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Learning generalized Nash equilibria in multi-agent dynamical systems via extremum seeking control[☆]

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ABSTRACT

In this paper, we consider the problem of learning a generalized Nash equilibrium (GNE) in strongly monotone games. First, we propose semi-decentralized and distributed continuous-time solution algorithms that use regular projections and first-order information to compute a GNE with and without a central coordinator. As the second main contribution, we design a data-driven variant of the former semi-decentralized algorithm where each agent estimates their individual pseudogradient via zeroth-order information, namely, measurements of their individual cost function values, as typical of extremum seeking control. Third, we generalize our setup and results for multi-agent systems with nonlinear dynamics. Finally, we apply our methods to connectivity control in robotic sensor networks and almost-decentralized wind farm optimization.

1. Introduction

Multi-agent optimization problems and games with self-interested decision-makers or agents appear in many engineering applications, such as demand-side management in smart grids (Mohsenian-Rad, Wong, Jatskevich, Schober, & Leon-Garcia, 2010; Saad, Han, Poor, & Basar, 2012), charging/discharging coordination for plug-in electric vehicles (Ma, Callaway, & Hiskens, 2011), thermostatically controlled loads (Li, Zhang, Lian, & Kalsi, 2015a, 2015b) and robotic formation control (Lin, Qu, & Simaan, 2014). Typically, in these games, the cost functions and the constraints of the agents are coupled together, e.g. due to common congestion penalties and shared resource capacity, respectively. Since the agents are self-interested, their interaction might be unstable. Thus, one main research area is that of finding (seeking) agent decisions that are self-enforceable, e.g. decisions such that no agent has an incentive to deviate from — the so-called Generalized Nash equilibrium (GNE) (Facchinei & Kanzow, 2010). From a control-theoretic perspective, in the presence of dynamical agents, the main challenge is to design distributed, possibly almost decentralized, control laws that ensure both the convergence of the agent decisions to a GNE and the asymptotic stability of the equilibrium for the agent dynamics.

Literature review: The literature on generalized Nash equilibrium problems (GNEPs) is vast, see Facchinei and Kanzow (2010) for a survey. What differentiates GNEPs from Nash equilibrium problems (NEPs) is the presence of shared constraints. Although the difference seems minor, it introduces several technical challenges. The main one is that the primal–dual Lagrangian reformulation of the GNEP, which is necessary to decouple the coupling constraints, *does not preserve the strong monotonicity of the* (extended) *pseudogradient* (Ryu & Boyd, 2016, p. 13), the usual background assumption in the NEP literature, as this is a sufficient condition for the projected pseudogradient descent to converge (Yi & Pavel, 2019, Lemma 5), Bauschke, Combettes, et al. (2011, Thm. 26.14). For a special class of GNEPs in so-called aggregative games this issue can be avoided. In these games, each cost function depends on the local decision and on the aggregate (e.g. average) of the decisions of all (other) agents. Various semi-distributed (Belgioioso & Grammatico, 2017) and decentralized (Belgioioso, Nedich, & Grammatico, 2020; Gadjov & Pavel, 2019) algorithms have been developed for NE seeking in aggregative games. The aforementioned technical challenge has only been addressed recently in Yi and Pavel (2019) by applying a preconditioning matrix on the operators. In turn, in Yi and Pavel (2019) the authors propose a GNE seeking algorithm in games with linear coupling constraints. In Franci, Staudigl, and Grammatico (2020), the authors overcome the lack of strong monotonicity by adopting an algorithm (the forward–backward–forward) with weaker assumptions on the projected pseudogradient, at the expense of one additional computation of the pseudogradient at each iteration.

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In most of the literature and all of previously mentioned work, GNE seeking algorithms are designed in discrete-time and for static agents, i.e., where the agent costs instantaneously reflect the chosen decisions. However, this is not the case when the cost functions depend on some internal states of the agents and not on their decisions (control inputs). Let us refer to this class of agents as dynamical agents. The two main approaches to reach a (G)NE for dynamical agents are passivity-based *first-order* algorithms and payoff-based *zeroth-order* algorithms. By using a passivity property, in Gadjov and Pavel (2018) and Romano and Pavel (2019), Pavel and co-authors design a control law that guarantees convergence to a Nash equilibrium (NE) in a multi-agent system with single and multi-integrator dynamics over a time-invariant network. With the same goal, in De Persis and Grammatico (2019b), De Persis and Grammatico relax the network connectivity assumption in Gadjov and Pavel (2018) by designing a network weight adaptation scheme. In Bianchi and Grammatico (2021), the authors extend the convergence results to GNEPs for the first time via a preconditioning approach as in Yi and Pavel (2019) and the use of non-Lipschitz continuous projections onto tangents cones.

In payoff-based algorithms, each agent can only measure the value of their cost function, but does not know its analytic form. Many of such algorithms are designed for NEPs with static agents with finite action spaces, e.g. Goto, Hatanaka, and Fujita (2012), Marden, Arslan, and Shamma (2009) and Marden and Shamma (2012). In the case of continuous action spaces, the measurements of the cost functions are often used to estimate the pseudo-gradients. Perhaps the most popular class of control algorithms that exploits this principle is that of extremum seeking control (ESC). The main idea is to use perturbation signals to "excite" the cost function and estimate its gradient which is then used in a gradient-descent-like algorithm (Dürr, Stanković, Ebenbauer, & Johansson, 2013; Guay & Dochain, 2015; Krstić & Wang, 2000). As ESC estimates only one value of the (pseudo)gradient in a time instant, it is not possible to adopt it in algorithms that require multiple (pseudo)gradient computations. ESC was used for non-generalized NE seeking in Frihauf, Krstic, and Basar (2011) where the proposed algorithm is proven to converge to a neighborhood of a NE for nonlinear dynamical agents. The results are extended in Liu and Krstić (2011) to include stochastic perturbation signals. In Poveda and Teel (2017), Poveda and Teel propose a framework for the synthesis of a hybrid controller which could also be used for NEPs with nonlinear dynamical agents. For a class of NEPs called N-cluster games, the authors in Ye, Hu, and Xu (2020) propose an ESC-based algorithm. The extension of these algorithms to GNEPs is nontrivial. Only for a special class of so-called population games, the authors in Poveda and Quijano (2015) propose an approach based on Shahshahani gradients. In fact, there is still no methodology on data-driven (zeroth-order) GNE learning in strongly monotone games for nonlinear dynamical agents. The reasons for that are technical: (i) the lack of strong monotonicity of the extended pseudogradient in primal–dual framework; (ii) additional pseudogradient computations necessary in other dynamics; (iii) the incompatibility of projections with available extremum seeking techniques. In fact, in Frihauf et al. (2011), Frihauf, Krstić and Başar specifically mention: "*Several challenges remain in the development of convergence proofs for Nash seeking players with projection*".

Contribution: Motivated by the above literature and open research problem, to the best of our knowledge, we consider and solve for the first time the problem of learning a GNE in strongly monotone games with nonlinear dynamical agents. Specifically, our main technical contributions are summarized next:

- We design novel continuous-time GNE seeking algorithms (Section 3.1, Section 3.2), which use projections onto fixed convex sets instead of projections onto state-dependent tangent cones as in Bianchi and Grammatico (2021). In this way, the state flow is Lipschitz continuous and admits solutions in the classical sense. We overcome the lack of strong monotonicity of the primal–dual pseudogradient thanks to a suitable preconditioning of the operators defining the optimality conditions.
- We design an extremum seeking scheme that learns a GNE in strongly monotone games with static agents who perform local computations and receive, broadcast information from a central coordinator (Section 3.3). Differently from Guay and Dochain (2015), where an optimization problem is considered, we study a noncooperative game. Furthermore, we prove that, with a time-scale separation, our algorithm learns a GNE in (strongly) monotone games with nonlinear dynamical agents (Section 4).

We also apply for the first time semi-decentralized GNE learning to the robot connectivity problem and to wind farm optimization (Section 5).

Notation: $\mathbb{R}$ denotes the set of real numbers. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ and $\|A\|$ denote its transpose and maximum singular value respectively. For vectors $x, y \in \mathbb{R}^n$, $x^\top y$ and $\|x\|$ denote the Euclidean inner product and norm, respectively. We denote the unit ball set as $\mathbb{B} := \{x \in \mathbb{R}^N \mid \|x\| \leq 1\}$. Given N vectors $x_1, \ldots, x_N$, $\mathrm{col}(x_1, \ldots x_N) := \left[x_1^\top, \ldots, x_N^\top\right]^\top$. Collective vectors are defined as $\boldsymbol{x} := \mathrm{col}(x_1, \ldots, x_N)$ and for each $i = 1, \ldots, N$, $\boldsymbol{x}_{-i} := \mathrm{col}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. Given N matrices $A_1, A_2, \ldots, A_N$, $\mathrm{diag}(A_1, \ldots, A_N)$ denotes the block diagonal matrix with A_i on its diagonal. For a function $v : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ differentiable in the first argument, we denote the partial gradient vector as $\nabla_x v(x, y) := \mathrm{col}\left(\frac{\partial v(x,y)}{\partial x_1}, \ldots, \frac{\partial v(x,y)}{\partial x_N}\right) \in \mathbb{R}^n$. Maximal and minimal eigenvalues of matrix A are denoted as $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ respectively. The mapping $\mathrm{proj}_S : \mathbb{R}^n \to S$ denotes the projection onto a closed convex set S, i.e., $\mathrm{proj}_S(v) := \mathrm{argmin}_{y\in S} \|y - v\|$. The set-valued mapping $\mathrm{N}_S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the normal cone operator for the set $S \subseteq \mathbb{R}^n$, i.e., $\mathrm{N}_S(x) := \varnothing$ if $x \notin S$, $\mathrm{N}_S(x) := \left\{v \in \mathbb{R}^n \middle| \sup_{z\in S} v^\top(z - x) \leq 0\right\}$ otherwise. Id is the identity operator. I_n is the identity matrix of dimension n and $\mathbf{0}_n$ is vector column of n zeros. The non-negative orthant is defined as $\mathbb{R}^n_+ := \{v \in \mathbb{R}^n \mid v \succeq \mathbf{0}_n\}$. For a set $\mathcal{M} := \{1, \ldots, M\}$ and a vector-valued function $\boldsymbol{\phi} := \mathrm{col}((\phi_i(\cdot))_{i\in\mathcal{M}}) : \mathbb{R} \to \mathbb{R}^M$, we denote $D^+\phi(t) := \mathrm{col}\left((\limsup_{h\to 0^+} \frac{\phi_i(t+h)-\phi_i(t)}{h})_{i\in\mathcal{M}}\right)$.

2. Multi-agent dynamical systems

We consider an N agents multi-agent system indexed by $i \in \mathcal{I} = \{1, 2, \ldots, N\}$, each with the following dynamics:

$$\dot{x}_i = f_i(x_i, u_i) \tag{1a}$$

$$y_i = h_i(x_i, \boldsymbol{x}_{-i}) \tag{1b}$$

where $x_i \in \mathcal{X}_i \subset \mathbb{R}^{n_i}$ is the state variable, $u_i \in \Omega_i \subset \mathbb{R}^{m_i}$ is the control input (decision variable), $y_i \in \mathbb{R}$ is the output variable which evaluates the cost function $h_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_{-i}} \to \mathbb{R}$, and $f_i : \mathcal{X}_i \times \Omega_i \to \mathbb{R}^{n_i}$ is the state flow mapping. Let us also define $n := \sum n_i$, $m := \sum m_i$ and $n_{-i} := \sum_{j\neq i} n_j$.

To ensure existence and uniqueness of the solutions to (1a), we make a common assumption in the nonlinear system literature (Khalil, 2002, Thm. 3.3):

Assumption 1 (*Local Lipschitz Continuity*). For each $i \in \mathcal{I}$, f_i is locally Lipschitz continuous. □

Furthermore, we assume that the decision variables of the agents are subject to local constraints $u_i \in \Omega_i$ and coupling constraints $A\boldsymbol{u} \leq b$, where $A \in \mathbb{R}^{q\times m}$, $b \in \mathbb{R}^q$, and $\boldsymbol{u} := \mathrm{col}\left((u_i)_{i\in\mathcal{I}}\right)$ collects all the control inputs. Let us denote the collection of local constraints as

$$\boldsymbol{\Omega} := \Omega_1 \times \cdots \times \Omega_N. \tag{2}$$

As the decision variables are also coupled together, the overall feasible decision set $\boldsymbol{\mathcal{U}}$ is contained in $\boldsymbol{\Omega}$, i.e.

$$\boldsymbol{\mathcal{U}} := \boldsymbol{\Omega} \cap \left\{\boldsymbol{u} \in \mathbb{R}^m \mid A\boldsymbol{u} \leq b\right\}, \tag{3}$$

Let us also denote the feasible set of each agent i as

$$\mathcal{U}_i(\boldsymbol{u}_{-i}) := \Omega_i \cap \left\{u_i \in \mathbb{R}^{m_i} \mid A\boldsymbol{u} \leq b\right\}. \tag{4}$$

In equilibrium seeking problems, we can consider only equilibrium points of the nonlinear systems as possible solutions. Here we consider the setting where agents have a continuum of possible equilibria. In order to characterize them, we assume they are input-dependent points, similarly to reference tracking problems. In fact, this motivates a common assumption amongst the extremum seeking literature (e.g. Guay and Dochain (2017, Equ. 3), Krstić and Wang (2000, Ass. 2.1), Poveda and Teel (2017, Ass. 2)), namely the existence of the steady-state mappings which characterizes the behavior of the systems for a constant input.

Standing Assumption 1 (*Steady-state Mapping*). *For each $i \in \mathcal{I}$, there is a differentiable mapping $\pi_i : \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ (called the steady-state mapping) such that for every $\overline{u}_i \in \Omega_i$, it holds that $f_i(\pi_i(\overline{u}_i), \overline{u}_i) = 0$.* □

By using the previous definition, let us also define the collective steady-state mappings

$$\begin{aligned}\pi(\boldsymbol{u}) &:= \mathrm{col}\left((\pi_i(u_i))_{i\in\mathcal{I}}\right),\\ \pi_{-i}(\boldsymbol{u}_{-i}) &:= \mathrm{col}\left(\left(\pi_j(u_j)\right)_{j\in\mathcal{I}\setminus\{i\}}\right).\end{aligned} \tag{5}$$

Another common assumption in ESC is the (local) exponential stability of the equilibrium points $\pi_i(u_i)$, under constant input ($\dot{u}_i = 0$) (Krstić & Wang, 2000, Ass. 2.2), (Frihauf et al., 2011, Ass. 4.2). Thus, with the change of coordinates $z_i := x_i - \pi_i(u_i)$, we adopt the following assumption throughout the paper:

Standing Assumption 2 (*Lyapunov Stability*). *For each $i \in \mathcal{I}$, there is a smooth Lyapunov function, $z_i \mapsto V_i(z_i, u_i)$, with Lipschitz continuous partial derivatives, i.e. for every constant $\overline{u}_i \in \mathcal{U}_i$, it holds that*

$$\begin{aligned}\underline{\alpha}_i\|z_i\|^2 \leq V_i(z_i, \overline{u}_i) &\leq \overline{\alpha}_i\|z_i\|^2\\ \frac{\partial V_i}{\partial z_i}(z_i, \overline{u}_i)^\top f_i(z_i + \pi_i(\overline{u}_i), \overline{u}_i) &\leq -\kappa_i\|z_i\|^2\\ \frac{\partial V_i}{\partial z_i}(0, \overline{u}_i) &= 0\end{aligned}$$

for some positive constants $\underline{\alpha}_i$, $\overline{\alpha}_i$ and κ_i. Moreover, for every constant $\overline{u}_i \in \mathcal{U}_i$, it holds that

$$\frac{\partial V_i}{\partial u_i}(0, \overline{u}_i) = 0. \quad \square$$

Formally, let the goal of each agent be to minimize their own steady-state cost function, i.e.,

$$\begin{aligned}\forall i \in \mathcal{I}: \quad &\min_{u_i\in\mathcal{U}_i(\boldsymbol{u}_{-i})} J_i(u_i, \boldsymbol{u}_{-i}), &(7)\\ := &\min_{u_i\in\mathcal{U}_i(\boldsymbol{u}_{-i})} h_i(\pi(u_i), \pi_{-i}(\boldsymbol{u}_{-i})), &(8)\end{aligned}$$

which depends on the decision variables of other agents as well. From a game-theoretic perspective, we actually consider the problem to compute a generalized Nash equilibrium (GNE), as formalized next.

Definition 1 (*Generalized Nash Equilibrium*). A set of control actions $\boldsymbol{u}^* := \mathrm{col}\left(u_i^*\right)_{i\in\mathcal{I}}$ is a generalized Nash equilibrium if, for all $i \in \mathcal{I}$,

$$u_i^* \in \operatorname*{argmin}_{v_i} J_i\left(v_i, \boldsymbol{u}_{-i}^*\right) \text{ s.t. } \left(v_i, \boldsymbol{u}_{-i}^*\right) \in \boldsymbol{\mathcal{U}}. \tag{9}$$

with $\boldsymbol{\mathcal{U}}$ as in (3) and J_i as in (8). □

In plain words, a set of inputs is a GNE if no agent can improve its steady-state cost function by unilaterally changing its input. To ensure the existence of the GNE, we postulate the following assumption (Facchinei & Kanzow, 2010, Thm. 3.3):

Standing Assumption 3 (*Regularity*). *For each $i \in \mathcal{I}$, the function J_i in (8) is continuous; the function $J_i(\cdot, \boldsymbol{u}_{-i})$ is convex for every $\boldsymbol{u}_{-i}$. For each $i \in \mathcal{I}$, the set Ω_i is non-empty, closed and convex; $\boldsymbol{\mathcal{U}}$ is non-empty and satisfies Slater's constraint qualification.* □

More precisely, we focus on a subclass of GNE called variational GNE (v-GNE) (Facchinei & Kanzow, 2010, Def. 3.10). A collective decision $\boldsymbol{u}^*$ is a v-GNE in (9) if and only if there exists a dual variable $\lambda^* \in \mathbb{R}^q$ such that the following KKT conditions are satisfied (Facchinei & Kanzow, 2010, Th. 4.8):

$$\mathbf{0}_{m+q} \in \begin{bmatrix} F\left(\boldsymbol{u}^*\right) + A^\top\lambda^* \\ -\left(A\boldsymbol{u}^* - b\right)\end{bmatrix} + \begin{bmatrix}\mathrm{N}_{\boldsymbol{\Omega}}\left(\boldsymbol{u}^*\right)\\ \mathrm{N}_{\mathbb{R}^q_+}\left(\lambda^*\right)\end{bmatrix}, \tag{10}$$

where by stacking the partial gradients $\nabla_{u_i}J_i(u_i, \boldsymbol{u}_{-i})$ into a vector, we have the pseudogradient mapping:

$$F(\boldsymbol{u}) := \mathrm{col}\left(\left(\nabla_{u_i}J_i\left(u_i, \boldsymbol{u}_{-i}\right)\right)_{i\in\mathcal{I}}\right). \tag{11}$$

Let us postulate additional common assumptions ((De Persis & Grammatico, 2019a, Std. Ass. 2), (De Persis & Grammatico, 2019b, Ass. 1)) in order to assure the convergence of the algorithm we propose later on.

Standing Assumption 4 (*Well-behavedness*). *For each $i \in \mathcal{I}$, J_i in (8) is twice differentiable, and its gradient ∇J_i is ℓ-Lipschitz continuous, with $\ell > 0$. The pseudogradient mapping F in (11) is μ-strongly monotone, i.e. for any pair $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $(\boldsymbol{u} - \boldsymbol{v})^\top(F(\boldsymbol{u}) - F(\boldsymbol{v})) \geq \mu\|\boldsymbol{u} - \boldsymbol{v}\|^2$, with $\mu > 0$.* □

3. Generalized Nash equilibrium seeking for static agents

Let us start from the case of static agents to highlight the proposed algorithm and its integration with the zeroth-order gradient scheme.

Assumption 2 (*Static Agents*). For each $i \in \mathcal{I}$, $x_i = u_i$ (in place of (1a)). □

We propose three control schemes for GNE seeking with static agents. In the first two, the agents have perfect information about the decisions of other agents and know the analytic expression of their partial gradient. The third scheme is data-driven, i.e. the agents have access to the output of their cost function only. Additionally, the first scheme assumes the existence of the central coordinator for dual variable calculation, while in the second one, the computation of the dual variable is distributed.

3.1. Gradient-based case with central coordinator

Our GNE seeking algorithm is based on the forward–backward splitting (Bauschke et al., 2011, Thm. 26.14), (Boţ & Csetnek, 2017, Thm. 12) applied to a variant of the KKT operator in (10). In fact, we emphasize that there is a fundamental issue in applying the forward–backward splitting directly to (10): the forward part of the monotone operator must be cocoercive (Bauschke et al., 2011, Def. 4.2). Thus, the standard approach must move all the non-cocoercive elements in the backward step, but this would make the backward step impossible to compute (non-causal equations). Other splitting methods that only require monotonicity of the operator require multiple evaluations of the pseudogradient (Bauschke et al., 2011, Th. 26.17), which makes them incompatible with ESC as the latter can only estimate one pseudogradient. To overcome these issues, we apply a continuous-time variant of the approach introduced in Yi and Pavel (2019), where a preconditioning matrix is used. However, differently from Bianchi and Grammatico (2021), we do not use projections onto tangent cones to enable the use of extremum seeking techniques later on. We refer to Appendix A for technical details.

In our proposed algorithm, each agent updates their decision, u_i, based on: (i) decisions of all other agents; (ii) the dual variable, which is computed by a central coordinator, indexed by 0, who is in a bidirectional computation with all of the agents:

$$\forall i \in \mathcal{I}: \dot{u}_i = -u_i + \mathrm{proj}_{\Omega_i}\left(u_i - \gamma_i(\nabla_{u_i} J_i(\boldsymbol{u}) + A_i^\top \lambda)\right)$$
$$\dot{\lambda} = -\lambda + \mathrm{proj}_{\mathbb{R}^q_+}(\lambda + \gamma_0(A\boldsymbol{u} - b + 2A\dot{\boldsymbol{u}})),$$

or in collective form

$$\begin{bmatrix} \dot{\boldsymbol{u}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{u} + \mathrm{proj}_{\boldsymbol{\Omega}}\left(\boldsymbol{u} - \Gamma(F(\boldsymbol{u}) + A^\top\lambda)\right) \\ -\lambda + \mathrm{proj}_{\mathbb{R}^q_+}(\lambda + \gamma_0(A\boldsymbol{u} - b + 2A\dot{\boldsymbol{u}})) \end{bmatrix}, \tag{12}$$

where A_i are the m_i columns of A which correspond to coupling constraints on u_i, $\lambda \in \mathbb{R}^q_+$, $(\gamma_i)_{i\in\mathcal{I}_0}$ are the step sizes chosen by the agents and the central coordinator; $\mathcal{I}_0 := \mathcal{I} \cup \{0\}$; $\Gamma = \mathrm{diag}\left((\gamma_i I_{m_i})_{i\in\mathcal{I}}\right)$. We note that the decision dynamics are primal–dual pseudogradient dynamics, while those of the dual variable resemble the dual ascent, here with the additional pricing term $2A\dot{\boldsymbol{u}}$. We are now ready to state our first convergence result:

Theorem 1 (*v-GNE Seeking*). *Let the Standing Assumptions and Assumption 2 hold and let $(\boldsymbol{u}(t), \lambda(t))_{t\geq 0}$ be the solution to (12). Then, there exist small enough $(\gamma_i)_{i\in\mathcal{I}_0}$ such that the pair $(\boldsymbol{u}(t), \lambda(t))_{t\geq 0}$ converges to some $(\boldsymbol{u}^*, \lambda^*) \in \boldsymbol{\mathcal{U}} \times \mathbb{R}^q_+$, where $\boldsymbol{u}^*$ is the v-GNE of the game in (7).* □

Proof. See Appendix A. ■

3.2. Gradient-based case without a central coordinator

Let us study the case where the agents communicate with each other in order to calculate the dual variable. The communication structure is described via a graph $\mathcal{G} := (\mathcal{I}, \mathcal{E})$, where the first member of the ordered pair is the set of nodes (agents) and the second member is the set of edges (communication links) $\mathcal{E} \in \mathcal{I} \times \mathcal{I}$. The weight of the communication link $w_{ij} \geq 0$ is equal to zero if there is no edge between nodes i and j. We make a common assumption for consensus algorithms (Bianchi & Grammatico, 2021; Yi & Pavel, 2019):

Assumption 3. The communication graph $\mathcal{G}$ is strongly connected, undirected and its Laplacian satisfies $L = L^\top$.

Each agent updates his decision variable u_i, dual variable estimate λ_i and auxiliary variable z_i as follows:

$$\dot{u}_i = -u_i + \mathrm{proj}_{\Omega_i}\left(u_i - \gamma_i\left(\nabla_{u_i} J_i(\boldsymbol{u}) + A_i^\top \lambda_i\right)\right)$$
$$\dot{z}_i = \gamma_i \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(z_i - z_j)$$
$$\dot{\lambda}_i = -\lambda_i + \mathrm{proj}_{\mathbb{R}^q_+}\Bigg(\lambda_i + \gamma_i\Big(A_i(u_i + 2\dot{u}_i) - \tfrac{b}{N} +$$
$$\textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(2\dot{z}_i - 2\dot{z}_j - z_i + z_j - \lambda_i + \lambda_j)\Big)\Bigg), \tag{13}$$

or, as in collective from:

$$\begin{bmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{z}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{u} + \mathrm{proj}_{\boldsymbol{\Omega}}\left(\boldsymbol{u} - \Gamma_m(F(\boldsymbol{u}) + \Lambda^\top\boldsymbol{\lambda})\right) \\ \overline{\Gamma}\bar{L}\boldsymbol{\lambda} \\ \boldsymbol{\lambda} + \mathrm{proj}_{\mathbb{R}^{Nq}_+}(-\boldsymbol{\lambda} + \overline{\Gamma}(\Lambda(\boldsymbol{u} + 2\dot{\boldsymbol{u}}) - \frac{\boldsymbol{b}}{N} \\ +\bar{L}(2\dot{\boldsymbol{z}} - \boldsymbol{z} - \boldsymbol{\lambda}))) \end{bmatrix}, \tag{14}$$

where $z_i \in \mathbb{R}^q$ is an auxiliary variable; $\bar{L} := L \otimes I_q$; $\mathcal{N}_i$ is a set of agents with whom agent i has a communication link; $\overline{\Gamma} := \mathrm{diag}\left((\gamma_i I_q)_{i\in\mathcal{I}}\right)$; $\Lambda := \mathrm{diag}\left((A_i)_{i\in\mathcal{I}}\right)$; $\boldsymbol{b} = \mathrm{col}(b, \ldots, b)$, where b repeats N times. We conclude the subsection with our convergence result:

Theorem 2 (*Distributed v-GNE Seeking*). *Let the Standing Assumptions, Assumptions 2 and 3 hold and let $(\boldsymbol{u}(t), \boldsymbol{z}(t), \boldsymbol{\lambda}(t))_{t\geq 0}$ be the solution to (14). Then, there exist small enough $(\gamma_i)_{i\in\mathcal{I}}$ such that $(\boldsymbol{u}(t), \boldsymbol{z}(t), \boldsymbol{\lambda}(t))_{t\geq 0}$ converges to some $(\boldsymbol{u}^*, \mathbf{1}_N \otimes z^*, \mathbf{1}_N \otimes \lambda^*) \in \boldsymbol{\mathcal{U}} \times \mathbb{R}^{Nq}_+ \times \mathbb{R}^{Nq}_+$, where $\boldsymbol{u}^*$ is the v-GNE of the game in (7).* □

Proof. See Appendix B. ■

3.3. Data-driven case

In this section, we consider that the agents have access to the cost output only. We emphasize that in this case, they neither know the actions of the other agents, nor they know the analytic expressions of their partial gradients. In fact, this is a standard setup used in extremum seeking (Guay & Dochain, 2017; Krstić & Wang, 2000; Poveda & Teel, 2017 among others). However, we assume that the agents can communicate with a central coordinator, to whom they send their decision variable and its derivative.

Let us first evaluate the time derivative of the cost output $l_i = J_i(u_i, \boldsymbol{u}_{-i})$ along the trajectories of $\boldsymbol{u}$:

$$\dot{l}_i = \theta_i^0(\boldsymbol{u}) + \theta_i^1(\boldsymbol{u})\dot{u}_i = [1, \dot{u}_i^\top]\theta_i(\boldsymbol{u}), \tag{15}$$

where we define

$$\theta_i^0 = \theta_i^0(\boldsymbol{u}) := \nabla_{\boldsymbol{u}_{-i}} J_i(u_i, \boldsymbol{u}_{-i})^\top \dot{\boldsymbol{u}}_{-i} \tag{16}$$
$$\theta_i^1 = \theta_i^1(\boldsymbol{u}) := \nabla_{u_i} J_i(u_i, \boldsymbol{u}_{-i})^\top \tag{17}$$
$$\theta_i = \theta_i(\boldsymbol{u}) := [\theta_i^0, \theta_i^{1\top}] \tag{18}$$

In (16), the variable θ_i^0 measures the influence of the decision variables of the other agents on the cost output of agent i. Instead, in (17), the variable θ_i^1 measures the effect of the decision variable of agent i on the cost output l_i, which is needed for (12). To estimate the local θ_i^0 and θ_i^1, we use a time-varying parameter estimation approach, as proposed in Guay and Dochain (2017) for centralized optimization. Let us provide a basic intuition first.

Let $\hat{l}_i$ and $\hat{\theta}_i$ be estimations of the output l_i and the variable θ_i respectively and let $e_i = l_i - \hat{l}_i$ be the estimation error and $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ the parameter estimation error. Then, the estimator model in (15) for agent i is given by

$$\dot{\hat{l}}_i = [1, \dot{u}_i^\top]\hat{\theta}_i + K_i e_i + c_i^\top \dot{\hat{\theta}}_i, \tag{19}$$

where K_i is a free design parameter. Note that the first two terms on the right-hand side resemble high gain observer schemes (Oh & Khalil, 1997). As the structure of the problem does not directly allow the use of high gain observers, it is necessary to introduce some other dynamics into the estimation. This is the primary role of the third term in (19). Therefore, the dynamics of $c_i(t)$ are chosen as

$$\dot{c}_i^\top = -K_i c_i^\top + [1, \dot{u}_i^\top]. \tag{20}$$

Let us also introduce an auxiliary variables $\tilde{\eta}_i = \eta_i - \hat{\eta}_i$, $\eta_i = e_i - c_i^\top \tilde{\theta}_i$, with dynamics $D^+\eta_i = -K_i\eta_i - c_i^\top D^+\theta$, and its estimate $\hat{\eta}_i$, with dynamics

$$\dot{\hat{\eta}}_i = -K_i\hat{\eta}_i. \tag{21}$$

It is also necessary to define a positive definite matrix variable $\Sigma_i \in \mathbb{R}^{(m_i+1)\times(m_i+1)}$ with dynamics given by

$$\dot{\Sigma}_i = c_i c_i^\top - \rho_i \Sigma_i + \sigma_i I \qquad \Sigma_i(0) = \Sigma_i^0, \tag{22}$$

where ρ_i, σ_i and Σ_i^0 are free design parameters. We note that, in Adetola and Guay (2008), $\dot{\Sigma}_i = c_i c_i^\top$, but this proved to be inconvenient in practical implementations, as the elements of Σ_i grow unbounded. Instead, as in (22), dynamics of Σ_i behave as a first-order system. The third term is added so that the matrix is always invertible. Eqs. (19)-(22) form the parameter update law in Adetola and Guay (2008):

$$\dot{\hat{\theta}}_i = \Sigma_i^{-1}(c_i(e_i - \hat{\eta}_i) - \sigma_i\hat{\theta}_i). \tag{23}$$

We are finally ready to propose our semi-decentralized v-GNE learning algorithm:

$$\forall i \in \mathcal{I} : \dot{u}_i = -u_i + \mathrm{proj}_{\Omega_i}\left(u_i - \gamma_i(\hat{\theta}_i^1 + A_i^\top \lambda) + d_i\right),$$
$$\dot{\lambda} = -\lambda + \mathrm{proj}_{\mathbb{R}_+^q}(\lambda + \gamma_0(A\boldsymbol{u} - b + 2A\dot{\boldsymbol{u}})),$$

where d_i represents the perturbation signal of agent i. In collective form, it can be written as

$$\begin{bmatrix} \dot{\boldsymbol{u}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{u} + \mathrm{proj}_{\Omega}\left(\boldsymbol{u} - \Gamma(\hat{\boldsymbol{\theta}}^1 + A^\top\lambda) + d\right) \\ -\lambda + \mathrm{proj}_{\mathbb{R}_+^q}(\lambda + \gamma_0(A\boldsymbol{u} - b + 2A\dot{\boldsymbol{u}})) \end{bmatrix}, \tag{24}$$

where $\hat{\boldsymbol{\theta}}^1$ col $\left(\left(\hat{\theta}_i^1\right)_{i\in\mathcal{I}}\right)$. For $\hat{\boldsymbol{\theta}}^1$ to successfully estimate $F(\boldsymbol{u})$, it is necessary to assume that the input signals are "exciting" enough. As in Guay and Dochain (2017, Ass. 5), we postulate a persistency of excitation (PE) assumption.

Assumption 4 (*Persistence of Excitation*)**.** For each $i \in \mathcal{I}$, there exist $\alpha_i, T_i > 0$, such that

$$\int_t^{t+T_i} c_i(\tau)c_i(\tau)^\top d\tau \geq \alpha_i I, \qquad \text{for all } t > 0, \tag{25}$$

where c_i is the solution to (20). □

We conclude the section with the convergence result. For any initial condition of the decision variables, there exist gains such that the control variables converge to an arbitrarily small neighborhood of a v-GNE.

Theorem 3 (*v-GNE Static Learning*)**.** *Let the Standing Assumptions and Assumptions 2 and 4 hold and let* $(s(t) := (\hat{\boldsymbol{\eta}}(t), \hat{\boldsymbol{\theta}}(t), \boldsymbol{u}(t), \lambda(t)))_{t\geq 0}$ *be the closed-loop solution to (19)–(24). Then, for any compact set* $\mathcal{K}$ *and any* $\varepsilon > 0$, *there exist small enough parameters* $(\frac{1}{K_i}, \frac{1}{\rho_i}, \sigma_i, \gamma_i)_{i\in\mathcal{I}}$ *and* γ_0, *such that for every solution with* $s(0) \in \mathcal{K}$, $\boldsymbol{u}(t)$ *converges to the set* $\{\boldsymbol{u}^*\} + \varepsilon\mathbb{B}$, *where* $\boldsymbol{u}^*$ *is a v-GNE of the game in (7).* □

Proof. See Appendix C. ■

4. Generalized Nash equilibrium learning for dynamical agents

In this section, we propose a control scheme for generalized Nash equilibrium learning for dynamical agents. We consider a data-driven scenario only, i.e. the agents have access to the cost output measurements and the information that is given to them by a central coordinator. They are not aware of the analytic expression of their steady-state cost function, nor of their pseudogradient, nor can they observe the states and decisions of the other agents.

For the multi-agent dynamical system

$$\epsilon\dot{\boldsymbol{x}} = \mathrm{col}\left((f_i(x_i, u_i))_{i\in\mathcal{I}}\right) = f(\boldsymbol{x}, \boldsymbol{u}), \tag{26}$$

where $\epsilon > 0$ is a time scale separation constant with the objective of reaching a neighborhood of a v-GNE, we propose the same control law as in (24), with the distinction that $\hat{\boldsymbol{\theta}}^1$ is estimated by a parameter estimation scheme (19) – (23), where we collect the measurements of the output y_i in (1b) instead of $J_i(u_i, \boldsymbol{u}_{-i})$ directly. Thus, the estimation error is hereby redefined as

$$e_i = y_i - \hat{l}_i. \tag{27}$$

We conclude the section with the most general theoretical result of the paper, namely, the convergence of the closed-loop dynamics to a neighborhood of a v-GNE.

Theorem 4 (*v-GNE Dynamic Learning*)**.** *Let the Standing Assumptions and Assumptions 1 and 4 hold and let* $(s(t) := (\hat{\boldsymbol{\eta}}(t), \hat{\boldsymbol{\theta}}(t), \boldsymbol{x}(t), \boldsymbol{u}(t), \lambda(t)))_{t\geq 0}$ *be the closed-loop solution to (19)–(24), (26), (27). Then, for any compact set* $\mathcal{K}$ *and any* $\varepsilon > 0$, *there exist small enough parameters* $(\frac{1}{K_i}, \frac{1}{\rho_i}, \sigma_i, \gamma_i)_{i\in\mathcal{I}}$, γ_0 *and* ϵ *such that every solution with* $s(0) \in \mathcal{K}$, $(\boldsymbol{x}(t), \boldsymbol{u}(t))$ *converges to an* ε *neighborhood of some* $(\pi(\boldsymbol{u}(t)), \boldsymbol{u}^*)$, *where* $\boldsymbol{u}^*$ *is a v-GNE of the game in (7).* □

Proof. See Appendix D. ■

5. Illustrative applications

5.1. Connectivity control in robotic swarms

The problem of connectivity control has been considered in Stankovic, Johansson, and Stipanovic (2011) as a Nash equilibrium problem. In many practical scenarios, multi-agent systems, besides their primary objective, are designed to uphold certain connectivity as their secondary objective. In what follows, we consider a similar problem in which each agent is tasked with finding a source of an unknown signal while maintaining certain connectivity. Unlike Stankovic et al. (2011), we require the existence of a central coordinator and we allow for coupled restrictions on the decisions variables. Moreover, we model the agents as unicycles with setpoint regulators, which does not require a constant angular velocity as in Stankovic et al. (2011).

Consider a multi-agent system consisting of unicycle vehicles, indexed by $i \in \{1, \ldots N\}$, where each one implements the feedback controller studied in Lee, Cho, Hwang-Bo, You, and Oh (2000) for target tracking, to have the following dynamics:

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\phi}_i \end{bmatrix} = \begin{bmatrix} -K_i^1\|r_i - u_i\| \cos(\phi_i) \cos\left(\phi_i - \arctan\frac{y_i}{x_i}\right) \\ -K_i^1\|r_i - u_i\| \cos(\phi_i) \sin\left(\phi_i - \arctan\frac{y_i}{x_i}\right) \\ -K_i^2\phi_i \end{bmatrix}, \tag{28}$$

where x_i, y_i are position variables, ϕ_i is the relative angle with respect to the setpoint, $K_i^1, K_i^2 > 0$ controller parameters, $r_i = \mathrm{col}(x_i, y_i)$ and $u_i = \mathrm{col}\left(u_i^x, u_i^y\right)$ is the input of the transformed system, which represents the coordinates of the setpoint input. Note that, in the new dynamics (28), we do not follow the global angle coordinate, but rather ϕ_i (local coordinate), as illustrated in Figure (Lee et al., 2000, Fig. 1). For each i, the steady-state mapping is then given by $\pi_i(u_i) = \mathrm{col}(u_i, 0)$.

Each agent is tasked with locating a source of a unique unknown signal. The strength of all signals abides by the inverse-square law, i.e. proportional to $1/r^2$. Therefore, the inverse of the signal strength can be used in the cost function. Additionally, the agents must not drift apart from each other too much, as

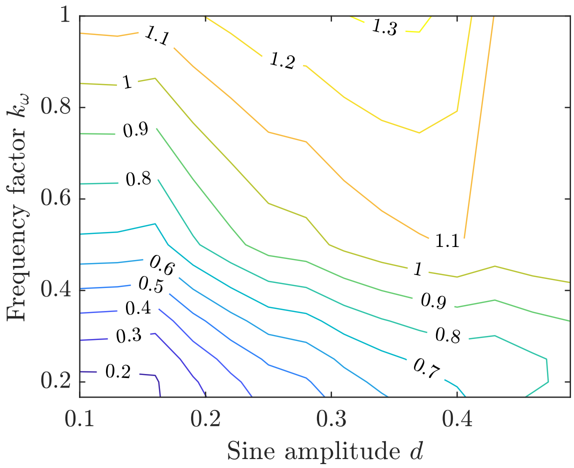

Fig. 1. Distance of the final average steady-state trajectory from the v-GNE for agent 4.

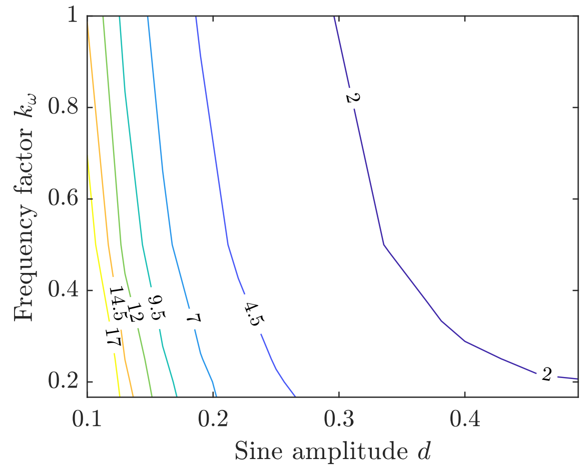

Fig. 2. Hours required to enter a ball of size $\varepsilon = 1.5$ centered around the v-GNE for agent 4.

they should provide quick assistance to each other in case of critical failure. This is enforced in two ways: by incorporating the signal strength of the fellows agents in the cost functions and by communicating with the central coordinator. Thus, we design the cost output and position constraints as

$$\forall i \in \mathcal{I}: \begin{cases} h_i(r_i) = \|r_i - r_i^s\|^2 + c\sum_{j\in\mathcal{I}_{-i}} \|r_i - r_j\|^2, \\ \left\|\operatorname{col}\left((u_i - u_j)_{j\in\mathcal{I}_{-i}}\right)\right\|_\infty \leq b \end{cases} \tag{29}$$

where $\mathcal{I}_{-i} := \mathcal{I}\setminus\{i\}$, $c, b > 0$ and r_i^s represents the position of the source assigned to agent i. The safe traversing area is described by a rectangle: $[x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$.

For our numerical simulations, we choose the parameters: $N = 4$, $(x_{\min}, x_{\max}) = (-16, 16)$, $(y_{\min}, y_{\max}) = (-6, 6)$, $r_1^s = (-4, -8)$, $r_2^s = (-12, -3)$, $r_3^s = (1, 7)$, $r_4^s = (16, 8)$, $(c, b) = (0.04, 14)$, $K_i = 100$, $k_i^\sigma = 100$, $\sigma_i = 10^{-6}$, $\Sigma_i(0) = 0.1I$, $\gamma_i = 0.002$, $\gamma_0 = 0.002$, $\epsilon = 0.1$, $K_1^i = 3$, $K_2^i = 6$, $(\varpi_1^1, \varpi_1^2) = (5.11, 6.38)$, $(\varpi_2^1, \varpi_2^2) = (4.42, 5.16)$, $(\varpi_3^1, \varpi_3^2) = (10.59, 11.91)$, $(\varpi_4^1, \varpi_4^2) = (14.65, 16.12)$. We run simulations for different values of perturbation amplitudes in range $[0.1, 0.5]$ and different values of the frequency factor k_ω in range $[0.17, 1]$. The numerical results are illustrated in Figs. 1–3. In Fig. 1, we see that smaller perturbations and frequency factors bring the system closer towards the v-GNE; however in Fig. 2, we see that the convergence rate slows down significantly. Thus, there is a trade-off between convergence speed and closeness to the solution. Moreover, we numerically test robustness to output noise on a representative example of agent trajectories. We simulate noise with zero mean and variance equal to 1 and 3. In Fig. 3, the shaded regions represent envelopes of the trajectories (ten simulations per variance). The darkest shade represents the case without noise, while the lightest represents the case with largest variance. We observe that the algorithm still converges to a neighborhood of the v-GNE.

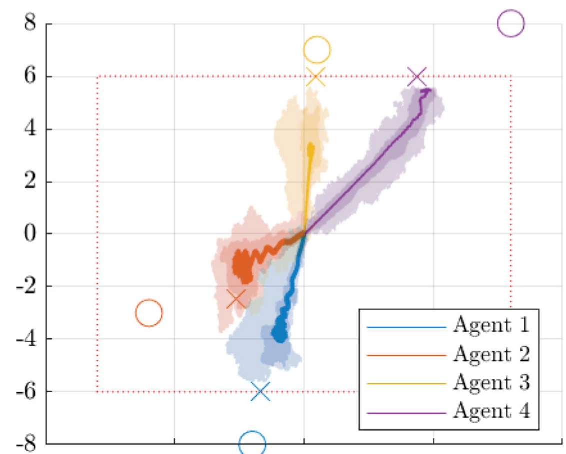

Fig. 3. State trajectories in the x–y plane for the case of $d_i = 0.49$ and various noise levels. Circle symbols represent locations of the sources, while the $\times$ symbols represent locations of the v-GNE.

5.2. Wind farm optimization

As one of the main sources of renewable energy, wind farms and their optimization have been addressed extensively from different perspectives such as power tracking of single turbines (Boukhezzar, Siguerdidjane, & Hand, 2006; Koutroulis & Kalaitzakis, 2006), power tracking via extremum seeking (Ghaffari, Krstic, & Seshagiri, 2014), power tracking with load reduction (Soliman, Malik, & Westwick, 2011; Soltani, Wisniewski, Brath, & Boyd, 2011), distributed optimization of wind farms (Barreiro-Gomez, Ocampo-Martinez, Bianchi, & Quijano, 2015; Marden, Ruben, & Pao, 2013) and distributed optimization via extremum seeking (Ebegbulem & Guay, 2017). While in the power tracking case, often the torque or some other related variable is taken as the control input, in the distributed optimization case, the axial induction factor (AIL) is usually taken as the control input.

In what follows, we consider a similar problem in which a wind farm tries to maximize its power output with AIL as the control input. The control variables are subject to local constraints (feasible values of AIL). Also, we require that the turbines experience a similar amount of mechanical stress. To do that, we impose that AILs of a row of wind turbines cannot differ too much from AILs of the succeeding row, which introduces coupling constraints to the optimization problem. Unlike the previously mentioned literature on distributed wind farm optimization, here we also allow for AIL dynamics in order to reflect the turbine time constant and its effect on the power output. One possible way of solving the problem would be via global optimization, where a central coordinator would minimize a global cost function and send AIL commands to the turbines. To avoid having a single critical node for computation and communication, an alternative approach is to pose the problem as a potential game, where the cost function

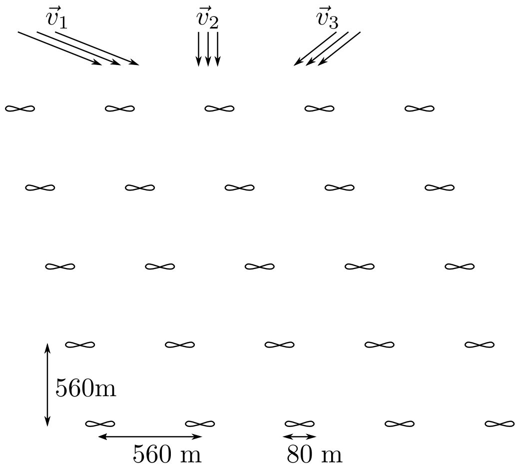

Fig. 4. Layout of the wind turbines and the wind directions.

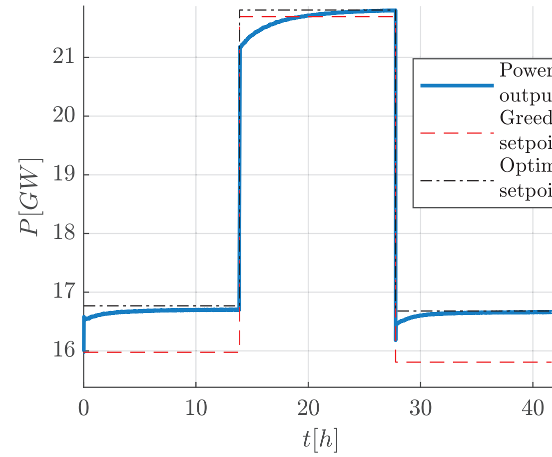

Fig. 5. Power generation with the proposed algorithm (solid line) compared to the greedy power output setpoint (dashed red) and the global optimal power setpoint (dot-dashed black).

of the turbines is "aligned" to a global utility function. In our case, the potential function would be the sum of all power outputs. We choose that the individual cost functions are equal to the potential function and each of the agents minimizes their cost function on their own, with limited information from the central coordinator. In this setup, a v-GNE corresponds to an optimal solution to the global power maximization problem.

Technically speaking, we consider N wind turbines, indexed by $i \in \{1, \ldots, N\}$, each with the following AIL dynamics and power output:

$$\dot{a}_i = -\tfrac{1}{\tau}(a_i - u_i) \tag{30}$$

$$y_i = -\sum_{i\in\mathcal{I}} P_i(\boldsymbol{a}) = -\tfrac{1}{2}\rho A \sum_{i\in\mathcal{I}} C_P\,(a_i)\, V_i(\boldsymbol{a})^3, \tag{31}$$

where a_i represent the state variable, u_i represent the control input, namely the AIL reference, y_i is the measured power output of the wind farm, which is broadcasted by the central coordinator, ρ is the air density, A is the surface area encompassed by the blades of a single turbine, $C_P\,(a_i) := a_i(1-a_i)^2$ is the power efficiency coefficient and V_i is the average wind speed experienced by wind turbine i, as in Marden et al. (2013, Equ. 5):

$$V_i(a) = U_\infty \left(1 - 2\sqrt{\sum \left(a_j c_{ji}\right)^2}\right). \tag{32}$$

The wind turbines are placed in R rows and C columns with coordinates x_i and their indices can be written as $i = i_c + i_r C$, where $i_c \in \{1, \ldots, C\}$ and $i_r \in \{0, \ldots, R-1\}$. They are tasked to maximize the wind farm power output under local constraints $a_i \in [a_{\min}, a_{\max}]$ and coupling constraints $|a_i - a_j| \le b$ for all i, j, where it holds that $j_r = i_r + 1$.

For our numerical simulation, we choose a similar setup as in Marden et al. (2013). The wind farm geometric setup is shown in Fig. 4 and the following parameters are chosen: $\rho = 1.225$, $U_\infty = 8$, $\tau = 10$, $\gamma_i = \gamma_0 = 0.05$, $\epsilon = 0.005$, $b = 0.03$, $a_{\min} = 0.1$, $a_{\max} = \frac{1}{3}$. We take the same parameter estimation scheme as in previous numerical simulation, apart for the perturbation frequencies that we randomly choose in the interval [3, 11] and perturbation amplitudes that we take as $\|d_i\| = 0.01$. All initial conditions, apart for a_i, were set to zero. The initial condition for a_i was set to $\frac{1}{3}$, which corresponds to the greedy strategy in Marden et al. (2013). In our simulations, we use three different wind directions. In the time interval [0, 50000), the wind was blowing with speed direction vector $\vec{v}_1 = (2, -1)$; in the time interval [50000, 100000), the wind was blowing with the speed direction vector $\vec{v}_2 = (0, -1)$; and finally, in the time interval [100000, 150000], the wind was blowing with speed direction vector $\vec{v}_3 = (-1, -1)$. We assume that the wind turbines instantly adjust their orientation towards the wind direction as this process is relatively fast compared with the GNE learning process. The simulation results are shown in Fig. 5. We can see that the wind turbines reach a neighborhood of the v-GNE, even with the delay introduced by AIL dynamics.

6. Conclusion

Generalized Nash equilibrium problems with nonlinear dynamical agents can be solved via a preconditioned forward–backward algorithm that uses estimates of the pseudogradient mapping if it is strongly monotone and Lipschitz continuous, and if the dynamical agents have a certain exponential stability property. Regular projections enable the use of a parameter estimation scheme. Numerical simulations show that there is a trade-off between closeness to the equilibrium solution and the speed of convergence.

Appendix A. Proof of Theorem 1

To prove the convergence of the algorithm, we show that equation in (12) is equivalent to a continuous-time preconditioned forward–backward algorithm, whose convergence is proven using well-known properties of monotone operators. First, we show the equivalence. Let us denote $\omega = \mathrm{col}(\boldsymbol{u}, \lambda)$. We write Eq. (12) as:

$$\dot{\omega} = -\omega + \mathrm{proj}_{\Omega\times\mathbb{R}^q_+}\left(\omega + \boldsymbol{\Gamma}\begin{bmatrix} -F(\boldsymbol{u}) - A^\top\lambda \\ A\boldsymbol{u} - b + 2A\dot{\boldsymbol{u}}\end{bmatrix}\right), \tag{A.1}$$

where $\boldsymbol{\Gamma} = \mathrm{diag}\left(\Gamma, \gamma_0 I_q\right)$. Next, by the property of projection operator in Bauschke et al. (2011, Prep. 6.47), Eq. (A.1) reads as

$$\dot{\omega} + \omega + \boldsymbol{\Gamma}\mathrm{N}_{\Omega\times\mathbb{R}^q_+}(\dot{\omega} + \omega) \ni \omega + \boldsymbol{\Gamma}\left[\begin{smallmatrix} -F(\boldsymbol{u}) - A^\top\lambda \\ A\boldsymbol{u} - b + 2A\dot{\boldsymbol{u}}\end{smallmatrix}\right]. \tag{A.2}$$

When the elements of the last matrix product in (A.2) are rearranged and the equation is premultiplied by $\boldsymbol{\Gamma}^{-1}$, the equations read as follows:

$$\left(\boldsymbol{\Gamma}^{-1} + \mathrm{N}_{\Omega\times\mathbb{R}^q_+}\right)(\dot{\omega} + \omega) \ni \hat{\Phi}\omega + \left[\begin{smallmatrix} -F(\boldsymbol{u}) \\ -b\end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ 2A\dot{\boldsymbol{u}}\end{smallmatrix}\right]$$

$$\left(\boldsymbol{\Gamma}^{-1} + \mathrm{N}_{\Omega\times\mathbb{R}^q_+}\right)(\dot{\omega} + \omega) \ni \Phi\omega - \left[\begin{smallmatrix} F(\boldsymbol{u}) \\ b\end{smallmatrix}\right] + \hat{A}(\omega + \dot{\omega}). \tag{A.3}$$

where we have used the notation
$\hat{\Phi} = \begin{bmatrix} \Gamma^{-1} & -A^\top \\ A & \gamma_0^{-1} I_q \end{bmatrix}$, $\Phi = \begin{bmatrix} \Gamma^{-1} & -A^\top \\ -A & \gamma_0^{-1} I_q \end{bmatrix}$, $\hat{A} = \begin{bmatrix} 0 & 0 \\ 2A & 0 \end{bmatrix}$.

Next, the following expression is valid for the matrices:

$$\Gamma^{-1} - \hat{A} = \Phi + \begin{bmatrix} 0 & A^\top \\ -A & 0 \end{bmatrix} = \Phi + \Psi. \tag{A.4}$$

From Eqs. (A.3) and (A.4), it follows

$$\left(\mathrm{Id} + \Phi^{-1}\left(\mathrm{N}_{\Omega \times \mathbb{R}^q_+} + \Psi\right)\right)(\dot{\omega} + \omega) \ni \omega - \Phi^{-1}\begin{bmatrix} F(\boldsymbol{u}) \\ b \end{bmatrix}.$$

By inverting the operator on the left side of the previous expression, we finally arrive to desired equation:

$$\begin{aligned} &\dot{\omega} = -\omega + \left(\mathrm{Id} + \Phi^{-1}\mathcal{A}\right)^{-1} \circ \left(\omega - \Phi^{-1}\mathcal{B}(\omega)\right) \Leftrightarrow \\ &\dot{\omega} = -\omega + J_{\Phi^{-1}\mathcal{A}}\left(\omega - \Phi^{-1}\mathcal{B}(\omega)\right), \end{aligned} \tag{A.5}$$

where $\mathcal{A} = \mathrm{N}_{\Omega \times \mathbb{R}^q_+} + \Psi$ and $\mathcal{B} = \mathrm{col}(F(\boldsymbol{u}), b)$. Eq. (A.5) represents a forward–backward algorithm preconditioned by matrix Φ^{-1}. Fixed points of the operator on the right-hand side of (A.5) ((Bauschke et al., 2011, Prep. 26.1, (iv)(a)), (Yi & Pavel, 2019, Lemma 1), (10)) represent GNE that are the solutions to the game in (7) and equilibrium points of dynamics in (24). Before proving convergence, we have to prove an additional result.

Lemma 1. *Let $T = (\mathrm{Id} + \mathcal{A})^{-1} \circ (\mathrm{Id} - \mathcal{B})$, where $\mathcal{A}$ is maximally monotone. Then it holds:*

$$(Tx - x^*)^\top (x - Tx) \geq (Tx - x^*)^\top (\mathcal{B}x - \mathcal{B}x^*).$$

for all $(x, x^*) \in \mathrm{dom}(T) \times \mathrm{fix}(T)$. □

Proof. Let us denote $x^* = Tx^* = J_{\mathcal{A}} y^*$ as the fixed point of operator T. Then it holds:

$$\begin{aligned} &(Tx - x^*)^\top (x - Tx - (\mathcal{B}x - \mathcal{B}x^*)) \\ &= (Tx - x^*)^\top (x - \mathcal{B}x - Tx + x^* - (x^* - \mathcal{B}x^*)) \\ &= (Tx - x^*)^\top (y - Tx + x^* - y^*) \\ &= (J_{\mathcal{A}}y - J_{\mathcal{A}}y^*)^\top ((\mathrm{Id} - J_{\mathcal{A}})y - (\mathrm{Id} - J_{\mathcal{A}})y^*) \\ &\geq 0, \end{aligned} \tag{A.6}$$

where the last equation holds due to properties of firmly nonexpansive operators. ■

Now we denote $\tilde{\mathcal{A}} = \Phi^{-1}\mathcal{A}$, $\tilde{\mathcal{B}} = \Phi^{-1}\mathcal{B}$ and $T = (\mathrm{Id} + \tilde{\mathcal{A}})^{-1} \circ (\mathrm{Id} - \tilde{\mathcal{B}})$. Then, the dynamics in (A.5) read as $\dot{\omega} = -\omega + T\omega$. We propose the Lyapunov function candidate

$$V(\omega) = \tfrac{1}{2}\|\omega - \omega^*\|^2, \tag{A.7}$$

where ω^* is a fixed point of operator T. Its derivative along the trajectory in (A.5) is then

$$\begin{aligned} \dot{V}(\omega) &= -(\omega - \omega^*)^\top (\omega - T\omega) \\ &= -\|\dot{\omega}\|^2 - (T\omega - \omega^*)^\top (\omega - T\omega) \end{aligned} \tag{A.8}$$

From Lemma 1, it follows that

$$\begin{aligned} \dot{V}(\omega) &\leq -\|\dot{\omega}\|^2 - (T\omega - \omega^*)^\top (\tilde{\mathcal{B}}\omega - \tilde{\mathcal{B}}\omega^*) \\ &\leq -\|\dot{\omega}\|^2 - (T\omega - \omega)^\top (\tilde{\mathcal{B}}\omega - \tilde{\mathcal{B}}\omega^*) - (\omega - \omega^*)^\top (\tilde{\mathcal{B}}\omega - \tilde{\mathcal{B}}\omega^*) \\ &\leq -\|\dot{\omega}\|^2 - (T\omega - \omega)^\top \Phi^{-1}(\mathcal{B}\omega - \mathcal{B}\omega^*) \\ &\quad - (\omega - \omega^*)^\top \Phi^{-1}(\mathcal{B}\omega - \mathcal{B}\omega^*). \end{aligned} \tag{A.9}$$

Bounds on the eigenvalues of Φ can be estimated with Gershgorin's theorem. For small enough step sizes, we denote the lower and upper bounds on the eigenvalues as $\sigma_{\min} = \frac{1}{\max_{i \in \mathcal{I}_0}(\gamma_i^{-1}) + \|A\|}$ and $\sigma_{\max} = \frac{1}{\min_{i \in \mathcal{I}_0}(\gamma_i^{-1}) - \|A\|}$, respectively. We bound (A.9) as

$$\begin{aligned} \dot{V}(\omega) \leq &- \|\dot{\omega}\|^2 + \sigma_{\max}\|\dot{\omega}\|\|\mathcal{B}\omega - \mathcal{B}\omega^*\| \\ &- \sigma_{\min}(\omega - \omega^*)^\top (\mathcal{B}\omega - \mathcal{B}\omega^*). \end{aligned} \tag{A.10}$$

Since $F(\boldsymbol{u})$ is strongly monotone and Lipschitz continuous, it is also cocoercive (Yi & Pavel, 2019, Lemma 5). Therefore, the operator $\mathcal{B}$ is $\frac{\mu}{L^2}$ cocoercive. Eq. (A.10) then becomes:

$$\begin{aligned} \dot{V}(\omega) &\leq -\|\dot{\omega}\|^2 + \sigma_{\max}\|\dot{\omega}\|\|\mathcal{B}\omega - \mathcal{B}\omega^*\| \\ &\quad - \tfrac{\sigma_{\min}}{2}\beta\|\mathcal{B}\omega - \mathcal{B}\omega^*\|^2 - \tfrac{\sigma_{\min}}{2}(\omega - \omega^*)^\top (\mathcal{B}\omega - \mathcal{B}\omega^*) \\ &\leq -\tfrac{1}{2}\|\dot{\omega}\|^2 - \tfrac{\sigma_{\min}}{2}(\omega - \omega^*)^\top (\mathcal{B}\omega - \mathcal{B}\omega^*) \\ &\quad - \tfrac{1}{2}\begin{bmatrix} \|\dot{\omega}\| & \|\mathcal{B}\omega - \mathcal{B}\omega^*\| \end{bmatrix}\begin{bmatrix} 1 & \sigma_{\max} \\ \sigma_{\max} & \beta\sigma_{\min} \end{bmatrix}\begin{bmatrix} \|\dot{\omega}\| \\ \|\mathcal{B}\omega - \mathcal{B}\omega^*\| \end{bmatrix} \end{aligned} \tag{A.11}$$

Since, it is always possible to choose parameters γ_i and γ_0 small enough such that $\beta\sigma_{\min} \geq \sigma_{\max}^2$ and the matrix in (A.11) is negative definite, the last equation reads as

$$\begin{aligned} \dot{V}(\omega) &\leq -\tfrac{1}{2}\|\dot{\omega}\|^2 - \tfrac{\sigma_{\min}}{2}(\omega - \omega^*)^\top (\mathcal{B}\omega - \mathcal{B}\omega^*) \\ &\leq -\tfrac{1}{2}\|\dot{\omega}\|^2 - \tfrac{\mu\sigma_{\min}}{2}\|\tilde{\boldsymbol{u}}\|^2, \end{aligned} \tag{A.12}$$

where $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}^*$ and the last line follows from strong monotonicity of $F(\boldsymbol{u}^*)$. The rest of the proof represents a La Salle argument. As the right-hand side is a sum of negative squares, it follows that $\dot{V}(\omega) \leq 0$ for all ω. Let us denote the following sets

$$\begin{aligned} &\zeta_c := \{\omega \in \Omega \times \mathbb{R}^q_+ \mid V(\omega) \leq c\} \\ &\zeta_o := \{\omega \in \zeta_c \mid \|\dot{\omega}\| = 0 \text{ and } \|\tilde{\boldsymbol{u}}\| = 0\} \\ &\mathcal{Z} := \{\omega \in \zeta_c \mid \dot{V}(\omega) = 0\} \\ &\mathcal{O} := \{\omega \in \zeta_c \mid \omega(0) \in \mathcal{Z} \implies \omega(t) \in \mathcal{Z}\ \forall t \in \mathbb{R}\}, \\ &\mathcal{A} := \{\omega \in \zeta_c \mid \dot{\omega} = \mathbf{0}\}, \end{aligned} \tag{A.13}$$

where ζ_c is a compact level set chosen such that it is nonempty, $\mathcal{Z}$ is the zero set of (A.8), ζ_o is the superset of $\mathcal{Z}$ which follows from (A.12), $\mathcal{O}$ is the maximum invariant set as explained in Khalil (2002, Chp. 4.2) and $\mathcal{A}$ is the equilibrium set. It holds that $\zeta_c \supseteq \zeta_o \supseteq \mathcal{Z} \supseteq \mathcal{O} \supseteq \mathcal{A}$. As ζ_c is invariant and the right-hand side equations of (12) are (locally) Lipschitz, by Khalil (2002, Thm. 3.3) we conclude that solutions to (12) exist and are unique. Next we note that $\dot{\omega} = 0 \Leftrightarrow \omega \in \mathrm{fix}(T)$. Therefore, the set $\mathcal{A}$ is the set of fixed points and it holds $\zeta_o \equiv \mathcal{Z} \equiv \mathcal{O} \equiv \mathcal{A}$. Hence, by La Salle's invariance principle (Khalil, 2002, Thm. 4.4), trajectories converge to the set $\mathcal{A}$. Additionally, as $\tilde{\boldsymbol{u}} = \mathbf{0}$ in $\mathcal{A}$, it follows that u^* is a singleton, which is not necessarily true for λ^*.

Appendix B. Proof of Theorem 2

Similarly to Theorem 1, it can be shown that the dynamics in (14) can be written as

$$\dot{\bar{\omega}} = -\bar{\omega} + J_{\bar{\Phi}^{-1}\bar{\mathcal{A}}}\left(\bar{\omega} - \Phi^{-1}\bar{\mathcal{B}}(\omega)\right), \tag{B.1}$$

where $\bar{\omega} := \mathrm{col}(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\lambda})$, $\bar{\mathcal{A}} := \mathrm{N}_{\Omega \times \mathbb{R}^{Nq} \times \mathbb{R}^{Nq}_+} + \bar{\Psi}$, $\bar{\mathcal{B}} := (F(\boldsymbol{u}),\ \mathbf{0}_{Nq},\ \frac{\boldsymbol{b}}{N} + \bar{L}\lambda)$,

$$\bar{\Psi} := \begin{bmatrix} 0 & 0 & \Lambda^\top \\ 0 & 0 & -\bar{L} \\ -\Lambda^\top & \bar{L} & 0 \end{bmatrix}, \quad \bar{\Phi} := \begin{bmatrix} \Gamma^{-1} & 0 & -\Lambda^\top \\ 0 & \bar{\Gamma}^{-1} & \bar{L} \\ -\Lambda & \bar{L} & \bar{\Gamma}^{-1} \end{bmatrix}.$$

Proof that an equilibrium point $\bar{\omega}^*$ of (B.1) exists and that $\boldsymbol{u}^*$ is the solution to the game in (7) is analogous to proof of Theorem 2 in Yi and Pavel (2019) and is omitted for brevity. Furthermore, as all of the operators and matrices hold the same properties as the ones in Theorem 1 of this paper, convergence is proven in the same manner.

Appendix C. Proof of Theorem 3

Let us consider a Lyapunov function candidate $V = V_\theta + V_\omega$, where V_θ represents a parameter estimation error and V_ω represents a primal–dual convergence error:

$$V_\theta(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}) = \sum_{i\in\mathcal{I}} \left(\tfrac{1}{2}\tilde{\eta}_i^\top \tilde{\eta}_i + \tfrac{1}{2}\tilde{\theta}_i^\top \Sigma_i \tilde{\theta}_i\right), \tag{C.1}$$

$$V_\omega(\omega) = \tfrac{1}{2}\|\omega - \omega^*\|^2. \tag{C.2}$$

Since we anticipate that the derivative of the projection function does not exist on some corner points, we use the Lyapunov theory for differential inclusions as in Blanchini and Miani (2008, Chp. 2), namely we use upper Dini derivatives (D^+) instead of regular time derivatives. For ease of notation, we use the regular derivatives whenever they are equal to Dini derivatives.

Outline of the proof: We first bound all of the positive terms in D^+V_θ with functions of variables $(\boldsymbol{\eta}, \hat{\boldsymbol{\theta}}, \omega)$, then we similarly bound all of the terms in D^+V_ω introduced by the parameter estimates. Finally, we use the quadratic terms of D^+V to show that the positive terms are majorized by the negative terms.

Parameter estimation term: We bound the Dini derivative of V_θ similarly to Guay and Dochain (2017, Thm. 1) and Guay, Vandermeulen, Dougherty, and McLellan (2018, Eq. 31) with the only difference that we let each agent choose their own parameters (σ_i, K_i, ρ_i). The Lyapunov derivative, similarly to Guay and Dochain (2017, p. 4, col. 1), reads as follows:

$$\begin{aligned} D^+V_\theta(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}) \le \sum_{i\in\mathcal{I}} \Bigg(& -\tilde{\eta}_i^\top \left(K_i - \tfrac{1+k_1\zeta_i}{2}\right)\tilde{\eta}_i - \tfrac{1}{2}\|e_i - \eta_i\|^2 + \tfrac{\sigma_i}{2}\theta_i^\top\theta_i \\ & - \tfrac{\rho_i'\gamma_i^1}{2}\tilde{\theta}_i^\top\tilde{\theta}_i + \tfrac{1}{2k_1}D^+\theta_i^\top D^+\theta_i + \tfrac{\gamma_i^2}{2k_2}D^+\theta_i^\top D^+\theta_i \Bigg) \\ \le & -k_a\|\tilde{\boldsymbol{\eta}}\|^2 - k_b\|\tilde{\boldsymbol{\theta}}\|^2 - \tfrac{1}{2}\|\boldsymbol{e} - \boldsymbol{\eta}\|^2 + k_c\|D^+\boldsymbol{\theta}\|^2 \\ & + \tfrac{\sigma}{2}\|\boldsymbol{\theta}\|^2, \end{aligned} \tag{C.3}$$

where $k_1, k_2 > 0$, $\zeta_i = c_i c_i^\top$, $0 < \gamma_i^1 \le \gamma_i^2$ are bounds for matrices Σ_i, $\rho_i' = \rho_i - k_2$ for all $i \in \mathcal{I}$, $k_a := \min_i\left(K_i - \frac{1+k_1\zeta_i}{2}\right)$, $k_b := \min_i\left(\frac{k_i^{\Sigma'}\gamma_i^1}{2}\right)$, $k_c := \max_i\left(\frac{1}{2k_1} + \frac{\gamma_i^2}{2k_2}\right)$ and $\sigma := \max_i \sigma_i$.[1] Assumption 4 is used in order to derive these expressions. We define the compact set $\zeta_c := \{(\omega, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\theta}}) \in \boldsymbol{\mathcal{U}} \times \mathbb{R}^q \times \mathbb{R}^N \times \mathbb{R}^m \mid V(\omega, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}) \le c\}$. Next, we bound the positive terms in (C.3). The analysis starts with $\boldsymbol{\theta} = \mathrm{col}\left(\boldsymbol{\theta}^0, \boldsymbol{\theta}^1\right)$, where

$$\begin{aligned} \boldsymbol{\theta}^0 &:= \mathrm{col}\left(\left(\nabla_{\boldsymbol{u}_{-i}} J_i\left(u_i, \boldsymbol{u}_{-i}\right)^\top \dot{\boldsymbol{u}}_{-i}\right)_{i\in\mathcal{I}}\right) \\ &:= J^0(\boldsymbol{u})\dot{\boldsymbol{u}}, \end{aligned} \tag{C.4}$$

$$\boldsymbol{\theta}^1 := F(\boldsymbol{u}). \tag{C.5}$$

We have that

$$\begin{aligned} \|\boldsymbol{\theta}^0\| &\le \|J^0(\boldsymbol{u})\|\|\dot{\boldsymbol{u}}\| = L^0\|\dot{\boldsymbol{u}}\|, \\ \|\boldsymbol{\theta}^1\| &= \|F(\boldsymbol{u})\| \le \|F(\boldsymbol{u}) - F(\boldsymbol{u}^*)\| + \|F(\boldsymbol{u}^*)\| \\ &\le \ell\|\tilde{\boldsymbol{u}}\| + \|F(\boldsymbol{u}^*)\|, \end{aligned}$$

where $L^0 := \max_{\boldsymbol{u}\in\mathcal{U}} \left\|J^0(\boldsymbol{u})\right\| < \infty$, since $\mathcal{U}$ is bounded. Then, we bound $\|\theta\|$ as follows

$$\begin{aligned} \|\theta\|^2 &\le {L^0}^2\|\dot{\boldsymbol{u}}\|^2 + (\ell\|\tilde{\boldsymbol{u}}\| + \|F(\boldsymbol{u}^*)\|)^2 \\ &\le L_1\|\dot{\boldsymbol{u}}\|^2 + L_2\|\tilde{\boldsymbol{u}}\|^2 + L_3\|F(\boldsymbol{u}^*)\|^2, \end{aligned}$$

for $L_1 := {L^0}^2$, $L_2 := 2\ell^2$ and $L_3 = 2$. In order to bound $\|D^+\boldsymbol{\theta}\|$, we observe the dini derivatives of $\boldsymbol{\theta}^0$ and $\boldsymbol{\theta}^1$:

$$\|D^+\boldsymbol{\theta}^0\| \le \|\dot{\boldsymbol{u}}^\top H_{J^0}\dot{\boldsymbol{u}}\| + \|J^0(\boldsymbol{u})D^+\dot{\boldsymbol{u}}\|, \tag{C.6}$$

$$\|D^+\boldsymbol{\theta}^1\| = \|\nabla F(\boldsymbol{u})\dot{\boldsymbol{u}}\| \le L\|\dot{\boldsymbol{u}}\|, \tag{C.7}$$

$$\begin{aligned} \|D^+\dot{\boldsymbol{u}}\| &= \left\|\dot{\boldsymbol{u}} + D^+\,\mathrm{proj}_\Omega\left(\boldsymbol{u} - \Gamma(\boldsymbol{\theta}^1 + A^\top\lambda) + d(t)\right)\right\| \\ &\le 2\|\dot{\boldsymbol{u}}\| + \sigma_{\max}(\Gamma)\|\dot{\hat{\boldsymbol{\theta}}}^1\| + \sigma_{\max}(\Gamma)\|A\|\|\dot{\lambda}\| + \|\dot{d}(t)\| \end{aligned}$$

Next, we bound $\|\dot{\hat{\boldsymbol{\theta}}}^1\|$ using the dynamics in (23):

$$\|\dot{\hat{\boldsymbol{\theta}}}\| \le \sum_{i\in\mathcal{I}} \left(\|\Sigma_i^{-1}c_i(e_i - \eta_i)\| + \sigma_i\|\Sigma_i^{-1}\|\|\theta_i - \tilde{\theta}_i\|\right)$$

On a compact set ζ_c, c_i and Σ_i are bounded, therefore, the last equation reads as:

$$\|\dot{\hat{\boldsymbol{\theta}}}\| \le L_3^*\|(\boldsymbol{e} - \boldsymbol{\eta})\| + L_4^*\|\boldsymbol{\theta}\| + L_5^*\|\tilde{\boldsymbol{\theta}}\|, \;\; L_3^*, L_4^*, L_5^* > 0$$

Now, bound on $\|D^+\dot{\boldsymbol{u}}\|$ equals to:

$$\begin{aligned} \|D^+\dot{\boldsymbol{u}}\| \le\; & 2\|\dot{\boldsymbol{u}}\| + \sigma_{\max}(\Gamma)L_3^*\|(\boldsymbol{e} - \boldsymbol{\eta})\| + \sigma_{\max}(\Gamma)L_4^*\|\boldsymbol{\theta}\| \\ & + \sigma_{\max}(\Gamma)L_5^*\|\tilde{\boldsymbol{\theta}}\| + \sigma_{\max}(\Gamma)\|A\|\|\dot{\lambda}\| + \|\dot{d}(t)\| \end{aligned} \tag{C.8}$$

On a compact set ζ_c, $\dot{\boldsymbol{u}}^\top H_{J^0}$ is bounded, therefore by combining (C.6), (C.7), (C.8) and the arithmetic mean–quadratic mean inequality, it follows:

$$\begin{aligned} \|D^+\boldsymbol{\theta}\|^2 \le\; & L_4\|\dot{\boldsymbol{u}}\|^2 + L_5\|\dot{\lambda}\|^2 + L_6\|\boldsymbol{\theta}\|^2 + L_7\|\tilde{\boldsymbol{\theta}}\|^2 + L_8\|\boldsymbol{e} - \boldsymbol{\eta}\|^2 \\ & + L_9\|\dot{d}(t)\|^2, \end{aligned}$$

for some positive L_4 to L_9. By using the previously calculated bounds, D^+V_θ reads as:

$$\begin{aligned} D^+V_\theta \le\; & -k_a\|\tilde{\eta}\|^2 - (k_b - k_cL_7)\|\tilde{\boldsymbol{\theta}}\|^2 - \left(\tfrac{1}{2} - k_cL_8\right)\|\boldsymbol{e} - \boldsymbol{\eta}\|^2 \\ & + (k_cL_4 + (\sigma + k_cL_6)L_1)\|\dot{\boldsymbol{u}}\|^2 + (\sigma + k_cL_6)L_2\|\tilde{\boldsymbol{u}}\|^2 \\ & + k_cL_5\|\dot{\lambda}\|^2 + k_cL_9\|\dot{d}(t)\|^2 + (\sigma + k_cL_6)L_3\|F(\boldsymbol{u}^*)\|. \end{aligned} \tag{C.9}$$

Primal–dual term: Unlike the full-information case, our agents use the estimate $\hat{\boldsymbol{\theta}}^1$ instead of $F(\boldsymbol{u})$. Therefore, by adding and subtracting the derivative of full-information case in the Dini derivative of (C.1), we have:

$$\begin{aligned} D^+V_\omega(\omega) = (\omega - \omega^*)^\top \Bigg[& -\omega + J_{\Phi^{-1}\mathcal{A}}\left(\omega - \Phi^{-1}\mathcal{B}(\omega)\right) \\ & - \mathrm{proj}_{\boldsymbol{\Omega}\times\mathbb{R}_+^q}\left(\omega + \boldsymbol{\Gamma}\begin{bmatrix} -F(\boldsymbol{u}) - A^\top\lambda \\ A\boldsymbol{u} - b + 2A\dot{\boldsymbol{u}} \end{bmatrix}\right) \\ & + \mathrm{proj}_{\boldsymbol{\Omega}\times\mathbb{R}_+^q}\left(\omega + \boldsymbol{\Gamma}\begin{bmatrix} -\boldsymbol{\theta}^1 - A^\top\lambda \\ A\boldsymbol{u} - b + 2A\dot{\boldsymbol{u}} \end{bmatrix} + \begin{bmatrix} d(t) \\ 0 \end{bmatrix}\right)\Bigg] \\ \le & -\tfrac{1}{2}\|\dot{\omega}\|^2 - \tfrac{\mu\sigma_{\min}}{2}\|\tilde{\boldsymbol{u}}\|^2 + \sigma_{\max}(\Gamma)\|\tilde{\boldsymbol{u}}\|\|\tilde{\boldsymbol{\theta}}\| + \|\tilde{\boldsymbol{u}}\|\|d(t)\| \\ \le & -\tfrac{1}{2}\|\dot{\omega}\|^2 - \left(\tfrac{\mu\sigma_{\min}}{2} - \tfrac{\sigma_{\max}(\Gamma)}{2k_3} - \tfrac{1}{2k_4}\right)\|\tilde{\boldsymbol{u}}\|^2 \\ & + \tfrac{\sigma_{\max}(\Gamma)k_3}{2}\|\tilde{\boldsymbol{\theta}}\|^2 + \tfrac{k_4}{2}\|d(t)\|^2, \end{aligned}$$

where the last line follows from the inequality

$$ab \le \tfrac{1}{2k}a^2 + \tfrac{k}{2}b^2, \quad \forall(a, b, k) \in (\mathbb{R}^2 \times \mathbb{R}_{>0}). \tag{C.10}$$

Complete Lyapunov candidate: Finally, the Dini derivative of V is bounded as follows:

$$\begin{aligned} D^+V_\omega + D^+V_\theta \le\; & -\left(\tfrac{1}{2} - k_cL_4 - (\sigma + k_cL_6)L_1\right)\|\dot{\boldsymbol{u}}\|^2 \\ & - \left(\tfrac{1}{2} - k_cL_5\right)\|\dot{\lambda}\|^2 - k_a\|\tilde{\eta}\|^2 - \left(\tfrac{1}{2} - k_cL_8\right)\|\boldsymbol{e} - \boldsymbol{\eta}\|^2 \\ & - \left(\tfrac{\mu\sigma_{\min}}{2} - \tfrac{\sigma_{\max}(\Gamma)}{2k_3} - \tfrac{1}{2k_4} - (\sigma - k_cL_6)L_2\right)\|\tilde{\boldsymbol{u}}\|^2 \\ & - \left(k_b - k_cL_7 - \tfrac{\sigma_{\max}(\Gamma)k_3}{2}\right)\|\tilde{\boldsymbol{\theta}}\|^2 \\ & + k_cL_9\|\dot{d}(t)\|^2 + (\sigma + k_cL_6)L_3\|F(\boldsymbol{u}^*)\|^2 + \tfrac{k_4}{2}\|d(t)\|^2 \\ \le\; & -b_1\|\dot{\boldsymbol{u}}\|^2 - b_2\|\dot{\lambda}\|^2 - k_a\|\tilde{\boldsymbol{\eta}}\|^2 b_3 - b_3\|\boldsymbol{e} - \boldsymbol{\eta}\|^2 - b_4\|\tilde{\boldsymbol{u}}\|^2 \end{aligned}$$

[1] The term $\frac{1}{2}\|\boldsymbol{e} - \boldsymbol{\eta}\|^2$ was omitted in Guay and Dochain (2017) (page 4, first column, second to last equation) and Guay et al. (2018), as it is not required.

$$- b_5\|\tilde{\boldsymbol{\theta}}\|^2 + k_c L_9 \|\dot{d}(t)\|^2 + (\sigma + k_c L_6) L_3 \|F(\boldsymbol{u}^*)\|^2 + \tfrac{k_4}{2}\|d(t)\|^2. \quad \text{(C.11)}$$

Now, we can make the last three norms arbitrarily small and b_1, b_2, b_3 positive by choosing k_c, σ and k_4 small enough, we can make b_4 positive by choosing $(\sigma_i)_{i\in\mathcal{I}}$ small enough, we can make b_5 positive by making k_b large enough. Of the mentioned parameters, only k_b and k_c cannot be chosen arbitrarily. To make k_b large enough, we chose $(K_i)_{i\in\mathcal{I}}$ and $(\rho_i)_{i\in\mathcal{I}}$ large enough, to make k_c small enough we have to chose parameters k_1 and k_2 small enough. Since the positive terms can be made arbitrarily small, we conclude that the Lyapunov derivative can be made negative on the boundary of the set ζ_c, which implies that the set is invariant. As the right-hand side of equations (19)–(22), (24) is (locally) Lipschitz on their domain, by Khalil (2002, Thm. 3.3), we conclude that their solutions exist and are unique. Furthermore, as ζ_c was chosen for arbitrary c, it follows that for any compact set K of initial conditions, it is possible to find such control parameters that for $(\boldsymbol{\eta}(0), \hat{\boldsymbol{\theta}}(0), \boldsymbol{u}(0)) \in K$, $(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}, \boldsymbol{u})$ converge to an arbitrarily small neighborhood of $(0, 0, \boldsymbol{u}^*)$, which concludes the proof. For λ we can only claim that this is bounded.

Appendix D. Proof of Theorem 4

We have to prove that there exists a timescale separation between the GNE learning scheme described in Section 3 and dynamics of the multi-agent system in (26) such that the interconnection is also stable. Let us consider a Lyapunov function candidate $V = V_\theta + V_\omega + V_z$, where V_θ and V_ω are the same as (C.1), (C.2) and V_z is formed using Standing Assumption 2 in the following way:

$$V_z(\boldsymbol{z}, \boldsymbol{u}) = \textstyle\sum_{i\in\mathcal{I}} V_i(z_i, u_i). \quad \text{(D.1)}$$

Outline of the proof: We first bound all of the terms in D^+V_z introduced by nonconstant inputs with functions of the variables $(\boldsymbol{\eta}, \hat{\boldsymbol{\theta}}, \omega, \boldsymbol{z})$, then we bound all of the terms in D^+V_θ introduced by the redefinition of error e_i in (27) in the same manner. At the end, we use the quadratic terms of the complete D^+V to show that the additional terms are majorized by the negative terms.

Multi-agent term: Let us do a change of variables $\boldsymbol{z} = \boldsymbol{x} - \pi(\boldsymbol{u})$ in (26). New dynamics read as

$$\epsilon\dot{\boldsymbol{z}} = f(\boldsymbol{z} + \pi(\boldsymbol{u}), \boldsymbol{u}) - \epsilon\nabla\pi(u)\dot{\boldsymbol{u}}. \quad \text{(D.2)}$$

Dini derivative of (D.1), by plugging in (D.2), reads as

$$\begin{aligned} D^+V_z(\boldsymbol{z}, \boldsymbol{u}) &= \nabla_{\boldsymbol{z}} V_z^\top \dot{\boldsymbol{z}} + \nabla_{\boldsymbol{u}} V_z^\top \dot{\boldsymbol{u}} \\ &= \tfrac{1}{\epsilon}\nabla_{\boldsymbol{z}} V_z(\boldsymbol{z}, \boldsymbol{u})^\top f(\boldsymbol{z} + \pi(\boldsymbol{u}), \boldsymbol{u}) \\ &\quad - \nabla_{\boldsymbol{z}} V_z(\boldsymbol{z}, \boldsymbol{u})^\top \nabla\pi(\boldsymbol{u})\dot{\boldsymbol{u}} + \nabla_{\boldsymbol{u}} V_z(\boldsymbol{z}, \boldsymbol{u})^\top \dot{\boldsymbol{u}} \end{aligned}$$

By using Standing Assumption 2 and inequality (C.10), we can further improve the bound:

$$\begin{aligned} D^+V_z(\boldsymbol{z}, \boldsymbol{u}) &\le -\tfrac{\kappa}{\epsilon}\|\boldsymbol{z}\|^2 + L_{10}\|\boldsymbol{z}\|\|\dot{\boldsymbol{u}}\| \\ &\le -\left(\tfrac{\kappa}{\epsilon} - \tfrac{L_{10}k_5}{2}\right)\|\boldsymbol{z}\|^2 + \tfrac{L_{10}}{2k_5}\|\dot{\boldsymbol{u}}\|^2, \end{aligned}$$

where $L_{10} > 0$ is the Lipschitz constant of the function $\max_{\boldsymbol{u}\in\mathcal{U}} \nabla_{\boldsymbol{u}} V_z(z, \boldsymbol{u}) - \nabla_{\boldsymbol{z}} V_z(z, \boldsymbol{u})^\top \nabla\pi(\boldsymbol{u})$ and $k_5 > 0$.

Parameter estimation term: GNE learning is identical as in the static case, apart from the measurements of the cost function. Let us denote

$$\begin{aligned} \boldsymbol{l} &:= \operatorname{col}\left((l_i)_{i\in\mathcal{I}}\right), \\ \boldsymbol{y} &:= \operatorname{col}\left((y_i)_{i\in\mathcal{I}}\right), \\ h(\boldsymbol{x}) &:= \operatorname{col}\left((h_i(u_i, \boldsymbol{u}_{-i}))_{i\in\mathcal{I}}\right). \end{aligned}$$

The difference in the measurement introduces an additional component in the bound for the derivative of the Lyapunov function of the parameter estimation term:

$$\begin{aligned} &\|\tilde{\boldsymbol{\eta}}\| \left\|\dot{\boldsymbol{y}} - \dot{\boldsymbol{l}}\right\| = \left\|\tilde{\boldsymbol{\eta}}\right\| \left\|\tfrac{d}{dt}\left(h(\boldsymbol{x}) - h(\pi(\boldsymbol{u}))\right)\right\| \\ &= \|\tilde{\boldsymbol{\eta}}\| \left\|\tfrac{1}{\epsilon}\nabla h(\boldsymbol{x}) f(\boldsymbol{x}, \boldsymbol{u}) - \nabla h(\pi(\boldsymbol{u}))\nabla\pi(\boldsymbol{u})\dot{\boldsymbol{u}}\right\| \\ &\le \tfrac{1}{\epsilon}\|\tilde{\boldsymbol{\eta}}\| \|\nabla h(\boldsymbol{x}) f(\boldsymbol{x}, \boldsymbol{u}) - \nabla h(\boldsymbol{x}) f(\pi(\boldsymbol{u}), \boldsymbol{u})\| \\ &\quad + \|\tilde{\boldsymbol{\eta}}\| \|\nabla h(\pi(\boldsymbol{u}))\nabla\pi(\boldsymbol{u})\dot{\boldsymbol{u}}\| \\ &\le \tfrac{L_{11}}{\epsilon}\|\tilde{\boldsymbol{\eta}}\|\|\boldsymbol{z}\| + L_{12}\|\tilde{\boldsymbol{\eta}}\|\|\dot{\boldsymbol{u}}\| \\ &\le \left(\tfrac{L_{11}k_6}{2\epsilon} + \tfrac{L_{12}k_7}{2}\right)\|\tilde{\boldsymbol{\eta}}\|^2 + \tfrac{L_{11}}{2\epsilon k_6}\|\boldsymbol{z}\|^2 + \tfrac{L_{12}}{2k_7}\|\dot{\boldsymbol{u}}\|^2, \end{aligned} \quad \text{(D.3)}$$

where $L_{11}, L_{12}, k_7, k_6 > 0$ and the second to last equation follows from the (local) Lipschitz continuity of the functions and the fact that the variables are bounded on a compact set $\zeta_c := \{(\boldsymbol{x}, \omega, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\theta}}) \in \boldsymbol{\mathcal{X}} \times \mathcal{U} \times \mathbb{R}^q \times \mathbb{R}^N \times \mathbb{R}^m \mid V(\boldsymbol{z}, \omega, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}) \le c\}$.

Complete Lyapunov candidate: Finally, the Dini derivative of the complete Lyapunov function candidate is:

$$\begin{aligned} &D^+V_\theta(\omega) + D^+V_\theta(\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\theta}}) + D^+V_z(\boldsymbol{z}, \boldsymbol{u}) \\ &\le -\left(\tfrac{1}{2} - k_c L_4 - (\sigma + k_c L_6)L_1 - \tfrac{L_{10}}{2k_5} - \tfrac{L_{12}}{2k_7}\right)\|\dot{\boldsymbol{u}}\|^2 \\ &\quad - \left(\tfrac{1}{2} - k_c L_5\right)\|\dot{\lambda}\|^2 - \left(k_a - \tfrac{L_{11}k_6}{2\epsilon} - \tfrac{L_{12}k_7}{2}\right)\|\tilde{\boldsymbol{\eta}}\|^2 \\ &\quad - \left(k_b - k_c L_7 - \tfrac{\sigma_{\max}(\Gamma)k_3}{2}\right)\|\tilde{\boldsymbol{\theta}}\|^2 \\ &\quad - \left(\tfrac{\mu\sigma_{\min}}{2} - \tfrac{\sigma_{\max}(\Gamma)}{2k_3} - \tfrac{1}{2k_4} - (\sigma - k_c L_6)L_2\right)\|\tilde{\boldsymbol{u}}\|^2 \\ &\quad - \left(\tfrac{\kappa}{\epsilon} - \tfrac{L_{10}k_5}{2} - \tfrac{L_{11}}{2\epsilon k_6}\right)\|\boldsymbol{z}\|^2 - \left(\tfrac{1}{2} - k_c L_8\right)\|\boldsymbol{e} - \boldsymbol{\eta}\|^2 \\ &\quad + k_c L_9\|\dot{d}(t)\|^2 + (\sigma + k_c L_6)L_3\|F(\boldsymbol{u}^*)\| + \tfrac{k_4}{2}\|d(t)\|^2 \\ &\le -b_1\|\dot{\boldsymbol{u}}\|^2 - b_2\|\dot{\lambda}\|^2 - b_3\|\tilde{\boldsymbol{\eta}}\|^2 - b_4\|\tilde{\boldsymbol{\theta}}\|^2 - b_5\|\tilde{\boldsymbol{u}}\|^2 - b_6\|\boldsymbol{z}\|^2 \\ &\quad - b_7\|\boldsymbol{e} - \boldsymbol{\eta}\|^2 + b_8\|F(\boldsymbol{u}^*)\|^2 + b_9\|\dot{d}(t)\|^2 + b_{10}\|d(t)\|^2. \end{aligned} \quad \text{(D.4)}$$

The rest follows analogously to the proof of Theorem 3 with addition of the parameter ϵ. We conclude that it is possible to find such control parameters that for $(\tilde{\boldsymbol{\eta}}(0), \tilde{\boldsymbol{\theta}}(0), \boldsymbol{u}(0), \boldsymbol{z}(0)) \in K$, $(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}, \boldsymbol{u}, \boldsymbol{z})$ converge to an arbitrarily small neighborhood of $(0, 0, \boldsymbol{u}^*, 0)$, which concludes the proof. For λ the dual variable we can only claim boundedness.

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