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Optimal safety standards for dike-ring areas

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Abstract in English

After the flood disaster in 1953 in the southwestern part of the Netherlands, Van Dantzig tried to solve the economic-decision problem concerning the optimal height of dikes. His formula with a fixed exceedance probability after each investment (Econometrica, 1956) is still in use today in cost benefit analysis of flood-protection measures. However, his solution is both incomplete and wrong. In the context of economic growth, not the exceedance probability but the expected yearly loss by flooding is the key variable in the real optimal safety strategy. Under some conditions, it is optimal to keep this expected loss within a constant interval. Therefore, when the potential damage increases by economic growth, the flooding probability has to decline in the course of time in order to keep the expected loss between the fixed boundaries. The paper gives the formulas for the optimal boundaries for a more complicated problem which is more in line with engineering experience. One condition is that the rate of return at the moment of investment (FYRR) has to be zero (or positive). Then the net present value (NPV) of a safety investment will be very positive or even infinite. Therefore, in case of economic growth the well-known NPV criterion in cost benefit analysis of a single project is not a sufficient criterion for investing.
An application of the model with the original figures for the dike ring Central Holland has been added as well as a recent application for dike-ring areas along the river Rhine.

Key words:
Optimal safety norms, cost benefit analysis, optimal height of dikes.

Abstract in Dutch

Na de Watersnoodramp in 1953 heeft Prof. D. van Dantzig geprobeerd om het economische beslissingsprobleem over de optimale hoogte van dijken op te lossen. Zijn formule (Econometrica 1956) werkt met een vaste waarde voor de overstromingskans direct na investeren en wordt nog steeds gebruikt in kosten-batenanalyses van maatregelen tegen overstroming. Zijn oplossing is echter zowel onvolledig als onjuist. Bij economische groei is niet de overstromingskans, maar de verwachte jaarlijkse schade de centrale variabele in de echte optimale veiligheidsstrategie. Onder enige voorwaarden is het optimaal om de verwachte schade binnen een vast interval te houden. Omdat economische groei de schade bij overstroming verhoogt, is een daling van de overstromingskansen nodig om de verwachte schade binnen dit vaste interval te houden. In het paper worden de formules voor de grenzen van dit interval afgeleid voor een ingewikkelder model, dat meer in overeenstemming is met de kosten van actie in de praktijk. Een voorwaarde is dat in het eerste jaar na investeren het rendement nul (of positief) moet zijn. Door de stijging van de baten zal de netto contante waarde van een veiligheidsproject zeer groot zijn of zelfs oneindig. Daarom is in het geval van

economische groei een positieve netto contante waarde nog niet voldoende om te besluiten tot uitvoering van een investeringsproject.

Het model is evenals dat van Van Dantzig toegepast op de dijkring Centraal Holland. Daarnaast zijn er voorbeelden gegeven uit de KBA voor het project Ruimte voor de Rivier (Eijgenraam, 2005 en CPB, 2005).[1]

Steekwoorden:

Optimale veiligheidsnormen, kosten-batenanalyse, hoogte van dijken.

.

[1] Het gehele model is in het Nederlands te vinden in hoofdstuk 2 en bijlage A van Eijgenraam (2005). In de hoofdstukken 4 en 5 van die publicatie is het model toegepast op 22 dijkringen in het rivierengebied voor het project Ruimte voor de Rivier. Ten opzichte van Eijgenraam (2005) zijn in deze publicatie additioneel: de beschouwingen over de middenkans (deels ook in CPB, 2005), de stabiliteitsvoorwaarden in paragraaf 3.5, de technische vooruitgang in paragraaf 3.6 en het geval dat de schade afhankelijk is van de waterhoogte bij overstromen in paragraaf 3.7, evenals de toepassing op dijkring 14 Centraal Holland in paragraaf 4.1

Contents

Summary

After the flood disaster in 1953 in the southwestern part of the Netherlands, Prof. D. van Dantzig tried to solve the economic-decision problem concerning the optimal height of dikes. His formula is based on a fixed exceedance probability directly after an investment (Econometrica, 1956) and is still in use today in cost benefit analysis of the optimal size of flood-protection measures, like the height of dikes. However, his solution is both incomplete and wrong. In the real optimal safety strategy, not the exceedance probability but the expected yearly loss by flooding is the key variable. Under some conditions, it is optimal to keep this expected loss in the future within a constant interval. Therefore, when the potential damage increases by economic growth, the flooding probability has to decline in the course of time in order to keep the expected loss between the fixed boundaries. The paper gives the formulas for the optimal boundaries for a more complicated case. One condition is that the rate of return at the moment of investment (FYRR) has to be zero (or positive). In case of a positive rate of growth of the expected damage, the net present value (NPV) of a safety investment which passes the FYRR criterion, will be very positive or even infinite. Therefore, the well-known NPV criterion in cost benefit analysis of a single project is not a sufficient criterion for investing in this type of projects.

We recalculate the application of Van Dantzig for the dike ring Central Holland, by far the most important dike ring in the Netherlands. It turns out that the real developments in the past 50 years were in line with his pessimistic scenario with respect to the development of the expected risk by flooding. It confirms the conclusion of Van Dantzig that the safety level chosen for Central Holland seems too low. The model has been developed for and applied to the dike-ring areas along the river Rhine. In general, the legal norms lay in the middle of the figures calculated for the individual dike rings, but these calculated figures show an enormous spread.

1 Introduction

After the flood disaster in the southern parts of the Netherlands in 1953, the Delta Commission asked Prof. D. van Dantzig to solve the economic-decision problem concerning the optimal height of dikes. His formula published in Econometrica (Van Dantzig, 1956) is still in use today in cost benefit analysis of the optimal size of flood protection measures, like the height of dikes.[2]

But firstly, his solution is not complete. An optimal investment strategy should give the answers to two related questions. The first is: when to invest? and the second is: how much to invest on the chosen point of time? Van Dantzig only tried to give a formal answer to the second question: how much? He did not really address the 'when' question for the, at that time, obvious reason that heightening the dikes was immediately necessary. Vrijling and Van Beurden (1990) posed the first question as well but did not succeed in finding an analytical solution. Yet, their model was simpler than that of Van Dantzig because it neglects the effect of economic growth on the increase of damage by flooding.

Recently, CPB Netherlands Bureau for Economic Policy Analysis was asked to perform a cost benefit analysis (CBA) on a project that aims to improve the safety of dike rings along the river Rhine and its branches towards the sea (Eijgenraam, 2005). In order to perform the correct calculations we tried to solve the 'when' question too. In the process, the first result we got was that Van Dantzig's answer to the 'how much' question is wrong.

Purpose of this paper is first to give the right answers to both questions in the original problem and second to give in the appendix a full mathematical proof for the solution of a model that is more general than Van Dantzig's.[3]

A general sketch of the problem is given in paragraph 2. In paragraph 3, we introduce the model. With the help of one supposition we can give a simple proof of the basic outcomes. In section 3.5 we enlarge the model of which a formal proof is given in the Appendix. In this section, we also analyse the meaning of mathematical and economic stability conditions. The next two sections show that richer interpretations of the mathematical model are possible by slightly different interpretations of the parameters. In paragraph 4, some real numerical examples have been added. Paragraph 5 concludes.

[2] See e.g., among many others, Brinkhuis et al. (2003).

[3] The author thanks Dr. J.H. van Schuppen (Free University Amsterdam and Centre for Mathematics and Informatics) for checking the mathematical proof in the Appendix.

2 Safety standards for dike-ring areas

A dike ring is an uninterrupted ring of water defences, like dikes or dunes, and high grounds (that are grounds which even under the most unlikely circumstances will not be flooded). The area within a dike ring is called a dike-ring area and is sometimes also referred to as 'polder'. In the Netherlands, dike rings are found in the north-western half of the country and along the big rivers Rhine and Meuse.[4] The question we want to answer is: What is – from an economic point of view – the optimal strategy for protecting a dike-ring area against flooding?

In designing such a strategy, the social costs of investing in water defences have to be balanced against the social benefits of avoiding damage by flooding. Because both in social costs and in social benefits, non-material issues are involved, the choice of safety standards is in the end a political decision. In the Netherlands, the Act on the Water Defences gives for different types of dike rings a standard for the maximum exceedance probability of the design-water level a dike section must sustain.[5] These exceedance probabilities range from 1/50 per year for small areas without dikes upstream along the Meuse, via 1/1250 per year for dike rings along the upper part of the Rhine, till 1/10000 per year for the most important dike rings in the provinces North and South Holland along the coast.

This approach is in principle in accordance with the formula of Van Dantzig (1956) concerning the optimal height of dikes in the presence of a constant rate of economic growth and a constant investment cost curve. According to his formula, the height of the dike after an investment should be as high that the resulting exceedance probability is the same as directly after the previous investment. With the possible exception of the first investment, all subsequent investments will have the same size and the same time span between them.

[4] See Zhou (1995) for an extensive description of flood protection in the Netherlands, both from a policy and an engineering point of view.

[5] In the discussion about safety standards in the Netherlands the word 'exceedance probability' is used in two different meanings. The first is the well-known statistical meaning of the cumulative probability of the occurrence of an event bigger than a certain value, see equation 1 in the next section. (In other types of studies sometimes the word 'survival rate' is used.)
The second meaning of exceedance is a civil engineering one. It refers to a water level that exceeds the top of the dike resulting in overflow and flooding. If the dike has been well constructed, this failure mechanism should be decisive for the answer to the question under which conditions a flooding would occur. Flooding as a result of other failure mechanisms (e.g. a collapse of the dike, for instance because the ground has been satiated with water) should have a probability that is an order of magnitude smaller than the probability of overflow. The legal standards in the Netherlands have been based on this premise. It has the advantage of providing a standard which is relatively easy to calculate and apply. Of course this is a great simplification of reality. In an other study in the Netherlands the actual probabilities of other failure mechanisms have been taken into account. Preliminary outcomes suggest that in reality the contribution of other failure mechanisms to the total probability of flooding is far from small. On the other hand the dikes are usually stronger than the design level indicates, for instance because always an extra height will be added to cope with a strong influence of wind.
Throughout this paper we follow the civil engineering and legal premise that the construction of dikes is well enough to secure that overflow (exceedance) is by far the most likely failure mechanism. The consequence is that when the water level exceeds the design-water level, the dike ring will be flooded with a probability 1, and as long as the water level is lower than the design-water level, there is no probability of flooding at all.
So, we use the word 'exceedance probability' in both meanings.

However, his solution can not be the optimal one in the presence of economic growth, which results in increasing potential damage by flooding. Because then the criterion would ‘equalise’ in the margin fixed amounts of investment costs with increasing expected losses by flooding. This is obviously impossible.

In the next section, we derive the really optimal solution for the same problem as Van Dantzig tried to solve. At the end we expand the model in an important way to bring the model more in line with engineering experience. A formal proof of the solution of the extended model is given in the Appendix A.

3 Deriving optimal safety standards for dike rings

3.1 Model

For the simple case of one dike ring, one failure mechanism: overflow, one type of water defence: dikes, social costs and benefits which can all be monetarised, and constant growth rates, the correct reasoning is in short as follows.

Expected loss

The expected yearly loss by flooding S_t is the product of the loss by flooding V_t (potential loss) times the probability of a flooding per year P_t.[6]

Under normal conditions the dike-ring area will be well protected, so the relevant probability distribution is an extreme value distribution for water levels. In practice an exponential distribution (with parameter α) fits the data reasonably well (Noortwijk et al., 2002). This distribution is supposed to shift to the right (with a constant speed of η centimetres per year) as a result of increasing water levels at sea compared to the surface of the land and by higher peak levels of the river discharges.[7] The resulting probability of flooding P is the probability that the water level exceeds the level of the dike H, resulting in a break of the dike. This deterministic description of the level and development of the flooding probability is the same as the one used by Van Dantzig:

$$P_t = 1 - F_t(H_t) = P_o\, e^{\alpha\eta t}\; e^{-\alpha(H_t - H_o)} \qquad for\ H_t \geq H_o \tag{1}$$

with:

	P_t	exceedance probability in year t
	F	exponential distribution function of water levels
	α	parameter exponential distribution for extreme water levels (1/cm)
	η	structural increase of the water level (cm/year)
	H	height of the dike (cm)

There is no problem in defining H_t as such that H_o can be put to zero. The height of the dike will then be measured from the level in year o. The level H_o is the relevant height compared to the local water level. This relevant height is supposed to be equal along the dike ring in the sense

[6] Zhou (1995) mentions that in the literature two definitions of the word 'risk' can be found. Some authors "define risk as the product of the probability of events and the magnitude of specific consequences (…). Others (…) define risk as the probability of the realisation of an adverse event. This definition is also adopted by the recently published Dictionary of Scientific and Technological Terms." To avoid confusion we will use the term expected yearly loss or in short expected loss.

[7] Since we are only interested in extreme values with an aggregated probability lower than 0,002, we don't bother for the necessary changes in the underlying distribution of all water levels during the shift. The extreme right side of the true distribution can always be approximated by an exponential distribution function. See also Van Dantzig (1956) p 282.

that the dike provides everywhere the same exceedance probability P_o.[8] Also α and η are supposed to be the same for the whole dike ring.

The development of the loss by flooding V is:

$$V_t = V_o \, e^{\gamma t} \, e^{\zeta (H_t - H_o)} \qquad (2)$$

with: V_t loss by flooding in year t (million euros)

γ rate of growth of wealth in the dike ring (perunes per year)

ζ increase of loss per cm dike heightening (1/cm)

The first two factors on the right hand side are the level and growth of the loss in case of flooding. V_o includes a valuation in money for non-material losses. It is also possible to give the loss V_o an extra weight in the actual calculation as a reflection of risk aversion, or to raise this loss in the dike-ring area to a power greater than 1 taking into account that flooding affects a lot of people at the same time causing a big disruption of society, both factual and emotional, see Van Dantzig (1956) paragraph 7.[9]

The third factor on the right hand side is an addition to Van Dantzig's model and is only relevant along rivers. Along rivers a dike has a slope which is equal to the slope of the river towards the sea. Compared with the sea level the top of the dikes upstream is higher than the top of the dikes downstream. When a flooding occurs, the resulting water level in the dike ring is assumed to come always as high as the lowest point of the dike above sea level. On this point the water runs over the dike back into the river or into another outlet. The height of the water level within the dike ring is one of the determinants of the amount of damage.[10] When the dike is heightened, the resulting damage within the dike ring will rise. This is expressed in a simple exponential way, which is, for the time being, a reasonable approximation for the change in the amount of loss within a relevant range for the heightening of the dike along the rivers. A maximum is not yet relevant, because in these dike rings the loss by flooding is still far from the total wealth in the dike-ring area.

Multiplying (1) and (2) leads to the formula for the expected loss S:

$$S_t = P_t V_t = S_o \, e^{\beta t} \, e^{-\theta H_t} \qquad (3)$$

[8] We discard the possibility that sections of the dikes are already higher than H_o and P_o. But in reality improvements are done only for the lowest sections of the dike and are not completely even spread along a dike ring.

[9] In the actual calculations in section 4 we use a risk neutral valuation of loss in comparison to investment costs and a linear valuation of the loss with respect to the size of the damage.

[10] One of the other determinants is the level of the water at the moment of the break. This can have an important effect on the size of the damage in dike rings along the sea or along rivers in small valleys. See for this extension of the model paragraph 3.7.

with $\beta = \alpha\eta + \gamma$

$\theta = \alpha - \zeta > o$

with: S_t expected loss at time t (mln euros)

Heightening of dikes

The expected loss S increases by β% per year as a result of growth of wealth and of rising frequencies of high water levels caused among others by climate change. To cope with these systematic changes more than one defence action u_t will be needed in the future. Here we name these defence actions: heightening of dikes. But it makes no difference for the mathematical model if we should use instead measures which lower water levels. Since the last type of measures is in general not possible along the coast, we continue to speak only of heightening dikes. Also, heightening of dikes is in general cheaper than lowering design-water levels by giving the river more space, e.g. by enlarging the distance between the dikes along a river.

At moments in time T_i, the dike will be heightened with u_t centimetres. The following definitions apply for the height of the dike H:[11]

$$H_{T_1}^{-} = H_o = o \tag{4}$$

$$\dot{H}_t = o \qquad when \;\; t \neq T_i \tag{5a}$$

$$= H_{T_i}^{+} - H_{T_i}^{-} = u_i > o \qquad when \;\; t = T_i \tag{5b}$$

$$H_z \quad is \; free \tag{6}$$

with H^- height of dike directly before a heightening

H^+ height of dike directly after a heightening

T_i time of investment number i

u heightening of the dike in cm

Equation (4) defines the starting point. Equation (5a) implies that there is no decline in protection level of the dike, provided it gets the proper maintenance. Since the heightening of the dike u_i is not continuous in time, $\dot{H}_t$ is also not continuous. Therefore H is not differentiable in $t = T_i$. There is no fixed endpoint H_z at a point of time z in the far future.

Relations by definition

Substitution of (5) in (3) leads to the following two relations in volume u and time span D:

[11] Variables with a dot are time derivates. To simplify notation we often use i as an index instead of T_i as long as it doesn't lead to confusion.

$$S(H_{T_i}^{+}) \equiv S_i^{+} = S(H_{T_i}^{-})\, e^{-\theta u_i} \equiv S_i^{-}\, e^{-\theta u_i} \tag{7}$$

and:

$$S_{i+1}^{-} = S_i^{+}\, e^{\beta(T_{i+1} - T_i)} = S_i^{+}\, e^{\beta D_{i+1}} = S_i^{-}\, e^{\beta D_{i+1} - \theta u_i} \tag{8}$$

with $\quad D_{i+1} = T_{i+1} - T_i \qquad\qquad (9)$

Equation (7) gives the relation between the expected losses just before and after a heightening of the dike. Equation (8) relates the expected loss directly before the next investment to the expected losses directly after and before the last investment.

Investment in heightening the dike

For investment costs (including the present value of future maintenance costs) we use a convex cost function, which is a more general specification than the linear specification used by Van Dantzig.

$$I_i(u_i, H_{T_i}^{-}) = I(u_i) \qquad when \quad u_i > o \tag{10a}$$

$$= o \qquad when \quad u = o \tag{10b}$$

$$\lim_{u \downarrow o} I(u) = I_F > o$$

$$I_u' > o \qquad and \qquad I_u'' \geq o \qquad for \quad u > o$$

Investment costs I(u) are in the simple case (10) not dependent on the height of the existing dike and not dependent on time. Further both the marginal and the fixed costs are strictly positive. In the formal proof in the appendix we make the investments costs dependent on the height of the existing dike (see section 3.5), which appears to be more in line with engineering experience. (See already Van Dantzig (1956) p 280.)

Criterion

Social welfare will be maximised by minimising the present value of the total cost of flooding (S) and investment (I) over the whole future:

$$\min_{U} C = \int_{o}^{\infty} S_t\, e^{-\delta t}\, dt + \sum_{i=1}^{\infty} I_i\, e^{-\delta T_i} \tag{11}$$

with	C	present value of all future losses by flooding and the investment cost, at time o (mln euros)
	δ	discount rate, strictly positive
	U	set of all possible combinations $\{T_i, u_i\}$

Later we will address the question under which conditions the integral and sum in (11) are convergent. Here we only use the general restriction: $\delta > o$.

Because the size of investment u_i at time T_i will influence the choice of the next moment T_{i+1} and the size of the next investment u_{i+1}, we have to use the Maximum Principle with jumps in the state-variable to find the necessary conditions for an optimum. These conditions must imply that marginal decreases in the cost of flooding equal marginal increases in the costs of investment. Yet, with some reasoning we can make a supposition about the form of the solution of this simple case, like Van Dantzig did. But instead of looking at *probabilities* of flooding, we rather should look at *expected costs* of flooding. Because it is expected costs that appear in the marginal conditions. With the supposition about the form, we can find the rest of the solution of (11) just by differentiation.

3.2 Periodical solution

The problem looks like replenishing a stock of 'safety-units'. In the course of time 'safety-units' gradually disappear in a fixed rate of change by a combination of growth of wealth and relatively rising flood levels. The optimal solution resembles therefore the well-known (s^-, S^+) strategy for replenishing stocks. So, the protection level will be high just after an investment resulting in a low expected yearly loss by flooding S^+. But safety will gradually decline afterwards. When a certain low level of safety with a high expected loss by flooding s^- is reached, we decide that a new action is profitable and we invest again. Figure 3.1 gives a sketch of this strategy. The numbers 1 to 3 point to the three different effects of a heightening.[12] The relative size of the fixed investment costs must play an important role in determining the width of the interval.

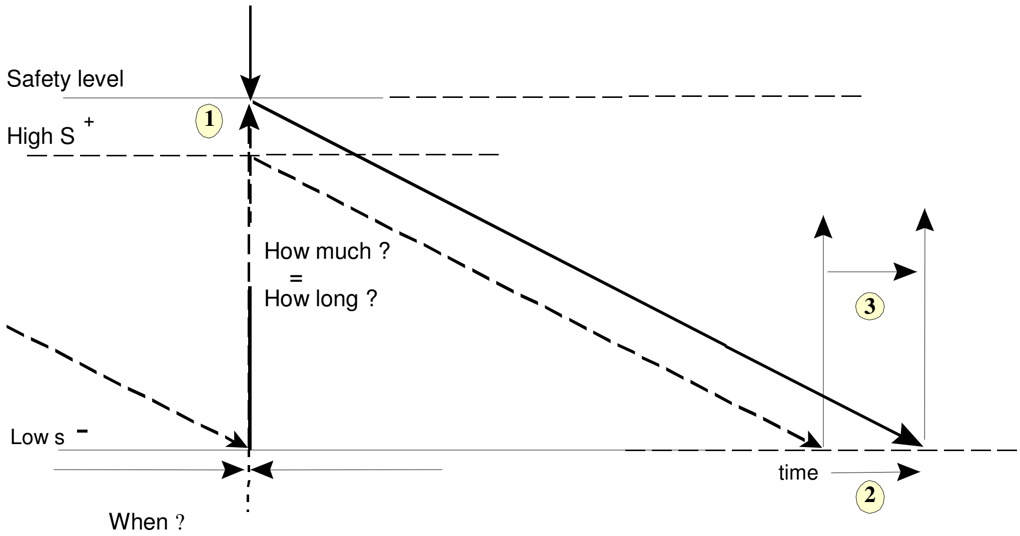
In order to get a simple analytical solution the strategy has to be periodical, at least after some time. When the future will exactly look the same every time the system is back on the re-order level s^-, we will also take the same optimal decision again to bring the expected loss back to the S^+ level.[13] In the simple case that all (growth) rates are constant in time, the constancy of the investment function implies that the marginal changes in the expected loss as a result of the (marginal) investment must also be constant in time. Therefore we suppose that the borders of

[12] These effects can also be identified in equation (21), which is the marginal condition concerning S^+.

[13] A difference with normal stock problems is that the diminishing of the stock is in this case not a stochastic, but a deterministic, smooth process.

the loss interval are constant, so $[s^-_i, S^+_i] = [s^-, S^+]$. Substitution of this result in (7) shows that also the optimal investment u is constant. In the appendix we show that a constant size for the optimal investment u is indeed a solution of the necessary conditions for an optimum. Further we will prove that this solution fulfils the sufficient conditions for a minimum and that this solution is unique. Here we take the constancy of the loss interval as the starting point for the rest of the solution of the model in a formal way.

Figure 3.1 Basic Strategy



Substitution of a fixed s^- in (8) gives:

$$\beta D = \theta u \tag{12}$$

So, with a fixed size of the optimal investment u, also the optimal time span between investments D will be fixed.

According to (11) the present value of the total costs of damage by flooding and investment in protection measures can for one investment period be written as follows, starting from a situation that the system is on the re-order level s^- till the moment the system is back on that level:

$$C_{period} = \int_o^D S^+ \, e^{\beta t} \, e^{-\delta t} dt + I(u) = BS^+ + I(u) \tag{13}$$

with: $$B = \frac{1}{\delta - \beta}\left(1 - e^{-(\delta - \beta)D}\right) \tag{14}$$

The first term of (13) gives the present value of the expected loss by flooding during a period D between two repetitive investments. According to (14) B is always positive, irrespective whether $\delta > \beta$ or $\delta < \beta$. Also $\delta = \beta$ is no problem, because then B = D. (13) is basically the same as formula (VD10) in Van Dantzig.[14]

Because u is finite and constant, also I(u) and S^+ are finite and constant. The present values of all costs during the repetitive periods form a geometrical series with discount factor exp(-δD), which is always smaller than one, provided that the interest rate δ is strictly positive. Therefore the sum of the series is finite:

$$C_{rep} = \frac{B\,S^{+} + I(u)}{1 - e^{-\delta D}} \tag{15}$$

The reason that the integral in (11) is always convergent, is that there are periodical investments which keep S^+ finite, so in the simple case no maximum condition on the value of β or minimal condition on the value of δ, other than $\delta > o$, is needed.

Comparison with the result of Van Dantzig

Formula (15) is not longer the same as Van Dantzig's formula (VD11). Van Dantzig's reasoning is to some extent comparable with the one above. But he treats the two causes for the increase of the expected loss by flooding in a different way. He proposes to repair the increase of the flooding probability by periodic investments and fixes therefore a boundary for P^+. The future increase in wealth, on the contrary, is completely dealt with by the first investment only. Above we already argued that this cannot be the optimal result. Now we can point to the inconsistency in another way. On the moment of the first investment Van Dantzig's idea about what to do at the beginning of the next period (namely only repairing the increase in flooding probability) is not the same as the outcome of his formula applied again at the beginning of the second period, because that includes repairing the increase in wealth as well.

Nevertheless, if we write Van Dantzig's result (VD11) in the same way as (15) above, we get:[15]

$$C_{rep,VD} = \frac{B\,S^{+}}{1 - e^{-(\delta-\gamma)D}} + \frac{I(u)}{1 - e^{-\delta D}} \tag{16}$$

Comparison of (16), Van Dantzig's formula, with the correct formula (15) shows that the nominator is the same, but that there is an essential difference in the first denominator, because in (16) it includes the parameter γ.[16] Besides (16) being not correct, it is also more restrictive,

[14] To avoid confusion we add VD to the numbers of the equations used by Van Dantzig.

[15] In (16) we have left out the terms for the first investment period in (VD11), because these are also not included in (15).

[16] Vrijling and Van Beurden follow the same reasoning as Van Dantzig, but in their case γ = o. In that case (16) and (15) coincide.

since in (16) the restriction δ > γ is needed for assuring that the integral in (11) is finite. This is certainly not always the case for relatively small areas like dike rings. Further this restriction is odd when we look to the effect on the expected loss S. Because in (3) it makes no difference for the increase in S whether η is high or γ is high, and both parameters appear in (11) only as part of β and have no other influence on the model we started with.

3.3 Total costs and necessary conditions for a minimum

At time t = o in general two situations are possible. Either the dike ring is in good condition with an expected loss by flooding S_o lower than the upper bound s^-, in which case we can wait before investing again. Or the dike ring turns out to have a backlog in safety, which backlog has to be repaired immediately. The first case is more general, because then the moment of first investment has yet to be determined.

Waiting time T_1 > o

If $S_o < s^-$, total costs are the costs according to (15) plus the costs in the waiting period till s^- has been reached. The moment T_1 of the first investment is determined by:

$$S_{T_1}^- = S_o\, e^{\beta T_1} = s^-$$

$$T_1 = \frac{1}{\beta}\ln\left(s^- / S_o\right) \qquad when \quad S_o < s^- \tag{17}$$

The costs of the expected loss during the waiting time are:

$$W_o = S_o \frac{1}{\delta - \beta}\left(1 - e^{-(\delta-\beta)T_1}\right) \tag{18}$$

Therefore total costs are:

$$C = S_o \frac{1}{\delta - \beta}\left(1 - e^{-(\delta-\beta)T_1}\right) + e^{-\delta T_1} C_{rep} \qquad when \quad S_o < s^- \tag{19}$$

Necessary conditions for a minimum

There are two instruments which can be used to minimise total costs in (19): the investment size u and the waiting time T_1. Putting the first derivatives of (19) to these two variables to zero – keeping in mind the definitional relations (7), (12) and (17) –, gives two optimality conditions.

The first, looking at the timing of the first investment, is:

$$\frac{\partial C}{\partial T_1} = o \qquad \Rightarrow \qquad s^- - S^+ = \delta I(u) \tag{20}$$

This condition is equivalent to the well-known First Year Rate of Return (FYRR). The benefits of the jump in safety at the moment of investment (on a yearly basis) must be equal (or bigger) than the total investment costs (on a yearly basis). Because this condition involves the decision to invest or not, all investment costs are relevant, including the fixed costs.

The second optimality condition looks at u. Since (18) does not contain u, differentiating (19) to u boils down to differentiating (15) to u:

$$\frac{\partial C}{\partial u} = \frac{\partial C_{rep}}{\partial u} = o \qquad \Rightarrow$$

$$I'_u = \frac{\theta}{\delta - \beta} S^+ + \frac{\theta}{\beta} \delta e^{-\delta D} \left(\frac{-1}{\delta - \beta} s^- + C_{rep} \right) \tag{21}$$

The three terms in (21) are the three numbered effects visible in figure 3.1 above. The marginal costs of investment must be equal to the marginal diminishing of the loss directly after investment, corrected for the effects of lengthening the time interval between investments. A marginal period with higher expected damage costs is added at the end of the interval and the total costs in further periods shift a bit to the future.[17] Since in (21) the decision to invest has already been taken, it is only marginal costs that matter in this condition.

Together with the definition equations relating S^+, s^-, u, D (and waiting time T_1), this system of equations (20) and (21) can be solved. Substituting (20) in (21) gives:

$$S^+ = \frac{1 - e^{-\delta D}}{\theta B} I'(u) \quad = P_t^+ V_t \tag{22}$$

Where (21) holds independently whether (20) – determining the moment of investment – holds or not, (22) is only valid in the optimum.[18] The right hand side of (22) shows clearly that where S^+ is a constant, P^+ can not be constant but should be declining, because V_t increases with time.

The result for u is:

$$u = \frac{1}{\theta} \ln\left(1 + \theta \frac{B\delta}{\left(1 - e^{-\delta D}\right)} \frac{I(u)}{I'_u(u)} \right) \tag{23}$$

[17] Figure 3.1 gives not completely correct picture of equation (21), because a higher safety level just after investment (lower level of S^+) implies a lower safety level just before a new investment (higher level of s^-). So the broadening of the interval is a symmetric process. Therefore In figure 3.1 the broadening of the interval can give a false feeling for the real outcome.

[18] This is important when the moment of investment is determined by other rules, e.g. the legal rule that the exceedance probability must not rise above a number specified in the Act. The same applies for condition (A.24) in the Appendix.

After substitution of (14) for B and (12) for D the resulting equation is an implicit function of u alone, which can easily be solved. Notice that only investment costs and the growth parameters appear in (23). Neither the level of potential loss, V_o, nor the exceedance probability in the base year, P_o, plays a role in determining the span of the optimal repetitive interval. The level of the expected loss, S_o, is only important for determining either the first moment of investment or the size of the first investment, when T_1 = o.

Investment at once T_1 = o

In case the expected loss S_o turns out to be on or above the upper limit s^- , immediate action is necessary, so T_1 = o.[19] The heightening X follows from a reasoning similar to (17):

$$S_o^+ = S_o\, e^{-\theta X}$$

$$X = \frac{1}{\theta} \ln \left(S_o / S_o^+ \right) \qquad when\ S_o \geq s^- \tag{24}$$

Since in general there is a backlog in this case, it holds that X > u and therefore the marginal costs will differ in case the investment costs are non-linear, which is allowed in this model. Since according to (22) this influences the value of S^+, we have in general:

$$I'(X) > I'(u) \qquad \Rightarrow \qquad S_o^+ > S^+ \tag{25}$$

This means that while the heightening X is in general bigger than the standard amount u, X is not big enough to bring the expected loss back to the bottom level S^+. As a result the period D_2 till the next investment will also be somewhat shorter than D:

$$S_o^+\, e^{\beta D_2} = s^-$$

$$D_2 = \frac{1}{\beta} \ln \left(s^- / S_o^+ \right) \qquad when\ S_o \geq\ s^- \tag{26}$$

The costs of expected loss in the period till the second investment are:

$$W_1 = S_o^+ \frac{1}{\delta - \beta} \left(1 - e^{-(\delta - \beta) D_2} \right) \tag{27}$$

[19] It may seem strange that the case with a backlog is of great practical importance, because we argued before that the increase in expected loss is a gradual, deterministic process. The problem is that the parameters of this process concerning the level and change of extreme values are - almost by definition - badly known, creating jumps in knowledge. The flood disaster in 1953 is a painful example. Another example is the shift in the estimated value of P_0 along the river Rhine after the high discharges in 1993 and 1995, lucky enough not resulting in a flooding. This shift in knowledge was the reason to start the project Room for the Rivers, which ultimately has led to this paper.

Therefore total costs are:

$$C = I(X) + S_o^+ \frac{1}{\delta - \beta}\left(1 - e^{-(\delta-\beta)D_2}\right) + e^{-\delta D_2}\, C_{rep} \qquad when \quad S_o \geq s^- \tag{28}$$

Differentiating (28) with respect to X gives the same equation as (21) with I'(X), S^+_o and D_2 instead of I'(u), S^+ and D, see further (30).

General cost function

Combining (19) and (28) gives a general formulation of the total cost function:

$$C = S_o \frac{1}{\delta - \beta}\left(1 - e^{-(\delta-\beta)T_1}\right) + e^{-\delta T_1}\left(I(X) + S_o^+ \frac{1}{\delta - \beta}\left(1 - e^{-(\delta-\beta)D_2}\right)\right) + e^{-\delta(T_1 + D_2)} C_{rep} \tag{29}$$

When $S_0 < s^-$, then T_1 follows from (17) and further: X = u, $S_o^+ = S^+$ and D_2 = D.
In the other case: T_1 = 0 and X, S_0^+ and D_2 follow from (24), (26) and the adjusted version of (21).

This completes the sketch of the solution of the minimisation problem (11).

3.4 Comparison with the result of Van Dantzig

When the investment function is linear, which is the case Van Dantzig looked at, it holds:

$$I'(X) = I'(u) \qquad \Rightarrow \qquad S_o^+ = S^+$$

Therefore we can use (22) as a starting point. After substituting (3) in (22) and rearranging the following formula for X results:

$$X = \frac{1}{\theta}\ln\left(\frac{\theta B S_o}{\left(1 - e^{-\delta D}\right) I'}\right) \tag{30}$$

Van Dantzig's reasoning is comparable with the one above, with the difference explained in formulas (15) and (16). If we write Van Dantzig's main result (VD14) in the same way as (30), we get, with ζ = o and therefore θ = α:

$$X_{VD} = \frac{1}{\alpha}\ln\left(\frac{\alpha B S_o}{\left(1 - e^{-(\delta-\gamma)D}\right) I'}\right) \tag{31}$$

Van Dantzig's result looks similar to (30), with δ in the exponent in the denominator replaced by (δ-γ). Besides (31) being not correct, it is also more restrictive than (30), since in (31) the restriction δ > γ is needed for assuring that the integral in (11) is finite. This is certainly not always the case for relatively small areas like dike rings, see also section 4.1. The implication of the correct formulation (30) is that the first heightening of the dikes is smaller than according to Van Dantzig's formula (31).

3.5 More general investment cost function and stability conditions

Investment costs depend on the height of the dike

Civil engineering data of investments costs assembled for the CBA of the river system show that the engineers use linear cost functions as long as they think that the same heightening technique can be applied. But every technique has its technical and economic limits, which basically depend on the height of the dike. Then a technique will be replaced by a technique with higher costs. Mostly fixed as well as variable costs will rise (Arcadis et al., 2004).

There is also a general argument of making the investment costs a function of the height of the dike. Heightening a dike is only possible in combination with broadening the dike. So where the influence of the investment on the exceedance probabilities is in one dimension, height, the costs of the investment are partly in line with a surface measure and that the more, the higher the dike already is.[20]

Therefore in our actual calculations we replace (10a) by:

$$I(u_i, H_{T_i}^{-}, T_i) = F(u_i)\, e^{\lambda (H_{T_i}^{-} + u_i)} = I_1(u_i)\, e^{\lambda H_{T_i}^{-}} \qquad when \quad u_i > o \tag{32}$$

Since I is already convex in H (including u), there seems no reason to chose for F a more complicated specification than linear in u and independent of T_i.[21] This choice for F is also in line with the underlying investment functions used by the engineers (Arcadis et al., 2004).

The consequence of a shifting cost function for the solution of the model is that the same shift occurs in the optimal loss-interval. It becomes: $(s^{-} e^{\lambda H_{T_i}^{-}},\ S^{+} e^{\lambda H_{T_i}^{-}})$ with constant periodical heightening u and constant time span D ; see the mathematical proof in the Appendix based on the Maximum Principle.

Convergence and stability conditions

However, in case of increasing investment costs a condition must hold to assure the existence of a proper solution of the problem, see (A.25):

[20] Also Van Dantzig (1956, p280) mentioned that the real cost function should depend on the height of the dike. He restricted his derivation to a linear function only because in his case the actual cost curve turned out to be nearly linear on the relevant interval.

[21] See paragraph 3.6 for an easy addition of a time trend in the cost function within the scope of this model.

$$\delta > \frac{\lambda}{\theta + \lambda}\beta \tag{33}$$

Condition (33) always holds when $\lambda = o \; or \; \delta \geq \beta$. But when $\lambda > o \; and \; \beta > \delta$ a problem can arise. Above we already argued that in general $\lambda > o$. Further, the growth rate of the expected loss β comprises, besides the rate of economic growth γ, the rate of deterioration of the water system $\alpha\eta$. In the area of the downstream rivers the component $\alpha\eta$ can be as high as 4 to 5% per year. So in this area β will be 6 to 7% per year and can therefore be higher than a discount rate δ chosen in advance, which will normally be in the same order of magnitude as a part of β, namely the rate of economic growth γ. So, it is not a priori clear whether condition (33) in practice holds or not for a value for δ chosen in advance, e.g. 4% (real, risk free) as prescribed in the Netherlands for CBA.

To get an idea of the economic meaning of (33), we write this condition in another way. First we multiply both sides with $(\theta+\lambda)/\theta$. Further, ζ turns out to be always small compared to α, so $\theta \approx \alpha$. Substitution in (33) gives:

$$\begin{gathered} \frac{\theta+\lambda}{\theta}\delta > \frac{\lambda}{\theta}(\alpha\eta+\gamma) \approx \lambda\eta + \frac{\lambda}{\theta}\gamma \qquad \Rightarrow \\ \delta > \lambda\eta + \frac{\lambda}{\theta}(\gamma - \delta) \end{gathered} \tag{34}$$

If the rate of growth in wealth γ is roughly equal to the rate of economic growth in the area (implying a constant capital-output ratio on the macro level in the long run), and if we suppose a dynamically efficient macro economic situation in which the rate of discount δ is bigger than or equal to the rate of economic growth (including population growth) then the second term on the right hand side is negative, but small.

The basic meaning of condition (33) turns out to be $\delta > \lambda\eta$ or, in words, that the discount rate (per year) must be bigger than the rate of increase of the investment costs per year; which is equal to the increase of the investment costs of another centimetre dike (λ) times the number of centimetres rise of the water level per year (η).

Economic stability

Besides the mathematical condition for convergence (33), we can also formulate an economic stability condition. That is that the yearly equivalent of all costs connected with flooding has to be a declining or at most stable fraction of the income generated in the area. Otherwise all flooding costs together form a rising part of income and in the long run that is obviously not a tenable situation.[22] The easiest point to check what this economic condition means, is the

[22] The matter of investment costs rising with the rising height of the dikes ($\lambda > o$), raises the question whether technical progress can act as a compensating effect on future developments. See on this question section 3.6.

moment directly after a new investment. On that moment all cost components have grown with a factor exp(λu) compared with the situation after the investment before. According to the model income has grown in that period by a factor exp(γD). Substituting the definitional relation (A.14) between u en D and using $\theta \approx \alpha$ as above, gives:

$$\gamma \geq \frac{\lambda}{\theta + \lambda}(\alpha\eta + \gamma)$$

$$\gamma \geq \lambda\eta \tag{35}$$

Combining the mathematical and the economic stability conditions gives in normal situations:

$$\delta \geq \gamma \geq \lambda\eta \; (\geq o) \tag{36}$$

If conditions in (36) do not hold, which is easily possible for relatively small areas like dike rings, then only condition (33) is necessary. This is always possible by choosing δ high enough. In the last part of this section we study the implications of an increase of δ for the safety of a dike ring.

Using some reasonable recent parameter values for dike ring 14 Central Holland $\lambda = 0.0066$ and $\eta = 0.6$, gives according to (35) a lower border for economic growth in this area of 0.4% per year to keep all costs connected with flooding under control. This doesn't look a problem. But when we compare these data with the figures for the other dike rings in the area of the downstream rivers in Eijgenraam (2005), it turns out that the likely parameter values for Central Holland form the most favourable combination of all normal dike rings in that area. When we make a pessimistic combination of parameters which are actually relevant for different dike rings in this area: $\lambda = 0.01$ and the middle climate scenario of the IPCC ($\eta = 1.36$ cm per year till the year 2050 in an unfavourable combination with wind), the following condition results:

$$\gamma > 0.01 * 1.36 \; = 1.36\%$$

This value for the lower bound for the rate of economic growth is higher than the economic growth in the lowest of the four economic scenarios up to 2040 recently published by CPB and the value is almost equal to that in another scenario (Mooij & Tang, 2003 and Huizinga & Smid, 2004). But in the high climate scenario of IPCC we get values for η of 1.5 till 2.2 cm per year for the rise in water level in this area. Within the context of this climate scenario the actual combinations for the dike rings 16 Alblasserwaard en Vijfheerenlanden and 35 Donge turn out to be as high as 1.85% per year. Only in the highest of the four recent economic growth scenarios the economic growth in the Netherlands is expected to be higher than 1.85% per year.

So, for these two dike rings economic stability is likely, but not certain in case of an unfavourable combination of plausible scenarios.

Higher value for δ

All parameters and variables in the model but one have a clear technical meaning and can therefore not be chosen freely. The exception is – to some extent – the discount rate, with the limitations given in (33) and to a lesser extent in (36). It is always possible to find a solution for the extended version of problem (11) by choosing δ high enough.

But the choice of the numerical value for δ has a direct, almost proportional influence on the optimal flooding probability. In the appendix we derive as an approximation, see (A.38):

$$S_{T_i}^{mean} \approx \delta \frac{1}{\theta} \frac{I_i(u)}{u} \left(1 + 0{,}5\lambda u\right) \tag{37}$$

with $S_{T_i}^{mean}$ average expected loss during the standard period after investment i

The influences of δ on the mean investment costs per centimetre and on u are not big, so the influence of the first factor on the right hand side dominates the effect of a change in δ. Because the loss by flooding, V_t, does not depend at all on the discount rate, also the mean optimal probability of flooding is almost proportional to the value chosen for δ.

Suppose we would find for a certain moment in time an optimal mean value for the flooding probability of 1/3500 per year in case of a discount rate of 4% per year, then this probability would rise to 1/2000 per year if we use a discount rate of 7% per year instead.

Policy recommendations in respect to the stability conditions

The policy recommendations concerning the stability conditions found above are the following. First we should look at equation (35): Is the rate of growth of wealth within the dike ring bigger than the rate of growth of the investment costs? When the answer is negative, abandoning of the dike ring should seriously be considered. The same consideration seems appropriate, when the discount rate necessary for convergence in equation (34) turns out to be (much) higher than the standard discount rate prescribed by the government as an expression of the rate of return on alternative investment projects.

3.6 Technical progress, relative prices and macro economic risk

Technical progress and relative prices

The matter of investment costs rising with the rising height of dikes (λ > o) raises the question of the possible influence of technical progress on future developments. Before we can do that, we have to realise that the analysis hitherto has been solely made in constant relative prices: all nominal values are deflated by the same, general price index. By doing so, we implicitly assume that the rate of technical progress is roughly the same between the sectors in the economy, which are relevant for this problem. So we already need a certain rate of technical progress in the civil engineering industry to cope with rising real labour costs as a consequence of the general real income growth and also to cope with the cost increase as a consequence of a higher population density. This higher density results in higher real costs for the preservation of e.g. environment, landscape, nature and cultural amenities. It is far from certain that technical progress in the civil engineering industry will be high enough to compensate on top of that at least a part of the effect represented by the parameter λ.

The Maximum Principle allows for an analysis with variables depending on time, but then in general a completely new derivation of all formulas is necessary because of this addition. However, if we choose a special specification, an analysis within the context of the model used so far is possible.

We define three different price indices for respectively the general price level Πg, the price level of civil engineering goods Πe and the price level for wealth Πv, which is predominantly the price index for the building industry. We allow the rates of technical progress in the two specific sectors to deviate systematically from the average rate of technical progress in the economy, resulting in systematically different rates of growth of the sector price indices compared with the general price index. So we have:

$$\frac{\Pi e_t}{\Pi g_t} = e^{\rho t} \qquad \frac{\Pi v_t}{\Pi e_t} = e^{\pi t} \tag{38}$$

Now we first write (11) explicitly in nominal terms with the proper corrections for the increase of the general price level and then substitute (38):

$$\begin{aligned} \min_U C &= \int_o^\infty S_t \, \Pi v_t \, e^{-\delta t} \Pi g_t^{-1} \, dt + \sum_{i=1}^\infty I_i \, \Pi e_{T_i} e^{-\delta T_i} \Pi g_{T_i}^{-1} \\ &= \int_o^\infty S_o \, e^{-\theta H_t} e^{(\beta+\pi)t} \, e^{-(\delta-\rho)t} \, dt + \sum_{i=1}^\infty I_i \, e^{-(\delta-\rho)T_i} \end{aligned} \tag{39}$$

By redefining δ and γ (included in β), we get back the same specification as in (11) resulting in the same derivation in appendix A. Substituting the redefinitions in the results already found, we get, expressed in the original parameters, instead of (33):

$$\delta > \frac{\lambda}{\theta + \lambda}(\beta + \pi) + \rho \qquad (40)$$

Therefore positive values for the relative price changes π and ρ (meaning less technical progress than on average in the economy) increase the necessary value of the *real* discount rate. In the same way for (36):

$$\delta \geq (\gamma + \pi + \rho) \geq (\lambda\eta + \rho) \geq \max(o, \rho) \qquad (41)$$

In a recent CBA of a real estate development project (Besseling et al., 2003) a value of 1.2% per year was used for the sum (π +ρ), so this value is not negligible for the choice of δ. Also important is whether ρ is negative, so alleviating the rise of investment costs, or positive and therefore aggravating the problem. Historical figures for the Netherlands show no clear indication for a systematic positive or negative value for ρ, so the best guess is ρ = o. This means that the rate of technical progress in the civil engineering industry will not be enough to compensate, even in part, the effect of the cost increase λη due to the increase of the height of the dike.

On the other hand, this analysis shows that we have to add an effect of a real relative price increase π to the growth rate of real wealth γ.

Macro economic risk

From a general perspective heightening of dikes can be seen as just one of the many possibilities to foster welfare by government investment. The investment portfolio of the general government is much diversified; therefore the combined value of the many risks is equal to a normal insurance premium or cancels out. There is one clear exception. The benefits of many projects are positively correlated with economic growth, as is also the case for the type of investment we discuss here. The correlation between the costs and economic growth is much smaller and can be neglected.

One of the familiar ways of dealing with this problem is to add a risk premium to the discount rate of the benefits, leaving the discount rate for the costs unaltered. Another familiar way is to lower the economic growth rate with a risk premium. With the same type of reasoning as in (39) it can be shown that using a higher discount rate for benefits than for costs is equivalent to subtracting the risk premium from the rate of economic growth γ.

Different value for γ (or β)

In the two sections above we have sketched reasons to choose for γ (and therefore for β) another value than the expected future rate of economic growth. One is the likely relative price increase of wealth and therefore positive, the other the distraction for macro economic risk.

Since β has no direct influence on condition (20), the First Year Rate of Return, the only influence of β is via the value of u, the size of the heightening. When in a particular situation the size of the action would be given in advance, the numerical value of β is of no importance at all for the solution of the problem at hand, as long as it is certain that β > o. In section 4.3 we will discuss the influence of a lower value for β, using the outcomes in Table 4.4 for a lower value of η, but the outcomes are basically the same if we had used a lower value for γ instead.

3.7 Loss by flooding depending on other variables

Loss by flooding depending on the water level

In equation (2) the loss by flooding depends on the lowest height of the dike ring. This is relevant along lowland rivers. But along highland rivers or the sea the loss by flooding is more dependent on the height of the water level at the time of flooding. The higher the water level, the more serious and widespread will be the flooding and therefore the bigger the damage. In this section we will show that the introduction of this phenomenon is possible within the scope of the model already developed.

Because each possible height of the water level has now not only its probability of occurrence but also an associated amount of loss by flooding, the probability of flooding and the damage can not longer be calculated independently. The expected loss is the integral of both factors. To simplify the derivation, we delete for a moment the increase of both variables in time and we do the same with the effect of the height of the dike on the maximum loss by flooding. These influences are not depending on the height of the water level and can be handled outside the integral.

Probability of an extreme water level

$$p(w) = P_o \alpha e^{-\alpha w} \qquad for\ w \geq H_o \tag{42}$$

with: p exponential probability function of (extreme) water levels
w water level

Loss by flooding

$$V(w) = V_o^* \frac{\alpha - \nu}{\alpha} e^{\nu w} \tag{43}$$

The scale factor can only be written in this way if ν < α.

Expected loss

$$S(H_t) = \int_{H_t}^{\infty} p(w)V(w)\,dw = S_o^* \int_{H_t}^{\infty} (\alpha - \nu) e^{-(\alpha-\nu)w}\,dw = S_o^* e^{-\theta H_t} \tag{44}$$

with $\theta = \alpha - \nu > o$

The reason for ν < α, is that integral (44) has to be convergent, or in other words, that the expected loss should be limited. This can only be the case when the loss by flooding in (43) increases more slowly than the probability on the accompanying water level in (42) decreases. In the end this is always the case, because the maximum loss inside a dike ring is in the end limited.

The other factors temporarily left out can easily be added to (44). Because this equation fits in format (3), also the case with a loss by flooding which is depending on the height of the water level at the time of flooding, can be handled by the model already developed.

Loss by flooding depending on the probability of flooding

Formula (44) is an example of a wider class of formulations, in which the loss by flooding is made dependent on the probability of flooding. The general idea is that floods with a small probability of occurrence are more extreme than floods with a higher probability of occurrence. The extreme character of the flooding manifests itself by an extreme amount of damage. Research on the different possibilities of flooding in the Netherlands with their accompanying damage suggests that this phenomenon can be relevant.

In this case the probability of flooding is the same as in the standard case, but the loss of flooding is formulated as follows (leaving out again the factors depending on time and the height of the dike):

$$V_t = V_o^* P_t^{-\mu} \tag{45}$$

To assure convergence it is necessary that μ < 1.

Although formula (45) seems preferable above (44), because the reasoning behind formula (45) is more general than that behind formula (44), this approach has the big disadvantage that the theory is vague. The consequence is, that it is not at all clear how to estimate the value of μ. The reasoning behind (44) implies that in case of a flooding with a low probability also all floods with a higher chance on realisation occur. But this is not necessarily true for other types of cases where the causes of the different floods are different. So it is not clear whether we should cumulate the losses in the different flooding scenario's or not. One has to specify the probabilities on the occurrence of more than one flood at the same time. So one has to be very cautious in using (45) in practice.

4 Numerical results[23]

4.1 Van Dantzig's (1956) results for Central Holland recalculated

To show the results of the new optimal safety strategy we give a recalculation of Van Dantzig's original calculation published in 1956 in Econometrica for dike ring 14 Central Holland.

Central Holland is the most important dike-ring area in the Netherlands, comprising big parts of the four largest cities in the Netherlands. This dike ring stretches from Amsterdam in the north to Rotterdam in the south and from The Hague in the west till the western parts of Utrecht in the east. The dike ring borders the sea in the west and the big rivers in the south. Van Dantzig's calculation was looking at the most critical section of the dike along the rivers in the south, which were, at that time, in open connection with the sea at Hook of Holland.

The outcome of his calculation was one of the arguments, but not the most important one, to fix the safety level of Central Holland in 1958 by law on an exceedance probability of not more than 1/10000 per year. Today this number is still the legal standard. Also the legal standards of all other dike rings in the Netherlands are in a loose way related to this one. So, this recalculation has more than only historical significance. Of course, for a real evaluation actual figures should be used, but unfortunately, reliable figures could not be found. Instead we present in section 4.2 the calculation for a smaller dike ring in the neighbourhood of Rotterdam: dike ring 16 Alblasserwaard en Vijfheerenlanden.

Data

The numbers for the parameters and variables mentioned by Van Dantzig are assembled in table 4.1.[24] Van Dantzig already multiplied the estimate for the amount of material damage by flooding by a factor 2 to take into account "ideal values" i.e. non-material damage, mentioning that it was clearly a political decision what value to take for this factor. V comprises also an estimate for indirect effects or 'consequential loss': "(for a first rough estimate by multiplying the actual value of the goods by a constant factor of 1.2)". About 'the doubtful constants' Van Dantzig wrote: "We have already mentioned the fact that several of the constants entering into the problem are rather badly known. (…) So the best thing we can do is to ascertain that our solution will hold under the most unfavourable circumstances which must be considered to be realistic. (…) So in order to remain on the safe side we must take the highest reasonable estimates of P_o, V_o, and η and the lowest ones of F'_u, α, and $(\delta - \gamma)$."

[23] The new method was developed and first applied to 22 dike rings along the rivers Rhine and Meuse, see par. 4.2.

[24] Since the Econometrica article was based on a presentation for the Econometric Society in August 1954, all nominal figures probably refer to 1953. All water levels refer to the situation at Hook of Holland near Rotterdam, where the main branch of the Rhine floats into the North Sea. One figure necessary for our recalculation is missing in the Econometrica article, namely the fixed costs of investment, because Van Dantzig only tried to answer the how much question. We derived the figure in table 4.1 for the fixed costs from the final report of the Delta Commission (1960) comprising Van Dantzig's last version of his report. There an amount of 110 mln NLG is mentioned for the first 115 cm heightening. Van Dantzig's 1956 figure for the time-span D was a civil engineering rule of thumb.

Table 4.1 Variables and parameters in Van Dantzig (1956, p284)

Name	Measure	Symbol	Value
Height above mean sea level, base	cm	H^-	425
Exceedance probability belonging to H_o	1/year	P_o	0.0038
Parameter exponential distribution water level	1/cm	α	0.026
Increase water level	cm/year	η	1
Damage by flooding in 1953	mln NLG	V_o	20000
Economic growth	1/year	γ	0.02
Effect of heightening on amount of damage	1/cm	ζ	–
Rate of interest (real)	1/year	δ	0.04
Variable costs of investment	mln NLG/cm	F'_u	0.42
Fixed costs of investment	mln NLG	F(o)	61.7
Parameter non-linear effect investment	1/cm	λ	–
Time span between periodical investments	year	D	75
Result			
Height above mean sea level with X (pessimistic combination)	cm	H^+	673
Height above mean sea level with X (reasonable combination)	cm	H^+	600

The combination of values in table 4.1 resulted in a calculated heightening of 248 cm on top of a base level of 4.25 meters above sea level (NAP). Van Dantzig continued: "The combination of these extreme values for all constants, however, is rather pessimistic. Several reasonable combinations of values lead to the conclusion that roughly 6.00 meters may be considered as a reasonable estimate of a sufficiently safe height." Unfortunately, the only exact result published in the Econometrica article has been based on the pessimistic combination in table 4.1.

In 1958 the legal design level chosen at Hook of Holland was 5.00 m, starting from a level of 3.85 m at that place, instead of 4.25 m, so a heightening of 115 cm.[25] Not surprisingly, in his final 1960 report Van Dantzig expressed his great disappointment about this legal choice, which was in his opinion clearly far to low.[26]

Results

Table 4.2 gives an overview of some original results and those of the recalculation.

[25] One of the reasons for the lower legal standard was, that the Delta Commission 'guesstimated' that the flooding probabilities would be in reality a factor 12,5 smaller than the exceedance probabilities of the design-water level a dike should sustain. So overflow would cause some hinder, but no break would occur.
This idea is in sharp contrast with the preliminary results of the recent VNK-project which has thrown light on many weaknesses in the surrounding water defences of dike rings, see also footnote 3. A well-known recent example in which construction failures turned out to be far more important than overflow, is the flooding in New Orleans. There the constructions already collapsed through the pressure of the water long before overflow could have taken place.

[26] We cannot use the calculations in the 1960-report because of even more mistakes than in the Econometrica article, e.g. a derivation for D while neglecting fixed investment costs, not surprisingly resulting in an optimal value D = o. Obviously this makes no sense. Another strange point is, that the value of η has been neglected in the calculation, using the argument that there should be a proper 'regeneration' of the dikes in the future correcting for the rising water levels. This means that also the increase in the near future had not been taken into account in the 1960-calculation of the first heightening.

Table 4.2 Comparison of results for the heightening of the dikes of Central Holland in 1953

	Van Dantzig's formula			New formulas	
	Reasonable combination	Published 1956	Recalculated	Data table 4.1	γ = 0.038 η =0.22
Growth of expected loss β in %	?	4.6	4.6	4.6	4.37
Heightening dike in 1953 in cm	175	248	244	236	233
Period in years		75	75	73	75
Periodical heightening in cm	?	75	75	129	127
Exceedance probabilities	Flooding once in … year				
Actual in 1953	263	263	263	263	263
Highest acceptable in 1953		23 650	21 500	4 150	4 200
Lowest after heightening in 1953	24 900	166 000	151 000	120 000	113 400
Middle in 1953				15 500	14 350
Legal standard Act of 1958	10 000			10 000	10 000

The second column 'Published' gives the 'pessimistic' results mentioned in Van Dantzig (1956), with the addition of figures that follow directly from his formulas and reasoning. The amount of the periodical heightening follows from the facts that the period chosen was 75 years and the increase of the water level was set at 1 cm per year, and Van Dantzig's reasoning of a constant exceedance probability after an investment. The exceedance probabilities follow from equation (1). The figures in the row 'lowest after heightening in 1953' follow from the sum of the actual dike level in 1953 and the preferred heightening in 1953. The figures in the row 'highest acceptable in 1953' follow by subtracting the periodical heightening from the preferred level after heightening in 1953. It turns out that the actual exceedance probability in 1953 was much higher than the highest acceptable one in 1953.

The next column 'Recalculated' gives the results of our own calculation with the formula and data of Van Dantzig. The results confirm that the article has been correctly interpreted. The small numerical differences with the column 'Published' must be the consequences of rounding off.

The first column 'reasonable combination' gives the results given by Van Dantzig that according to his text (p284): "… may be considered as a reasonable estimate of a sufficiently safe height." Since at that time probably η, the increase of the water level per year, was the most 'doubtful constant', some of the outcomes in this column cannot be recalculated, because the 'reasonable' estimate used for η is unknown.[27]

[27] In his 1960 report Van Dantzig's outcome is also 6 m above NAP, but then starting from 3.85 m, so a heightening of 215 cm. In that report he combines this 6 m with an exceedance probability of 1/125000 per year. This exceedance probability differs a lot from his 'reasonable estimate' cited in table 4.2. But this figure is roughly in line with the exceedance probability belonging to a heightening of 215 cm in Table 4.2, see the row lowest after heightening in 1953 in combination with an optimal heightening of 233 till 236 cm. The correspondence is not exact because in 1960 for some parameters slightly different values have been used.

Calculation with the new formulas and the data in table 4.1 gives results which are in the same order of magnitude as Van Dantzig's with respect to the first heightening and the period between the periodical heightenings. The first heightening is indeed smaller, as it should be according to the difference between equations (30) and (31). D is now the outcome of a calculation and confirms that Van Dantzig used a really good guesstimate for that parameter. The periodical heightening is now clearly higher because heightening is also necessary to compensate for the increase in wealth. It should be noted that the maximum acceptable exceedance probability just before the next investment 73 years later would be much lower than the 1/4150 in 1953. It is also possible to calculate the middle exceedance probability in 1953, see (A.52). This gives the best idea of a central optimal safety level for that year.

The last column gives a calculation with the new formulas and with growth rates that are more in line with the realisations in the past 50 years. The rate of economic growth in the Netherlands (γ) has been much more, roughly 3.8% per year.[28] But the rise of the sea level at Hook of Holland (η) has been much lower than expected, about 22 cm per century.[29] In combination this gives a new growth rate of the expected loss (β) which is only slightly lower than in the 'pessimistic combination of extreme values' of Van Dantzig. Since expected loss is the central variable, the results based on the realisations differ not much from the 'pessimistic case' of Van Dantzig!

Therefore, Van Dantzig's 'reasonable estimate' turns out to be far too optimistic. This is even more true for the legal standard. A second observation is that his formula would not even have been applicable to this combination of real data, because the value for (δ - γ) is not more than 0.002.[30] This figure would have led to an explosion of the first heightening X compared to the values in table 4.2, instead of X being a bit smaller as correctly shown in the one but last column.

Policy advice

Unfortunately, there are no reliable, recent data available for the dike ring Central Holland. Anyway, the calculation above confirms that the safety level chosen for Central Holland in the Act of the Water defences seems very low. The clear policy advice is to recalculate the economic optimal safety level with recent figures and to start a political debate about new standards for the safety levels for dike rings in general.

[28] Note that besides using the rate of real economic growth as a proxy for the growth of wealth, no correction has been made for the difference in the price index of wealth compared to the general price index, see section 3.6. This difference can have been in the order of magnitude of 1 till 1.5% per year.

[29] In fact, new insights have led to an even more optimistic view on the present situation. It turns out that even the complete extreme value distribution at Hook of Holland is in 2001 still the same as in 1960. Better insights in the processes at hand in combination with local changes in the environment have offset the relative rise in the sea level at Hook of Holland mentioned in Table 4.2. Therefore a height of 5.10 m above sea level (NAP) at Hook of Holland is still the level belonging to an exceedance probability of 1/10000.

[30] If we had used a slightly different reference period, the value for δ - β would even have been negative, resulting in Van Dantzig's formula in a negative height of the dike. This result clearly makes no sense.

4.2 Application for the project 'Room for the River'

Data

This model has been developed for a Cost Benefit Analysis of the project Room for the River (Eijgenraam, 2005). This project aims to bring the safety levels of the dike rings along the river Rhine back to their legal values by the year 2015. The reason that the actual safety level is lower than the legal standard, is a recent heightening of the design discharge volume that is connected to the safety level. This heightening was caused by the very big river discharges in 1993 and 1995, almost resulting in flooding.

The study area has been divided in dike rings along the upstream rivers in the east and dike rings in the area of the downstream rivers in the west. The difference between the areas is that in the area of the downstream rivers besides the river discharges also the sea and the wind play an important role in the estimation of the exceedance probabilities. The example chosen here is dike ring 16 Alblasserwaard en Vijfheerenlanden near to the spot that was crucial in the original calculation of the safety of Central Holland. It has the same weak underground with peat, so the investment costs are highly dependent on the height of the dike. On top of that this dike ring will experience a very fast rate of increase of the water level. The number of inhabitants is 210 thousand and the length of the surrounding dikes along the rivers is 85 kilometres.

Table 4.3 Variables and parameters dike ring 16 Alblasserwaard en Vijfheerenlanden, prices 2003

Name	Measure	Symbol	Value
Exceedance probability belonging to H_o	1/year	P_o	0.0011
Parameter exponential distribution water level	1/cm	α	0.0574
Increase water level	cm/year	η	0.76
Damage by flooding in 2002	mld euro	V_o	22.7
Economic growth	1/year	γ	0.02
Effect of heightening on amount of damage	1/cm	ζ	0.002
Rate of interest (real)	1/year	δ	0.04
Variable costs of investment	mln euro/cm	F'_u	2.1304
Fixed costs of investment	mln euro	F(o)	324.63
Parameter non-linear effect investment costs	1/cm	λ	0.0100

The actual exceedance probability in 2001 is about twice the legal figure of the maximum exceedance probability of 1/2000 per year. Till 2015 the actual figure will probably double again before the project has been carried out. We suppose that the investment takes place completely in the year 2015, because completion in an earlier year is not possible. The parameter of the extreme value distribution of water levels is much higher of than that at Hook of Holland. The future rise in the relative water level is in accordance with the mean climate scenario of IPCC and a favourable combination with wind. The potential damage by flooding is almost 23 mld euro, which comprises the material damage plus an amount of 5000 euro per inhabitant for nuisance costs of evacuation. In contrast to Van Dantzig no value has been included for the loss of lives and other serious personal damage. Since population growth is expected to halt in the future, economic growth will be not more than about 2% per year. Because the dike ring is situated near the sea, the slope of the surface of the dike ring (ζ) is small. The investment figures include a fairly high estimate for extra maintenance of 25% of the proper investment costs. On top of that some modifications on the investment costs have been made because the parameters α and η have not exactly the same value along the whole dike ring, for details see Eijgenraam, 2005, appendix B.

Results

The results are summarised in table 4.4. Starting with the first column the first result is, that – without the foreseen projects – there is a backlog in safety in 2015, resulting in an immediate need for investment. The backlog to reach the highest acceptable border of the exceedance probability (1/950) turns out to be relatively small: 10 cm (not visible in the table). So in 2015 we would only be about 10 years too late. Because the investment curve is bending upwards, the actual first heightening has an optimal value of 61 cm, which is a bit smaller than the sum of the backlog and the periodical heightening of 52 cm. Consequently, the length of the period till the second investment of 53 years is one year shorter than the standard period of 54 years. Also, after the first heightening the value for the upper bound of the safety interval (1/16850) will not fully be reached.

Table 4.4 Results for Alblasserwaard en Vijfheerenlanden with recent data

Name	Variable	Data table 4.3			η =0.22, known	
		Best guess	V +50%	I +50%	In advance	Afterwards
Growth of expected loss in %	β	6.36	6.36	6.36	3.26	3.26
Heightening dike in 2015 in cm	X	61	67	55	46	61
Next period in years	D_2	53	52	53	84	113
Periodical heightening in cm	u	52	52	52	42	42
Period in years	D	54	54	54	84	84
Exceedance probability in 2015		Flooding once in … thousand years				
Legal standard		2.00	2.00	2.00	2.00	2.00
Actual		0.50	0.50	0.50	0.50	0.50
Middle		1.95	2.90	1.30	2.00	2.00
Standard Highest	P^-	0.95	1.40	0.60	1.05	1.05
Standard Lowest	P^+	16.85	25.25	11.25	10.65	10.65
Total costs flooding & investment (NPV, mln euros)	C	1323	1435	1828	913	967

There is, coincidently, a very good correspondence between the legal standard for this dike ring (1/2000) and the middle optimal exceedance probability in 2015 (1/1950). So there is no reason to change the legal standard for this dike ring in the next two decennia, unless completely different new information about crucial parameters emerges in the future.

But the size of the probability interval is enormous, ranging in 2015 from 1/1000 till 1/17000. The reason is the very high growth rate of the expected loss of 6.4% per year and that during a period of more than 50 years till the next investment. The length of the period illustrates the enormous influence of the amount of fixed costs on the results.

4.3 Sensitivity analysis

Other values for variables

The next two columns in table 4.4 give the results in case the potential loss (V) or the investment costs (I, both fixed and variable) are 50% higher than the figures in table 4.3. Reasons to use a higher amount for the loss by flooding can be an underestimation of the damage caused by flooding, e.g. contamination of the soil, a valuation for the loss of lives or a valuation of risk aversion. Reason for using a higher amount of investment costs could be an underestimation of the real costs because no precise design was made to estimate the investment costs.

None of these changes has an influence on the value of the standard heightening u, see (23), or the length of the standard period D, because the relation between the fixed and marginal costs has not been altered. But both changes have of course an influence on the absolute height of the dike and on the absolute safety level. The first heightening X is 6 cm higher or lower than in the reference case and the period D_2 till the next investment is (less than) one year shorter or longer than in the reference case. The resulting probabilities in the reference case are about 50% smaller than with higher investment costs, and the probabilities with the higher damage costs are about 50% smaller than in the reference case.

In both these situations total cost will increase, but less than the impulse may suggest. The reason is the adjustment of the optimal policy. The resulting cost increase is 8% when the potential loss is 50% higher. But total cost increase by 38% when the investment costs turn out to be 50% higher. The reason is that from the total costs of 1323 mln euro in the first column, not less than 850 mln euro are connected with the first investment X. Already the 50% of this amount alone is 425 mln euro. The NPV of the rest of the increase is only 80 mln euro.

According to the reasons mentioned above for the two variants it is quite possible that both apply at the same time. Increasing both the damage by flooding and all investments costs by the same percentage does not change the original results at all, with the obvious exception of total costs which then increase by the same percentage. This can easily be understood by realising that changing all monetary values by the same percentage is like calculating in e.g. dollars instead of euros. Of course this leads not to another number of centimetres or probabilities.

Other expectations on future developments

The two variables used in the sensitivity analysis above are related to the present situation. So it is relatively easy to improve their estimates by doing more research or by taking political decisions on the weight of non-monetarised values like human lives. For parameters involving the future these kinds of improvement are in principle not possible. For these parameters we should make a clear distinction between a better estimate known in advance (ex ante), comparable with the first two variants, and new knowledge that comes shortly after the investment has already taken place (ex post).

The figure for the rise in the relative water level of 0.22 cm per year used in the last two columns of table 4.4 belongs to the low climate scenario of the IPCC. The drop by 54 cm per century compared to the reference case of 76 cm per century has an important effect on the growth rate of the expected loss per year. This growth rate drops by 3.10 percentage points, from 6.36% per year to 3.26% per year. Consequently, the period between the investments increases and the size of the heightening drops. This has also an effect on the total costs, as can be seen in the column ‘known in advance’. This drop in costs in comparison with the reference case is in general not dependent on the policy chosen, because Dutch politics can not change anything at all on the rise of the water level. Government has only control over the decisions concerning the height of the dike.

Suppose we only know the figures in the two climate scenarios, but we have no knowledge of their probability of realisation. It is standard policy in the Netherlands to base decisions of the kind discussed here, on the mean climate scenario. What is the value of the mistake when afterwards the low scenario turns out to have been the realistic one?

In this case we do the same calculation as for the low growth scenario, but now with the size for the first investment X fixed on the value of 61 cm calculated in the reference case. Ex post this turns out to have been an overinvestment, because the proper optimal investment should have been 46 cm. Therefore the real period till the next investment, 113 year, will turn out to be much longer than we thought ex ante, namely 53 year. But from these 60 years of difference only 29 years can be ascribed to the overinvestment. The other 31 years are the result of the other growth rate of the water level. During the whole period there is also an extra high level of safety, diminishing the expected loss by flooding. As a result the NPV of total costs turns out tot be only 54 mln euro or 6 percent higher than if we would have known the improved information on the change of the water level in advance. This example shows clearly that it is no real problem to choose for a safe strategy.

In practice we should always invest earlier than the outcome of the calculation indicates, due to the inherent and unavoidable imprecisions in the data used in combination with the asymmetric form of the loss function. It is likely that the real rise in costs in case of (unknown) underinvestment is higher than the rise in costs in case of (unknown) overinvestment.

4.4 Summary of results for dike rings along the river Rhine

Table 4.5 and Table 4.6 show the main results concerning exceedance probabilities for 22 dike rings along the river Rhine, studied in the project Room for the Rivers. The first table gives the results for the dike rings ‘upstream’, meaning that for these areas only the threat from the river is important. For the second group of six dike rings the influences of the sea and the wind are also quite important. Because of the structure of the ground also the costs of heightening dikes are considerably higher than in the upstream area.

Comparing the actual exceedance probabilities in 2001 (column 4) with the legal norm (column 3) shows that in 2001 in 19 of the 22 dike rings the legal maximum has been exceeded. According to the law at least for these 19 dike rings action is necessary.

Comparing the expected actual exceedance probabilities in 2015 (without taking any action before that date) (column 5) with the optimal maximum for the exceedance probability (column 6) makes clear that in 2015 this maximum will be exceeded in 14 out of 21 dike rings. The same is visible in column 7 where the optimal year for the first coming investment is shown, supposing that 2015 will be the first year in which investment projects can be completed. Column 7 also shows that for two dike rings there can only be a few years delay. So, according to the optimal investment strategy in 16 out of 21 dike rings action is needed around 2015.

Table 4.5 Yearly exceedance probabilities for dike rings upstream

1 Nr.	2 Name dike-ring area	3 Legal norm 1/year	4 Actual in 2001	5 Actual expected in 2015	6 Optimal maximum in 2015	7 Optimal year first investment	8 Middle in 2015
38	Bommelerwaard	1/1250	1/1010	1/870	1/840	2017	1/1300
40	Heerewaarden (Waalkant)	1/2000	1/1550	1/1340	–	–	–
41	Land van Maas en Waal	1/1250	1/995	1/860	1/1620	2015	1/2850
42	Ooij en Millingen	1/1250	1/715	1/610	1/470	2024	1/900
43	Betuwe, Tieler- en Culemborgerwaard	1/1250	1/645	1/550	1/580	2015	1/1000
44	Kromme Rijn	1/1250	1/2565	1/2230	1/11630	2015	1/18250
45	Gelderse Vallei	1/1250	1/4140	1/3560	1/19970	2015	1/29600
47	Arnhemse en Velperbroek	1/1250	1/470	1/400	1/1030	2015	1/1800
48	Rijn en IJssel	1/1250	1/550	1/470	1/980	2015	1/1300
49	IJsselland	1/1250	1/565	1/490	1/290	2033	1/450
50	Zutphen	1/1250	1/425	1/370	1/2030	2015	1/4350
51	Gorssel	1/1250	1/320	1/280	1/260	2018	1/400
52	Oost Veluwe	1/1250	1/525	1/450	1/680	2015	1/1200
53	Salland	1/1250	1/315	1/270	1/1540	2015	1/3050
10	Mastenbroek	1/2000	1/2630	1/2270	1/730	2053	1/1400
11	IJsseldelta	1/2000	1/785	1/680	1/310	2042	1/650

Source: CPB (2005)

Table 4.6 Yearly exceedance probabilities for dike rings upstream

1 Nr.	2 Name dike-ring area	3 Legal norm 1/year	4 Actual in 2001	5 Actual expected in 2015	6 Optimal maximum in 2015	7 Optimal year first investment	8 Middle in 2015
15	Lopiker- en Krimpenerwaard	1/2000	1/730	1/430	1/1130	2015	1/2150
16	Alblasserwaard en Vijfheerenlanden	1/2000	1/905	1/490	1/930	2015	1/1950
22	Eiland van Dordrecht	1/2000	1/1800	1/980	1/990	2015	1/2650
23	Biesbosch (Noordwaard)	1/2000	1/730	1/400	1/30	2059	1/50
24	Land van Altena	1/2000	1/530	1/280	1/410	2015	1/850
35	Donge	1/2000	1/510	1/300	1/810	2015	1/1300

Source: CPB (2005)

For an overall assessment of the present urgency for action before 2015 it doesn't make much difference which criterion we use: the present legal standards or the outcome of the calculation.

Revision of the legal standards seems appropriate

After completion of all actions proposed in the official Plan of Action 'Room for the River' all dike rings will be at least back on their legal safety level mentioned in column 3. But along all dike rings there will be places where the standard will be just met. Comparing the legal standard (column 3) with the maximum optimal exceedance probability in column 6 shows that for most dike rings the actual safety level is then within the optimal interval. Therefore for these dike rings there would be no immediate need for further action.

However, for 5 dike rings the legal safety level stays far below the lowest boundary of the optimal safety interval.(compare column 3 with column 6). These are the dike rings 45 Gelderse Vallei, 44 Kromme Rijn, 50 Zutphen, 41 Land van Maas en Waal and 53 Salland. For these dike rings it can be worthwhile to do more than precisely what is necessary to comply with the present standards in the law.

But the urgency of action doesn't say much about the desirability of a change in the legal safety standards. The figures in the last columns seem to give a good impression of efficient legal standards. They allow for a time span for reaction of about 20 years between the moment that the level is exceeded till the moment the system is in order again. Of course, it is a matter of policy to take also into account other considerations, for instance matters of equity or measures in the field of spatial regulations.

4.5 Check on the results

There is a simple, intuitive check possible on the results in the last columns above, which also has the advantage of not using a monetary valuation. We use only the number of inhabitants (N) as a measure of loss and the length of the dikes (l) as a measure of costs. We may expect that the optimal probability of flooding is a decreasing function of the number of inhabitants and an increasing function of the length of the dike. More formally, we take the simple formulas for the 'middle' probability (A.52) and the mean expected loss (A.39) as starting point:

$$P_j^{middle} = \frac{S_j^{mean}}{V_j} \cong const \frac{I_j(u_j)}{V_j} \approx const \frac{\ell_j}{N_j} \tag{46}$$

with:

	P^{middle}	'middle' optimal probability of flooding
	j	dike-ring area, j = 1,…, 20
	S^{mean}	mean expected loss in a standard optimal investment period
	l	length of the dike in km
	N	number of inhabitants
	const	approximately the same value for each dike ring

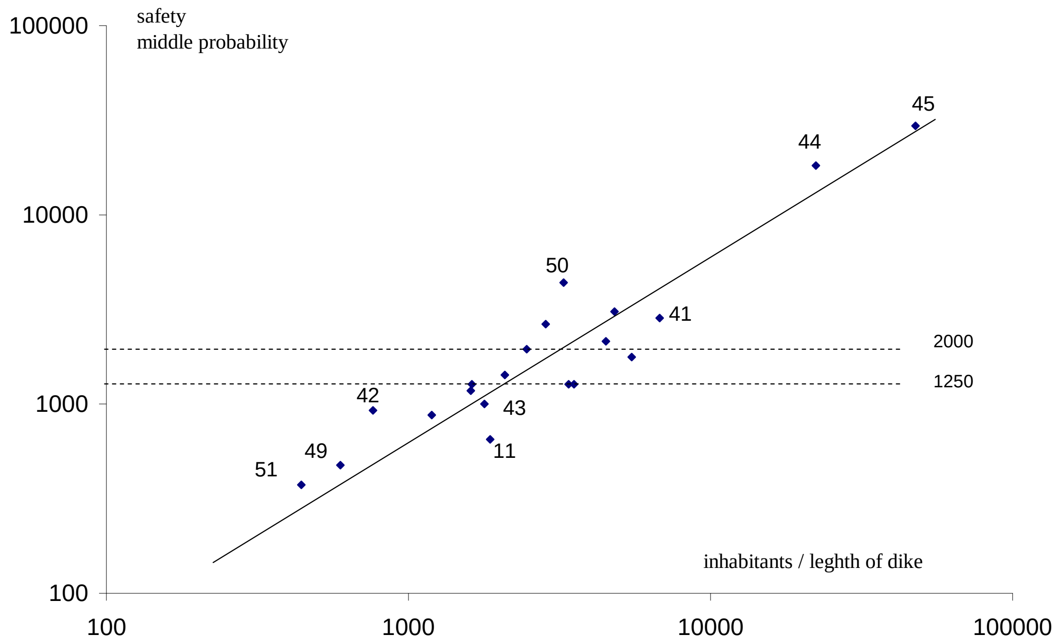
For all dike rings investment costs are thought to be proportional to the length of the dike and the potential loss of flooding is thought to be proportional to the number of inhabitants. Of course, both assumptions are not correct. But the relation found above is in accordance with intuition and we can check the goodness of fit. In Eijgenraam (2005) the calculations have been done for 20 normal dike rings along the river Rhine. Figure 4.1 shows the results of the equation above (after taking logarithms and changing signs).

If the assumptions would be exactly right, all points would lay on a straight line. Taking the great differences between dike rings into account, the result of the simple check is very good.[31] Regression analysis shows that the estimated line is indeed straight.[32] The value of the constant has no meaning as such because this value depends on the units of measurement used.

[31] In fact the potential loss per inhabitant can differ a factor 2 till 3 between the dike rings in Eijgenraam (2005) and the investment costs per kilometre can even differ a factor 5.

[32] The estimated constant in the line in figure 4.1 is 0.447 with a standard error of 20%. This is 1/1.56 for the constant in equation (45). The numerical value of the constant depends on the measures of the variables and has therefore no intrinsic meaning.

Figure 4.1 **Middle optimal safety and the number of inhabitant per kilometre dike (logarithmic)**



The two dike-ring areas in the top at the right (numbers 44 and 45) have short dikes along the river and contain both a big city. Their middle optimal probabilities of flooding are in 2015 respectively 1/18000 and 1/30000 (CPB, 2005). The ones at the bottom at the right are predominantly agricultural areas with villages and small cities and with long dikes because they lay amidst branches of the big rivers. The two highest middle probabilities of flooding are around 1/400. The horizontal lines at the levels of 1/1250 and 1/2000 show the legal standards for protection in respectively the area of the upstream rivers (like the dike ring numbers 44 and 45) and the area of the downstream rivers (like dike ring 16 Alblasserwaard en Vijfheerenlanden in paragraph 4.2).

It is remarkable how big the differences are between the legal standards and the middle optimal values. But the spread is to both sides. For 5 dike rings the calculated middle optimal probabilities of flooding are less than half of the legal norm, suggesting that the legal norm provides too little safety for those areas. But also for 5 dike rings the middle optimal probabilities of flooding are twice the legal norm, suggesting that the legal norm for those dike rings is too costly. How big the differences in safety levels between dike rings may be, is in the end a political decision, because it is the outcome of a weighing of efficiency against equity in safety.

A wild guess for an appropriate order of magnitude for the optimal safety of New Orleans

The main cause for flooding in New Orleans differs much from the situation in the Netherlands where tropical hurricanes do not exist. But on the other hand, the dikes seem not to differ that much from those in the western part of the Netherlands. Also the economic situation will be comparable with urbanised areas in the Netherlands, which is one of the most densely populated areas in the world. In comparison with the 20 dike rings studied in the Netherlands the value of the loss per inhabitant in New Orleans will probably be on the high side, because it is a city and therefore including a lot of public goods. But also the investment cost will probably on the high end of the ones in the Netherlands, because these high figures refer to areas in the delta with a soil of clay and peat, as is the case in New Orleans. So, to some extent, these two influences cancel out. Overall and having no knowledge at all on the specific situation at hand in New Orleans, the formula found in the regression of the 20 dike rings in the Netherlands forms maybe a not too bad starting point for a wild guess for an appropriate order of magnitude for the optimal safety of New Orleans. But one has to be aware of the fact that in this calculation the loss by flooding only includes material damage. The reason is that along rivers a dangerous situation can be foreseen a few days ahead, making an appropriate evacuation possible. The same looks possible in case of tropical hurricanes.

Let us assume that there are roughly 600 thousand people living in the area and that the length of the dikes is about 120 km. The number of inhabitants per km dike is than 5000. The constant is the regression is 1/1.56. This leads to a value for the middle optimal probability of 1/7800. So for New Orleans a reasonable range for the middle of the optimal probability interval seems to be between 1/4000 and 1/10000. These two figures are the same as the legal safety standards in the Netherlands for dike rings bordering the sea and are the highest in the country. Anyway, figures in this order of magnitude differ greatly from 1/200, which seems to have been the safety standard for New Orleans.

The calculation also shows that economic reasoning does not lead to socially unwanted low safety levels as was sometimes suggested in publications by engineers in the Netherlands in commenting on the low safety level in New Orleans.

5 Conclusions

In deriving the optimal safety levels for dike-ring areas the answers on two related questions on investment: ‘when?’ and ‘how much?’ have to be solved simultaneously. Van Dantzig (1956 and 1960) did not address the first question and his solution for the second question is not the optimal one and can even lead to very odd outcomes.

Not the exceedance probability, but the expected yearly loss by flooding is the key variable in an optimal safety strategy in case of economic growth.
The new model can handle:

- Economic growth
- Rise of the water level
- Increase of investment costs
- Different developments of relative prices
- Actual loss by flooding depending on the height of the dike and/or the level of the water

Under certain general conditions it is optimal to keep the expected yearly loss within an interval. The borders of the loss interval are moving in line with investment costs.

The ‘when?’ criterion is the well-known First Year Rate of Return (FYRR). Because the yearly costs of investment stay the same after installation and the benefits of protection increase, the net present value (NPV) will be very positive or infinite for a safety investment which passes the FYRR criterion, see (A.54) to (A.56). Therefore, the well-known NPV criterion in cost benefit analysis of a single project is not a sufficient criterion for investing in projects with benefits depending on economic growth.

When costs are increasing with the height of the dikes, a minimum value for the discount rate is needed to assure convergence. Choosing a higher discount rate results in a higher probability of flooding and a higher expected loss of flooding. When the discount rate necessary for convergence exceeds the common discount rate used as a proxy for the rate of return of alternative investment projects, abandoning of the dike-ring area has to be considered.

Recalculation for the dike ring Central Holland confirms again that the legal safety level chosen for this most vital dike ring seems too low. The clear policy advice is to recalculate the economic optimal safety level of all dike rings in the Netherlands with recent figures and to start a political debate about new legal standards for the safety levels for dike-ring areas in general.

Appendix A[33]

We use criterion function (11) with an endpoint z in combination with the specifications for expected loss in (3) and investment costs (including the net present value of maintenance costs) in (32). In addition there are the initial condition (4), the differential equations (5), the terminal condition (6) and the definitions (7) till (9).[34]

According to the Maximum Principle finding a solution of (11) is the same as finding a minimum for the two Hamiltonians (A.1) and (A.2) under some necessary conditions.[35] Later we will check whether the solution is really a minimum and that the solution is unique.

Hamiltonians

Since there is no fixed endpoint H_z, see (6), the problem is a so-called normal problem with the first adjoint variable: $\psi_o = 1$. Because there is only one control variable, there is only one adjoint variable ψ. Therefore we can use the subscript for indicating time or a jump.

At non-jump points the Hamiltonian is:

$$Ham = S_t \, e^{-\delta t} \tag{A.1}$$

At jump points T_i:

$$IHam = +I_i \, e^{-\delta T_i} + \psi_i^+ \, u_i \tag{A.2}$$

where: ψ adjoint variable

ψ can be interpreted as the marginal benefit of an extra high dike in the initial situation (a larger H_o^-) on total future costs.

Necessary conditions

Applying the list of necessary conditions gives:

[33] The author thanks Dr. J.H. van Schuppen (Free University Amsterdam and Centre for Mathematics and Informatics) for checking the mathematical proof in the Appendix.

[34] To facilitate a check on the mathematical derivation we give the correspondence between the numbered equations in S & S and the equations in this paper: (63) = (5a); (64) = (5b); (65) = (4); (66c) = (6); (68) = (11); (69) and (72) = see text in Hamiltonians; (70) = (A.1); (71) = (A.3); (73c) = (A.5); (74) = (A.7); (75) = (A.8); (76) not present; (77) = (A.6).

[35] Nota bene that (11) is the minimum of a function, while all theory books give the formula for a maximum (of utility or wealth). Therefore signs have sometimes to be changed. The adjoint variable ψ changes its sign automatically with S en I, so on this point there is no visible change in sign.

At non-jump points:

$$\frac{\partial}{\partial H} S_t\, e^{-\delta t} = -\dot{\psi}_t \tag{A.3}$$

$$\psi_z = \frac{\partial}{\partial H}\left(\frac{1}{\delta} S_z\, e^{-\delta z}\right) \tag{A.4}$$

$$\psi_z = o \tag{A.5}$$

At jump points i:[36]

$$\left(S(H_{T_i}^{+}) - S(H_{T_i}^{-})\right) e^{-\delta T_i} + \delta I_i\, e^{-\delta T_i} \quad \begin{array}{ll} \leq o & when\ T_i = o \\ = o & when\ T_i\ \varepsilon\,(o,z) \\ \geq o & when\ T_i = z \end{array} \tag{A.6}$$

$$\psi_i^{+} - \psi_i^{-} = -\lambda\, I_i\, e^{-\delta T_i} \tag{A.7}$$

$$\left\{\frac{\partial}{\partial u} I_i\, e^{-\delta T_i} + \psi_i^{+}\right\}(u - u_i) \leq o \qquad for\ u \geq o \tag{A.8}$$

Upper Border Loss interval

(A.6) gives the first border of the loss interval:

$$S(H_{T_i}^{+}) - S(H_{T_i}^{-}) + \delta I_i \quad \begin{array}{ll} \leq o & when\ T_i = o \\ = o & when\ T_i\ \varepsilon\,(o,z) \\ \geq o & when\ T_i = z \end{array} \tag{A.9}$$

This is the well-known return criterion: First Year Rate of Return equals zero. This criterion determines the timing of a new investment.

Substitution of (7) in (A.9) gives for $T_i > o$:[37]

$$S(H_{T_i}^{-}) = \frac{\delta I_i}{1 - e^{-\theta u_i}} = \frac{\delta I_1(u_i)\, e^{\lambda H_{T_i}^{-}}}{1 - e^{-\theta u_i}} \tag{A.10}$$

Constant periodical heightening of the dike fulfils the necessary conditions

We will show that for $T_i > o$, the following solution fits condition (A.10) and condition (A.21) and is therefore an admissible solution of the problem. The proposed solution is:

$$u_i = u \qquad \forall i\ \ with\ \ T_i > o \tag{A.11}$$

[36] In (A.6) the inequality signs are interchanged compared to a situation leading to a maximum.

[37] See for t = 0 paragraph 3.3.

Substitution of (A.11) in the right hand side of (A.10) shows that the lower border of the loss interval is proportional to investment costs and via (9) the same applies for the upper border for all i with $T_i > o$.

$$S(H_{T_i}^-) = s^- \; e^{\lambda H_{T_i}^-} \; = S(H_{T_{i-1}}^-)\, e^{\lambda u} \tag{A.12}$$

$$S(H_{T_i}^+) = S^+ \, e^{\lambda H_{T_i}^-} \tag{A.13}$$

Substitution of (A.12) and (A.13) in (10) gives:

$$\lambda u = \beta \, D_{i+1} - \theta \, u \qquad \Rightarrow$$

$$\beta D = (\theta + \lambda) u \tag{A.14}$$

Therefore substitution of (A.11) in (A.10) is possible and leads to a loss interval $(s^- \; e^{\lambda H_{T_i}^-} \, , \; S^+ e^{\lambda H_{T_i}^-})$ with constant periodical heightening u with constant time span D .

The fact that the proposed solution (A.11) is so far an admissible solution, does not rule out the possibility that there might be other types of solutions. But later on we will prove that this solution not only fulfils the sufficient conditions, but also that the solution is unique.

Lower Border Loss interval

Substitution of u = o and u = $2\,u_i$ in (A.8) leads to:

$$+ I_i' \; e^{-\delta T_i} + \psi_i^+ = o \tag{A.15}$$

Combination of (A.4) and (A.5) gives:

$$\psi_z = \frac{\partial}{\partial H} \left(\frac{1}{\delta} \, S_z \, e^{-\delta z} \right) = o \tag{A.16}$$

Since $\frac{\partial}{\partial H} S_z = -\theta \, S_z \neq o$, (A.16) can only be true if $z \to \infty$ and also

$$\lim_{z \to \infty} \; S_z \, e^{-\delta z} = o$$

This is the case if S_z is limited. For the time being we suppose that and use $z = \infty$.
Integration of (A.3) using the results linked to (A.16) gives:

$$\psi_t\ (continu) = \int_t^{\infty} \frac{\partial}{\partial H}\ S_\tau\ e^{-\delta\tau}\ d\tau \tag{A.17}$$

Summation of (A.7) gives

$$\psi_{T_i}^{+}\ (jump) = \sum_{j=1}^{\infty} \lambda\, I_{i+j}\ e^{-\delta T_{i+j}} \tag{A.18}$$

Substitution of (A.17) and (A.18) in (A.15) gives:

$$-I_i'\, e^{-\delta T_i} = \int_{T_i}^{\infty} \frac{\partial}{\partial H}\ S_\tau\ e^{-\delta\tau}\ d\tau + \psi_{T_i}^{+}\ (jump)$$

$$I_i' = \int_{T_i}^{\infty} -\frac{\partial}{\partial H}\ S_\tau\ e^{-\delta(\tau - T_i)}\ d\tau - \psi_{T_i}^{+}\ (jump) e^{\delta T_i} \tag{A.19}$$

(A.19) says that the marginal cost of improvement must be equal to the marginal benefit of the improvement in the future diminished by the raise in future costs caused by the improvement. In other words: the net present value of the improvement must be zero.

Use of (3) gives:

$$\frac{\partial S_\tau}{\partial H} = -\theta\, S_\tau = -\theta\, S_{T_i}^{+}\ e^{-\theta\left(H_\tau - H_{T_i}^{+}\right)}\ e^{\ \beta(\tau - T_i)} = -\theta\, S_{T_{i+j}}^{+}\ e^{\beta(\tau - T_{i+j})} \tag{A.20}$$

In the right hand side $T_{i+j} \le \tau < T_{i+j+1}$.

Twice substitution of (A.20) in (A.19) gives:

$$I_i' = \int_{T_i}^{\infty} \theta\ S_{T_i}^{+}\ e^{-\theta\left(H_\tau - H_{T_i}^{+}\right)}\ e^{(\beta-\delta)(\tau - T_i)}\ d\tau - \psi_{T_i}^{+}\ (jump) e^{\delta T_i}$$

$$= \theta \sum_{j=o}^{\infty} e^{-\delta\left(T_{i+j} - T_i\right)} \int_{T_{i+j}}^{T_{i+j+1}} S_{T_{i+j}}^{+}\ e^{(\beta-\delta)\left(\tau - T_{i+j}\right)}\ d\tau - \psi_{T_i}^{+}\ (jump) e^{\delta T_i}$$

$$= \theta \sum_{j=o}^{\infty} e^{-\delta\left(T_{i+j} - T_i\right)}\ S_{T_{i+j}}^{+}\ \frac{1}{\beta - \delta} \left(e^{(\beta-\delta)(T_{i+j+1} - T_{i+j})} - 1 \right) - \psi_{T_i}^{+}\ (jump) e^{\delta T_i}$$

$$= \frac{\theta}{\delta - \beta} \sum_{j=o}^{\infty} e^{-\delta\left(T_{i+j} - T_i\right)}\ S_{T_{i+j}}^{+} \left(1 - e^{-(\delta-\beta) D_{i+j+1}}\right) - \psi_{T_i}^{+}\ (jump) e^{\delta T_i} \tag{A.21}$$

(A.21) combines (A.3), (A.4), (A.5), (A.7) and (A.8).

Substitution of (A.11) till (A.14) in (A.21) and dividing both sides by $e^{\lambda H_{T_i}^{-}}$ gives:

$$I_1'(u) = \frac{\theta}{\delta-\beta}\left(1-e^{-(\delta-\beta)D}\right) S^+ \sum_{j=o}^{\infty} e^{-\delta jD}\, e^{\lambda ju} - \lambda e^{\lambda u-\delta D} \sum_{j=o}^{\infty} I_1\, e^{\lambda ju}\, e^{-\delta jD} \qquad \text{(A.22)}$$

With constant u the left hand side of (A.22), the derivative of the investment costs, is constant and therefore also the right hand side. This is the case because the summation runs till ∞. In that case the benefit of the same investment u is equal because each time the life span is the same. (A.22) gives:

$$I_1'(u) = \left\{ \frac{\theta}{\delta-\beta} S^+ \left(1-e^{-(\delta-\beta)D}\right) - \lambda e^{\lambda u-\delta D}\, I_1 \right\} \left(1-e^{\lambda u-\delta D}\right)^{-1} \qquad \text{(A.23)}$$

or:

$$S^+ = \frac{\delta-\beta}{\theta}\left(1-e^{-(\delta-\beta)D}\right)^{-1} \left\{ I_1' \left(1-e^{\lambda u-\delta D}\right) + \lambda e^{\lambda u-\delta D}\, I_1 \right\} \qquad \text{(A.24)}$$

This is only possible when:

$$e^{\lambda u-\delta D} < 1 \qquad \Rightarrow \qquad \lambda u-\delta D < o \qquad \Rightarrow \qquad \delta > \lambda \frac{u}{D}$$

Substitution of (A.14) gives:

$$\delta > \frac{\lambda}{\theta+\lambda}\,\beta \qquad \text{(A.25)}$$

This is always the case when $\lambda = o \; or \, \delta \geq \beta$. But for the quite likely case that $\lambda > o \wedge \beta > \delta$, see the main text about equation (33).

Check and conclusions

Because we stretched out the time horizon in (A.16) till infinity, we have to check whether or not (11) is still convergent.

When $\lambda = o$, this is always the case, because after some start-up period it always holds

$$S_t(H) \leq s^- \qquad \text{for} \quad t > T_1$$

The expected loss will always be finite because there will always be investments. So, as far as convergence of the integral in (11) is concerned, there is no need for restrictions, see also (14) in the main text. In this case I(u) is constant too and finite and therefore the sum is convergent.

When $\lambda > o$, (A.25) is necessary and sufficient to assure convergence of (11).

Concluding: There are four variables: the two borders of the loss-interval $(s^- e^{\lambda H^-_{T_i}},\ S^+ e^{\lambda H^-_{T_i}})$, the standard heightening u and the time span of a standard period D. Values for these four variables can in general be found by solving the system of two definition equations (9) and (A.14) and two conditions for an optimum (A.10) and (A.24), provided that (A.25) holds.

Start-up period

With the help of the solution for the periodical interval the complete solution can be found by straightforward differentiation according to the line of reasoning in the main text from equation (13) till (28). Since λ has no influence on the timing condition, the only changes with λ > o are in (26) and the equation for X. The more general equations are:

$$S_o^+ = S_o\, e^{-\theta X}$$

$$X = \frac{1}{\theta}\ln\left(S_o / S_o^+\right) = \frac{1}{\theta+\lambda}\ln\left(S_o / \left(S_o^+ e^{-\lambda X}\right)\right) \qquad for\ S_o \geq s^- \tag{A.26}$$

D_2 follows from:

$$S_o^+ e^{\beta D_2} = s^- e^{\lambda X}$$

$$D_2 = \frac{1}{\beta}\ln\left(s^- / \left(S_o^+ e^{-\lambda X}\right)\right) \qquad for\ S_o \geq\ s^- \tag{A.27}$$

We assume throughout that:[38]

$$S_o^+ e^{-\lambda X} < s^- \tag{A.28}$$

Total costs become:

$$C = I(X) + S_o^+ \frac{1}{\delta-\beta}\left(1 - e^{-(\delta-\beta)D_2}\right) + e^{\lambda X - \delta D_2}\, C_{rep} \qquad for\, S_o \geq s^- \tag{A.29}$$

Partial differentiation of (A.29) to X, given (A.26) and (A.27), gives the equation for S_o^+:

$$S_o^+ e^{-\lambda X} = \frac{(\delta-\beta)}{\theta}\left(F'(X) + \lambda F(X)\right) + \frac{\lambda(\delta-\beta)+\theta\delta}{\theta\beta}\left(s^- - (\delta-\beta)C_{rep}\right)e^{-\delta D_2} \tag{A.30}$$

For λ = o the equivalent of (21) appears.

[38] If that is not the case, more steps are needed to reach the optimal interval with minimal costs. Then extra information is needed, e.g. on the minimal time span between two investments, to find the right solution.

Sufficient conditions and Uniqueness

A solution which fits the necessary conditions, is not automatically a minimum. To be a maximum, it is according to S&S (1987, chapter 3, theorem 8 (p 198)) sufficient when both Hamiltonians are concave. Because this problem has been formulated as cost minimisation, here it is sufficient when both are convex.

The first sufficient condition is that in the optimum the following Hamiltonian is convex in H for every t:

$$Ham = S_t \, e^{-\delta t} \tag{A.31}$$

According to (3) S is a downward sloped exponential function of H and therefore strictly convex. Together with convexity of the Impulse-Hamiltonian this also guarantees that the minimum is unique.

The second sufficient condition is that the following Hamiltonian is convex in H and u for every t:

$$IHam = + I_i \, e^{-\delta T_i} + \psi_{T_i} \; u_i \tag{A.32}$$

When $\lambda > 0$, I is an upward sloping exponential function of H and therefore strictly convex. When $\lambda = o$, H is missing in (A.32) and the condition is automatically fulfilled.
Since ψ in (A.32) must be considered as an independent variable, (A.32) is convex in u if and only if I is convex in u. Convexity of I_1 for $u > o$ (e.g. linearity) is a sufficient condition for assuring that the solution found before is an minimum and also to assure that it is unique.

However, convexity of (10a) or (32) is not a necessary condition for a minimum. But there are limits to the specification of the investment function to assure there is a definite minimum. If

$$\lim_{u \to \infty} \; I'_u(u) = o \tag{A.33}$$

than it is clear that (23) gives not a finite solution for u. A minimal condition to the investment function to assure the existence of a minimum is

$$I'_u(u) > c > o \qquad for\ u > o \tag{A.34}$$

This doesn’t rule out concavity of the investment function, but (A.34) poses limits to it.

Means and Indicators for standards

Mean optimal expected loss

We give the formula for the mean expected loss during a standard investment period after investment i. In this calculation we do not use a discount factor as in (13).

$$S_{T_i}^{mean} = \frac{I}{D} \int_o^D S_{T_i}^{+} \ e^{\beta t} \ dt$$

$$= \frac{I}{D} \ S_{T_i}^{+} \ \frac{1}{\beta}\left(e^{\beta D} - 1\right) = \frac{s_{T_{i+1}}^{-} - S_{T_i}^{+}}{\beta D} \tag{A.35}$$

Using (A.9), (A.12) and (A.14) gives:

$$S_{T_i}^{mean} = \frac{s_{T_i}^{-} \ e^{\lambda u} - S_{T_i}^{+}}{(\theta + \lambda)u}$$

$$= \frac{s_{T_i}^{-} \left(e^{\lambda u} - 1\right) + \delta I_i}{(\theta + \lambda)u} \tag{A.36}$$

Multiplying both sides with (θ + λ)/θ and rearranging gives:

$$S_{T_i}^{mean} = \delta \frac{1}{\theta} \frac{I_i(u)}{u} + \frac{s_{T_i}^{-} \left(e^{\lambda u} - 1\right) - \lambda u \, S_{T_i}^{mean}}{\theta u} \tag{A.37}$$

For not too large values of λu (A.37) can be approximated by:

$$S_{T_i}^{mean} \approx \delta \frac{1}{\theta} \frac{I_i(u)}{u} (1 + 0{,}5 \lambda u) \tag{A.38}$$

If $\lambda = o$, both the exact result (A.36) and the approximation (A.38) boil down to the easy interpretable result:

$$S^{mean} = \delta \frac{1}{\theta} \frac{I(u)}{u} \tag{A.39}$$

The last factor is the average cost per centimetre optimal heightening. $1/\theta$ is a number of centimetres. The last two factors form an amount of investment, which is transformed in an equivalent amount per year by the discount factor.

Ignoring for the moment the small correction ζ for the increase in loss by the heightening of the dike, $1/\theta$ is equal to $1/\alpha$. This is the average number of centimetres by which the height of the water by flooding exceeds the height of the dike.[39] Because the conditional distribution of an exponential distribution is again an exponential distribution with the same basic parameter, this average exceedance height is independent of the height of the dike. The last two factors are the investment costs to prevent the expected loss till the mean level of water exceeding the height of the dike.

The practical significance of (A.35) is that the average expected loss is only depending on the average costs of protection per centimetre. Average costs are far less dependent on the nature of the preferred specific measure at a time and location than fixed and variable costs. Where the size of the loss interval is very sensitive for the relative size of fixed and variable costs, the central value of the loss interval is not. Since in general the average costs are also not very sensible for not too large variations in u, the exact guesstimates for the future developments play hardly any role in the determination of the mean expected loss within an investment period. Because the boundaries of the loss interval are constant between investment actions, the mean gives for the whole period a good impression of the middle of the optimal expected loss interval. So, this mean can be helpful in defining a norm which is not too dependent on the nature of a specific investment action. But still, according to (A.38) there are jumps directly after an investment, which can lie between 0 and a maximum of 60% of the previous value.

Limit solution if there would be no 'fixed costs'

Since S^{mean} seems to be a good starting point for defining practical norms for safety levels, it can be important to prove that this central value also emerges in an other way, namely as the limit of the loss interval in case there are no 'fixed costs'. No 'fixed costs' should be interpreted in the broad sense that the size or the timing of an action has no influence on the unit costs of that action, so:

$$\frac{J(y,H,t)}{y(H,t)} = \frac{\partial J(y,H,t)}{\partial y} = b_o e^{\lambda H_t} \qquad with \quad H_o = o \tag{A.40}$$

with J investment costs per unit of time

y rate of investment per unit of time

Because there are no size effects on unit costs, the interval shrinks to a line S*(t) and investment will be a continuous process. Since the time interval D goes to zero, it is not easy possible to take the limit directly. Therefore we have to apply the maximum principle again, but then in the continuous case using the investment function (A.40).

[39] This result can already be found in Van Dantzig (1960).

We retain the definitions of S in (3), and of H_o and H_∞ in (4) en (6). The rest of the model is the following:

$$\min_U C = \int_o^\infty (S_t + J_t) e^{-\delta t} dt \tag{A.41}$$

$$\dot{H}_t = y_t \tag{A.42}$$

Because it is a normal problem the Hamiltonian is:

$$Ham = (S_t + J_t) e^{-\delta t} + \psi_t \, y_t \tag{A.43}$$

Necessary conditions for an optimal control are:

$$\frac{\partial}{\partial H} Ham = -\dot{\psi}_t \tag{A.44}$$

$$\frac{\partial}{\partial y} Ham = o \tag{A.45}$$

Starting with (A.45) gives:

$$-\psi_t = b_o e^{\lambda H_t} e^{-\delta t} \tag{A.46}$$

Differentiating (A.46) with respect to t and equalising the resulting right hand side to the left hand side of (A.44) gives:

$$\begin{aligned} &(-\theta S_t + \lambda J_t) e^{-\delta t} = -\delta b_o e^{\lambda H_t} e^{-\delta t} + \lambda b_o \, y_t \, e^{\lambda H_t} e^{-\delta t} \\ &S_t^* = \frac{\delta}{\theta} b_o e^{\lambda H_t} = \frac{\delta}{\theta} \frac{J_t(y_t)}{y_t} \end{aligned} \tag{A.47}$$

In course of time, the actual development of S should be equal to the optimal development of S*. So the rate of change of both variables should be equal. Using (3) and (A.47) gives:

$$S_t^* = S_o^+ e^{\beta t} e^{-\theta H_t} = \frac{\delta b_o}{\theta} e^{\lambda H_t} \qquad \Rightarrow$$

$$\beta - \theta \frac{\partial H_t}{\partial t} = \lambda \frac{\partial H_t}{\partial t} \tag{A.48}$$

$$\beta = (\theta + \lambda) y_t$$

$$y = \frac{\beta}{\theta + \lambda} \tag{A.49}$$

Even in the case that D↓ o, equation (A.14) still holds in its continuous equivalent (A.49). This equation determines the optimal rate of investment per unit of time y. The level S* has no influence at all on the value of y, which turns out to be constant.[40]

When λ = o, equation (A.47) is equal to equation (A.39). Therefore we reach the conclusion that the limit value S* of the loss interval (S^{+}_{Ti}, s^{-}_{Ti+1}) in case there are no size effects on yearly unit costs ('fixed costs approach to zero') and no effect of the height of the dike on investment costs, is the same as the (limit of the) mean value for the expected loss during that interval, S^{mean}. When λ > o, equation (A.47) is the continuous equivalent of the step functions (A.36) or (A.38). When the steps become smaller and sooner after each other, equation (A.47) is the obvious limit.

This result has an important practical application. If we would have different measures with different size effects on the costs of the measures but with roughly the same unit costs for an optimal investment size, then the mean value for the expected loss during the optimal time interval will also be roughly the same. Therefore the mean optimal expected loss is less dependent on the nature of a specific measure than the size of the interval. This makes the mean optimal expected loss a good starting point for a definition of a safety standard.

Mean exceedance probability

Because exceedance probabilities are easier to interpret and to use in regulations than amounts of loss, we will speak in the examples about probabilities. But here we have choices to make, taking also the mean of the potential losses into account. If we define them both in the same way as (A.35), their product is not equal to (A.35). If we want to preserve the identity, which has advantages, then we have to define one of them as a weighted average.[41] It seems appropriate to define the mean of the potential loss by flooding in the same way as (A.35) and the average of the flooding probabilities as a weighted mean with the values of the potential loss as weights.

The formula for the mean potential loss within a standard investment period is as follows. In this calculation we do not use a discount factor as in (13).

[40] We do not bother about the determination of S^{+}_{0}, since we are only interested in the limit as such. The case without fixed costs has no practical meaning. But in principle the same kind of reasoning can be applied on the start-up period as has been done for the normal case, resulting in the possibility of waiting or in making a jump at the moment of start.

[41] The problem is similar to the definition of price and volume indices of aggregates in the National Accounts (NA). The common definition of the volume index in NA is the value index over a period (which is the simple division of the values at both ends of the interval) divided by the price index used over the same period.

$$V_{T_i}^{mean} = \frac{I}{D} \int_o^D V_{T_i} \; e^{\gamma t} \; dt$$

$$= V_{T_i} \frac{\left(e^{\gamma D} - 1\right)}{\gamma D} \tag{A.50}$$

When D approaches to zero, the quotient in the right hand side goes to 1 and the mean value of the potential loss approaches to the value at the beginning of the period.

The weighted mean of the probabilities is in the same way as follows:

$$P_{T_i}^{wmean} = \frac{I}{D} \int_o^D \frac{V_t}{V_{T_i}^{mean}} P_t \quad dt$$

$$= \frac{S_{T_i}^{mean}}{V_{T_i}^{mean}} \tag{A.51}$$

So, if we use the weighted mean for the probabilities, we preserve the original relation between expected loss, probabilities and loss by flooding also for the means.

The central value as an indicator

However, there is a clear difference between the different means in giving a good rough impression of a central value of the variable. Since the optimal boundaries for the expected loss are constant in time, the mean expected loss is also a good indicator for the middle of the interval during the whole period. On the other hand, the potential loss and the optimal probabilities increase respectively decrease exponentially, see (22). Therefore their means do not give a good rough idea of their value during the whole period. Therefore the weighted mean of the probabilities can not serve as a good indicator for a standard.

A better indicator for the middle of the interval at every moment in time seems to be:

$$P_t^{middle} = \frac{S_{T_i}^{mean}}{V_t} \tag{A.52}$$

The variable $P^{middle}(t)$ moves along with V(t) and stays therefore roughly in the middle of the interval (P^- (T_{i+1}) , P^+ (T_i)). So it can serve as a rough indicator for the actual central value of the interval during the whole period between consecutive investments.

The weighted mean of this ‘middle’ probability over an investment period is the same as the weighted mean of the real probabilities (A.51):

$$P_{T_i}^{wmean(middle)} = \frac{I}{D}\int_o^D \frac{V_t}{V_{T_i}^{mean}} P_t^{middle}\ dt = \frac{S_{T_i}^{mean}}{V_{T_i}^{mean}}\left(\frac{I}{D}\int_o^D dt\right) =$$

$$= \frac{S_{T_i}^{mean}}{V_{T_i}^{mean}} \tag{A.53}$$

So, P^{middle} can serve as an indicator for the 'middle' value for the optimal probability interval at any moment in time.

NPV of a single investment

We can calculate the net present value (NPV) of a single investment. We begin on the moment of investment with $T_i > o$ with the present value of the difference in loss by the i^e investment:

$$W_i(\infty) = \int_o^\infty (S_{i,t}^+ - S_{i,t}^-)\, e^{-\delta t}\ dt$$

$$= (S_i^+ - s_i^-)\int_o^\infty e^{(\beta-\delta)t}\ dt \tag{A.54}$$

$$= (S_i^+ - s_i^-)\frac{1}{\delta-\beta} \qquad\qquad if\ \ \delta > \beta$$

In contrast with the integration over a finite period, the last condition is needed here to secure that the integral is convergent. Using the FYRR (A.9), (A.54) can also be written as:

$$W_i(\infty) = \frac{-\delta}{\delta-\beta} I_i \tag{A.55}$$

The NPV of project i is

$$Y_i = I_i - \frac{\delta}{\delta-\beta} I_i = \frac{-\beta}{\delta-\beta} I_i \tag{A.56}$$

It looks as if the NPV of a project would be negative. But the expression means a diminishing of costs. The consequence of (A.54) and (A.56) is that the NPV of a single investment is: either infinite when $\beta \geq \delta$, like in the area of the downstream rivers with $0.06 < \beta < 0.075$; or finite when $\beta < \delta$, but much bigger than the investment costs.

Suppose $\beta = 0.03$ *and* $\delta = 0.04$, as in the area of the upstream rivers, than the NPV of an optimal project is according to (A.56) three times as big as investment costs. The NPV is therefore not a sufficient condition for investment for a project with constant future costs and increasing future benefits. The optimal strategy gives the right criteria (A.9) en (A.23).

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