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Grenander functionals and Cauchy's formula

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Abstract

Let $\hat{f}_n$ be the nonparametric maximum likelihood estimator of a decreasing density. Grenander characterized this as the left-continuous slope of the least concave majorant of the empirical distribution function. For a sample from the uniform distribution, the asymptotic distribution of the L_2-distance of the Grenander estimator to the uniform density was derived in an article by Groeneboom and Pyke by using a representation of the Grenander estimator in terms of conditioned Poisson and gamma random variables. This representation was also used in an article by Groeneboom and Lopuhaä to prove a central limit result of Sparre Andersen on the number of jumps of the Grenander estimator. Here we extend this to the proof of the main result on the L_2-distance of the Grenander estimator to the uniform density and also prove a similar asymptotic normality results for the entropy functional. Cauchy's formula and saddle point methods are the main tools in our development.

KEYWORDS

Cauchy's formula, Grenander estimator, integral statistics, saddle points

This article is dedicated to the memory of Ronald Pyke.

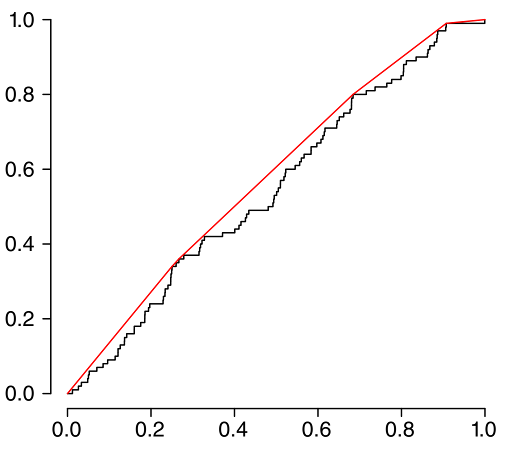
1 | INTRODUCTION

The Grenander estimator is the (nonparametric) maximum likelihood estimator of a monotone decreasing density. It was introduced in Grenander (1957), where it was proved that it is the left-continuous slope of the least concave majorant of the empirical distribution function. Some properties and limit results are discussed in Groeneboom and Jongbloed (2014) and also in Groeneboom and Jongbloed (2018) in the special issue on nonparametric inference under shape constraints of the journal *Statistical Science*. The Grenander estimator is shown in Figure 1 for a sample of size $n = 100$ from the uniform distribution on $[0, 1]$. It can be improved by using boundary penalties (in fact, the estimator is inconsistent at the boundary points 0 and 1), but this is not the concern of the present article.

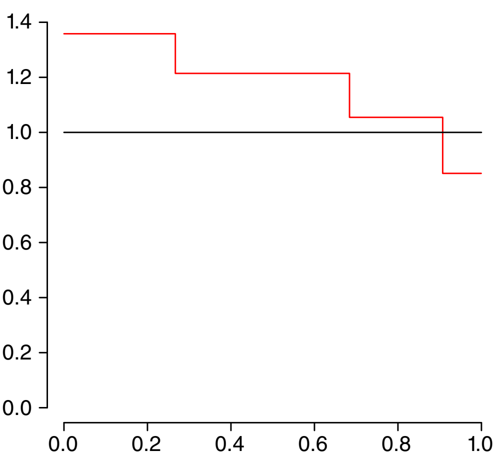
The Grenander estimator is a piecewise constant function with downward jumps at locations that correspond to the changes of slope ("kinks") of the least concave majorant of the empirical distribution function. Although the Grenander estimator is defined as the left-continuous slope of the empirical distribution function, we can make the Grenander estimator right-continuous by taking the limits on the right at its points of jump. This does not change the probability mass of the induced (absolutely continuous) probability distribution, which is absolutely continuous w.r.t. Lebesgue measure.

The number of jumps of the Grenander estimator is of order $\log n$ if the sample is from a uniform distribution (see Section 2), if the sample comes from a strictly decreasing smooth density like the exponential density, then the number of jumps is of order $cn^{1/3}$, for some constant $c > 0$. The limit behavior of the Grenander estimator for these situations is rather different. For a sample from the uniform distribution, we have for $t \in (0, 1)$:

$$\sqrt{n}\left\{\hat{f}_n(t) - 1\right\} \xrightarrow{\mathcal{D}} S_t, \tag{1}$$



(A) empirical distribution function (black) and its least concave majorant (red)



(B) Grenander estimator

FIGURE 1 Left: the empirical distribution function and its least concave majorant and right: the Grenander estimator, on the basis of a sample of size $n = 100$ from the uniform distribution on $[0, 1]$ [Color figure can be viewed at wileyonlinelibrary.com] where S_t is the slope of the least concave majorant of the standard Brownian Bridge on $[0, 1]$, see remark 2.2, p. 543 of Groeneboom (1985). The density of S_t is a function of the standard normal distribution function Φ and the standard normal density ϕ, see (3.11) in Groeneboom (1983).

In contrast with the result (1), we get for a sample from a decreasing density f on $[0, \infty)$ at a point $t \in (0, \infty)$, where f is differentiable and $f'(t) < 0$, the following result, due to Prakasa Rao in Prakasa Rao (1969):

$$n^{1/3}\left|4f(t)f'(y)\right|^{-1/3}\left\{\hat{f}_n(t) - f(t)\right\} \xrightarrow{\mathcal{D}} Z, \tag{2}$$

where $Z = \mathrm{argmax}_t\{W(t) - t^2\}$, that is, Z is the (almost surely unique) location of the maximum of two-sided Brownian motion minus the parabola $y(t) = t^2$. For further details, see, for example, Groeneboom and Jongbloed (2014) and Groeneboom and Jongbloed (2018).

Recently, integrated functionals of a monotone density were studied in Mukherjee and Sen (2019). Using the same notation as in Mukherjee and Sen (2019) the following functionals were studied:

$$\mu(h, f) = \int_0^1 h(f(x))\, dx,$$

where f is a nonincreasing function on $\mathbb{R}_+$ and h satisfies some regularity conditions. In the case that the underlying distribution is uniform, the following central limit result is proved:

Theorem 1 (theorem 3.2 in Mukherjee & Sen, 2019). *Let f be the uniform density on* $[0, 1]$. *Moreover, let $h \in C^4([0, \infty))$ and let $h''(1) \neq 0$. Then:*

$$\frac{n\left\{\mu(h, \hat{f}_n) - \mu(h, f)\right\} - \frac{1}{2}h''(1)\log n}{\sqrt{\frac{3}{4}h''(1)^2 \log n}} \xrightarrow{\mathcal{D}} N(0, 1), \tag{3}$$

where $N(0, 1)$ denotes the standard normal distribution.

We prove an analogous result for analytic functions h, defined on the positive open complex half plane. Hence, our functions h have a lot more smoothness, but are on the other hand defined on the *open* complex half plane, which makes the result applicable to functions that are not covered by the conditions in Mukherjee and Sen (2019). We assume that h satisfies the following condition.

Condition 1. The function h is analytic on the complex half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and satisfies the following conditions.

(i) $h''(1) \neq 0$.

(ii) For $t \in \mathbb{R}$:

$$|h(1 - it)| = O(t^2), \quad |t| \to \infty, \tag{4}$$

and

(iii) For $t \in \mathbb{R}$:

$$\left|h''(1 - it)\right| = O(1), \quad |t| \to \infty. \tag{5}$$

We now have the following result.

Theorem 2. *Let f be the uniform density on* $[0,1]$. *Moreover, let h satisfy Condition* 1 *and let $\hat{f}_n$ be the Grenander estimator. Then*

$$\frac{n\left\{\mu(h,\hat{f}_n)-\mu(h,f)\right\}-\frac{1}{2}h''(1)\log n}{\sqrt{\frac{3}{4}h''(1)^2\log n}}\xrightarrow{\mathcal{D}}N(0,1), \tag{6}$$

where $N(0,1)$ *denotes the standard normal distribution.*

The result is a corollary to theorem 6 in Section 4, see the remark at the end of Section 4. Examples of the application of Theorem 2 are:

Example 1. Let $h(z)=(z-1)^2$. Then $h'(z)=2(z-1)$ and $h^{''}(z)=2$. The function h is obviously analytic on the positive complex half plane. Condition 1 is fulfilled, so we get

$$\frac{1}{\sqrt{3\log n}}\left\{n\int_0^1\left\{\hat{f}_n(t)-1\right\}^2\,dt-\log n\right\}\xrightarrow{\mathcal{D}}N(0,1),\qquad n\to\infty. \tag{7}$$

This is the main result of Groeneboom and Pyke (1983). Since in Mukherjee and Sen (2019), theorem 3.2 is deduced from this result, we also get theorem 3.2 in Mukherjee and Sen (2019) back from Theorem 2.

Example 2. Let $h(z)=z\log z$. Then $h'(z)=1+\log(z)$ and $h^{''}(z)=1/z$. The function h is again analytic on the positive complex half plane. Condition 1 is fulfilled, and we get:

$$\frac{1}{\sqrt{\frac{3}{4}\log n}}\left\{n\int_0^1\hat{f}_n(t)\log\hat{f}_n(t)\,dt-\frac{1}{2}\log n\right\}\xrightarrow{\mathcal{D}}N(0,1),\qquad n\to\infty. \tag{8}$$

For this example, the conditions of theorem 3.2 in Mukherjee and Sen (2019) are not satisfied. The result follows from Theorem 2 and can be applied to the theory on a likelihood ratio test for monotonicity in Chan, Tang, and Yam (2018).

To derive limit results for the uniform distribution, a special representation in terms of gamma and Poisson random variables was given in Groeneboom and Pyke (1983), with the purpose of proving a limit result for a two-sample rank statistic introduced in the dissertation of Behnen (1974) and also for a test statistic in the combination of tests in Scholz (1983). We describe this representation now.

Let $X_1,\dots,X_n$ be a sample from the uniform distribution, and let $0=\xi_{n0}<\xi_{n1}<\dots<\xi_{nm}<\xi_{n,m+1}=1$ be the locations of the jumps of the Grenander estimator $\hat{f}_n$ for this sample, augmented with the points 0 and 1. Note that $[\xi_{n,0},\xi_{n,1}]$, $(\xi_{n,1},\xi_{n2}]$, $(\xi_{n,2},\xi_{n3}]$, …, $(\xi_{n,m},1]$ are the successive intervals of constancy of the Grenander estimator if we take the estimator to be left-continuous.

Furthermore, let D_{ni}, J_{ni}, and Q_{nj} be defined by:

$$D_{ni}=\xi_{ni}-\xi_{n,i-1},\quad i=1,\dots,m+1,$$

$$J_{ni}=n\left\{\mathbb{F}_n(\xi_{ni})-\mathbb{F}_n(\xi_{n,i-1})\right\},\quad i=1,\dots,m+1,$$

$$Q_{nj}=\#\left\{i:J_{ni}=j\right\},$$

where $\mathbb{F}_n$ is the empirical distribution function of the sample $X_1, \dots, X_n$, and where m is the number of jumps of the Grenander estimator.

Next, let $\{N_j : j \geq 1\}$ be independent Poisson random variables with $\mathbb{E}N_j = 1/j$, and let, for each i, $\{S_{ji},\ i, j \geq 1\}$ be a collection of independent gamma random variables, independent of the N_j, where S_{ji} is Gamma(j, 1) (the sum of j independent standard exponentials). We define:

$$S_n = \sum_{j=1}^{n} \sum_{i=1}^{N_j} S_{ji}, \quad T_n = \sum_{j=1}^{n} jN_j. \tag{9}$$

and

$$\boldsymbol{S}^{(n)} = \left(S_{11}, \dots, S_{1,N_1}, \dots, S_{n1}, \dots, S_{n,N_n}\right), \quad \boldsymbol{N}^{(n)} = (N_1, \dots, N_n).$$

Note that there are N_1 induced spacings ξ_{ni} (intervals of constancy of $\hat{f}_n$) of length 1, N_2 induced spacings ξ_{ni} (intervals of constancy of $\hat{f}_n$), consisting of two consecutive original spacings between locations of jumps of the least concave majorant, and so forth, where N_j can be zero.

We now have the following representation theorem:

Theorem 3 (theorem 2.1 in Groeneboom & Pyke, 1983).

$$\left(nD_{n1}, \dots, nD_{n,m+1}; Q_{n1}, \dots, Q_{n,m}\right) \stackrel{\mathcal{D}}{=} \left(\boldsymbol{S}^{(n)}, \boldsymbol{N}^{(n)} | S_n = n, T_n = n\right).$$

Remark 1. For specificity, the random variables $S_{ji},\ i = 1, \dots, N_j$, and $D_{ji},\ i = 1, \dots Q_{nj}$, are ordered in theorem 2.1 in Groeneboom and Pyke (1983). There is also a zero-step spacing introduced in Groeneboom and Pyke (1983), but this does not seem to be necessary.

Using this representation, we can reduce the proofs of the limit behavior of global functionals of the Grenander estimator to a theorem for gamma and Poisson random variables, under the condition $(T_n, S_n) = (n, n)$. For convenience in later proofs, we also use a further standardization of (S_n, T_n):

$$V_n = n^{-1/2} \left\{ S_n - \sum_{j=1}^{n} jN_j \right\} = n^{-1/2} \left\{S_n - T_n\right\}, \quad W_n = \frac{T_n}{n}, \tag{10}$$

where S_n and T_n are defined by (9). A conditional central limit theorem for functionals of the Grenander estimator can then be proved under the condition:

$$V_n = 0, \quad W_n = 1. \tag{11}$$

The infinitely divisible limit distribution of the pair (V_n, W_n) was given in lemma 3.1 of Groeneboom and Pyke (1983), but unfortunately lemma 3.1 of Groeneboom and Pyke (1983) contains a rather silly error (the u^2 in the representation of the characteristic function should be u). The correct version of this result is given in Lemma 1, where also the origin of the error is explained. The proof in Groeneboom and Lopuhaä (1993) does not use the result on the limit distribution of (V_n, W_n), so is not influenced by the erroneous Lemma 3.1 in Groeneboom and Pyke (1983).

We use methods different from those in Groeneboom and Pyke (1983). The conditional central limit theorem was proved in Groeneboom and Pyke (1983) using Le Cam's article (Le Cam, 1958; an article apparently published without his permission, as became clear in a conversation of the author with him). In the present context, where we clearly have to deal with nonstandard asymptotics, Le Cam (1958) does not seem to be the right tool to use. We replace this by a direct analysis of the characteristic function. The crucial tools here are Cauchy's formula and saddle point methods, using contour integration in the complex plane. To illustrate our method in a simple setting, we give a shortened version of the proof in Groeneboom and Lopuhaä (1993) of Sparre Andersen's result (Sparre Andersen, 1954) in Section 2.

2 | SPARRE ANDERSEN'S RESULT

To illustrate our method in the simplest setting, we give a short version of the proof in Groeneboom and Lopuhaä (1993) of the following remarkable result of Sparre Andersen in Sparre Andersen (1954).

Theorem 4 (Sparre Andersen's result). *Let $X_1,\dots,X_n$ be a sample from the Uniform$(0,1)$ distribution and let N_{jumps} be the number of jumps of the Grenander estimator for this sample. Then*

$$\frac{N_{\text{jumps}}-\log n}{\sqrt{\log n}} \xrightarrow{\mathcal{D}} N(0,1),$$

where $N(0,1)$ is the standard normal distribution.

Proof. Let, for the sample $X_1,\dots,X_n$, U_n be defined by:

$$U_n = \frac{\sum_{j=1}^n N_j - \log n}{\sqrt{\log n}},$$

and let T_n be defined as in (9). Using (part of) the representation, introduced in Section 1, we only have to prove that $(U_n|T_n = n)$ tends in law to a standard normal distribution. To this end, we consider the conditional characteristic function

$$\mathbb{E}\left\{e^{isU_n}|T_n = n\right\}.$$

Lemma 3.2 of Groeneboom and Pyke (1983) implies:

$$\mathbb{P}\{T_n = n\} = \exp\left\{-\sum_{j=1}^n \frac{1}{j}\right\}.$$

Hence we get, by Fourier inversion and using the notation $b_n = \sqrt{\log n}$,

$$\mathbb{E}\left\{e^{isU_n}|T_n = n\right\} = \exp\left\{\sum_{j=1}^n \frac{1}{j}\right\}\frac{1}{2\pi}\int_{u=-\pi}^{\pi} \mathbb{E}\, e^{isU_n+iuT_n-inu}\,du$$

$$= \exp\left\{\sum_{j=1}^n \frac{1}{j}\right\}\frac{1}{2\pi}\int_{u=-\pi}^{\pi} \exp\left\{-isb_n + \sum_{j=1}^n \frac{1}{j}\left(e^{iju+is/b_n}-1\right) - inu\right\}\,du$$

$$= e^{-isb_n}\frac{1}{2\pi}\int_{u=-\pi}^{\pi} \exp\left\{e^{is/b_n}\sum_{j=1}^n \frac{1}{j}e^{iju} - inu\right\}\,du.$$

Denoting the contour $u \mapsto e^{iu}$, $u \in [-\pi, \pi)$, by C, we write this in the form

$$e^{-isb_n}\frac{1}{2\pi i}\int_C \exp\left\{\delta_n \sum_{j=1}^n \frac{z^j}{j}\right\} z^{-n-1}\,dz, \tag{12}$$

where δ_n is given by:

$$\delta_n = e^{is/b_n}.$$

The expression in (12), multiplied by e^{isb_n}, is by Cauchy's formula equal to the coefficient of z^n in the power series around $z = 0$ of the function:

$$z \mapsto \exp\left\{\delta_n \sum_{j=1}^n \frac{z^j}{j}\right\}.$$

Comparing this with the power series of the function $z \mapsto (1-z)^{-\delta_n}$, we see that the coefficient of z^n is the same in both series. This coefficient is:

$$(-1)^n\binom{-\delta_n}{n} = \prod\left(1+\frac{\delta_n-1}{j}\right) = \exp\left\{\sum_{j=1}^n \log\left(1+\frac{\delta_n-1}{j}\right)\right\}.$$

Hence we get:

$$\begin{aligned}
\mathbb{E}\left\{e^{isU_n}|T_n=n\right\} &= \exp\left\{-isb_n+\sum_{j=1}^n \log\left(1+\frac{\delta_n-1}{j}\right)\right\}\\
&= \exp\left\{-isb_n+\sum_{j=1}^n \log\left(1+\frac{e^{is/b_n}-1}{j}\right)\right\}\\
&= \exp\left\{-isb_n+\sum_{j=1}^n \frac{e^{is/b_n}-1}{j}+o(1)\right\} = \exp\left\{-\frac{s^2}{2b_n^2}\sum_{j=1}^n \frac{1}{j}+o(1)\right\}\\
&= \exp\left\{-\frac{1}{2}s^2+o(1)\right\}.
\end{aligned}$$

■

3 | THE LIMIT DISTRIBUTION OF THE CONDITIONING VARIABLES (V_n, W_n)

Let the pair (V_n, W_n) be defined by (10). We prove the following lemma, which corrects lemma 3.1 in Groeneboom and Pyke (1983).

Lemma 1. *The pair (V_n, W_n) converges in distribution to (V, W), where (V, W) has the infinitely divisible characteristic function*

$$\phi_{(V,W)}(t,u) = \exp\left\{\int_0^1 \frac{e^{-\left(\frac{1}{2}t^2-iu\right)y}-1}{y}\,dy\right\}.$$

Proof. We have:

$$
\begin{aligned}
\mathbb{E}\exp\{itV_n+iuW_n\} &= \exp\left\{\sum_{j=1}^n \frac{e^{iuj/n}\phi_{S_{j1}-j}(tn^{-1/2})-1}{j}\right\}\\
&= \exp\left\{\sum_{j=1}^n \frac{e^{ij(u/n-tn^{-1/2})}\left(1-itn^{-1/2}\right)^{-j}-1}{j}\right\},
\end{aligned}
\tag{13}
$$

where $\phi_{S_{j1}-j}$ is the characteristic function of the centered gamma variable $S_{j1}-j$, see (3.9) of Groeneboom and Pyke (1983).

Writing $y_{j,n}=j/n$, and noting that for $y\in(0,1)$:

$$
\frac{e^{iny(u/n-tn^{-1/2})}\left(1-itn^{-1/2}\right)^{-ny}-1}{y}=\frac{e^{iyu-\frac12 t^2y}-1}{y}-\frac13 ie^{-\frac12 t^2y+iuy}t^3n^{-1/2}+O\left(n^{-1}\right),
$$

and that the limit of the expression on the left for $y\downarrow 0$ is equal to:

$$
i(-t\sqrt{n}+u)-n\log(1-it/\sqrt{n})=iu-\frac12 t^2+O\left(n^{-1/2}\right),\quad n\to\infty,
$$

we can write the exponent in the form

$$
\sum_{j=1}^n \frac{e^{-\frac12 t^2y_{j,n}+iuy_{j,n}}-1}{ny_{j,n}}+O\left(n^{-1/2}\right),
$$

(it is here that the mistake was made in Groeneboom & Pyke, 1983, in the formula after (3.9) on p. 333), hence we get a Riemann sum converging to the integral

$$
\int_0^1 \frac{e^{-\frac12 t^2y+iuy}-1}{y}\,dy.
$$

The infinite divisibility of the limit distribution is shown below. ■

Remark 2. In Groeneboom and Pyke (1983) first, the limit distribution of U_n is computed, using moment conditions (going up to the eighth moments). Next, the limit distribution of (V_n,W_n) is computed and it is stated that this distribution is infinite divisible and has no normal component, implying that therefore the limit (V_n,W_n) has to be independent of the limit of U_n.

The s^2u^2 in the exponent of the characteristic function of the limit distribution of (V_n,W_n) in lemma 3.1 of Groeneboom and Pyke (1983) should be s^2u. The incorrect u^2 arose on p. 333 of Groeneboom and Pyke (1983), where the limit of the characteristic function of the rescaled gamma random variable $(S_j-j)/\sqrt{n}$ was given by $\exp\{-s^2u^2/2\}$ instead of $\exp\{-s^2u/2\}$. This also invalidates the ensuing remarks on p. 333 of Groeneboom and Pyke (1983). We correct these remarks below.

The distribution of (V,W) is infinitely divisible, as we now show. A general characterization of infinitely divisible distributions in $\mathbb{R}^d$ is given in Sato (2001) and given below for convenience.

Theorem 5 (theorem 1.3 in Sato, 2001, Lévy-Khintchine representation). *If the distribution μ is infinitely divisible, then its characteristic function $\hat{\mu}(\mathbf{s}) = \int_{\mathbb{R}^d} \exp\{i\langle \mathbf{s}, \mathbf{z}\rangle\}\, d\mu(\mathbf{z})$ is given by:*

$$\hat{\mu}(z) = \exp\left\{-\frac{1}{2}\mathbf{s}^T\mathbf{A}\mathbf{s} + \int_{R^d}\left(e^{i\langle \mathbf{s},\mathbf{z}\rangle} - 1 - i\langle \mathbf{s}, \mathbf{z}\rangle 1_{\{\|\mathbf{z}\|\le 1\}}(\mathbf{z})\right)\, d\nu(\mathbf{z}) + i\langle \boldsymbol{\delta}, \mathbf{s}\rangle\right\}, \tag{14}$$

where $\mathbf{A}$ is a symmetric nonnegative-definite $d \times d$ matrix, $\|\cdot\|$ is the Euclidean norm, ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d}\left(\|\mathbf{z}\|^2 \wedge 1\right)\, d\nu(\mathbf{z}) < \infty$, and where $\boldsymbol{\delta} \in \mathbb{R}^d$. The representation (14) by $\mathbf{A}$, ν, and $\boldsymbol{\delta}$ is unique. Conversely, for any choice of $\mathbf{A}$, ν, and $\boldsymbol{\delta}$ satisfying the conditions above, there exists an infinite divisible distribution μ having characteristic function (14).

In the present situation, we can take $\mathbf{A}$ the 2×2 matrix with zeroes, $\boldsymbol{\delta} = (0, 0)^T$ and define ν by the density

$$\frac{\partial^2 \nu(v, w)}{\partial v \partial w} = \phi\left(v/\sqrt{w}\right) w^{-3/2} 1_{(0,1)}(w),$$

where ϕ is the standard normal density. With these choices of $\mathbf{A}$, $\boldsymbol{\delta}$, and ν we get, using the notation $\mathbf{s} = (t, u)^T$ and $\mathbf{z} = (v, w)^T$,

$$\exp\left\{-\frac{1}{2}\mathbf{s}^T A\mathbf{s} + \int_{R^2}\left(e^{i\langle \mathbf{s},\mathbf{z}\rangle} - 1 - i\langle \mathbf{s}, \mathbf{z}\rangle 1_{\{\|\mathbf{z}\|\le 1\}}(\mathbf{z})\right)\, d\nu(\mathbf{z}) + i\langle \boldsymbol{\delta}, \mathbf{s}\rangle\right\}$$
$$= \exp\left\{\int_{y=0}^1 \frac{e^{-\frac{1}{2}t^2 y + iuy} - 1}{y}\, dy\right\}.$$

We note that the computer package Mathematica evaluates the characteristic function for (V, W) in the following way:

$$\exp\left\{\int_{y=0}^1 \frac{e^{-\frac{1}{2}t^2 y + iuy} - 1}{y}\, dy\right\} = \exp\left\{-\gamma - \Gamma(0, \frac{1}{2}t^2 - iu) - \log\left(\frac{1}{2}t^2 - iu\right)\right\}, \tag{15}$$

where γ is Euler's gamma and $\Gamma(0, \frac{1}{2}t^2 - iu)$ is the complementary incomplete gamma function, defined by:

$$\Gamma\left(0, \frac{1}{2}t^2 - iu\right) = \exp\left\{-\frac{1}{2}t^2 + iu\right\}\int_0^\infty \exp\left\{-\left(\frac{1}{2}t^2 - iu\right)x\right\}(1 + x)^{-1}\, dx, \quad t \ne 0.$$

see, for example, (2.01) on p. 109 of Olver (1974).

4 | CENTRAL LIMIT THEOREM FOR $\int h(\hat{f}_n(x))\, dx$

In this section, we use the notation

$$b_n = \sqrt{\frac{3}{4}h''(1)^2 \log n}, \quad c_n = \sqrt{n}. \tag{16}$$

Using the conditioning of Section 1, the statistic $\int h(\hat{f}_n(x))\,dx$ has the following representation:

$$\int h(\hat{f}_n(x))\,dx = n^{-1}\sum_{j=1}^{n}\sum_{i=1}^{N_j} h\left(\frac{j}{S_{ji}}\right) S_{ji},$$

where the Poisson random variables N_j and the gamma random variables S_{ji} are defined as in (9), and where we condition on $(V_n, W_n) = (0,1)$, where (V_n, W_n) is defined by (10). We define

$$U_n = \frac{1}{\sqrt{\frac{3}{4}h''(1)^2\log n}}\left\{\sum_{j=1}^{n}\sum_{i=1}^{N_j}\left[\left(h\left(\frac{j}{S_{ji}}\right) - h(1)\right)S_{ji} + h'(1)\left(S_{ji} - j\right)\right] - \frac{1}{2}h''(1)\log n\right\}. \tag{17}$$

Remark 3. The terms $h'(1)\left(S_{ji} - j\right)$ are present in U_n as variance reducing terms and give, after the summation over i and j, a zero contribution to U_n if $(V_n, W_n) = (0,1)$. Also note that

$$h(1)\sum_{j=1}^{n}\sum_{i=1}^{N_j} S_{ji} = nh(1) = n\mu(h,f),$$

if the condition $(V_n, W_n) = (0,1)$ is satisfied.

We assume that the function h satisfies Condition 1 and first consider the conditional density of V_n, given $W_n = 1$.

Lemma 2. *The conditional density of V_n, given $W_n = 1$, is the density of a centered and standardized Gamma$(n,1)$ variable:*

$$f_{V_n|W_n=1}(x) = \Gamma(n)^{-1}n^{n-1/2}e^{-n-x\sqrt{n}}\left(1 + x/\sqrt{n}\right)^{n-1}, \quad x \in \mathbb{R}.$$

The density $f_{V_n|W_n=1}(x)$ converges uniformly to the standard normal density, as $n \to \infty$.

Proof. By Fourier inversion, the conditional characteristic function is given by:

$$\begin{aligned}
&\frac{1}{\mathbb{P}\{W_n = 1\}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbb{E}\exp\left\{itV_n + niuW_n - niu\right\}\,du\\
&= \frac{1}{\mathbb{P}\{W_n = 1\}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left\{\sum_{j=1}^{n}\frac{e^{ij(u-t/c_n)}(1 - it/c_n)^{-j} - 1}{j} - niu\right\}\,du,
\end{aligned}$$

where $c_n = \sqrt{n}$, see (13). Denoting the contour $w \mapsto e^{iw}$, $w \in [-\pi,\pi)$, by C, we write this in the form

$$\begin{aligned}
&P\{W_n = 1\}^{-1}\exp\left\{-\sum_{j=1}^{n}\frac{1}{j}\right\}\frac{1}{2\pi i}\int_C \exp\left\{\sum_{j=1}^{n}\frac{(\beta_n(t)z)^j}{j}\right\}z^{-n-1}\,dz\\
&= \frac{1}{2\pi i}\int_C \exp\left\{\sum_{j=1}^{n}\frac{(\beta_n(t)z)^j}{j}\right\}z^{-n-1}\,dz,
\end{aligned} \tag{18}$$

where $\beta_n(t)$ is given by:

$$\beta_n(t) = e^{-\frac{it}{c_n}}\left(1-\frac{it}{c_n}\right)^{-1},$$

and where we use:

$$\mathbb{P}\{W_n = 1\} = \exp\left\{-\sum_{j=1}^{n}\frac{1}{j}\right\},$$

by lemma 3.2 of Groeneboom and Pyke (1983). An application of Cauchy's formula yields:

$$\frac{1}{2\pi i}\int_C \exp\left\{\sum_{j=1}^{n}\frac{(\beta_n(t)z)^j}{j}\right\} z^{-n-1}\,dz = \beta_n(t)^n,$$

where we use that the coefficient of z^n in the power series around $z=0$ of the function $z \mapsto \exp\{\sum_{j=1}^{n}(\beta_n(t)z)^j/j\}$ is the same as the coefficient of z^n in the power series of the function $z \mapsto (1-\beta_n(t)z)^{-n}$.

Hence,

$$\mathbb{E}\left\{e^{itV_n}|W_n = 1\right\} = \frac{1}{\mathbb{P}\{W_n = 1\}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbb{E}\exp\left\{itV_n + niuW_n - niu\right\}\,du$$
$$= \beta_n(t)^n = \exp\left\{-nit/c_n\right\}\left(1-\frac{it}{c_n}\right)^{-n}.$$

This is just the characteristic function of the sum of n standardized exponential variables, and hence its density tends uniformly to the standard normal density by theorem 2 on p. 516 of Feller (1971). ■

Hence, in particular, we get:

$$\mathbb{P}\{W_n = 1\}f_{V_n|W_n=1}(0) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\sum_{j=1}^{n}1/j\right\}(1+o(1)) = \frac{e^{-\gamma}}{n\sqrt{2\pi}}(1+o(1)),\quad n\to\infty,$$

where γ is Euler's gamma. The characteristic function of $(U_n|V_n = 0, W_n = 1)$ is therefore given by

$$\frac{1}{4\pi^2}\int_{t=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathbb{E}e^{isU_n+itV_v+iuW_n-inu}\,du\,dt\Big/\left(\mathbb{P}\{W_n = 1\}f_{V_n|W_n=1}(0)\right)$$
$$\sim \frac{ne^{\gamma}}{(2\pi)^{3/2}}\int_{t=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathbb{E}e^{isU_n+itV_v+iuW_n-inu}\,du\,dt. \tag{19}$$

We now consider the characteristic function ϕ_{nj}, defined by:

$$\phi_{nj}(s,t) = E\exp\left\{is\frac{\left\{h(j/S_{j1}) - h(1)\right\}S_{j1} + h'(1)\left(S_{j1}-j\right)}{b_n} + it\frac{S_{j1}-j}{c_n}\right\}. \tag{20}$$

which involves the components of the random variables U_n and V_n. Its asymptotic behavior is determined using a saddle point method.

Lemma 3. *The characteristic function (20) satisfies*

$$\phi_{nj}(s,t) \sim \frac{1}{j}\exp\left\{-\frac{jt^2}{2n}+O\left(\frac{j}{nb_n}\right)\right\}\left\{1+\frac{ish''(1)}{2b_n}-\frac{3s^2h''(1)^2}{8b_n^2}\right\},\quad j\to\infty. \tag{21}$$

uniformly for t in a bounded interval.

Proof. After a change of variables, $\phi_{nj}(s,t)$ can be written:

$$\begin{aligned}&\frac{j^j}{\Gamma(j)}\int_{x=0}^{\infty}\exp\left[j\left\{is\frac{\{h(1/x)-h(1)\}x+h'(1)(x-1)}{b_n}+\frac{it(x-1)}{c_n}-x+\log x\right\}\right]x^{-1}\,dx\\ &=\frac{j^j}{\Gamma(j)}\int_{x=0}^{\infty}\exp\left\{jf_{n,s,t}(x)\right\}x^{-1}\,dx,\end{aligned}$$

where $f_{n,s,t}$ is defined on the right half plane by:

$$f_{n,s,t}(z)=is\frac{\{h(1/z)-h(1)\}z+h'(1)(z-1)\}}{b_n}+\frac{it(z-1)}{c_n}-z+\log z,\quad \mathrm{Re}(z)>0, \tag{22}$$

The derivative of $f_{n,s,t}$ is given by

$$f'_{n,s,t}(z)=\frac{is\{h(1/z)-h(1)\}}{b_n}-\frac{ish'(1/z)}{b_n z}+\frac{ish'(1)}{b_n}+\frac{it}{c_n}-1+\frac{1}{z}. \tag{23}$$

A saddle point is given by the equation

$$f'_{n,s,t}(z)=0.$$

Multiplying both sides of this equation with z, we get the equation

$$z=g_n(z),\quad g_n(z)\stackrel{\text{def}}{=}\left(1-\frac{it}{c_n}\right)^{-1}\left\{1+\frac{is\{h(1/z)-h(1)\}z}{b_n}-\frac{ish'(1/z)}{b_n}+\frac{ish'(1)z}{b_n}\right\}. \tag{24}$$

This equation has, for sufficiently large n, a unique solution in a neighborhood of $z_0(t)=\left(1-\frac{it}{c_n}\right)^{-1}$, as is clear from the following properties.

(i)

$$g_n'(z)=\left(1-\frac{it}{c_n}\right)^{-1}b_n^{-1}\left\{ish(1)+ish(1/z)+ish'(1)-\frac{ish'(1/z)}{z}+\frac{ish''(1/z)}{z^2}\right\}\to 0,\ n\to\infty.$$

(ii) For $z_0(t)=\left(1-\frac{it}{c_n}\right)^{-1}$ we have:

$$g_n(z_0(t))-z_0(t)=-\frac{is\left\{h(1)-h(1-it/c_n)-h'(1)/c_n+(1-it/c_n)h'(1-it/c_n)\right\}}{b_n(1-it/c_n)^2}\to 0,\ n\to\infty,$$

see, for example, (3.3), p. 56 of Dieudonné (1968). Hence, the saddle point is given by a solution of Equation (24) and can be found by the simple iteration $z_{k+1} = g_n(z_k)$, starting at $z_0(t)$. We can, however, also take $z_0(t)$ itself instead of the real saddle point for the asymptotic expansion, since this gives us the same terms in the expansion we need.

The value of $f_{n,s,t}$ at $z_0(t)$ has the following expansion:

$$f_{n,s,t}(z_0(t)) = -1 - \frac{t^2 + ist^2h''(1)/(2b_n) + o\left(b_n^{-1}\right)}{2n} = -1 - \frac{t^2}{2n} + O\left(\frac{1}{nb_n}\right). \tag{25}$$

Furthermore,

$$f''_{n,s,t}(z) = \frac{-b_n z + ish''(1/z)}{b_n z^3}.$$

hence,

$$f''_{n,s,t}(z_0(t)) = -1 + \frac{ish''(1)}{b_n} + O\left(c_n^{-1}\right).$$

It follows that we get:

$$\frac{\alpha_n}{|f''_{n,s,t}(z_0(t))|^{1/2}} \approx 1 + \frac{ish''(1)}{2b_n} - \frac{3h''(1)^2s^2}{8b_n^2} + O\left(c_n^{-1}\right) = 1 + \frac{ish''(1)}{2b_n} - \frac{s^2}{2\log n} + O\left(c_n^{-1}\right),$$

where $\alpha_n = \exp\left\{ish''(1)/(2b_n)\right\}$ is a complex number with absolute value 1, corresponding to the argument of the main axis of the saddle point (note that the argument of this axis is $\frac{1}{2}\pi - \frac{1}{2}\arg f''_{n,s,t}(z_0(t))$, see de Bruijn, 1981, p. 84).

Evaluating the integrand at $z_0(t)$, and applying Stirling's formula on $\Gamma(j)$, we obtain the following asymptotic representation:

$$\begin{aligned}\phi_{nj}(s,t) &= \frac{j^j}{\Gamma(j)}\int_{x=0}^{\infty} \exp\left\{jf_{n,s,t}(x)\right\} x^{-1}\,dx \sim \frac{\alpha_n\sqrt{2\pi}e^{j\{f_{n,s,t}(z_0(t))+1\}}}{\sqrt{2\pi j}\sqrt{j|f''_{n,s,t}(z_0(t))|}} \\ &\sim \frac{1}{j}\exp\left\{-\frac{jt^2}{2n} + O\left(\frac{j}{nb_n}\right)\right\}\left\{1 + \frac{ish''(1)}{2b_n} - \frac{3s^2h''(1)^2}{8b_n^2}\right\}, \quad j \to \infty. \end{aligned} \tag{26}$$

see, for example, de Bruijn (1981), (5.10.3) on p. 92 for the first asymptotic equivalence. The second asymptotic equivalence holds uniformly for t in a bounded interval. Note that this corresponds to changing the path of integration for x to a path in the complex plane, going through the saddle point. ∎

We can now prove the following property of the characteristic function of (U_n, V_n, W_n). This is "almost" the Fourier inversion for (V_n, W_n), but we still have to extend the inversion for V_n to the whole real line. Cauchy's formula is an essential ingredient of the proof of Lemma 4.

Lemma 4. *Let U_n be defined by (17) and let h satisfy Condition 1. Then, for each $M > 0$:*

$$\frac{ne^{\gamma}}{(2\pi)^{3/2}}\int_{t=-M}^{M}\int_{-\pi}^{\pi} \mathbb{E}e^{isU_n+itV_v+niuW_n-inu}\,du\,dt \to \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}}\int_{t=-M}^{M} e^{-\frac{1}{2}t^2}\,dt, \quad n \to \infty,$$

where γ is Euler's gamma.

Proof. Let ϕ_{nj} be defined by (20). Since, by (16) and (17),

$$U_n = \frac{1}{\sqrt{\frac{3}{4}h''(1)^2 \log n}} \left\{ \sum_{j=1}^{n} \sum_{i=1}^{N_j} \left[\left(h\left(\frac{j}{S_{ji}}\right) - h(1) \right) S_{ji} + h'(1)\left(S_{ji} - j\right) \right] - \frac{1}{2}h''(1)\log n \right\}$$
$$= b_n^{-1} \left\{ \sum_{j=1}^{n} \sum_{i=1}^{N_j} \left[\left(h\left(\frac{j}{S_{ji}}\right) - h(1) \right) S_{ji} + h'(1)\left(S_{ji} - j\right) \right] - \frac{1}{2}h''(1)\log n \right\},$$

where

$$b_n = \sqrt{\frac{3}{4}h''(1)^2 \log n},$$

we get, evaluating the probabilities for the Poisson random variables N_j in the third line:

$$\mathbb{E}\exp\left\{isU_n + itV_n + iuW_n\right\} = \exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n\right\} \mathbb{E}\left\{ \prod_{j=1}^{n} \left\{\phi_{nj}(s,t)\right\}^{N_j} e^{iuN_j/n} \right\}$$
$$= \exp\left\{ -\frac{1}{2}ish''(1)(\log n)/b_n + \sum_{j=1}^{n} \frac{1}{j}\left\{\phi_{nj}(s,t)e^{iju/n} - 1\right\} \right\}$$
$$= \exp\left[-\frac{1}{2}ish''(1)(\log n)/b_n + \sum_{i=1}^{n} \frac{1}{j}\left\{ \exp\left(-\frac{jt^2}{2n} + \frac{iju}{n}\right) - 1 \right\} \right.$$
$$\left. + \left\{ \frac{ish''(1)}{2b_n} - \frac{3s^2h''(1)^2}{8b_n^2} \right\} \sum_{j=1}^{n} \frac{1}{j} \exp\left(-\frac{jt^2}{2n} + \frac{iju}{n}\right) + o(1) \right]. \tag{27}$$

As in the proof of Lemma 2, we consider the contour $w \mapsto e^{iw}$, $w \in [-\pi, \pi)$, and denote this contour by C. Hence, integrating (27) w.r.t. u and changing variables we get:

$$\frac{1}{2\pi}\int_{u=-\pi}^{\pi} \mathbb{E}\exp\left\{isU_n + itV_n + inuW_n - inu\right\}\,du \tag{28}$$

$$= \exp\left\{ -\frac{1}{2}ish''(1)(\log n)/b_n - \sum_{j=1}^{n}\frac{1}{j} \right\} \frac{1}{2\pi i}\int_C \exp\left\{ \sum_{j=1}^{n} \frac{\delta_n(\beta_n(t)z)^j}{j} \right\} z^{-n-1}\,dz, \tag{29}$$

where

$$\beta_n(t) \sim \exp\left(-\frac{t^2}{2n}\right), \tag{30}$$

and

$$\delta_n(t) = \frac{\alpha_n}{\sqrt{|f''_{n,s,t}(z_0(t))|}} = 1 + \frac{ish''(1)}{2b_n} - \frac{3s^2h''(1)^2}{8b_n^2} + O\left(b_n^{-3}\right), \tag{31}$$

and where α_n is a complex number of absolute value 1, corresponding to the angle of the main axis through the (approximate) saddle point $z_0(t)$.

Hence we get, by Cauchy's formula, for $t \in [-M, M]$ and M arbitrarily large,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{E}\exp\{isU_n + itV_n + inuW_n - niu\}\; du \\
&\quad\sim \exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \sum_{j=1}^{n}\frac{1}{j}\right\}(-1)^n\binom{-\delta_n(t)}{n}\beta_n(t)^n \\
&= \exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \sum_{j=1}^{n}\frac{1}{j}\right\}\prod_{j=1}^{n}\left(1+\frac{\delta_n(t)-1}{j}\right)\beta_n(t)^n \\
&\quad\sim \exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \sum_{j=1}^{n}\frac{1}{j}\right\}\exp\left\{-\frac{1}{2}t^2 + \sum_{j=1}^{n}\log\left(1+\frac{\delta_n(t)-1}{j}\right)\right\} \\
&= \exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \log n - \gamma\right\}\exp\left\{-\frac{1}{2}t^2 + \{\delta_n(t)-1\}\sum_{j=1}^{n}\frac{1}{j} + o(1)\right\} \\
&= \frac{1}{n}\exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \gamma\right\}\exp\left\{-\frac{1}{2}t^2 + \left\{\frac{ish''(1)}{2b_n} - \frac{3s^2h''(1)^2}{8b_n^2}\right\}\sum_{j=1}^{n}\frac{1}{j} + o(1)\right\}.
\end{aligned} \tag{32}$$

Thus:

$$\begin{aligned}
&\frac{ne^{\gamma}}{(2\pi)^{3/2}}\int_{t=-M}^{M}\int_{-\pi}^{\pi}\mathbb{E}e^{isU_n+itV_v+niuW_n-inu}\; du\; dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{t=-M}^{M}\exp\left\{-\frac{1}{2}t^2 - \frac{3s^2h''(1)^2}{8b_n^2}\sum_{j=1}^{n}\frac{1}{j} + o(1)\right\}\; dt \\
&= \exp\left\{-\frac{1}{2}s^2 + o(1)\right\}\frac{1}{\sqrt{2\pi}}\int_{t=-M}^{M}\exp\left\{-\frac{1}{2}t^2\right\}\; dt.
\end{aligned}$$

■

We still have to prove that the remaining part of integral w.r.t. the integration variable t can be made arbitrarily small by choosing M large. To this end, we split the remaining region into two regions: $A_1 = \{t \in \mathbb{R} : M < |t| \le \delta n^{1/2}\}$ and $A_2 = \{t \in \mathbb{R} : |t| > \delta n^{1/2}\}$. This split-up is familiar from inversion theorems for densities, see, for example, the proof of theorem 2 on p. 516 of Feller (1971). We start with the region $A_1 = \{t \in \mathbb{R} : M < |t| \le \delta n^{1/2}\}$.

Lemma 5. *Let h satisfy Condition* 1. *Then there exists for each* $\varepsilon > 0$ *an* $M > 0$ *and* $\delta > 0$ *such that*

$$\left|\int_{t:M<|t|\le\delta n^{1/2}}\int_{u=-\pi}^{\pi}\mathbb{E}\exp\{isU_n + itV_n + inuW_n - niu\}\; du\; dt\right| < \varepsilon,$$

Proof. We consider again the expansion of the function $f_{n,s,t}$ defined by (22) at the point $z_0(t) = (1 - it/c_n)^{-1}$. We get:

$$f_{n,t,u}(z_0(t)) = -1 - \frac{t^2}{2n} - \frac{ist^2h''(\theta)}{2nb_n},$$

where θ is a point on the line segment between 1 and $1/(1-it/c_n)$. Likewise

$$f''_{n,t,u}(z_0(t)) = -1 + \frac{ish''(1)}{b_n} + \frac{2i\theta'}{c_n},$$

where θ' is a point on the line segment between 1 and $1/(1-it/c_n)$.

Hence we have a local expansion

$$\phi_{nj}(s,t) \sim \frac{1}{j}\exp\left\{-\frac{jt^2}{2n} + O\left(\frac{jt^2}{nb_n}\right)\right\}\left\{1 + \frac{ish''(1)}{2b_n} - \frac{3s^2h''(1)^2}{8b_n^2} + O\left(c_n^{-1}\right)\right\}, \quad j \to \infty. \tag{33}$$

as in (26). This means that we can follow the same steps as in the proof of Lemma 4 and that we can choose M and $\delta > 0$ in such a way that

$$\begin{aligned}
&\left|\frac{ne^{\gamma}}{(2\pi)^{3/2}}\int_{M<|t|\le\delta n^{1/2}}\int_{-\pi}^{\pi}\mathbb{E}e^{isU_n+itV_\nu+niuW_n-inu}\,du\,dt\right| \\
&\quad\le \exp\left\{-\frac{1}{2}s^2 + o(1)\right\}\frac{1}{\sqrt{2\pi}}\int_{M<|t|\le\delta n^{1/2}}\exp\left\{-\frac{1}{2}t^2+\frac{1}{4}t^2\right\}\,dt \\
&\quad= \exp\left\{-\frac{1}{2}s^2 + o(1)\right\}\frac{1}{\sqrt{2\pi}}\int_{M<|t|\le\delta n^{1/2}}\exp\left\{-\frac{1}{4}t^2\right\}\,dt < \varepsilon.
\end{aligned}$$

■

The following lemma deals with the region $A_2 = \{t \in \mathbb{R} : |t| > \delta n^{1/2}\}$.

Lemma 6. *Let h satisfy Condition 1. Then, for each $\delta > 0$:*

$$\int_{|t|>\delta c_n}\int_{u=-\pi}^{\pi}\mathbb{E}\exp\{isU_n + itV_n + inuW_n - niu\}\,du\,dt \to 0, \quad n \to \infty.$$

Proof. We consider the characteristic function:

$$\bar{\phi}_{nj}(s,t) = E\exp\left\{is\frac{\{h(j/S_{j1}) - h(1)\}S_{j1} + h'(1)\left(S_{j1}-j\right)}{b_n} + it\left(S_{j1}-j\right)\right\},$$

so we replace c_n by 1 in (20). This means that, for the saddle point analysis, the constant c_n is replaced by 1 in the function (22). Hence we now define

$$\bar{f}_{n,s,t}(z) = is\frac{\{h(1/z)-h(1)\}z + h'(1)(z-1)\}}{b_n} + it(z-1) - z + \log z, \quad \mathrm{Re}(z) > 0. \tag{34}$$

The saddle point equation (24) now turns into

$$z = g_n(z), \quad g_n(z) \overset{\text{def}}{=} (1-it)^{-1}\left\{1 + \frac{is\{h(1/z)-h(1)\}z}{b_n} - \frac{ish'(1/z)}{b_n} + \frac{ish'(1)z}{b_n}\right\} \tag{35}$$

and has a unique solution in a neighborhood of $(1-it)^{-1}$ for the same reasons as before.

We define:

$$z_0(t) = 1/(1 - it).$$

Then:

$$\bar{f}_{n,s,t}(z_0(t)) = -\frac{t^2+1}{1-it} - \log(1-it) - is\frac{h(1) - h(1-it) + sth'(1)}{b_n(1-it)}$$
$$= -\frac{t^2+1}{1-it} + O\left(b_n^{-1}|t|\right),$$

and

$$\bar{f}''_{n,s,t}(z_0(t)) = (t+i)^2 - \frac{s(t+i)h''(1-it)}{b_n}$$
$$= (t+i)^2 + O\left(b_n^{-1}|t|\right),$$

uniformly in $|t| > \delta$, using Condition 1. This implies that, uniformly for $|t| > \delta$,

$$\phi_{nj}(s,t) = \frac{j^j}{\Gamma(j)}\int_{x=0}^{\infty} \exp\left\{jf_{n,s,t}(x)\right\} x^{-1}\,dx \sim \frac{\alpha_n(t)e^{j\{\bar{f}_{n,s,t}(z_0(t))+1\}}}{j\sqrt{|\bar{f}''_{n,s,t}(z_0(t))|}}$$
$$= \frac{\alpha_n(t)\exp\left\{-j\left(it + \log(1-it) + O\left(b_n^{-1}\right)\right)\right\}}{j\left|(t+i)^2 + O\left(b_n^{-1}|t|\right)\right|^{1/2}}, \quad j \to \infty, \tag{36}$$

where $\alpha_n(t)$ is a complex number with absolute value 1 and argument $\frac{1}{2}\pi - \frac{1}{2}\arg\bar{f}''_{n,s,t}(z_0(t))$.

Hence we find, using Cauchy's formula again, as in the proof of Lemma 4, and using Condition 1,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{E}\exp\left\{isU_n + itc_nV_n + inuW_n - niu\right\}\,du$$
$$\sim \exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \sum_{j=1}^{n}\frac{1}{j}\right\}(-1)^n\binom{-\delta_n(t)}{n}\beta_n(t)^n$$
$$= \exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \sum_{j=1}^{n}\frac{1}{j}\right\}\prod_{j=1}^{n}\left(1 + \frac{\delta_n(t)-1}{j}\right)\beta_n(t)^n,$$

where

$$\beta_n(t) = \exp\left\{-(it + \log(1-it) + O\left(b_n^{-1}\right)\right\},$$

and

$$\delta_n(t) = \frac{\alpha_n}{\left|(t+i)^2 + O\left(b_n^{-1}|t|\right)\right|^{1/2}}.$$

Hence

$$\exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \sum_{j=1}^{n}\frac{1}{j}\right\}\prod_{j=1}^{n}\left(1+\frac{\delta_n(t)-1}{j}\right)\beta_n(t)^n$$
$$= \frac{1}{n}\exp\left\{-\frac{1}{2}ish''(1)(\log n)/b_n - \gamma\right\}$$
$$\cdot\exp\left\{-n\left\{it+\log(1-it)+O\left(b_n^{-1}\right)\right\}+\frac{\alpha_n\sum_{j=1}^{n}1/j}{\left|(t+i)^2+O\left(b_n^{-1}|t|\right)\right|^{1/2}}\right\},$$

implying

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbb{E}\exp\left\{isU_n+itc_nV_n+inuW_n-niu\right\}\,du\right|$$
$$\le \frac{1}{n}\exp\left\{-n\left\{\frac{1}{2}\log(1+t^2)+O\left(b_n^{-1}\right)\right\}+\frac{\sum_{j=1}^{n}1/j}{\left|(t+i)^2+O\left(b_n^{-1}|t|\right)\right|^{1/2}}\right\}$$
$$\le \left|1+t^2\right|^{-n/4},$$

for large n, uniformly for $|t| > \delta$, using Condition 1. It now follows that

$$\left|\frac{1}{2\pi}\int_{|t|>\delta c_n}\int_{u=-\pi}^{\pi}\mathbb{E}\exp\left\{isU_n+itV_n+inuW_n-niu\right\}\,du\,dt\right|$$
$$\left|\frac{1}{2\pi}\int_{|t|>\delta}\int_{u=-\pi}^{\pi}\mathbb{E}\exp\left\{isU_n+itc_nV_n+inuW_n-niu\right\}\,du\,dt\right|$$
$$\le \int_{|t|>\delta}(1+t^2)^{-n/4}\,dt \to 0,\qquad n\to\infty.$$

■

This leads to the main result of this section.

Theorem 6. *Let h satisfy Condition* 1. *Then* $U_n|(V_n, W_n) = (0, 1)$ *converges in law to a standard normal distribution.*

Proof. The preceding lemma's imply

$$\frac{ne^{\gamma}}{(2\pi)^{3/2}}\int_{t=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathbb{E}e^{isU_n+itV_v+niuW_n-inu}\,du\,dt \to \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}}\int_{t=-\infty}^{\infty}e^{-\frac{1}{2}t^2}\,dt = e^{-\frac{1}{2}s^2},\qquad n\to\infty,$$

where γ is Euler's gamma. Hence

$$\mathbb{E}\left\{e^{isU_n}|(V_n,W_n)=(0,1)\right\}\sim\frac{ne^{\gamma}}{(2\pi)^{3/2}}\int_{t=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathbb{E}e^{isU_n+itV_v+niuW_n-inu}\,du\,dt\to e^{-\frac{1}{2}s^2},\qquad n\to\infty.$$

■

Theorem 2 now follows from the conditional representation from Section 1, definition (17) of U_n, Remark 3, and Theorem 6.

5 | CONCLUSION

We derived a general theorem (Theorem 2) for integrals of the Grenander estimator when the distribution is uniform from a representation in terms of Poisson and gamma random variables in Groeneboom and Pyke (1983). The result implies the main result of Groeneboom and Pyke (1983) and gives also the limit behavior of the entropy functional. We corrected the limit distribution of the conditioning variables given in lemma 3.1 of Groeneboom and Pyke (1983) in Section 3. The methods used are rather different from the methods in Groeneboom and Pyke (1983), where a result in Le Cam (1958) was used.

Our main result was inspired by Mukherjee and Sen (2019) who derived a similar result from Groeneboom and Pyke (1983) under different conditions. The main tools are Cauchy's formula and the saddle point method for integrals of analytic functions of a complex variable. A simple version of the approach is given in Section 2 to illustrate the method without the complications of the saddle point method.

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REFERENCES

Behnen, K. (1974). *Güteeigenschaften von Rangtests unter Bindungen*. Habilitationsschrift: University of Freiburg.

Chan K. C. G., Tang C. F., & Yam S. C. P. Likelihood ratio test for monotonicity of density. 2018.

de Bruijn, N. G. (1981). *Asymptotic methods in analysis* (3rd ed.). New York, NY: Dover Publications, Inc.

Dieudonné, J. (1968). *Calcul infinitésimal*. Paris, France: Hermann.

Feller, W. (1971). *An introduction to probability theory and its applications* (Vol. *II*, 2nd ed.). New York, NY/London, UK/Sydney, Australia: John Wiley & Sons, Inc.

Grenander, U. (1957). On the theory of mortality measurement II. *Skand Aktuarietidskr*, *39*, 125–153.

Groeneboom, P. (1985). *Estimating a monotone density*. In *Proceedings of the Berkeley conference in honor of Jerzy Neyman and Jack Kiefer Wadsworth Statist./Probab. Ser* (Vol. *II*, pp. 539–555). Berkeley, CA: Wadsworth.

Groeneboom, P., & Lopuhaä, H. P. (1993). Isotonic estimators of monotone densities and distribution functions: Basic facts. *Statist Neerlandica*, *47*, 175–183. https://doi.org/10.1111/j.1467-9574.1993.tb01415.x

Groeneboom, P. (1983). The concave majorant of Brownian motion. *Ann Probab*, *11*, 1016–1027. http://links.jstor.org/sici?sici=0091-1798(198311)11:4<1016:TCMOBM>2.0.CO;2-7&origin=MSN

Groeneboom, P., & Jongbloed, G. (2014). *Nonparametric estimation under shape constraints*. Cambridge, MA: Cambridge University Press.

Groeneboom, P., & Jongbloed, G. (2018). Some developments in the theory of shape constrained inference. *Statistical Science*, *33*(4), 473–492. https://doi.org/10.1214/18-STS657

Groeneboom, P., & Pyke, R. (1983). Asymptotic normality of statistics based on the convex minorants of empirical distribution functions. *The Annals of Probability*, *11*, 328–345. http://links.jstor.org/sici?sici=0091-1798(198305)11:2<328:ANOSBO>2.0.CO;2-Q&origin=MSN

Le Cam L. (1958). Un théorème sur la division d'un intervalle par des points pris au hasard (Vol. 7, (3/4), pp. 7–16). Paris, France: Public Institute Statistics University Press.

Mukherjee R., &Sen B. Estimation of integrated functionals of a monotone density. 2019. https://arxiv.org/abs/1808.07915.

Olver, F. W. J. (1974). *Asymptotics and special functions*. New York, NY/London, UK: Academic Press[A subsidiary of Harcourt Brace Jovanovich, Publishers].

Prakasa Rao, B. L. S. (1969). Estimation of a unimodal density. *Sankhya Series A*, *31*, 23–36.

Sato, K.-I. (2001). *Basic results on Lévy processes*. In *Lévy processes* (pp. 3–37). Boston, MA: Birkhäuser Boston.

Scholz, F.-W. (1983). *Combining independent* P-*values*. In *A festschrift for Erich L. Lehmann Wadsworth Statistics/Probability Series* (pp. 379–394). Belmont, CA: Wadsworth.

Sparre Andersen, E. (1954). On the fluctuations of sums of random variables. *Mathematica Scandinavica*, *2*, 195–223.

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