In- and out-of-plane response of a stretched string due to an in-plane harmonic excitation

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Abstract

In this paper a stretched string which is attached between a fixed support and a vibrating support will be considered. It is assumed that the vibrating support only vibrates in a plane. The vibrating support causes a parametrical and vertical (that is, perpendicular to the X-axis) excitation of the string. For an excitation-frequency near an eigenvalue $\mu_m = m\pi$ of the unperturbed system (m is an odd positive integer) an 1:1 resonance can occur between the m-th modes in- and out-of plane. Whereas, when the excitation-frequency is near μ_m (but m is now an even number) the interaction between a parametric and a normal mode in- and out-of-plane can occur. The study in this paper will be focused on the existence and the stability of (almost) periodic solutions. Several cases with and without damping will be investigated. The effect of the frequency and the amplitude of the excitation force on the existence and the stability of the (almost) periodic solutions are studied analytically by using the averaging method.

1 Introduction

The vibrations of the stay-cables of cable-stayed-bridges have recently obtained a lot of attention in the literature. Usually the stay-cables are attached to a pylon at one end and at the other end to the bridge deck. Therefore the cable-stayed-bridge can become prone to vibrations due to wind, rain, traffic, etc. The cable vibrations have been extensively analyzed in the literature. Pinto da Costa et.al. [1] studied the oscillations of bridge-stay-cables induced by periodic motions of the deck and/or the towers. They showed that reasonably small anchoraged amplitudes may lead to important cable oscillations when conditions are met for the lower-order classical or parametrical resonance of the cables. Jones and Scanlan [2] studied the wind effects on cable-supported bridges and provided an overview of the basic steps in the process of a typical aerodynamic analysis and design. The other papers related to this subject can be found in Refs. [3, 4, 5].
The stay cable could be modelled as a stretched string. Most of the studies of oscillations of

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a stretched string have mainly focused on the in-plane motion of the string [6, 7, 8, 9]. On the other hand, whenever the excitation-frequency falls in a certain resonance range the string movement in the plane can become unstable, and can lead to stable non-planar oscillations [10, 11, 12, 13, 14, 15]. These phenomena have been shown by Matsumoto [16]. He did an experiment to observe the aerodynamic characteristics of an "inclined circular cylinder" of a cable-stayed-bridge when the movement was induced by rain or rain-wind. One of the results is that the flow stagnation point of the inclined circular cylinder determines the aerodynamically most sensitive vibrational direction so that the stay cable can vibrate in a coupled mode in and out of the cable-plane. This shows that the study of non-planar motion of a string in resonance can be of interest to engineers.

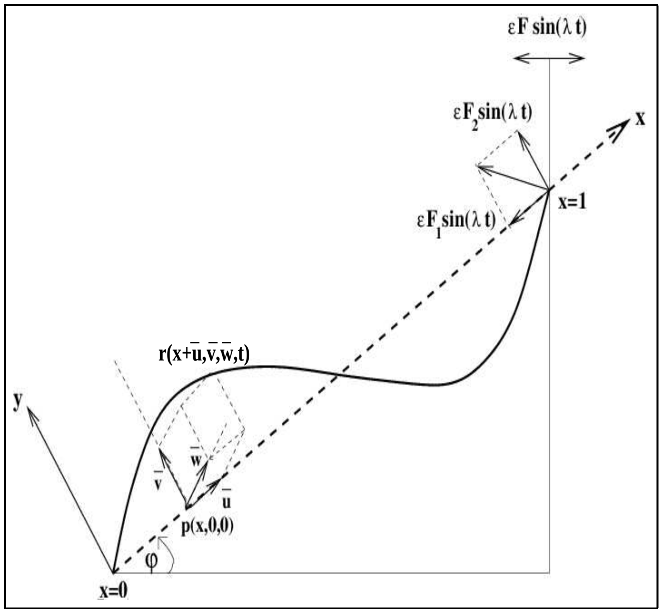


Figure 1: The inclined string in the dynamic state including normal and parametrical excitation at $x = 1$.

In the papers [17, 18] the in- and out-of-plane excitation of an inclined elastic cable has been investigated. Numerical results are presented to describe the transverse responses in- and out-of-plane due to an external forcing which is distributed along the cable. The numerical results, however, can not explain the periodic motion completely. In this paper the dynamics of a stretched string will be studied analytically. As a model the string is attached to a fixed support and to a vibrating support (see: Fig. 1). Here, it is assumed that the vibrating support at $x = 1$ can not vibrate in the w-direction. Due to the inclination the vibrating support produces an excitation along the string (the so-called parametric excitation) and an excitation in vertical direction (the so-called normal excitation). It should be observed that both excitations act only at the end point.
The mathematical model for this system is an extension of the model as given in [20]. The equations of motion for this model are given by:

$$\begin{aligned} v_{\bar{t}\bar{t}} - v_{xx} &= \varepsilon\Big(-\alpha_1 v_{\bar{t}} + v_{xx}\big[\frac{1}{2}\int_0^1 (v_x^2 + w_x^2)dx + F_1 \sin(\lambda\bar{t})\big]\Big), \\ w_{\bar{t}\bar{t}} - w_{xx} &= \varepsilon\Big(-\alpha_2 w_{\bar{t}} + w_{xx}\Big[\frac{1}{2}\int_0^1 (v_x^2 + w_x^2)dx + F_1 \sin(\lambda\bar{t})\Big]\Big), \end{aligned} \tag{1.1}$$

for $0 < x < 1$ and $\bar{t} > 0$, with initial conditions and boundary conditions:

$$\begin{aligned}
&\text{IC's:} \quad v(x,0) = \bar{v}_0(x), v_{\bar{t}}(x,0) = \bar{v}_1(x), w(x,0) = \bar{w}_0(x), w_{\bar{t}}(x,0) = \bar{w}_1(x),\\
&\text{BC's:} \quad v(0,\bar{t}) = w(0,\bar{t}) = w(1,\bar{t}) = 0, v(1,\bar{t}) = \varepsilon F_2 \sin(\lambda \bar{t}),
\end{aligned} \tag{1.2}$$

where $0 < \varepsilon << 1$, $\alpha_1, \alpha_2 \geq 0$, $\lambda > 0$, $v(x,\bar{t})$ and $w(x,\bar{t})$ are the transversal displacement in- and out-of-plane, respectively. In (1.1) gravity is not considered, implying that there is no sag. Hence the parametric excitation only applies to elastic elongation. Therefore, this model is only of particular relevance for shorter cable-stayed-bridges. In this paper the averaging method [19] will be applied to study the existence and the stability of (almost) periodic solutions of (1.1). Bifurcation diagrams will be presented to show the complicated behaviour of the solutions.

2 Analysis of the problem

Consider the two-point boundary value problem for the partial differential equations (1.1). To have a problem with homogeneous boundary conditions the following transformation is introduced:

$$v(x,\bar{t}) = \bar{v}(x,\bar{t}) + \varepsilon F_2 x \sin(\lambda \bar{t}) \quad \text{and} \quad w(x,\bar{t}) = \bar{w}(x,\bar{t}). \tag{2.1}$$

Substitution of (2.1) into (1.1) and (1.2) yields (up to $O(\varepsilon)$):

$$\begin{aligned}
&\bar{v}_{\bar{t}\bar{t}} - \bar{v}_{xx} = \varepsilon\Big(-\alpha_1 \bar{v}_{\bar{t}} + \lambda^2 F_2 x \sin(\lambda \bar{t}) + \bar{v}_{xx}\Big[\frac{1}{2}\int_0^1 (\bar{v}_x^2 + \bar{w}_x^2)dx + F_1 \sin(\lambda \bar{t})\Big]\Big),\\
&\bar{w}_{\bar{t}\bar{t}} - \bar{w}_{xx} = \varepsilon\Big(-\alpha_2 \bar{w}_{\bar{t}} + \bar{w}_{xx}\Big[\frac{1}{2}\int_0^1 (\bar{v}_x^2 + \bar{w}_x^2)dx + F_1 \sin(\lambda \bar{t})\Big]\Big),\\
&\bar{v}(x,0) = \bar{v}_0(x),\ \bar{w}(x,0) = \bar{w}_0(x),\ \bar{v}_{\bar{t}}(x,0) = \bar{v}_1(x,0) - \varepsilon\lambda F_2 x,\ \bar{w}_{\bar{t}}(x,0) = \bar{w}_1(x),\\
&\bar{v}(0,\bar{t}) = \bar{v}(1,\bar{t}) = \bar{w}(0,\bar{t}) = \bar{w}(1,\bar{t}) = 0.
\end{aligned} \tag{2.2}$$

Notice that in the first equation of (2.2) there are terms describing normal and parametric forcing, whereas in the second equation there is only a parametric forcing term.
By considering the boundary conditions it follows that system (2.2) can be transformed to an infinite system of ordinary differential equations (ODEs) by expanding the solutions $\bar{v}(x,\bar{t})$ and $\bar{w}(x,\bar{t})$ in eigenfunction-series, that is,

$$\bar{v}(x,\bar{t}) = \sum_{n=1}^{\infty} v_n(\bar{t}) \sin(\mu_n x) \quad \text{and} \quad \bar{w}(x,\bar{t}) = \sum_{n=1}^{\infty} w_n(\bar{t}) \sin(\mu_n x), \tag{2.3}$$

where $\mu_n = n\pi$, $n = 1,2,3,\ldots$. By substitution of (2.3) into (2.2) and then by using the orthogonality properties of the eigenfunctions, one obtains an infinite dimensional system of ODEs for $v_n(\bar{t})$ and $w_n(\bar{t})$:

$$\begin{aligned}
&\ddot{w}_n(\bar{t}) + \mu_n^2 w_n(\bar{t}) = -\varepsilon\Big(\alpha_2 \dot{w}_n(\bar{t}) + \mu_n^2 w_n(\bar{t})\Big[\frac{1}{4}\sum_{k=1}^{\infty} \mu_k^2 (v_k^2(\bar{t}) + w_k^2(\bar{t})) + F_1 \sin(\lambda \bar{t})\Big]\Big),\\
&\ddot{v}_n(\bar{t}) + \mu_n^2 v_n(\bar{t}) = -\varepsilon\Big(\alpha_1 \dot{v}_n(\bar{t}) + \mu_n^2 v_n(\bar{t})\Big[\frac{1}{4}\sum_{k=1}^{\infty} \mu_k^2 (v_k^2(\bar{t}) + w_k^2(\bar{t})) + F_1 \sin(\lambda \bar{t})\Big] -\\
&\qquad\qquad \lambda^2 F_2 d_n \sin(\lambda \bar{t})\Big),\\
&v_n(0) = 2\int_0^2 \bar{v}_0(x) \sin(\mu_n x)dx, \quad \dot{v}_n(0) = 2\int_0^1 (\bar{v}_1(x) - \varepsilon\lambda F_2 x) \sin(\mu_n x)dx,\\
&w_n(0) = 2\int_0^2 \bar{w}_0(x) \sin(\mu_n x)dx, \quad \dot{w}_n(0) = 2\int_0^1 \bar{w}_1(x) \sin(\mu_n x)dx,
\end{aligned} \tag{2.4}$$

where $d_n = (-1)^{(n+1)}\frac{2}{n\pi}$, $n = 1, 2, 3, \ldots$. The dot represents differentiation with respect to $\bar{t}$. By considering the values of excitation-frequency λ, there are three possibilities, namely:

(i) $\lambda \neq \mu_n + O(\varepsilon)$ for all n. Then the parametric and normal forcing terms do not influence the $O(1)$ approximations of the solutions on a time scale of order ε^{-1}. The origin will be stable and no $O(1)$ time varying motion will occur due to the forcing terms on time-scales of $O(\varepsilon^{-1})$.

(ii) $\lambda = \mu_m + O(\varepsilon)$, m is an odd number. Then the normal forcing is important in the equation for the m-th mode, whereas the parametric term is still not important. This possibility implies that an $O(1)$ periodic response due to the parametric excitation can be expected.

(iii) $\lambda = \mu_m + O(\varepsilon)$, m is an even number. Then both parametric and normal forcing are important in the equations for the m-th mode and the $\frac{1}{2}$m-th mode. Interactions between these modes (including $O(1)$ periodic responses) can be expected.

When λ is near μ_m, where $m = (2s-1)$ and s is a positive integer, then the normal excitation will influence the m-th modes in- and out-of plane up to $O(1)$ but the other modes are only influenced up to order $O(\varepsilon)$ on time-scales of ε^{-1}. Hence, the truncation method can be applied to (2.4). Assuming that initially (up to order $O(\varepsilon)$) there is no energy in the system and by applying the truncation method to (2.4), one obtains as **model 1** the following problem:

$$\begin{aligned}
\ddot{w}_m(\bar{t}) + \mu_m^2 w_m(\bar{t}) &= -\varepsilon\Big(\alpha_2 \dot{w}_m(\bar{t}) + \mu_m^2 w_m(\bar{t})\Big[\frac{1}{4}\mu_m^2(v_m^2(\bar{t}) + w_m^2(\bar{t})) + F_1\sin(\lambda\bar{t})\Big]\Big),\\
\ddot{v}_m(\bar{t}) + \mu_m^2 v_m(\bar{t}) &= -\varepsilon\Big(\alpha_1 \dot{v}_m(\bar{t}) + \mu_m^2 v_m(\bar{t})\Big[\frac{1}{4}\mu_m^2(v_m^2(\bar{t}) + w_m^2(\bar{t})) + F_1\sin(\lambda\bar{t})\Big] -\\
&\quad \lambda^2 F_2 d_m \sin(\lambda\bar{t})\Big).
\end{aligned} \tag{2.5}$$

When λ is near μ_m with $m = 2s$ and s is a positive integer, then the normal excitation will excite the m-th modes in- and out-of-plane, whereas the parametric excitation excites the s-th modes in both planes. Again by assuming that the system initially has no energy (up to $O(\varepsilon)$) and by applying truncation method to (2.4), one obtains as **model 2** the following problem:

$$\begin{aligned}
\ddot{v}_m(\bar{t}) + \mu_m^2 v_m(\bar{t}) &= -\varepsilon\Big(\alpha_1 \dot{v}_m(\bar{t}) + \mu_m^2 v_m(\bar{t})\Big[\frac{1}{4}\mu_m^2(v_m^2(\bar{t}) + w_m^2(\bar{t})) + \frac{1}{4}\mu_s^2(v_s^2(\bar{t}) +\\
&\quad w_s^2(\bar{t})) + F_1\sin(\lambda\bar{t})\Big] - \lambda^2 F_2 d_m \sin(\lambda\bar{t})\Big),\\
\ddot{w}_m(\bar{t}) + \mu_m^2 w_m(\bar{t}) &= -\varepsilon\Big(\alpha_2 \dot{w}_m(\bar{t}) + \mu_m^2 w_m(\bar{t})\Big[\frac{1}{4}\mu_m^2(v_m^2(\bar{t}) + w_m^2(\bar{t})) + \frac{1}{4}\mu_s^2(v_s^2(\bar{t}) +\\
&\quad w_s^2(\bar{t})) + F_1\sin(\lambda\bar{t})\Big]\Big),\\
\ddot{v}_s(\bar{t}) + \mu_s^2 v_s(\bar{t}) &= -\varepsilon\Big(\alpha_1 \dot{v}_s(\bar{t}) + \mu_s^2 v_s(\bar{t})\Big[\frac{1}{4}\mu_m^2(v_m^2(\bar{t}) + w_m^2(\bar{t})) + \frac{1}{4}\mu_s^2(v_s^2(\bar{t}) +\\
&\quad w_s^2(\bar{t})) + F_1\sin(\lambda\bar{t})\Big] - \lambda^2 F_2 d_s \sin(\lambda\bar{t})\Big),\\
\ddot{w}_s(\bar{t}) + \mu_s^2 w_s(\bar{t}) &= -\varepsilon\Big(\alpha_2 \dot{w}_s(\bar{t}) + \mu_s^2 w_s(\bar{t})\Big[\frac{1}{4}\mu_m^2(v_m^2(\bar{t}) + w_m^2(\bar{t})) + \frac{1}{4}\mu_s^2(v_s^2(\bar{t}) +\\
&\quad w_s^2(\bar{t})) + F_1\sin(\lambda\bar{t})\Big]\Big).
\end{aligned} \tag{2.6}$$

In this paper the study of the model 1 and of the model 2 will be focused on the O(1) periodic solutions. The existence and the stability of the periodic solutions will be determined by using the averaging method and the linearisation method, respectively.

3 Analysis of model 1

System (2.5) describes the interaction between the m-th modes in- and out-of-plane. Because the frequency of the normal excitation is near the natural frequency μ_m then the motion in the plane of the string will always exist. When the motion in the out-of-plane also exists the string begins to whirl, and so-called whirling (non-planar) motion occurs.
Let us set $\lambda \bar{t} = 2t$, where $\lambda = \mu_m + \eta\varepsilon$ with m is an odd number and $\eta = O(1)$. System (2.5) then becomes

$$\begin{aligned} v_m''(t) + 4v_m(t) &= -\frac{\varepsilon}{\mu_m}\Big(2\alpha_1 v_m'(t) + 4v_m(t)\Big[\gamma_m(v_m^2(t) + w_m^2(t)) + F_1\mu_m \sin(2t) - \\ &\quad 2\eta\Big] - 8F_2\sin(2t)\Big) + O(\varepsilon^2), \\ w_m''(t) + 4w_m(t) &= -\frac{\varepsilon}{\mu_m}\Big(2\alpha_2 w_m'(t) + 4w_m(t)\Big[\gamma_m(v_m^2(t) + w_m^2(t)) + F_1\mu_m \sin(2t) - \\ &\quad 2\eta\Big]\Big) + O(\varepsilon^2), \end{aligned} \tag{3.1}$$

where $\gamma_m = \frac{1}{4}\mu_m^3$. A prime represents differentiation with respect to t. System (3.1) in fact represents an $1:1$ internal resonance case. The steady-states of system (3.1) can be found analytically from the averaged system. To apply the averaging method the following transformations $(v_m(t), v_m'(t)) \to (A_m(t), B_m(t))$ and $(w_m(t), w_m'(t)) \to (C_m(t), D_m(t))$ are introduced:

$$\begin{aligned} v_m(t) &= A_m(t)\sin(2t) + B_m(t)\cos(2t), & w_m(t) &= C_m(t)\sin(2t) + D_m(t)\cos(2t), \\ v_m'(t) &= 2(A_m(t)\cos(2t) - B_m(t)\sin(2t)), & w_m'(t) &= 2(C_m(t)\cos(2t) - D_m(t)\sin(2t)). \end{aligned} \tag{3.2}$$

By substituting (3.2) into (3.1) and after averaging one obtains:

$$\begin{aligned} \bar{C}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}_2\bar{C}_m + \bar{D}_m\Big[\frac{1}{4}((\bar{A}_m^2 + \bar{B}_m^2) + 3(\bar{C}_m^2 + \bar{D}_m^2)) - 2\bar{\eta}\Big] + \frac{1}{2}(\bar{A}_m\bar{C}_m + \bar{B}_m\bar{D}_m)\bar{B}_m\Big), \\ \bar{D}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}_2\bar{D}_m - \bar{C}_m\Big[\frac{1}{4}((\bar{A}_m^2 + \bar{B}_m^2) + 3(\bar{C}_m^2 + \bar{D}_m^2)) - 2\bar{\eta}\Big] - \frac{1}{2}(\bar{A}_m\bar{C}_m + \bar{B}_m\bar{D}_m)\bar{A}_m\Big), \\ \bar{A}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}_1\bar{A}_m + \bar{B}_m\Big[\frac{1}{4}(3(\bar{A}_m^2 + \bar{B}_m^2) + (\bar{C}_m^2 + \bar{D}_m^2)) - 2\bar{\eta}\Big] + \frac{1}{2}(\bar{A}_m\bar{C}_m + \bar{B}_m\bar{D}_m)\bar{D}_m\Big), \\ \bar{B}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}_1\bar{B}_m - \bar{A}_m\Big[\frac{1}{4}(3(\bar{A}_m^2 + \bar{B}_m^2) + (\bar{C}_m^2 + \bar{D}_m^2)) - 2\bar{\eta}\Big] - \frac{1}{2}(\bar{A}_m\bar{C}_m + \bar{B}_m\bar{D}_m)\bar{C}_m + \\ &\quad 2\beta\Big), \end{aligned} \tag{3.3}$$

where $\bar{\varepsilon} = \frac{\gamma_m}{\mu_m}\varepsilon$, $\bar{\eta} = \frac{\eta}{\gamma_m}$, $\beta = \frac{F_2}{\gamma_m}$, $\bar{\alpha}_i = \frac{\alpha_i}{\gamma_m}$, $i = 1$ or 2, and where $\bar{A}_m$, $\bar{B}_m$, $\bar{C}_m$, $\bar{D}_m$ are the averaged approximations of A_m, B_m, C_m, and D_m respectively. In what follows the stability of the critical points and then the dependence on the parameters $\bar{\eta}$, β, and $\bar{\alpha}_i$, $i = 1$ or 2, will be analyzed. The analysis can be restricted to the case $\beta \geq 0$, since for $\beta < 0$ a simple transformation ($\bar{A}_m := -\bar{A}_m$, $\bar{B}_m := -\bar{B}_m$, $\bar{C}_m := -\bar{C}_m$, and $\bar{D}_m := -\bar{D}_m$) leads to system (3.3) with $\beta \geq 0$. In this section the study will be divided into two cases: (i) $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ (no damping), and (ii) $\bar{\alpha}_1, \bar{\alpha}_2 > 0$ (positive damping).

3.1 The case without damping, i.e. $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$

The critical points of system (3.3) with $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ satisfy the following algebraic equations:

$$\begin{aligned}
&\bar{B}_m\left[\frac{1}{4}(3(\bar{A}_m^2+\bar{B}_m^2)+(\bar{C}_m^2+\bar{D}_m^2))-2\bar{\eta}\right]+\frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{D}_m=0,\\
&\bar{A}_m\left[\frac{1}{4}(3(\bar{A}_m^2+\bar{B}_m^2)+(\bar{C}_m^2+\bar{D}_m^2))-2\bar{\eta}\right]+\frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{C}_m-2\beta=0,\\
&\bar{D}_m\left[\frac{1}{4}((\bar{A}_m^2+\bar{B}_m^2)+3(\bar{C}_m^2+\bar{D}_m^2))-2\bar{\eta}\right]+\frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{B}_m=0,\\
&\bar{C}_m\left[\frac{1}{4}((\bar{A}_m^2+\bar{B}_m^2)+3(\bar{C}_m^2+\bar{D}_m^2))-2\bar{\eta}\right]+\frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{A}_m=0.
\end{aligned} \tag{3.4}$$

It turns out (see: Appendix 6.1.1) that if a solution of (3.4) exists, then it has to satisfy $\bar{B}_m = \bar{C}_m = 0$ and $\bar{A}_m \neq 0$. Substituting $\bar{B}_m = \bar{C}_m = 0$ into system (3.4) then gives

$$\begin{aligned}
&\bar{A}_m\left[\frac{1}{4}(3\bar{A}_m^2+\bar{D}_m^2)-2\bar{\eta}\right]-2\beta=0,\\
&\bar{D}_m\left[\frac{1}{4}(\bar{A}_m^2+3\bar{D}_m^2)-2\bar{\eta}\right]=0.
\end{aligned} \tag{3.5}$$

From the second equation in (3.5) it follows that $\bar{D}_m$ can be zero. Therefore, for the critical points of system (3.3) with $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ three types can be distinguished:

$$\begin{aligned}
&\text{CP-type 1:} \quad (\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m)=(\tilde{A}_m,0,0,0),\\
&\text{CP-type 2:} \quad (\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m)=(\tilde{A}_m,0,0,\tilde{D}_m),\\
&\text{CP-type 3:} \quad (\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m)=(\tilde{A}_m,0,0,-\tilde{D}_m),
\end{aligned} \tag{3.6}$$

where $\tilde{D}_m > 0$. The first type represents motion in the plane of the string while the second and the third type represent a whirling motion composed of periodic solutions in-and out-of-plane. However, the second and the third type represent the same kind of motion (and stability). The difference between these solutions is only a difference in phase. Therefore, the behaviour around the critical points of type 3 can be determined straight-forwardly from the behaviour around the critical points of type 2.
The first type of critical points are on the $\bar{A}_m$ -axis and can be studied as solutions of a cubical equation. The dependence of $\bar{A}_m$ on the detuning parameter in the excitation-frequency is well-known in the literature (see for instance in Refs. [9, 14]). In what follows a $(\bar{\eta},\beta)$-diagram will be constructed in which an overview of all possible critical points is given. Starting with $\tilde{D}_m \neq 0$, system (3.5) can be simplified to:

$$\begin{aligned}
&\tilde{A}_m^3-2\bar{\eta}\tilde{A}_m-3\beta=0,\\
&\tilde{D}_m^2=\frac{1}{3}(8\bar{\eta}-\tilde{A}_m^2),\\
&\text{Cond}_1(\tilde{A}_m)=8\bar{\eta}-\tilde{A}_m^2>0.
\end{aligned} \tag{3.7}$$

Looking at the second equation in (3.7) it follows that a positive value of $\tilde{D}_m$ corresponds to a critical point of type 2 and a negative value to a critical point of type 3. System (3.7) defines domains in the $(\bar{\eta},\beta)$-plane where one, two, or three critical points of type 2 exist. The critical points of type 3 follow from these of type 2 (as mentioned before). These domains are found by determining the boundary curves which follow from the equality $Cond_1(\tilde{A}_m) = 0$ (where $\tilde{A}_m$ is a solution of the first equation in (3.7)), and the radicant of the cubic equation in standard form $\Delta_2 = (\frac{3}{2}\beta)^2 + (-\frac{2}{3}\overline{\eta})^3 = 0$. The $Cond_1 = 0$-curve divides the $(\overline{\eta},\beta)$-plane into two domains in which a critical point of type 2 exists or not, whereas the $\Delta_2 = 0$-curve divides the domain where a CP-type 2 exists into two domains with one and three critical points. When one looks separately at the case $\tilde{D}_m = 0$ one obtains the following cubic equation for $\tilde{A}_m$:

$$\tilde{A}_m^3 - \frac{8}{3}\overline{\eta}\tilde{A}_m - \frac{8}{3}\beta = 0. \tag{3.8}$$

This equation differs from the first equation in (3.7). The curve defined by the radicant $\Delta_1 = (\frac{4}{3}\beta)^2 + (-\frac{8}{9}\overline{\eta})^3 = 0$ divides the $(\overline{\eta},\beta)$-plane into two domains in which one or three critical points of type 1 exist. Whereas on the curve $\Delta_1 = 0$ in the $(\overline{\eta},\beta)$-plane there are two critical points of type 1. As illustration a bifurcation diagram is given in Fig.2. In this figure there are four domains. The type and the number of critical points in each of these domains and on the boundary curves are listed in Table 1.

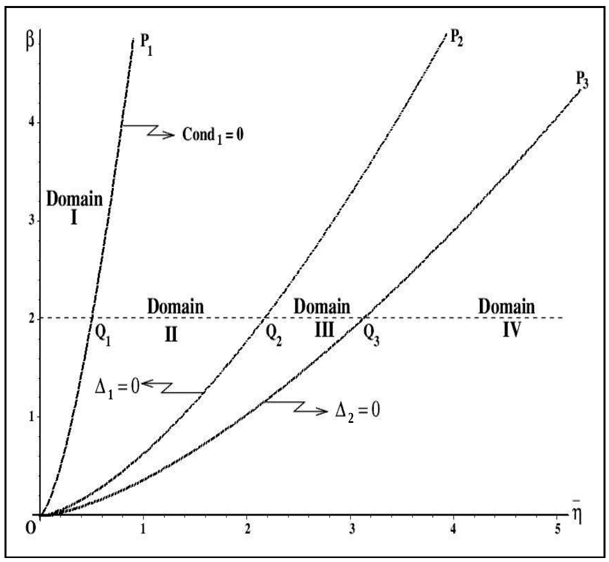


Figure 2: Four domains with real solutions of the equations (3.7) and (3.8): $\Delta_1 = (\frac{4}{3}\beta)^2 + (-\frac{8}{9}\overline{\eta})^3$ and $\Delta_2 = (\frac{3}{2}\beta)^2 + (-\frac{2}{3}\overline{\eta})^3$.

The stability of a critical point can be determined from the eigenvalues of the Jacobian matrix of system (3.3) in a neighbourhood of this critical point. The stability of the critical points of type 1 in Fig. 2 is as follows: in domain I a stable solution; in domain II an unstable solution; on the curve $\overline{OP_1}$ a degenerate stable solution; and in the domains III and IV a stable solution for the smallest amplitude, and the other solutions are unstable. The stability of the critical points of type 2 is as follows: in the domains II and III stable solutions; a degenerate stable solution on the curve $\overline{OP_3}$; and in domain IV a stable solution for the solution with the largest amplitude and the other solutions are unstable. The stability of the critical points of type 3 is exactly the same as these of type 2. It is of interest to know what happens if one varies a parameter while the other parameters are kept fixed. Of particular interest are the response-curves $R_3 = \tilde{A}_m^2$ and $R_4 = \tilde{D}_m^2$ as function of this varying parameter. The results are presented in Fig. 3 and in Fig. 4, where the solid and dashed lines stand for the stable and unstable periodic solutions respectively.
Looking at Fig. 3 it can clearly be seen that for $\overline{\eta} < Q_1 = 0.5000$ there is only one stable critical point which is of type 1. At $\overline{\eta} = Q_1$ a pitchfork bifurcation occurs and by increasing

Table 1: The number of critical points of system (3.3) as described in Fig. 2.

Domain/Curve	CP-type 1	CP-type 2	CP-type 3	Total
I and $\overline{OP_1}$	1	0	0	1
II	1	1	1	3
III	3	1	1	5
IV	3	3	3	9
$\overline{OP_2}$	2	1	1	4
$\overline{OP_3}$	3	2	2	7

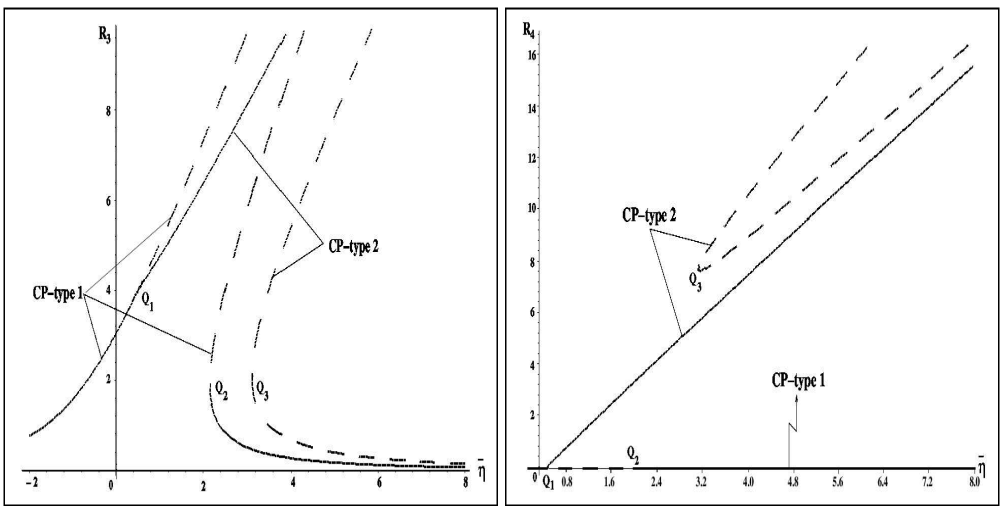


Figure 3: The stability response-curves $R_3 = \tilde{A}_m^2$ and $R_4 = \tilde{D}_m^2$ of system (3.3) with respect to $\bar{\eta}$ for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ and $\beta = 2.0$. The solid and dashed line represent stable and unstable solutions, respectively.

$\bar{\eta}$ it follows that the critical point of type 1 losses its stability (via a degenerate case) and two stable non-planar motions occur. When $\bar{\eta}$ increases to $Q_2 = 2.1634$ two new critical points of type 1 appear, a stable and an unstable one. Analogously, at $Q_3 = 3.1201$ four new unstable non-planar motions turn up leading in total to nine critical points in domain IV. A remarkable result is that for small energy differences (that is, the difference between the amplitudes of the in-plane motion and the in-plane component of non-planar motion) can lead to non-planar motion for which the amplitudes of the out-plane component become large. This indicates that the stability of the planar motion can be rather unstable.

In Fig. 4 the influence of the forcing amplitude on the solution is shown. Looking at this figure one sees that the amplitude of the out-of-plane component decreases gradually to zero for increasing values of β, while the amplitude of the in-plane component grows for increasing β. This means that if the amplitude of the vibrating support is very large compared to the excitation-frequency λ, the motion of the string is usually in plane. It turns out that the non-planar motion of system (3.3) for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ and $\bar{\eta} = 2$ dies out

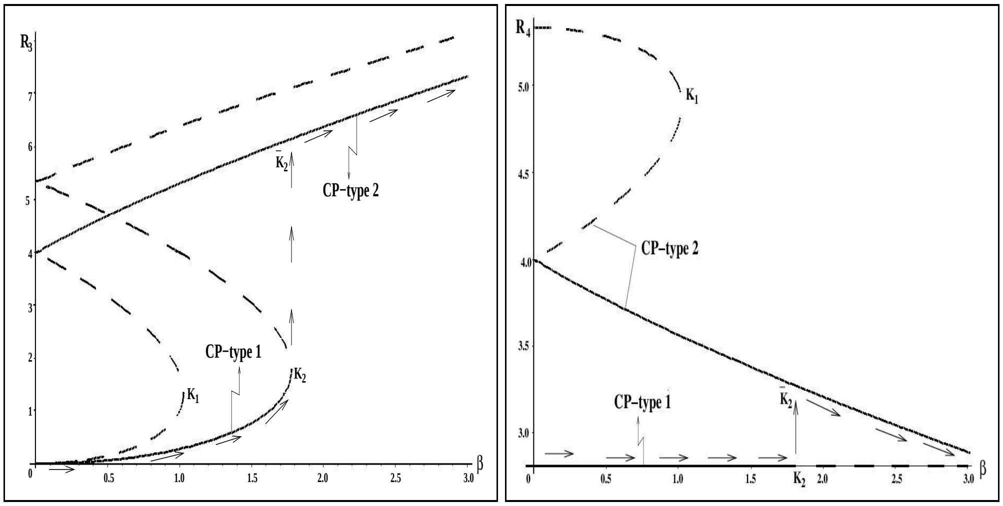


Figure 4: The stability response-curves $R_3 = \tilde{A}_m^2$ and $R_4 = \tilde{D}_m^2$ of system (3.3) with respect to β for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ and $\bar{\eta} = 2.0$. The solid and dashed line represent stable and unstable solutions, respectively.

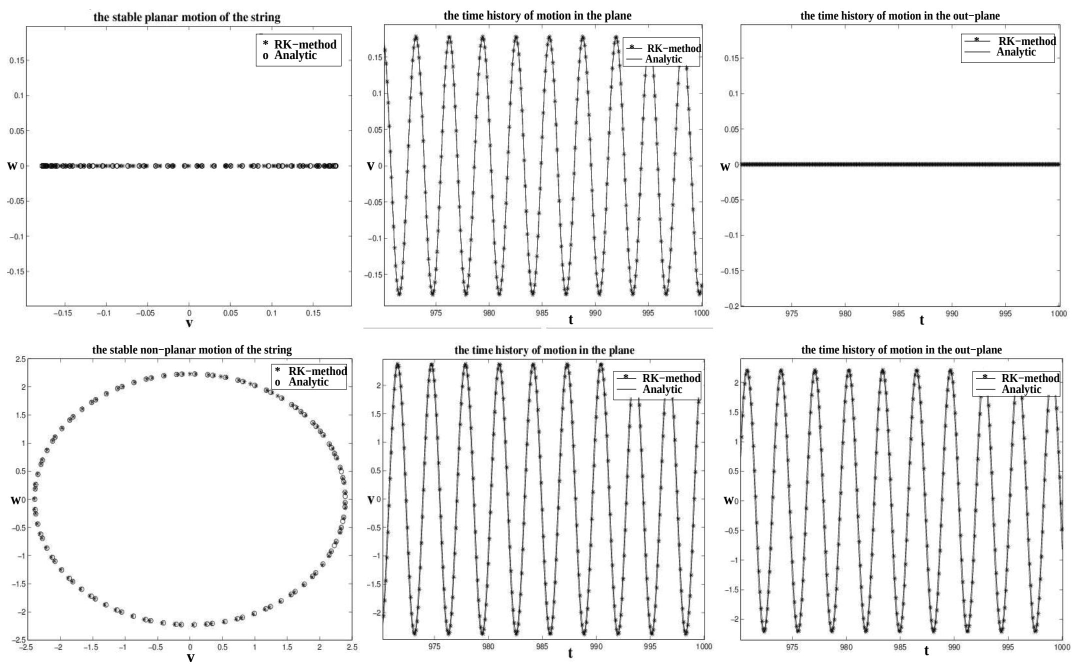


Figure 5: The stable motions of the string in the 1 : 1 resonance case for the first mode, and $\varepsilon = 0.0025$, $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, $\bar{\eta} = 0.6450$, and $\beta = 0.4561$. In (i)-(iii) planar motion is presented, and in (iv)-(vi) non-planar motion is given.

after $\beta = 16.00$. Moreover, in Fig. 4 a jump upward at $\beta = K_2$ from planar to the non-planar motion can occur for increasing values of β up to $\beta = 16.00$. For $\beta > 16.00$ the motion will be planar again. The analytically obtained results are compared to numerical results obtained by using the Runge-Kutta method (see Fig. 5). This figure shows that the analytical results are the same as the numerical results on long time-scales. In the upper part of Fig. 5 the stable in-plane motion is presented, whereas in the lower part of Fig. 5 the non-planar motion is given.

3.2 The case with (positive) damping in both planes, i.e. $\bar{\alpha}_1, \bar{\alpha}_2 > 0$

The critical points of system (3.3) for $\bar{\alpha}_1$, $\bar{\alpha}_2 > 0$ satisfy the following algebraic equations:

$$\begin{aligned}
&\bar{\alpha}_1\bar{A}_m + \bar{B}_m\left[\frac{1}{4}(3(\bar{A}_m^2+\bar{B}_m^2)+(\bar{C}_m^2+\bar{D}_m^2))-2\bar{\eta}\right]+\frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{D}_m=0,\\
&\bar{\alpha}_1\bar{B}_m - \bar{A}_m\left[\frac{1}{4}(3(\bar{A}_m^2+\bar{B}_m^2)+(\bar{C}_m^2+\bar{D}_m^2))-2\bar{\eta}\right]-\frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{C}_m+2\beta=0,\\
&\bar{\alpha}_2\bar{C}_m + \bar{D}_m\left[\frac{1}{4}((\bar{A}_m^2+\bar{B}_m^2)+3(\bar{C}_m^2+\bar{D}_m^2))-2\bar{\eta}\right]+\frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{B}_m=0,\\
&\bar{\alpha}_2\bar{D}_m - \bar{C}_m\left[\frac{1}{4}((\bar{A}_m^2+\bar{B}_m^2)+3(\bar{C}_m^2+\bar{D}_m^2))-2\bar{\eta}\right]-\frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{A}_m=0. \qquad (3.9)
\end{aligned}$$

In Appendix 6.1.2 it has been shown that when a solution of system (3.9) exists then $\bar{B}_m \neq 0$ and that if $\bar{C}_m = 0$ then $\bar{D}_m = 0$. Therefore, the critical points of system (3.3) can be classified into two types:

$$\begin{aligned}
&\text{CP-type 1: } (\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m)=(\tilde{A}_m,\tilde{B}_m,0,0), \text{ with } \tilde{B}_m\neq 0,\\
&\text{CP-type 2: } (\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m)=(\tilde{A}_m,\tilde{B}_m,\tilde{C}_m,\tilde{D}_m), \quad \text{with } \tilde{C}_m\neq 0 \text{ and } \tilde{B}_m\neq 0.
\end{aligned} \qquad (3.10)$$

Starting with the analysis of the critical point of type 2 it follows from Appendix 6.2.1 that (3.9) can be reduced to:

$$\begin{aligned}
&(R_3+R_4)(R_3+3R_4)-\frac{16}{3}\bar{\eta}(2R_3+3R_4)+\frac{64}{3}\bar{\eta}^2+\frac{16}{3}\bar{\alpha}_2^2=0,\\
&(\bar{\alpha}_1R_3+\bar{\alpha}_2R_4)^2+(R_3-R_4)^2\left[\frac{3}{4}(R_3+R_4)-2\bar{\eta}\right]^2-4\beta^2R_3=0,\\
&\text{Cond}_2(R_3,R_4)=(R_3+3R_4)-8\bar{\eta}<0,\\
&\text{Cond}_3(R_3)=R_3-4\bar{\alpha}_2>0, \qquad (3.11)
\end{aligned}$$

where $R_3 = \tilde{A}_m^2+\tilde{B}_m^2$ and $R_4 = \tilde{C}_m^2+\tilde{D}_m^2$. The first two equations in (3.11) can be reduced further to (as also has been shown in Appendix 6.2.1) :

$$\begin{aligned}
&a_oR_3^4+[a_1(\bar{\alpha}_1-\bar{\alpha}_2)-4\beta^2]R_3^3+[a_2(\bar{\alpha}_1-\bar{\alpha}_2)+a_3]R_3^2+[a_4(\bar{\alpha}_1-\bar{\alpha}_2)+a_5]R_3+\\
&\qquad a_6(\bar{\alpha}_1-\bar{\alpha}_2)+a_7=0,\\
&R_4=\frac{8}{3}\bar{\eta}-\frac{2}{3}R_3\pm\frac{1}{3}\sqrt{R_3^2-16\bar{\alpha}_2^2},\\
&\text{Cond}_4(R_3)=8\bar{\eta}-2R_3\pm\sqrt{R_3^2-16\bar{\alpha}_2^2}>0, \qquad (3.12)
\end{aligned}$$

where a_j, $j=0,1,2,\ldots,7$ are defined in Appendix 6.2.1. First, we consider the case $\bar{\alpha}_1 = \bar{\alpha}_2 > 0$. In this case the first equation of (3.12) can be simplified to the standard cubic equation:

$$X^3+\kappa_{12}X+\delta_{12}=0, \qquad (3.13)$$

where:

$$\kappa_{12} \;=\; -\frac{1}{48\beta^4}(a_3^2+12\beta^2a_5),$$

$$\delta_{12} = -\frac{1}{864\beta^6}\left[a_3^3 + 18\beta^2(a_3a_5 + 12\beta^2 a_7)\right],$$
$$X = R_3 - \frac{1}{12\beta^2}a_3.$$

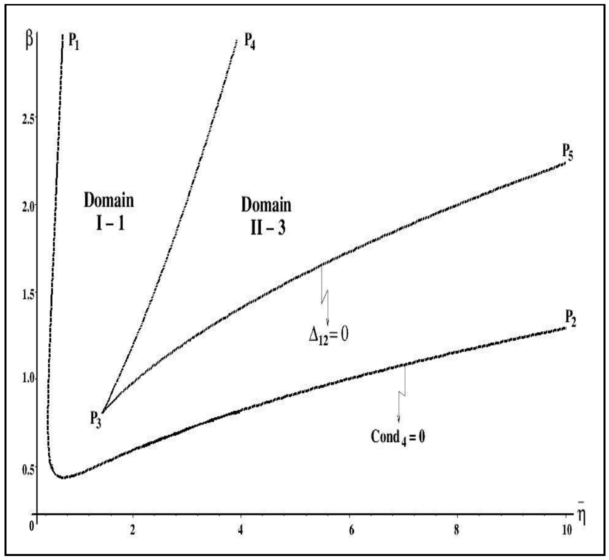


Figure 6: The domains with real solutions of (3.13) in which the conditions 2-4 are satisfied for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.50$: $\Delta_{12} = (\frac{1}{2}\delta_{12})^2 + (\frac{1}{3}\kappa_{12})^3$.

In Fig. 6 an overview is given of all real solutions of (3.13) (and satisfying Cond_4) for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.50$. In this figure two domains I-1 and II-3 are given in which one and three critical points of type 2 exist respectively. When one considers separately the critical points of type 1, that is $\bar{C}_m = \bar{D}_m = 0$, one obtains the following equation for R_3:

$$Y^3 + \kappa_{11}Y + \delta_{11} = 0, \tag{3.14}$$

where:

$$\kappa_{11} = -\frac{16}{9}(\frac{4}{3}\bar{\eta}^2 - \bar{\alpha}_1^2),$$
$$\delta_{11} = -\frac{64}{9}(\beta^2 - \frac{4}{9}\bar{\alpha}_1^2\bar{\eta} - \frac{16}{81}\bar{\eta}^3),$$
$$Y = R_3 - \frac{16}{9}\bar{\eta}.$$

By setting the radicant$\Delta_{11} = (\frac{1}{2}\delta_{11})^2 + (\frac{1}{3}\kappa_{11})^3$ of (3.14) equal to zero one obtains two curves which are indicated by $\overline{P_2P_6}$ and $\overline{P_6P_8}$ in Fig. 7. It should be noted that the boundary curves $\overline{P_2P_6}$ and $\overline{P_6P_8}$ are independent of α_2. This means that the position of the critical points of type 1 does not depend on the damping coefficient in the out-of-plane. However, the stability depends on α_2. In Domain I system (3.3) has only one stable critical point of type 1. In domain II system (3.3) has two critical points, one of which is a critical point of type 1, and the other is a critical point of type 2. In the other domains and on the boundary curves, the type and the number of critical points are given in Table 2. The stability of these critical points will be studied later on in this section.

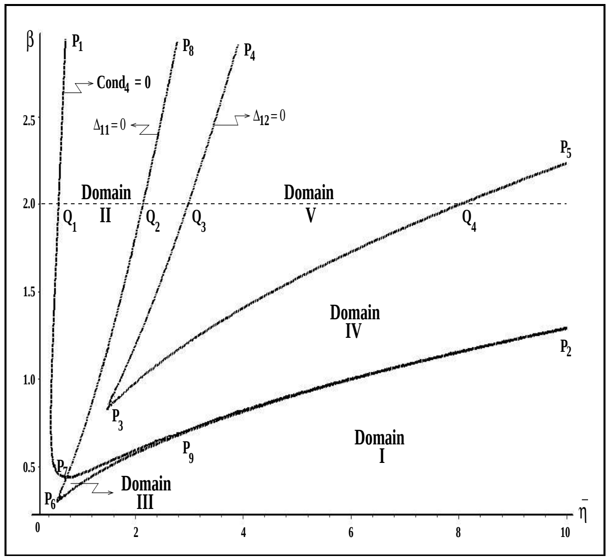


Figure 7: The domains with real solutions of system (3.9) for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.50$: $\Delta_{11} = (\frac{1}{2}\delta_{11})^2 + (\frac{1}{3}\kappa_{11})^3$ and $\Delta_{12} = (\frac{1}{2}\delta_{12})^2 + (\frac{1}{3}\kappa_{12})^3$.

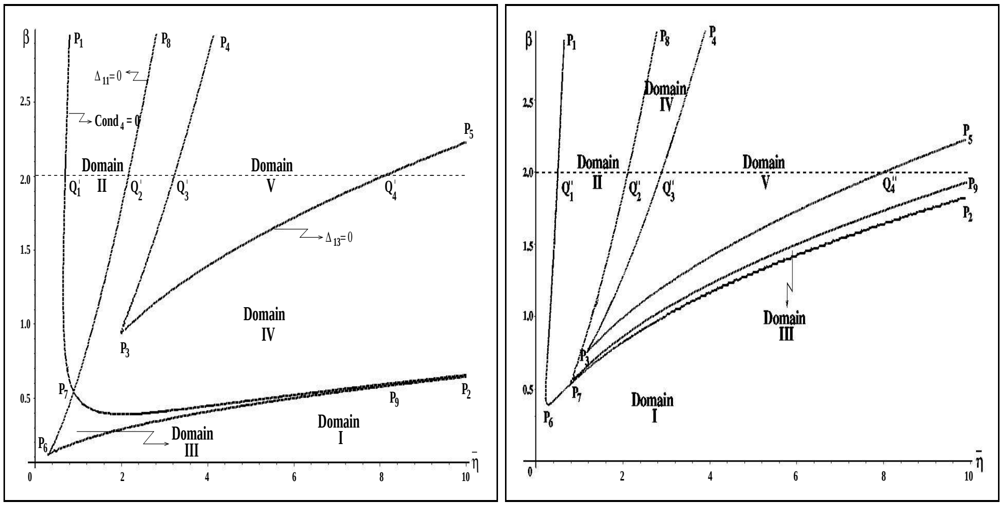
Table 2: The number of critical points of system (3.3) as described in Fig. 7.

Domain/Curve	CP-type 1	CP-type 2	Total
I and $\overline{P_1P_7}$	1	0	1
II	1	1	2
III and $\overline{P_7P_9}$	3	0	3
IV	3	1	4
V	3	3	6
$\overline{P_2P_6}$ and $\overline{P_6P_7}$	2	0	2
$\overline{P_7P_8}$	2	1	3
$\overline{P_3P_4}$ and $\overline{P_3P_5}$	3	2	5

For $\bar{\alpha}_1 \neq \bar{\alpha}_2$, the solutions of the first equation in (3.12) can be determined from the following equation:

$$\bar{z}^3 + \kappa_{13}\bar{z} + \delta_{13} = 0, \tag{3.15}$$

where κ_{13}, δ_{13}, and the relationship between R_3 and $\bar{z}$ are defined in Appendix 6.2.1. An overview of all real solutions of (3.12) for $\bar{\alpha}_1 \neq \bar{\alpha}_2$ can be found from (3.15). As illustration we consider the case $\bar{\alpha}_1 = 0.25$ and $\bar{\alpha}_2 = 0.75$, and the case $\bar{\alpha}_1 = 0.75$ and $\bar{\alpha}_2 = 0.25$. The resulting boundary curves are presented in Fig. 8. Figure 8(i) is quite similar in appearance as Fig. 7; so, the type and the number of critical points for system (3.3) in this figure are given in Table 2. Figure 8(ii), however, is quite different from Fig. 7. For this case the type and the number of critical points of system (3.3) are given in Table 3.
According to (3.12) non-planar motion will exist for those parameters for which $\text{Cond}_4 > 0$.



(i) $\bar{\alpha}_1 = 0.25$ and $\bar{\alpha}_2 = 0.75$ (ii) $\bar{\alpha}_1 = 0.75$ and $\bar{\alpha}_2 = 0.25$

Figure 8: The domains with real solutions of system (3.9) as illustration for the case $\bar{\alpha}_1 \neq \bar{\alpha}_2$: $\Delta_{13} = (\frac{1}{2}\delta_{13})^2 + (\frac{1}{3}\kappa_{13})^3$.

Table 3: The number and the type of critical points of system (3.3) as described in Fig. 8(ii), with $\bar{\alpha}_1 = 0.75$ and $\bar{\alpha}_2 = 0.25$.

Domain/Curve	CP-type 1	CP-type 2	Total
I, $\overline{P_1P_6}$, and $\overline{P_6P_7}$	1	0	1
II and $\overline{P_2P_7}$	1	1	2
III	1	2	3
IV	3	1	4
V	3	3	6
$\overline{P_7P_8}$ and $\overline{P_7P_9}$	2	1	3
$\overline{P_3P_4}$ and $\overline{P_3P_5}$	3	2	5

When $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are kept fixed, it can be observed from Fig. 8 that the domain of existence of non-planar motion in the case $\bar{\alpha}_1 > \bar{\alpha}_2$ is smaller than in the case $\bar{\alpha}_1 < \bar{\alpha}_2$. This means that the damping in the in-plane direction is more effective to reduce non-planar motion than the damping in the out-of-plane direction.

As in the case without damping, the stability of the critical points of system (3.3) is analyzed by the linearisation method. For $\bar{\alpha}_1 = \bar{\alpha}_2$ the results of the stability analysis are presented in Fig. 9 and in Fig. 10. In Fig. 9 the response-curves $R_3 = \tilde{A}_m^2 + \tilde{B}_m^2$ and $R_4 = \tilde{C}_m^2 + \tilde{D}_m^2$ are plotted as function of the detuning parameter $\bar{\eta}$ for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.5$ and $\beta = 2.0$. The presence of jump phenomena can be observed in this figure. Starting with a negative value of $\bar{\eta}$ in Domain I in Fig. 7 for $\beta = 2.0$ and following the line indicated and parallel to the $\bar{\eta}$-axis, one encounters the points Q_i, $i = 1-4$. These points are also indicated in Fig. 9.

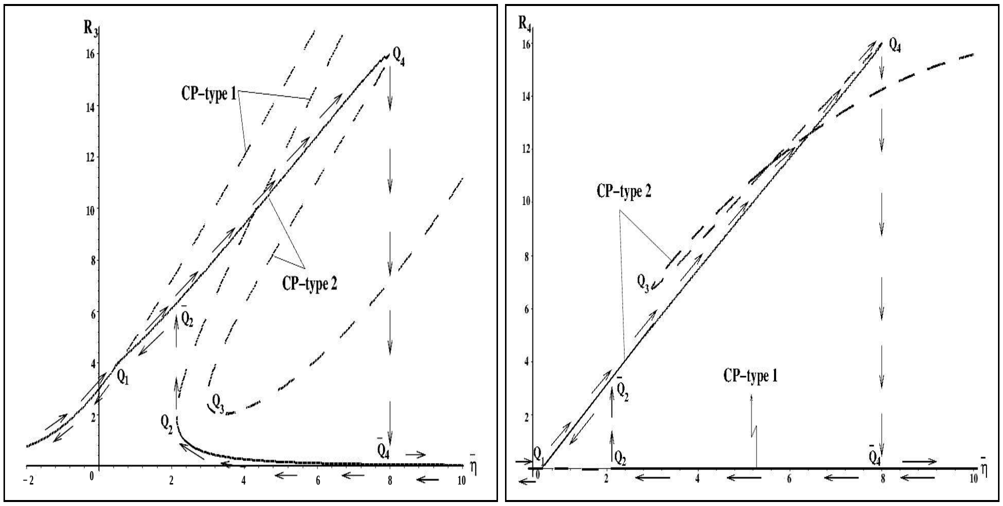


Figure 9: The stability response-curves of system (3.3) with respect to the detuning parameter $\bar{\eta}$ for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.50$ and $\beta = 2.0$. The solid and dashed line represent stable and unstable solutions, respectively.

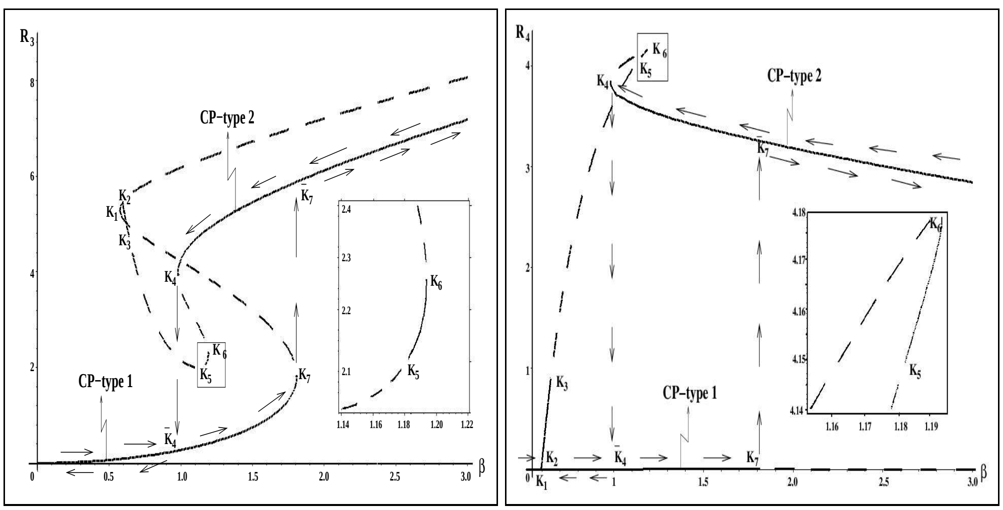


Figure 10: The stability response-curves of system (3.3) with respect to the excitation-amplitude β for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.50$ and $\bar{\eta} = 2.0$. The solid and dashed line represent stable and unstable solutions, respectively.

When $\bar{\eta} < Q_1 = 0.5764$, initially only stable planar motion is possible. At the point Q_1 the transversally excited mode in the out-of-plane direction turns up. Then as $\bar{\eta}$ increases, the motion of the string smoothly changes from planar to non-planar motion. When one arrives at $Q_4 = 8.0177$ the non-planar motion becomes unstable and then any further increase of $\bar{\eta}$ causes a spontaneous jump downward from non-planar motion back to the planar motion. During this jump the amplitude of the in- and the out-of-plane components suddenly decay. And after that only the planar motion will exist. Reversing the procedure and starting with a large value of $\bar{\eta}$, a jump phenomenon occurs at $Q_2 = 2.1415$, where now the amplitudes of the in- and the out-of-plane components increase. This phenomenon changes the motion from planar to non-planar motion. If one decreases the value of $\bar{\eta}$ further, then the motion changes smoothly back to planar motion after the point Q_1. We note that for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.50$, $\beta = 2.0$, and large values of $\bar{\eta}$ (approximately $\bar{\eta} > 24.0007$) non-planar motion will disappear.

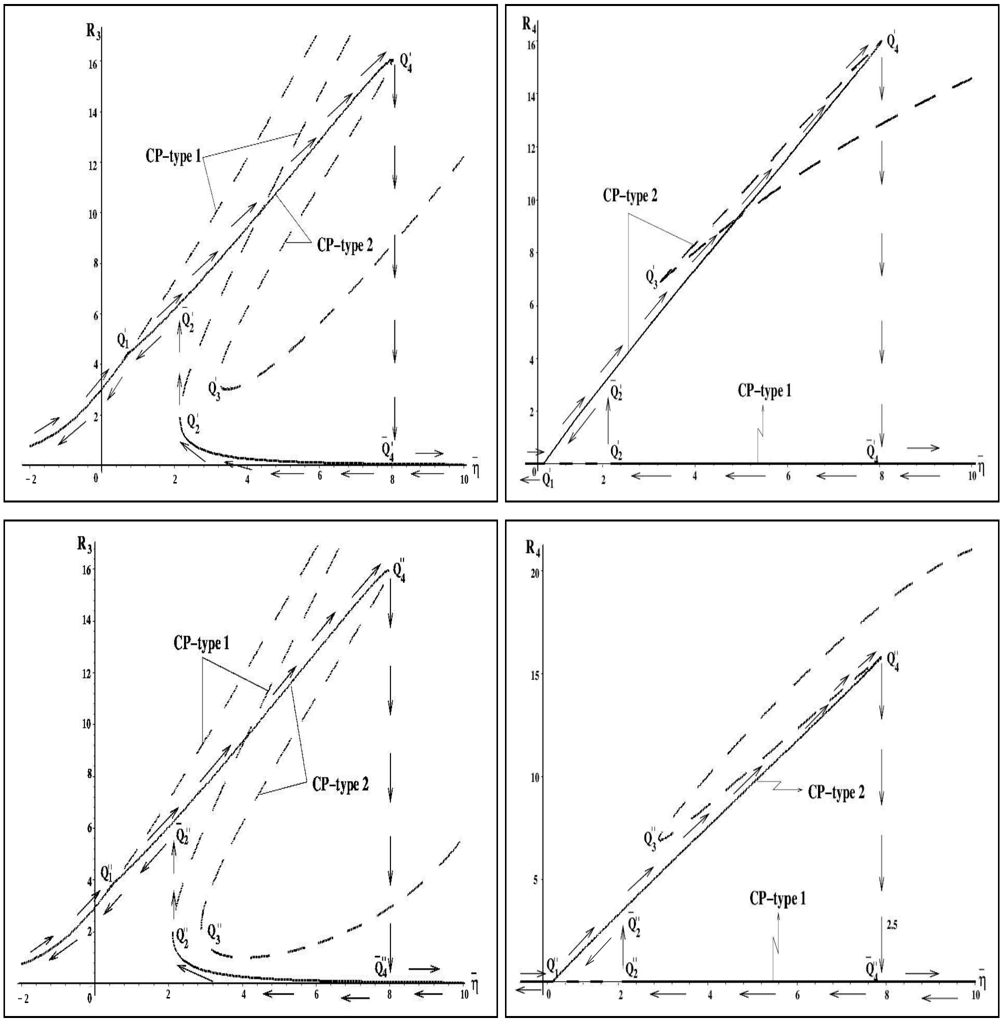
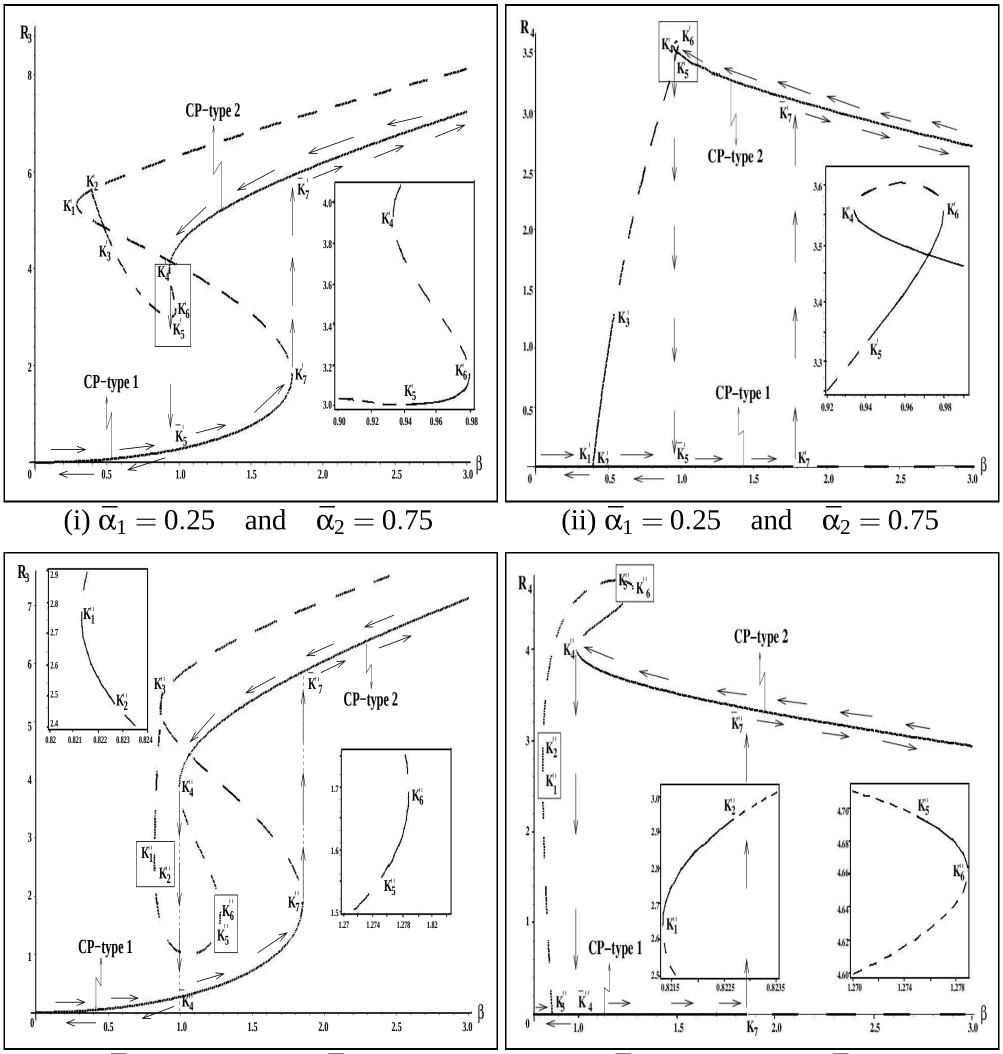


Figure 11: The stability response-curves of system (3.3) with respect to the detuning parameter $\bar{\eta}$ for $\beta = 2$ and $\bar{\alpha}_1 \neq \bar{\alpha}_2$. In the upper figures $\bar{\alpha}_1 = 0.25$ and $\bar{\alpha}_2 = 0.75$ ($\bar{\alpha}_1 < \bar{\alpha}_2$) and in the lower figure $\bar{\alpha}_1 = 0.75$ and $\bar{\alpha}_2 = 0.25$ ($\bar{\alpha}_1 > \bar{\alpha}_2$). The solid and dashed line represent stable and unstable solutions, respectively.

In Fig. 10 the response-curves R_3 and R_4 are given as function of the excitation-amplitude β for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.50$ and $\bar{\eta} = 2.0$. In this figure non-planar motion can be observed for $\beta > K_2 = 0.5946$. In contrast to the results in Fig. 9 for varying values of $\bar{\eta}$, it is of interest to observe the additional stable critical point of type 2 which is indicated by the $\overline{K_2K_3}$ curve. When one follows the solution along this curve apparently at the end $\beta = K_3$ a jump from non-planar motion to planar motion occurs with an accompanying decrease in amplitude of the in-plane and out-of-plane components. However, by reversing the procedure, a different phenomenon occurs, that is, a jump from non-planar motion to planar motion occurs at $\beta = K_2$ followed by a sudden decrease in amplitude of the in-plane component but by a smooth decrease in amplitude of the out-of-plane component.

For $\bar{\alpha}_1 \neq \bar{\alpha}_2$ the response-curves R_3 and R_4 as function of the detuning parameter $\bar{\eta}$ and



(i) $\bar{\alpha}_1 = 0.25$ and $\bar{\alpha}_2 = 0.75$ (ii) $\bar{\alpha}_1 = 0.25$ and $\bar{\alpha}_2 = 0.75$

(iii) $\bar{\alpha}_1 = 0.75$ and $\bar{\alpha}_2 = 0.25$ (iv) $\bar{\alpha}_1 = 0.75$ and $\bar{\alpha}_2 = 0.25$

Figure 12: The stability response-curves of system (3.3) with respect to the excitation-amplitude β for $\bar{\alpha}_1 \neq \bar{\alpha}_2$ and $\bar{\eta} = 2.0$. In the upper figures $\bar{\alpha}_1 = 0.25$ and $\bar{\alpha}_2 = 0.75$ ($\bar{\alpha}_1 < \bar{\alpha}_2$) and in the lower figures $\bar{\alpha}_1 = 0.75$ and $\bar{\alpha}_2 = 0.25$ ($\bar{\alpha}_1 > \bar{\alpha}_2$). The solid and dashed line represent stable and unstable solutions, respectively.

the excitation-amplitude β are presented in Fig. 11 and in Fig. 12, respectively. In these figures jump phenomena can readily be observed. A jump downward or a jump upward from planar motion to non-planar motion or vice versa can be observed by varying $\bar{\eta}$ or β. For $\beta = 2$ and for increasing values of $\bar{\eta}$ the numerical calculations indicate that for $\bar{\alpha}_1 = 0.25$ and $\bar{\alpha}_2 = 0.75$ non-planar motion will die out approximately after $\bar{\eta} > 95.9681$ and after $\bar{\eta} > 11.9960$ for $\bar{\alpha}_1 = 0.75$ and $\bar{\alpha}_2 = 0.25$. This shows that the string can oscillate in-plane for large values of $\bar{\alpha}_1$.

As in the case without damping, the analytical results can be compared with numerical results which are obtained by for instance using a Runge-Kutta method. Here, we only compare the predicted stable motion of the string in the case $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.2580$, $\beta = 0.4561$, and $\bar{\eta} = 0.3225$. The results are presented in Fig. 13. This figure shows that the analytical results based on the averaging method are comparable with the numerical results on long

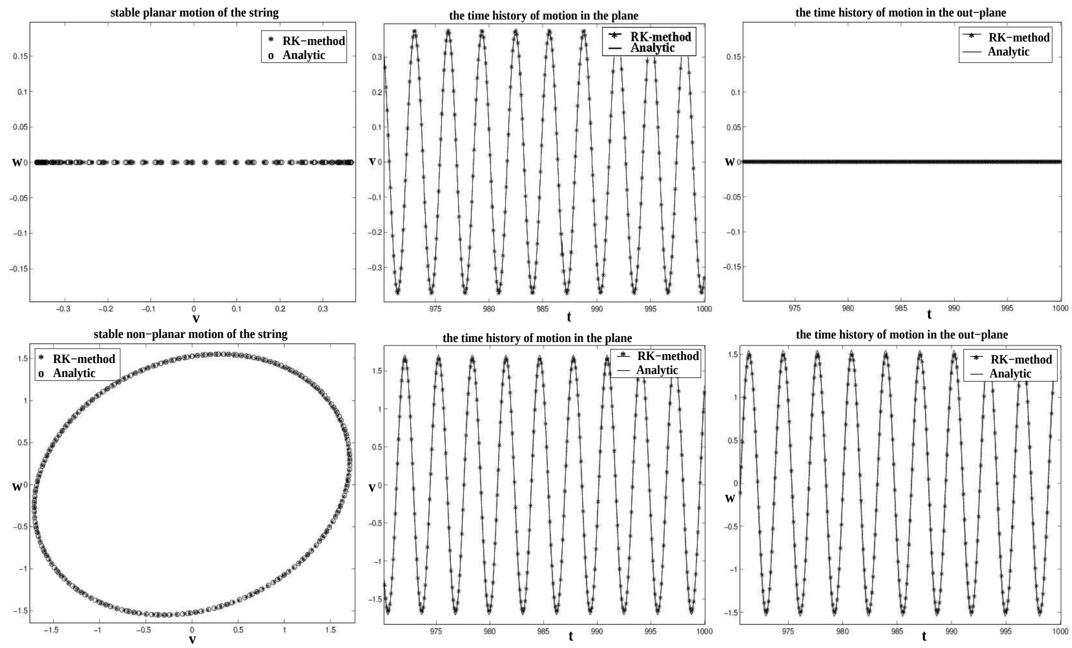


Figure 13: The stable motions of the string in the 1:1 resonance case for the first mode with $\bar{\varepsilon} = 0.0025$, $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.2580$, $\beta = 0.4561$, and $\bar{\eta} = 0.3225$. The upper figure presents planar motion and the lower figure presents non-planar motion.

time-scales. In the same way the analytical and the numerical results for $\bar{\alpha}_1 \neq \bar{\alpha}_2$ are close to each other on long time-scales.

4 Analysis of model 2

In this section system (2.6) will be studied, which describes the $O(1)$ responses of the string due to the transversal excitation and parametric excitation of the string. It is assumed that both F_1 and F_2 are nonzero (since the cases $F_1 = 0$ or $F_2 = 0$ already have been studied in the previous the previous section).
As in section 3 for model 1 we introduce $\lambda \bar{t} = 2t$, where $\lambda = \mu_m + \varepsilon\eta$ with $\mu_m = 2\mu_s$ (m=2s) and $\eta = O(1)$. system (2.6) then becomes (up to order ε):

$$
\begin{aligned}
v_m''(t) + 4v_m(t) &= -\frac{\varepsilon}{\mu_m}\Big(2\alpha_1 v_m'(t) + 4v_m(t)\Big[\gamma_m((v_m^2(t) + w_m^2(t)) + \frac{1}{4}(v_s^2(t) + w_s^2(t))) + \\
&\qquad F_1\mu_m \sin(2t) - 2\eta\Big] + 8F_2 \sin(2t)\Big), \\
w_m''(t) + 4w_m(t) &= -\frac{\varepsilon}{\mu_m}\Big(2\alpha_2 w_m'(t) + 4w_m(t)\Big[\gamma_m((v_m^2(t) + w_m^2(t)) + \frac{1}{4}(v_s^2(t) + w_s^2(t))) + \\
&\qquad F_1\mu_m \sin(2t) - 2\eta\Big]\Big), \\
v_s''(t) + v_s(t) &= -\frac{\varepsilon}{\mu_m}\Big(2\alpha_1 v_s'(t) + v_s(t)\Big[\gamma_m((v_m^2(t) + w_m^2(t)) + \frac{1}{4}(v_s^2(t) + w_s^2(t))) + \\
&\qquad F_1\mu_m \sin(2t) - 2\eta\Big] - 4F_2\mu_m d_s \sin(2t)\Big),
\end{aligned}
$$

$$w_s''(t)+w_s(t) = -\frac{\varepsilon}{\mu_m}\Big(2\alpha_2 w_s'(t)+w_s(t)\Big[\gamma_m((v_m^2(t)+w_m^2(t))+\frac{1}{4}(v_s^2(t)+w_s^2(t)))+ F_1\mu_m\sin(2t)-2\eta\Big]\Big). \tag{4.1}$$

By introducing a similar transformation as (3.2), and then by using the averaging method, one obtains as averaged system for (4.1):

$$\begin{aligned}
\bar{A}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}_1\bar{A}_m+\bar{B}_m\Big[\frac{3}{4}(\bar{A}_m^2+\bar{B}_m^2)+\frac{1}{4}(\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{8}(\bar{A}_s^2+\bar{B}_s^2+\bar{C}_s^2+\bar{D}_s^2)-2\bar{\eta}\Big]+\\
&\quad \frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{D}_m\Big),\\
\bar{B}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}_1\bar{B}_m-\bar{A}_m\Big[\frac{3}{4}(\bar{A}_m^2+\bar{B}_m^2)+\frac{1}{4}(\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{8}(\bar{A}_s^2+\bar{B}_s^2+\bar{C}_s^2+\bar{D}_s^2)-2\bar{\eta}\Big]-\\
&\quad \frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{C}_m-2\beta\Big),\\
\bar{C}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}_2\bar{C}_m+\bar{D}_m\Big[\frac{1}{4}(\bar{A}_m^2+\bar{B}_m^2)+\frac{3}{4}(\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{8}(\bar{A}_s^2+\bar{B}_s^2+\bar{C}_s^2+\bar{D}_s^2)-2\bar{\eta}\Big]+\\
&\quad \frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{B}_m\Big),\\
\bar{D}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}_2\bar{D}_m-\bar{C}_m\Big[\frac{1}{4}(\bar{A}_m^2+\bar{B}_m^2)+\frac{3}{4}(\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{8}(\bar{A}_s^2+\bar{B}_s^2+\bar{C}_s^2+\bar{D}_s^2)-2\bar{\eta}\Big]-\\
&\quad \frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{A}_m\Big),\\
\bar{A}_s'(t) &= -\bar{\varepsilon}\Big((\bar{\alpha}_1+\sigma)\bar{A}_s+\bar{B}_s\Big[\frac{1}{4}(\bar{A}_m^2+\bar{B}_m^2+\bar{C}_m^2+\bar{D}_m^2)+\frac{3}{32}(\bar{A}_s^2+\bar{B}_s^2)+\frac{1}{32}(\bar{C}_s^2+\bar{D}_s^2)\\
&\quad -\bar{\eta}\Big]+\frac{1}{16}(\bar{A}_s\bar{C}_s+\bar{B}_s\bar{D}_s)\bar{D}_s\Big),\\
\bar{B}_s'(t) &= -\bar{\varepsilon}\Big((\bar{\alpha}_1-\sigma)\bar{B}_s-\bar{A}_s\Big[\frac{1}{4}(\bar{A}_m^2+\bar{B}_m^2+\bar{C}_m^2+\bar{D}_m^2)+\frac{3}{32}(\bar{A}_s^2+\bar{B}_s^2)+\frac{1}{32}(\bar{C}_s^2+\bar{D}_s^2)\\
&\quad -\bar{\eta}\Big]-\frac{1}{16}(\bar{A}_s\bar{C}_s+\bar{B}_s\bar{D}_s)\bar{C}_s\Big),\\
\bar{C}_s'(t) &= -\bar{\varepsilon}\Big((\bar{\alpha}_2+\sigma)\bar{C}_s+\bar{D}_s\Big[\frac{1}{4}(\bar{A}_m^2+\bar{B}_m^2+\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{32}(\bar{A}_s^2+\bar{B}_s^2)+\frac{3}{32}(\bar{C}_s^2+\bar{D}_s^2)\\
&\quad -\bar{\eta}\Big]+\frac{1}{16}(\bar{A}_s\bar{C}_s+\bar{B}_s\bar{D}_s)\bar{B}_s\Big),\\
\bar{D}_s'(t) &= -\bar{\varepsilon}\Big((\bar{\alpha}_2-\sigma)\bar{D}_s-\bar{C}_s\Big[\frac{1}{4}(\bar{A}_m^2+\bar{B}_m^2+\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{32}(\bar{A}_s^2+\bar{B}_s^2)+\frac{3}{32}(\bar{C}_s^2+\bar{D}_s^2)\\
&\quad -\bar{\eta}\Big]-\frac{1}{16}(\bar{A}_s\bar{C}_s+\bar{B}_s\bar{D}_s)\bar{A}_s\Big),
\end{aligned} \tag{4.2}$$

where $\bar{\varepsilon}=\frac{\gamma_m}{\mu_m}\varepsilon$, $\bar{\eta}=\frac{\eta}{\gamma_m}$, $\beta=\frac{F_2}{\gamma_m}$, $\sigma=\frac{\mu_m}{4\gamma_m}F_1$, $\bar{\alpha}_i=\frac{\alpha_i}{\gamma_m}$, $i=1$ or 2, and where $\bar{A}_m(t)$, $\bar{B}_m(t)$, $\bar{C}_m(t)$, $\bar{D}_m(t)$, $\bar{A}_s(t)$, $\bar{B}_s(t)$, $\bar{C}_s(t)$, and $\bar{D}_s(t)$ are the averaged approximations of $A_m(t)$, $B_m(t)$, $C_m(t)$, $D_m(t)$, $A_s(t)$, $B_s(t)$, $C_s(t)$, and $D_m(t)$, respectively. It follows from the last four equations in (4.2) that (see also [20]):

$$\frac{d}{dt}(\bar{A}_s\bar{D}_s-\bar{B}_s\bar{C}_s)(t)=-\bar{\varepsilon}(\bar{\alpha}_1+\bar{\alpha}_2)(\bar{A}_s\bar{D}_s-\bar{B}_s\bar{C}_s)(t). \tag{4.3}$$

A first integral of system (4.2) then follows from (4.3), yielding

$$\overline{\mathbf{G}}(\bar{A}_s(t),\bar{B}_s(t),\bar{C}_s(t),\bar{D}_s(t))=(\bar{A}_s\bar{D}_s(t)-\bar{B}_s\bar{C}_s(t))e^{\bar{\varepsilon}(\bar{\alpha}_1+\bar{\alpha}_2)t}. \tag{4.4}$$

Based on (4.4) $\overline{\mathbf{G}}_E$ is defined by:

$$\begin{aligned}\overline{\mathbf{G}}_E &= \Big\{(\bar{A}_s(t),\bar{B}_s(t),\bar{C}_s(t),\bar{D}_s(t),\bar{A}_m(t),\bar{B}_m(t),\bar{C}_m(t),\bar{D}_m(t)):\\ &\quad |\,\overline{\mathbf{G}}(\bar{A}_s(t),\bar{B}_s(t),\bar{C}_s(t),\bar{D}_s(t))\,| = E, \quad E \geq 0\Big\}.\end{aligned} \tag{4.5}$$

Because of the properties of a first integral, the "hyperplanes" $\overline{\mathbf{G}}_E$ are invariants of system (4.2). Again from (4.4) one can easily deduce that for $\bar{\alpha}_1+\bar{\alpha}_2>0$ the solutions $\bar{A}_s(t)$, $\bar{B}_s(t)$, $\bar{C}_s(t)$, and $\bar{D}_s(t)$ of system (4.2) satisfy $(\bar{A}_s(t)\bar{D}_s(t)-\bar{B}_s(t)\bar{C}_s(t)) \to 0$ as $t\to\infty$. This implies that the critical points of system (4.2) can only be found in the hyperplane $\overline{\mathbf{G}}_0$. It follows from [20] that if $\bar{\alpha}_1 \neq \bar{\alpha}_2$, then one only obtains $\bar{A}_s=\bar{B}_s=0$ or $\bar{C}_s=\bar{D}_s=0$. In the other words, there is no interaction between the parametric in- and out-of-plane modes. To reduce the amount of computations we will restrict the analysis to the most interesting case $\bar{\alpha}_1=\bar{\alpha}_2=\bar{\alpha}>0$. For this case it turns out from (4.3) that the stability of the critical points of system (4.2) in the $(\bar{A}_s,\bar{B}_s,\bar{C}_s,\bar{D}_s,\bar{A}_m,\bar{A}_s,\bar{B}_m,\bar{C}_m,\bar{D}_m)$-space is exactly the same as in the hyperplanes $\overline{\mathbf{G}}_0$. Hence, the study of the existence and the stability of periodic solutions of system (4.2) can be restricted to the hyperplane $\overline{\mathbf{G}}_0$.
Now one defines the hyperplane $\overline{\mathbf{H}}_k$ as a partition of the hyperplane $\overline{\mathbf{G}}_0$ as follows:

$$\overline{\mathbf{H}}_k = \begin{cases} \Big\{(\bar{A}_s(t),\bar{B}_s(t),\bar{C}_s(t),\bar{D}_s(t),\bar{A}_m(t),\bar{B}_m(t),\bar{C}_m(t),\bar{D}_m(t)) \in \overline{\mathbf{G}}_0 \;\mid\; \bar{A}_s(t)=k\bar{C}_s(t) \\ \qquad \text{and } \bar{B}_s(t)=k\bar{D}_s(t)\Big\};\ \text{k is a real number}, \\ \Big\{(\bar{A}_s(t),\bar{B}_s(t),\bar{C}_s(t),\bar{D}_s,\bar{A}_m(t),\bar{B}_m(t),\bar{C}_m(t),\bar{D}_m(t)) \in \overline{\mathbf{G}}_0 \;\mid\; \bar{C}_s=0 \text{ and} \\ \qquad \bar{D}_s(t)=0\Big\};\ k=\pm\infty. \end{cases} \tag{4.6}$$

The behaviour of the solutions of the critical points of system (4.2) in the hyperplane $\overline{\mathbf{H}}_k$ is described by:

$$\begin{aligned}
\bar{A}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}\bar{A}_m+\bar{B}_m\Big[\frac{1}{4}(3(\bar{A}_m^2+\bar{B}_m^2)+(\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{2}(k^2+1)(\bar{C}_s^2+\bar{D}_s^2))-2\bar{\eta}\Big]+\\
&\quad \frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{D}_m\Big),\\
\bar{B}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}\bar{B}_m-\bar{A}_m\Big[\frac{1}{4}(3(\bar{A}_m^2+\bar{B}_m^2)+(\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{2}(k^2+1)(\bar{C}_s^2+\bar{D}_s^2))-2\bar{\eta}\Big]-\\
&\quad \frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{C}_m-2\beta\Big),\\
\bar{C}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}\bar{C}_m+\bar{D}_m\Big[\frac{1}{4}((\bar{A}_m^2+\bar{B}_m^2)+3(\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{2}(k^2+1)(\bar{C}_s^2+\bar{D}_s^2))-2\bar{\eta}\Big]+\\
&\quad \frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{B}_m\Big),\\
\bar{D}_m'(t) &= -\bar{\varepsilon}\Big(\bar{\alpha}\bar{D}_m-\bar{C}_m\Big[\frac{1}{4}((\bar{A}_m^2+\bar{B}_m^2)+3(\bar{C}_m^2+\bar{D}_m^2)+\frac{1}{2}(k^2+1)(\bar{C}_s^2+\bar{D}_s^2))-2\bar{\eta}\Big]-\\
&\quad \frac{1}{2}(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)\bar{A}_m\Big),\\
\bar{C}_s'(t) &= -\bar{\varepsilon}\Big((\bar{\alpha}+\sigma)\bar{C}_s+\bar{D}_s\Big[\frac{1}{4}((\bar{A}_m^2+\bar{B}_m^2+\bar{C}_m^2+\bar{D}_m^2)+\frac{3}{8}(k^2+1)(\bar{C}_s^2+\bar{D}_s^2))-\bar{\eta}\Big]\Big),\\
\bar{D}_s'(t) &= -\bar{\varepsilon}\Big((\bar{\alpha}-\sigma)\bar{D}_s-\bar{C}_s\Big[\frac{1}{4}((\bar{A}_m^2+\bar{B}_m^2+\bar{C}_m^2+\bar{D}_m^2)+\frac{3}{8}(k^2+1)(\bar{C}_s^2+\bar{D}_s^2))-\bar{\eta}\Big]\Big),
\end{aligned} \tag{4.7}$$

It should be observed that the behaviour of the solutions of the critical points of system (4.2) in the hyperplane $\overline{\mathbf{G}}_0$ corresponds to the behaviour of the solutions in the hyperplane $\overline{\mathbf{H}}_k$.

The periodic solutions of system (4.7) and their stability will be studied for the case $\bar{\alpha} > 0$. From the applicational point of view this is the most interesting case.
The critical points of system (4.7) for $\bar{\alpha} > 0$ satisfy (after introducing the rescalings $\bar{A}_m = \sqrt{\bar{\alpha}}\tilde{A}_m$, $\bar{B}_m = \sqrt{\bar{\alpha}}\tilde{B}_m$, $\bar{C}_m = \sqrt{\bar{\alpha}}\tilde{C}_m$, $\bar{D}_m = \sqrt{\bar{\alpha}}\tilde{D}_m$, $\bar{C}_s = \sqrt{\bar{\alpha}}\tilde{C}_s$, and $\bar{D}_s = \sqrt{\bar{\alpha}}\tilde{D}_s$):

$$
\begin{aligned}
&\tilde{A}_m + \tilde{B}_m\Big[\frac{1}{4}(3(\tilde{A}_m^2+\tilde{B}_m^2)+(\tilde{C}_m^2+\tilde{D}_m^2)+\frac{1}{2}(k^2+1)(\tilde{C}_s^2+\tilde{D}_s^2))-2\tilde{\eta}\Big]+\frac{1}{2}(\tilde{A}_m\tilde{C}_m+\\
&\qquad\qquad \tilde{B}_m\tilde{D}_m)\tilde{D}_m = 0,\\
&\tilde{B}_m - \tilde{A}_m\Big[\frac{1}{4}(3(\tilde{A}_m^2+\tilde{B}_m^2)+(\tilde{C}_m^2+\tilde{D}_m^2)+\frac{1}{2}(k^2+1)(\tilde{C}_s^2+\tilde{D}_s^2))-2\tilde{\eta}\Big]-\frac{1}{2}(\tilde{A}_m\tilde{C}_m+\\
&\qquad\qquad \tilde{B}_m\tilde{D}_m)\tilde{C}_m - 2\tilde{\beta} = 0,\\
&\tilde{C}_m + \tilde{D}_m\Big[\frac{1}{4}((\tilde{A}_m^2+\tilde{B}_m^2)+3(\tilde{C}_m^2+\tilde{D}_m^2)+\frac{1}{2}(k^2+1)(\tilde{C}_s^2+\tilde{D}_s^2))-2\tilde{\eta}\Big]+\frac{1}{2}(\tilde{A}_m\tilde{C}_m+\\
&\qquad\qquad \tilde{B}_m\tilde{D}_m)\tilde{B}_m = 0,\\
&\tilde{D}_m - \tilde{C}_m\Big[\frac{1}{4}((\tilde{A}_m^2+\tilde{B}_m^2)+3(\tilde{C}_m^2+\tilde{D}_m^2)+\frac{1}{2}(k^2+1)(\tilde{C}_s^2+\tilde{D}_s^2))-2\tilde{\eta}\Big]-\frac{1}{2}(\tilde{A}_m\tilde{C}_m+\\
&\qquad\qquad \bar{B}_m\tilde{D}_m)\tilde{A}_m = 0,\\
&(1+\tilde{\sigma})\tilde{C}_s + \tilde{D}_s\Big[\frac{1}{4}((\tilde{A}_m^2+\tilde{B}_m^2+\tilde{C}_m^2+\tilde{D}_m^2)+\frac{3}{8}(k^2+1)(\tilde{C}_s^2+\tilde{D}_s^2))-\tilde{\eta}\Big]=0,\\
&(1-\tilde{\sigma})\tilde{D}_s - \tilde{C}_s\Big[\frac{1}{4}((\tilde{A}_m^2+\tilde{B}_m^2+\tilde{C}_m^2+\tilde{D}_m^2)+\frac{3}{8}(k^2+1)(\tilde{C}_s^2+\tilde{D}_s^2))-\tilde{\eta}\Big]=0, \qquad (4.8)
\end{aligned}
$$

where $\tilde{\eta} = \frac{\bar{\eta}}{\bar{\alpha}}$, $\tilde{\sigma} = \frac{\sigma}{\bar{\alpha}}$, and $\tilde{\beta} = \frac{\beta}{\bar{\alpha}\sqrt{\bar{\alpha}}}$. The dependence of the solutions of system (4.8) on the parameters $\tilde{\sigma}$ and $\tilde{\beta}$ can be restricted to the case $\tilde{\sigma} > 0$ and $\tilde{\beta} > 0$, since the case $\tilde{\sigma} < 0$ and $\tilde{\beta} < 0$ leads by a simple transformation as defined in [20] and in section 3 to system (4.8) with $\tilde{\sigma} > 0$ and $\tilde{\beta} > 0$. As in subsection 3.2 it can be shown that when a solution of system (4.8) exists then $\tilde{B}_m \neq 0$, and that when $\tilde{C}_m = 0$ then also $\tilde{D}_m = 0$. When a solution of system (4.8) exists, then $\tilde{C}_s = \tilde{D}_s = 0$ can be a solution of system (4.8). It follows from the fifth and the sixth equation in (4.8) that the non-trivial solutions $\tilde{C}_s$ and $\tilde{D}_s$ are given by $\tilde{C}_s = \pm\sqrt{\frac{\tilde{\sigma}-1}{\tilde{\sigma}+1}}\tilde{D}_s$ under the conditions $\frac{1}{4}((\tilde{A}_m^2+\tilde{B}_m^2+\tilde{C}_m^2+\tilde{D}_m^2)+\frac{3}{8}(k^2+1)(\tilde{C}_s^2+\tilde{D}_s^2))-\tilde{\eta} = \mp\sqrt{\tilde{\sigma}^2-1}$, respectively. As we know from [20] the critical points related to $\tilde{C}_s = \sqrt{\frac{\tilde{\sigma}-1}{\tilde{\sigma}+1}}\tilde{D}_s$ are unstable. These unstable points will not be studied in this paper. Therefore, the type of critical points of system (4.7) are classified as follows:

$$
\begin{aligned}
&\text{CP-type 1:}\quad (\bar{C}_s,\bar{D}_s,\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m) = (0,0,\sqrt{\bar{\alpha}}\tilde{A}_m,\sqrt{\bar{\alpha}}\tilde{B}_m,0,0),\\
&\text{CP-type 2:}\quad (\bar{C}_s,\bar{D}_s,\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m) = (0,0,\sqrt{\bar{\alpha}}\tilde{A}_m,\sqrt{\bar{\alpha}}\tilde{B}_m,\sqrt{\bar{\alpha}}\tilde{C}_m,\sqrt{\bar{\alpha}}\tilde{D}_m),\\
&\text{CP-type 3:}\quad (\bar{C}_s,\bar{D}_s,\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m) = (-\sqrt{\frac{\bar{\alpha}(\tilde{\sigma}-1)}{\tilde{\sigma}+1}}\tilde{D}_s,\sqrt{\bar{\alpha}}\tilde{D}_s,\sqrt{\bar{\alpha}}\tilde{A}_m,\sqrt{\bar{\alpha}}\tilde{B}_m,0,0),\\
&\text{CP-type 4:}\quad (\bar{C}_s,\bar{D}_s,\bar{A}_m,\bar{B}_m,\bar{C}_m,\bar{D}_m) = (-\sqrt{\frac{\bar{\alpha}(\tilde{\sigma}-1)}{\tilde{\sigma}+1}}\tilde{D}_s,\sqrt{\bar{\alpha}}\tilde{D}_s,\sqrt{\bar{\alpha}}\tilde{A}_m,\sqrt{\bar{\alpha}}\tilde{B}_m,\\
&\qquad\qquad\qquad \sqrt{\bar{\alpha}}\tilde{C}_m,\sqrt{\bar{\alpha}}\tilde{D}_m). \qquad (4.9)
\end{aligned}
$$

It is clear that critical points of type 3 and 4 can be expected for $\tilde{\sigma} \geq 1$. The existence of critical points of type 1 and of type 2 can be determined as in subsection 3.2. Again it can be shown that their existence is independent of the parametric excitation. However, the behaviour of the solutions around these critical points might depend on it.
Substitution of the CP-type 3 as given in (4.9) into system (4.8) finally leads to

$$X^3 + \kappa_{21}X + \delta_{21} = 0,$$

$$\text{Cond}_8(X) = \frac{44}{15}\tilde{\eta} + \frac{92}{15}\sqrt{\tilde{\sigma}^2 - 1} - X > 0,$$
$$R_3 = X + \frac{16}{15}\tilde{\eta} - \frac{32}{15}\sqrt{\tilde{\sigma}^2 - 1},$$
$$R_2 = \frac{8}{3(1+k^2)}\text{Cond}_8(X), \tag{4.10}$$

where $R_2 = \tilde{A}_s^2 + \tilde{B}_s^2$, $R_3 = \tilde{A}_m^2 + \tilde{B}_m^2$, k is defined in (4.6), κ_{21} and δ_{21} are defined in Appendix 6.2.2. In a similar way, but now by using the CP-type 4 as given in (4.9), one obtains $R_3 = Y - \frac{1}{3}c_2$, $R_4 = \tilde{C}_m^2 + \tilde{D}_m^2 = \frac{1}{5}\text{Cond}_9(R_3)$, and $R_2 = \frac{8}{5(k^2+1)}\text{Cond}_{10}(R_3)$, where Y satisfies:

$$Y^3 + \kappa_{22}Y + \delta_{22} = 0,$$
$$\text{Cond}_9(R_3) = 8\tilde{\eta} - 16\sqrt{\tilde{\sigma}^2 - 1} - 2R_3 \pm 3\sqrt{R_3^2 - 16} > 0,$$
$$\text{Cond}_{10}(R_3) = 4\tilde{\eta} + 12\sqrt{\tilde{\sigma}^2 - 1} - R_3 \mp \sqrt{R_3^2 - 16} > 0, \tag{4.11}$$

where c_2, κ_{22}, and δ_{22} are defined in Appendix 6.2.2. The $+$ sign in $\text{Cond}_9(R_3)$ is related to the $-$ sign in $\text{Cond}_{10}(R_3)$ and vice versa. As indicated in the previous section, an overview of the number of real solutions of types 1-4 of system (4.8) and their dependence on $\tilde{\eta}$ and $\tilde{\beta}$ is given in Fig. 14.

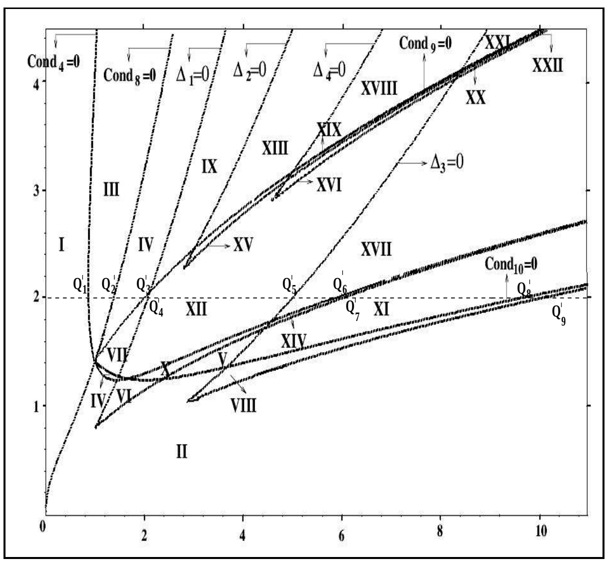


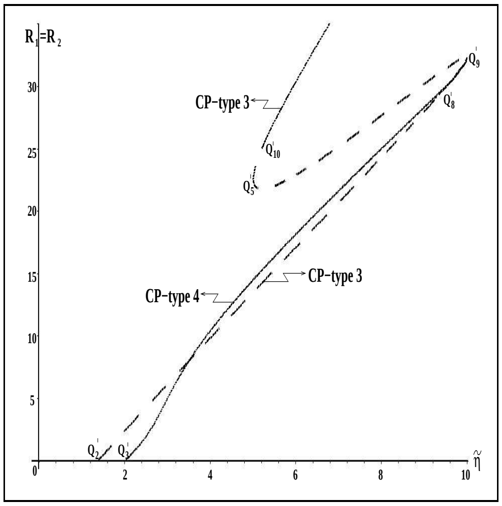
Figure 14: Twenty-two domains with real solutions of system (4.8) for $\bar{\alpha} > 0$ and $\tilde{\sigma} = 1$: $\Delta_3 = \frac{1}{4}\delta_{21}^2 + \frac{1}{27}\kappa_{21}^3$ and $\Delta_4 = \frac{1}{2}\delta_{22}^2 + \frac{1}{27}\kappa_{22}^3$. The vertical axis is the $\tilde{\beta}$ axis and the horizontal axis is the $\tilde{\eta}$ axis.

In contrast to the case without a parametric excitation, it now turns out that the non-planar motion consisting of the transversally excited modes (corresponding to CP-type 2) becomes unstable when the non-planar motion corresponding to CP-type 3 and/or CP-type 4 exists as is shown in Fig. 9 and Fig. 15. This is caused by the fact that the effect of the parametric excitation is not negligible to the damping. Moreover, the parametric excitation also reduces the domain of existence of this motion, but it enlarges the domain of existence of the planar motion (see Fig. 7 and Fig. 14). The number of critical points of system (4.7) in the domains in Fig. 14 is given in Table 4.

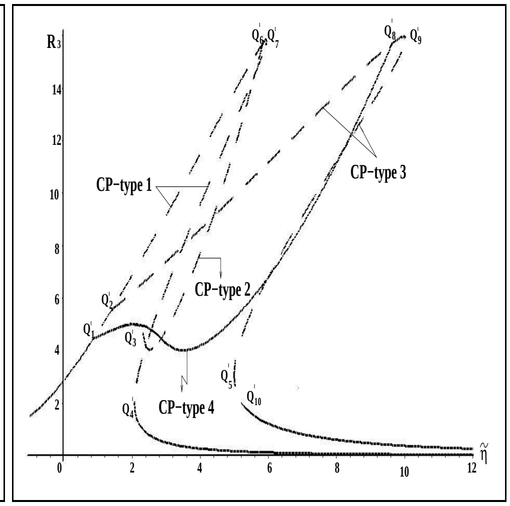
Table 4: The number of critical points of system (4.7) for $\bar{\alpha} > 0$ and $\tilde{\sigma} = 1$ as described in Fig. 14.

Domain/Curve	CP-type 1	CP-type 2	CP-type 3	CP-type 4	Total
I	1	0	0	0	1
II	1	0	1	0	2
III	1	1	0	0	2
IV	1	1	1	0	3
V	1	0	1	1	3
VI	3	0	1	0	4
VII	1	1	1	1	4
VIII	1	0	3	0	4
IX	3	1	1	0	5
X	3	0	1	1	5
XI	1	0	3	1	5
XII	3	1	1	1	6
XIII	3	3	1	0	7
XIV	3	0	3	1	7
XV	3	3	1	1	8
XVI	3	1	1	3	8
XVII	3	1	3	1	8
XVIII	3	3	1	2	9
XIX	3	3	1	3	10
XX	3	1	3	3	10
XXI	3	3	3	2	11
XXII	3	3	3	3	12

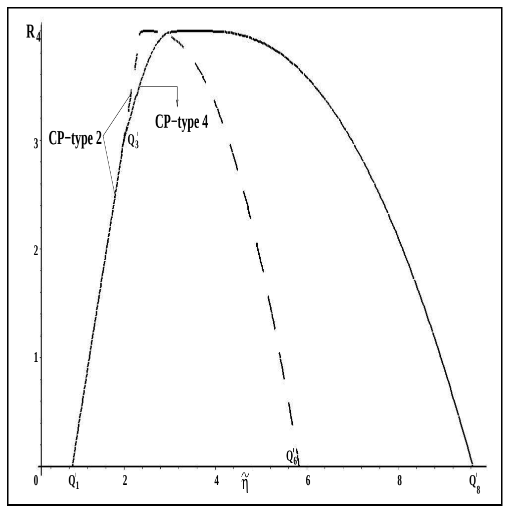
In Fig. 15 the stability response-curves for system (4.7) are plotted as function of $\tilde{\eta}$. In this figure jumps from planar motion to non-planar motion can be observed. Let us first consider the case for which $\tilde{\eta}$ increases from zero. One notes that when $\tilde{\eta}$ is smaller than Q_1', only a stable planar motion exists. At $\tilde{\eta} = Q_2'$ the motion will be transformed smoothly to the stable non-planar motion consisting of the transversally excited modes up to Q_3'. Between Q_3' and Q_8' the stable motion transforms smoothly to the stable non-planar motion consisting of the transversally excited modes and the parametrically excited modes in-plane and out-of-plane (CP-type 4). For increasing values of $\tilde{\eta}$ the stable motion becomes stable non-planar motion consisting of the parametrically excited modes in both planes and the transversally excited mode in-plane (CP-type 3). A jump occurs at $\tilde{\eta} = Q_9'$ followed by a decrease in the amplitudes R_1, R_2, and R_3. By reversing the procedure a jump upward occurs at $\tilde{\eta} = Q_4'$ or $\tilde{\eta} = Q_2'$ after which the stable motion changes smoothly. As an illustration the types of stable motions are presented in Fig. 16.



(i) $R_1 = \tilde{A}_s^2 + \tilde{B}_s^2$, $R_2 = \tilde{C}_s^2 + \tilde{D}_s^2$



(ii) $R_3 = \tilde{A}_m^2 + \tilde{B}_m^2$

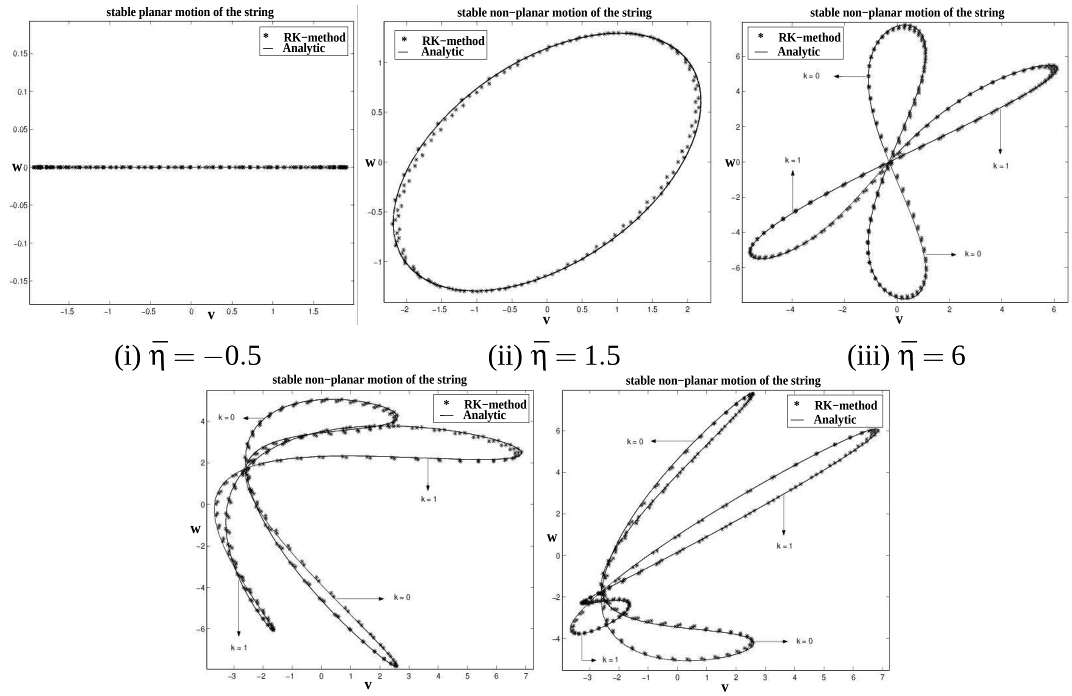


(iii) $R_4 = \tilde{C}_m^2 + \tilde{D}_m^2$

Figure 15: The stability response-curves for system (4.7) for $\bar{\alpha} > 0$, $\tilde{\sigma} = 1$, and $\tilde{\beta} = 2$. The solid and dashed line represent stable and unstable solutions, respectively.

5 Conclusions

In this paper a stretched string which is attached between a fixed support and a vibrating support is studied. The vibrations of the support lead to parametrical and normal (external) excitations of the string. By using a modified Kirchhoff approach (see also the previous paper) the model problem can be reduced to a coupled system of partial differential equations which describes the in- and out-of-plane vibrations of the string. There are two interesting cases which are related to the frequency λ of the vibrating support. In the first case λ is near an eigenfrequency μ_m of the string (where $\mu_m = m\pi$ and m is odd), and in the second case λ is near $2\mu_s = 2s\pi$. In the first case an interaction between the m-th modes (the transversally excited modes) in-plane and out-of-plane occurs. This interaction is described by a system of two nonlinearly coupled, second order ordinary differential equations. By using the averaging method the periodic solutions (and their stability) of these equations are found. Then,



(i) $\bar{\eta} = -0.5$ (ii) $\bar{\eta} = 1.5$ (iii) $\bar{\eta} = 6$

(iv) $\bar{\eta} = 6$ (v) $\bar{\eta} = 6$

Figure 16: Stable motions of the string for $\bar{\alpha} > 0$, $\tilde{\sigma} = 1$, and $\tilde{\beta} = 2$. Middle and right part of the upper figure are motions corresponding to CP-type 2 and CP-type 3, respectively, and in the lower figure the motions correspond to CP-type 4.

in a parameter plane the existence of all types of periodic solutions and their stability have been determined. In the second case (that is, for λ near $2\mu_s$) an interaction between the s-th modes (the parametrically excited modes) and the m-th modes (the transversally excited modes with $m = 2s$) in-plane and out-of-plane occurs. This interaction is described by a non-linear system of four second order, ordinary differential equations. As in the previous case, the stability of all types of periodic solutions has been determined. Stable non-planar motion consisting of the transversally excited modes in-plane and out-of-plane can only occur when positive damping is present. A number of jump phenomena from planar motion to non-planar motion and from non-planar motion to non-planar motion can occur in both cases for λ. These jumps are usually followed by an increase or decrease in the amplitudes of the periodic solutions.

6 Appendix

6.1 On the determination of the critical points

6.1.1 The case : $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ for system (3.3)

Proof $\bar{A}_m \neq 0$:
Let $\bar{A}_m = 0$ be a solution of (3.4). Substituting this into the first and the third equation of (3.4) yields:

$$\bar{B}_m\left[\frac{1}{4}(3\bar{B}_m^2 + \bar{C}_m^2 + 3\bar{D}_m^2) - 2\bar{\eta}\right] = 0,$$

$$\bar{D}_m\left[\frac{3}{4}(\bar{B}_m^2+\bar{C}_m^2+\bar{D}_m^2)-2\bar{\eta}\right]=0. \tag{A.1}$$

From (A.1) it follows that at least one of the $\bar{B}_m$, $\bar{C}_m$, or $\bar{D}_m$ is equal to zero. Substituting this into the second equation of (3.4) gives $\beta = 0$. This contradicts the assumption that $\beta = \frac{F_2}{\gamma_m} \neq 0$. So, if a solution of (3.4) exists, then $\bar{A}_m \neq 0$.

Proof $\bar{B}_m = 0$:
Let $\bar{B}_m \neq 0$ be a solution of system (3.4). By multiplying the first and the second equation of (3.4) with $\bar{A}_m$ and $\bar{B}_m$ respectively, and then by subtracting the so-obtained equations, it follows that:

$$(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)(\bar{A}_m\bar{D}_m-\bar{B}_m\bar{C}_m)+4\beta\bar{B}_m=0. \tag{A.2}$$

Since $\beta \neq 0$ (A.2) then implies that at least one of the $\bar{C}_m$ or $\bar{D}_m$ is not equal to zero. On the other hand the last two equations in (3.4) imply that $\bar{C}_m = 0$ if and only if $\bar{D}_m = 0$. Therefore, by multiplying the third and the fourth equation of (3.4) with $\bar{C}_m$ and $\bar{D}_m$ respectively, and by subtracting the so-obtained equations, it follows that

$$(\bar{A}_m\bar{C}_m+\bar{B}_m\bar{D}_m)(\bar{A}_m\bar{D}_m-\bar{B}_m\bar{C}_m)=0. \tag{A.3}$$

It then follows from (A.2) and (A.3) that $4\beta\bar{B}_m = 0$, implying $\bar{B}_m = 0$ (since $\beta = 0$). So, if a solution of (3.4) exists, then $\bar{B}_m = 0$.

Proof $\bar{C}_m = 0$:
Let $\bar{C}_m \neq 0$ be a solution of system (3.4), then the first equation of (3.4) implies that $\bar{D}_m = 0$ (by using the fact that $\bar{A}_m \neq 0$, $\bar{B}_m = 0$). By substituting this into the second and fourth equation of (3.4) it follows that:

$$\begin{aligned}\bar{A}_m\left[\frac{3}{4}(\bar{A}_m^2+\bar{C}_m^2)-2\bar{\eta}\right]-2\beta&=0,\\ \bar{C}_m\left[\frac{3}{4}(\bar{A}_m^2+\bar{C}_m^2)-2\bar{\eta}\right]&=0.\end{aligned} \tag{A.4}$$

Because both $\bar{A}_m$ and $\bar{C}_m$ are not equal to zero, it follows that $\beta = 0$, which is a contradiction. So, if a solution of (3.4) exists, then $\bar{C}_m$ should be equal to zero.

6.1.2 The case : $\bar{\alpha}_1, \bar{\alpha}_2 > 0$ for system (3.3)

Proof $\bar{B}_m \neq 0$ **:**
Let $\bar{B}_m = 0$ be a solution of system (3.9). Since $\beta \neq 0$ it then follows from the second equation in (3.9) that $\bar{A}_m \neq 0$. From the first equation in (3.9) it can then be deduced that $\bar{C}_m\bar{D}_m = -2\bar{\alpha}_1$. By multiplying the third and the fourth equation in (3.9) with $\bar{C}_m$ and $\bar{D}_m$ respectively, and by adding the so-obtained equations it follows that (using the fact that $\bar{C}_m\bar{D}_m = -2\bar{\alpha}_1$):

$$\bar{\alpha}_2(\bar{C}_m^2+\bar{D}_m^2)+\bar{\alpha}_1\bar{A}_m^2=0. \tag{A.5}$$

Since $\bar{\alpha}_1, \bar{\alpha}_2 > 0$ it follows from (A.5) that $\bar{A}_m = \bar{C}_m = \bar{D}_m = 0$. Then, by substituting $\bar{A}_m = \bar{B}_m = \bar{C}_m = \bar{D}_m = 0$ into the second equation of (3.9) it follows that $\beta = 0$, which contradicts the assumption that $\beta \neq 0$. So, if a solution of (3.9) exists, then $\bar{B}_m \neq 0$.

Proof If $\bar{C}_m = 0$ then $\bar{D}_m = 0$:
Let $\bar{C}_m = 0$ be a solution of system (3.9). Then, it follows from the fourth equation in (3.9)

that $\bar{D}_m = 0$ or $2\bar{\alpha}_2 = \bar{A}_m\bar{B}_m$. For $\bar{D}_m \neq 0$ it then follows that $2\bar{\alpha}_2 = \bar{A}_m\bar{B}_m$, and since $\bar{\alpha}_2 > 0$ it follows that $\bar{A}_m$ and $\bar{B}_m$ are both nonzero, and that $\bar{A}_m\bar{B}_m > 0$. By multiplying the third equation in (3.9) with $-\bar{B}_m$ and by adding this to the first equation in (3.9) one obtains: $\bar{\alpha}_1\bar{A}_m + \frac{1}{2}\bar{A}_m^2\bar{B}_m = 0 \Leftrightarrow \bar{A}_m = 0$ or $\bar{A}_m\bar{B}_m = -2\bar{\alpha}_1 < 0$, which contradicts the fact that (for $\bar{D}_m \neq 0$) $\bar{A}_m\bar{B}_m > 0$. So, if a solution of (3.9) exists with $\bar{C}_m = 0$, then $\bar{D}_m$ also should be equal to zero.

6.2 Critical Points

6.2.1 Model 1 ($\lambda = \mu_m + O(\varepsilon)$, m is an odd number)

When at least one of the components $\tilde{A}_m$ and $\tilde{D}_m$ of the critical point of type 2 is equal to zero, the properties of the critical point can easily be derived. Now we consider the case for which all of the components of the critical points of type 2 are not equal to zero. By multiplying the first and second equation in (3.9) with $\tilde{A}_m$ and $\tilde{B}_m$ respectively, and then by adding the so-obtained equations it follows that:

$$\bar{\alpha}_1(\tilde{A}_m^2+\tilde{B}_m^2)+\frac{1}{2}(\tilde{A}_m\tilde{C}_m+\tilde{B}_m\tilde{D}_m)(\tilde{A}_m\tilde{D}_m-\tilde{B}_m\tilde{C}_m)+2\beta\tilde{B}_m = 0. \tag{B.1}$$

Again by multiplying the first and second equation in (3.9) but now with $\tilde{B}_m$ and $\tilde{A}_m$ respectively, and then by subtracting the so-obtained equations it follows that:

$$(\tilde{A}_m^2+\tilde{B}_m^2)\left[\frac{1}{4}(3(\tilde{A}_m^2+\tilde{B}_m^2)+(\tilde{C}_m^2+\tilde{D}_m^2))-2\bar{\eta}\right]+\frac{1}{2}(\tilde{A}_m\tilde{C}_m+\tilde{B}_m\tilde{D}_m)^2-2\beta\tilde{A}_m = 0. \tag{B.2}$$

Similarly, it follows from the third and fourth equation in (3.9) that

$$\begin{aligned}&(\tilde{C}_m^2+\tilde{D}_m^2)\left[\frac{1}{4}((\tilde{A}_m^2+\tilde{B}_m^2)+3(\tilde{C}_m^2+\tilde{D}_m^2))-2\bar{\eta}\right]+\frac{1}{2}(\tilde{A}_m\tilde{C}_m+\tilde{B}_m\tilde{D}_m)^2 = 0, \quad \text{and}\\ &\bar{\alpha}_2(\tilde{C}_m^2+\tilde{D}_m^2)-\frac{1}{2}(\tilde{A}_m\tilde{C}_m+\tilde{B}_m\tilde{D}_m)(\tilde{A}_m\tilde{D}_m-\tilde{B}_m\tilde{C}_m) = 0.\end{aligned} \tag{B.3}$$

From (B.3) it follows that a critical point of type 2 can be expected if the following condition is satisfied:

$$\text{Cond}_2 = (\tilde{A}_m^2+\tilde{B}_m^2)+3(\tilde{C}_m^2+\tilde{D}_m^2)-8\bar{\eta} < 0. \tag{B.4}$$

Accordingly, to have a critical point of type 2 it is assumed that Cond_2 holds. By adding (B.1) and the second equation in (B.3) one obtains

$$\bar{\alpha}_1(\tilde{A}_m^2+\tilde{B}_m^2)+\bar{\alpha}_2(\tilde{C}_m^2+\tilde{D}_m^2) = -2\beta\tilde{B}_m, \tag{B.5}$$

and by subtracting (B.2) from the first equation in (B.3), it follows that

$$\left[(\tilde{A}_m^2+\tilde{B}_m^2)-(\tilde{C}_m^2+\tilde{D}_m^2)\right]\left[\frac{3}{4}((\tilde{A}_m^2+\tilde{B}_m^2)+(\tilde{C}_m^2+\tilde{D}_m^2))-2\bar{\eta}\right] = 2\beta\tilde{A}_m. \tag{B.6}$$

Let us rewrite (B.3) as

$$\begin{aligned}&-2(\tilde{C}_m^2+\tilde{D}_m^2)\left[\frac{1}{4}((\tilde{A}_m^2+\tilde{B}_m^2)+3(\tilde{C}_m^2+\tilde{D}_m^2))-2\bar{\eta}\right] = (\tilde{A}_m\tilde{C}_m+\tilde{B}_m\tilde{D}_m)^2,\\ &2\bar{\alpha}_2(\tilde{C}_m^2+\tilde{D}_m^2) = (\tilde{A}_m\tilde{C}_m+\tilde{B}_m\tilde{D}_m)(\tilde{A}_m\tilde{D}_m-\tilde{B}_m\tilde{C}_m).\end{aligned} \tag{B.7}$$

By squaring both sides of the first and second equation in (B.7), and then by adding the so-obtained equations, one obtains (by using the fact that $\tilde{C}_m^2+\tilde{D}_m^2 \neq 0$):

$$\begin{aligned}4(\tilde{C}_m^2+\tilde{D}_m^2)\left(\bar{\alpha}_2^2+\left[\frac{1}{4}((\tilde{A}_m^2+\tilde{B}_m^2)+3(\tilde{C}_m^2+\tilde{D}_m^2))-2\bar{\eta}\right]^2\right) = (\tilde{A}_m\tilde{C}_m+{}\\ \tilde{B}_m\tilde{D}_m)^2(\tilde{A}_m^2+\tilde{B}_m^2).\end{aligned} \tag{B.8}$$

From (B.8) and the first equation in (B.7) it follows that (after some simplifications):

$$\left[\frac{1}{4}(2(\tilde{A}_m^2+\tilde{B}_m^2)+3(\tilde{C}_m^2+\tilde{D}_m^2))-2\bar{\eta}\right]^2=\frac{1}{16}(\tilde{A}_m^2+\tilde{B}_m^2)^2-\bar{\alpha}_2^2. \tag{B.9}$$

To have a critical point of type 2 it follows from (B.9) that $\tilde{A}_m$ and $\tilde{B}_m$ have to satisfy

$$\mathrm{Cond}_3=(\tilde{A}_m^2+\tilde{B}_m^2)-4\bar{\alpha}_2\geq 0. \tag{B.10}$$

By squaring both sides of (B.5) and (B.6), and then by adding the so-obtained equations, one obtains (after some simplifications):

$$\begin{aligned}&\left[(\tilde{A}_m^2+\tilde{B}_m^2)-(\tilde{C}_m^2+\tilde{D}_m^2)\right]^2\left[\frac{3}{4}((\tilde{A}_m^2+\tilde{B}_m^2)+(\tilde{C}_m^2+\tilde{D}_m^2))-2\bar{\eta}\right]^2+\left[\bar{\alpha}_1(\tilde{A}_m^2+\tilde{B}_m^2)+\right.\\&\left.\bar{\alpha}_2(\tilde{C}_m^2+\tilde{D}_m^2)\right]^2-4\beta^2(\tilde{A}_m^2+\tilde{B}_m^2)=0.\end{aligned} \tag{B.11}$$

By putting $R_3=\tilde{A}_m^2+\tilde{B}_m^2$ and $R_4=\tilde{C}_m^2+\tilde{D}_m^2$ it follows that (B.9) and (B.11) can be simplified to

$$\begin{aligned}&a_0R_3^4+\left[a_1(\bar{\alpha}_1-\bar{\alpha}_2)-4\beta^2\right]R_3^3+\left[a_2(\bar{\alpha}_1-\bar{\alpha}_2)+a_3\right]R_3^2+\left[a_4(\bar{\alpha}_1-\bar{\alpha}_2)+a_5\right]R_3+\\&\qquad a_6(\bar{\alpha}_1-\bar{\alpha}_2)+a_7=0,\\&R_4=\frac{8}{3}\bar{\eta}-\frac{2}{3}R_3\pm\frac{1}{3}\sqrt{(R_3^2-16\bar{\alpha}_2^2)},\end{aligned} \tag{B.12}$$

where:

$$\begin{aligned}a_0&=(\bar{\alpha}_1-\bar{\alpha}_2)^2,\\a_1&=-\frac{4}{3}(3\bar{\alpha}_1-7\bar{\alpha}_2)\bar{\eta},\\a_2&=\frac{1}{36}\left[16(9\bar{\alpha}_1-73\bar{\alpha}_2)\bar{\eta}^2+81\bar{\alpha}_1^3-135\bar{\alpha}_1^2\bar{\alpha}_2-81\bar{\alpha}_1\bar{\alpha}_2^2+249\bar{\alpha}_2^3\right],\\a_3&=\frac{16}{9}(4\bar{\alpha}_1\bar{\alpha}_2\bar{\eta}^2+9\beta^2\bar{\eta}+9\bar{\alpha}_1\bar{\alpha}_2^3),\\a_4&=\frac{2}{9}\left[96\bar{\alpha}_2\bar{\eta}^3+4(27\bar{\alpha}_1^2+104\bar{\alpha}_1\bar{\alpha}_2+35\bar{\alpha}_2^2)\bar{\alpha}_2\bar{\eta}-27\beta^2(3\bar{\alpha}_1+\bar{\alpha}_2)\right],\\a_5&=-\frac{16}{9}(16\bar{\alpha}_2^2\bar{\eta}^3+9\beta^2\bar{\eta}^2+52\bar{\alpha}_1\bar{\alpha}_2\bar{\eta}-\frac{27}{4}\beta^2\bar{\alpha}_1\bar{\alpha}_2),\\a_6&=-\frac{16}{9}\bar{\alpha}_2\left[4(17\bar{\alpha}_1+5\bar{\alpha}_2)\bar{\alpha}_2\bar{\eta}^2+54\beta^2\bar{\eta}+4\bar{\alpha}_2^3(4\bar{\alpha}_1+\bar{\alpha}_2)\right],\\a_7&=\frac{16}{9}\left[\bar{\alpha}_2^2(16\bar{\eta}^4+104\bar{\alpha}_1^2\bar{\eta}^2-72\beta^2\bar{\eta}+25\bar{\alpha}_1^2\bar{\alpha}_2^2)+\frac{81}{4}\beta^4\right].\end{aligned}$$

If one assumes additionally that $\bar{\alpha}_1=\bar{\alpha}_2=\alpha$, then the first equation in (B.12) can be simplified to a standard cubic equation:

$$X^3+\kappa_{12}X+\delta_{12}=0, \tag{B.13}$$

where:

$$\begin{aligned}X&=R_3-\frac{1}{12\beta^2}a_3,\\\kappa_{12}&=-\frac{1}{48\beta^4}(12\beta^2a_5+a_3^2)),\\\delta_{12}&=-\frac{1}{864\beta^6}(18\beta^2a_3a_5+216\beta^4a_7+a_3^3).\end{aligned}$$

For $\bar{\alpha}_1 \neq \bar{\alpha}_2$ a simplification of the first equation in (B.12) can be obtained by introducing a new variable $Z = R_3 + \frac{1}{4a_0}(a_1(\bar{\alpha}_1 - \bar{\alpha}_2) - 4\beta^2)$. Substituting this variable Z into the first equation of (B.12), one obtains the standard quartic equation in Z:

$$Z^4 + p_1 Z^2 + q_1 Z + r_1 = 0, \tag{B.14}$$

where:

$$\begin{aligned}
p_1 &= \frac{3\beta^2}{a_0^2}\Big((\bar{\alpha}_1 - \bar{\alpha}_2)a_1 - 2\beta^2\Big) + \frac{1}{a_0}\Big((\bar{\alpha}_1 - \bar{\alpha}_2)a_2 + a_3 - \frac{3}{8}a_1^2\Big),\\
q_1 &= \frac{2\beta^4}{a_0^3}\Big(3(\bar{\alpha}_1 - \bar{\alpha}_2)a_1 - 4\beta^2\Big) + \frac{1}{a_0^2}\Big((\bar{\alpha}_1 - \bar{\alpha}_2)(\frac{1}{8}a_1^3 - \frac{1}{2}a_1 a_2 + 2\beta^2 a_2) + \beta^2(2a_3 -\\
&\quad \frac{3}{2}a_1^2)\Big) + \frac{1}{a_0}\Big((\bar{\alpha}_1 - \bar{\alpha}_2)a_4 + a_5 - \frac{1}{2}a_1 a_2\Big),\\
r_1 &= \frac{3\beta^6}{a_0^4}\Big((\bar{\alpha}_1 - \bar{\alpha}_2)a_1 - \beta^2\Big) + \frac{1}{a_0^3}\Big((\bar{\alpha}_1 - \bar{\alpha}_2)\beta^2(\frac{3}{16}a_1^3 - \frac{1}{2}a_1 a_3 + \beta^2 a_2) + \beta^4(a_3 -\\
&\quad \frac{9}{8}a_1^2)\Big) + \frac{1}{a_0^2}\Big((\bar{\alpha}_1 - \bar{\alpha}_2)(\frac{1}{16}a_1^2 a_2 + \beta^2 a_4 - \frac{1}{4}a_1 a_5) + \frac{1}{16}a_1^2(a_3 - \frac{3}{16}a_1^2) +\\
&\quad \beta^2(a_5 - \frac{1}{2}a_1 a_2)\Big) + \frac{1}{a_0}\Big((\bar{\alpha}_1 - \bar{\alpha}_2)a_6 + a_7 - \frac{1}{4}a_1 a_4\Big).
\end{aligned}$$

The four solutions Z_i, $i = 1,2,3$, and 4 of (B.14) can be given as:

$$\begin{aligned}
Z_{1,2} &= \frac{1}{2}(\sqrt{z_1} \pm \sqrt{z_2} \pm \sqrt{z_3}),\\
Z_{3,4} &= \frac{1}{2}(-\sqrt{z_1} \pm \sqrt{z_2} \mp \sqrt{z_3}),
\end{aligned} \tag{B.15}$$

where z_i, $i = 1,2$, and 3 are solutions of the cubic resolvent:

$$z^3 + 2p_1 z^2 + (p_1^2 - 4r_1)z - q_1^2 = 0. \tag{B.16}$$

From (B.15) and $z_1 z_2 z_3 = q_1^2 > 0$ the relation between the solutions of (B.14) and (B.16) follows and is given in Table (5).
By introducing a new variable $\bar{z} = z + \frac{2}{3}p_1$ (B.16) becomes a standard cubic equation:

$$\bar{z}^3 + \kappa_{13}\bar{z} + \delta_{13} = 0, \tag{B.17}$$

where $\kappa_{13} = -(4r_1 + \frac{1}{3}p_1^2)$ and $\delta_{13} = -(\frac{2}{27}p_1^3 + q_1^2 - \frac{8}{3}p_1 r_1)$.

6.2.2 Model 2 ($\lambda = \mu_m + O(\varepsilon)$, m is an even number)

It can readily be seen that $\tilde{C}_s = \tilde{D}_s = 0$ is a solution of system (4.8). For $\tilde{C}_s = \tilde{D}_s = 0$ system (4.8) reduces to an identical form as system (3.9). Hence, the solutions of the reduced system (4.8) can be found directly from system (3.9). In order to have non-trivial solutions for $\tilde{C}_s$ and $\tilde{D}_s$ it follows from the last two equations of (4.8) that $\tilde{\sigma}^2 - 1 \geq 0$. By considering $\tilde{C}_s \neq 0$ and $\tilde{D}_s \neq 0$, it also follows from the last two equations of (4.8) that:

$$\begin{aligned}
&\frac{1}{4}(\tilde{A}_m^2 + \tilde{B}_m^2 + \tilde{C}_m^2 + \tilde{D}_m^2) + \frac{3}{32}(k^2+1)(\tilde{C}_s^2 + \tilde{D}_s^2) - \tilde{\eta} = -\sqrt{\tilde{\sigma}^2 - 1}, \quad \text{or}\\
&\frac{1}{4}(\tilde{A}_m^2 + \tilde{B}_m^2 + \tilde{C}_m^2 + \tilde{D}_m^2) + \frac{3}{32}(k^2+1)(\tilde{C}_s^2 + \tilde{D}_s^2) - \tilde{\eta} = \sqrt{\tilde{\sigma}^2 - 1},
\end{aligned} \tag{B.18}$$

Table 5: The three cases which arise from the relation between the solutions of (B.14) and (B.16).

solutions of the cubic resolvent (B.16)	solutions of the standard quartic equation (B.14)
all positive real solutions	all real solutions
one positive and two negative real solutions	two pairs of conjugate complex solutions
one real and two conjugate complex solutions	two real and two conjugate complex solutions

and that $\tilde{C}_s = \sqrt{\frac{\tilde{\sigma}-1}{\tilde{\sigma}+1}}\tilde{D}_s$ and $\tilde{C}_s = -\sqrt{\frac{\tilde{\sigma}-1}{\tilde{\sigma}+1}}\tilde{D}_s$, correspondingly. However, the solution $\tilde{C}_s = \sqrt{\frac{\tilde{\sigma}-1}{\tilde{\sigma}+1}}\tilde{D}_s$ produces an unstable periodic solution (see also [20]). Hence, this solution will not be studied. Substitution of the second equation in (B.18) and $\tilde{C}_s = -\sqrt{\frac{\tilde{\sigma}-1}{\tilde{\sigma}+1}}\tilde{D}_s$ into system (4.8) gives

$$\begin{aligned}
&\tilde{A}_m + \tilde{B}_m\Big[\frac{5}{12}(\tilde{A}_m^2+\tilde{B}_m^2) - \frac{1}{12}(\tilde{C}_m^2+\tilde{D}_m^2) + \frac{4}{3}\sqrt{\tilde{\sigma}_s^2-1} - \frac{2}{3}\tilde{\eta}\Big] + \frac{1}{2}(\tilde{A}_m\tilde{C}_m + \\
&\qquad \tilde{B}_m\tilde{D}_m)\tilde{D}_m = 0,\\
&\tilde{B}_m - \tilde{A}_m\Big[\frac{5}{12}(\tilde{A}_m^2+\tilde{B}_m^2) - \frac{1}{12}(\tilde{C}_m^2+\tilde{D}_m^2) + \frac{4}{3}\sqrt{\tilde{\sigma}_s^2-1} - \frac{2}{3}\tilde{\eta}\Big] - \frac{1}{2}(\tilde{A}_m\tilde{C}_m + \\
&\qquad \tilde{B}_m\tilde{D}_m)\tilde{C}_m - 2\tilde{\beta} = 0,\\
&\tilde{C}_m + \tilde{D}_m\Big[-\frac{1}{12}(\tilde{A}_m^2+\tilde{B}_m^2) + \frac{5}{12}(\tilde{C}_m^2+\tilde{D}_m^2) + \frac{4}{3}\sqrt{\tilde{\sigma}^2-1} - \frac{2}{3}\tilde{\eta}\Big] + \frac{1}{2}(\tilde{A}_m\tilde{C}_m + \\
&\qquad \tilde{B}_m\tilde{D}_m)\tilde{B}_m = 0,\\
&\tilde{D}_m - \tilde{C}_m\Big[-\frac{1}{12}(\tilde{A}_m^2+\tilde{B}_m^2) + \frac{5}{12}(\tilde{C}_m^2+\tilde{D}_m^2) + \frac{4}{3}\sqrt{\tilde{\sigma}^2-1} - \frac{2}{3}\tilde{\eta}\Big] - \frac{1}{2}(\tilde{A}_m\tilde{C}_m + \\
&\qquad \tilde{B}_m\tilde{D}_m)\tilde{A}_m = 0.
\end{aligned} \tag{B.19}$$

It is easy to see that $\tilde{C}_m = \tilde{D}_m = 0$ satisfies the last two equations of (B.19). Substituting this into the first equation in (B.19) yields $\tilde{A}_m = -\frac{R_3}{24}(5R_3 + 16\sqrt{\tilde{\sigma}^2-1} - 8\tilde{\eta})$ and $\tilde{B}_m = \frac{R_3}{2\tilde{\beta}}$, where R_3 satisfies:

$$\begin{aligned}
&X^3 + \kappa_{21}X + \delta_{21} = 0,\\
&R_3 = X - \frac{1}{3}b_2,
\end{aligned} \tag{B.20}$$

$\kappa_{21} = b_1 - \frac{1}{3}b_2^2$, $\delta_{21} = -\frac{1}{27}(9b_1b_2 - 2b_2^3 - 27b_0)$, $b_0 = -\frac{576}{25}\tilde{\beta}^2$, $b_1 = \frac{16}{25}(16\tilde{\sigma}^2 + 4\tilde{\eta}^2 - 16\sqrt{\tilde{\sigma}^2-1} - 7)$, and $b_2 = \frac{16}{5}(2\sqrt{\tilde{\sigma}^2-1} - \tilde{\eta})$. Solutions of (B.20) are only solutions of system (4.8) when the condition $\mathrm{Cond}_8(X) = \frac{44}{15}\tilde{\eta} + \frac{92}{15}\sqrt{\tilde{\sigma}^2-1} - X > 0$ has been satisfied.

The solution $\tilde{C}_s$ satisfies $R_2 = \tilde{C}_s^2 + \tilde{D}_s^2 = \frac{8}{3(k^2+1)}\text{Cond}_8(X)$. For $\tilde{C}_m \neq 0$ and $\tilde{D}_m \neq 0$ the solutions of system (B.19) can be obtained in a similar way as has been shown in (B.1)-(B.13), and then by substituting these solutions into the second equation in (B.18) the solutions for $\tilde{C}_s$ are found. After some calculations, the following equations are obtained:

$$\begin{aligned}
&Y^3 + \kappa_{22}Y + \delta_{22} = 0,\\
&R_3 = Y - \frac{1}{3}c_2,\\
&R_4 = \frac{1}{5}(8\tilde{\eta} - 16\sqrt{\tilde{\sigma}^2-1} - 2R_3 \pm 3\sqrt{R_3^2 - 16}),\\
&R_2 = \frac{8}{5(k^2+1)}(4\tilde{\eta} + 12\sqrt{\tilde{\sigma}^2-1} - R_3 \mp \sqrt{R_3^2-16}),\\
&\text{Cond}_9(R_3) = 8\tilde{\eta} - 16\sqrt{\tilde{\sigma}^2-1} - 2R_3 \pm 3\sqrt{R_3^2-16},\\
&\text{Cond}_{10}(R_3) = 4\tilde{\eta} + 12\sqrt{\tilde{\sigma}^2-1} - R_3 \mp \sqrt{R_3^2-16},
\end{aligned} \tag{B.21}$$

where $\kappa_{22} = c_1 - \frac{1}{3}c_2^2$, $\delta_{22} = -\frac{1}{27}(9c_1c_2 - 2c_2^3 - 27c_0)$, and

$$\begin{aligned}
c_0 &= -\frac{64}{25\tilde{\beta}^2}\Big[16\tilde{\eta}^4 - 128\tilde{\eta}^3\sqrt{\tilde{\sigma}^2-1} + 8\tilde{\eta}^2(48\tilde{\sigma}^2 - 19) - 8\tilde{\eta}(64\tilde{\sigma}^2\sqrt{\tilde{\sigma}^2-1} +\\
&\quad 52\sqrt{\tilde{\sigma}^2-1} + 25\tilde{\beta}^2) + (16\tilde{\sigma}^2+33)(16\tilde{\sigma}^2-7) + \frac{25\tilde{\beta}^2}{4}(64\sqrt{\tilde{\sigma}^2-1} + 25\tilde{\beta}^2\Big],\\
c_1 &= \frac{4}{25\tilde{\beta}^2}\Big[16\tilde{\eta}^3 + (25\tilde{\beta}^2 - 96\sqrt{\tilde{\sigma}^2-1})\tilde{\eta}^2 + 4(48\tilde{\sigma}^2 - 25\tilde{\beta}^2\sqrt{\tilde{\sigma}^2-1} - 19)\tilde{\eta} -\\
&\quad 8(16\tilde{\sigma}^2+13)\sqrt{\tilde{\sigma}^2-1} + 25\tilde{\beta}^2(4\tilde{\sigma}^2 - \frac{3}{4}\tilde{\beta}^2)\Big],\\
c_2 &= -\frac{4}{25\tilde{\beta}^2}\Big[4\tilde{\eta}^2 + (25\tilde{\beta}^2 - 16\sqrt{\tilde{\sigma}^2-1})\tilde{\eta} + 16\tilde{\sigma}^2 + 9 - 50\tilde{\beta}^2\sqrt{\tilde{\sigma}^2-1}\Big].
\end{aligned}$$

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