tag{39}$$

or, since $n_3 = -1$ at $\partial\mathbb{D}_0$ and $n_3 = +1$ at $\partial\mathbb{D}_1$,

$$\int_{\partial\mathbb{D}_0} \mathbf{q}_A^t\mathbf{N}\mathbf{q}_B\mathrm{d}^2\mathbf{x}_{\mathrm{H}} = \int_{\partial\mathbb{D}_1} \mathbf{q}_A^t\mathbf{N}\mathbf{q}_B\mathrm{d}^2\mathbf{x}_{\mathrm{H}}. \tag{40}$$

Since this holds for any choice of the domain $\mathbb{D}$, we infer that the quantity

$$\int_{\mathbb{A}} \mathbf{q}_A^t\mathbf{N}\mathbf{q}_B\mathrm{d}^2\mathbf{x}_{\mathrm{H}}, \tag{41}$$

with $\mathbb{A}$ denoting a horizontal plane at arbitrary depth x_3, is a unified propagation invariant (i.e. it is independent of the depth x_3 of $\mathbb{A}$). On the other hand, when the medium parameters are each other's adjoints in the two states, we find in a similar way from eq. (37) that the quantity

$$\int_{\mathbb{A}} \mathbf{q}_A^{\dagger}\mathbf{K}\mathbf{q}_B\mathrm{d}^2\mathbf{x}_{\mathrm{H}} \tag{42}$$

is a unified propagation invariant. Propagation invariants have been extensively used in the analysis of symmetry properties of reflection and transmission responses and for the design of efficient numerical modelling schemes for acoustic and elastodynamic wavefields (Haines 1988; Kennett *et al.* 1990; Koketsu *et al.* 1991; Takenaka *et al.* 1993).

4 MATRIX–VECTOR WAVEFIELD REPRESENTATIONS

4.1 Representation of the convolution type

A wavefield representation is obtained by replacing one of the states in a reciprocity theorem by a Green's state (Knopoff 1956; de Hoop 1958; Gangi 1970; Pao & Varatharajulu 1976). Here we derive a unified matrix–vector wavefield representation from the matrix–vector reciprocity theorem of the convolution type (eq. 36).

We introduce the Green's matrix $\mathbf{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ (with the same dimensions as matrix $\boldsymbol{\mathcal{A}}$) as the solution of the unified matrix–vector wave eq. (1), with the source vector $\mathbf{d}$ replaced by a diagonal point-source matrix. Hence

$$\partial_3\mathbf{G} = \boldsymbol{\mathcal{A}}\mathbf{G} + \mathbf{I}\delta(\mathbf{x} - \mathbf{x}_A), \tag{43}$$

where $\mathbf{I}$ is an identity matrix and $\mathbf{x}_A$ defines the position of the point source. We let $\mathbf{G}$ represent the forward propagating solution of eq. (43), which corresponds to imposing causality in the time domain, that is, $\mathbf{G}(\mathbf{x}, \mathbf{x}_A, t) = \mathbf{O}$ for $t < 0$, where $\mathbf{O}$ is a zero matrix (the relation between functions in the time- and frequency domain is defined by the Fourier transform, eq. A7). Before we derive the unified wavefield representation, we first derive a reciprocity relation for the Green's matrix. To this end we define a second forward propagating Green's matrix $\mathbf{G}(\mathbf{x}, \mathbf{x}_B, \omega)$, with its point source at $\mathbf{x}_B$. We assume that $\mathbf{x}_A$ and $\mathbf{x}_B$ are both situated in $\mathbb{D}$. We replace $\mathbf{q}_A$ and $\mathbf{q}_B$ in reciprocity theorem (36) by $\mathbf{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ and $\mathbf{G}(\mathbf{x}, \mathbf{x}_B, \omega)$, respectively. Accordingly, we replace $\mathbf{d}_A$ and $\mathbf{d}_B$ by $\mathbf{I}\delta(\mathbf{x} - \mathbf{x}_A)$ and $\mathbf{I}\delta(\mathbf{x} - \mathbf{x}_B)$, respectively. Both Green's matrices are defined in the same medium, hence, $\boldsymbol{\mathcal{A}}_A = \boldsymbol{\mathcal{A}}_B$. This implies that the second integral on the right-hand side of eq. (36) vanishes. When Neumann or Dirichlet boundary conditions apply on $\partial\mathbb{D}$, or when the medium outside $\partial\mathbb{D}$ is homogeneous, the first integral on the right-hand side of eq. (36) vanishes as well. We thus obtain

$$\mathbf{N}\mathbf{G}(\mathbf{x}_A, \mathbf{x}_B, \omega) + \mathbf{G}^t(\mathbf{x}_B, \mathbf{x}_A, \omega)\mathbf{N} = \mathbf{O}. \tag{44}$$

Using $\mathbf{N}^{-1} = -\mathbf{N}$ this gives

$$\mathbf{G}(\mathbf{x}_A, \mathbf{x}_B, \omega) = \mathbf{N}\mathbf{G}^t(\mathbf{x}_B, \mathbf{x}_A, \omega)\mathbf{N}, \tag{45}$$

which is the unified source–receiver reciprocity relation for the Green's matrix.

Next, we use the same reciprocity theorem to derive a representation for the actual wavefield vector $\mathbf{q}$. We let state B be the actual state (i.e. actual wavefield, source and medium parameters). For convenience we drop the subscript B from $\mathbf{q}_B$, $\mathbf{d}_B$ and $\boldsymbol{\mathcal{A}}_B$. For state A we choose again the Green's state. Hence, we replace $\mathbf{q}_A$ by $\mathbf{G}(\mathbf{x}, \mathbf{x}_A, \omega)$ and $\mathbf{d}_A$ by $\mathbf{I}\delta(\mathbf{x} - \mathbf{x}_A)$. Operator $\boldsymbol{\mathcal{A}}_A$ may be defined in a reference medium or in the actual medium. Making these substitutions in eq. (36), pre-multiplying all terms by $-\mathbf{N}$, using eq. (45) and $-\mathbf{NN} = \mathbf{I}$, we obtain

$$\chi(\mathbf{x}_A)\mathbf{q}(\mathbf{x}_A, \omega) = \int_{\mathbb{D}} \mathbf{G}(\mathbf{x}_A, \mathbf{x}, \omega)\mathbf{d}(\mathbf{x}, \omega)\mathrm{d}^3\mathbf{x} - \int_{\partial\mathbb{D}} \mathbf{G}(\mathbf{x}_A, \mathbf{x}, \omega)\mathbf{q}(\mathbf{x}, \omega)n_3\mathrm{d}^2\mathbf{x}_\mathrm{H} + \int_{\mathbb{D}} \mathbf{G}(\mathbf{x}_A, \mathbf{x}, \omega)\{\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}_A\}\mathbf{q}(\mathbf{x}, \omega)\mathrm{d}^3\mathbf{x}, \tag{46}$$

where $\chi(\mathbf{x}_A)$ is the characteristic function, defined as

$$\chi(\mathbf{x}_A) = \begin{cases} 1, & \text{for } \mathbf{x}_A \text{ inside } \mathbb{D}, \\ \frac{1}{2}, & \text{for } \mathbf{x}_A \text{ on } \partial\mathbb{D}, \\ 0, & \text{for } \mathbf{x}_A \text{ outside } \mathbb{D}. \end{cases} \tag{47}$$

The left-hand side of eq. (46) is the actual wavefield vector $\mathbf{q}$, observed at $\mathbf{x}_A$ (when $\mathbf{x}_A$ is inside $\mathbb{D}$). The right-hand side contains, respectively, a contribution from the source distribution $\mathbf{d}(\mathbf{x}, \omega)$ inside $\mathbb{D}$, a contribution from the wavefield $\mathbf{q}(\mathbf{x}, \omega)$ at the boundary $\partial\mathbb{D}$, and a contribution from the contrast operator $\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}_A$, applied to the wavefield $\mathbf{q}(\mathbf{x}, \omega)$ inside $\mathbb{D}$. This unified matrix–vector wavefield representation holds for all wave phenomena listed in Table 1.

This representation can often be simplified, which leads to different applications. For example, when the medium outside the domain $\mathbb{D}$ is homogeneous, source free and identical in both states, the boundary integral on the right-hand side vanishes. The remaining representation (with $\boldsymbol{\mathcal{A}}_A$ defined in a reference medium) forms a basis for the analysis of forward scattering problems. On the other hand, when $\boldsymbol{\mathcal{A}}_A$ is defined in the actual medium (i.e. $\boldsymbol{\mathcal{A}}_A = \boldsymbol{\mathcal{A}}$) and the domain $\mathbb{D}$ is source free, only the boundary integral on the right-hand side remains. In this case, eq. (46) is a generalization of the Kirchhoff-Helmholtz integral (Morse & Feshbach 1953; Born & Wolf 1965; Pao & Varatharajulu 1976; Berkhout 1982; Frazer & Sen 1985), which finds applications in forward wavefield extrapolation problems.

4.2 Representation of the correlation type

Representations of the correlation type find their application in inverse source problems (Porter & Devaney 1982; de Hoop 1995), inverse scattering problems (Devaney 1982; Bojarski 1983; Bleistein 1984; Oristaglio 1989), imaging (Porter 1970; Schneider 1978; Berkhout 1982; Maynard *et al.* 1985; Esmersoy & Oristaglio 1988; Lindsey & Braun 2004), time-reversal acoustics (Fink & Prada 2001), and Green's function retrieval from ambient noise (Derode *et al.* 2003; Wapenaar 2003; Weaver & Lobkis 2004). There are several ways to approach the representation of the correlation type. The homogeneous Green's function representation (Porter 1970; Oristaglio 1989) elegantly covers most of the aforementioned applications for scalar wavefields. It is obtained by replacing both states in the reciprocity theorem of the correlation type by Green's states. Here we derive a unified representation for the homogeneous Green's matrix by substituting two Green's matrices into the matrix–vector reciprocity theorem of the correlation type (eq. 37).

Before we discuss the homogeneous Green's matrix, we introduce the Green's matrix of the adjoint medium, $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A, \omega)$, as the forward propagating solution of the following matrix–vector wave equation

$$\partial_3\bar{\mathbf{G}} = \bar{\boldsymbol{\mathcal{A}}}\bar{\mathbf{G}} + \mathbf{I}\delta(\mathbf{x} - \mathbf{x}_A). \tag{48}$$

Pre- and post multiplying all terms by $\mathbf{J}$ and subsequently using eq. (28) gives

$$\partial_3\mathbf{J}\bar{\mathbf{G}}\mathbf{J} = \boldsymbol{\mathcal{A}}^*\mathbf{J}\bar{\mathbf{G}}\mathbf{J} + \mathbf{JJ}\delta(\mathbf{x} - \mathbf{x}_A). \tag{49}$$

Taking the complex conjugate of all terms and using $\mathbf{JJ} = \mathbf{I}$ gives

$$\partial_3\mathbf{J}\bar{\mathbf{G}}^*\mathbf{J} = \boldsymbol{\mathcal{A}}\mathbf{J}\bar{\mathbf{G}}^*\mathbf{J} + \mathbf{I}\delta(\mathbf{x} - \mathbf{x}_A). \tag{50}$$

Subtracting all terms in this equation from the corresponding terms in eq. (43) we obtain

$$\partial_3\mathbf{G}_\mathrm{h}(\mathbf{x}, \mathbf{x}_A, \omega) = \boldsymbol{\mathcal{A}}\mathbf{G}_\mathrm{h}(\mathbf{x}, \mathbf{x}_A, \omega), \tag{51}$$

with

$$\mathbf{G}_\mathrm{h}(\mathbf{x}, \mathbf{x}_A, \omega) = \mathbf{G}(\mathbf{x}, \mathbf{x}_A, \omega) - \mathbf{J}\bar{\mathbf{G}}^*(\mathbf{x}, \mathbf{x}_A, \omega)\mathbf{J}. \tag{52}$$

Because $\mathbf{G}_\mathrm{h}(\mathbf{x}, \mathbf{x}_A, \omega)$ obeys a matrix–vector wave equation without a source term, we call it the homogeneous Green's matrix. The second term on the right-hand side represents a backward propagating wavefield in the adjoint medium.

Next, we use the correlation-type reciprocity theorem (eq. 37) to derive a representation for the homogeneous Green's matrix $\mathbf{G}_\mathrm{h}$. For state A we choose the Green's matrix in the adjoint medium, hence, we replace $\mathbf{q}_A$ by $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{x}_A, \omega)$, $\mathbf{d}_A$ by $\mathbf{I}\delta(\mathbf{x} - \mathbf{x}_A)$, and $\boldsymbol{\mathcal{A}}_A$ by $\bar{\boldsymbol{\mathcal{A}}}$. For state B we choose the Green's matrix in the actual medium, hence, we replace $\mathbf{q}_B$ by $\mathbf{G}(\mathbf{x}, \mathbf{x}_B, \omega)$, $\mathbf{d}_B$ by $\mathbf{I}\delta(\mathbf{x} - \mathbf{x}_B)$, and $\boldsymbol{\mathcal{A}}_B$ by $\boldsymbol{\mathcal{A}}$. With these choices the contrast operator $\bar{\boldsymbol{\mathcal{A}}}_A - \boldsymbol{\mathcal{A}}_B = \bar{\bar{\boldsymbol{\mathcal{A}}}} - \boldsymbol{\mathcal{A}}$ vanishes. Making these substitutions in eq. (37), taking $\mathbf{x}_A$ and $\mathbf{x}_B$ both inside $\mathbb{D}$, pre-multiplying all terms by $\mathbf{K}$, using eqs (45) and (52), $\mathbf{KK} = \mathbf{I}$ and $\mathbf{K} = \mathbf{JN} = -\mathbf{NJ}$, we obtain

$$\mathbf{G}_{\mathrm{h}}(\mathbf{x}_A, \mathbf{x}_B, \omega) = \int_{\partial\mathbb{D}} \mathbf{K}\bar{\mathbf{G}}^{\dagger}(\mathbf{x}, \mathbf{x}_A, \omega)\mathbf{K}\mathbf{G}(\mathbf{x}, \mathbf{x}_B, \omega) n_3 \mathrm{d}^2\mathbf{x}_{\mathrm{H}}. \tag{53}$$

This unified homogeneous Green's matrix representation holds for all wave phenomena listed in Table 1. It forms the basis for generalizing the applications mentioned at the beginning of this section.

5 CONCLUSIONS

A unified matrix–vector wave equation is presented for acoustic, quantum mechanical, electromagnetic, elastodynamic, poroelastodynamic, piezoelectric and seismoelectric waves. For most cases a 3-D inhomogeneous, anisotropic, dissipative medium is considered. The unified equation may be used as a basis for generalizing various applications of the elastodynamic matrix–vector wave equation, such as the analysis of surface waves, the derivation of long-wave effective medium parameters, the derivation of propagator matrices, decomposition into downgoing and upgoing waves, modelling wide-angle propagation in laterally variant media, etc.

The operator matrix in the matrix–vector wave equation obeys unified symmetry relations. These symmetry relations underly unified reciprocity theorems of the convolution and correlation type, which, in turn, form the basis for representations of the wave vector and the homogeneous Green's matrix. Reciprocity theorems and representations find applications in forward modelling problems, inverse source and inverse scattering problems, imaging, time-reversal methods and Green's function retrieval from ambient noise. The unified treatment in this paper provides a starting point for generalizing these applications to a broad range of wave phenomena.

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APPENDIX A: ACOUSTIC WAVES

The basic equations for acoustic wave propagation are the linearized equation of motion

$$\partial_t m_i + \partial_i p = f_i \tag{A1}$$

and the linearized deformation equation

$$-\partial_t \Theta + \partial_i v_i = q \tag{A2}$$

(de Hoop 1995; Willis 2012). Here $m_i = m_i(\mathbf{x}, t)$ is the momentum density as a function of spatial position $\mathbf{x}$ and time t, $p = p(\mathbf{x}, t)$ is the acoustic pressure, $\Theta = \Theta(\mathbf{x}, t)$ the cubic dilatation, $v_i = v_i(\mathbf{x}, t)$ the particle velocity, and $f_i = f_i(\mathbf{x}, t)$ and $q = q(\mathbf{x}, t)$ represent the sources in terms of external force density and volume-injection rate density, respectively (the function q should not be confused with vector $\mathbf{q}$ and its components $\mathbf{q}_1$ and $\mathbf{q}_2$ in eqs (1) and (2)). Operator ∂_i stands for differentiation in the x_i-direction. Lower-case Latin subscripts (except t) take on the values 1, 2 and 3, and Einstein's summation convention applies to repeated subscripts. Operator ∂_t stands for the temporal differential operator $\partial/\partial t$. The constitutive relations for an inhomogeneous, anisotropic fluid are given by

$$m_i = \rho_{ij} v_j, \tag{A3}$$

$$\Theta = -\kappa p, \tag{A4}$$

where $\rho_{ij} = \rho_{ij}(\mathbf{x})$ and $\kappa = \kappa(\mathbf{x})$ are the mass density and compressibility, respectively. To account for anisotropy, the mass density is defined as a tensor. Although ideal fluids are by definition isotropic, inhomogeneities at the micro scale can often be represented by effective anisotropic parameters at the macro scale. For example, a periodic stratified fluid can, in the long wavelength limit, be represented by a homogeneous fluid with an effective transverse isotropic mass density tensor and an effective isotropic compressibility (Schoenberg & Sen 1983). The mass density tensor is symmetric, that is, $\rho_{ij} = \rho_{ji}$. Substituting the constitutive relations (A3) and (A4) into eqs (A1) and (A2) yields

$$\rho_{ij} \partial_t v_j + \partial_i p = f_i, \tag{A5}$$

$$\kappa \partial_t p + \partial_i v_i = q. \tag{A6}$$

We define the temporal Fourier transform of a space- and time-dependent function $h(\mathbf{x}, t)$ as

$$h(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} h(\mathbf{x}, t) \exp(i\omega t) \mathrm{d}t, \tag{A7}$$

where i is the imaginary unit. For notational convenience, we use the same symbol (here h) for quantities in the time domain and in the frequency domain. We use eq. (A7) to transform eqs (A5) and (A6) to the frequency domain. The time derivatives are thus replaced by $-i\omega$, hence

$$-i\omega\rho_{ij} v_j + \partial_i p = f_i, \tag{A8}$$

$$-i\omega\kappa p + \partial_i v_i = q, \tag{A9}$$

with $p = p(\mathbf{x}, \omega)$, $v_i = v_i(\mathbf{x}, \omega)$, $f_i = f_i(\mathbf{x}, \omega)$ and $q = q(\mathbf{x}, \omega)$. In a lossless medium, the parameters $\rho_{ij}(\mathbf{x})$ and $\kappa(\mathbf{x})$ are real-valued and frequency independent. To account for losses, we replace them by complex-valued, frequency-dependent parameters $\rho_{ij} = \rho_{ij}(\mathbf{x}, \omega)$ and $\kappa = \kappa(\mathbf{x}, \omega)$ (de Hoop 1995; Carcione 2007).

The quantities p and v_3 constitute the power-flux density j in the x_3-direction, via

$$j = \frac{1}{4}(p^* v_3 + v_3^* p). \tag{A10}$$

We choose these quantities for the 1×1 vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in eqs (3) and (4), hence

$$\mathbf{q}_1 = p, \quad \mathbf{q}_2 = v_3. \tag{A11}$$

To arrive at a set of equations for these quantities, we need to eliminate the remaining wavefield quantities, v_1 and v_2, from eqs (A8) and (A9). To this end, we first introduce the inverse of the mass density tensor, the so-called specific volume tensor $\vartheta_{hi} = \vartheta_{hi}(\mathbf{x}, \omega)$, via

$$\vartheta_{hi}\rho_{ij} = \delta_{hj}, \tag{A12}$$

with δ_{hj} denoting the Kronecker delta. On account of the symmetry of the mass density tensor and eq. (A12), the specific volume tensor is symmetric as well, hence $\vartheta_{hi} = \vartheta_{ih}$. Applying ϑ_{hi} to both sides of eq. (A8), using eq. (A12), gives

$$-i\omega v_h + \vartheta_{hi}\partial_i p = \vartheta_{hi} f_i. \tag{A13}$$

We separate the derivatives in the x_3-direction from the lateral derivatives in eqs (A13) and (A9), according to

$$\partial_3 p = \vartheta_{33}^{-1}\big(-\vartheta_{3\beta}\partial_\beta p + i\omega v_3 + \vartheta_{3i} f_i\big), \tag{A14}$$

$$\partial_3 v_3 = i\omega\kappa p - \partial_\alpha v_\alpha + q. \tag{A15}$$

Einstein's summation convention applies also to repeated Greek subscripts (which take on the values 1 and 2). The particle velocity v_α needs to be eliminated from eq. (A15). From eq. (A13) we obtain

$$v_\alpha = \frac{1}{i\omega}\big(\vartheta_{\alpha\beta}\partial_\beta p + \vartheta_{\alpha 3}\partial_3 p - \vartheta_{\alpha i} f_i\big). \tag{A16}$$

Substituting eq. (A16) into eq. (A15), using eq. (A14), we obtain

$$\begin{aligned}\partial_3 v_3 &= i\omega\kappa p - \frac{1}{i\omega}\partial_\alpha\big(\vartheta_{\alpha\beta}\partial_\beta p + \vartheta_{\alpha 3}\partial_3 p - \vartheta_{\alpha i} f_i\big) + q\\ &= i\omega\kappa p - \frac{1}{i\omega}\partial_\alpha\Big(\vartheta_{\alpha\beta}\partial_\beta p + \vartheta_{\alpha 3}\vartheta_{33}^{-1}\big(-\vartheta_{3\beta}\partial_\beta p + i\omega v_3 + \vartheta_{3i} f_i\big) - \vartheta_{\alpha i} f_i\Big) + q.\end{aligned} \tag{A17}$$

We define

$$b_{\alpha i} = \vartheta_{\alpha i} - \vartheta_{\alpha 3}\vartheta_{33}^{-1}\vartheta_{3i}, \tag{A18}$$

with $b_{\alpha 3} = 0$ and $b_{\alpha\beta} = b_{\beta\alpha}$ on account of $\vartheta_{hi} = \vartheta_{ih}$. Eqs (A14) and (A17) have the form of eqs (3) and (4), with $\mathbf{q}_1$ and $\mathbf{q}_2$ defined in eq. (A11), 1×1 vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ defined as

$$\mathbf{d}_1 = \vartheta_{33}^{-1}\vartheta_{3i} f_i, \quad \mathbf{d}_2 = \frac{1}{i\omega}\partial_\alpha(b_{\alpha\beta} f_\beta) + q, \tag{A19}$$

and 1×1 operator matrices $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ defined as

$$\boldsymbol{\mathcal{A}}_{11} = -\vartheta_{33}^{-1}\vartheta_{3\beta}\partial_\beta, \tag{A20}$$

$$\boldsymbol{\mathcal{A}}_{12} = i\omega\vartheta_{33}^{-1}, \tag{A21}$$

$$\boldsymbol{\mathcal{A}}_{21} = i\omega\kappa - \frac{1}{i\omega}\partial_\alpha b_{\alpha\beta}\partial_\beta, \tag{A22}$$

$$\boldsymbol{\mathcal{A}}_{22} = -\partial_\alpha\vartheta_{\alpha 3}\vartheta_{33}^{-1}. \tag{A23}$$

The notation in the right-hand side of these equations should be understood in the sense that differential operators act on all factors to the right of it. For example, operator $\partial_\alpha b_{\alpha\beta}\partial_\beta$, applied via eq. (4) to p, stands for $\partial_\alpha(b_{\alpha\beta}\partial_\beta p)$, etc. Operators $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ obey the symmetry relations (16) – (18). We define adjoint acoustic medium parameters as $\bar{\kappa} = \kappa^*$, $\bar{\vartheta}_{hi} = \vartheta_{hi}^*$ and hence $\bar{b}_{\alpha\beta} = b_{\alpha\beta}^*$. Operators $\bar{\boldsymbol{\mathcal{A}}}_{11}$,

$\bar{\mathcal{A}}_{12}$, $\bar{\mathcal{A}}_{21}$ and $\bar{\mathcal{A}}_{22}$ in the adjoint medium are defined as in eqs (A20)–(A23), but with κ, ϑ_{hi} and $b_{\alpha\beta}$ replaced by $\bar{\kappa}$, $\bar{\vartheta}_{hi}$ and $\bar{b}_{\alpha\beta}$, respectively. These operators obey relations (19)–(22).

For the special case of an isotropic fluid we have $\vartheta_{hi} = \frac{1}{\rho}\delta_{hi}$, with ρ denoting the mass density of the isotropic fluid. For this situation eqs (A20)–(A23) reduce to the well-known expressions

$$\mathcal{A}_{11} = \mathcal{A}_{22} = 0, \tag{A24}$$

$$\mathcal{A}_{12} = i\omega\rho, \tag{A25}$$

$$\mathcal{A}_{21} = i\omega\kappa - \frac{1}{i\omega}\partial_\alpha \frac{1}{\rho}\partial_\alpha, \tag{A26}$$

(Corones 1975; Ursin 1983; Fishman & McCoy 1984; Wapenaar & Berkhout 1989; de Hoop 1996b).

APPENDIX B: QUANTUM MECHANICAL WAVES

Schrödinger's wave equation for a particle with mass m in a potential $V = V(\mathbf{x})$ is given by (Messiah 1961; Merzbacher 1961)

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_i\partial_i\psi + V\psi, \tag{B1}$$

where $\psi = \psi(\mathbf{x}, t)$ is the wave function and $\hbar = h/2\pi$, with h Planck's constant. We use eq. (A7) to transform this equation to the space–frequency domain, which means we can replace ∂_t by $-i\omega$. This gives

$$\hbar\omega\psi = -\frac{\hbar^2}{2m}\partial_i\partial_i\psi + V\psi, \tag{B2}$$

with $\psi = \psi(\mathbf{x}, \omega)$. To account for losses, we replace $V(\mathbf{x})$ by a complex-valued, frequency-dependent function $V(\mathbf{x}, \omega)$. The quantities ψ and $\frac{2\hbar}{mi}\partial_3\psi$ constitute the probability current density j in the x_3-direction, via

$$j = \frac{1}{4}\frac{2\hbar}{mi}\left(\psi^*\partial_3\psi - \psi\partial_3\psi^*\right). \tag{B3}$$

We choose these quantities for the 1×1 vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in eqs (3) and (4), hence

$$\mathbf{q}_1 = \psi, \quad \mathbf{q}_2 = \frac{2\hbar}{mi}\partial_3\psi. \tag{B4}$$

To arrive at a set of equations for these quantities, we first recast eq. (B2) (using the fact that $\hbar$ and m are constants) as

$$\partial_3\left(\frac{2\hbar}{mi}\partial_3\psi\right) = 4i\left(\omega - \frac{V}{\hbar}\right)\psi - \frac{2\hbar}{mi}\partial_\alpha\partial_\alpha\psi. \tag{B5}$$

This equation, together with the trivial equation

$$\partial_3\psi = \frac{mi}{2\hbar}\left(\frac{2\hbar}{mi}\partial_3\psi\right), \tag{B6}$$

have the form of eqs (4) and (3), with $\mathbf{q}_1$ and $\mathbf{q}_2$ defined in eq. (B4), $\mathbf{d}_1 = \mathbf{d}_2 = 0$, and 1×1 operator matrices $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{21}$ and $\mathcal{A}_{22}$ defined as

$$\mathcal{A}_{11} = \mathcal{A}_{22} = 0, \tag{B7}$$

$$\mathcal{A}_{12} = \frac{mi}{2\hbar}, \tag{B8}$$

$$\mathcal{A}_{21} = 4i\left(\omega - \frac{V}{\hbar}\right) - \frac{2\hbar}{mi}\partial_\alpha\partial_\alpha. \tag{B9}$$

Operators $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$ obey the symmetry relations (17) and (18). We define the adjoint potential as $\bar{V} = V^*$. Operators $\bar{\mathcal{A}}_{12}$ and $\bar{\mathcal{A}}_{21}$ for the adjoint potential obey relations (20) and (21).

APPENDIX C: ELECTROMAGNETIC WAVES

In the space–frequency domain, the Maxwell equations for electromagnetic wave propagation read (Landau & Lifshitz 1960; de Hoop 1995)

$$-i\omega D_i + J_i - \epsilon_{ijk}\partial_j H_k = -J_i^e, \tag{C1}$$

$$-i\omega B_k + \epsilon_{klm}\partial_l E_m = -J_k^m, \tag{C2}$$

where $E_m = E_m(\mathbf{x}, \omega)$ is the electric field strength, $H_k = H_k(\mathbf{x}, \omega)$ the magnetic field strength, $D_i = D_i(\mathbf{x}, \omega)$ the electric flux density, $B_k = B_k(\mathbf{x}, \omega)$ the magnetic flux density, $J_i = J_i(\mathbf{x}, \omega)$ the induced electric current density, $J_i^e = J_i^e(\mathbf{x}, \omega)$ and $J_k^m = J_k^m(\mathbf{x}, \omega)$ are source functions in terms of external electric and magnetic current densities and, finally, ϵ_{ijk} is the alternating tensor (or Levi-Civita tensor), with $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = -\epsilon_{213} = -\epsilon_{321} = -\epsilon_{132} = 1$ and all other elements being equal to 0. The constitutive relations for an inhomogeneous, anisotropic, dissipative medium are given by

$$D_i = \varepsilon_{ik}E_k = \varepsilon_0\varepsilon_{\mathrm{r},ik}E_k, \tag{C3}$$

$$B_k = \mu_{km}H_m = \mu_0\mu_{\mathrm{r},km}H_m, \tag{C4}$$

$$J_i = \sigma_{ik}E_k, \tag{C5}$$

where $\varepsilon_{ik} = \varepsilon_{ik}(\mathbf{x}, \omega)$, $\mu_{km} = \mu_{km}(\mathbf{x}, \omega)$ and $\sigma_{ik} = \sigma_{ik}(\mathbf{x}, \omega)$ are the permittivity, permeability and conductivity tensors, respectively. The subscripts 0 refer to the parameters in vacuum and the subscripts r denote relative parameters for the anisotropic medium. These tensors obey the symmetry relations $\varepsilon_{ik} = \varepsilon_{ki}$, $\mu_{km} = \mu_{mk}$ and $\sigma_{ik} = \sigma_{ki}$, respectively. Substituting the constitutive relations (C3)–(C5) into Maxwell's electromagnetic field eqs (C1) and (C2) yields

$$-i\omega\mathcal{E}_{ik}E_k - \epsilon_{ijk}\partial_j H_k = -J_i^e, \tag{C6}$$

$$-i\omega\mu_{km}H_m + \epsilon_{klm}\partial_l E_m = -J_k^m, \tag{C7}$$

with

$$\mathcal{E}_{ik} = \varepsilon_{ik} - \frac{\sigma_{ik}}{i\omega}. \tag{C8}$$

A matrix–vector wave equation for electromagnetic waves in an isotropic stratified medium is given by Ursin (1983) and van Stralen (1997). This has been extended for an anisotropic stratified medium by Løseth & Ursin (2007). Here we derive the matrix–vector wave equation for electromagnetic waves in a 3-D inhomogeneous, anisotropic, dissipative medium.

The quantities

$$\mathbf{E}_0 = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \quad \text{and} \quad \mathbf{H}_0 = \begin{pmatrix} H_2 \\ -H_1 \end{pmatrix} \tag{C9}$$

constitute the power-flux density j in the x_3-direction, via

$$j = \frac{1}{4}(\mathbf{E}_0^\dagger\mathbf{H}_0 + \mathbf{H}_0^\dagger\mathbf{E}_0) = \frac{1}{4}(E_1^*H_2 - E_2^*H_1 + H_2^*E_1 - H_1^*E_2). \tag{C10}$$

We choose these quantities for the 2×1 vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in eqs (3) and (4), hence

$$\mathbf{q}_1 = \mathbf{E}_0, \quad \mathbf{q}_2 = \mathbf{H}_0. \tag{C11}$$

To arrive at a set of equations for these quantities, we need to eliminate the remaining wavefield quantities, E_3 and H_3, from eqs (C6) and (C7). We start by rewriting these equations as

$$-i\omega\boldsymbol{\mathcal{E}}_1\mathbf{E}_0 - i\omega\boldsymbol{\mathcal{E}}_3E_3 + \partial_3\mathbf{H}_0 - \nabla_2H_3 = -\mathbf{J}_0^e, \tag{C12}$$

$$-i\omega\boldsymbol{\mathcal{E}}_3^t\mathbf{E}_0 - i\omega\mathcal{E}_{33}E_3 + \nabla_1^t\mathbf{H}_0 = -J_3^e, \tag{C13}$$

$$-i\omega\boldsymbol{\mu}_1\mathbf{H}_0 - i\omega\boldsymbol{\mu}_3H_3 + \partial_3\mathbf{E}_0 - \nabla_1E_3 = -\mathbf{J}_0^m, \tag{C14}$$

$$-i\omega\boldsymbol{\mu}_3^t\mathbf{H}_0 - i\omega\mu_{33}H_3 + \nabla_2^t\mathbf{E}_0 = -J_3^m, \tag{C15}$$

with

$$\boldsymbol{\mathcal{E}}_1 = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{12} & \mathcal{E}_{22} \end{pmatrix}, \quad \boldsymbol{\mathcal{E}}_3 = \begin{pmatrix} \mathcal{E}_{13} \\ \mathcal{E}_{23} \end{pmatrix}, \quad \boldsymbol{\mu}_1 = \begin{pmatrix} \mu_{22} & -\mu_{12} \\ -\mu_{12} & \mu_{11} \end{pmatrix}, \quad \boldsymbol{\mu}_3 = \begin{pmatrix} \mu_{23} \\ -\mu_{13} \end{pmatrix}, \tag{C16}$$

$$\nabla_1 = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}, \quad \nabla_2 = \begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix}, \quad \mathbf{J}_0^e = \begin{pmatrix} J_1^e \\ J_2^e \end{pmatrix}, \quad \mathbf{J}_0^m = \begin{pmatrix} J_2^m \\ -J_1^m \end{pmatrix}. \tag{C17}$$

Eq. (10) implies

$$\nabla_1^t = \begin{pmatrix} -\partial_1 & -\partial_2 \end{pmatrix}, \quad \nabla_2^t = \begin{pmatrix} -\partial_2 & \partial_1 \end{pmatrix}. \tag{C18}$$

We separate the derivatives in the x_3-direction from the lateral derivatives in eqs (C14) and (C12), according to

$$\partial_3 \mathbf{E}_0 = i\omega \boldsymbol{\mu}_1 \mathbf{H}_0 + i\omega \boldsymbol{\mu}_3 H_3 + \nabla_1 E_3 - \mathbf{J}_0^m, \tag{C19}$$

$$\partial_3 \mathbf{H}_0 = i\omega \boldsymbol{\mathcal{E}}_1 \mathbf{E}_0 + i\omega \boldsymbol{\mathcal{E}}_3 E_3 + \nabla_2 H_3 - \mathbf{J}_0^e. \tag{C20}$$

The field components E_3 and H_3 need to be eliminated. From eqs (C13) and (C15) we obtain

$$E_3 = \mathcal{E}_{33}^{-1}\Big(-\boldsymbol{\mathcal{E}}_3^t \mathbf{E}_0 + \frac{1}{i\omega}\nabla_1^t \mathbf{H}_0 + \frac{1}{i\omega} J_3^e\Big), \tag{C21}$$

$$H_3 = \mu_{33}^{-1}\Big(-\boldsymbol{\mu}_3^t \mathbf{H}_0 + \frac{1}{i\omega}\nabla_2^t \mathbf{E}_0 + \frac{1}{i\omega} J_3^m\Big). \tag{C22}$$

Substituting eqs (C21) and (C22) into eqs (C19) and (C20) we obtain

$$\partial_3 \mathbf{E}_0 = \Big(\boldsymbol{\mu}_3 \mu_{33}^{-1}\nabla_2^t - \nabla_1 \mathcal{E}_{33}^{-1}\boldsymbol{\mathcal{E}}_3^t\Big)\mathbf{E}_0 + \Big(i\omega\boldsymbol{\mu}_1 - i\omega\boldsymbol{\mu}_3\mu_{33}^{-1}\boldsymbol{\mu}_3^t + \frac{1}{i\omega}\nabla_1\mathcal{E}_{33}^{-1}\nabla_1^t\Big)\mathbf{H}_0 + \frac{1}{i\omega}\nabla_1(\mathcal{E}_{33}^{-1}J_3^e) - \mathbf{J}_0^m + \boldsymbol{\mu}_3\mu_{33}^{-1}J_3^m, \tag{C23}$$

$$\partial_3 \mathbf{H}_0 = \Big(i\omega\boldsymbol{\mathcal{E}}_1 - i\omega\boldsymbol{\mathcal{E}}_3\mathcal{E}_{33}^{-1}\boldsymbol{\mathcal{E}}_3^t + \frac{1}{i\omega}\nabla_2\mu_{33}^{-1}\nabla_2^t\Big)\mathbf{E}_0 + \Big(\boldsymbol{\mathcal{E}}_3\mathcal{E}_{33}^{-1}\nabla_1^t - \nabla_2\mu_{33}^{-1}\boldsymbol{\mu}_3^t\Big)\mathbf{H}_0 - \mathbf{J}_0^e + \boldsymbol{\mathcal{E}}_3\mathcal{E}_{33}^{-1}J_3^e + \frac{1}{i\omega}\nabla_2(\mu_{33}^{-1}J_3^m). \tag{C24}$$

Eqs (C23) and (C24) have the form of eqs (3) and (4), with $\mathbf{q}_1$ and $\mathbf{q}_2$ defined in eq. (C11), 2×1 vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ defined as

$$\mathbf{d}_1 = \frac{1}{i\omega}\nabla_1(\mathcal{E}_{33}^{-1}J_3^e) - \mathbf{J}_0^m + \boldsymbol{\mu}_3\mu_{33}^{-1}J_3^m, \tag{C25}$$

$$\mathbf{d}_2 = -\mathbf{J}_0^e + \boldsymbol{\mathcal{E}}_3\mathcal{E}_{33}^{-1}J_3^e + \frac{1}{i\omega}\nabla_2(\mu_{33}^{-1}J_3^m), \tag{C26}$$

and 2×2 operator matrices $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ defined as

$$\boldsymbol{\mathcal{A}}_{11} = \boldsymbol{\mu}_3\mu_{33}^{-1}\nabla_2^t - \nabla_1\mathcal{E}_{33}^{-1}\boldsymbol{\mathcal{E}}_3^t, \tag{C27}$$

$$\boldsymbol{\mathcal{A}}_{12} = i\omega(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_3\mu_{33}^{-1}\boldsymbol{\mu}_3^t) + \frac{1}{i\omega}\nabla_1\mathcal{E}_{33}^{-1}\nabla_1^t, \tag{C28}$$

$$\boldsymbol{\mathcal{A}}_{21} = i\omega(\boldsymbol{\mathcal{E}}_1 - \boldsymbol{\mathcal{E}}_3\mathcal{E}_{33}^{-1}\boldsymbol{\mathcal{E}}_3^t) + \frac{1}{i\omega}\nabla_2\mu_{33}^{-1}\nabla_2^t, \tag{C29}$$

$$\boldsymbol{\mathcal{A}}_{22} = \boldsymbol{\mathcal{E}}_3\mathcal{E}_{33}^{-1}\nabla_1^t - \nabla_2\mu_{33}^{-1}\boldsymbol{\mu}_3^t. \tag{C30}$$

These operators obey the symmetry relations (16)–(18). We define adjoint electromagnetic medium parameters as $\bar{\varepsilon}_{ik} = \varepsilon_{ik}^*$, $\bar{\mu}_{km} = \mu_{km}^*$ and $\bar{\sigma}_{ik} = -\sigma_{ik}^*$. Using eq. (C8) it follows that $\bar{\mathcal{E}}_{ik} = \mathcal{E}_{ik}^*$. Similar relations hold for $\boldsymbol{\mathcal{E}}_1$, $\boldsymbol{\mathcal{E}}_3$, $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_3$, which contain the parameters $\mathcal{E}_{ik}$ and μ_{km}. Operators $\bar{\boldsymbol{\mathcal{A}}}_{11}$, $\bar{\boldsymbol{\mathcal{A}}}_{12}$, $\bar{\boldsymbol{\mathcal{A}}}_{21}$ and $\bar{\boldsymbol{\mathcal{A}}}_{22}$ in the adjoint medium obey relations (19)–(22). Explicit expressions for the operator matrices in an isotropic medium are given in the supplemental material, section 1.

APPENDIX D: ELASTODYNAMIC WAVES

In the space-frequency domain, the elastodynamic equations of motion and deformation read (Achenbach 1973; Aki & Richards 1980; de Hoop 1995; Willis 2012)

$$-i\omega m_i - \partial_j \tau_{ij} = f_i, \tag{D1}$$

$$i\omega e_{kl} + \frac{1}{2}(\partial_k v_l + \partial_l v_k) = h_{kl}, \tag{D2}$$

where $m_i = m_i(\mathbf{x}, \omega)$ is the momentum density, $\tau_{ij} = \tau_{ij}(\mathbf{x}, \omega)$ the stress tensor, $e_{kl} = e_{kl}(\mathbf{x}, \omega)$ the strain tensor, $v_k = v_k(\mathbf{x}, \omega)$ the particle velocity, and $f_i = f_i(\mathbf{x}, \omega)$ and $h_{kl} = h_{kl}(\mathbf{x}, \omega)$ are source functions in terms of external force density and deformation-rate density, respectively. The stress, strain and deformation-rate tensors obey the symmetry relations $\tau_{ij} = \tau_{ji}$, $e_{kl} = e_{lk}$ and $h_{kl} = h_{lk}$. The constitutive relations for an inhomogeneous, anisotropic, dissipative solid are given by

$$m_i = \rho_{ij} v_j, \tag{D3}$$

$$e_{kl} = s_{klmn}\tau_{mn}, \tag{D4}$$

where $\rho_{ij} = \rho_{ij}(\mathbf{x}, \omega)$ and $s_{klmn} = s_{klmn}(\mathbf{x}, \omega)$ are the mass density and compliance tensors, respectively. These tensors obey the symmetry relations $\rho_{ij} = \rho_{ji}$ and $s_{klmn} = s_{lkmn} = s_{klnm} = s_{mnkl}$, respectively (Aki & Richards 1980; Dahlen & Tromp 1998). Substituting the constitutive relations (D3) and (D4) into eqs (D1) and (D2) yields

$$-i\omega\rho_{ij} v_j - \partial_j \tau_{ij} = f_i, \tag{D5}$$

$$i\omega s_{klmn}\tau_{mn} + \frac{1}{2}(\partial_k v_l + \partial_l v_k) = h_{kl}. \tag{D6}$$

We introduce the stiffness tensor $c_{ijkl} = c_{ijkl}(\mathbf{x}, \omega)$ as the inverse of the compliance tensor s_{klmn}, according to

$$c_{ijkl}s_{klmn} = s_{ijkl}c_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}). \tag{D7}$$

The stiffness tensor obeys the symmetry relation $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. Multiplying all terms in eq. (D6) by c_{ijkl}, using the symmetry relations $\tau_{ij} = \tau_{ji}$ and $c_{ijkl} = c_{ijlk}$, we obtain an alternative form of eq. (D6), according to

$$i\omega\tau_{ij} + c_{ijkl}\partial_l v_k = c_{ijkl}h_{kl}. \tag{D8}$$

A matrix–vector wave equation for elastodynamic waves in an inhomogeneous anisotropic medium is given by Woodhouse (1974). Here we review this derivation, which also serves as a starting point for the derivation of the matrix–vector wave equations for poroelastodynamic waves (Appendix E), piezoelectric waves (Appendix F) and seismoelectric waves (Appendix G). The quantities $-\boldsymbol{\tau}_3$ and $\mathbf{v}$ (which are 3×1 vectors, with $(\boldsymbol{\tau}_3)_i = \tau_{i3}$ and $(\mathbf{v})_i = v_i$) constitute the power-flux density j in the x_3-direction, via

$$j = \frac{1}{4}(-\boldsymbol{\tau}_3^\dagger\mathbf{v} - \mathbf{v}^\dagger\boldsymbol{\tau}_3) = \frac{1}{4}(-\tau_{i3}^* v_i - v_i^*\tau_{i3}). \tag{D9}$$

We choose these quantities for the 3×1 vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in eqs (3) and (4), hence

$$\mathbf{q}_1 = -\boldsymbol{\tau}_3, \quad \mathbf{q}_2 = \mathbf{v}. \tag{D10}$$

To arrive at a set of equations for these quantities, we need to eliminate the remaining wavefield quantities, 3×1 vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ (with $(\boldsymbol{\tau}_j)_i = \tau_{ij}$), from eqs (D5) and (D8). We start by rewriting these equations as

$$-i\omega\boldsymbol{\rho}\mathbf{v} - \partial_j\boldsymbol{\tau}_j = \mathbf{f}, \tag{D11}$$

$$i\omega\boldsymbol{\tau}_j + \mathbf{C}_{jl}\partial_l\mathbf{v} = \mathbf{C}_{jl}\mathbf{h}_l, \tag{D12}$$

where $\boldsymbol{\rho}$ and $\mathbf{C}_{jl}$ are 3×3 matrices, with $(\boldsymbol{\rho})_{ij} = \rho_{ij}$, $\boldsymbol{\rho}^t = \boldsymbol{\rho}$, $(\mathbf{C}_{jl})_{ik} = c_{ijkl}$, $\mathbf{C}_{jl}^t = \mathbf{C}_{lj}$, and where $\mathbf{f}$ and $\mathbf{h}_l$ are 3×1 vectors, with $(\mathbf{f})_i = f_i$ and $(\mathbf{h}_l)_k = h_{kl}$. We separate the derivatives in the x_3-direction from the lateral derivatives in eqs (D11) and (D12), according to

$$-\partial_3\boldsymbol{\tau}_3 = i\omega\boldsymbol{\rho}\mathbf{v} + \partial_\alpha\boldsymbol{\tau}_\alpha + \mathbf{f}, \tag{D13}$$

$$\partial_3\mathbf{v} = \mathbf{C}_{33}^{-1}\Big(-i\omega\boldsymbol{\tau}_3 - \mathbf{C}_{3\beta}\partial_\beta\mathbf{v} + \mathbf{C}_{3l}\mathbf{h}_l\Big). \tag{D14}$$

The field components $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ need to be eliminated. From eq. (D12) we obtain

$$\boldsymbol{\tau}_\alpha = -\frac{1}{i\omega}\big(\mathbf{C}_{\alpha\beta}\partial_\beta\mathbf{v} + \mathbf{C}_{\alpha 3}\partial_3\mathbf{v} - \mathbf{C}_{\alpha l}\mathbf{h}_l\big). \tag{D15}$$

Substituting eq. (D14) into (D15) and the result into eq. (D13), we obtain

$$-\partial_3\boldsymbol{\tau}_3 = \partial_\alpha\big(\mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}\boldsymbol{\tau}_3\big) + i\omega\boldsymbol{\rho}\mathbf{v} - \frac{1}{i\omega}\partial_\alpha\Big(\mathbf{U}_{\alpha\beta}\partial_\beta\mathbf{v} - \mathbf{U}_{\alpha l}\mathbf{h}_l\Big) + \mathbf{f}, \tag{D16}$$

with

$$\mathbf{U}_{\alpha l} = \mathbf{C}_{\alpha l} - \mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}\mathbf{C}_{3l}, \tag{D17}$$

where $\mathbf{U}_{\alpha 3} = \mathbf{O}$ and $\mathbf{U}^t_{\alpha\beta} = \mathbf{U}_{\beta\alpha}$ on account of $\mathbf{C}^t_{jl} = \mathbf{C}_{lj}$. Eqs (D16) and (D14) have the form of eqs (3) and (4), with $\mathbf{q}_1$ and $\mathbf{q}_2$ defined in eq. (D10), 3×1 vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ defined as

$$\mathbf{d}_1 = \mathbf{f} + \frac{1}{i\omega}\partial_\alpha\big(\mathbf{U}_{\alpha\beta}\mathbf{h}_\beta\big), \tag{D18}$$

$$\mathbf{d}_2 = \mathbf{C}_{33}^{-1}\mathbf{C}_{3l}\mathbf{h}_l, \tag{D19}$$

and 3×3 operator matrices $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ defined as

$$\boldsymbol{\mathcal{A}}_{11} = -\partial_\alpha\mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}, \tag{D20}$$

$$\boldsymbol{\mathcal{A}}_{12} = i\omega\boldsymbol{\rho} - \frac{1}{i\omega}\partial_\alpha\mathbf{U}_{\alpha\beta}\partial_\beta, \tag{D21}$$

$$\boldsymbol{\mathcal{A}}_{21} = i\omega\mathbf{C}_{33}^{-1}, \tag{D22}$$

$$\boldsymbol{\mathcal{A}}_{22} = -\mathbf{C}_{33}^{-1}\mathbf{C}_{3\beta}\partial_\beta. \tag{D23}$$

These operators obey the symmetry relations (16)–(18). We define adjoint elastodynamic medium parameters as $\bar{c}_{ijkl} = c^*_{ijkl}$ and $\bar{\rho}_{ij} = \rho^*_{ij}$. Similar relations hold for $\mathbf{C}_{jl}$, $\mathbf{U}_{\alpha\beta}$ and $\boldsymbol{\rho}$, which contain the parameters c_{ijkl} and ρ_{ij}. Operators $\bar{\boldsymbol{\mathcal{A}}}_{11}$, $\bar{\boldsymbol{\mathcal{A}}}_{12}$, $\bar{\boldsymbol{\mathcal{A}}}_{21}$ and $\bar{\boldsymbol{\mathcal{A}}}_{22}$ in the adjoint medium obey relations (19)–(22). Explicit expressions for the operator matrices in an isotropic medium are given in the supplemental material, Section 2.

APPENDIX E: POROELASTODYNAMIC WAVES

In the space–frequency domain, the basic equations for poroelastodynamic wave propagation in an inhomogeneous, anisotropic, dissipative, fluid-saturated porous solid read (Biot 1956a,b; Pride *et al.* 1992; Pride & Haartsen 1996)

$$-i\omega\rho^b_{ij}v^s_j - i\omega\rho^f_{ij}w_j - \partial_j\tau^b_{ij} = f^b_i, \tag{E1}$$

$$-\frac{i\omega}{\eta}k_{ij}\rho^f_{jl}v^s_l + w_i + \frac{1}{\eta}k_{ij}\partial_j p^f = \frac{1}{\eta}k_{ij}f^f_j, \tag{E2}$$

$$i\omega\tau^b_{ij} + c_{ijkl}\partial_l v^s_k + C_{ij}\partial_k w_k = c_{ijkl}h^b_{kl} + C_{ij}q^f, \tag{E3}$$

$$-i\omega p^f + C_{kl}\partial_l v^s_k + M\partial_k w_k = C_{kl}h^b_{kl} + Mq^f, \tag{E4}$$

with

$$w_j = \phi(v^f_j - v^s_j), \tag{E5}$$

$$v^b_j = \phi v^f_j + (1-\phi)v^s_j = v^s_j + w_j, \tag{E6}$$

$$\tau^b_{ij} = \phi\tau^f_{ij} + (1-\phi)\tau^s_{ij} = -\phi\delta_{ij}p^f + (1-\phi)\tau^s_{ij}, \tag{E7}$$

$$f^b_i = \phi f^f_i + (1-\phi)f^s_i, \tag{E8}$$

$$\rho^b_{ij} = \phi\rho^f_{ij} + (1-\phi)\rho^s_{ij}. \tag{E9}$$

Superscripts b, f and s stand for bulk, fluid and solid, respectively. The wavefield quantity $v_j = v_j(\mathbf{x}, \omega)$ is the averaged particle velocity in the bulk, fluid or solid (depending on the superscript), $w_j = w_j(\mathbf{x}, \omega)$ is the filtration velocity, $\tau_{ij} = \tau_{ij}(\mathbf{x}, \omega)$ the averaged stress in the bulk, fluid or solid and $p^f = p^f(\mathbf{x}, \omega)$ the averaged fluid pressure. The stress tensors are symmetric, i.e., $\tau_{ij} = \tau_{ji}$. The medium parameter $\rho_{ij} = \rho_{ij}(\mathbf{x}, \omega)$ is the mass density of the bulk, fluid or solid (depending on the superscript). Furthermore, $k_{ij} = k_{ij}(\mathbf{x}, \omega)$ is the dynamic permeability tensor, $\eta = \eta(\mathbf{x}, \omega)$ is the fluid viscosity parameter and $\phi = \phi(\mathbf{x})$ the porosity. Moreover, $c_{ijkl} = c_{ijkl}(\mathbf{x}, \omega)$, $C_{ij} = C_{ij}(\mathbf{x}, \omega)$ and $M = M(\mathbf{x}, \omega)$ are stiffness parameters of the porous solid. The medium parameters obey the following symmetry relations $\rho_{ij} = \rho_{ji}$, $k_{ij} = k_{ji}$, $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$ and $C_{ij} = C_{ji}$. The source function $f_i = f_i(\mathbf{x}, \omega)$ is the volume density of external force on the bulk, fluid or solid. For many source types the forces on the bulk and fluid are equal but in the following they will be treated distinctly. The source functions $h^b_{kl} = h^b_{kl}(\mathbf{x}, \omega)$ and $q^f = q^f(\mathbf{x}, \omega)$ are the volume densities of external deformation rate on the bulk and volume-injection rate in the fluid (Wapenaar & Berkhout 1989; Pride 1994; de Hoop 1995; Grobbe 2016). The deformation rate tensor is symmetric, that is, $h^b_{kl} = h^b_{lk}$. For later convenience, we eliminate $\partial_k w_k$ from eq. (E3), using eq. (E4). This yields

$$i\omega\tau^b_{ij} + c'_{ijkl}\partial_l v^s_k + \frac{i\omega}{M}C_{ij}p^f = c'_{ijkl}h^b_{kl}, \tag{E10}$$

with $c'_{ijkl} = c'_{ijkl}(\mathbf{x}, \omega)$ defined as

$$c'_{ijkl} = c_{ijkl} - \frac{1}{M}C_{ij}C_{kl}. \tag{E11}$$

A matrix–vector wave equation for normal-incidence poroelastodynamic waves in a stratified isotropic medium is given by Norris (1993) and Gurevich & Lopatnikov (1995). This has been extended for oblique-incidence poroelastodynamic waves in a stratified anisotropic medium, separately for P-SV and SH propagation, by Gelinsky & Shapiro (1997). Here we derive the matrix–vector wave equation for poroelastodynamic waves in a 3D inhomogeneous, anisotropic, dissipative, fluid-saturated porous solid. The quantities $-\boldsymbol{\tau}^b_3$, p^f, $\mathbf{v}^s$ and w_3 (with $(\boldsymbol{\tau}^b_j)_i = \tau^b_{ij}$ and $(\mathbf{v}^s)_i = v^s_i$) constitute the power-flux density j in the x_3-direction, via

$$j = \frac{1}{4}(-(\boldsymbol{\tau}^b_3)^\dagger\mathbf{v}^s + p^{f*}w_3 - (\mathbf{v}^s)^\dagger\boldsymbol{\tau}^b_3 + w^*_3 p^f) = \frac{1}{4}(-\tau^{b*}_{i3}v^s_i + p^{f*}w_3 - v^{s*}_i\tau^b_{i3} + w^*_3 p^f). \tag{E12}$$

We choose these quantities for the 4×1 vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in eqs (3) and (4), hence

$$\mathbf{q}_1 = \begin{pmatrix} -\boldsymbol{\tau}^b_3 \\ p^f \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} \mathbf{v}^s \\ w_3 \end{pmatrix}. \tag{E13}$$

To arrive at a set of equations for these quantities, we need to eliminate the remaining wavefield quantities from eqs (E1), (E2), (E10) and (E4). We start by rewriting these equations as

$$-i\omega\boldsymbol{\rho}^b\mathbf{v}^s - i\omega\boldsymbol{\rho}^f\mathbf{i}_j w_j - \partial_j\boldsymbol{\tau}^b_j = \mathbf{f}^b, \tag{E14}$$

$$-\frac{i\omega}{\eta}\mathbf{k}\boldsymbol{\rho}^f\mathbf{v}^s + \mathbf{i}_j w_j + \frac{1}{\eta}\mathbf{k}\mathbf{i}_j\partial_j p^f = \frac{1}{\eta}\mathbf{k}\mathbf{i}_j f^f_j, \tag{E15}$$

$$i\omega\boldsymbol{\tau}^b_j + \mathbf{C}_{jl}\partial_l\mathbf{v}^s + \frac{i\omega}{M}\mathbf{c}_j p^f = \mathbf{C}_{jl}\mathbf{h}^b_l, \tag{E16}$$

$$-i\omega p^f + \mathbf{c}^t_l\partial_l\mathbf{v}^s + M\partial_k w_k = \mathbf{c}^t_l\mathbf{h}^b_l + Mq^f, \tag{E17}$$

where $\boldsymbol{\rho}$, $\mathbf{k}$ and $\mathbf{C}_{jl}$ are 3×3 matrices, with $(\boldsymbol{\rho})_{ij} = \rho_{ij}$, $\boldsymbol{\rho}^t = \boldsymbol{\rho}$, $(\mathbf{k})_{ij} = k_{ij}$, $\mathbf{k}^t = \mathbf{k}$, $(\mathbf{C}_{jl})_{ik} = c'_{ijkl}$, $\mathbf{C}^t_{jl} = \mathbf{C}_{lj}$, and where $\mathbf{c}_j$, $\mathbf{f}^b$, $\mathbf{h}^b_l$ and $\mathbf{i}_j$ are 3×1 vectors, with $(\mathbf{c}_j)_i = C_{ij}$, $(\mathbf{f}^b)_i = f^b_i$, $(\mathbf{h}^b_l)_k = h^b_{kl}$, and $(\mathbf{i}_j)_i = \delta_{ij}$. Eqs (E14)–(E17) form the starting point for deriving matrix–vector equations in the form of eqs (3) and (4), with $\mathbf{q}_1$ and $\mathbf{q}_2$ defined in eq. (E13). The other quantities ($\boldsymbol{\tau}^b_1$, $\boldsymbol{\tau}^b_2$, w_1 and w_2) need to be eliminated. The detailed derivation can be found in the supplemental material, Section 3. The 4×1 vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are defined as

$$\mathbf{d}_1 = \begin{pmatrix} \frac{i\omega}{\eta}\boldsymbol{\rho}^f\mathbf{i}_\alpha\mathbf{i}^t_\alpha\mathbf{k}\big(\mathbf{I} + \frac{1}{b}\mathbf{i}_3\mathbf{i}^t_3\mathbf{k}^{-1}\mathbf{i}_\gamma\mathbf{i}^t_\gamma\mathbf{k}\big)\mathbf{i}_j f^f_j - \frac{i\omega}{\eta b}\boldsymbol{\rho}^f\mathbf{i}_\alpha\mathbf{i}^t_\alpha\mathbf{k}\mathbf{i}_3 f^f_3 + \mathbf{f}^b + \frac{1}{i\omega}\partial_\alpha\big(\mathbf{U}_{\alpha\beta}\mathbf{h}^b_\beta\big) \\ \frac{1}{b}\big(-\mathbf{i}^t_3\mathbf{k}^{-1}\mathbf{i}_\alpha\mathbf{i}^t_\alpha\mathbf{k}\mathbf{i}_j f^f_j + f^f_3\big) \end{pmatrix}, \tag{E18}$$

$$\mathbf{d}_2 = \begin{pmatrix} \mathbf{C}^{-1}_{33}\mathbf{C}_{3l}\mathbf{h}^b_l \\ -\partial_\alpha\Big(\frac{1}{\eta}\mathbf{i}^t_\alpha\mathbf{k}\big(\mathbf{I} + \frac{1}{b}\mathbf{i}_3\mathbf{i}^t_3\mathbf{k}^{-1}\mathbf{i}_\gamma\mathbf{i}^t_\gamma\mathbf{k}\big)\mathbf{i}_j f^f_j - \frac{1}{\eta b}\mathbf{i}^t_\alpha\mathbf{k}\mathbf{i}_3 f^f_3\Big) + \frac{1}{M}\mathbf{u}^t_\alpha\mathbf{h}^b_\alpha + q^f \end{pmatrix} \tag{E19}$$

and the 4×4 operator matrices $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ as

$$\boldsymbol{\mathcal{A}}_{11} = \begin{pmatrix} \boldsymbol{\mathcal{A}}^{11}_{11} & \boldsymbol{\mathcal{A}}^{12}_{11} \\ \boldsymbol{\mathcal{A}}^{21}_{11} & \boldsymbol{\mathcal{A}}^{22}_{11} \end{pmatrix}, \quad \boldsymbol{\mathcal{A}}_{12} = \begin{pmatrix} \boldsymbol{\mathcal{A}}^{11}_{12} & \boldsymbol{\mathcal{A}}^{12}_{12} \\ (\boldsymbol{\mathcal{A}}^{12}_{12})^t & \boldsymbol{\mathcal{A}}^{22}_{12} \end{pmatrix}, \tag{E20}$$

$$\boldsymbol{\mathcal{A}}_{21} = \begin{pmatrix} \boldsymbol{\mathcal{A}}_{21}^{11} & \boldsymbol{\mathcal{A}}_{21}^{12} \\ (\boldsymbol{\mathcal{A}}_{21}^{12})^t & \boldsymbol{\mathcal{A}}_{21}^{22} \end{pmatrix}, \quad \boldsymbol{\mathcal{A}}_{22} = -\boldsymbol{\mathcal{A}}_{11}^t. \tag{E21}$$

Here

$$\boldsymbol{\mathcal{A}}_{11}^{11} = -\partial_\alpha \mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}, \tag{E22}$$

$$\boldsymbol{\mathcal{A}}_{11}^{12} = -\frac{i\omega}{\eta}\boldsymbol{\rho}^f\Big(\mathbf{I} - \frac{1}{b}\mathbf{k}\mathbf{i}_3\mathbf{i}_3^t\mathbf{k}^{-1}\mathbf{i}_3\mathbf{i}_3^t\Big)\mathbf{k}\mathbf{i}_\beta\partial_\beta - \partial_\alpha\frac{1}{M}\mathbf{u}_\alpha, \tag{E23}$$

$$\boldsymbol{\mathcal{A}}_{11}^{21} = \mathbf{0}^t, \tag{E24}$$

$$\boldsymbol{\mathcal{A}}_{11}^{22} = -\frac{1}{b}\mathbf{i}_3^t\mathbf{k}^{-1}\mathbf{i}_3\mathbf{i}_3^t\mathbf{k}\mathbf{i}_\beta\partial_\beta, \tag{E25}$$

$$\boldsymbol{\mathcal{A}}_{12}^{11} = i\omega\boldsymbol{\rho}^b - \frac{1}{i\omega}\partial_\alpha\mathbf{U}_{\alpha\beta}\partial_\beta - \frac{\omega^2}{\eta}\boldsymbol{\rho}^f\mathbf{i}_\alpha\mathbf{i}_\alpha^t\Big(\mathbf{k} - \frac{1}{b}\mathbf{k}\mathbf{i}_3\mathbf{i}_3^t\mathbf{k}^{-1}\mathbf{i}_3\mathbf{i}_3^t\mathbf{k}\Big)\mathbf{i}_\beta\mathbf{i}_\beta^t\boldsymbol{\rho}^f, \tag{E26}$$

$$\boldsymbol{\mathcal{A}}_{12}^{12} = \frac{i\omega}{b}\boldsymbol{\rho}^f\mathbf{k}\mathbf{i}_3\mathbf{i}_3^t\mathbf{k}^{-1}\mathbf{i}_3, \tag{E27}$$

$$\boldsymbol{\mathcal{A}}_{12}^{22} = -\frac{\eta}{b}\mathbf{i}_3^t\mathbf{k}^{-1}\mathbf{i}_3, \tag{E28}$$

$$\boldsymbol{\mathcal{A}}_{21}^{11} = i\omega\mathbf{C}_{33}^{-1}, \tag{E29}$$

$$\boldsymbol{\mathcal{A}}_{21}^{12} = -\frac{i\omega}{M}\mathbf{C}_{33}^{-1}\mathbf{c}_3, \tag{E30}$$

$$\boldsymbol{\mathcal{A}}_{21}^{22} = \frac{i\omega}{M^2}\mathbf{c}_3^t\mathbf{C}_{33}^{-1}\mathbf{c}_3 + \frac{i\omega}{M} + \partial_\alpha\frac{1}{\eta}\Big(\mathbf{i}_\alpha^t\mathbf{k}\mathbf{i}_\beta - \frac{1}{b}\mathbf{i}_\alpha^t\mathbf{k}\mathbf{i}_3\mathbf{i}_3^t\mathbf{k}^{-1}\mathbf{i}_3\mathbf{i}_3^t\mathbf{k}\mathbf{i}_\beta\Big)\partial_\beta, \tag{E31}$$

with **0** denoting a zero vector and

$$\mathbf{U}_{\alpha\beta} = \mathbf{C}_{\alpha\beta} - \mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}\mathbf{C}_{3\beta}, \tag{E32}$$

$$\mathbf{u}_\alpha = \mathbf{c}_\alpha - \mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}\mathbf{c}_3, \tag{E33}$$

$$b = 1 - \mathbf{i}_3^t\mathbf{k}^{-1}\mathbf{i}_\alpha\mathbf{i}_\alpha^t\mathbf{k}\mathbf{i}_3. \tag{E34}$$

Operators $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ obey the symmetry relations formulated in eqs (16)–(18). We defined the adjoints of the medium parameters c_{ijkl} and ρ_{ij} in Appendix D (where ρ_{ij} now has superscript b or f). Moreover, we define $\bar{k}_{ij} = k_{ij}^*$, $\bar{\eta} = -\eta^*$, $\bar{C}_{ij} = C_{ij}^*$ and $\bar{M} = M^*$. Similar relations hold for $\mathbf{C}_{jl}$, $\boldsymbol{\rho}^b$, $\boldsymbol{\rho}^f$, $\mathbf{k}$ and $\mathbf{c}_j$, which contain the parameters $c'_{ijkl} = c_{ijkl} - \frac{1}{M}C_{ij}C_{kl}$, ρ_{ij}^b, ρ_{ij}^f, k_{ij} and C_{ij}. Operators $\bar{\boldsymbol{\mathcal{A}}}_{11}$, $\bar{\boldsymbol{\mathcal{A}}}_{12}$, $\bar{\boldsymbol{\mathcal{A}}}_{21}$ and $\bar{\boldsymbol{\mathcal{A}}}_{22}$ in the adjoint medium obey relations (19)–(22). Explicit expressions for the operator matrices in an isotropic medium are given in the supplemental material, section 3.

APPENDIX F: PIEZOELECTRIC WAVES

Piezoelectric waves are governed by the equations for electromagnetic waves (Appendix C) and elastodynamic waves (Appendix D), in which two of the constitutive relations need to be modified to account for the coupling between the two wave types. For piezoelectric waves, the modified constitutive relations are (Auld 1973)

$$D_i = \varepsilon_{ik}E_k + d_{ijk}\tau_{jk}, \tag{F1}$$

$$e_{kl} = d_{klm} E_m + s_{klmn} \tau_{mn}. \tag{F2}$$

The field quantities and medium parameters (except d_{ijk}) have been defined in Appendices C and D. Parameters ε_{ik} in eq. (F1) and s_{klmn} in eq. (F2) are defined under constant stress and constant electric field, respectively. The coupling tensor $d_{ijk} = d_{ijk}(\mathbf{x}, \omega)$ obeys the symmetry relation $d_{ijk} = d_{jik} = d_{ikj}$. Eq. (F1) replaces constitutive relation (C3) and is substituted, together with constitutive relations (C4) and (C5), into the Maxwell eqs (C1) and (C2). Eq. (F2) replaces stress–strain relation (D4) and is substituted, together with constitutive relation (D3), into eqs (D2) and (D1). Subsequently, all terms in the latter equation are multiplied by c_{ijkl}, using eq. (D7) as well as the symmetry relations $\tau_{ij} = \tau_{ji}$ and $c_{ijkl} = c_{ijlk}$. The basic equations for coupled electromagnetic and elastodynamic waves thus read

$$-i\omega\mathcal{E}_{ik}E_k - \epsilon_{ijk}\partial_j H_k - i\omega d_{ijk}\tau_{jk} = -J_i^e, \tag{F3}$$

$$-i\omega\mu_{km}H_m + \epsilon_{klm}\partial_l E_m = -J_k^m, \tag{F4}$$

$$-i\omega\rho_{ij}v_j - \partial_j\tau_{ij} = f_i, \tag{F5}$$

$$i\omega\tau_{ij} + c_{ijkl}\partial_l v_k + i\omega c_{ijkl}d_{klm}E_m = c_{ijkl}h_{kl}, \tag{F6}$$

with $\mathcal{E}_{ik} = \varepsilon_{ik} - \frac{\sigma_{ik}}{i\omega}$.

A matrix–vector wave equation in the quasi-static approximation for 2-D piezoelectric waves in an anisotropic stratified medium is given by Honein *et al.* (1991), Wang & Rokhlin (2002) and Zhao *et al.* (2012). Here we derive the exact matrix–vector wave equation for piezoelectric waves in a 3-D inhomogeneous, anisotropic, dissipative, piezoelectric medium. The quantities $-\boldsymbol{\tau}_3$, $\mathbf{E}_0$, $\mathbf{v}$ and $\mathbf{H}_0$ (with $\mathbf{E}_0$ and $\mathbf{H}_0$ defined in Appendix C and $\boldsymbol{\tau}_3$ and $\mathbf{v}$ defined in Appendix D) constitute the power-flux density j in the x_3-direction, via

$$j = \frac{1}{4}(-\boldsymbol{\tau}_3^\dagger\mathbf{v} + \mathbf{E}_0^\dagger\mathbf{H}_0 - \mathbf{v}^\dagger\boldsymbol{\tau}_3 + \mathbf{H}_0^\dagger\mathbf{E}_0) = \frac{1}{4}(-\tau_{i3}^* v_i + E_1^* H_2 - E_2^* H_1 - v_i^*\tau_{i3} + H_2^* E_1 - H_1^* E_2). \tag{F7}$$

We choose these quantities for the 5×1 vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in eqs (3) and (4), hence

$$\mathbf{q}_1 = \begin{pmatrix} -\boldsymbol{\tau}_3 \\ \mathbf{E}_0 \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} \mathbf{v} \\ \mathbf{H}_0 \end{pmatrix}. \tag{F8}$$

To arrive at a set of equations for these quantities, we need to eliminate the remaining wavefield quantities from eqs (F3) to (F6). Using the notation introduced in Appendices C and D, we rewrite eqs (F3)–(F6) as

$$-i\omega\boldsymbol{\mathcal{E}}_1\mathbf{E}_0 - i\omega\boldsymbol{\mathcal{E}}_3 E_3 + \partial_3\mathbf{H}_0 - \nabla_2 H_3 - i\omega\mathbf{D}_{1k}^t\boldsymbol{\tau}_k = -\mathbf{J}_0^e, \tag{F9}$$

$$-i\omega\boldsymbol{\mathcal{E}}_3^t\mathbf{E}_0 - i\omega\mathcal{E}_{33}E_3 + \nabla_1^t\mathbf{H}_0 - i\omega\mathbf{D}_{3k}^t\boldsymbol{\tau}_k = -J_3^e, \tag{F10}$$

$$-i\omega\boldsymbol{\mu}_1\mathbf{H}_0 - i\omega\boldsymbol{\mu}_3 H_3 + \partial_3\mathbf{E}_0 - \nabla_1 E_3 = -\mathbf{J}_0^m, \tag{F11}$$

$$-i\omega\boldsymbol{\mu}_3^t\mathbf{H}_0 - i\omega\mu_{33}H_3 + \nabla_2^t\mathbf{E}_0 = -J_3^m, \tag{F12}$$

$$-i\omega\boldsymbol{\rho}\mathbf{v} - \partial_j\boldsymbol{\tau}_j = \mathbf{f}, \tag{F13}$$

$$i\omega\boldsymbol{\tau}_j + \mathbf{C}_{jl}\partial_l\mathbf{v} + i\omega\mathbf{C}_{jl}\big(\mathbf{D}_{1l}\mathbf{E}_0 + \mathbf{D}_{3l}E_3\big) = \mathbf{C}_{jl}\mathbf{h}_l, \tag{F14}$$

with

$$\mathbf{D}_{1k} = \begin{pmatrix} d_{11k} & d_{12k} \\ d_{21k} & d_{22k} \\ d_{31k} & d_{32k} \end{pmatrix}, \quad \mathbf{D}_{3k} = \begin{pmatrix} d_{13k} \\ d_{23k} \\ d_{33k} \end{pmatrix}. \tag{F15}$$

Eqs (F9)–(F14) form the starting point for deriving matrix–vector equations in the form of eqs (3) and (4), with $\mathbf{q}_1$ and $\mathbf{q}_2$ defined in eq. (F8). The other quantities ($\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, E_3 and H_3) need to be eliminated. The detailed derivation can be found in the supplemental material, section 4.

The 5 × 1 vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are defined as

$$\mathbf{d}_1 = \begin{pmatrix} \mathbf{f} + \frac{1}{i\omega}\partial_\alpha(\mathbf{U}'_{\alpha\beta}\mathbf{h}_\beta) - \frac{1}{i\omega}\partial_\alpha\big((\mathcal{E}'_{33})^{-1}\mathbf{U}_{\alpha\beta}\mathbf{D}_{3\beta}J^e_3\big) \\ \frac{1}{i\omega}\nabla_1\big((\mathcal{E}'_{33})^{-1}(J^e_3 - \mathbf{D}^t_{3\alpha}\mathbf{U}_{\alpha\beta}\mathbf{h}_\beta)\big) - \mathbf{J}^m_0 + \boldsymbol{\mu}_3\mu^{-1}_{33}J^m_3 \end{pmatrix}, \tag{F16}$$

$$\mathbf{d}_2 = \begin{pmatrix} \mathbf{C}^{-1}_{33}\big(\mathbf{C}'_{3m}\mathbf{h}_m - (\mathcal{E}'_{33})^{-1}\mathbf{C}_{3l}\mathbf{D}_{3l}J^e_3\big) \\ -\mathbf{J}^e_0 + (\mathcal{E}'_{33})^{-1}\boldsymbol{\mathcal{E}}'_3J^e_3 + \frac{1}{i\omega}\nabla_2(\mu^{-1}_{33}J^m_3) + (\mathbf{D}'_{1\alpha})^t\mathbf{U}_{\alpha\beta}\mathbf{h}_\beta \end{pmatrix} \tag{F17}$$

and the 5 × 5 operator matrices $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ have the form defined in eqs (E20) and (E21), where

$$\boldsymbol{\mathcal{A}}^{11}_{11} = -\partial_\alpha(\mathbf{C}'_{3\alpha})^t\mathbf{C}^{-1}_{33}, \tag{F18}$$

$$\boldsymbol{\mathcal{A}}^{12}_{11} = -\partial_\alpha\mathbf{U}_{\alpha\beta}\mathbf{D}'_{1\beta}, \tag{F19}$$

$$\boldsymbol{\mathcal{A}}^{21}_{11} = \nabla_1(\mathcal{E}'_{33})^{-1}\mathbf{D}^t_{3k}\mathbf{C}_{k3}\mathbf{C}^{-1}_{33}, \tag{F20}$$

$$\boldsymbol{\mathcal{A}}^{22}_{11} = \boldsymbol{\mu}_3\mu^{-1}_{33}\nabla^t_2 - \nabla_1(\mathcal{E}'_{33})^{-1}(\boldsymbol{\mathcal{E}}'_3)^t, \tag{F21}$$

$$\boldsymbol{\mathcal{A}}^{11}_{12} = i\omega\boldsymbol{\rho} - \frac{1}{i\omega}\partial_\alpha\mathbf{U}'_{\alpha\beta}\partial_\beta, \tag{F22}$$

$$\boldsymbol{\mathcal{A}}^{12}_{12} = -\frac{1}{i\omega}\partial_\alpha(\mathcal{E}'_{33})^{-1}\mathbf{U}_{\alpha\beta}\mathbf{D}_{3\beta}\nabla^t_1, \tag{F23}$$

$$\boldsymbol{\mathcal{A}}^{22}_{12} = i\omega\big(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_3\mu^{-1}_{33}\boldsymbol{\mu}^t_3\big) + \frac{1}{i\omega}\nabla_1(\mathcal{E}'_{33})^{-1}\nabla^t_1, \tag{F24}$$

$$\boldsymbol{\mathcal{A}}^{11}_{21} = i\omega\big(\mathbf{C}^{-1}_{33} - (\mathcal{E}'_{33})^{-1}\mathbf{C}^{-1}_{33}\mathbf{C}_{3l}\mathbf{D}_{3l}\mathbf{D}^t_{3k}\mathbf{C}_{k3}\mathbf{C}^{-1}_{33}\big), \tag{F25}$$

$$\boldsymbol{\mathcal{A}}^{12}_{21} = -i\omega\mathbf{C}^{-1}_{33}\mathbf{C}_{3l}\mathbf{D}'_{1l}, \tag{F26}$$

$$\boldsymbol{\mathcal{A}}^{22}_{21} = i\omega\big(\boldsymbol{\mathcal{E}}'_1 - \boldsymbol{\mathcal{E}}'_3(\mathcal{E}'_{33})^{-1}(\boldsymbol{\mathcal{E}}'_3)^t\big) + \frac{1}{i\omega}\nabla_2\mu^{-1}_{33}\nabla^t_2, \tag{F27}$$

with

$$\mathbf{U}_{\alpha\beta} = \mathbf{C}_{\alpha\beta} - \mathbf{C}_{\alpha3}\mathbf{C}^{-1}_{33}\mathbf{C}_{3\beta}, \tag{F28}$$

$$\boldsymbol{\mathcal{E}}'_1 = \boldsymbol{\mathcal{E}}_1 - \mathbf{D}^t_{1\alpha}\mathbf{U}_{\alpha\beta}\mathbf{D}_{1\beta}, \tag{F29}$$

$$\boldsymbol{\mathcal{E}}'_3 = \boldsymbol{\mathcal{E}}_3 - \mathbf{D}^t_{1\alpha}\mathbf{U}_{\alpha\beta}\mathbf{D}_{3\beta}, \tag{F30}$$

$$\mathcal{E}'_{33} = \mathcal{E}_{33} - \mathbf{D}^t_{3\alpha}\mathbf{U}_{\alpha\beta}\mathbf{D}_{3\beta}, \tag{F31}$$

$$\mathbf{U}'_{\alpha\beta} = \mathbf{U}_{\alpha\beta} + (\mathcal{E}'_{33})^{-1}\mathbf{U}_{\alpha\gamma}\mathbf{D}_{3\gamma}\mathbf{D}^t_{3\delta}\mathbf{U}_{\delta\beta}, \tag{F32}$$

$$\mathbf{C}'_{3m} = \mathbf{C}_{3m} + (\mathcal{E}'_{33})^{-1}\mathbf{C}_{3l}\mathbf{D}_{3l}\mathbf{D}^t_{3\alpha}\mathbf{U}_{\alpha m}, \tag{F33}$$

$$\mathbf{D}'_{1l} = \mathbf{D}_{1l} - (\mathcal{E}'_{33})^{-1}\mathbf{D}_{3l}(\boldsymbol{\mathcal{E}}'_3)^t, \tag{F34}$$

with $\mathbf{U}_{\alpha 3} = \mathbf{O}$. Operators $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ obey the symmetry relations formulated in eqs (16)–(18). We defined the adjoints of the medium parameters $\mathcal{E}_{ik}$, μ_{km}, c_{ijkl} and ρ_{ij} in Appendices C and D. Moreover, we define $\bar{d}_{ijk} = d^*_{ijk}$. Similar relations hold for $\boldsymbol{\mathcal{E}}_1$, $\boldsymbol{\mathcal{E}}_3$, $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_3$, $\mathbf{C}_{jl}$, $\boldsymbol{\rho}$, $\mathbf{D}_{1k}$ and $\mathbf{D}_{3k}$, which contain the parameters $\mathcal{E}_{ik}$, μ_{km}, c_{ijkl}, ρ_{ij} and d_{ijk}. Operators $\bar{\boldsymbol{\mathcal{A}}}_{11}$, $\bar{\boldsymbol{\mathcal{A}}}_{12}$, $\bar{\boldsymbol{\mathcal{A}}}_{21}$ and $\bar{\boldsymbol{\mathcal{A}}}_{22}$ in the adjoint medium obey relations (19)–(22).

APPENDIX G: SEISMOELECTRIC WAVES

Seismoelectric waves are governed by the equations for electromagnetic waves (Appendix C) and poroelastodynamic waves (Appendix E), in which two of the constitutive relations need to be modified to account for the coupling between the two wave types. In this appendix we consider an isotropic medium (the derivation for the anisotropic situation is disproportionally long). For seismoelectric waves, the modified constitutive relations are (Pride 1994; Pride & Haartsen 1996)

$$J_i = \sigma E_i + L(-\partial_i p^f + i\omega\rho^f v_i^s + f_i^f), \tag{G1}$$

$$w_i = L E_i + \frac{k}{\eta}(-\partial_i p^f + i\omega\rho^f v_i^s + f_i^f). \tag{G2}$$

The field quantities, sources and medium parameters (except L) have been defined in Appendices C and E (except that tensors are now replaced by scalars). Here $L = L(\mathbf{x}, \omega)$ accounts for the coupling between the elastodynamic and electromagnetic waves and vice versa. Eqs (G1) and (G2) contain the same coupling coefficient L (due to Onsager's reciprocity relation, Pride (1994)). Eq. (G1) replaces the isotropic version of constitutive relation (C5) and is substituted, together with the isotropic versions of constitutive relations (C3) and (C4), into the Maxwell eqs (C1) and (C2). Eq. (G2) replaces the isotropic version of eq. (E2). The basic equations for coupled electromagnetic and poroelastodynamic waves thus read

$$-i\omega\rho^b v_i^s - i\omega\rho^f w_i - \partial_j\tau_{ij}^b = f_i^b, \tag{G3}$$

$$-i\omega\rho^f v_i^s + \frac{\eta}{k}(w_i - L E_i) + \partial_i p^f = f_i^f, \tag{G4}$$

$$i\omega\tau_{ij}^b + c_{ijkl}\partial_l v_k^s + C\delta_{ij}\partial_k w_k = c_{ijkl}h_{kl}^b + C\delta_{ij}q^f, \tag{G5}$$

$$-i\omega p^f + C\delta_{kl}\partial_l v_k^s + M\partial_k w_k = C\delta_{kl}h_{kl}^b + Mq^f, \tag{G6}$$

$$-i\omega\varepsilon E_i + \sigma E_i + L(-\partial_i p^f + i\omega\rho^f v_i^s) - \epsilon_{ijk}\partial_j H_k = -J_i^e - L f_i^f, \tag{G7}$$

$$-i\omega\mu H_k + \epsilon_{klm}\partial_l E_m = -J_k^m. \tag{G8}$$

For the isotropic medium we have

$$c_{ijkl} = (K_G - \frac{2}{3}G_{\rm fr})\delta_{ij}\delta_{kl} + G_{\rm fr}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{G9}$$

where $G_{\rm fr}$ is the shear modulus of the framework of the grains when the fluid is absent and K_G is the Gassmann modulus (Pride *et al.* 1992). The permittivity and permeability are defined as $\varepsilon = \varepsilon_0\varepsilon_{\rm r}$ and $\mu = \mu_0\mu_{\rm r}$. The subscripts 0 refer to the parameters in vacuum and the subscripts r denote relative parameters. For $\varepsilon_{\rm r}$ and $\mu_{\rm r}$ we have (Pride 1994)

$$\varepsilon_{\rm r} = \frac{\phi}{\alpha_\infty}(\kappa^f - \kappa^s) + \kappa^s, \tag{G10}$$

$$\mu_{\rm r} \approx 1, \tag{G11}$$

where κ^f and κ^s are the dielectric parameters of the fluid and solid, respectively, and α_∞ is the tortuosity at infinite frequency. For later convenience, we eliminate $\partial_k w_k$ from eq. (G5), using eq. (G6). This yields

$$i\omega\tau_{ij}^b + c'_{ijkl}\partial_l v_k^s + i\omega\frac{C}{M}\delta_{ij}p^f = c'_{ijkl}h_{kl}^b, \tag{G12}$$

with $c'_{ijkl} = c'_{ijkl}(\mathbf{x}, \omega)$ defined as

$$c'_{ijkl} = c_{ijkl} - \frac{C^2}{M}\delta_{ij}\delta_{kl}. \tag{G13}$$

Also for later convenience, we add L times eq. (G4) to eq. (G7) in order to compensate for the term $L(-\partial_i p^f + i\omega\rho^f v_i^s)$. This yields

$$-i\omega\mathcal{E}E_i + \frac{\eta}{k}Lw_i - \epsilon_{ijk}\partial_j H_k = -J_i^e, \tag{G14}$$

with

$$\mathcal{E} = \varepsilon - \frac{1}{i\omega}\big(\sigma - \frac{\eta}{k}L^2\big). \tag{G15}$$

A matrix–vector wave equation for oblique-incidence seismoelectric waves in a stratified isotropic medium, separately for P-SV-TM and SH-TE propagation, is given by Haartsen & Pride (1997), White & Zhou (2006) and Grobbe (2016). Here we derive the matrix–vector wave equation for a 3-D inhomogeneous, isotropic, dissipative, fluid-saturated porous solid. The quantities $-\boldsymbol{\tau}_3^b$, p^f, $\mathbf{E}_0$, $\mathbf{v}^s$, w_3 and $\mathbf{H}_0$ constitute the power-flux density j in the x_3-direction, via

$$\begin{aligned} j &= \frac{1}{4}(-(\boldsymbol{\tau}_3^b)^\dagger\mathbf{v}^s + p^{f*}w_3 + \mathbf{E}_0^\dagger\mathbf{H}_0 - (\mathbf{v}^s)^\dagger\boldsymbol{\tau}_3^b + w_3^* p^f + \mathbf{H}_0^\dagger\mathbf{E}_0) \\ &= \frac{1}{4}(-\tau_{i3}^{b*}v_i^s + p^{f*}w_3 + E_1^*H_2 - E_2^*H_1 - v_i^{s*}\tau_{i3}^b + w_3^*p^f + H_2^*E_1 - H_1^*E_2). \end{aligned} \tag{G16}$$

We choose these quantities for the 6×1 vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in eqs (3) and (4), hence

$$\mathbf{q}_1 = \begin{pmatrix} -\boldsymbol{\tau}_3^b \\ p^f \\ \mathbf{E}_0 \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} \mathbf{v}^s \\ w_3 \\ \mathbf{H}_0 \end{pmatrix}. \tag{G17}$$

To arrive at a set of equations for these quantities, we need to eliminate the remaining wavefield quantities from eqs (G3), (G4), (G12), (G6), (G14) and (G8). We start by rewriting these equations as

$$-i\omega\rho^b\mathbf{v}^s - i\omega\rho^f\mathbf{i}_j w_j - \partial_j\boldsymbol{\tau}_j^b = \mathbf{f}^b, \tag{G18}$$

$$-i\omega\rho^f\mathbf{i}_i^t\mathbf{v}^s + \frac{\eta}{k}\big(w_i - L(\mathbf{j}_i^t\mathbf{E}_0 + \delta_{3i}E_3)\big) + \partial_i p^f = f_i^f, \tag{G19}$$

$$i\omega\boldsymbol{\tau}_j^b + \mathbf{C}_{jl}\partial_l\mathbf{v}^s + i\omega\frac{C}{M}\mathbf{i}_j p^f = \mathbf{C}_{jl}\mathbf{h}_l^b, \tag{G20}$$

$$-i\omega p^f + C\mathbf{i}_l^t\partial_l\mathbf{v}^s + M\partial_k w_k = C\mathbf{i}_l^t\mathbf{h}_l^b + Mq^f, \tag{G21}$$

$$-i\omega\mathcal{E}\mathbf{E}_0 + \frac{\eta}{k}L\mathbf{j}_\alpha w_\alpha + \partial_3\mathbf{H}_0 - \nabla_2 H_3 = -\mathbf{J}_0^e, \tag{G22}$$

$$-i\omega\mathcal{E}E_3 + \frac{\eta}{k}Lw_3 + \nabla_1^t\mathbf{H}_0 = -J_3^e, \tag{G23}$$

$$-i\omega\mu\mathbf{H}_0 + \partial_3\mathbf{E}_0 - \nabla_1 E_3 = -\mathbf{J}_0^m, \tag{G24}$$

$$-i\omega\mu H_3 + \nabla_2^t\mathbf{E}_0 = -J_3^m, \tag{G25}$$

with most of the vectors and matrices defined in Appendices C and E. In addition, $\mathbf{j}_i$ is a 2×1 unit vector, with $(\mathbf{j}_i)_\beta = \delta_{\beta i}$. Eqs (G18)–(G25) form the starting point for deriving matrix–vector equations in the form of eqs (3) and (4), with $\mathbf{q}_1$ and $\mathbf{q}_2$ defined in eq. (G17). The other quantities ($\boldsymbol{\tau}_1^b$, $\boldsymbol{\tau}_2^b$, E_3, w_1, w_2 and H_3) need to be eliminated. The detailed derivation can be found in the supplemental material, Section 5. The 6×1 vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are defined as

$$\mathbf{d}_1 = \begin{pmatrix} \mathbf{f}^b + i\omega\rho^f\frac{k}{\eta}\mathbf{i}_\alpha f_\alpha^f + \frac{1}{i\omega}\partial_\alpha\big(\mathbf{U}_{\alpha\beta}\mathbf{h}_\beta^b\big) \\ \frac{1}{i\omega\mathcal{E}}\frac{\eta}{k}LJ_3^e + f_3^f \\ -\mathbf{J}_0^m + \nabla_1\big(\frac{1}{i\omega\mathcal{E}}J_3^e\big) \end{pmatrix}, \tag{G26}$$

$$\mathbf{d}_2 = \begin{pmatrix} \mathbf{C}_{33}^{-1}\mathbf{C}_{3l}\mathbf{h}_l^b \\ q^f + \frac{1}{M}\mathbf{u}_\alpha^t\mathbf{h}_\alpha^b - \partial_\beta\left(\frac{k}{\eta}f_\beta^f\right) \\ -\mathbf{J}_0^e + \nabla_2\left(\frac{1}{i\omega\mu}J_3^m\right) - L\mathbf{j}_\alpha f_\alpha^f \end{pmatrix} \tag{G27}$$

and the 6 × 6 operator matrices $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ as

$$\boldsymbol{\mathcal{A}}_{11} = \begin{pmatrix} \boldsymbol{\mathcal{A}}_{11}^{11} & \boldsymbol{\mathcal{A}}_{11}^{12} & \boldsymbol{\mathcal{A}}_{11}^{13} \\ \mathbf{0}^t & 0 & \mathbf{0}^t \\ \mathbf{O} & \mathbf{0} & \mathbf{O} \end{pmatrix}, \quad \boldsymbol{\mathcal{A}}_{12} = \begin{pmatrix} \boldsymbol{\mathcal{A}}_{12}^{11} & \boldsymbol{\mathcal{A}}_{12}^{12} & \mathbf{O} \\ (\boldsymbol{\mathcal{A}}_{12}^{12})^t & \boldsymbol{\mathcal{A}}_{12}^{22} & \boldsymbol{\mathcal{A}}_{12}^{23} \\ \mathbf{O} & (\boldsymbol{\mathcal{A}}_{12}^{23})^t & \boldsymbol{\mathcal{A}}_{12}^{33} \end{pmatrix}, \tag{G28}$$

$$\boldsymbol{\mathcal{A}}_{21} = \begin{pmatrix} \boldsymbol{\mathcal{A}}_{21}^{11} & \boldsymbol{\mathcal{A}}_{21}^{12} & \mathbf{O} \\ (\boldsymbol{\mathcal{A}}_{21}^{12})^t & \boldsymbol{\mathcal{A}}_{21}^{22} & \boldsymbol{\mathcal{A}}_{21}^{23} \\ \mathbf{O} & (\boldsymbol{\mathcal{A}}_{21}^{23})^t & \boldsymbol{\mathcal{A}}_{21}^{33} \end{pmatrix}, \quad \boldsymbol{\mathcal{A}}_{22} = -\boldsymbol{\mathcal{A}}_{11}^t. \tag{G29}$$

Here

$$\boldsymbol{\mathcal{A}}_{11}^{11} = -\partial_\alpha\mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}, \tag{G30}$$

$$\boldsymbol{\mathcal{A}}_{11}^{12} = -i\omega\rho^f\frac{k}{\eta}\mathbf{i}_\alpha\partial_\alpha - \partial_\alpha\frac{1}{M}\mathbf{u}_\alpha, \tag{G31}$$

$$\boldsymbol{\mathcal{A}}_{11}^{13} = i\omega\rho^f L\mathbf{i}_\alpha\mathbf{j}_\alpha^t, \tag{G32}$$

$$\boldsymbol{\mathcal{A}}_{12}^{11} = -\frac{1}{i\omega}\partial_\alpha\mathbf{U}_{\alpha\beta}\partial_\beta + i\omega\left(\rho^b\mathbf{I}_3 + i\omega(\rho^f)^2\frac{k}{\eta}\mathbf{i}_\alpha\mathbf{i}_\alpha^t\right), \tag{G33}$$

$$\boldsymbol{\mathcal{A}}_{12}^{12} = i\omega\rho^f\mathbf{i}_3, \tag{G34}$$

$$\boldsymbol{\mathcal{A}}_{12}^{22} = -\frac{\eta}{k}\left(1 - \frac{1}{i\omega\mathcal{E}}\frac{\eta}{k}L^2\right), \tag{G35}$$

$$\boldsymbol{\mathcal{A}}_{12}^{23} = \frac{1}{i\omega\mathcal{E}}\frac{\eta}{k}L\nabla_1^t, \tag{G36}$$

$$\boldsymbol{\mathcal{A}}_{12}^{33} = i\omega\mu\mathbf{I}_2 + \nabla_1\frac{1}{i\omega\mathcal{E}}\nabla_1^t, \tag{G37}$$

$$\boldsymbol{\mathcal{A}}_{21}^{11} = i\omega\mathbf{C}_{33}^{-1}, \tag{G38}$$

$$\boldsymbol{\mathcal{A}}_{21}^{12} = -i\omega\frac{C}{M}\mathbf{C}_{33}^{-1}\mathbf{i}_3, \tag{G39}$$

$$\boldsymbol{\mathcal{A}}_{21}^{22} = i\omega\frac{C^2}{M^2}\mathbf{i}_3^t\mathbf{C}_{33}^{-1}\mathbf{i}_3 + \frac{i\omega}{M} + \partial_\beta\frac{k}{\eta}\partial_\beta, \tag{G40}$$

$$\boldsymbol{\mathcal{A}}_{21}^{23} = -\partial_\beta L\mathbf{j}_\beta^t, \tag{G41}$$

$$\boldsymbol{\mathcal{A}}_{21}^{33} = \left(i\omega\mathcal{E} - \frac{\eta}{k}L^2\right)\mathbf{I}_2 + \nabla_2\frac{1}{i\omega\mu}\nabla_2^t, \tag{G42}$$

where $\mathbf{I}_3$ is a 3 × 3 identity matrix, $\mathbf{I}_2$ a 2 × 2 identity matrix and

$$\mathbf{U}_{\alpha\beta} = \mathbf{C}_{\alpha\beta} - \mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}\mathbf{C}_{3\beta}, \tag{G43}$$

$$\mathbf{u}_\alpha = C(\mathbf{i}_\alpha - \mathbf{C}_{\alpha 3}\mathbf{C}_{33}^{-1}\mathbf{i}_3). \tag{G44}$$

Operators $\boldsymbol{\mathcal{A}}_{11}$, $\boldsymbol{\mathcal{A}}_{12}$, $\boldsymbol{\mathcal{A}}_{21}$ and $\boldsymbol{\mathcal{A}}_{22}$ obey the symmetry relations formulated in eqs (16)–(18). We define the adjoints of the medium parameters ε, μ, σ, c_{ijkl}, ρ (with superscript b or f), k, η, C and M similar as in Appendices C, D and E but for the isotropic situation. Moreover, we define $\bar{L} = -L^*$. Operators $\bar{\boldsymbol{\mathcal{A}}}_{11}$, $\bar{\boldsymbol{\mathcal{A}}}_{12}$, $\bar{\boldsymbol{\mathcal{A}}}_{21}$ and $\bar{\boldsymbol{\mathcal{A}}}_{22}$ in the adjoint medium obey relations (19)–(22). Explicit expressions for the operator matrices are given in the supplemental material, Section 5.

APPENDIX H: SYMMETRY PROPERTIES OF THE OPERATOR MATRIX IN THE WAVENUMBER-FREQUENCY DOMAIN

We derive symmetry properties of the operator matrix for the special case of a laterally invariant medium (or potential) in the wavenumber-frequency domain. We define the spatial Fourier transform of a space- and frequency-dependent quantity $h(\mathbf{x}, \omega)$ as

$$\tilde{h}(k_\alpha, x_3, \omega) = \int_{\mathbb{A}} h(\mathbf{x}, \omega)\exp(-ik_\alpha x_\alpha)\mathrm{d}^2\mathbf{x}_\mathrm{H}, \tag{H1}$$

with k_α for $\alpha = 1, 2$ representing the horizontal wavenumbers. Lateral derivatives $\partial_\alpha h(\mathbf{x}, \omega)$ in the space-frequency domain are replaced by products $ik_\alpha\tilde{h}(k_\alpha, x_3, \omega)$ in the wavenumber-frequency domain. We denote the Fourier transform of ∂_α as $\partial_\alpha \Rightarrow ik_\alpha$. Similarly, for an operator matrix $\boldsymbol{\mathcal{U}}$ in a laterally invariant medium, containing the differential operator ∂_α, we denote the Fourier transform as $\boldsymbol{\mathcal{U}}(\partial_\alpha) \Rightarrow \tilde{\boldsymbol{\mathcal{U}}}(k_\alpha)$. Using eqs (10) and (15), we find

$$\{\boldsymbol{\mathcal{U}}(\partial_\alpha)\}^t \Rightarrow \{\tilde{\boldsymbol{\mathcal{U}}}(-k_\alpha)\}^t, \tag{H2}$$

$$\{\boldsymbol{\mathcal{U}}(\partial_\alpha)\}^* \Rightarrow \{\tilde{\boldsymbol{\mathcal{U}}}(-k_\alpha)\}^*, \tag{H3}$$

$$\{\boldsymbol{\mathcal{U}}(\partial_\alpha)\}^\dagger \Rightarrow \{\tilde{\boldsymbol{\mathcal{U}}}(k_\alpha)\}^\dagger. \tag{H4}$$

We use eq. (H1) to transform eq. (1) to the wavenumber-frequency domain, according to $\partial_3\tilde{\mathbf{q}} = \tilde{\boldsymbol{\mathcal{A}}}\tilde{\mathbf{q}} + \tilde{\mathbf{d}}$. We find the symmetry properties of $\tilde{\boldsymbol{\mathcal{A}}}(k_\alpha, x_3, \omega)$ by applying eqs (H2)–(H4) to the left-hand sides of eqs (27)–(29). This gives

$$\{\tilde{\boldsymbol{\mathcal{A}}}(-k_\alpha, x_3, \omega)\}^t\mathbf{N} = -\mathbf{N}\tilde{\boldsymbol{\mathcal{A}}}(k_\alpha, x_3, \omega), \tag{H5}$$

$$\{\tilde{\boldsymbol{\mathcal{A}}}(-k_\alpha, x_3, \omega)\}^*\mathbf{J} = \mathbf{J}\tilde{\bar{\boldsymbol{\mathcal{A}}}}(k_\alpha, x_3, \omega), \tag{H6}$$

$$\{\tilde{\boldsymbol{\mathcal{A}}}(k_\alpha, x_3, \omega)\}^\dagger\mathbf{K} = -\mathbf{K}\tilde{\bar{\boldsymbol{\mathcal{A}}}}(k_\alpha, x_3, \omega). \tag{H7}$$