Network-Decentralized Control with Collision Avoidance for Multi-Agent Systems

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Abstract

An often preferred method to control a *multi-agent system* is by a network-decentralised controller. Network-decentralised means that each agent only has knowledge about its own state and the state of its neighbouring agents. In this thesis the multi-agent system consists of holonomic robots moving in a 2-dimensional configuration space. Each agent is equipped with a collision avoidance algorithm which does not need unmeasurable information from other agents, hence it is completely decentralised. The collision avoidance algorithm generates a reference velocity which is always away and (counter-clockwise) around a possible collision. Cooperation of the agents is reached by estimating their position and moving in an assigned formation by only communicating with neighbouring agents, hence the strategy is network-decentralised. The communication network of the agents is modelled as a graph and described by the associated incidence matrix. This thesis combines for the first time the network-decentralised estimation method proposed by Giordano et al. [1] together with the network-decentralized control method proposed by Blanchini et al. [2] and a collision avoidance algorithm. When anonymity in coordination is possible the agents are able to switch target location. Each agent solves a local optimisation problem to minimise the total distance travelled by itself and its neighbouring agents. The agent with the highest cost savings is allowed to reallocate the target locations. The results obtained in the simulations are very promising and suggest that the proposed algorithms can be successfully implemented to efficiently coordinate agents and avoid collisions in real-world applications.

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Extended Summary

The use of multiple robots to perform a complex task is getting more and more popular. The dynamic interaction of multiple robots (or entities, or units, or *agents* in general) gives rise to an overall *multi-agent system*, resulting from the interplay of the dynamics of each individual agent and of their interactions. The interactions among units occur according to a given interconnection topology: in the associated graph representation, each agent is a node and interacts only with its neighbours (the nodes that are linked to it). To control multi-agent systems, resorting to a centralised strategy based on the state of the whole system is often impossible or disadvantageous, due to the large scale or to the geographical sparsity of the system. In these cases, an effective method to control a multi-agent system is by a network-decentralised controller. With this controller every agent only has information about its own state and the state of its neighbouring agents. There are several reasons why a network-decentralised controller is preferred, like limited resources to process all the data from every agent or privacy issues that arise when a single controller is not allowed to access all the information.

In this thesis we consider multi-agent systems where each agent is holonomic, circle shaped and has a mass that is time-varying because the agent is able to pick up objects which can change its total mass. The agents can all have a different speed limit and are all controlled by a backstepping controller that includes a saturation function, to ensure that the speed constraints are always satisfied. All the agents are placed in a 2 dimensional configuration space where their objective is to reach a given formation. They have to reach this formation in a network-decentralised way while avoiding collision, by cooperating with one another. Also, the agents are able to exchange their target destinations, whenever it does not matter which agent performs what task, as long as the task is performed: this feature is called anonymity in coordination.

First, to avoid collisions, a completely decentralised collision avoidance method is proposed. Every agent has a so-called collision detection zone that aims in the direction where it (the agent) wants to move and the radius of this collision detection zone is dependent on the speed of the agent. When another agent or obstacle is detected, the entire surrounding of the agent is scanned to be able to prevent a possible collision. Around the closest possible collision point a circle is constructed, the line from the agent to the collision point is rotated with a specified angle then the desired velocity of the agent is in the direction of where this line crosses the constructed circle. This desired velocity is always directed away from the possible collision, therefore all collisions are avoided. Remarkably, since the collision avoidance manoeuvre

is local, the needed computation time does not increase with the number of agents in the system: this *scalability* makes the method particularly interesting and efficient when the system consists of many agents.

Secondly, cooperation of the agents is reached by allowing them to exchange information so as to estimate their position and move in an assigned formation. The communication network of the agents is modelled as a graph and described by the associated incidence matrix. The incidence matrix is used to estimate the absolute position and orientation of all the agents in a network-decentralised way, namely, exclusively based on *local* information. Then, to achieve an observer-based decentralised control for the system of agents, this thesis combines for the first time the network-decentralised estimation method proposed by Giordano et al. [1] together with the network-decentralised control method proposed by Blanchini et al. [2]. This network-decentralised control method prioritizes the formation of the agents above the target destination, thus preventing selfish behaviours that may otherwise lead to deadlocks. In particular, it is possible to tune a coefficient to privilege formation reaching (collective performance) or individual target reaching. With the proposed method, the agents are of course steered each to its target position, but they are also induced to move in the desired formation even before their target is reached. The agents are now able to avoid collision, estimate their position and orientation and are able to cooperate (i.e. to move in the assigned formation), all in a network-decentralised fashion. The agents can all have different masses and speed limits but they will reach the desired target position at the same time while moving in formation.

Finally, when anonymity in coordination is possible, the agents are able to coordinate at best and to suitably swap their target destination with their neighbours by minimizing a local cost function. The local cost function to minimise is, for each agent, the total distance travelled by the agent itself and by its neighbouring agents. To minimise this function, each agent runs an optimisation algorithm and calculates how to optimally reallocate its own target position as well as the target position of all the neighbouring agents. The cost an agent can save with its proposed reallocation is called the bid and the bid is placed in the auction held by every connected agent. The agent 'sells' its target location to the agent with the highest bid in its auction who is then allowed to reallocate the target locations.

All the methods discussed above can be effectively integrated. The proposed collision avoidance algorithm is performed in a fully decentralised fashion, hence it can be naturally embedded in a network-decentralised control and estimation framework. Also the proposed algorithm to exploit anonymity in coordination is fully decentralised, based on local cost functions and local information.

In the thesis, the proposed methods are combined and tested in a computer simulation. The simulation shows that the proposed methods for collision avoidance, cooperation and anonymity in coordination can be successfully combined and give rise to the expected behaviour: the agents are able to cooperate in order to efficiently reach the desired goal, they are able to coordinate so as to swap their target destinations if this is suitable, and they are able to move without any collision. The results obtained in simulation are very promising and suggest that the proposed algorithms can be successfully implemented to efficiently coordinate agents and avoid collisions in real-world applications.

Preface

I can not believe it is only 5 months ago since I walked in the office of my then to become thesis supervisor Dr. Ing. G. Giordano. I was left disorientated by a seven month struggle with a thesis subject which looked endless at that time. Giulia (Giordano) had the difficult task to get me back on track again to get my Masters degree in Systems & Control. Her trust, feedback and cooperation laid the foundation of this thesis which I am now handing in with pride. I am very grateful that Giulia was my supervisor during my final months as a student at the TU Delft.

Despite the fact we had to separate our ways, my gratitude also goes to Dr. Ir. M.B. Mendel for his valuable lessons in mechanics and economics. Those lessons provided me with a different point of view on mechanics and economics which directly and indirectly contributed to my final thesis.

The inspiration for some of the material in this thesis came from 'everyday things'. The collision avoidance algorithm for example is inspired by a traffic roundabout and human behaviour on a ski-slope. On a ski-slope a skier only looks in the direction he is going i.e. downhill (most of the times) this was the inspiration for the agent to only 'look' in the direction that he wants to go. Besides this, the framework of the thesis is inspired by a colony of ants, all individual ants working together to achieve the team goal. This shows that complex problems can be transformed in to day to day things, or vice versa depending on the taste of the reader.

Now without further ado I like to wish you a pleasant journey through the following chapters.

Chapter 1

Introduction

The use of multiple robots to perform certain complex tasks is getting more and more popular. Many robots operating in a warehouse to sort packages is no longer a futuristic scene: in the Alibaba warehouse for example there are many unmanned vehicles cooperating to distribute packages [4]. A system resulting from the interaction of several robots, vehicles, or units in general (termed agents) is called a multi-agent system. In the past two decades there has been a significant increase in the interest of the control community towards multi-agent systems. The increasing interest in multi-agent systems is due to the fact that often one complex agent can be replaced by many simpler agents: this leads to an overall system that is more flexible and resilient, and can reduce the cost and the needed resources [5]. However, the efficient coordination of multi-agent systems poses new challenges to control engineers. There are two approaches to control a multi-agent system: a centralised and a decentralised approach. A centralised controller processes all the information of all the individual agents and calculates the desired control action affecting each of them. Conversely, a decentralised controller is based on several control units, each acting on a single agent and deciding its strategy based on local information only. A particular type of decentralised control is network-decentralised control. In a network-decentralised controller the only information available to each local control unit comes from the agents that are located within communication distance. Every agent makes its decisions based on its own measurements and local information only.

The idea of network-decentralised control has been first introduced by Iftar and Davison in the papers [6], [7] and [8]. Several further developments have been provided in [9], [10] and [11]. The concept has been mathematically formalised in [12] and fully developed in [2], [13] and [14].

This thesis will focus on the network-decentralised control approach for the following reasons: as the number agents to be controlled increases, the computation time for a centralised controller to calculate all the trajectories of all the agents increases drastically; moreover, privacy laws (or reasons) could prevent a centralised controller to access the information about all the agents (which may be owned by different companies) and to have control authority on all of them; also, the large scale of the system and the geographical sparsity could make a centralised controller physically impossible, due to physical constraints like walls, mountains or water that prevent the central controller from receiving all the information that is needed to compute a centralised control strategy. The objective of the network-decentralised controller is to make the agents cooperate and pursue a *global* complex collaborative task, by enforcing exclusively *local* actions that are decided based on *local* information.

The goal of this thesis is to provide a unified approach for network-decentralised control and estimation in multi-agent systems, which embeds a collision avoidance protocol. The proposed approach allows effective cooperation among the agents and can also incorporate a local optimisation method to optimally allocate the target position to the agents when anonymity in coordination can be exploited (namely, when the agents are interchangeable). In particular, cooperation in this thesis is focussed on reaching the desired target while keeping or making a desired formation. Agents moving in formation are usefull for many interesting applications e.g. surveillance, search or rescue missions [15]. An effective cooperation requires that the agents must reach this formation without causing a collision. And finally cooperation includes that the agents have anonymity in coordination: this means that it does not matter which agent goes where as long as the formation is obtained. This must all be achieved in a network-decentralised fashion.

Nature can often be inspiring to devise efficient strategies in engineering, and this is the case also for decentralisation and for global coordination based on myriads of local actions. A good example of these principles in nature is a colony of ants. The ants only see and feel what is in their direct surroundings i.e. they only have local information. They are not colliding into each other. They are cooperating to make a certain formation, for example the ant bridge in Figure 1-1 and it does not matter which ant performs what action.

In this thesis a novel completely decentralised collision avoidance method is proposed which can be combined with a network-decentralised position estimation as in [1] and a network-decentralised control method as in [2], [13] and [14]. Completely decentralised means that in order to avoid collision, no information about other agents is needed with exception of the shape and distance which can be measured. Also a method is proposed to locally reallocate the destinations of every agent in order to minimize the total distance travelled by the agents. This satisfies the anonymity in coordination for the agents. The proposed methods are tested with computer simulations using Matlab and Simulink.

Figure 1-1: Bridge of ants as a fascinating natural example of how a global target can be effectively reached through the coordination of myriads of local actions based on local information. This is a source of inspiration for man-made decentralised strategies in engineering. [Credit: Christopher Reid, Matthew Lutz and New Jersey Institute of Technology]

The thesis is structured as follows.

- Chapter 2 introduces the considered multi-agent model, where each agent is seen as a circle with a mass moving in a 2 dimensional configuration space. The considered network-decentralised backstepping control strategy is introduced to control the agent.
- Chapter 3 presents the proposed novel collision avoidance protocol to avoid collisions in a completely decentralised way.
- Chapter 4 discusses several examples of collision avoidance, to illustrate the effectiveness and the limits of the proposed approach.
- Chapter 5 presents the considered approach for network-decentralised control of the agents and network-decentralised estimation of their position and orientation.
- Chapter 6 discusses further examples to showcase the behaviour of the proposed collision avoidance strategy when it is embedded in a network-decentralised control and estimation framework. It is shown that the integration of the collision avoidance protocol with a network-decentralised control can rule out selfish behaviours that would otherwise lead to deadlocks, thus overcoming all the limits that had previously emerged.
- Chapter 7 proposes a completely novel network-decentralised algorithm to exploit anonymity in coordination in order to minimise the total distance travelled by the agents, based on the solution of local optimisation problems with local cost functions and local information the agents locally reallocate the target locations.

Chapter 2

Multi-Agent Model

This chapter describes the properties of the individual agents that compose the multi-agent system considered in this thesis. The backstepping control strategy that will be used to control the multi-agent system is also briefly presented.

2-1 The Agent

Contrary to many theoretical multi-agent systems, the agents in this thesis are not considered to be a massless point. Here, each agent is circle shaped with a radius of 0.15 meter and has a mass $m(t)$ as is shown in Figure 2-1. The agent is not constrained to move in any direction in $\mathbb{R}^2$ i.e. the agent is holonomic. The mass is considered time varying because agents are able to drop or pick up objects which changes their total mass.

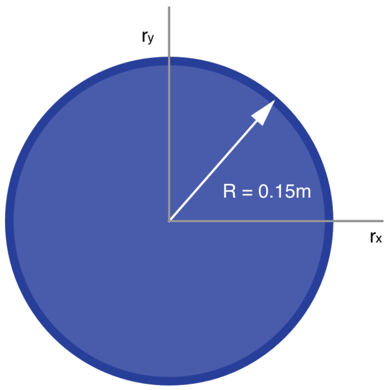

Figure 2-1: Model of an agent: R is the radius of the agent, while r_x and r_y are its local coordinates (x-axis and y-axis)

The equations of motion of the ith agent can be described by the following state space system

$$\underbrace{\begin{bmatrix} \dot{r}_{x,i} \\ \dot{r}_{y,i} \\ \ddot{r}_{x,i} \\ \ddot{r}_{y,i} \end{bmatrix}}_{[\dot{x}_i \;\; \ddot{x}_i]^T} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_i} \underbrace{\begin{bmatrix} r_{x,i} \\ r_{y,i} \\ \dot{r}_{x,i} \\ \dot{r}_{y,i} \end{bmatrix}}_{[x_i \;\; \dot{x}_i]^T} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_i(t) & 0 \\ 0 & 1/m_i(t) \end{bmatrix}}_{B_i} u_i(t) \tag{2-1}$$

where $r_{x,i}$ and $r_{y,i}$ are the local coordinates of the agent as shown in Figure 2-1 and $u_i(t)$ is the input force acting on the mass $m_i(t)$. The speed of the ith agent is limited individually and its upper limit is denoted by $v_{lim,i}$. A formal definition of speed $v_i(t)$ is given by Equation 2-2; note that the speed should not be confused with the velocity, which is defined by the expression in Equation 2-3.

$$v_i(t) \triangleq \sqrt{\dot{r}_{x,i}^2 + \dot{r}_{y,i}^2} \tag{2-2}$$

$$\text{velocity} \triangleq \dot{x}_i = \dot{r}_i = \begin{bmatrix} \dot{r}_{x,i} \\ \dot{r}_{y,i} \end{bmatrix} \tag{2-3}$$

2-2 Backstepping Control

Now that the model of the agent is known, a controller for the agent can be designed. First, recall that the agent's speed is limited, therefore the controller must be able to cope with this limitation by including a proper saturation function. Second, it is worth stressing that the agent's mass is time-varying in general, and this must be taken into account by the controller. Third, note that a force input is available for control purposes, while it is typically much easier to design a control for the velocity. To overcome this issue, a backstepping controller can be used, which is appropriate to deal with all the above aspects. In particular, a backstepping controller allows to design a velocity control that provides the reference velocity, and to simply track this reference velocity by means of the available force input. Two assumptions for this controller are that the mass $m_i(t)$ of the ith agent at time t is known or can be measured and that the velocity, as defined by Equation 2-3, of the agent can be measured. As mentioned above, the idea of a backstepping controller is to stabilize the lowest derivative with a feedback law and then step back until all states are stabilized and the control input is reached. For the agent as described in the previous section this means that the first step is to calculate a reference velocity ($\bar{\dot{x}}_i$) which is given by Equation 2-4.

$$\bar{\dot{x}}_i = \underset{v_{lim,i}}{\text{sat}} \left[(\bar{r}_i - x_i(t)) k_1 \right] \tag{2-4}$$

where $\bar{r}_i$ is the target destination and k_1 is a gain. The second step is from the velocity to the input $u_i(t)$, this step is shown in the blue rectangle in Figure 2-2. This final step linearises the system. The method of backstepping was developed by P.V. Kokotovic and can be used to control non-linear systems by using a non-linear feedback which linearizes the system, it can even be used to control so-called chaos [16]. Another advantage of this method is that the collision avoidance algorithm and the decentralised control method proposed in the next chapters will both generate a reference velocity for the agents. This reference velocity can

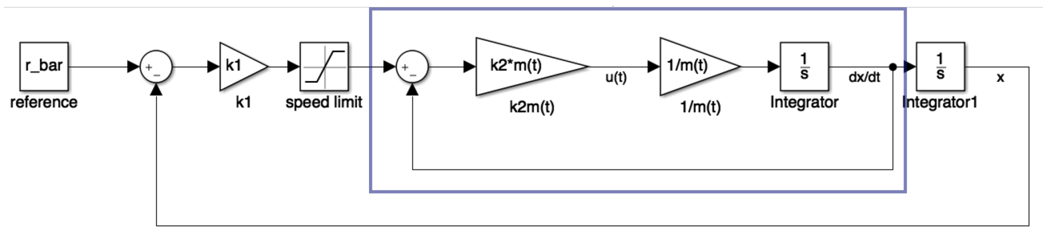

Figure 2-2: Block scheme of the backstepping control

then directly be compared with the actual velocity. The control input $u_i(t)$ as in Figure 2-2 is given by:

$$u_i(t) = \left(\underset{v_{lim,i}}{\text{sat}} \left[(\bar{r}_i - x_i(t))k_1 \right] - \dot{x}_i \right) k_2 m_i(t) \tag{2-5}$$

where $x_i(t)$ is the location of the agent at time t and k_1 and k_2 are gains which can be used to change the eigenvalues of the system. The state space equation for every agent with this input $u_i(t)$ now becomes

$$\begin{bmatrix} \dot{x}_i \\ \ddot{x}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0}_2 & I_2 \\ -k_1 k_2 I_2 & -k_2 I_2 \end{bmatrix} \begin{bmatrix} x_i \\ \dot{x}_i \end{bmatrix} + \begin{bmatrix} 0 \\ I_2 \end{bmatrix} k_1 k_2 \bar{r}_i \tag{2-6}$$

where $\mathbf{0}_2$ is the zero matrix of size 2×2 and I_2 is the identity matrix of size 2×2. The four eigenvalues of the system are given by:

$$\lambda_{1,2} = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1 k_2}}{2} \tag{2-7}$$

λ_1 and λ_2 both have an algebraic multiplicity equal to 2. The behaviour of the agent with a backstepping controller is not dependent on the time varying mass of the agent so the system is linear time invariant. Therefore, the considered controller cancels out all the differences among the agents that are due to the fact that their mass is not equal.

2-3 Constrained Configuration Space

The agents are placed in a bounded space (such as, for instance, a room) called the configuration space, which is constrained by four barriers or walls in a square. The control action to prevent an agent from hitting the barrier is to multiply by zero the reference velocity in the direction of the barrier when the agent is within a certain range of the barrier. This method can be extended so that the barrier is repellent, which means that the agent is pushed away from the barrier in a preferred direction.

Of course, in this type of scenario an agent can face two different types of obstacles: one of the barriers or one of the other agents. There are several ways the agent can make a distinction between a barrier and another agent. Some examples are that other agents can communicate where a barrier can not; that the agent already knows the location of the walls; or that the barrier can communicate as well, but then it would communicate that it is a barrier.

Chapter 3

Collision Avoidance Protocol

Multi-agent systems where robots operate in the same environment are an important research area within robotics nowadays [17]. Those agents, each performing a specific task, must be able to avoid collision. Since a centralized collision avoidance algorithm usually requires a large amount of resources and does not scale well with the number of agents [18], a decentralised collision avoidance algorithm is preferred. To prevent the agents from colliding into each other a completely decentralised collision avoidance method is proposed in Section 3-2. This method can directly be embedded in a network-decentralised frame work as described in [1], [2], [13] and [14].

3-1 Current Collision Avoidance Algorithms

First a formal definition of a collision in a simulation is given: the definition of a collision in a simulation is when two or more agents occupy the same space at the same time. One of the most popular decentralised collision avoidance algorithms nowadays is the 'Reciprocal Velocity Obstacle' which is an extension of the Velocity Obstacle concept described in [3]. In order to avoid collision using the Velocity Obstacle method the agent needs to have the other agents' position, velocity and shape. The need of knowing the velocity of the other agents is a serious limitation, since it requires either communication among the agents, or the presence of a centralised supervisor that knows the velocity of all the agents and can provide this information to all other agents. Then, under the assumption that the other agents make a similar collision avoidance reasoning, the Velocity Obstacle (VO) algorithm is guaranteed to generate safe motions by calculating a velocity vector in which no collision will occur. This is the so called collision free velocity. A brief explanation on how this Velocity Obstacle is created will now be given, for a detailed explanation the reader is referred to [3].

Consider Figure 3-1, agent A having velocity v_A encounters a moving obstacle B with velocity v_B. With the information of the velocities of A and B a Collision Cone ($CC_{A,B}$) can be created. This collision cone is the gray area depicted in Figure 3-2, in this figure $\hat{A}$ is agent A reduced to a point and $\hat{B}$ is obstacle B enlarged with the radius of A. Now by translating

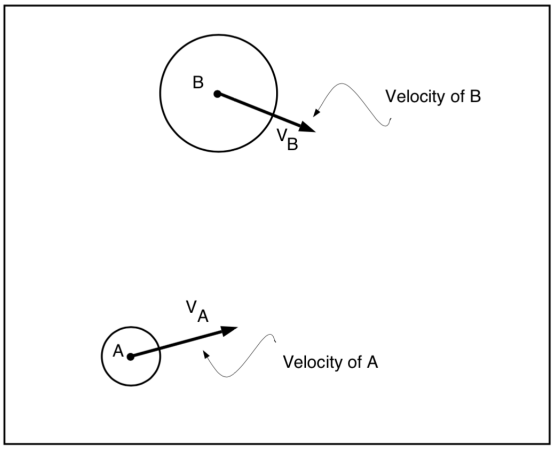

Figure 3-1: Agent A and moving obstacle B source: [3]

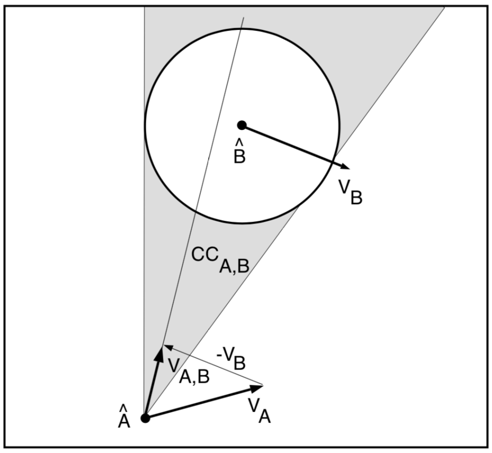

Figure 3-2: The relative velocity $v_{A,B}$ and the collision cone $CC_{A,B}$ source: [3]

the collision cone by v_B the Velocity Obstacle VO is obtained. The velocity obstacle is the dark gray area as shown in Figure 3-3. A velocity of agent A that lies within this velocity obstacle will result in a collision where a velocity in the white area will avoid collision. Figure

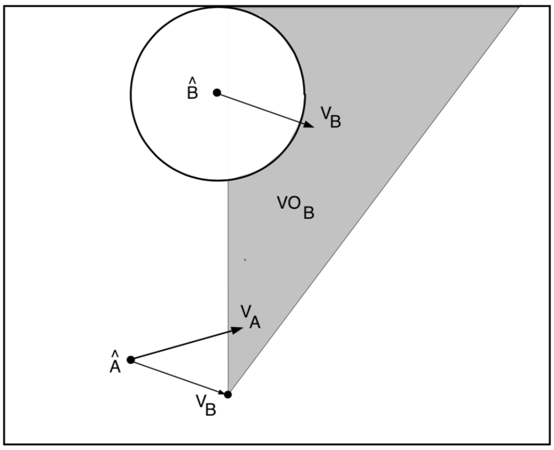

Figure 3-3: The velocity obstacle VO_B source: [3]

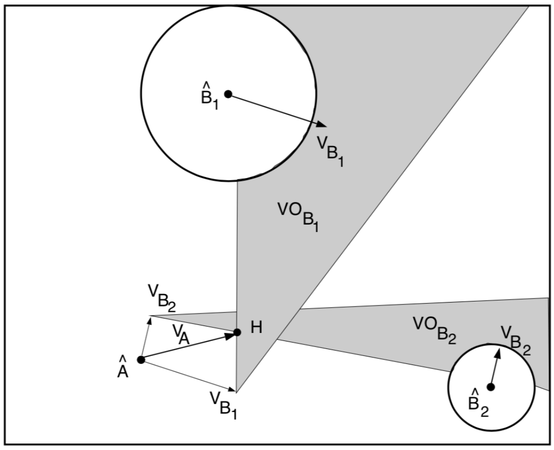

Figure 3-4: The velocity obstacles for B_1 and B_2 source: [3]

3-4 shows the velocity obstacles when there are multiple moving objects in the configuration space. Now to avoid collision the agent must have a velocity that is outside all the velocity obstacles. The agent calculates for every time step a feasible velocity that avoids collision and steers the agent to its target.

An improvement to the Velocity Obstacle is the Reciprocal Velocity Obstacle where smoother collision avoidance trajectories are calculated. The Reciprocal Velocity Obstacle is described extensively in [19]. The reciprocal velocity obstacle formulation has some limitations, particularly that it frequently causes agents to end up in a 'reciprocal dance' as they cannot reach agreement on which side to pass each other [20]. Another limitation is that as the number of agents within possible collision distance increases the computation time for the collision free velocity increases.

In this thesis a new collision avoidance algorithm is proposed which only needs the other agents' position and shape, which can both be measured. The computation time for this algorithm does not increase when the number of agents increases.

The approach proposed in this thesis bears some resemblance to another human-inspired collision avoidance method, proposed in [21]. The metod in [21] makes the agents perform a decentralised collision avoidance manoeuvre based on the so-called virtual rectabouts (namely, rectangular roundabouts): each agent involved in the possible collision re-plans its path independently by moving along the sides of a virtual rectabout that lies across the conflicting positions of the two agents' routes. The approach does not depend on predefined priority schemes and its advantage with respect to centralised approaches is that it involves only local information, without the need of inter-agent communication or of centralised coordination.

The method proposed in this thesis has all the advantages of that in [21], but allows the agents to follow more natural and smooth trajectories, instead of moving along the sides of virtual rectangles in the configuration space.

3-2 Proposed New Collision Avoidance Method

For the proposed Collision Avoidance Algorithm there is no need for communication between agents, they independently perform a collision avoidance manoeuvre when another agent is detected. Besides the assumption that agents are not actively trying to hit one another, there is the assumption that all agents prefer to move to the right (counter clock-wise) to prevent a collision. Just like a car in most countries must pass a roundabout on the right.

A comprehensive explanation of the collision avoidance algorithm will now be given. An agent moving to a target location in a two dimensional space detects a possible collision in the zone depicted in red in Figure 3-5, which is called the collision detection zone. When a possible collision is detected the agent scans the area depicted in gray and red and will make a move around the obstacle which is closest to the agent until the collision detection zone is obstacle free. The collision detection zone is dependent on the speed of the agent: the higher the speed of the agent, the larger the radius of the collision detection zone. The relation between the radius of the collision detection zone R_{cz} and the agent speed $v(t)$ is given by:

$$R_{cz}(t) = c + \beta v(t) \tag{3-1}$$

where both c and β are constants which can be adjusted.

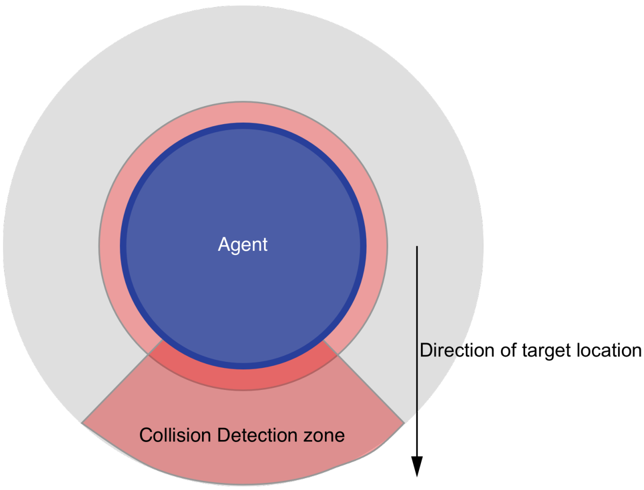

Figure 3-5: Collision zone of the agent

The angle θ that sets the range of the collision detection zone is chosen so that the collision detection zone is wider than the diameter of the agent. The circle shaped red area around the agents is also part of the collision detection zone. This makes sure that if an obstacle is very close the agent can detect this and respond. Figure 3-6 and Figure 3-7 illustrate how the collision avoidance manoeuvre works when an agent detects a single obstacle. The blue agent detects an obstacle, which is the green agent, the agent then scans its entire surroundings and the grey zone in Figure 3-5 now becomes a collision zone. Only the green agent is in the red collision zone so a circle is constructed with the possible collision point as its center, whose boundary is the red circumference (the interior is white) in the figure. The red circle is called the preferred distance from the obstacle and its radius, R_{dist}, is a little bit smaller than the radius of the collision detection zone. The line from the blue agent to the possible collision point is rotated with an angle α and the intersection between this rotated line and the red circumference gives the desired velocity for the agent to prevent a collision and to continue its course. Figure 3-7 shows what happens if the green agent is closer to the blue agent: the reference velocity is always directed away from the obstacle, so that an agent will never have an input that can potentially cause a collision. Table 3-1 shows all the parameters involved in the proposed collision avoidance method, along with their description.

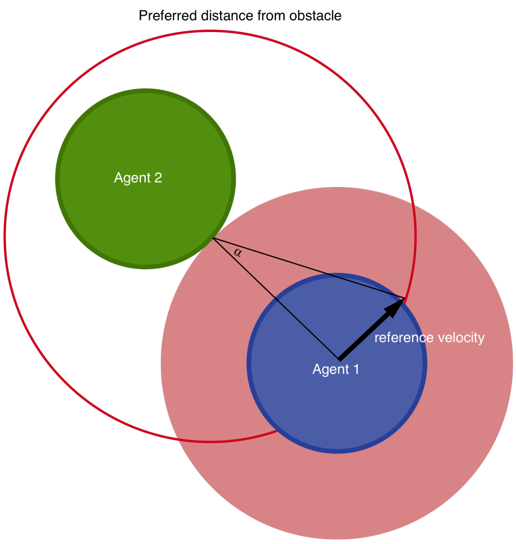

Figure 3-6: Collision manoeuvre of the agent

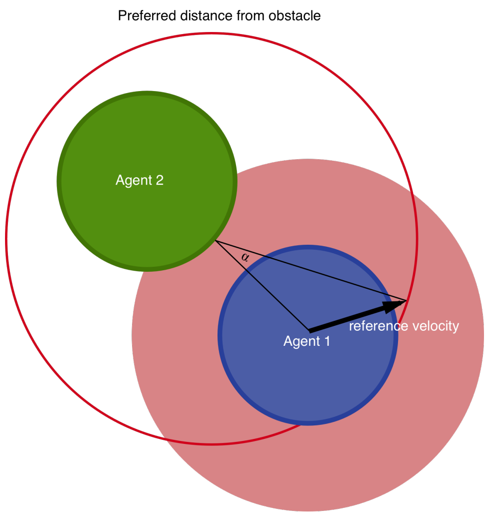

Figure 3-7: Another collision manoeuvre of the agent

Parameter	Description
α	Rotation angle for Collision avoidance
R_{cz}	Collision detection radius
c	Constant in equation $R_{cz}(t) = c + \beta v(t)$
β	Constant in equation $R_{cz}(t) = c + \beta v(t)$
θ	Range collision detection zone
R_{dist}	Preferred distance from obstacle

Table 3-1: Table of the used parameters in collision avoidance

Figure 3-8 shows how two agents avoid each other when they are moving from bottom left to top right and vice versa. When the agents meet in the middle they both move to the right to pass each other. When the agents have different speed limits the faster moving agent has a larger detection zone so it has more distance to perform a collision avoidance manoeuvre. An example of this is shown in Figure 3-9 where the red agent has a maximum speed of twice the blue agent's speed. Its collision detection zone is now larger than the blue agent's collision detection zone, this results in a smooth collision avoidance manoeuvre where the blue agent barely has to change its direction. Another advantage of this feature is that the high speed agent now has more distance to adjust his direction to avoid a collision.

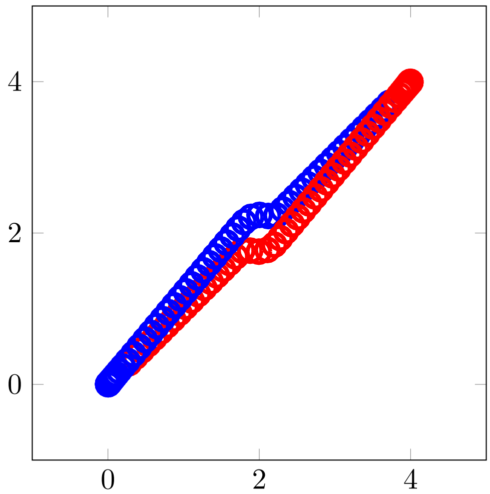

Figure 3-8: Collision avoidance manoeuvre of two agents with the same speed limit

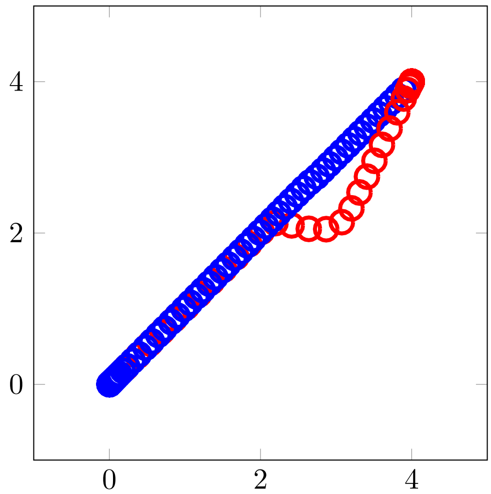

Figure 3-9: Collision avoidance manoeuvre of two agents with different speeds limit

The effect of a non-circular collision detection zone is not investigated and can be interesting for future work.

3-3 Deadlock and Livelock

An important issue in collision avoidance algorithms is the occurrence of deadlocks and livelocks. In this section a brief explanation of the two phenomena are given. First of all a lock is when an agent forces another agent to stop moving. Then, a deadlock is when an agent is locked and the locking agent is either directly or indirectly locked by said agent. This can be explained at best by a figure. Consider Figure 3-10, agent 1 waits for agent 2 to move, agent 2 waits for agent 3 to move and agent 3 waits again for agent 1 to move: hence, none of the agents will move, because they are all locked. Note that deadlocks are not likely to occur with the proposed collision avoidance method but the figure serves as an illustration to describe a deadlock. Similar to the deadlock is the livelock, here the agents can move but they are not progressing towards their goal. Agents in a corridor both moving to the same side of the corridor over and over again to try to pass each other provide a good example of a livelock. Deadlock and livelock detection and how to deal with them is a research area on its own, although several rules can be implemented for an agent to break from the livelock

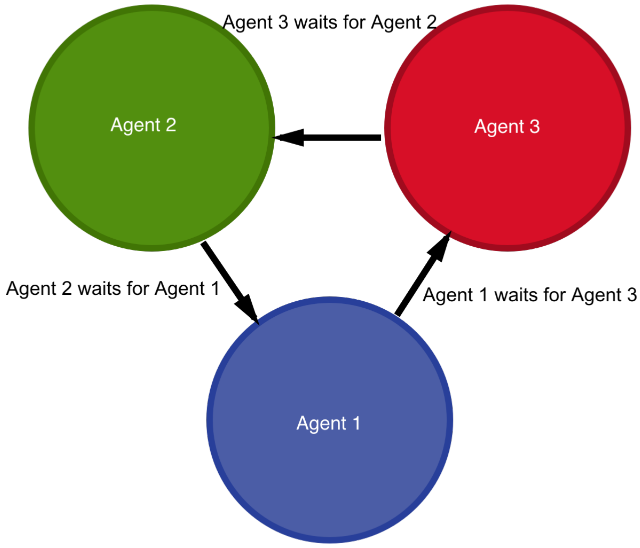

Figure 3-10: Possible Deadlock Situation

or deadlock (like increasing the collision distance or temporarily making a left turn). In the thesis, the focus will be on preventing the lock thanks to the cooperation of the agents which will be achieved by a network-decentralised controller as explained in Section 5-4.

Chapter 4

Collision Avoidance Case-Studies

In this chapter the strengths and weaknesses of the collision avoidance method are shown by treating several different cases. In all the cases 10 agents are placed in a constrained space. The agents must reach a target position from a starting position: both of these positions are represented by a plus sign. The figures are screen-shots of the final state where all the agents have reached their target position, the (dotted) lines represent the paths the agents travelled. To make a mathematical model of the circle shaped agent the circumference of the agent is discretized. Each agent has 10 points on the border of its body which can be detected by other agents. To simulate the movement of 10 agents for 5 seconds in Simulink, a 3,1 GHz Intel Core i5 Macbook Pro 2017 took less than 1 second calculation time.

4-1 Random Starting and Target Locations

The first case-study is a simulation where all agents start at a random position and must move to a random destination. The random starting and final positions are shown in Table 4-2. The agents are not cooperating or communicating to reach their own destination. The agents all successfully avoid collision and reach their target location. The trajectories of agent 1 and 6 around agent 5 and 9 respectively clearly show the collision avoidance manoeuvre. Table 4-1 shows all the values of the parameters used in the simulation.

4-2 Circle to Circle Simulation

The second case-study is a simulation of agents starting in a circle and willing to reach the opposite of their starting position. Without communication the agents all meet in the middle where they must avoid collision without any form of cooperation. Figure 4-2 clearly shows that the agents meet in the middle and perform a collision avoidance manoeuvre, some agents actually reach the congested middle where other agents will move around like agents number 1 and 5. There are no major oscillations and the agents all reach their goal while successfully

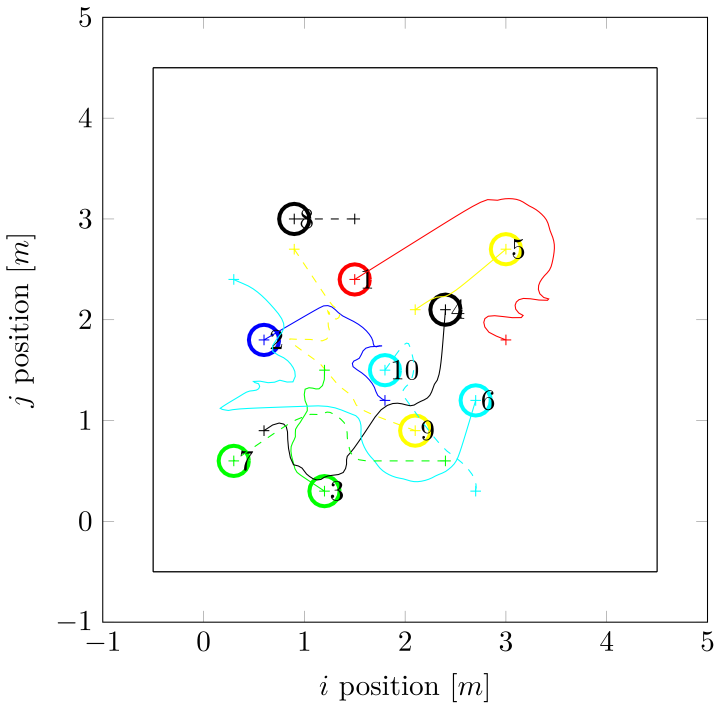

Figure 4-1: Simulated trajectories of 10 agents starting at a random location and moving to a random location

Parameter	Description	Value
m	Mass	1
k_1	Gain k_1 in Figure 2-2	12
k_2	Gain k_2 in Figure 2-2	8
α	Rotation angle for Collision avoidance	5 degree
c	Constant in equation $R_{cz}(t) = c + \beta v(t)$	0.26
β	Constant in equation $R_{cz}(t) = c + \beta v(t)$	0.39
θ	Range collision detection zone	$[-60, 60]$ degree
R_{dist}	Preferred distance from obstacle	$R_{cz} - 0.08$ m
h	Simulation sample time	0.01
v_{lim}	Speed limit	1 m/s

Table 4-1: Table of the parameter values used for the simulation of the agents in Figure 4-1, Figure 4-2 and Figure 4-3

avoiding each other. The parameters of the simulation are shown in Table 4-1. The starting positions and final positions lie on a circle with radius of 4 equally dived in 10 parts where agent 1 starts at $(4, 0)$ and ends at $(-4, 0)$.

Agent	Starting Coordinates (i, j)	Target Coordinates (i, j)
1	$(3, 1.8)$	$(1.5, 2.4)$
2	$(1.8, 1.2)$	$(0.6, 1.8)$
3	$(1.2, 1.5)$	$(1.2, 0.3)$
4	$(0.6, 0.9)$	$(2.4, 2.1)$
5	$(2.1, 2.1)$	$(3, 2.7)$
6	$(0.3, 2.4)$	$(2.7, 1.2)$
7	$(2.4, 0.6)$	$(0.3, 0.6)$
8	$(1.5, 3)$	$(0.9, 0.3)$
9	$(0.9, 2.7)$	$(2.1, 0.9)$
10	$(2.7, 0.3)$	$(1.8, 1.5)$

Table 4-2: Starting and Target locations for the agents in case in Figure 4-1

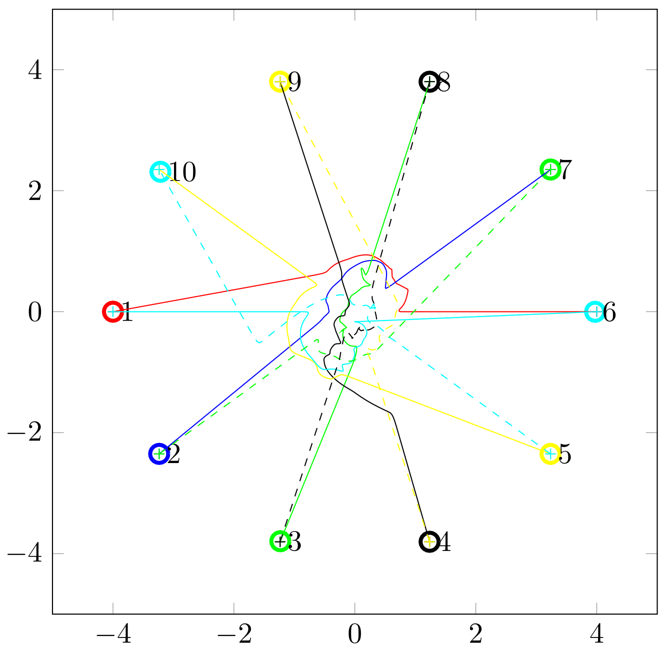

Figure 4-2: Simulated trajectories of 10 agents moving from a point on a circle to the opposite point on the circle

4-3 Simulation With Different Speed Limits

Figure 4-3 shows a simulation where the agents all have different speed limits. The agents do not collide and will all reach their target destination. This case-study demonstrates that the collision avoidance manoeuvre is not limited to agents all having the same speed limit. The parameters used for the simulation are again given by Table 4-1 except from the individual speed limits those are shown in Table 4-5. The starting and target position are summarized in Table 4-4.

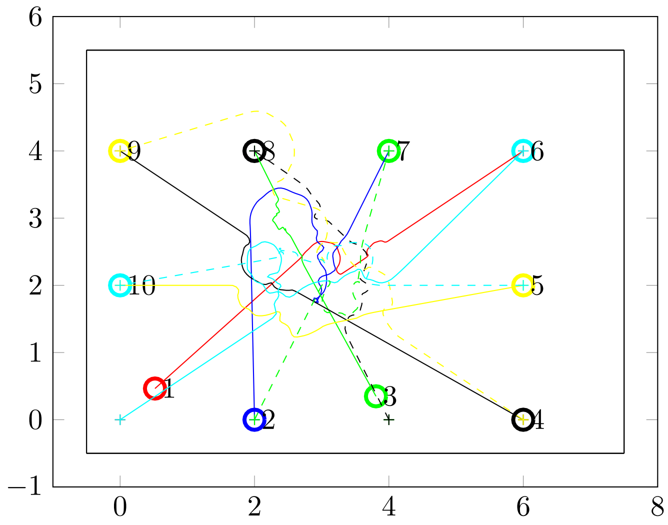

Figure 4-3: Simulated trajectories of 10 agents with different speed limits

Agent	Speed limit [m/s]
1	0.5
2	1.4
3	0.3
4	1.2
5	1.3
6	2.2
7	2.5
8	1.6
9	1
10	2.4

Table 4-3: Different speed limits of the agents in Figure 4-3

Agent	Starting Coordinates (i, j)	Target Coordinates (i, j)
1	$(6, 4)$	$(0, 0)$
2	$(4, 4)$	$(6, 0)$
3	$(3, 4)$	$(4, 0)$
4	$(0, 4)$	$(6, 0)$
5	$(0, 2)$	$(6, 2)$
6	$(0, 0)$	$(6, 4)$
7	$(2, 0)$	$(4, 4)$
8	$(4, 0)$	$(2, 4)$
9	$(6, 0)$	$(0, 4)$
10	$(6, 2)$	$(0, 2)$

Table 4-4: Starting and Target locations of the agents in Figure 4-3

4-4 Simulation with Different c and β

In this section the effects of varying the values of the c and β parameters in the function expressing the radius of the collision detection zone

$$R_{cz}(t) = c + \beta v(t) \tag{4-1}$$

are shown. The agents in the simulations shown by Figure 4-4, 4-5, 4-6 and 4-7 all have the speed limits shown by Table 4-5. The speed limits of the simulations shown Figure 4-8 and 4-9 are given by Table 4-6. All the other parameter values are again shown in Table 4-1 except for c and β. It is clear from the figures that increasing c increases the preferred distance from the obstacle independent from the speed of the agent whereas increasing β increases the preferred distance dependent on the speed. This can result that an agent with a low speed barely has to change its direction, this is clearly visible in Figure 4-8 and 4-9.

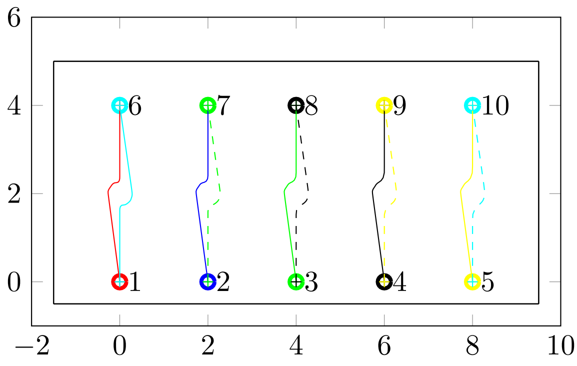

Figure 4-4: Simulated trajectories of 10 agents with $c = 0.39$ and $\beta = 0.39$ with speed limits shown in Table 4-5

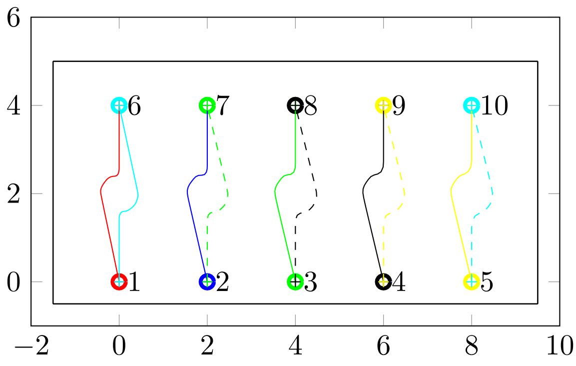

Figure 4-5: Simulated trajectories of 10 agents with $c = 0.78$ and $\beta = 0.39$ with speed limits shown in Table 4-5

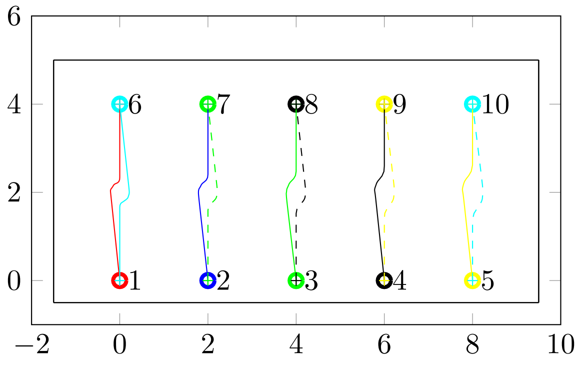

Figure 4-6: Simulated trajectories of 10 agents with $c = 0.195$ and $\beta = 0.65$ with speed limits shown in Table 4-5

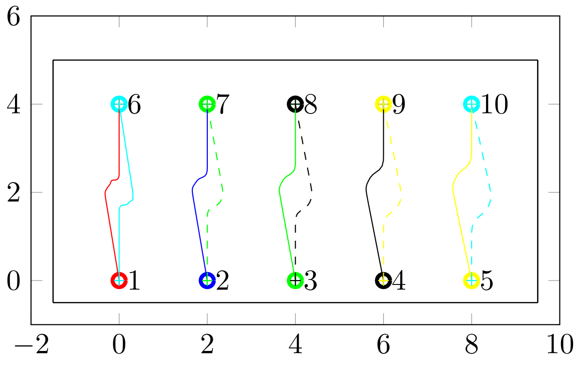

Figure 4-7: Simulated trajectories of 10 agents with $c = 0.195$ and $\beta = 0.975$ with speed limits shown in Table 4-5

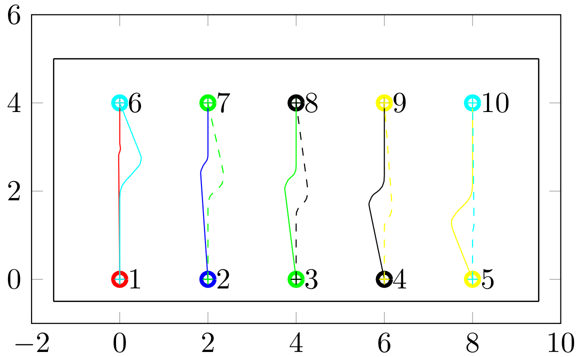

Figure 4-8: Simulated trajectories of 10 agents with $c = 0.195$ and $\beta = 0.78$ with speed limits shown in Table 4-6

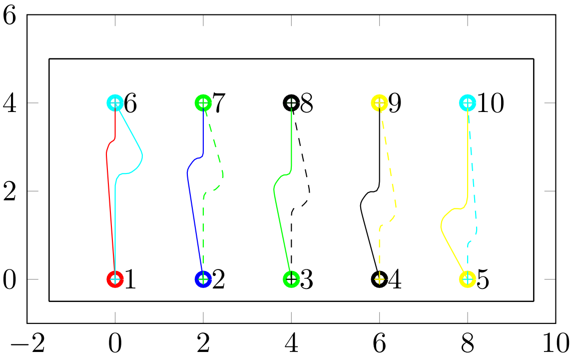

Figure 4-9: Simulated trajectories of 10 agents with $c = 0.78$ and $\beta = 0.195$ with speed limits shown in Table 4-6

Agent	Speed limit [m/s]
1	0.5
2	0.75
3	1
4	1.25
5	1.5
6	0.5
7	0.75
8	1
9	1.25
10	1.5

Table 4-5: Different speed limits of the agents in Figures 4-4, 4-5, 4-6 and 4-7

Agent	Speed limit [m/s]
1	0.5
2	0.75
3	1
4	1.25
5	1.5
6	1.5
7	1.25
8	1
9	0.75
10	0.5

Table 4-6: Different speed limits of the agents in Figures 4-8 and 4-9

4-5 Livelock Simulation

The final case-study is a simulation of a livelock. Agent two and three in Figure 4-10 already reached their destination and agent one (the red agent) has a target location at the bottom left. Agent one detects agent two (the blue agent) and makes a collision avoidance manoeuvre. Then agent one detects agent three (the green agent) and will make a collision avoidance manoeuvre to avoid it. At a certain point during this manoeuvre the collision detection zone in the direction of the target location does not detect a collision any more and agent one wants to continue its course thereby encountering agent two again. This cycle continuous indefinitely and is known as a livelock since the agent keeps moving, although it does not progress towards its target. As told in the previous chapter, there are several ways to break from this livelock but the focus will be on preventing the livelock by letting the agents cooperate. The parameter values used for this simulation is the same as stated in Table 4-1 and the starting and target locations are given by Table 4-7.

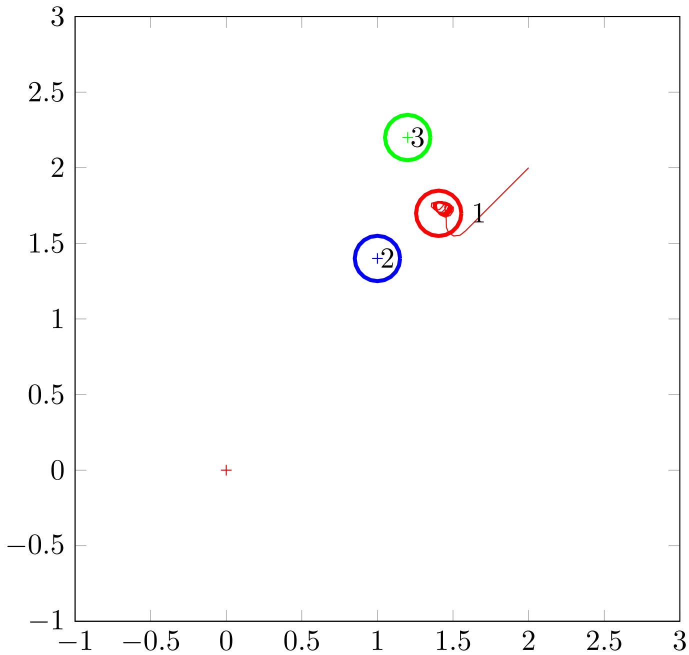

Figure 4-10: Simulation of a livelock

Agent	Starting Coordinates (i, j)	Target Coordinates (i, j)
1	$(2, 2)$	$(0, 0)$
2	$(1, 1.5)$	$(1, 1.5)$
3	$(1.2, 2.2)$	$(1.2, 2.2)$

Table 4-7: Starting and Target locations of the agents in Figure 4-10

Chapter 5

Network-Decentralised Control and Estimation

As discussed in the introduction there can be several reasons why a decentralised controller is preferred above a centralized controller. For example, the computation time for a centralised controller to calculate collision free trajectories increases drastically when the number of agents increases. There can be privacy reasons which prohibit a central controller to monitor all information or a connection to all agents is simply not possible by physical barriers like mountains or water. The way to describe the connection between agents is by the use of graph theory whose main basic notions will be introduced in Section 5-1. In Section 5-2 a method for decentralised position estimation as described by Giordano et al. in the paper [1] is explained. Different orientations of the agents in the configuration space are taken into account in Section 5-3. And finally in Section 5-4 a network-decentralised control method as described in [2], [13] and [14] for steering the agents to their target destination is treated.

5-1 Graph Theory

The interaction between people, computers, cells or generic units, able to affect one another or to exchange information with one another, can be represented by a set of points and lines. These units are called agents. An agent is represented by a point and is called a node n, all nodes are contained in the set $\mathcal{N}$. A connection between two nodes represented by a line is called an edge e, all edges are contained in the set $\mathcal{E}$. A graph is described by the ordered pair $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ together with an incidence matrix that associates each node with each edge. Often the direction of the connection or communication is of importance and such graphs are called directed graphs or digraphs [22]. Digraphs can be recognized by an arrow representing the communication direction on the edge connecting two nodes. The incidence matrix for directed graphs that are presented in the following section will be used for the network-decentralised position estimation and network-decentralised control. The incidence matrix of a directed graph is constructed as follows:

$$H_{ne} = \begin{cases} -1 \text{ if edge } e \text{ is a link and node } n \text{ is the tail of } e \\ 1 \text{ if edge } e \text{ is a link and node } n \text{ is the head of } e \\ 0 \text{ otherwise} \end{cases} \tag{5-1}$$

Consider Figure 5-1 where a graph consisting of 4 nodes and 8 edges is shown, agent 1 is connected to agent 2 and 3, agent 3 is connected to agent 1, 2 and 4 and so on. The incidence matrix H belonging to this graph is shown in Equation 5-2. The columns of the incidence matrix are associated with the edges and the rows are associated with the nodes.

$$H = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{matrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{matrix} \end{array} \tag{5-2}$$

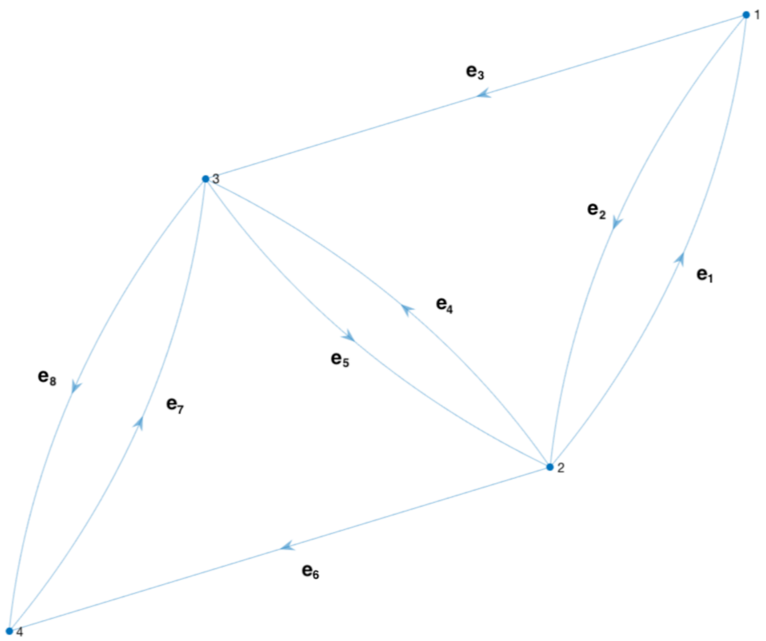

Figure 5-1: Digraph plot of 4 nodes and 8 edges

For the Internet Protocol confirmation of a connecting agent is needed, this is called the Three-Way Handshake. Actual data transmission is therefore only possible if there is symmetry in the directed graph i.e. there is an edge going in and out from and to the same agent. Edges e_1 and e_2 form such a symmetry pair. Removing the non symmetric connections from the graph in Figure 5-1 leads to the graph in Figure 5-2, which has the incidence matrix H_r shown in Equation 5-3.

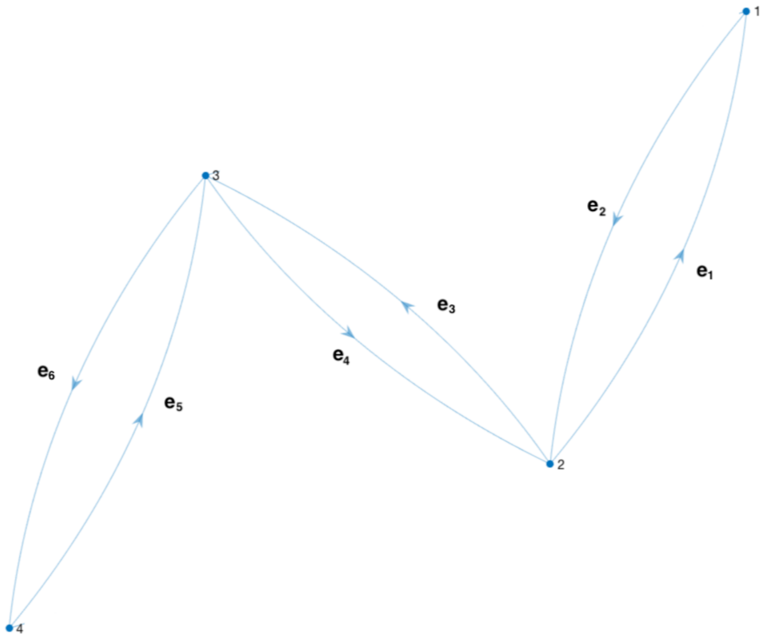

Figure 5-2: Reduced digraph plot of 4 nodes and 6 edges

$$H_r = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{5-3}$$

From now on we will consider graphs were all the connections are symmetric. The corresponding incidence matrix will be used to construct the decentralised position estimation strategy and the decentralised control input that are discussed in the next sections. The way an agent establishes a connection with other agents is by adjusting its transmitting/receiving radius. An agent increases or decreases its transmitting radius until n_{con} number of connection are made, at least. Since the agents are moving in the configuration space they establish new connections with other agents. Hence the (reduced) incidence matrix can vary over time, the time varying incidence matrix $H(t)$ has dimensions $N \times M_t$, with N the number of agents (nodes) and M_t the number of edges at time t.

5-2 Network Decentralised Position Estimation

This section highlights some parts of the network decentralised state estimation method described by Giordano et al. in the paper [1]. The method is used to allow each of the agents to estimate its own position by communicating with other nearby agents and measuring their distance. Consider the system consisting of N agents described by Equation 5-4.

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_N(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_N \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix}}_{B} u(t) \tag{5-4}$$

where the dynamics of each agent can be described by

$$\dot{x}_i = A_i x_i + B_i u, \; i = 1, ..., N \tag{5-5}$$

with N the number of agents. Moreover, the system has output

$$y_i = C_i x_i \tag{5-6}$$

where C_i is the individual output matrix. The overall output is

$$y = Cx \tag{5-7}$$

where $y = [y_1 \cdots y_N]^T$, while $C = \text{diag}(C_1, \cdots, C_N)$ is the overall output matrix. A Luenberger state observer [23] is given by Equation 5-8 where z is the estimated state, y are the measurements and L chosen such that the error defined by $e = x - z$ is acceptably small (precisely, it asymptotically goes to zero when the time goes to infinity).

$$\dot{z} = Az + Bu + L(y - Cz) \tag{5-8}$$

For an agent in a decentralised system the information available are the measured distances (d) to the connected agents and the estimated positions ($C_i z_i$) of the connected agents. So for agent (node) 2 in Figure 5-1 the available information are the distances between agent 2 and 1 (d_{21}), agent 2 and 3 (d_{23}) and the estimated positions of the two connected agents $C_1 z_1$ and $C_3 z_3$. This means that the measurement y in the Luenberger observer for agent 2 is

$$y_2 = \begin{bmatrix} C_1 z_1 - d_{21} \\ C_3 z_3 - d_{23} \end{bmatrix} \tag{5-9}$$

The measured distance can be described as a difference between the actual positions of the agents,

$$d_{21} = C_1 x_1 - C_2 x_2. \tag{5-10}$$

The distances between all agents of Figure 5-1 are all included in the matrix shown in Equation 5-11.

$$\underbrace{\begin{bmatrix} d_{12} \\ d_{21} \\ d_{23} \\ d_{32} \\ d_{34} \\ d_{43} \end{bmatrix}}_{d} = \underbrace{\begin{bmatrix} -C_1 & C_2 & 0 & 0 \\ C_1 & -C_2 & 0 & 0 \\ 0 & -C_2 & C_3 & 0 \\ 0 & C_2 & -C_3 & 0 \\ 0 & 0 & -C_3 & C_4 \\ 0 & 0 & C_3 & -C_4 \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{5-11}$$

If $C_1 = C_2 = C_3 = C_4$ it can be recognized that the $\mathcal{C}$ matrix is in this case

$$\mathcal{C} = H_r(t)^T \otimes -C_1 \tag{5-12}$$

where $\otimes$ is the Kronecker product. The observer for agent 1 is given by the following equation,

$$\dot{z}_1 = A_1 z_1 + Bu + L_1(C_2 z_2 - d_{12} - C_1 z_1) \tag{5-13}$$

Where $(-C_1 z_1 + C_2 z_2)$ has the same structure as the first row of $\mathcal{C}$. This means that the $L(y - Cz)$ part of the the Luenberger observer can be written as:

$$L(y - Cz) = L(\underbrace{\mathcal{C}x}_{d} - \mathcal{C}z) = L\mathcal{C}(x - z) \tag{5-14}$$

with an appropriate choice for the observer gain L. The error dynamics ($\dot{e} = \dot{x} - \dot{z}$) of the system containing all agents can now be written as

$$\dot{e} = A(x - z) - L(y - Cz) = (A - L\mathcal{C})e \tag{5-15}$$

The L matrix belonging to the graph of Figure 5-1 is given by Equation 5-16 .

$$L = \begin{bmatrix} L_{11} & -L_{11} & 0 & 0 & 0 & 0 \\ -L_{22} & L_{22} & L_{23} & -L_{23} & 0 & 0 \\ 0 & 0 & -L_{34} & L_{34} & L_{35} & -L_{35} \\ 0 & 0 & 0 & 0 & -L_{46} & L_{46} \end{bmatrix} \tag{5-16}$$

where L has the same block structure as $\mathcal{C}^T$ which means that the observer is network-decentralised [1]. Now if the graph is strongly connected, meaning that there is a path from an agent to every other agent, and if there is at least one agent who knows its exact location called the anchor, then under the assumption that $A_i = A_1$, the estimation error e converges to zero [1]. These assumptions and requirements are all satisfied by the agents described in this thesis.

5-3 Network Decentralised Orientation Estimation

In the previous section it was assumed that the distance measurements between agent 1 and 2 are equal but opposite from each other. For example if $d_{12} = [1\ 1]^T$ then $d_{21} = [-1\ -1]^T$ this means that the orientation of the agents in the configuration space is assumed to be the same. This assumption does not necessarily hold when agents are randomly placed in a configuration space. Consider Figure 5-3 where 3 agents are randomly placed in a configuration space, they all have different orientations. The blue line is the blue agent's x-direction and the blue dotted line is the blue agent's y-direction. This means that the distance measurements are no longer equal but opposite from each other but they are rotated. To reach a consensus in orientation an agent must receive the estimated orientation of the other agent and also the distance measurement of the other agent. Agent 1 connected to agent 2 and to an anchor called agent 0 will 'measure' the following information:

$$y_1 = \begin{bmatrix} \hat{\theta}_0 + \angle(-d_{01}) - \angle(d_{10}) \\ \hat{\theta}_2 + \angle(-d_{21}) - \angle(d_{12}) \end{bmatrix} \tag{5-17}$$

where $\hat{\theta}_i$ is the estimated orientation of agent i in radians in the range $(-\pi, \pi]$ and $\angle(d_{10})$ is the angle in radians of d_{10}. The measurement can be outside the range of $(-\pi, \pi]$, it can

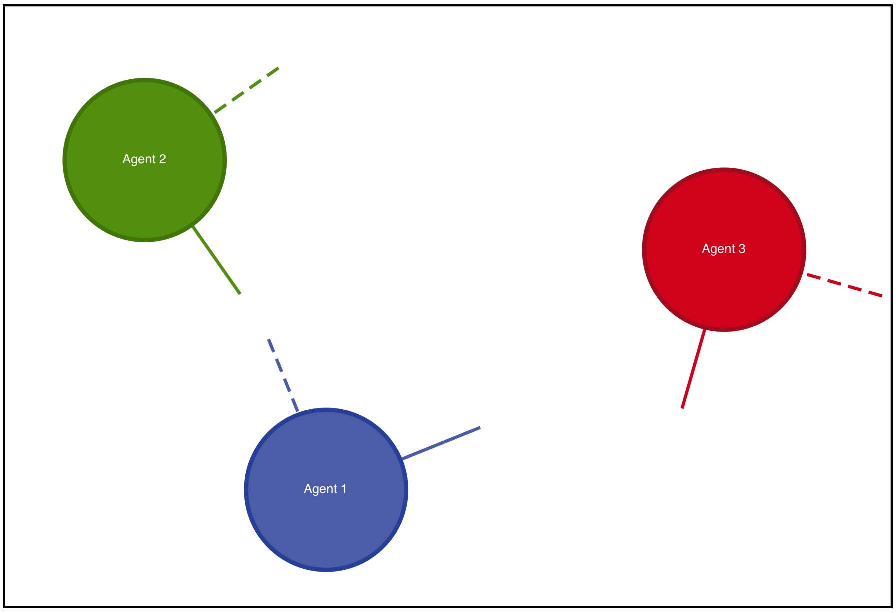

Figure 5-3: Different orientations of three agents

for example be $-1\frac{1}{4}\pi$ but this is equal to $\frac{3}{4}\pi$. The measurement must therefore be mapped in the range $(-\pi, \pi]$. It can be recognized that the difference in the angle measurements is equal to the difference in the actual states of the agents so,

$$\angle(-d_{01}) - \angle(d_{10}) = \theta_0 - \theta_1 \tag{5-18}$$

if the right hand side of Equation 5-18 is mapped in the range $(-\pi, \pi]$ radian. With this new insight a state estimator for the orientation has the following form:

$$\dot{\hat{\theta}}_1 = A\hat{\theta}_1 + L\left(\begin{bmatrix}\hat{\theta}_0 + \theta_0 - \theta_1\\ \hat{\theta}_2 + \theta_2 - \theta_1\end{bmatrix} - \begin{bmatrix}\hat{\theta}_1\\ \hat{\theta}_1\end{bmatrix}\right) \tag{5-19}$$

And now it is not hard to recognize the similarity with the state observer described in the previous section. A new $\mathcal{C}$-matrix can be constructed by taking the Kronecker product of $H(t)$ and C_1 with $C_1 = 1$, which is therefore equal to $H(t)$. A rotation matrix defined by Equation 5-20 can be used to rotate the distance measurement of the agent and the input received from the network-decentralised controller discussed in the next section.

$$R_i = \begin{bmatrix}\cos\hat{\theta}_i & -\sin\hat{\theta}_i\\ \sin\hat{\theta}_i & \cos\hat{\theta}_i\end{bmatrix} \tag{5-20}$$

It is important to stress that a correct position estimation can only be achieved when the orientations have converged to a consensus, therefore the network-decentralised orientation estimation is crucial.

5-4 Network Decentralised Control

With the network-decentralised state observer the agents are able to determine their absolute position. To make the agents cooperate i.e. move in a desired formation a control action will be constructed to achieve the formation. First the agents will only move to the formation and not yet to their destination. Consider Figure 5-4 where the distances from agent 1 and agent 2 to their target location are respectively $[0\ 6]^T$ and $[0\ 5]^T$, these are called the reference errors $e_r = \bar{r} - r(t)$. When the reference error of both agents are equal they have reached the desired formation which will be the first part of the controller. The input for agent 1 is a function of the reference error of agent 1 and agent 2 called $\bar{r}_{new}$ and is defined by

$$\bar{r}_{new,1} = e_{r,1} - e_{r,2} \tag{5-21}$$

For agent 2 this would be

$$\bar{r}_{new,2} = e_{r,2} - e_{r,1} \tag{5-22}$$

The agents will move to the location as shown by Figure 5-5. They both have an equal error (or distance from target location) of $[0\ 5.5]^T$ so $\bar{r}_{new,i} = 0$ and they are in the desired formation. Applying this method to the system of agents where the connections between the

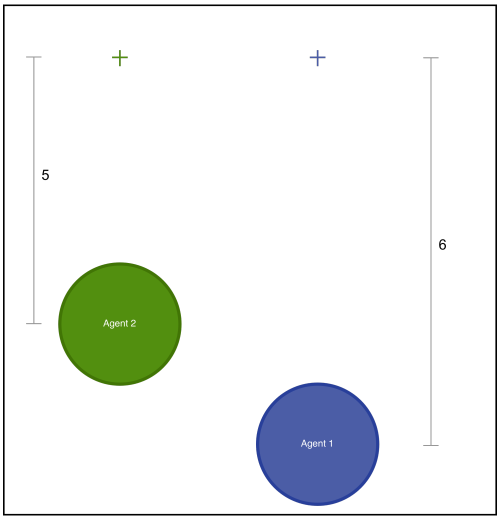

Figure 5-4: Starting situation for agent 1 and 2

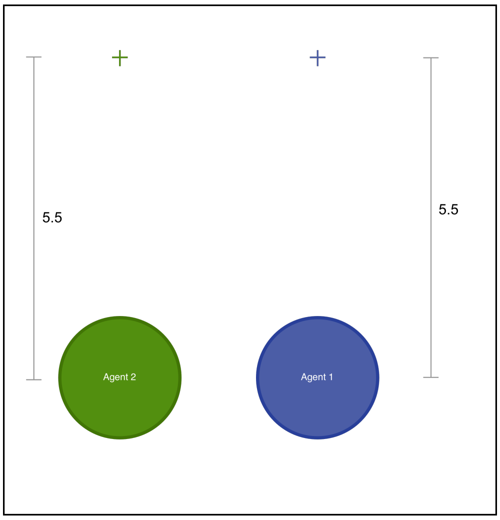

Figure 5-5: Final situation agent 1 and 2 with formation control

agents are as shown by Figure 5-2 yields a $\bar{r}_{new}$ defined by Equation 5-23

$$\bar{r}_{new} = \underbrace{\begin{bmatrix} I_2 & -I_2 & 0 & 0 \\ -I_2 & 2I_2 & -I_2 & 0 \\ 0 & -I_2 & 2I_2 & -I_2 \\ 0 & 0 & -I_2 & I_2 \end{bmatrix}}_{\frac{1}{2}G^T(t)G(t)} \underbrace{\begin{bmatrix} \bar{r}_1 - r_i(t) \\ \bar{r}_2 - r_2(t) \\ \bar{r}_3 - r_3(t) \\ \bar{r}_4 - r_4(t) \end{bmatrix}}_{\bar{r} - r(t)} \tag{5-23}$$

where

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5-24}$$

I_2 is a 2×2 matrix because the configuration space is 2 dimensional, if a third dimension is added this matrix would have size 3×3. The matrix $G(t)$, used to create the under braced part in Equation 5-23, can be constructed using the incidence matrix $H(t)$ by

$$G(t) = H(t) \otimes I_2 \tag{5-25}$$

As mentioned before this control input only steers the agents to the desired formation not the destination. To reach the destination the identity matrix I_{2N} of size $2N \times 2N$ times a positive constant ρ is added to $\frac{1}{2}G^T(t)G(t)$. A low ρ prioritizes the formation and a high ρ prioritizes the target location. The new matrix is given by Equation 5-26.

$$\frac{1}{2}G^T(t)G(t) + \rho I_{2N} = \begin{bmatrix} I_2+\rho & -I_2 & 0 & 0 \\ -I_2 & 2I_2+\rho & -I_2 & 0 \\ 0 & -I_2 & 2I_2+\rho & -I_2 \\ 0 & 0 & -I_2 & I_2+\rho \end{bmatrix} \tag{5-26}$$

Another advantage of this diagonal is that an unconnected agent always has a reference signal that is able to steer the agent to its target location. The next section shows how this new input is implemented in the backstepping controller used to control the agents.

5-5 Overview of Control Strategies and Implication on Collision Avoidance Algorithm

The different control strategies which will be used in the simulation cases of the next chapter are explained here along with the possible implication for the collision avoidance algorithm.

5-5-1 Individual Force Control

Although the individual force control strategy is already treated in Section 2-2 a brief summary is given to get a comprehensive view of all possibilities. The individual agent is only interested in its own target location so the reference velocity for agent i is given by:

$$\bar{\dot{r}}_i = \underset{v_{lim,i}}{\text{sat}} \left[(\bar{r}_i - r_i(t))k_1 \right] \tag{5-27}$$

The force $u_i(t)$ now applied to agent i is then given by the following equation

$$u_i(t) = k_2 m_i(t) \left(\underset{v_{lim,i}}{\text{sat}} \left[(\bar{r}_i - r_i(t))k_1 \right] - \dot{r}_i(t) \right) \tag{5-28}$$

There are no implications for the collision avoidance algorithm, the agent keeps looking for a possible collision in the direction of the target location.

5-5-2 Connection Based Force Control

The connection based force control strategy makes use of the information of connected agents. The reference velocity for the system of N agents is calculated as follows:

$$\bar{\dot{r}} = \underset{v_{lim}}{\text{sat}} \left[k_1 \left(\frac{1}{2}G^T(t)G(t) + \rho I_{2N} \right) (\bar{r} - r(t)) \right] \tag{5-29}$$

where each agent, therefore each row of $\bar{\dot{r}}$ can have a different speed limit. The input force applied to the system of agents is then

$$u(t) = k_2 m(t)(\bar{\dot{r}} - \dot{r}) \tag{5-30}$$

with

$$m(t) = \begin{bmatrix} m_1(t) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_N(t) \end{bmatrix} \tag{5-31}$$

For this method there is an implication for the collision avoidance algorithm. An agent can be in a situation where it receives an input which is not in the direction of the target location. This means that the collision detection zone is not pointing in the direction the agent wants to go, to overcome this the collision detection zone is no longer dependent on the target location but it depends on the direction of the input generated by the connection based force control.

Chapter 6

Cooperative Collision Avoidance: Case-Studies

In this chapter several case-studies are examined to show the performance of the proposed network-decentralised control strategy embedding the collision avoidance method introduced in this thesis.

6-1 Cooperation in Corridor

The first case-study concerns two agents placed in a small corridor. The red agent must reach the top of the corridor and the blue agent must reach the point $(0, 3)$. When the agents are not cooperating as in Figure 6-1, the blue agent is blocking the path for the red agent, hence a deadlock occurs. There is no way for the red agent to go around the blue agent without cooperation. This cooperation is achieved using the network-decentralised control method: in this case, the trajectories of the agents are shown in Figure 6-2. Both agents prioritize the formation, this results in the blue agent clearing the path for the red agent. Without cooperation the red agent would not have reached the target location, while cooperation allows the agents to avoid the formation of a deadlock. This is one of the biggest advantages of the proposed network-decentralised control enforcing cooperative collision avoidance. Table 6-1 shows all the parameter values used in the simulation.

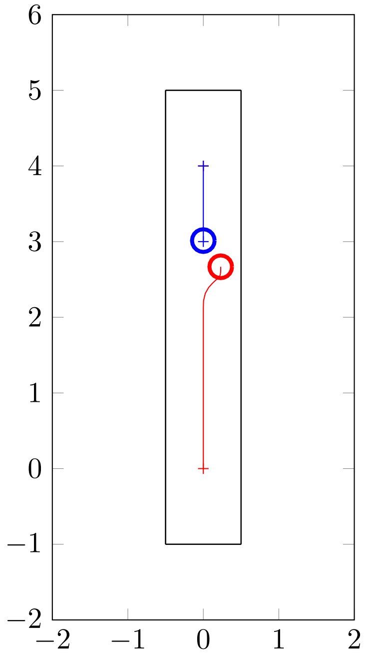

Figure 6-1: Two unconnected agents trying to pass each other in small corridor: this situation gives rise to a deadlock.

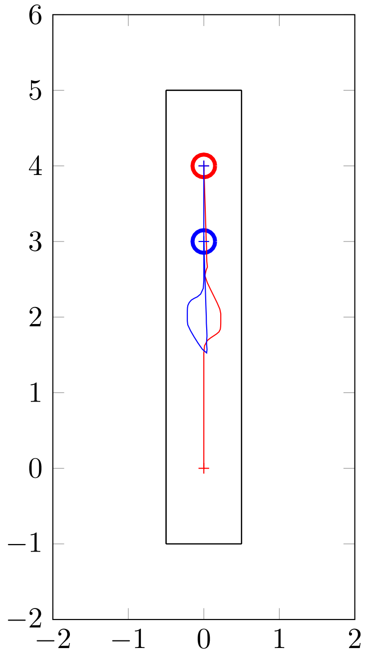

Figure 6-2: Two connected agents passing each other in small corridor: this time, no deadlock occurs and the two agents can successfully reach their targets

Parameter	Description	Value
m	Mass	1
k_1	Gain k_1 in Figure 2-2	20
k_2	Gain k_2 in Figure 2-2	12
α	Rotation angle for Collision avoidance	5 degree
c	Constant in equation $R_{cz}(t) = c + \beta v(t)$	0.36
β	Constant in equation $R_{cz}(t) = c + \beta v(t)$	0.48
θ	Range collision detection zone	$[-60, 60]$ degree
R_{dist}	Preferred distance from obstacle	$R_{cz} - 0.18$ m
h	Simulation sample time	0.01
v_{lim}	Speed limit	1 m/s
ρ	constant in network-decentralized controller	0.2

Table 6-1: Table of the parameter values used for the simulation in Figure 6-1 and Figure 6-2

6-2 Keep Formation

As mentioned in Section 5-4, the essence of the controller is to prioritize the assigned formation. This section shows a simulation where the agents must make a line formation. The agents first move to the formation, then they move as if they are one to the target location. The agents all have the same speed limit, in the next section a case where the agents have different speed limits is shown. Figure 6-3 shows how the agents move, it is clear to see by the straight lines that the agents first make a formation and then move as one to their destination. The same parameter values are used as in Table 6-1 with exception of the parameters mentioned in Table 6-2. The starting positions and target locations are shown in Table 6-3.

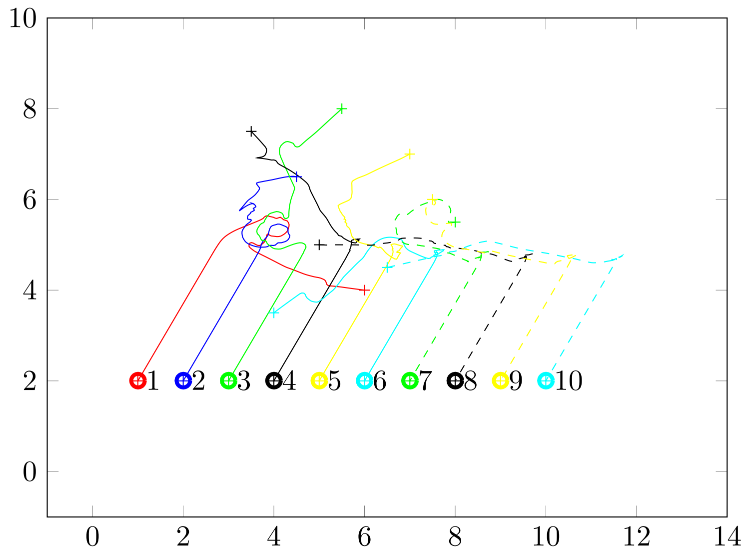

Figure 6-3: Agents moving in formation from a random starting position to a line

Parameter	Description	Value
v_{lim}	Speed limit	0.5 m/s
n_{con}	Number of connections	5

Table 6-2: Table of different parameters used for the simulation in Figure 6-3 and 6-4

6-3 Keep Formation With different Speed Limits

The next case-study is similar to the one in the previous section, the only difference is that the agents have different speed limits. Despite having different speed limits, the agents move in a formation to their destination. The simulation is shown in Figure 6-4. It is again clear to see by the straight lines that the agents move as one to their destination. The same parameters and starting positions are used as in the previous section. The speed limits of the agents are shown in Table 6-4.

Agent	Starting Coordinates (i, j)	Target Coordinates (i, j)
1	$(6, 4)$	$(1, 2)$
2	$(4.5, 6.5)$	$(2, 2)$
3	$(5.5, 8)$	$(3, 2)$
4	$(3.5, 7.5)$	$(4, 2)$
5	$(7, 7)$	$(5, 2)$
6	$(4, 3.5)$	$(6, 2)$
7	$(8, 5.5)$	$(7, 2)$
8	$(5, 5)$	$(8, 2)$
9	$(7.5, 6)$	$(9, 2)$
10	$(6.5, 4.5)$	$(10, 2)$

Table 6-3: Starting and Target locations of the agents in Figure 6-3 and 6-4

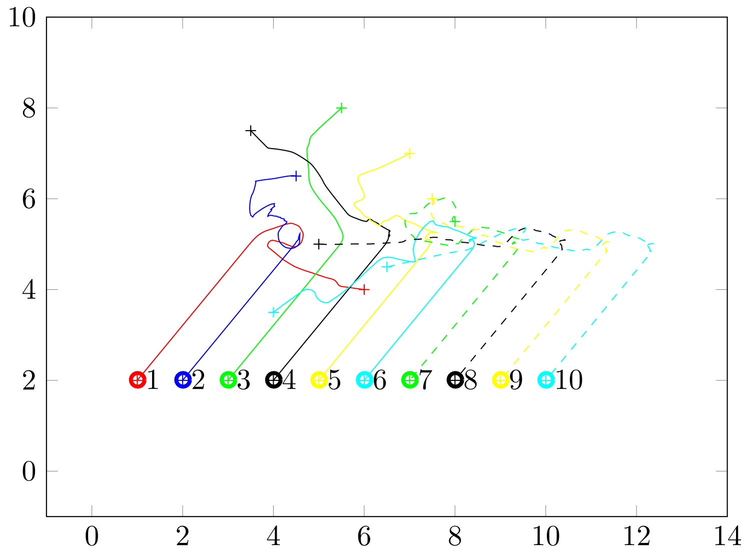

Figure 6-4: Agents moving in formation from a random starting position to a line having different speed limits

Agent	Speed limit [m/s]
1	0.5
2	0.65
3	0.35
4	0.6
5	0.65
6	1.1
7	1.25
8	0.8
9	0.9
10	1.2

Table 6-4: Different speed limits of the agents in Figure 6-4

6-4 Faster than without connected control

Another interesting case is the one where a network-decentralised controller steers the agents faster to their target location. The agents are placed in a square and their starting and target coordinates are shown in Table 6-5. Figure 6-5 shows how agents without network-decentralised control would behave to reach their target. They all meet in the middle and form a congestion where a lot of collision avoidance manoeuvres must be made. As a result that the agents take 25.09 seconds to reach their target location. With network-decentralised control however this only takes 14.03 seconds. The agents avoid the middle and form new smaller middle points thereby making less collision avoidance manoeuvres. The simulated trajectories with network-decentralised control are shown in Figure 6-6. The parameters used for the simulation are shown in Table 6-1

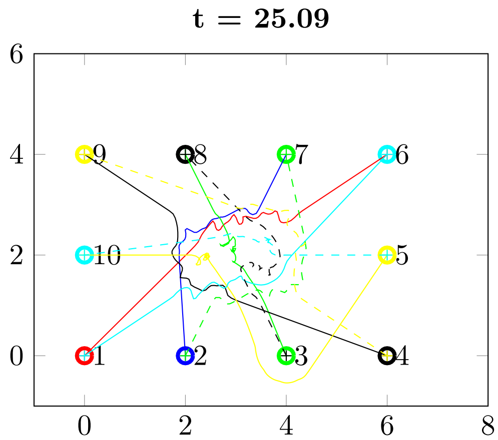

Figure 6-5: Unconnected ($n_{con} = 0$) agents moving to the opposite in a square thereby all meeting in the center

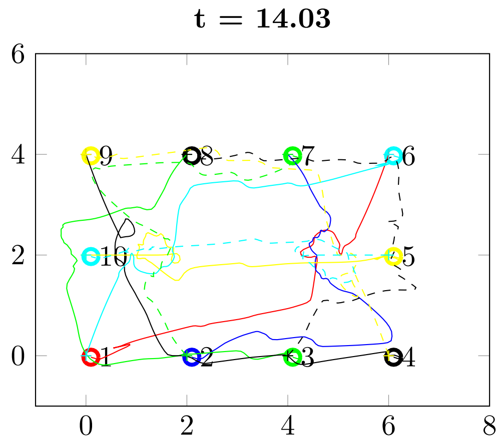

Figure 6-6: Agents connected up to 3 other agents ($n_{con} = 3$)moving to the opposite in a square

Agent	Starting Coordinates (i, j)	Target Coordinates (i, j)
1	$(6, 4)$	$(0, 0)$
2	$(4, 4)$	$(6, 0)$
3	$(3, 4)$	$(4, 0)$
4	$(0, 4)$	$(6, 0)$
5	$(0, 2)$	$(6, 2)$
6	$(0, 0)$	$(6, 4)$
7	$(2, 0)$	$(4, 4)$
8	$(4, 0)$	$(2, 4)$
9	$(6, 0)$	$(0, 4)$
10	$(6, 2)$	$(0, 2)$

Table 6-5: Starting and Target locations of the agents in Figure 6-5 and Figure 6-6

6-5 Simulation of 50 Agents

The simulation of 50 agents in Simulink for 5 seconds takes approximately 22 seconds where as 10 agents took about 1 second. The reason for this major increase in simulation time is because datasets of size 100×100 are saved in the Matlab workspace every 0.01 second which takes a lot of time. The 7 snapshots of a simulation where agents are randomly placed in a configuration space to form 5 circles are shown in Figure 6-7, 6-8, 6-9, 6-10, 6-11, 6-12 and 6-13. The agents successfully avoid each other and move in a formation to the desired target location. The same parameter values are used as in Table 6-1.

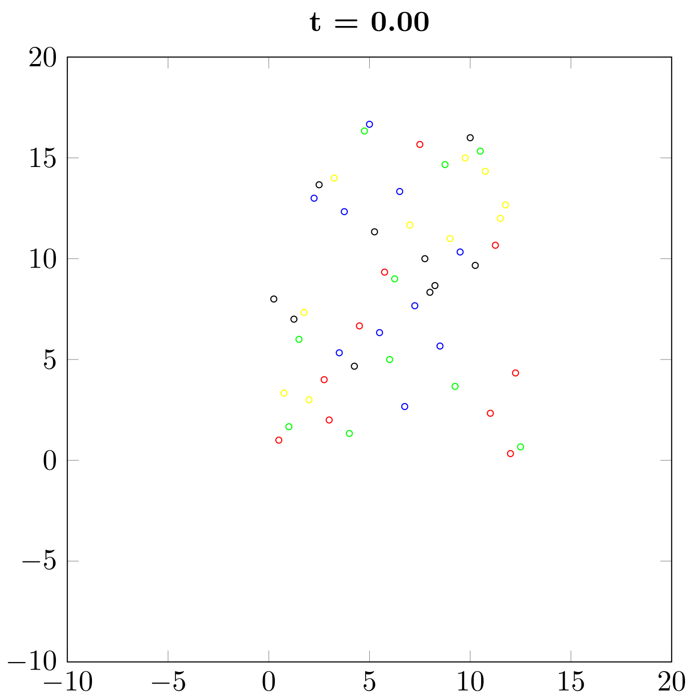

Figure 6-7: 50 agents moving in a configuration space to form 5 circles in the bottom left at $t = 0$ with $n_{con} = 5$

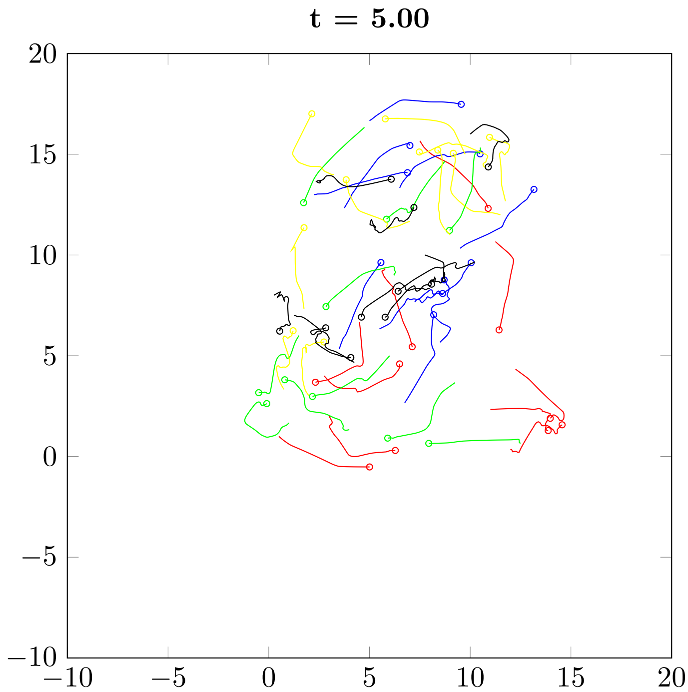

Figure 6-8: 50 agents moving in a configuration space to form 5 circles in the bottom left at $t = 5$ with $n_{con} = 5$

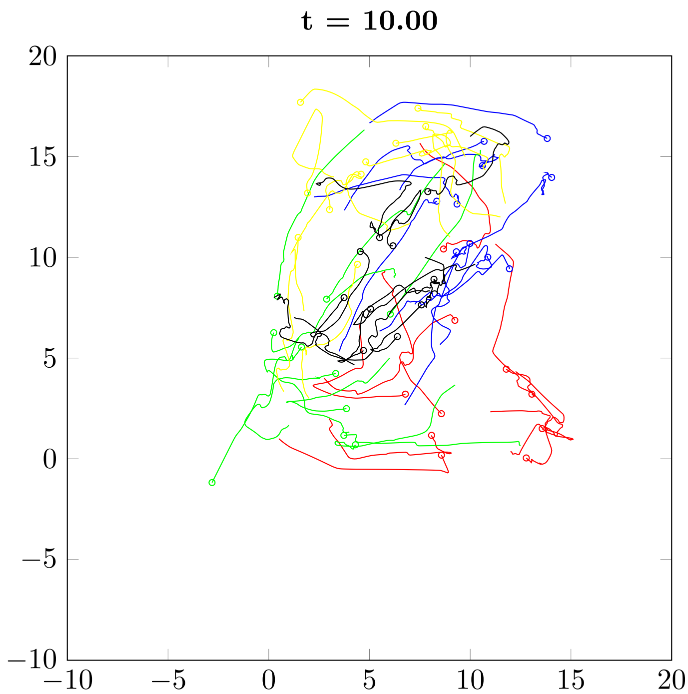

Figure 6-9: 50 agents moving in a configuration space to form 5 circles in the bottom left at $t = 10$ with $n_{con} = 5$

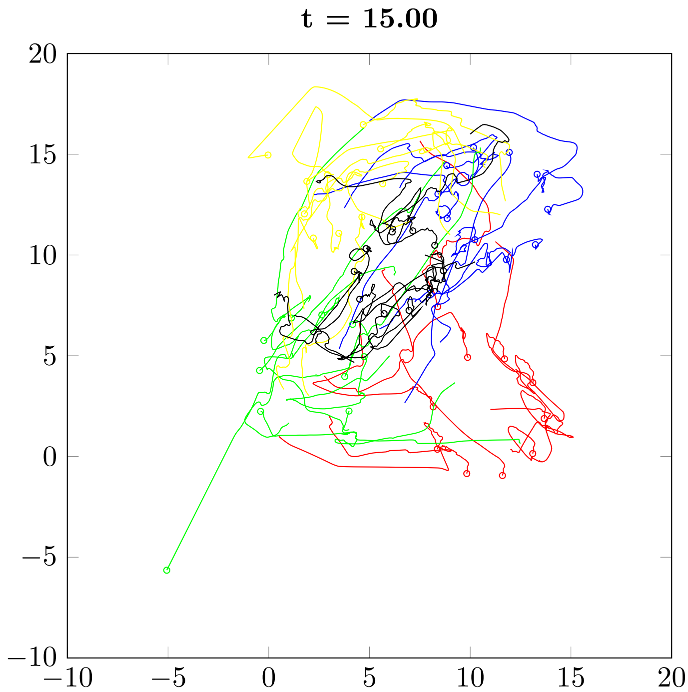

Figure 6-10: 50 agents moving in a configuration space to form 5 circles in the bottom left at $t = 15$ with $n_{con} = 5$

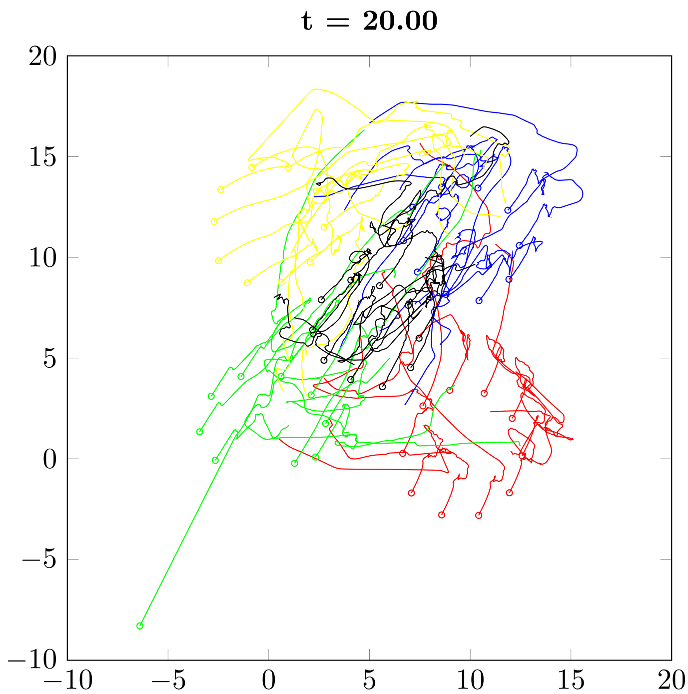

Figure 6-11: 50 agents moving in a configuration space to form 5 circles in the bottom left at $t = 20$ with $n_{con} = 5$

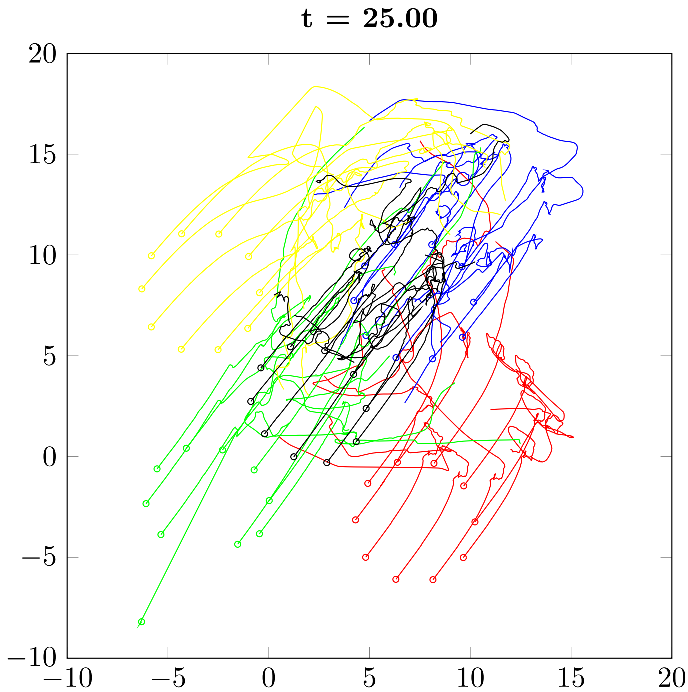

Figure 6-12: 50 agents moving in a configuration space to form 5 circles in the bottom left at $t = 25$ with $n_{con} = 5$

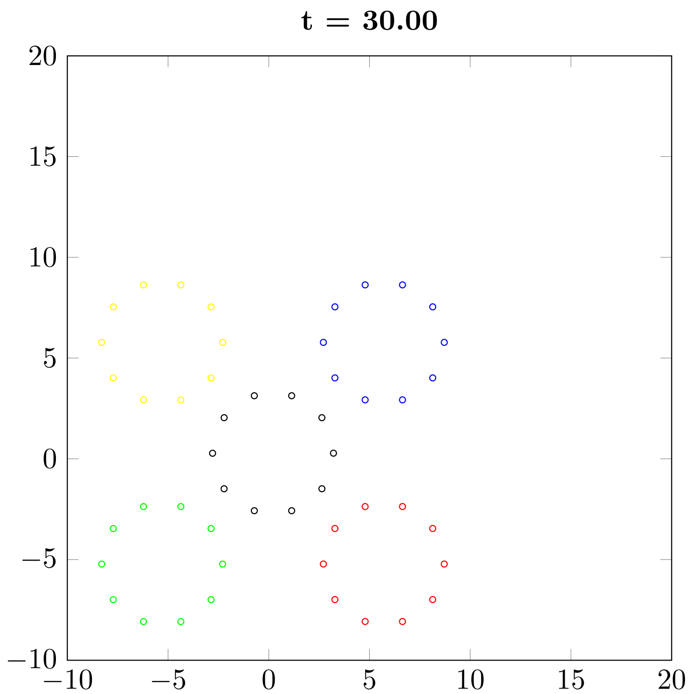

Figure 6-13: 50 agents moving in a configuration space to form 5 circles in the bottom left at $t = 30$ with $n_{con} = 5$

Chapter 7

Anonymity in Coordination

The agents are able to avoid collision, estimate their location and are able to move in a formation in a network-decentralized way. But how do we choose which agent goes where? Often anonymity in coordination can be exploited, which means that it does not matter which agents goes where in the formation as long as the formation is formed. To determine which agent goes where each agent solves a local optimisation problem to minimize the distance travelled. Opposed to the multi-agent optimisation problem described in [24] the optimisation problem is non-convex.

7-1 Problem Definition

The agents who are all assigned a target destination are allowed to change their destination with connected agents. The global goal of this local distribution of target destinations is to minimize the total distance travelled by the system of N agents. Or mathematically the following cost function must be minimized:

$$\min_T C = \sum_{i=1}^{N} \sqrt{(T\bar{r}_x - r_{x,i})^2 + (T\bar{r}_y - r_{y,i})^2} \tag{7-1}$$

where T is a matrix which distributes the target location to every agent, making sure that every target location is assigned only once. There are several ways to solve this global optimization problem such as checking every possible solution for the N location. This has $N!$ (N factorial) solutions, when there are 15 agents there are already 1307674368000 possibilities. A more efficient method is to use the Hungarian Algorithm as explained in [25]. The number of operations for this method scales with N^3 and is therefore preferred when $N > 5$ because $5! = 120$ and $5^3 = 125$.

The agents however are interconnected according to a communication topology $H(t)$ (the incidence matrix), so each agent is connected to at most 4 other agents. Of course they know their own target destination so the number of target locations to distribute is 5. The next section proposes a method for the agents to minimize the distance travelled by exploiting anonymity in coordination in a network-decentralised way (because the cost function is local and a local optimisation problem is solved).

7-2 Network-decentralised Optimization Method

As discussed in Chapter 5 the agents receive information about the error $\bar{r} - r(t)$ and the estimated position of the connected agents $C_i z_i$. This information will also be used for the optimization method run by each agent. Each agent runs an optimization such that the following local cost function is minimized

$$\min_{T_i} C_i = \sum_{k \in c_i} \sqrt{(T_i \bar{r}_{x,k} - r_{x,k})^2 + (T_i \bar{r}_{y,k} - r_{y,k})^2} \tag{7-2}$$

where c_i is the set with agent i and the agents connected to it; for instance, for agent 1 in Figure 7-1 c_1 is given by:

$$c_1 = \{1, 2, 3, 7\} \tag{7-3}$$

The difference between the value of the cost function C_i before optimization and after optimization $C_i(T_{i,opt})$ (the cost as function as $T_{i,opt}$) is called the Bid B_i:

$$B_i = C_i - C_i(T_{i,opt}) \tag{7-4}$$

The bid is placed in an auction held by each interconnected agent. For instance, for agent 1 in Figure 7-1 the Auction is given by

$$\text{Auction}_1 = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_7 \end{bmatrix} \tag{7-5}$$

Just as in a real auction, the agent 'sells' its target location to the agent with the highest bid. Suppose in the graph in Figure 7-1 agent 2, 3 and 7 all take the bid of agent 1 then agent 1 is allowed to allocate their target locations and a new optimization (if necessary) with those agents is performed. If agent 1 did not take its own bid, it would not perform an optimization since another agent would reallocate its position. This optimization can be executed multiple times per second, depending on the computational power of the on-board computer of the agent.

7-3 Result of Anonymity in Coordination

Each agent individually solves a local optimisation problem, where it minimises the distance travelled by itself and by the agents connected to it. Therefore, it is not the goal of the agents to reach the global optimum, in terms of minimum total distance travelled. However, this is often reached any ways. The outcome is dependent on the number of agents, how they are connected and how the locations are distributed among the agents. However, anonymity in coordination is reached, the agents are able to distribute their target locations in what is locally the optimal distribution. An example where agents distribute the locations is shown in Figure 7-1 and Figure 7-2, the blue lines represent the connection at time $t = 0$ and $t =$ final. In Figure 7-1 the agents are assigned a random target, all different from their final positions as shown in Figure 7-2 as shown by the plus signs. The locations of the agents in Figure 7-2 are the final positions where the plus signs that are not encircled mark their starting positions. By solving the local optimization problem they distribute their destination to reach, in this case, the global minimum of total distance travelled. It can be interesting to investigate what the minimum number of connections must be to guarantee a global minimum in a connected graph.

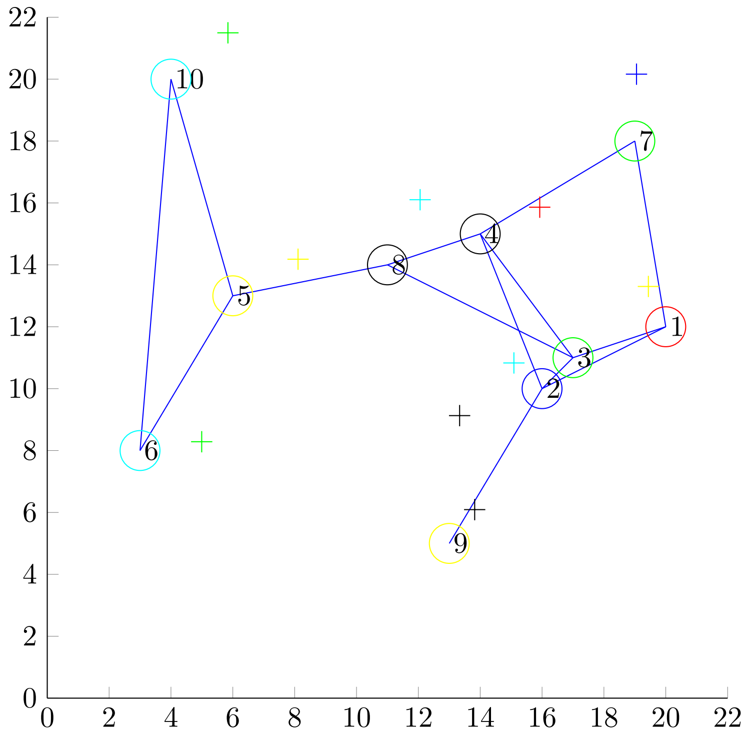

Figure 7-1: Network between agents at time $t = 0$ where the lines represent the connection between agents

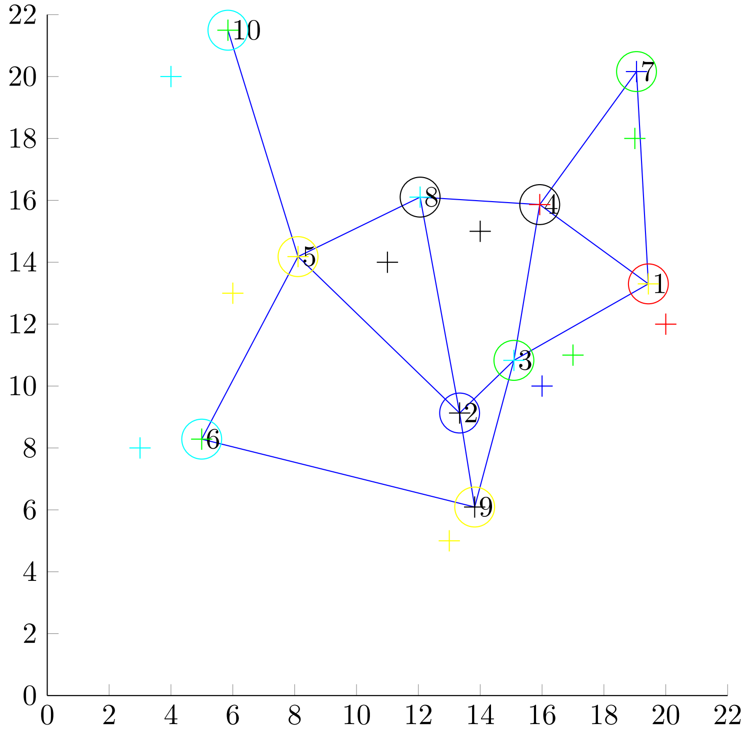

Figure 7-2: Final positions of agents with anonymity in coordination at time $t = \text{final}$

Chapter 8

Conclusion

By multiple simulations it is shown that the proposed collision avoidance algorithm avoids collisions by generating a reference velocity which is always away and around from the possible collision. This collision avoidance method can be combined with a network-decentralised state estimator. The algorithm can also be combined with a network-decentralised controller. The network decentralised-controller generates an input signal which makes the agents move in a formation as if they were one system. Special about this controller is that the agents do not have to be homogeneous, simulations with different speed limits have shown this. The network decentralised-controller improves the collision avoidance algorithm since agents are helping one another to reach their destination. A good example of this is the simulation in a corridor in Section 6-1. The chances on a deadlock or a livelock are therefore greatly reduced. Finally a local optimization method based on a bidding and auction process with connected agents reallocates the target destination. The target destination are reallocated such that at time t the total distance travelled of the connected agents will be minimized. A global optimum can not be guaranteed since a collision avoidance manoeuvre can prevent an agent from choosing the shortest path and only local information is used for the optimization. The proposed collision avoidance method in combination with the network-decentralised controller, observer and optimization is able to avoid collision, keep a formation and satisfies the anonymity in coordination requirement.

Chapter 9

Recommendations

Combining the collision avoidance algorithm with the network-decentralised estimator and controller showed great promise in the simulation. To test this method on a real life application is therefore the most important recommendation. Priority attention in the test must go to achieving consensus in orientation of the agents: this can be particularly challenging when sensor noise is present.

Another topic of interest is to extend the collision avoidance algorithm to deal with non-holonomic agents. Using a larger preferred distance from the obstacle is probably needed to increase the available space for a collision avoidance manoeuvre. The addition of a non-circular collision detection zone as in [26] might also improve the collision avoidance algorithm. Inspiration for a collision avoidance algorithm which is applicable to non-holonomic agents can be found in [27]. Combining the non-holonomic collision avoidance algorithm with the network-decentralised formation controller is interesting for future work.

Finally it can be investigated what the conditions are to reach a global minimum by solving multiple local minimum optimisations at the agents.

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Mechanics of Economics

The Lagrangian and Hamiltonian formulation in economics

Master of Science Thesis

For the degree of Master of Science in Systems and Control at Delft University of Technology

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Abstract

In mechanical and electrical engineering the equations of motions of a system can be derived using a Lagrangian or Hamiltonian and by the principle of least action. This thesis presents an economic equivalent of the Lagrangian and Hamiltonian formulation based on the minimization of cost by the rational behaving agent, the principle of least cost. The Lagrangian contains the cost belonging to the path of an agent, the path is a configuration of goods q and the flow of the goods per unit time $\dot{q}$. The Hamiltonian is the income as a function of price (Euro per good) and goods q just as the mechanical Hamiltonian is a function of momentum and generalized coordinate q. The economic equivalent of a force (F) is a rental with units Euro per good per unit time. It is shown that existing economic conservation laws, such as 'income is ρ times wealth, can also be obtained by the economic Lagrangian and Hamiltonian method. This method, with the exponential discount factor $e^{-\rho t}$, however, can not be used to obtain a second order model which describes the equations of motion of an agent, the obtained system is unstable. Also, when price (p) is defined as a constant m times demand $\dot{q}$, just like momentum $p = m\dot{q}$, then Newton's second law of $\dot{p} = F$ does not hold for economic systems, increasing the price does not increase demand, in fact, in general it decreases demand. A small modification of the Hamiltonian method leads to Pontryagin's Maximum Principle which is an important principle in the field optimal control theory. For the principle, the price (or momentum) does not have to be a function of the flow $\dot{q}$ and can therefore be very useful when optimizing economic problems when they are stated as the proposed economic Lagrangian method. Although a similar method is already used, the units are often not clear, there is no distinction between price and rental. Dissipating costs like adjustment cost are not treated correctly this, in combination with the unclear units used, results in differential equations which do not necessarily have an economic meaning. So the method is useful for obtaining conservation laws and solving optimization problems in economics but the mechanical equivalent of a mass spring system (a second order model) can not be derived with this.

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"As far as the theorems of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."[1]

— *Albert Einstein*

Chapter 1

Introduction

The fields of engineering and economics have been closely related throughout their histories. For example the physicist, engineer and mathematician Simon Stevin is remembered in the history of accounting for introducing the Italian method of double-entry bookkeeping into Northern Europe [2]. Stevin understood that, by influence of his bookkeeping studies, a perpetual mobile is impossible. The first appearance of Smith's invisible hand, which is famous for its explanation why prices are pushed to an equilibrium, was in a physics context: *"Heavy bodies descend, and lighter substances fly upwards by the necessity of their own nature; nor was the invisible hand of Jupiter ever apprehended to be employed in these matters"* (Smith 1967 p. 49). And many more examples can be found how engineering and economics have influenced each other. As a control engineer, the author of this thesis wants to explore the possibilities of applying control theory on economic problems. To achieve this a connection between mechanics and economics will be made by use of the Lagrangian and Hamiltonian formulation. With this formulation mechanical equivalents of electrical elements are already obtained, an inductor behaves like a mass, a capacitor like a spring and a resistor like a damper. Can such elements also be found in an economic system?

Chapter 2

Basic Economic Concepts

In this chapter some basic economic and financial concepts which will be used through out the thesis will be explained.

2-1 Rational Behaving Agent

One of the most important assumption of many economic and financial theories is that when a decision is to be made, the decision-maker always chooses the one which contributes to his goal. The goal can be the maximization of monetary and material value but can also be purely emotional. Such a decision-maker can refer to an individual, a firm, a household, a government etcetera and will from now on be called an agent. The agent who makes decisions which contribute to his goal is said to *behave rational.* A rational behaving firm wants to minimize his costs to reach a certain goal and the rational behaving consumer wants to maximize utility. Utility is described by Jeremy Bentham (founder of utilitarianism) as the sum of all pleasure that results from an action, minus the suffering of anyone involved in the action. Both utility and profit are expressed in monetary value to be able to compare them.

2-2 Time Value of Money

Because of the possibility to invest or save money, a Euro today does not have the same value as a Euro a year from now. An agent receives an amount of money N, which he can store in an account which pays an interest of 10% so after a year the account has a value of $1.1N$. Money received in the future must be discounted with the so called discount factor, this can contain interest rates, inflation, opportunity costs and so on. A commonly used discounting method is by multiplying the received amount of money by $e^{-\rho t}$, where ρ is the discount factor and t is the time. There are other methods of discounting like hyperbolic discounting which value future payments even less then regular discounting and are said to more closely describe human behaviour.

2-3 Supply and Demand

The demand and supply curve are two well known economic concepts, they are the forces which makes an economy work [3]. Figure 2-1 is a graphical interpretation of how supply and demand together result in a *market price*. Contrary to (most) mechanical graphs the free variable is the horizontal axes i.e. the price. If the price decreases, the demand increases and if the price increase the supply increases. An important distinction between the demand graph used by economics and the demand graph as it will be used in this thesis is that variable on the horizontal axes: quantity demanded in a unit of time will be expressed as the quantity per unit of time: $\dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t}$, the flow of goods per unit of time.

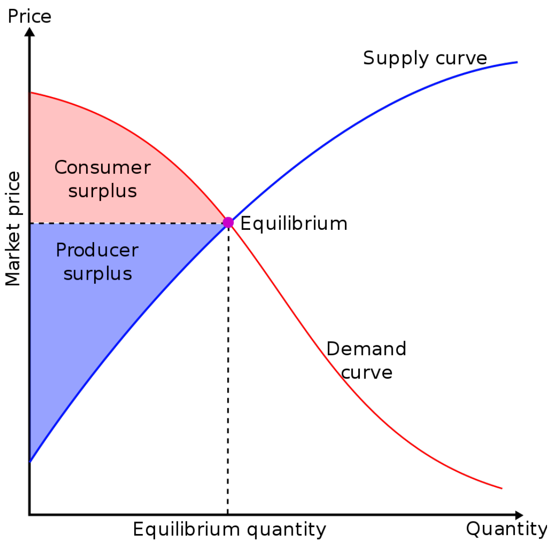

Figure 2-1: Supply and Demand Curve [4]

When supply equals demand the market is in equilibrium, both suppliers profit and consumers utility are maximized. This is clearly depicted in Figure 2-1. In an efficient markets prices and demand are moving to a stable equilibrium as if they are pushed by an invisible hand. This approach to economics is called neoclassical economics. Some economic events however are better described as if markets tend towards disequilibrium, an example of this are the boom and bust cycles in the stock market which are described by the reflexivity theory by G. Soros [5].

Chapter 3

Fundamentals of Mechanics and Economics

The principle of critical action, which is often called the principle of least action, and Hamilton's principle are fundamental principles used to derive the equations of motion of a mechanical and electrical systems. Although the two principles are regarded as being synonymous they are slightly different. Hamilton's principle involves varying the integral of the Lagrangian over all curves with fixed endpoint and fixed time. The principle of least action involves variation of

$$\int_a^b \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \mathrm{d}t \tag{3-1}$$

over all curves with fixed energy[6]. The principle of least action is also used in quantum mechanics [7]. Now the question rises if there is such a principle in economics to derive economical equations of motion similar to the mechanical equations of motions.

3-1 The Economic Agent and the Free Particle

One of the most fundamental concepts of mechanics is that of a free particle [8]. If al the generalized coordinates q and generalized velocities $\frac{\mathrm{d}}{\mathrm{d}t}q = \dot{q}$ are given at some instant, the accelerations $\ddot{q}$ are uniquely defined. In economics the free particle is the economic agent whose state is defined by its capital (and labour) q and the change in capital over time $\dot{q}$. In economic textbooks capital and labour are often denoted with K and L, $\dot{q}$ is the change in capital per unit time. Integration of $\dot{q}$ and $\ddot{q}$ makes it possible to determine the path of the agent which will be called the *Economic Activity*, Figure 3-1 shows a possible path of an agent who wants to reach a position $q(t_2)$ starting at $q(t_1)$, q_1 and q_2 can for example be chickens and eggs.

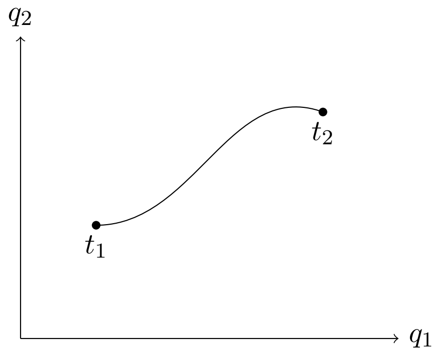

Figure 3-1: Economic Activity of an Agent

3-2 Principle of Least Cost

The cost for the economic activity can be determined by a function of $q(t)$, $\dot{q}(t)$ and time t, this can be written as

$$\text{Cost} = f(q(t), \dot{q}(t), t) \tag{3-2}$$

Now the economic activity, or path, the rational behaving agent takes when going from a state at t_1 to a state at t_2 is the one for which the accrued cost are minimized. Let $q(t)$ be the path for which the accrued cost are minimized and $\delta q(t)$ be a small variation of this path. And let the variation at initial and final positions be

$$\delta q(t_1) = \delta q(t_2) = 0 \tag{3-3}$$

This is like travelling from Rotterdam to Berlin where an agent has the option of taking multiple routes (the variation) but the start and final locations are not varied ($\delta q(t_1) = \delta q(t_2) = 0$). The change in accrued cost S when taking a small variation of the optimal path is given by

$$\delta S = \int_{t_1}^{t_2} f(q + \delta q, \dot{q} + \delta \dot{q}, t) \text{ d}t - \int_{t_1}^{t_2} f(q, \dot{q}, t) \text{ d}t \tag{3-4}$$

When this difference is expanded in powers of δq and $\delta \dot{q}$, the leading terms are of the first order and the first order necessary condition for S to have a minimum (or maximum or a saddle-point) is that these terms should be zero, or mathematically

$$\delta S = \delta \int_{t_1}^{t_2} f(q, \dot{q}, t) \; dt = 0 \tag{3-5}$$

The second order condition, known as the Legendre-Clebsch condition to verify if the function is indeed a minimum and not a maximum or saddle-point is not important when deriving the mechanical equations of motion since:

> *The motion of a system from time t_1 to t_2 is such that the line integral, called the action, has a stationary value for the actual path of the motion.[9]*

In economics this second order condition is important since the agent wants to minimize cost. If the Legendre-Clebsch condition holds this principle will in economics be called the principle of least cost, in chapter 7 it will be shown that this second order condition is actually contained in the Hamiltonian formulation. The function $f(q, \dot{q}, t)$ is in mechanics known as the Lagrangian ($\mathcal{L}$) with energy measured in Joule $[J]$ as its unit. The Lagrangian in mechanics usually consists of two parts, the kinetic energy $\mathcal{T}$ and the potential energy V such that,

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \tag{3-6}$$

In economics the Lagrangian is the running cost in €$/T$, where T is a given period of time, and the action S are the accrued cost in [€]. The kinetic energy $\mathcal{T}$ are cost per unit of time and the potential energy $\mathcal{V}$ can be a net profit per unit of time.

3-3 Minimizing the Accrued Cost

So the problem for the economic agent is to minimize a cost function which depends on the path, $f(q\dot{q})$. Since most economic problems which are minimized over a period of time are subjected to a (constant) discount factor ρ because of the time value of money. The un-discounted accrued cost dependent on the path will be described as $\mathcal{F}(q, \dot{q})$. The minimization problem with a constant discount factor is now described by:

$$\text{Minimize } S \text{ with } S = \int_0^T e^{-\rho t}\mathcal{F}(q, \dot{q})\mathrm{d}t \tag{3-7}$$

subjected to constraints, where S is the accrued cost of the firm and ρ is the discount factor. In similar fashion a typical economic problem for a consumer (or society) is to **maximize** an un-discounted utility function $U(q, \dot{q})$ over a period of time:

$$\text{Maximize } \mathcal{U} \text{ with } \mathcal{U} = \int_0^T e^{-\rho t}U(q, \dot{q})\mathrm{d}t \tag{3-8}$$

subjected to constraints, $\mathcal{U}$ is the (present) accrued utility of a consumer (or society) expressed in monetary value. The Euler-Lagrange equations will now be derived for the cost minimizing firm. By the principle of least cost the optimal path where the accrued costs are minimized when going from a state $q(t_1)$ to $q(t_2)$ is when a small variation δ of the path results in no variation in the accrued cost, $\delta S = 0$ so,

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\ dt = 0 \tag{3-9}$$

or, effecting the variation,

$$\int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q} \right)\ dt = 0 \tag{3-10}$$

and using integration by parts we obtain,

$$\delta S = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q\ dt = 0 \tag{3-11}$$

since the variation at the beginning and final position are zero, $\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} = 0$, the integral must vanish for al variations of δq, so the function is minimized if,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{3-12}$$

The next section described the economic meaning of this equation.

3-4 Price, Rental and the No Arbitrage Principle

In mechanics the partial derivatives of the Lagrangian are defined as the momentum (p) and the Force (F):

$$p \triangleq \frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{3-13}$$

$$F \triangleq \frac{\partial \mathcal{L}}{\partial q} \tag{3-14}$$

the economic definition of these partial derivatives are *price* (p) in Euro per good [€/#] and rental cost F in Euro per good per unit of time [€/#T]. With these definitions the economical interpretation of the Euler- Lagrange equation 3-12 is that the change in price is equal to the rental cost:

$$\dot{p}(t) = F(t) \tag{3-15}$$

or

$$p(t) = p(0) + \int_0^t F(t)\mathrm{d}t \tag{3-16}$$

Equation 3-16 is a fundamental economic concept first published by Hotelling in 1931 [10], it is known as Hotelling's rule for non-renewable resources where $F(t)$ is an opportunity cost or resource rent. It states that the price tomorrow is equal to the price today plus resource rent, nowadays this relation is well known as the *no arbitrage* principle. This is best explained in discrete time for a stock paying dividend when the cost price is defined as:

$$p_c \triangleq \frac{\partial \mathcal{F}}{\partial \dot{q}} \tag{3-17}$$

that is the un-discounted price of the cost function

$$S = \int_0^T e^{-\rho t}\mathcal{F}(q, \dot{q})\mathrm{d}t \tag{3-18}$$

Since the rental is usually negative, just like potential energy is negative in the mechanical Lagrangian formulation, i.e. the economic agent receives an amount of Euro per good per unit of time, the *net rental* R of the capital good will be defined as

$$R(t) = -\frac{\partial \mathcal{F}}{\partial q} \tag{3-19}$$

$$\tag{3-20}$$

then the discrete time price of a stock paying dividend is according to the equations obtained by the principle of leas cost in discrete time,

$$p_c[t+1] = p_c[t] + \rho p_c[t] - R[t] \tag{3-21}$$

The cost price tomorrow is equal to the cost price today plus opportunity cost ($\rho p_c[t]$) minus the dividend $R[t]$ paid to the stockholder. Which is exactly the no arbitrage rule for stocks paying dividends.

Equation 3-15 has more important economic and financial meanings, when both sides are divided by $p(t)$:

$$\frac{\dot{p}(t)}{p(t)} = \frac{F(t)}{p(t)} \tag{3-22}$$

$$\frac{e^{-\rho t}\left(\dot{p}_c(t) - \rho p_c(t)\right)}{e^{-\rho t} p_c(t)} = \frac{-e^{-\rho t} R(t)}{e^{-\rho t} p_c(t)} \tag{3-23}$$

$$\frac{\dot{p}_c(t)}{p_c(t)} = \rho + \frac{-R(t)}{p_c(t)} \tag{3-24}$$

$\frac{R(t)}{p_c(t)}$ is called the internal rate of return (IRR) of an investment and can be a decisive factor when choosing between several investment projects [11].

3-5 Rayleigh Dissipation and Adjustment cost

Mechanical problems can contain a frictional force, such forces can not be derived from a Lagrangian. Usually the frictional force is proportional to the velocity of the particle so that it has the following form:

$$F_f = -cv \tag{3-25}$$

With F_f the friction force, c the friction coefficient and v the velocity of the particle ($\dot{q}$). Frictional forces of that type may be derived in terms of a function $\mathcal{D}$, known as Rayleigh's dissipation function and is defined as [9]

$$\mathcal{D} = \frac{1}{2} \sum_i \left(c_x v_{ix}^2 + c_y v_{iy}^2 + c_z v_{iz}^2 \right) \tag{3-26}$$

Dissipation functions also arise in economic problems when there are *adjustment costs*. Adjustment cost, just as friction, cannot be recovered from the system and works in both directions i.e. installing and deinstalling an investment. It turns out that a quadratic cost function is a good approximation to costs such as machine set-up costs [12]. An example of an economic problem with dissipation is given in Section 3-6. The difference in economic problems is that an adjustment cost *increases* the cost price where in mechanical problem an friction decreases the momentum, the economic Lagrange equations with dissipation are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{D}}{\partial \dot{q}} = 0 \tag{3-27}$$

3-6 Example from Income, Wealth and the Maximum Principle

In chapter 1 of Income, Wealth and the Maximum Principle[13] a widget entrepreneur is used as an example for which the calculus of variation can be used to obtain a simple investment policy. The example in the book can use some insights from an engineering point of view. The price defined in the book by P of a widget is given by the linear demand function

$$P(q) = \bar{P} - bq \tag{3-28}$$

Here q is quantity demanded in a month not per month. The revenue (or net income since no costs are made) per month can be written as

$$\mathcal{V} = \bar{P}q - bq^2 \tag{3-29}$$

The letter $\mathcal{V}$ is used to emphasize that this is equivalent to what is in Lagrangian mechanic known as the potential. The definition of price for P can be confusing since is has units €/(#T) so we must think of it as a net rental. One machine produces one widget and changing the number of machines ($I = \dot{q}$) has a cost price $c(t)$. A new machine can not operate immediately but has an adjustment time. Weitzman reasons that if I new machines arrive at the beginning of a period, the manager must spend a total time of τI on new robot adjustments, so τ has dimension $[T^2/\#]$. At adjustment time s, the manager will have adjusted s/τ machines meaning that the remaining $I - s/\tau$ are waiting to be adjusted. The total loss of widget production per period from the downtime is therefore

$$\int_0^{\tau I} (I - \frac{s}{\tau})\mathrm{d}t = \alpha I^2 \tag{3-30}$$

where $\alpha = \tau/2$ is called a cost-of-adjustment coefficient. But carefully checking the dimension of all the variables in Equation 3-30 gives that αI^2 must be the number of widgets [#] and does not have the required units of Euro per time. By changing the dimension of τ such that αI^2 has dimension [€] and by multiplying with the discount (or opportunity) factor ρ we obtain a cost with dimension Euro per unit of time. Multiplying with $e^{-\rho t}$ to make the adjustment cost time dependent, we obtain a time dependent adjustment cost of $e^{-\rho t}\rho\alpha I^2$ in [€/T]. The optimization problem for the entrepreneur is now

$$\text{minimize} \int_0^{\infty} \left(c(t)I(t) + \rho\alpha I(t)^2 + bq(t)^2 - \bar{P}q(t) \right) e^{-\rho t}\mathrm{d}t \tag{3-31}$$

instead of the optimization problem defined in the book:

$$\text{maximize} \int_0^{\infty} \left(-cI(t) - \alpha I(t)^2 - bq(t)^2 + \bar{P}q(t) \right) e^{-\rho t}\mathrm{d}t \tag{3-32}$$

Since we have identified the adjustment cost as a dissipation factor the solution to the Lagrange equation is

$$\dot{c} = 2bq - \bar{P} + \rho c + 2\rho\alpha I \tag{3-33}$$

If the firm was operating under no arbitrage conditions, the inventory value $\dot{c}(t)$ (or value for which the entrepreneur would sell the machine) was changing by the right hand side of the above equation. Now since the machines do not depreciated i.e. $\dot{c}(t) = 0$,

$$\rho = \frac{\bar{P} - 2bq}{c + 2\alpha I} \tag{3-34}$$

where the right hanside of this equation is the internal rate of return and a sound investment policy would be to (des)invest until the internal rate of return is equal to the opportunity cost i.e. until Equation 3-34 is satisfied for maximum profit. This is also defined in the book as the stationary rate of return on capital but the relation with the Lagrange equation is missing. Moreover Weitzman states on page 19 of [13] that

$$-\bar{P} + 2bq = 2\alpha\dot{I} - \rho(c + 2\alpha I) \tag{3-35}$$

is a necessary optimality condition when solving the problem using the maximum principle, but this equation has no mechanical or economic meaning. The constant 2α is treated like a mass to simulate a resistance to acceleration: $\dot{I}$. A possibility to account for adjustment time is to put a constraint on $\dot{q}$.

3-7 Example from Advanced Macro Economics

In section 9.2 of Advanced Macro Economics by Romer [14] a similar example of a profit maximizing firm with adjustment cost is given. To prevent ambiguity the variables in this thesis are changed, Table 3-1 shows the translation between this thesis and Advanced Macro Economics.

Description	Romer	Thesis
Discount factor	r	ρ
Capital	κ	q
Investments	I	I
Adjustment costs	$C(I)$	$\mathcal{D}(I)$
Investment cost	1	c
Shadow costs	q	λ
Industry wide capital	K	Q
Net income	$\pi(K)$	$R(Q)$

Table 3-1: Table of variable translation to Advanced Macro Economics

The adjustment cost $\mathcal{D}(I)$ satisfies $\mathcal{D}(0) = 0, \frac{\partial \mathcal{D}(0)}{\partial I} = 0, \frac{\partial^2 \mathcal{D}(\cdot)}{\partial I^2} > 0$ and $\dot{q} = I$. The objective is to maximize the present value

$$V = \int_0^\infty e^{-\rho t} \left(R(Q)q(t) - cI(t) - \mathcal{D}(I(t))\right) \mathrm{d}t \tag{3-36}$$

where Q is the industry-wide capital stock, we immediately notice that the adjustment cost does not have the correct dimension of Euro per unit of time. The shadow price $\lambda(t)$ is defined as

$$\lambda = c + \frac{\partial \mathcal{D}}{\partial I} \tag{3-37}$$

Which can be interpreted, according to Romer, that the cost of acquiring a unit of capital is equal to the purchase price c plus the marginal adjustment cost $\frac{\partial \mathcal{D}}{\partial I}$. Then he continues that the equation

$$R(Q) = \rho \left(c + \frac{\partial \mathcal{D}}{\partial I})\right) - \dot{c} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{D}}{\partial I}\right) = \rho\lambda - \dot{\lambda} \tag{3-38}$$

states that the marginal revenue $R(Q)$ is equal to its user cost $\rho\lambda - \dot{\lambda}$ (page 416). The adjustment costs are treated like kinetic energy with a mass to represent the resistance to $\dot{I}$ or $\ddot{q}$, the acceleration. This 'economic mass' does not behave like a mechanical mass i.e. when a larger rental (force) is applied on the mass, the acceleration $\dot{I}$ is proportional to this rental this is not necessary true in economics. Adjustment time does not have to decrease when a larger rental is applied on the 'economic mass'. Another method would be to treat the adjustment cost as a friction force and to constrain $\dot{I}$. When the definition of cost is adjusted in the same way as in the previous section the Euler-Lagrange equation with dissipation becomes

$$\dot{p}_c(t) = \rho p_c - R(Q) + \rho \frac{\partial \mathcal{D}}{\partial I} \tag{3-39}$$

with $p_c = c$ So the investment policy is here again to (des)invest until marginal revenue $(R(Q))$ equals marginal cost $\rho \left(p_c + \frac{\partial \mathcal{D}}{\partial I} \right)$.

We can alo look at this equation as how much value does the investment have for the firm, invest as long as

$$\frac{R(Q)}{\rho} - \frac{\partial \mathcal{D}}{\partial I} > c \tag{3-40}$$

So describing the adjustment cost as a friction a meaningful equation is obtained. To include adjustment time a constraint on $\dot{q}$ can be implemented.

Chapter 4

Hamilton's Equations

We have seen that when the Lagrangian formulation is used properly it can have great value in economic problems, another (and more powerful [9]) formulation used in mechanics is the Hamiltonian formulation. In this chapter the Hamiltonian is derived from the Lagrangian and several useful properties of the Hamiltonian are explained. In Chapter 5 the Hamiltonian formulation is used for deriving conservation laws in theoretical economic problems. And in Chapter 7 t is shown that a little modification of the Hamiltonian leads to Pontryagin's Maximum Principle which is an important principle in the field of optimal control theory.

4-1 Hamilton's Equations

The passage from one set of independent variables to another can be realized by a *Legendre Transformation* see the paper *Making Sense of the Legendre Transform[15] by Zia et al.* for a detailed description of the Legendre transform. The total differential of the Lagrangian as a function of capital $q(t)$, flow of capital $\dot{q}(t)$ and time t is,

$$\mathrm{d}\mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial q}\mathrm{d}q_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\mathrm{d}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \tag{4-1}$$

which is equal by the definitions of price and momentum to

$$\mathrm{d}\mathcal{L} = \sum_i \dot{p}_i\mathrm{d}q_i + \sum_i p_i\mathrm{d}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \tag{4-2}$$

Writing the second term of the right hand side of Equation 4-2 as

$$\sum_i p_i\mathrm{d}\dot{q}_i = \mathrm{d}\sum_i p_i\dot{q}_i - \sum_i \dot{q}_i\mathrm{d}p_i \tag{4-3}$$

With this relation the following equation can be obtained from Equation 4-2

$$\mathrm{d}\left(\sum p_i\dot{q}_i - \mathcal{L}\right) = -\sum_i \dot{p}_i\mathrm{d}q_i + \sum_i \dot{q}_i\mathrm{d}p_i - \frac{\partial \mathcal{L}}{\partial t} \tag{4-4}$$

$$\mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L} \tag{4-5}$$

$\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}$ is called the energy of the system, when this is expressed in terms of coordinates q and momenta p it is called the Hamiltonian ($\mathcal{H}$) of the system. The Hamiltonian is the kinetic energy plus the potential energy:

$$\mathcal{H} = \mathcal{T} + V \tag{4-6}$$

Or economically it are the costs plus profits. The equations of motions of a Hamiltonian system are given by Equation 4-7, because of their simplicity an symmetry they are also called canonical equations.

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial q} \\ \frac{\partial \mathcal{H}}{\partial p} \end{bmatrix} \tag{4-7}$$

The time derivative therefore takes the following special form

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial q}\dot{q} + \frac{\partial \mathcal{H}}{\partial p}\dot{p} = \frac{\partial \mathcal{H}}{\partial t} \tag{4-8}$$

And

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{4-9}$$

The canonical formulation of the Hamiltonian has a correlation with what is in economics known as the envelope theorem, in particular interest are Shephard's lemma and Hotelling's lemma for (static) microeconomic problems.

4-2 The Hamiltonian and Income

The Hamiltonian is in economics encountered when optimizing a cost function and it is has the unfortunate name of a *current value Hamiltonian* (see any economic textbook about optimization). Unfortunate, since the units of the Hamiltonian are Euro per unit of time and not Euro as the word value would suggest. The term value is used for the integral of the Hamiltonian over a period. The economic description in this thesis of the Hamiltonian is the income in Euro per unit time. This because when a new investment $\dot{q}$ is made, this increases the value of the firm with the price p of the investment. Notice that there is no restriction on the income about how this investment is added to the firm. Now we have reached another economic theorem known as Modigliani-Miller theorem which states that in perfect conditions the value of the firm is unaffected by its choice of capital structure (self funded or borrowed, unlevered or levered)[16]. This definition of income is not new in economics, in fact the Marshall-Haig-Kuznets' definition of income (p. 82 of [17]) is the underlined part of Equation 4-10.

$$\mathcal{H}_c = e^{-\rho t} \underbrace{\left(U - \frac{\partial U}{\partial \dot{q}}\dot{q} \right)}_{\text{Income}} \tag{4-10}$$

or in words when the discount factor is omitted,

$$\text{Income} = \text{Utility} + \text{Utility value of investment} \tag{4-11}$$

because the society is maximizing the utility function the consumer Hamiltonian $\mathcal{H}_c$ is the negative of the firms Hamiltonian $\mathcal{H}$ and the supply price is defined by

$$p_s \triangleq -\frac{\partial U}{\partial \dot{q}} \tag{4-12}$$

because the consumer is maximizing utility not minimizing cost. The integral of the firms income is the value V,

$$V(0,t) = \int_0^t \mathcal{H}\mathrm{d}t \tag{4-13}$$

The integral of the Consumer Hamiltonian is the Wealth

$$\mathcal{W}(0,t) = \int_0^t \mathcal{H}_c\mathrm{d}t \tag{4-14}$$

There are some very useful properties of the Hamiltonian which are described in the next sections.

4-3 Poisson Brackets

The use of Poisson brackets is very useful in finding an integral of the motion. [Section is based on paragraph 42 of [8]] Let $f(q,p,t)$ be a function of capital, price and time then the time derivative is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} \tag{4-15}$$

substituting $\dot{q}$ and $\dot{p}$ given by the Hamiltonian equation leads to the expression

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + [\mathcal{H}, f] \tag{4-16}$$

Where

$$[\mathcal{H}, f] = \frac{\partial \mathcal{H}}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial \mathcal{H}}{\partial q}\frac{\partial f}{\partial p} \tag{4-17}$$

and is called the Poisson bracket of $\mathcal{H}$ and f. For f to be an integral of the motion ($\frac{\mathrm{d}f}{\mathrm{d}t} = 0$ the following condition must hold

$$\frac{\partial f}{\partial t} + [\mathcal{H}, f] = 0 \tag{4-18}$$

An integral of the motion is a function of dynamic variables which remain constant during the motion of a system. An important property of the Poisson bracket is that, if the functions f and g are two integrals of the motion, their Poisson bracket is an integral of the motion

$$[f, g] = \text{constant} \tag{4-19}$$

See Landau page 137 for a proof of this and for properties of the Poisson bracket.

4-4 Hamilton-Jacobi equation

Another formulation which often used in classical mechanics is the Hamilton - Jacobi equation which will be derived in this section. Suppose the variation $\delta q(t_2)$ in Equation 4-20 is not equal to zero but equal to δq, a small variation of the number of goods q at time t_2 . And the integral term of the equation vanishes by minimization.

$$\delta S = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q\ dt = 0 \tag{4-20}$$

then (p. 138 of [8])

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \tag{4-21}$$

Which states that increasing the inventory of an agent with one asset (q) the increase in accrued cost is equal to the price. From this relation it follows that the partial derivatives of the cost with respect to the coordinates are equal to

$$\frac{\partial S}{\partial q} = p \tag{4-22}$$

It is important to notice that this relation only holds when the no arbitrage condition is true i.e.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{4-23}$$

And by

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i \tag{4-24}$$

From this we obtain [8]

$$\frac{\partial S}{\partial t} + \mathcal{H}(p, q, t) = 0 \tag{4-25}$$

replacing the prices by $\frac{\partial S}{\partial q}$ the following equations is obtained,

$$\frac{\partial S}{\partial t} + \mathcal{H}(q, \frac{\partial S}{\partial q}, t) = 0 \tag{4-26}$$

Which is the Hamilton - Jacobi Equation and a complete integral of this can be found by seperating the variables, see paragraph 48 of Landau [8].

Chapter 5

Two Conservation Laws in Economics

The first example is based on *Law of Conservation of the Capital-Output Ratio in Closed von Neumann Systems by Paul A. Samuelson*[18] and *Two Conservation Laws in Theoretical Economics by Paul A. Samuelson*[19]. In this article Samuelson derives a conservation law in a closed von Neumann system with intertemporal efficiency. The second example is based on *The Invariance Principle and Income-Wealth Conservation Laws by R. Sato*[17].

5-1 Von Neumann model for a two Commodity Economy

This section is written in spirit of section 7.5 from *Symmetry and Economic Invariance - R. Sato* [20]. The closed Von Neumann model can best be explained for an economy with two commodities (capital goods) with help of Figure 5-1. Starting at an initial capital $(K_1(0), K_2(0))$ the curve t_1 are the possible combinations which can be efficiently reached at time $t = 1$. When the point P_1 is reached at time $t = 1$ the efficient reachable curve is t_2. If at $t = 1$, the economy was not in point P_1 there would be a different efficient curve t_2' (not shown) and the envelope of all these curves is the curve T. The objective is to maximize $K_1(T)$ given $K_2(T)$, so if the economy reached P_1 at $(K_1(1), K_2(1))$, the maximum obtainable efficient allocations is the coordinate $K_1(2), K_2(2)$. But from the envelope it is clear that there is a better $(\bar{K}_1(2), K_2(2))$ possible. So mathematically the objective is:

$$\text{maximize} \int_0^T \dot{K}_1(t)\ \mathrm{d}t \quad \text{s.t.} \quad F(K, \dot{K}) = 0 = C(t) \tag{5-1}$$

For given initial and boundary conditions and $F(K, \dot{K})$ the efficient transformation function. Another way of describing the problem is that the system starting at an initial capital (vector) $K(0)$ must reach any terminal capital (vector) $K(T)$ in minimum elapsed time. This is similar to the challenge issued by Bernoulli in 1696, the Brachystochrone problem ('shortest time problem') which according to H.J. Sussman and J.C Willems marks the birth of optimal contol [21]. The transformation function is assumed to be homogeneous-first-degree, concave

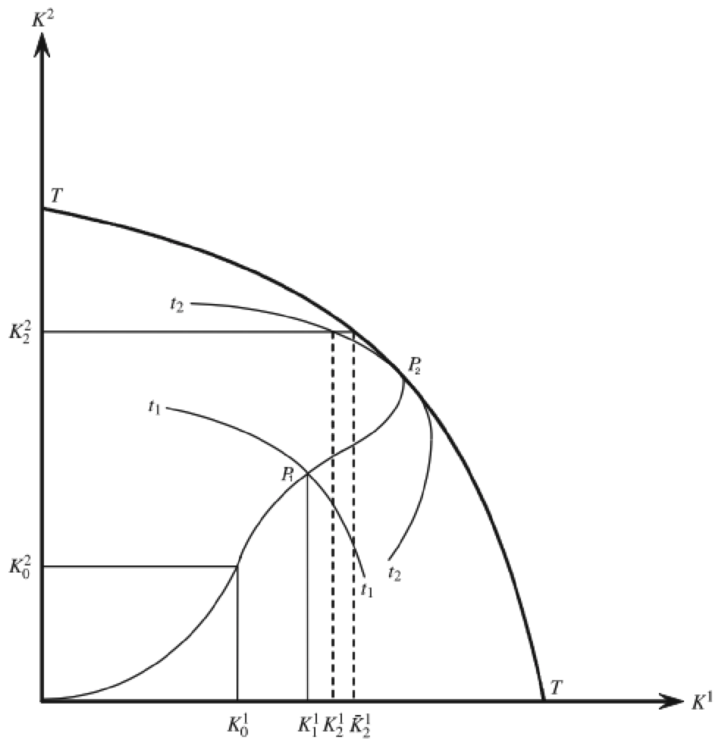

Figure 5-1: Representation of Von Neumann Model [20]

and smoothly differentiable with the following properties [18]

$$-\frac{\partial F}{\partial \dot{K}} = P, \qquad \text{the (cost) price of the capital good} \tag{5-2}$$

$$\frac{\partial F}{\partial K} = R, \qquad \text{the (net) rental of the capital good} \tag{5-3}$$

$$Y = P\dot{K} = RK, \qquad \text{national product = national income} \tag{5-4}$$

$$W = PK, \qquad \text{national wealth} \tag{5-5}$$

$$r = \frac{\dot{W}}{W}, \qquad \text{the interest rate when } C(t) \triangleq 0 \tag{5-6}$$

In his paper Samuelson derives conservation laws for income growth:

$$Y(t) = e^{\int_0^t r(s)\mathrm{d}s} Y(0) \tag{5-7}$$

and wealth growth

$$W(t) = e^{\int_0^t r(s)\mathrm{d}s} W(0) \tag{5-8}$$

from which he derives the fundamental economic law of conservation of capital-output ratio,

$$\frac{W(t)}{Y(t)} = \frac{W(0)}{Y(0)} \tag{5-9}$$

these conclusions can also be reached when the mechanical approach described in this thesis is used. The great difference is that it is no longer a least time problem but a least cost.

5-2 Mechanical Approach

Before the Lagrangian of the problem is stated, some properties of differentiating integrals is needed. Suppose a function f is continuous on $[a, b]$ then by the fundamental theorem of calculus [22]:

$$\text{if } g(x) = \int_a^x f(t)\mathrm{d}t, \text{ then } g'(x) = f(x). \tag{5-10}$$

So the derivative of the function $f(t) = x(t)e^{-\int_0^t r(s)\mathrm{d}s}$ is by use of the chain rule for exponential functions

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = e^{-\int_0^t r(s)\mathrm{d}s}\frac{\mathrm{d}x(t)}{\mathrm{d}t} + x(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\int_0^t r(s)\mathrm{d}s}\right) \tag{5-11}$$

$$= e^{-\int_0^t r(s)\mathrm{d}s}\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t} - r(t)x(t)\right) \tag{5-12}$$

Now with this information we have the following cost Lagrangian:

$$\mathcal{L} = e^{-\int_0^t \rho(s)\mathrm{d}s} f(q, \dot{q}) \tag{5-13}$$

And the function to minimize is the accrued cost S over the period $[0, T]$ where S is

$$S = \int_0^T e^{-\int_0^t \rho(s)\mathrm{d}s}\mathcal{F}(q, \dot{q})\mathrm{d}t \tag{5-14}$$

With the cost price and net rental definitions as defined in chapter 3-4.

$$p_c(t) = \frac{\partial \mathcal{F}}{\partial \dot{q}} \tag{5-15}$$

$$R(t) = -\frac{\partial \mathcal{F}}{\partial q} \tag{5-16}$$

The reason this Lagrangian is used is that it gives the right (same) equations of motion as in Samuelson's constraint problem which are

$$\dot{p}_c(t) - \rho(t)p_c(t) + R(t) = 0 \tag{5-17}$$

The Hamiltonian, or discounted income, in the von Neumann economy is

$$\mathcal{H} = e^{-\int_0^t \rho(s)\mathrm{d}s} p_c\dot{q} - \mathcal{L} \tag{5-18}$$

By the property of the Hamiltonian:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{5-19}$$

since the von Neumann economy is a consumptionless economy $f(q, \dot{q}) = 0$, the Lagrangian is zero and so is the time derivative of the Hamiltonian which means that the discounted income is constant.

$$\mathcal{H} = e^{-\int_0^t \rho(s)\mathrm{d}s} p_c\dot{q} - \mathcal{L} = \text{constant} \tag{5-20}$$

$$= e^{-\int_0^t \rho(s)\mathrm{d}s} p_c\dot{q} - 0 = \text{constant} \tag{5-21}$$

so the income at $t = 0$ $Y(0) = p(0)\dot{q}(0)$ which is the equal to Samuelson's definition of income grows with factor $e^{\int_0^t \rho(s)\mathrm{d}s}$ or:

$$Y(t) = Y(0)e^{\int_0^t \rho(s)\mathrm{d}s} \tag{5-22}$$

This conservation laws says that: Total income along any optimal path grows in a closed system at the variable compound interest yield of the system[19]. The second conservation law of capital-output ration can be derived using Liouville's theorem and will now be shown. It can be shown that $\mathcal{W}(t) = p(t)q(t)$ is a conserved density in phase space, where $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. The following relation must be true:

$$\frac{\partial \mathcal{W}}{\partial t} = -[\mathcal{W}, \mathcal{H}] \tag{5-23}$$

where $[\mathcal{W}, \mathcal{H}]$ is the Poisson bracket of $\mathcal{W}$ and $\mathcal{H}$. This boils down to

$$\frac{\partial \mathcal{W}}{\partial t} = -\frac{\partial \mathcal{W}}{\partial p}\frac{\partial \mathcal{H}}{\partial q} + \frac{\partial \mathcal{W}}{\partial q}\frac{\partial \mathcal{H}}{\partial p} \tag{5-24}$$

$$= e^{-\int_0^t \rho(s)\mathrm{d}s}(-qR + p_c\dot{q}) \tag{5-25}$$

$$0 = 0 \tag{5-26}$$

Because $-qR + p_c\dot{q} = 0$, national income equals national product. So $p(t)q(t)$ is a conserved density in phase space and wealth is growing with the common rate of interest, $W(t) = e^{\rho t}p_c q$. It is interesting that the wealth is defined by Samuelson as the integral of the Hamiltonian times the common rate of interest $e^{-\int_0^t \rho(s)\mathrm{d}s}$

$$\int \mathcal{H}\mathrm{d}t = \int p\mathrm{d}q - \int \mathcal{L}\mathrm{d}t \tag{5-27}$$

$$= \int p\mathrm{d}q \tag{5-28}$$

$$= e^{-\int_0^t \rho(s)\mathrm{d}s}p_c q \tag{5-29}$$

$$\mathcal{W}e^{\int_0^t \rho(s)\mathrm{d}s} = p_c q \tag{5-30}$$

but when considering the utility function, wealth is the integral of the Lagrangian, not the Hamiltonian.

5-3 Conservation of Utility Income

Let utility U depend on consumption of many goods which in turn depend on a vector of capital goods q and a vector of investments $\dot{q}$, so that

$$U = U(q, \dot{q}) \tag{5-31}$$

Let $U(q, \dot{q})$ be a strictly concave function with existent partial derivatives. The society's problem is to maximize the welfare functional

$$W = \int_0^\infty e^{-\rho t}U(q, \dot{q})\ \mathrm{d}t \tag{5-32}$$

The supply price of the capital is defined as

$$p_s = -\frac{\partial U}{\partial \dot{q}} \tag{5-33}$$

And the income in utility terms according to the Marshall-Haig-Kuznets' definition is (p. 82 of [17])

$$Y = U - \frac{\partial U}{\partial \dot{q}}\dot{q} \tag{5-34}$$

Which is clearly the negative Hamiltonian of $U(q, \dot{q})$. With (Landau p. 133 [8]):

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{5-35}$$

it is easy to identify the following relations

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\rho t}\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right)\right] \tag{5-36}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\rho t}\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right)\right] = e^{-\rho t}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right) - \rho\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right)\right] \tag{5-37}$$

$$= \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} = \rho e^{-\rho t} U(q,\dot{q}) \tag{5-38}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right) = \rho\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right) + \rho U(q,\dot{q}) \tag{5-39}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right) = \rho\left(\frac{\partial U}{\partial \dot{q}}\dot{q}\right) \tag{5-40}$$

Or in economic terms:

$$\text{(rate of change in utility income at } t) = \rho \times \text{(utility value of investment at } t) \tag{5-41}$$

When the discount factor $\rho > 0$ we can derive one of the most important conservation laws in neo-classical economics: the income-wealth conservation[23],

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\rho t}\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right)\right] = \rho e^{-\rho t} U(q,\dot{q}) \tag{5-42}$$

Integrating both sides between t and ∞ yields,

$$-e^{-\rho t}\left(\frac{\partial U}{\partial \dot{q}}\dot{q} - U(q,\dot{q})\right) = \int_t^\infty \rho e^{-\rho s} U(q,\dot{q})\,\mathrm{d}s \tag{5-43}$$

This holds since the left hand side of Equation 5-43 goes to zero if $t \to \infty$, multiplying both sides with $e^{-\rho t}$ gives the relation utility income is ρ times utility wealth,

$$\left(U(q,\dot{q}) - \frac{\partial U}{\partial \dot{q}}\dot{q}\right) = \rho \int_t^\infty e^{-\rho(s-t)} U(s)\mathrm{d}s \tag{5-44}$$

$$\text{Income} = \rho \times \text{Wealth} \tag{5-45}$$

Chapter 6

Price and Momentum

So far we have seen that there are some useful similarities between Lagrangian mechanics and economic systems. Now for a mechanical model to represent an economic model, price p must be a function of the flow $\dot{q}$ just like momentum is a function of generalized velocity $\dot{q}$ and the laws of motion must hold for the economic model.

6-1 Prices as Function of Demand

One of the most well known figures in economics is figure 6-1, it shows the relation between price (P), supply (S) and demand (D). A high price corresponds to a low demand and high supply, a low price corresponds to a high demand and low supply. The change in demand curve, depicted by an arrow from D_1 to D_2, is called a shift in demand curve which can be an effect of an increase in income of the consumer. Although the horizontal axes is labelled quantity, this should be quantity per unit of time. A consumer does not demand 5 apples ($5q$), the consumer demands 5 apples per month ($5\dot{q}$). Contrary to the usual formulation the

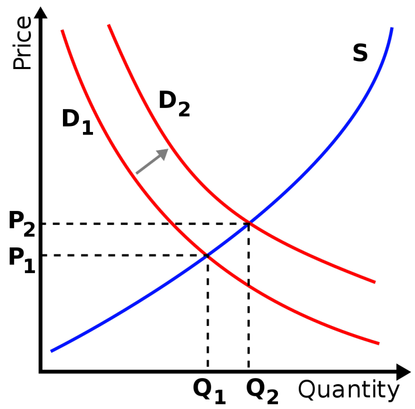

Figure 6-1: Supply and demand curve with shifted demand[24]

variable is on the vertical axis, this means that demand and supply are functions of price and the equilibrium price is where supply equals demand, at P_1 or P_2. Now consider the simple supply and demand graph of Figure 6-2, the direction of the supply line will be denoted with m such that the price is given by:

$$p = m\dot{q} \tag{6-1}$$

when m is considered constant i.e. there are no technological changes or any other influences which can change the supply curve, then a shift in demand (the red dotted line) means that the price increases. Equation 6-1 is price, as a function of demand $\dot{q}$ in mechanics this equation is known as momentum. So a possible economic equivalent of mechanical momentum is price. But do the laws of motion hold for price?

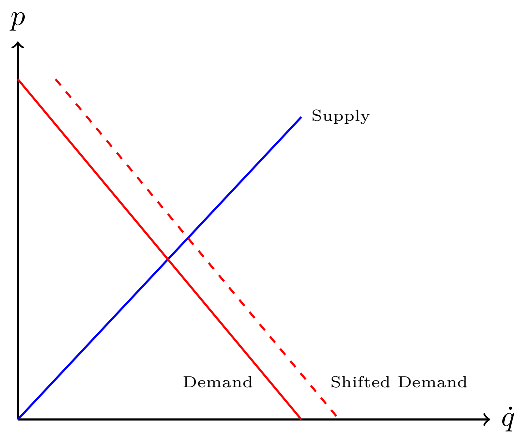

Figure 6-2: Simple Supply and Demand curve with shifted demand

6-2 Newton's Second Law

Newton's second law states that the rate of change of the momentum of a particle is proportional to the force F acting on the particle [25]:

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{6-2}$$

Or in economic terms, the rate of change in price is proportional to the net rental cost (or cost of carry) acting on it, which is an economically sound statement. When the price is a linear time invariant function of demand ($\dot{q}$) so price is

$$p = m\dot{q} \tag{6-3}$$

then a high demand corresponds to a high price and low demand corresponds to a low price, just like a high velocity corresponds to a high momentum and low velocity corresponds to a low momentum. The difference between the economic and mechanical model is that increasing the price results in a decrease in demand, where increasing momentum results in an increase in generalized velocity. Economically this is only true when the underlying good is a Giffen good: a product for which the demand increases when the price increases and vice versa. A similar problem occurs on the supply side, increasing the supply would decrease the price but increasing the price would increase supply. From the previous section it is clear that the price increases when there is an increase in demand, this means that the 'force' F, defined as net rental cost, should represent a force which shifts demand curves. Finding a constant m in reality can be hard because the supply curve is usually not time invariant, technological changes and governmental influences for example can change the supply curve over time, this means that obtaining a linear time invariant model of an economic system will be hard. The acceleration $\ddot{q}$ is dependent on the constant m so this tells something about the time it takes to reach the new price and m would in this sense be an adjustment time or a resistance to price change.

6-3 Second Order Model

Suppose an economic model is obtained where the price behaves just like momentum as stated in the previous section. And there is a something that behaves like a spring such that the cost Lagrangian is given by:

$$\mathcal{L} = e^{-\rho t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right) \tag{6-4}$$

The discount factor is in the Lagrangian because of the time value of money. The differential equation belonging to this Lagrangian is

$$m\ddot{q} - \rho m\dot{q} + kq = 0 \tag{6-5}$$

which has eigenvalues $\lambda_{1,2}$ at

$$\lambda_{1,2} = \frac{\rho m \pm \sqrt{(\rho m)^2 - 4mk}}{2m} \tag{6-6}$$

since the discount factor is (almost) always positive the eigenvalues are located in the right half plane, this is an unstable system and does not represent an economic problem in general. To compare economic and mechanical models Equation 6-4 can not be used, a different reference frame where prices can be zero and negative and where the discount takes a different (probably complex) form might be a possibility. But just because prices oscillate they do not have to be the equivalent of a mass spring system. A sinus input on a mass for example can have the same oscillating effect on momentum, this does not make it a mass spring system.

Chapter 7

Optimal Control From Lagrangian and Hamiltonian to the Maximum Principle

In this chapter it is shown how a small modification of the Lagrangian and Hamiltonian leads to Pontryagin's Maximum Principle, an important principle in the field of optimal control theory which can also be applied on economic problems. This chapter is based on *300 Years of Optimal Control: From The Brachystochrone to the Maximum Principle by J. Willems and H. Sussmann*[21].

7-1 Lagrangian Revisited

So far we have dealt with optimization problems by either minimizing or maximizing the following form:

$$\text{minimize } S = \int_{t_1}^{t_2} \mathcal{L}(q(t), u(t), t)\mathrm{d}t \tag{7-1}$$

subjected tot initial conditions $q(t_1)$ boundary conditions $q(t_2)$ and $\dot{q}(t) = u(t)$ to make clear that $u(t)$ and $q(t)$ are treated as independent variables. The minimization takes place over all curves where in optimal control problems a set of curves, determined by dynamical constraints, are minimized. For example the set of curves that satisfy the a differential equation:

$$\dot{q}(t) = f(q(t), u(t), t) \tag{7-2}$$

for a control function $u(t)$. Such constraints arise both in macro- and micro economic problems [26], for example the capital accumulation equation in the macro-economic Ramsey-Cass-Koopmans model is

$$\dot{q} = f(q) - (n + \delta)q - C \tag{7-3}$$

the change in capital (per worker) $\dot{q}$ is the output for a given capital, $f(q)$ which is a production function, minus the depreciation rate δ of capital minus consumption (C) and n is a constant growth rate in the model. a micro-economic example is

$$\dot{q} = I - \delta q \tag{7-4}$$

where I are investments fulfilling the role as a control variable and δq is again the depreciation. Problems where the Lagrangian $\mathcal{L} \triangleq 1$,so where are interesting action occurs because of the dynamics of f are called minimum time problems, like the original Von Neumann model where the agent must reach a certain configuration of capital goods in minimum time.. The Euler-Lagrange equations which follow from the variational problem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial u}(q, \dot{q}, t)\right) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}, t) = 0 \tag{7-5}$$

only gave conditions for the first variation of S, the expenditure (or Action), to be zero. An additional necessary condition for a minimum was found by Legendre which is:

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(q, \dot{q}, t) \geq 0 \tag{7-6}$$

Also known as the Legendre-Clebsch condition. In the Lagrangian equation $\dot{q}$ is now a function of the input and the state q,

$$\dot{q}(t) = f(q(t), u(t), t) \tag{7-7}$$

We have now changed the original Lagrangian equation which is the first step to the Maximum principle.

7-2 Classical Hamiltonian to Control Hamiltonian

In the classical Hamiltonian defined by:

$$\mathcal{H}(q, p, t) = p\dot{q} - \mathcal{L} \tag{7-8}$$

the variable $\dot{q}$ is not treated as an independent variable but as a function of p, q and t and as shown in the previous section defining the demand,$\dot{q}$, as a function of the price does not give the right equations of motion with the Lagrangian used in this thesis. But fortunately in the control Hamiltonian defined by Equation 7-9, the input variable is treated as an independent variable.

$$H(q, u, p, t) = pu - \mathcal{L}(q, u, t) \tag{7-9}$$

and we have

$$p(t) \triangleq \frac{\partial \mathcal{L}}{\partial u}(q, \dot{q}, t) \tag{7-10}$$

So $\frac{\partial H}{\partial p} = u$ and

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}(q, \dot{q}, p, t) \tag{7-11}$$

also $\frac{\partial H}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q}$ so

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}(q, \dot{q}, p, t) \tag{7-12}$$

And finally we have,

$$\frac{\partial H}{\partial u} = p - \frac{\partial \mathcal{L}}{\partial u} = 0 \tag{7-13}$$

Since $H(q, u, p, t)$ is equal to $-\mathcal{L}(q, u, t)$ plus a linear term, the Legendre condition for a minimum, Equation 7-6, becomes

$$\frac{\partial^2 H}{\partial u^2}(q, \dot{q}, t) \leq 0 \tag{7-14}$$

by Equation 7-13 it is clear that Equation 7-14 is true so the control Hamiltonian must have a maximum as a function of u. The advantages is that this new control Hamiltonian is equivalent to the Euler-Lagrange system but the transformation :

$$p(t) \triangleq \frac{\partial \mathcal{L}}{\partial u}(q, \dot{q}, t) \tag{7-15}$$

does not have to be invertible. Which is, again, very useful for economic problems. This formulation will lead to Pontryagin's Maximum Principle.

7-3 Pontryagin's Maximum Principle

Applications of Pontryagin's Maximum Principle to economic problems date back to (at least) 1968, where K. Shell applied the principle on an economic growth model [27], but also more recently its usefulness in economic problems is shown [28]. Equation 7-9 is almost Pontryagin's Maximum Principle, a 'abnormal multiplier' p_0 must be added to the function and u must be written as the constraint function to obtain:

$$H(q, u, p, p_0, t) = < p, f(p, q, t) > -p_0\mathcal{L}(q, u, t) \tag{7-16}$$

$<, >$ is the Euclidean inner product. Now to solve the dynamic constraint minimization problem for a control parameter u in the admissible control set U $(u \in U)$ in the fixed time interval $[a, b]$ and for a constant $p_0 \geq 0$, several conditions must hold (for the optimal p^*):

$$(p(t), p_0) \neq (0, 0) \forall\ t \in [a, b] \tag{7-17}$$

$$\dot{q}(t) = \frac{\partial H}{\partial p} \text{ for } t \in [a, b] \tag{7-18}$$

$$\dot{p}(t) = -\frac{\partial H}{\partial q} \text{ for } t \in [a, b] \tag{7-19}$$

(7-20)

For further information about Pontryagin's Maximum Principle the author refers to *300 Years of Optimal Control: From The Brachystochrone to the Maximum Principle by J. Willems and H. Sussmann*[21] since this is beyond the scope of this thesis.

A simple example from p. 122 of Caputo M.R., Foundations of Dynamic Economic Analysis Optimal Control Theory and Applications. [29] shows that this formulation has great use in economic problem. In the example the control problem to be solved for an agent is to determine the optimal inventory accumulation policy given

$$\min_u S = \int_o^T c_1[u(t)]^2 + c_2 q(t)\mathrm{d}t \tag{7-21}$$

$$\text{s.t. } \dot{q}(t) = u(t), \tag{7-22}$$

$$q(0) = 0,\ q(T) = q_T \tag{7-23}$$

Where c_1 and c_2 are defined as costs and q is as usual the capital. So the control Hamiltonian that can be constructed from this problem with $p_0 = 1$,

$$H(q, u, p) = pu - c_1[u(t)]^2 - c_2 q(t) \tag{7-24}$$

Without solving the optimal control problem we can already identify our definition of cost price $p = 2c_1 u$ (or shadow price or marginal price as it is often called since it is dependent on the variable $u(t)$). And our definition of rental (or storage) cost $F = c_2$, what Caputo indeed calls the cost of storing the capital good but it is not mentioned in what units this is measured. We know now by the definitions at the beginning of this thesis that this must be in Euro per good per unit time.

Chapter 8

Conclusion

Some great insights can be obtained by the proper Lagrangian and Hamiltonian approach in economic systems, so are the concept of energy and the principle of least action less abstract when considering their economical equivalents. And economic concepts such as rental can be described by their mechanical equivalent. The Hamiltonian and Lagrangian formulation can be used to obtain conservation laws in theoretical economic problems and Pontryagin's maximum principle can be used as a tool for economic optimization problems where the units of the variables used in the Lagrangian are now clearly defined. But when price is described as a function of the flow $\dot{q}$ i.e. a function of demand, the Lagrangian description with its definitions used in this thesis and a discount function of $e^{-\rho t}$ one does not obtain a economical equivalent of a mass spring system (a second order model). The obtained second order model is unstable. So in order to find an economic Lagrangian which is equivalent to the mechanical one, one must find a way to include a new kind of discount factor in the equation i.e. if you want to find a second order model, because observing an oscillating price does not automatically make it a mass spring system. Another point of interest is when the price is described as $p = m\dot{q}$ then increasing the price would increase demand, which is only true when the good is a Giffen good. The more common effect is that when price increases, demand decreases. So with this method the economic equivalents of a mass, spring and damper can not be found.

Chapter 9

Recommendations

Although a second order model which behaves like a mas spring system can not be obtained using this method, it can be a very powerful tool when optimizing economic problems. The economic optimization books read by the author did not have a good explanation why it worked for economic problems and the dimension analyse was often not there. This thesis can be a foundation on how to properly formulate economic problems so that it can be solved using Pontryagin's maximum principle, some more in depth research is needed here. Also if one want to obtain a second order with price as a function of demand $\dot{q}$ and a rental dependent on q such that a mass spring like second order model is obtained, a different discount factor is needed and some different definitions of the variables such that the laws of motion also hold for economic problems. However, the added economic value of this is not clear to the author since economics is anything but linear and time invariant.

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