Orthogonality relations of q-Meixner polynomials

with the use of spectral analysis

C. van Doesburg

Master thesis
Analysis

$$\phi_\gamma(x;a,b) = \frac{(-a\gamma)_\infty}{(a)_\infty}\, {}_2\varphi_1\left(\begin{matrix} -\frac{1}{\gamma}, -bx \\ b \end{matrix}; -a\gamma\right)$$

$$w(x) = \frac{(-qx)_\infty |x|}{(-ax,-bx)_\infty}$$

$$((\mu_1,\mu_2)) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\mu_1+\delta}^{\mu_2-\delta}\langle G(\lambda+i\varepsilon)f,g\rangle - \langle G(\lambda -$$

TU Delft

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Preface

A long journey of studying and struggling for years ends with this thesis 'An orthogonality relation of q-Meixner polynomials' in the field of functional analysis. My journey begun at the University of Amsterdam, where I studied mathematics. After three years I switched to study applied mathematics at the Delft University of Technology. Although the applied part made it much better for me to understand the mathematics, it was the field of analysis that sounded the most interesting for my bachelor thesis about Tauberian theorems.

After my bachelor I was sure about the subject of my master thesis. It had to be analysis again. I found Wolter Groenevelt to be prepared to be my supervisor. Years ago I started my journey in Amsterdam with *Analyse I* by him as lecturer. Now I will finish my journey with him as well. He needed a lot of patience with me, because I was not the fastest student. But he was always there to help me. Therefore I sincerely want to thank him for his supervision.

I also want to thank Cor Kraaikamp for being part of my thesis again after joining the committee of my bachelor thesis. And I thank Jan van Neerven, the chairman of my thesis committee, for all the lectures I took from him. He was very inspiring to me.

And last but not least I want to thank Zahra for her love and support. Finishing this thesis with such a great girlfriend made it much easier.

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1

Introduction

In functional analysis the study of spectral theory is an extension of the study of eigenvectors and eigenvalues of matrices in an infinite dimensional space. It also gives measures to describe operators and therefore we obtain a way to describe orthogonality. In this thesis the goal is to find orthogonality relations of special functions with the application of spectral theory.

Chapter 2 contains the basics of spectral theory that is needed in the remaining chapters. Allthough we are not going into quantum mechanics, spectral theory is used widely in this field. Therefore we restrict our attention to Hilbert spaces, because they posses inner products which allows us to talk about length and angle. The spectral measure of a bounded symmetric operator can be expressed in terms of the resolvent operator and this is a powerful tool in finding orthogonality relations. We also want to know whether a symmetric operator is (essentially) self-adjoint or not, by the use of deficiency indices. The final theorem in chapter two is the spectral theorem for unbounded operators, which ensures a unique measure such that the operator can expressed in terms of that measure.

The fundament of chapter 3 is the three-term recurrence relation of orthonormal polynomials such that we can introduce tridiagonal Jacobi operators and its eigenvalue problem. The main theorem in this chapter is Favard's theorem, which states that a set of polynomials that satisfy a three-term recurrence relation is a set of orthonormal polynomials. The spectral theorem now returns to a Jacobi operator a unique spectral measure. To obtain the spectral measure we define the Green kernel, which is the resolvent operator of the Jacobi operator. Finally we notice the importance of self-adjoint Jacobi operators by connecting it to the moment problem.

In the 4th and final chapter we study an explicit operator that generalizes a Jacobi operator, we introduce the q-Meixner polynomials, which are polynomials defined in terms of basic hypergeometric series. The method of finding asymptoticaly free solutions and using the Green kernel to obtain the spectral measure of the q-Meixner polynomials will be discribed. For this we need a suitable domain where the corresponding Jacobi operator is self-adjoint. To do the analysis we use a difference operator L instead. The chapter ends with some orthogonality relations of q-Meixner polynomials, the dual polynomials and the big q-Laguerre polynomials.

2

Basics of spectral theory

In this chapter we mention some definitions and properties about linear operators and we give the basics of spectral theory needed for the next chapters. Readers who are familiar in the area of spectral theory may rather continue to chapter 3. During this paper we use the convention $\mathbb{N} := \{0, 1, 2, \ldots\}$.

2.1. Hilbert spaces and bounded operators

To be able to talk about orthogonality we need a space with an inner product.

Definition 2.1. Let X be a complex vector space. A mapping $X \times X \to \mathbb{C}$ is called an **inner product** if for all $u, v, w \in X$ and $a, b \in \mathbb{C}$ we have

1. $\langle av + bw, u\rangle = a\langle v, u\rangle + b\langle w, u\rangle$
2. $\langle u, v\rangle = \overline{\langle v, u\rangle}$
3. $\langle u, u\rangle \geq 0$ and $\langle v, v\rangle = 0 \Leftrightarrow v = 0$.

If such a mapping exists, then X is called an **inner product space**, or **Pre-Hilbert space**. As a consequence of properties 1. and 2., the inner product is **sesquilinear**, i.e $\langle u, \alpha v\rangle = \overline{\alpha}\langle u, v\rangle$ for all $u, v \in X$ and $\alpha \in \mathbb{C}$.

We define $\|v\| := \sqrt{\langle v, v\rangle}$ as the **associated norm**. The **Cauchy-Schwarz inequality** now states that

$$|\langle u, v\rangle| \leq \|u\|\|v\|, \tag{2.1}$$

with equality if and only if u and v are linearly dependent.

Definition 2.2. Let X be a normed vector space. If every Cauchy sequence converges in X, then X is called **complete**. A complete inner product space is called a **Hilbert space**.

Example 2.3. Consider $v, w \in \mathbb{C}^n$ with standard inner product $\langle v, w\rangle = \sum_{i=1}^{\infty} v_i \overline{w_i}$ and $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$. The inner product together with completeness yields $\mathbb{C}^n$ as a Hilbert space.

We will assume that all Hilbert spaces in this thesis are **separable**, i.e. that every Hilbert space contains a countable dense subset. An example of a separable space is the real line $\mathbb{R}$. A countable dense subset is $\mathbb{Q} \subset \mathbb{R}$, the set of rationals. In general, all finite-dimensional spaces are seperable. A Hilbert space is separable if and only if there exists a countable set of orthonormal basis vectors. So we may always assume for any Hilbert space that there exists a basis.

Example 2.4. Let $\ell^2(\mathbb{N})$ denote the space of square summable sequences $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$, i.e.

$$\sum_{k\in\mathbb{N}} |a_k|^2 < \infty.$$

This is a separable Hilbert space with orthonormal basis the standard vectors e_k, so $(e_k)_l = \delta_{kl}$. Indeed for $x_i \in \mathbb{C}$, $0 \le i \le n$, the sequence $\{x_0, x_1, \ldots, x_n, 0, 0, \ldots\} \in \ell^2(\mathbb{N})$.

Theorem 2.5 (Bessels inequality)**.** *Let $\mathcal{H}$ be a Hilbert space and $f_1, f_2, \ldots$ an orthonormal sequence in $\mathcal{H}$. Then for every $f \in \mathcal{H}$ one has*

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le \|f\|^2.$$

The next theorem describes in which case the inequality of Theorem 2.5 is actually an equality. A proof can be found in [6].

Theorem 2.6. *Let $\mathcal{H}$ be an Hilbert space and $f_1, f_2, \ldots$ an orthonormal sequence in $\mathcal{H}$. Then the following are equivalent:*

1. *The sequence $f_1, f_2, \ldots$ is an orthonormal basis of $\mathcal{H}$.*
2. *If $f \in \mathcal{H}$ and $f \perp f_k$ for $k = 1, 2, \ldots$, then $f = 0$.*
3. $\mathcal{H} = \overline{\text{span}\{f_k \mid k = 1, 2, \ldots\}}$
4. $f = \sum_k \langle f, f_k\rangle f_k \qquad \forall f \in \mathcal{H}$
5. $\langle f, g\rangle = \sum_k \langle f, f_k\rangle \langle g, f_k\rangle \qquad \forall f, g \in \mathcal{H}$
6. $\|f\|^2 = \sum_k |\langle f, f_k\rangle|^2 \qquad \forall f \in \mathcal{H}$

The last assertion is called ***Parseval's identity****.*

To explain the concept of moments in the next example we need some knowledge about measures.

Definition 2.7. A **σ-algebra** is a set $\mathcal{B}$ of subsets of Ω such that

1. $\Omega \in \mathcal{B}$;
2. If $A \in \mathcal{B}$, then $A^C \in \mathcal{B}$;
3. If $A_1, A_2, \ldots \in \mathcal{B}$, then $\cup_{i\in\mathbb{N}} A_i \in \mathcal{B}$.

Let (Ω, d) be a metric space. $\mathcal{B}$ is called the **Borel σ-algebra** if it is the smallest σ-algebra that contains all open subsets of Ω.

Definition 2.8. Let (Ω, d) be a metric space. A **complex Borel measure** on Ω is a map

$\mu : \mathcal{B}(\Omega) \to \mathbb{C}$ such that

1. $\mu(\emptyset) = 0$
2. If $A_1, A_2, \ldots \in \mathcal{B}$ are mutually disjoint, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

In particular, if μ maps to $\mathbb{R}_{\ge 0}$, then μ is called a **positive (Borel) measure**.

Example 2.9. Let μ be a positive Borel measure on the real line $\mathbb{R}$, such that all moments exist, i.e. $\int_{\mathbb{R}} |x|^m \, \mathrm{d}\mu(x) < \infty$ for all $m \in \mathbb{N}$. Without loss of generality we assume that μ is a probability measure, i.e. $\int_{\mathbb{R}} \mathrm{d}\mu(x) = 1$. By $L^2(\mu)$ we denote the space of **square integrable functions** on $\mathbb{R}$, i.e. $\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}\mu(x) < \infty$. More formally, $L^2(\mu)$ consists of all equivalent classes such that f and g with $\int_{\mathbb{R}} |f(x) - g(x)|^2 \, \mathrm{d}\mu(x) = 0$ belong to the same class. Then the space $L^2(\mu)$ is a Hilbert space with respect to the inner product $\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} \, \mathrm{d}\mu(x)$.

Consider the operator T from a normed vector space X to a normed vector space Y. Then T is **bounded** if there exists an $M > 0$ such that for every $u \in X$ it holds that $\|Tu\| \leq M\|u\|$. The smallest M such that this inequality holds is called the **norm** of T, denoted by $\|T\|$. It can be shown that $\|T\| = \sup_{\|u\|=1} \|Tu\|$.

Proposition 2.10. *Let X and Y be two normed vector spaces. Then $T : X \to Y$ is a bounded linear operator if and only if T is continuous.*

Proof. Because T is bounded, we know that $\|T\| < \infty$. Furthermore we note that for $x, y \in X$,

$$\|Tx - Ty\| = \|T(x - y)\| \leq \|T\|\|x - y\|.$$

As an immediate consequence, T is continuous.
Now assume that T is not bounded. This means that $\|T\| = \sup_{\|u\|=1} \|Tu\|$ is not finite. Therefore we can find a sequence $x_n \in X$ with $\|x_n\| = 1$ such that $\|Tx_n\| \geq n$. But that implies that $\left\|\frac{1}{n}x_n\right\| = \frac{1}{n} \mapsto 0$ as $n \to \infty$, while $\left\|T(\frac{1}{n}x_n)\right\| \geq 1 \not\to 0$. It follows that T is not continuous. □

Definition 2.11. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces and consider $T : \mathscr{H} \to \mathscr{K}$ to be a bounded linear mapping. Then the mapping $T^* : \mathscr{K} \to \mathscr{H}$, defined by

$$\langle Tx, y\rangle = \langle x, T^* y\rangle,$$

for all $x \in \mathscr{H}, y \in \mathscr{K}$, is called the **adjoint** of T. If $TT^* = \mathbf{1}_{\mathscr{K}}$ and $T^*T = \mathbf{1}_{\mathscr{H}}$, we call T **unitary**. Moreover, if $\mathscr{H} = \mathscr{K}$ then T is called

1. **symmetric** if $T = T^*$;
2. a **projection** if $T^2 = T$.

In this thesis, the definition of symmetric is just reserved for bounded operators T. For unbounded operators we will develop an extended definition in section 2.4.

2.2. Spectral decomposition

The set of eigenvalues of a bounded linear operator $T : \mathscr{X} \to \mathscr{X}$ with $\mathscr{X}$ a complex Banach space, is part of the **spectrum** $\sigma(T)$. The spectrum is defined as $\sigma(T) = \mathbb{C}\setminus\rho(T)$ with $\rho(T)$ the **resolvent set**, the set of $\lambda \in \mathbb{C}$ such that the $\lambda I - T$ has a bounded inverse which is densely defined. Therefore the **resolvent operator**, or just the resolvent, $R(\lambda) = (\lambda I - T)^{-1}$ is bounded on $\rho(T)$.

The spectrum can be decomposed into mutually disjoint sets $\sigma_p(T), \sigma_c(T), \sigma_r(T) \subseteq \mathbb{C}$, with

1. $\lambda \in \sigma_p(T)$ iff $\lambda I - T$ is not one-to-one, i.e. λ is an eigenvalue of T;
2. $\lambda \in \sigma_c(T)$ iff $\lambda I - T$ is one-to-one, $\mathrm{rge}(\lambda I - T)$ is dense in $\mathscr{X}$, but the inverse is unbounded;
3. $\lambda \in \sigma_r(T)$ iff $\lambda I - T$ is one-to-one, but $\mathrm{rge}(\lambda I - T)$ is not dense in $\mathscr{X}$.

The sets are called **point spectrum**, **continuous spectrum** and **residual spectrum** respectively.

Proposition 2.12. *Let T be a bounded operator. Then $\sigma(T)$ is a compact subset of $\mathbb{C}$ contained in the closed disk of radius $\|T\|$.*

Proof. Let T be a bounded operator. Using Neumanns expansion, $(I-S)^{-1} = \sum_{k=0}^{\infty} S^k$ with $\|S\| < 1$, we can see that

$$(\lambda I - T)^{-1} = \lambda^{-1}(I - \lambda^{-1}T)^{-1} = \lambda^{-1}\sum_{k=0}^{\infty}\left(\frac{T}{\lambda}\right)^k,$$

only if $\|T\| < |\lambda|$, and consequently $\lambda \in \rho(T)$. So if $\lambda \in \sigma(T)$, then $|\lambda| \le \|T\|$ and so $\sigma(T)$ is bounded by $\|T\|$.

Next, we claim that $\rho(T)$ is open. Let $\lambda \in \rho(T)$. Then $(\lambda I - T)^{-1}$ is nonzero and bounded, so $0 < \|(\lambda I - T)^{-1}\| < \infty$. Define $\delta = \|(\lambda I - T)^{-1}\|^{-1}$ as the radius of the open ball $B_\delta(0)$ with the origin as centre. The claim has been proven if we can show that the open ball with radius δ and centre λ, $B_\delta(\lambda) \subseteq \rho(T)$. Take $\gamma \in B_\delta(0)$, then $|\gamma| < \delta$. This implies $\|\gamma(\lambda I - T)^{-1}\| < 1$. By Neumanns expansion we see that $(I - \gamma(\lambda I - T)^{-1})$ has a bounded inverse and therefore

$$(\lambda - \gamma)I - T = (\lambda I - T)(I - \gamma(\lambda I - T)^{-1})$$

also has a bounded inverse. And so $\lambda - \gamma \in \rho(T)$. Now

$$B_\delta(\lambda) = \{\mu \in \mathbb{C} : |\lambda - \mu| < \delta\} = \{\lambda - \gamma \in \mathbb{C} : |\gamma| < \delta\} \subseteq \rho(T)$$

which proves that $\rho(T)$ is open. □

A consequence of the previous proof is that $\rho(T)$ is non-empty. Without proof we mention that $\sigma(T)$ is non-empty.

Theorem 2.13. *The spectrum $\sigma(T)$ of a bounded operator T is non-empty.*

Proposition 2.14. *The residual spectrum $\sigma_r(T)$ of a symmetric operator T is empty.*

Proof. Let $T : \mathscr{X} \to \mathscr{X}$ be symmetric and assume $\lambda \in \sigma_r(T)$. Then $\operatorname{rge}(\lambda I - T)$ is not dense in $\mathscr{X}$ and we can decompose the space $\mathscr{X}$ into

$$\mathscr{X} = \ker(\lambda I - T)^* \oplus \overline{\operatorname{rge}(\lambda I - T)}, \tag{2.2}$$

with $\ker(\lambda I - T)^* = \ker\left(\overline{\lambda} I - T^*\right) \neq \{0\}$. Therefore, there exists a non-trivial $x \in \mathscr{X}$ such that $T^*x = \overline{\lambda}x$ and so

$$\langle Tx, x\rangle = \left\langle x, T^*x\right\rangle = \lambda\langle x, x\rangle.$$

It follows that $\langle Tx - \lambda x, x\rangle = 0$. But for a fixed $x \in \mathscr{X}$ this means that $Tx - \lambda x = 0$ and so $\lambda \in \sigma_p(T)$. This is a contradiction with the fact that $\sigma_r(T)$ and $\sigma_p(T)$ are mutually disjoint. □

Proposition 2.15. *Let T on $\mathscr{X}$ be symmetric. Then the spectrum is real. Furthermore,*

$$\sigma(T) \subset [-\|T\|, \|T\|].$$

Proof. Consider the numerical range $W(T) = \{\langle Tx, x\rangle \mid \|x\| = 1\}$. Then for $\|x\| \le 1$,

$$|\langle Tx, x\rangle| \le \|Tx\|\|x\| \le \|Tx\| \le \|T\|,$$

so $\langle Tx, x\rangle$ is bounded by boundedness of T. It follows that $W(T)$ is bounded. Notice that $W(T) \subseteq \mathbb{R}$, because $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$ and therefore $\langle Tx, x\rangle \in \mathbb{R}$.

Next, we want to show that $\sigma(T) \subset \overline{W(T)}$. Assume that $\lambda \notin \overline{W(T)}$ and let $d := \inf\{|\lambda - \mu| : \mu \in W(T)\}$. Then $d > 0$ by clossedness. It follows, for $\|x\| = 1$, that

$$0 < d \leq |\lambda - \langle Tx, x\rangle| = |\langle \lambda x - Tx, x\rangle| \leq \|(\lambda I - T)x\| \cdot \|x\| = \|(\lambda I - T)x\|.$$

This shows that there does not exist an $x \neq 0$ such that $(\lambda I - T)x = 0$. Therefore $\ker(\lambda I - T) = \varnothing$ and thus the map $\lambda I - T$ is injective. The mapping $\lambda - T : \mathscr{X} \to \mathrm{rge}(\lambda I - T)$ is an isomorfism.
The range is dense in $\mathscr{X}$. Indeed, if not, then there exists an $x_0 \in \mathrm{rge}(\lambda I - T)^{\perp}$ with $\|x_0\| = 1$, such that

$$0 = \langle (T - \lambda I)x_0, x_0\rangle = \langle Tx_0, x_0\rangle - \langle \lambda x_0, x_0\rangle = \langle Tx_0, x_0\rangle - \lambda.$$

It follows that $\lambda = \langle Tx_0, x_0\rangle \in W(T)$, which is a contradiction to $\lambda \notin \overline{W(T)}$. Therefore is the range dense in $\mathscr{H}$. This shows that $\lambda \in \rho(T)$ and thus $\lambda \notin \sigma(T)$. So $\sigma(T) \subseteq W(T) \subseteq \mathbb{R}$.

The inclusion is a corollary of Proposition 2.12. □

2.3. The spectral theorem for bounded symmetric operators

Let V be a subspace of the Hilbert space $\mathscr{H}$. Then $\mathscr{H}$ can be decomposed into $\mathscr{H} = V \oplus V^{\perp}$. Assume that $h \in \mathscr{H}$ can be decomposed into $h = f + g$ with $f \in V$ and $g \in V^{\perp}$, then $p_V : \mathscr{H} \to \mathscr{H}, h \mapsto f$ is called an **orthogonal projection**.

Definition 2.16. Let $\mathscr{P}(\mathscr{H})$ be the set of orthogonal projections on a Hilbert space $\mathscr{H}$ and let $\mathscr{B}$ the Borel σ-algebra of a set S. We say that the mapping $p : \mathscr{B} \to \mathscr{P}(\mathscr{H})$ is a **projection valued measure** on the Borel space $(S, \mathscr{B})$ if

1. $p(S) = I$ and $p(\emptyset) = 0$;
2. If $\{A_i\}$ is a countable collection of pairwise disjoint elements of $\mathscr{B}$, then $p(A_i)$ is a pairwise orthogonal collection of projections, and $p(\cup A_i) = \sum_i p(A_i)$.

The definition of a projection valued measure is similar to the definition of a measure. The only difference is that projection valued measures uses self-adjoint projections rather than real numbers.

Now we are able to introduce the concept of the resolution of the identity.

Definition 2.17. A **resolution of the identity**, say E, of a Hilbert space $\mathscr{H}$ is a projection valued Borel measure on $\mathbb{R}$ such that for all Borel sets $A, B \subseteq \mathbb{R}$ we have

1. $E(A)$ is a symmetric projection;
2. $E(A \cap B) = E(A)E(B)$;
3. $E(\varnothing) = \mathbf{0}, E(\mathbb{R}) = \mathbf{I}$;
4. $A \cap B = \varnothing$ implies $E(A \cup B) = E(A) + E(B)$;

In some literature will be spoken of a spectral family instead, see for example [3].

Proposition 2.18. *Let E be a resolution of the identity of a Hilbert space $\mathscr{H}$. For all $u, v \in \mathscr{H}$ the map $A \mapsto E_{u,v}(A) = \langle E(A)u, v\rangle$ is a complex Borel measure.*

Now we are able to formulate the most important theorem in spectral theory, restricted to symmetric operators. Later, in Theorem 2.31, we will formulate the spectral theorem in more general setting. This is in line with [3], where a proof can be found.

Theorem 2.19 (Spectral theorem for symmetric operators)**.** *Let T be a bounded symmetric linear operator on $\mathscr{H}$, then there exists a unique resolution of the identity E such that $T = \int_{\mathbb{R}} t\,\mathrm{d}E(t)$, i.e. $\langle Tu, v\rangle = \int_{\mathbb{R}} t\,\mathrm{d}E_{u,v}(t)$. Moreover, E is supported on the spectrum $\sigma(T) \subset \mathbb{R}$, which is contained in the interval $[-\|T\|, \|T\|]$. Moreover, any of the spectral projections $E(A)$, $A \subseteq \mathbb{R}$ a Borel set, commutes with operator T.*

With the spectral theorem in mind we define for any continuous function f its **functional calculus** for symmetric operator T as

$$f(T) := \int_{\mathbb{R}} f(t)\,\mathrm{d}E(t),$$

i.e. $\langle f(T)u, v\rangle = \int_{\mathbb{R}} f(t)\,\mathrm{d}E_{u,v}(t)$. The resolution of the identity in the theorem obtained from T is called the **spectral measure**. Notice that the spectral measure is indeed a (projection valued) measure.

If we look at the spectral theorem, Theorem 2.19, it may be clear that it now becomes the issue to find the spectral measure associated with the given bounded symmetric operator T. It turns out that the spectral measure can be expressed in terms of the resolvent operator $R(z) = (zI - T)^{-1}$ with $z \in \mathbb{C}$.

Theorem 2.20. *Let $u, v \in \mathscr{H}$ be fixed. The spectral measure of the open interval $(a, b) \subset \mathbb{R}$ is given by*

$$E_{u,v}((a,b)) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \langle R(x + i\varepsilon)u, v\rangle - \langle R(x - i\varepsilon)u, v\rangle\,\mathrm{d}x. \tag{2.3}$$

Theorem 2.20 gives a powerful tool to explicitly obtain the spectral measure of the operator T if we have its resolvent operator. In the next section we will explain this in more details.

Proof. See [2]. □

2.4. Unbounded self-adjoint operators

In the case of unbounded linear operators, things get tricky at some point. An important difference with bounded operators is that unbounded operators are not defined on the whole Hilbert space $\mathscr{H}$. Therefore we need a slightly different definition, see [1].

Definition 2.21. An **unbounded linear operator** T from $\mathscr{H}$ to $\mathscr{H}$ is a pair $(\mathscr{D}(T), T)$ consisting of a subspace $\mathscr{D}(T) \subset \mathscr{H}$, the domain of T, and a linear transformation $T : \mathscr{D}(T) \subset \mathscr{H} \to \mathscr{H}$.

The notions of kernel and range are identical to bounded operators. If the operator is bounded on the domain, then it can be extended to a bounded operator on the whole space $\mathscr{H}$. Therefore we use Definition 2.21 only for operators that are unbounded on their domain. As completeness of this paragraph, we define the spectrum for unbounded operators.

Definition 2.22. Let $T : \mathscr{D}(T) \subset \mathscr{H} \to \mathscr{H}$ be densely defined. Then the **resolvent set** of T is defined as

$$\rho(T) := \left\{\lambda \in \mathbb{C} \mid \lambda I - T : \mathscr{D}(T) \subset \mathscr{H} \to \mathscr{H} \text{ has a densely defined bounded inverse } (\lambda I - T)^{-1}\right\}.$$

As in the bounded case, $\sigma(T) := \mathbb{C} \setminus \rho(T)$ defines the spectrum of T.

The notion of a graph of the operator T is similar to the bounded case as well. We define the **graph** as

$$\mathscr{G}(T) = \{(x, Tx) | x \in \mathscr{D}_T\}.$$

A natural inner product on this space is defined by

$$\langle (u, v), (u' v') \rangle := \langle u, u' \rangle + \langle v, v' \rangle.$$

Definition 2.23. The operator T is called **closed** if its graph is closed. If the closure of $\mathscr{G}(T)$ is the graph of an operator, then T is called **closable**.

A helpful theorem is the closed graph theorem. It tells us that if T is linear, then T is continuous if and only if T is closed. If both T and S are unbounded operators on $\mathscr{H}$, $\mathscr{D}(S) \subset \mathscr{D}(T)$ and $Su = Tu$ for all $u \in \mathscr{D}(S)$, notation $S \subset T$, then T is called an **extension** of S and so

$$\mathscr{G}(S) \subset \mathscr{G}(T).$$

If, furthermore, $\mathscr{G}(T)$ is closed and for all extensions $\hat{S}$ of S it holds that

$$\mathscr{G}(T) \subseteq \mathscr{G}(\hat{S}),$$

then T is called the **minimal closed extension** of S. We now assume that $\mathscr{D}(T)$ is dense in $\mathscr{H}$. Then we can define the adjoint operator as follows.

Definition 2.24. Let $T : \mathscr{D}(T) \subset \mathscr{H} \to \mathscr{H}$ be a densely defined linear operator on a Hilbert space $\mathscr{H}$ and consider the map $\phi_v : u \mapsto \langle Tu, v \rangle$. We define its **adjoint** T^* by

$$\mathscr{D}(T^*) := \left\{v \in \mathscr{H} \mid \phi_v \text{ is continuous on } \mathscr{D}(T)\right\}$$

and

$$\langle Tu, v \rangle = \langle u, T^* v \rangle, \qquad \forall\ u \in \mathscr{D}(T),\ v \in \mathscr{D}(T^*).$$

The requirement of a densely defined operator in the definition of the adjoint is essential. In fact, if the operator is not densely defined, then the adjoint can not uniquely be extended.

Proposition 2.25. *If $T : \mathscr{D}(T) \subset \mathscr{H} \to \mathscr{H}$ is a densely defined unbounded linear operator, then*

1. *T^* is closed;*
2. *T is closable if and only if T^* is densely defined; in this case $\overline{T} = (T^*)^*$;*
3. *if T is closable, then $(\overline{T})^* = T^*$.*

Proof. see for details [1]. □

Lemma 2.26. *The graph of the adjoint of a densely defined operator T can be written as*

$$\mathscr{G}(T^*) = \{(Tu, -u) | u \in \mathscr{D}(T)\}^{\perp}. \tag{2.4}$$

Proof. Let $A = \{(Tu, -u) | u \in \mathscr{D}(T)\}$ and consider $v \in \mathscr{D}(T^*)$. Then

$$\langle Tu, v\rangle = \left\langle u, T^* v\right\rangle \quad \Leftrightarrow \quad \left\langle (Tu, -u), (v, T^* v)\right\rangle = 0,$$

which shows that all the elements of the graph of T^* are orthogonal to the elements of A.

"$\subseteq$". Take $(w, T^* w) \in \mathscr{G}(T^*)$ and let the sequence $(u_n, v_n)_{n=0}^{\infty} \subseteq A$ converge to a point $(u, v) \in \overline{A}$. Then

$$\left\langle (u, v), (w, T^* w)\right\rangle = \lim_{n\to\infty} \left\langle (u_n, v_n), (w, T^* w)\right\rangle = 0$$

and therefore $(w, T^* w) \in A^{\perp}$.

"$\supseteq$". Let $(u, v) \in A^{\perp}$. Then for all $w \in \mathscr{D}(T)$ we see that

$$0 = \langle (Tw, -w), (u, v)\rangle = \langle Tw, u\rangle - \langle w, v\rangle .$$

So $\langle Tw, u\rangle = \langle w, v\rangle$ for all $w \in \mathscr{D}(T)$ and thus $|\langle Tw, u\rangle| = |\langle w, v\rangle| \leq \|w\| \|v\|$. This shows that the mapping $w \mapsto \langle Tw, u\rangle$ is bounded and continuous on $\mathscr{D}(T)$. Consequently $u \in \mathscr{D}(T^*)$ and $T^* u = v$, thus $(u, v) \in \mathscr{G}(T^*)$. Now the identity (2.4) has been proved. □

By (2.4) we see that the graph of the adjoint T^* is an orthogonal complement and is therefore closed. Consequently, T^* is closed.

Now we can define symmetric and self-adjoint unbounded operators.

Definition 2.27. We call a densely defined operator $T : \mathscr{D}(T) \subseteq \mathscr{H} \to \mathscr{H}$ **symmetric** if

$$T \subset T^*.$$

This is equivalent to $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all $u, v \in \mathscr{D}(T)$.

Remark. If T is symmetric, then $\mathscr{D}(T) \subset \mathscr{D}(T^*)$ and so the domain of T^* is dense in $\mathscr{H}$ as well. It follows that T^* has also an adjoint, T^{**}. The operator T^{**} is the minimal closed extension of T. Indeed,

$$\mathscr{G}(T^{**}) = \left\{(u, T^{**} u) | u \in \mathscr{D}(T^{**})\right\} = \left\{(T^* u, -u) | u \in \mathscr{D}(T^*)\right\}^{\perp} \tag{2.5}$$

$$= \{(u, Tu) | u \in \mathscr{D}(T)\}^{\perp\perp} = \overline{\{(u, Tu) | u \in \mathscr{D}(T)\}} = \overline{\mathscr{G}(T)}, \tag{2.6}$$

whereby we used the identity of (2.4). We see that the graph of T^{**} is the closure of the graph of T and thus T^{**} is the minimal closed extension of T. This proves the second statement of Proposition 2.25.

To show that T^{**} is symmetric we first notice that $\mathscr{D}(T) \subset \mathscr{D}(T^{**})$ and so T^{**} is densely defined. Secondly

$$T^{**} \subseteq T^*,$$

because for $\phi_v : u \mapsto \langle Tu, v\rangle$,

$$\mathscr{D}(T^{**}) = \left\{v \in \mathscr{H} \mid \phi_v \text{ is continuous on } \mathscr{D}(T^*)\right\} \subseteq \left\{v \in \mathscr{H} \mid \phi_v \text{ is continuous on } \mathscr{D}(T)\right\} = \mathscr{D}(T^*).$$

So we can conclude that every symmetric operator has a closed symmetric extension. If there does not exists a proper symmetric extension of T, then T is called **maximal symmetric**.

Definition 2.28. A densely defined operator $T : \mathscr{D}(T) \to \mathscr{H}$ is called **self-adjoint** if $T = T^*$. That is, if $\mathscr{D}(T) \subset \mathscr{D}(T^*)$ and $\mathscr{D}(T^*) \subset \mathscr{D}(T)$.

We have already seen that the adjoint operator is closed, so every self-adjoint operator is closed as well. Every self-adjoint operator is maximal symmetric. A symmetric operator T is called **essentially self-adjoint** if $\overline{T}$ is self-adjoint. So $T \subset T^{**} = T^*$. In general, a densely defined symmetric operator does not have self-adjoint extensions. Next we will discuss deficiency indices, which will be used to denote the difference between operators being maximal symmetric and being self-adjoint.

Definition 2.29. Define for $z \in \mathbb{C}\backslash\mathbb{R}$ the eigenspace

$$N_z := \left\{v \in \mathscr{D}(T^*) \mid T^* v = zv\right\}.$$

Put $n_+ = \dim N_i$ and $n_- = \dim N_{-i}$. The pair (n_+, n_-) are the **deficiency indices** for a densely defined symmetric operator T.

Remark. Because $\dim N_z$ is constant for $\Im(z) > 0$ and $\Im(z) < 0$, we are sure that $\dim N_z = \dim N_i$ and so n_+ and n_- are constants too.

Note that

$$T^* v = iv \quad \Leftrightarrow \quad \overline{T^* v} = \overline{iv} \quad \Leftrightarrow \quad T^* \overline{v} = -i\overline{v},$$

and therefore $v \in N_i$ iff $\overline{v} \in N_{-i}$. So if T commutes with the complex conjugation, this implies that $n_+ = n_-$. Furthermore, if T is self-adjoint, the eigenvalues are real by Proposition 2.15 and it follows that $N_i = N_{-i} = \emptyset$, implying that $n_+ = n_- = 0$.

Proposition 2.30. *Let $(\mathscr{D}(T), T)$ be a densely defined symmetric operator.*

1. *$\mathscr{D}(T^*) = \mathscr{D}(T^{**}) \oplus N_i \oplus N_{-i}$, as an orthogonal direct sum with respect to the graph norm of T^* from $\langle u, v\rangle_{T^*} = \langle u, v\rangle + \langle T^* u, T^* v\rangle$. As a direct sum, $\mathscr{D}(T^*) = \mathscr{D}(T^{**}) + N_z + N_{\bar{z}}$ for general $z \in \mathbb{C}\backslash\mathbb{R}$.*
2. *Let U be an isometric bijection $U : N_i \to N_{-i}$ and define $(\mathscr{D}(S), S)$ by*

   $$\mathscr{D}(S) = \left\{u + v + Uv \mid u \in \mathscr{D}(T^{**}), v \in N_i\right\}, \quad Sw = T^* w,$$

   then $(\mathscr{D}(S), S)$ is a self-adjoint extension of $(\mathscr{D}(T), T)$ and every self-adjoint extension of T arises in this way.

Proof. See for [2]. □

A way to characterize the domain of S is making use of the sesquilinear form

$$B(u, v) = \langle T^* u, v\rangle - \langle u, T^* v\rangle, \qquad u, v \in \mathscr{D}(T^*). \tag{2.7}$$

Then $\mathscr{D}(S) = \left\{u \in \mathscr{D}(T^*) \mid B(u, v) = 0, \forall v \in \mathscr{D}(S)\right\}$.

2.5. The spectral theorem for unbounded self-adjoint operators

In Theorem 2.19 the spectral theorem for bounded symmetric operators was given. Now we can formulate the spectral theorem in the general case.

Theorem 2.31 (Spectral theorem)**.** *Let T be an unbounded self-adjoint operator. There exists a unique resolution of the identity E such that $T = \int_{\mathbb{R}} t \, \mathrm{d}E(t)$, i.e. $\langle Tu, v\rangle = \int_{\mathbb{R}} t \, \mathrm{d}E_{u,v}(t)$ for $u \in \mathscr{D}(T), v \in \mathscr{H}$. Furthermore, for any bounded operator S that satisfies $ST \subset TS$ we have $E(A)S = SE(A)$, wit $A \subset \mathbb{R}$ a Borel set. Moreover, inversion formula (2.3) remains valid.*

Proof. See for [2]. □

We can now define $f(T)$ for any measurable function f by

$$\langle f(T)u, v\rangle := \int_{\mathbb{R}} f(t) \, \mathrm{d}E_{u,v}(t), \qquad u \in \mathscr{D}(f(T)), v \in \mathscr{H},$$

where $\mathscr{D}(f(T)) = \left\{u \in \mathscr{H} \mid \int_{\mathbb{R}} |f(t)|^2 \, \mathrm{d}E_{u,v}(t) < \infty\right\}$ is the domain of $f(T)$. So $f(T)$ is a densely defined closed operator. If $f \in L^\infty$, then $f(T)$ is a continuous operator, by the closed graph theorem. This in particular can be applied to $f(x) = (x-z)^{-1}$, $z \in \rho(T)$, which gives the resolvent operator. So

$$\langle R(z)u, v\rangle = \int_{\mathbb{R}} (x-z)^{-1} \, \mathrm{d}E_{u,v}(t). \tag{2.8}$$

We are going to use this way of writing the resolvent operator in section 3.2.

3

Jacobi operators

3.1. Orthogonal polynomials

Consider again the Hilbert space $L^2(\mu)$ of square μ-integral functions as in example 2.9. Assume that all moments exists, which means that $\int_{\mathbb{R}} |x|^n \,\mathrm{d}\mu(x) < \infty$ for all n. Then all polynomials are integrable. By using the Gram Smith orthogonalisation process to the sequence $\{1, x, x^2, x^3, \ldots\}$ we obtain pairwise orthogonal polynomials $\{p_0, p_1, \ldots\}$. Two situations may occur: the polynomials are linearly dependent or the polynomials are linearly independent in $L^2(\mu)$.

Assume the polynomials are linearly dependent, which means that there are finitely many nonzero a_i such that

$$a_0 p_0 + a_1 p_1 + a_2 p_2 + \ldots a_n p_n = 0,$$

with n the greatest integer such that $a_n \neq 0$. By

$$\begin{aligned} 0 = \langle p_n, 0\rangle &= \langle p_n, a_1 p_1 + a_2 p_2 + a_3 p_3 + \ldots a_n p_n\rangle \\ &= a_n \langle p_n, p_n\rangle \end{aligned}$$

it follows that $\langle p_n, p_n\rangle = 0$, i.e.

$$\int_{\mathbb{R}} |p_n(x)|^2 \,\mathrm{d}\mu(x) = 0,$$

while $p_n \neq 0$. This can only be true when μ is a finite sum of Dirac measures at the zeros of p_n.

Remark. As a reminder, a **Dirac measure** is a measure δ_x on a set X defined as

$$\delta_x(A) := \begin{cases} 0, & x \notin A. \\ 1, & x \in A, \end{cases}$$

for a measurable subset $A \subseteq X$.

In the dependent situation we only have finitely many orthogonal polynomials and so we can proof the results with linear algebra. From now on we exclude the case that the polynomials are linearly dependent.

Definition 3.1. A sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ with $\deg(p_n) = n$ is a set of orthonormal polynomials with respect to μ if $\int_{\mathbb{R}} p_n(x) p_m(x) \,\mathrm{d}\mu(x) = \delta_{n,m}$.

Remark. The polynomials p_n on $\mathbb{R}$ are real-valued, because the set $\{1, x, x^2, \ldots\}$ is real-valued. Therefore the coefficients of p_n are real. As a consequence of the Gram-Schmidt procedure, $p_0(x) = 1$ and the leading coefficient is positive. It follows that μ is a probability measure.

The product $p_n(x)p_m(x)$ is a polynomial of degree $n+m$, say $\sum_{k=0}^{n+m} d_k x^k$ with coefficients $d_k \in \mathbb{R}$, yielding

$$\int_{\mathbb{R}} p_n(x)p_m(x)\,\mathrm{d}\mu(x) = \sum_{k=0}^{n+m} d_k \int_{\mathbb{R}} x^k\,\mathrm{d}\mu(x) = \sum_{k=0}^{n+m} d_k m_k,$$

with moments m_k. This results in the identity $\sum_{k=0}^{n+m} d_k m_k = \delta_{n,m}$. Apparently, the orthogonality relation is totally determined by the moments m_k.

As a very useful tool later in this section, we define the **Stieltjes transform** of the measure μ by

$$w(z) := \int_{\mathbb{R}} (x-z)^{-1}\,\mathrm{d}\mu(x) \qquad z \in \mathbb{C}\backslash\mathbb{R}.$$

By rewriting the Stieltjes transform we see that, formally,

$$w(z) = \frac{-1}{z}\int_{\mathbb{R}} \frac{1}{1-x/z}\,\mathrm{d}\mu(x) = \frac{-1}{z}\sum_{k=0}^{\infty}\int_{\mathbb{R}} \left(\frac{x}{z}\right)^k \mathrm{d}\mu(x) = -\sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}}. \tag{3.1}$$

If $\operatorname{supp}(\mu) \subseteq [-A, A]$, then

$$|m_k| = \left|\int_{\mathbb{R}} x^k\,\mathrm{d}\mu(x)\right| \le \int_{-A}^{A} |x|^k\,\mathrm{d}\mu(x) = 2\int_0^A x^k\,\mathrm{d}\mu(x) \le 2A^k. \tag{3.2}$$

We can use this to imply that

$$\sum_{k=0}^{\infty}\left|\frac{m_k}{z^{k+1}}\right| \le \sum_{k=0}^{\infty}\frac{2A^k}{|z^{k+1}|} = \frac{2}{|z|}\sum_{k=0}^{\infty}\left(\frac{A}{|z|}\right)^k = \frac{2}{|z|-A},$$

which proves that the series in (3.1) is absolutely convergent and thus convergent. Therefore the Stieltjes transform is totally determined by the moments of μ.

Proposition 3.2. *Let μ be a probability measure with finite moments, and let*

$$w(z) = \int_{\mathbb{R}} (x-z)^{-1}\,\mathrm{d}\mu(x) \qquad z \in \mathbb{C}\backslash\mathbb{R}$$

be its Stieltjes transform, then

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_a^b \Im(w(x+i\varepsilon))\,\mathrm{d}x = \mu((a,b)) + \frac{1}{2}\mu(\{a\}) + \frac{1}{2}\mu(\{b\}).$$

Let p be a polynomial with real coefficients. Then

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_a^b \Im(p(x+i\varepsilon)w(x+i\varepsilon))\,\mathrm{d}x = \int_{(a,b)} p(x)\,\mathrm{d}\mu(x) + \frac{1}{2}p(a)\mu(\{a\}) + \frac{1}{2}p(b)\mu(\{b\}). \tag{3.3}$$

Proof. See Koelink, [8]. □

The proposition gives rise to the identity

$$\mu((a,b)) = \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\frac{1}{\pi}\int_{a+\delta}^{b-\delta}\Im(w(x+i\varepsilon))\,\mathrm{d}x, \tag{3.4}$$

which is similar to the Stieltjes Perron formula (2.3). We need the extension of the inversion formula (3.3) to find a spectral measure later in (3.21).

Considering the Hilbert space $L^2(\mu)$, it turns out that a set of orthonormal polynomials posseses a very important property, which will lead to the definition of Jacobi operators in the next section.

Theorem 3.3 (Three-term recurrence relation)**.** *Let $\{p_k\}_{k=0}^{\infty}$ be a set of orthonormal polynomials in $L^2(\mu)$, then there exist sequences $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$ with $a_k, b_k \in \mathbb{R}$ and $a_k > 0$, such that*

$$xp_k(x) = a_k p_{k+1}(x) + b_k p_k(x) + a_{k-1}p_{k-1}(x), \qquad k \geq 1 \tag{3.5}$$

$$xp_0(x) = a_0 p_1(x) + b_0 p_0(x) \tag{3.6}$$

Moreover, if μ is compactly supported, then the coefficients a_k and b_k are bounded.

Proof. First we notice that $\deg(xp_k(x)) = k+1$, so there exist constants $c_i^k \in \mathbb{R}$ such that

$$xp_k(x) = \sum_{i=0}^{k+1} c_i^k p_i(x). \tag{3.7}$$

Multiplying both sides by p_j and integrating over $\mathbb{R}$ we obtain

$$\begin{aligned}\int_{\mathbb{R}} p_j(x)xp_k(x)\,\mathrm{d}\mu(x) &= \int_{\mathbb{R}}\sum_{i=0}^{k+1} c_i^k p_i(x)p_j(x)\,\mathrm{d}\mu(x)\\ &= \int_{\mathbb{R}} c_j^k\,\mathrm{d}\mu(x) = c_j^k,\end{aligned}$$

using the orthogonality property of p_i and the fact that μ is a probability measure.
In a similar way we can look at the same expression $c_j^k = \int_{\mathbb{R}} xp_j(x)p_k(x)\,\mathrm{d}\mu(x)$ and observe that the polynomial $xp_j(x)$ has degree $j+1$ so we see that $c_j^k = 0$ whenever $j+1 < k$, by the orthogonality property again. Therefore (3.7) can be written as

$$xp_k(x) = c_{k-1}^k p_{k-1}(x) + c_k^k p_k(x) + c_{k+1}^k p_{k+1}(x).$$

So we can easily see that

$$b_k := c_k^k = \int_{\mathbb{R}} x(p_k(x))^2\,\mathrm{d}\mu(x). \tag{3.8}$$

The other coefficients are

$$\begin{aligned}a_k := c_{k+1}^k &= \int_{\mathbb{R}} xp_{k+1}(x)p_k(x)\,\mathrm{d}\mu(x)\\ a_{k-1} := c_{k-1}^k &= \int_{\mathbb{R}} xp_{k-1}(x)p_k(x)\,\mathrm{d}\mu(x).\end{aligned} \tag{3.9}$$

To show that the coefficients are bounded if μ is compactly supported we notice that

$$\begin{aligned}|a_k| = \left|\int_{\mathbb{R}} p_{k+1}(x)xp_k(x)\,\mathrm{d}\mu(x)\right| &\leq \int_{\mathbb{R}} |p_{k+1}(x)||p_k(x)|\,\mathrm{d}\mu(x)\sup_{x\in\operatorname{supp}(\mu)}|x|\\ &\leq \|p_{k+1}\|_{L^2(\mu)}\|p_k\|_{L^2(\mu)}\sup_{x\in\operatorname{supp}(\mu)}|x| = \sup_{x\in\operatorname{supp}(\mu)}|x| < \infty.\end{aligned}$$

We used the triangle inequality, the Cauchy Schwarz inequality and the property that p_k is normalised. The fact that μ is compactly supported leads to the last inequality. The same kind of argument leads to the boundedness of b_k:

$$\begin{aligned}|b_k| &= \left|\int_{\mathbb{R}} x(p_k(x))^2\,\mathrm{d}\mu(x)\right| \le \left|\int_{\mathbb{R}} (p_k(x))^2\,\mathrm{d}\mu(x)\right| \sup_{x\in\operatorname{supp}(\mu)} |x| \\ &\le \|p_k\|^2_{L^2(\mu)} \sup_{x\in\operatorname{supp}(\mu)} |x| = \sup_{x\in\operatorname{supp}(\mu)} |x| < \infty.\end{aligned}$$

This completes the proof. □

So with every set of orthonormal polynomials we can find sequences $\{a_k\}$ and $\{b_k\}$ such that (3.5) and (3.6) hold. But the converse is also true, given the sequences $\{a_k\}$ and $\{b_k\}$, and $p_0(x) = 1$, we can determine the rest of the p_k as a solution, by (3.5) and (3.6). This will be shown in theorem 3.9.

We can also just use initial conditions on p_0 and p_1 to obtain a solution of (3.5) only. Let $\{r_k\}_{k=0}^{\infty}$ be defined as the solution of (3.5) with initial conditions $r_0(x) = 0$ and $r_1(x) = a_0^{-1}$. By construction, the polynomial r_k has degree $k-1$ and we call them the **associated polynomials**. Notice that (3.6) is not valid. The associated polynomials can be written in terms of the polynomials p_k.

Lemma 3.4. *Let $\{p_k\}_{k=0}^{\infty}$ be a set of orthonormal polynomials in $L^2(\mu)$. The associated polynomial r_k can be written as*

$$r_k(x) = \int_{\mathbb{R}} \frac{p_k(x) - p_k(y)}{x-y}\,\mathrm{d}\mu(y). \tag{3.10}$$

Proof. Let the right hand side of (3.10) be defined as $q_k(x)$. Now we have to show that q_k satisfies (3.5) together with the initial conditions. Then, by definition, q_k equals r_k.
To show that (3.5) is satisfied, we see that

$$\begin{aligned} xq_k(x) &= \int_{\mathbb{R}} \frac{xp_k(x) - xp_k(y)}{x-y}\,\mathrm{d}\mu(y) = \int_{\mathbb{R}} \frac{xp_k(x) - yp_k(y) + yp_k(y) - xp_k(y)}{x-y}\,\mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}} \frac{xp_k(x) - yp_k(y)}{x-y}\,\mathrm{d}\mu(y) - \int_{\mathbb{R}} \frac{xp_k(y) - yp_k(y)}{x-y}\,\mathrm{d}\mu(y) \\ &= a_k q_{k+1}(x) + b_k q_k(x) + a_{k-1} q_{k-1}(x) - \int_{\mathbb{R}} p_k(y)\,\mathrm{d}\mu(y) \end{aligned}$$

using Theorem 3.3 for p_k in the last equality. Noticing that $p_0(x) = 1$ is orthogonal to all p_k, $k \ge 1$, the integral is zero for $k \ge 1$. Thus q_k satisfies (3.5). For the initial conditions we observe that

$$q_0(x) = \int_{\mathbb{R}} \frac{p_0(x) - p_0(y)}{x-y}\,\mathrm{d}\mu(y) = \int_{\mathbb{R}} \frac{1-1}{x-y}\,\mathrm{d}\mu(y) = 0$$

and using (3.6) to get $p_1 = a_0^{-1}(x - b_0)$ we obtain

$$q_1(x) = \int_{\mathbb{R}} \frac{p_1(x) - p_1(y)}{x-y}\,\mathrm{d}\mu(y) = \int_{\mathbb{R}} \frac{x - b_0 - y + b_0}{a_0(x-y)}\,\mathrm{d}\mu(y) = a_0^{-1}.$$

So all conditions are satisfied and q_k equals r_k. □

As we now have obtained two solutions of (3.5), we are able to find an expression of the k-th coefficient of the function $(x-z)^{-1}$. Looking at the following simple equality for fixed $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}} \frac{p_k(x)}{x-z}\,\mathrm{d}\mu(x) &= \int_{\mathbb{R}} \frac{p_k(x) - p_k(z) + p_k(z)}{x-z}\,\mathrm{d}\mu(x) \\ &= r_k(z) + \int_{\mathbb{R}} \frac{p_k(z)}{x-z}\,\mathrm{d}\mu(x) = r_k(z) + p_k(z)w(z), \end{aligned} \tag{3.11}$$

we can easily prove the following corollary.

Corollary 3.5. *Let $z \in \mathbb{C} \backslash \mathbb{R}$ be fixed. The k-th coefficient with respect to the orthonormal set $\{p_k\}_{k=0}^{\infty}$ in $L^2(\mu)$ of $\rho_z : x \mapsto (x-z)^{-1}$ is given by $r_k(z) + p_k(z)w(z)$. Hence,*

$$\sum_{k=0}^{\infty} \left| r_k(z) + p_k(z)w(z) \right|^2 \le \int_{\mathbb{R}} |x-z|^{-2} \, \mathrm{d}\mu(x) < \infty.$$

Proof. Let d_i be the i-th coefficient such that $(x-z)^{-1} = \sum_{i=0}^{\infty} d_i p_i(x)$. Now multiplying both sides by p_k and integrating over $\mathbb{R}$ gives

$$\int_{\mathbb{R}} \frac{p_k}{x-z} \, \mathrm{d}\mu(x) = \int_{\mathbb{R}} \sum_{i=0}^{\infty} d_i p_i(x) p_k(x) \, \mathrm{d}\mu(x) = \int_{\mathbb{R}} d_k \, \mathrm{d}\mu(x) = d_k.$$

Using the equality obtained in (3.11), we see that

$$d_k = r_k(z) + p_k(z)w(z).$$

Using Bessels inequality, see Theorem 2.5, and realising that $d_k = \langle \rho_z, p_k \rangle$, yields

$$\sum_{k=0}^{\infty} |d_k| \le \|\rho_z\|^2$$

and the second assertion follows. □

As a consequence for $\{r_k(z) + p_k(z)w(z)\}_{k=0}^{\infty}$ to be in $\ell^2(\mathbb{N})$, the limit

$$\lim_{k \to \infty} r_k(z) + p_k(z)w(z) = 0$$

should hold. This yields a requirement for $w(z)$,

$$w(z) = -\lim_{k \to \infty} \frac{r_k(z)}{p_k(z)}, \qquad z \in \mathbb{C} \backslash \mathbb{R},$$

assuming the limit exists.

The next lemma assures that the fraction $\frac{r_k(z)}{p_k(z)}$ has no poles outside the real line. This means that the limit indeed exists for $z \in \mathbb{C} \backslash \mathbb{R}$.

Lemma 3.6. *The zeros of $p_k(z)$ in $\mathbb{C}$ are real and simple.*

Proof. First notice that $p_k(z)$ has degree k, so there are at most k real zeros. Let $x_1, \ldots, x_m$ be the distinct real zeros with odd multiplicity ($m \le k$), i.e. where the polynomial changes sign. Then for $x \in \mathbb{R}$,

$$p_k(x)(x-x_1)(x-x_2)\cdots(x-x_m)$$

is even and so does not change sign. Therefore

$$\int_{\mathbb{R}} p_k(x)(x-x_1)(x-x_2)\cdots(x-x_m) \, \mathrm{d}\mu(x) \tag{3.12}$$

is not equal to zero.
Now assume $m < k$, then the polynomial $(x-x_1)(x-x_2)\cdots(x-x_m)$ can be written as linear combination of the orthogonal polynomials $p_i(x)$ of degree smaller than k. By orthogonality between p_i and p_k it follows that (3.12) equals zero if $m < k$ which ends up with a contraduction. Thus $m = k$, which means that there are k distinct real zeros of odd multiplicity and thus the multiplicity needs to be 1. □

3.2. Jacobi operators

For solving (3.5) and (3.6) we want to make use of **Jacobi operators**. These are tridiagonal infinite matrices of the form

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & 0 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & 0 & a_2 & b_3 & a_3 & 0 & \dots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}$$

with $a_i, b_i \in \mathbb{R}$ and $a_i > 0$. The reason why we exclude the case $a_i = 0$ is that if there exists a unique m such that $a_m = 0$, then the Jacobi matrix can be split into two matrices. The first matrix has only nonzero entries on the first $m \times m$ block and analysis can be done using linear algebra. Therefore we are only interested in this type.

We want to define Jacobi operators on the Hilbert space $\ell^2(\mathbb{N})$. When we define the Jacobi operators first on the standard orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of $\ell^2(\mathbb{N})$, we will see the same structure as in Theorem 3.3:

$$Je_k := \begin{cases} a_k e_{k+1} + b_k e_k + a_{k-1} e_{k-1}, & k \geq 1, \\ a_0 e_1 + b_0 e_0, & k = 0. \end{cases}$$

This means that for every probability measure with finite moments we can obtain unique coefficients $\{a_n\}$ and $\{b_n\}$, which is similar to having a Jacobi operator J. Because the linear subspace $\mathscr{D}(\mathbb{N})$ of finite linear combinations of e_k is dense in $\ell^2(\mathbb{N})$, we would like to first extend J to $\mathscr{D}(\mathbb{N})$ by defining for $v = \sum_{j=0}^n c_j e_j$ in $\mathscr{D}(\mathbb{N})$,

$$Jv := \sum_{j=0}^{n} c_j Je_j,$$

justified by the linearity of J. To show that J is symmetric, we first want to observe that for $l = k-1 \geq 1$, $\langle Je_k, e_l\rangle = \langle e_k, Je_l\rangle$. Indeed,

$$\begin{aligned} \langle Je_k, e_l\rangle &= a_k\langle e_{k+1}, e_l\rangle + b_k\langle e_k, e_l\rangle + a_{k-1}\langle e_{k-1}, e_l\rangle \\ &= a_{l-1}\langle e_k, e_{l-1}\rangle + b_k\langle Je_k, e_l\rangle + a_l\langle e_k, e_{l+1}\rangle = \langle e_k, Je_l\rangle, \end{aligned}$$

using the property $\langle e_i, e_j\rangle = \delta_{ij}$ for all $i, j \in \mathbb{N}$. With the same technique we can easily see that the identity holds for $l = k$ and $l = k+1$. For other relations between l and k we get $\langle Je_k, e_l\rangle = 0 = \langle e_k, Je_l\rangle$. For the special case when either of the k or l is zero can also easily be seen that the relation holds.

As a consequence, symmetry holds for $v = \sum_{j=0}^n c_j e_j$ and $w = \sum_{i=0}^m d_i e_i$ in $\mathscr{D}(\mathbb{N})$. Indeed,

$$\begin{aligned} \langle Jv, w\rangle &= \sum_{j=0}^{n}\sum_{i=0}^{m} c_j \overline{d_i}\langle Je_j, e_i\rangle \\ &= \sum_{j=0}^{n}\sum_{i=0}^{m} c_j \overline{d_i}\langle e_j, Je_i\rangle \\ &= \langle v, Jw\rangle. \end{aligned} \tag{3.13}$$

This shows that J is a densely defined symmetric operator. If J is bounded on $\mathscr{D}(\mathbb{N})$, it is continuous and therefore can be extended to a bounded operator on $\ell^2(\mathbb{N})$.

Lemma 3.7. *There exists polynomials P_k of degree k with real coefficients such that $e_k = P_k(J)e_0$. In particular, e_0 is a cyclic vector for the action of J, i.e. the linear subspace $\{J^k e_0 \mid k \in \mathbb{N}\}$ is dense in $\ell^2(\mathbb{N})$.*

Proof. Trivially, $e_0 = P_0(J)e_0$ for polynomial $P_0(x) = 1$. Using (3.6), we see that $Je_0 = a_0 e_1 + b_0 e_0$, which can be rewritten as

$$e_1 = \frac{Je_0 - b_0 e_0}{a_0},$$

with $a_0 > 0$. So $e_1 = P_1(J)e_0$ for polynomial $P_1(x) = a_0^{-1}x - b_0/a_0$ of degree 1.
Using induction, assume $e_i = P_i(J)e_0$ for $0 \le i \le k$. Then $Je_k = a_k e_{k+1} + b_k e_k + a_{k-1}e_{k-1}$ leads to

$$\begin{aligned} e_{k+1} &= \frac{Je_k - b_k e_k - a_{k-1}e_{k-1}}{a_k} \\ &= \frac{JP_k(J)e_0 - b_k P_k(J)e_0 - a_{k-1}P_{k-1}(J)e_0}{a_k} \end{aligned}$$

which is the sum of polynomials with degree up to $k+1$, and thus $e_{k+1} = P_{k+1}(J)e_0$.
To see that the linear subspace is dense in $\ell^2(\mathbb{N})$ we remark that $e_k \in \{J^k e_0 \mid k \in \mathbb{N}\}$ and the set $\{e_k\}_{k=0}^{\infty}$ form an orthonormal basis of $\ell^2(\mathbb{N})$. □

Lemma 3.8. *If the sequences $\{a_k\}$ and $\{b_k\}$ are bounded, say $\sup_k |a_k| + \sup_k |b_k| \le M < \infty$, then J extends to a bounded symmetric operator with $\|J\| \le 2M$. On the other hand, if J is bounded, then the sequences $\{a_k\}$ and $\{b_k\}$ are bounded.*

For a proof, see [8].

Assume that J is continuous and thus bounded, on the domain $\mathcal{D}(\mathbb{N})$. Because $\mathcal{D}(\mathbb{N})$ is dense in $\ell(\mathbb{N})$, we can extend J, which, together with being symmetric, leads to self-adjointness of J on $\ell(\mathbb{N})$. Using the spectral theorem, Theorem 2.19, there exists a unique spectral measure E such that

$$\langle Jv, w\rangle = \int_{\mathbb{R}} t \, \mathrm{d}E_{v,w}(t), \qquad v, w \in \ell(\mathbb{Z}_{\ge 0}).$$

For $v = w = e_0$ we can define the positive Borel measure $\mu(A) := E_{e_0,e_0}(A) = \langle E(A)e_0, e_0\rangle$, as in Proposition 2.18. Moreover, the spectral theorem claims that $\mathrm{supp}(\mu) \subseteq [-\|J\|, \|J\|]$. By (3.2) we have an upperbound $|m_k| \le 2\|J\|^k$. Therefore the moments of μ are finite. This shows that every bounded Jacobi operator J corresponds with its unique compactly supported probability measure μ, which will be needed in the next theorem. Using the fact that J is self-adjoint and $E(A)$ commutes with J, yields

$$\begin{aligned} \langle E(A)e_k, e_l\rangle &= \langle E(A)P_k(J)e_0, P_l(J)e_0\rangle \\ &= \langle P_l(J)P_k(J)E(A)e_0, e_0\rangle = \int_A P_k(x)P_l(x)\, \mathrm{d}\mu(x). \end{aligned} \tag{3.14}$$

The P_k's are as in Lemma 3.7.

Theorem 3.9 (Favard's theorem to bounded coefficients in the three-term recurrence relation)**.** *Let J be a bounded Jacobi operator, then there exists a unique compactly supported probability measure μ such that for any polynomial P the map $U : P(J)e_0 \mapsto P$ extends to a unitary operator $\ell^2(\mathbb{N}) \to L^2(\mu)$ with $UJ = MU$, where $M : L^2(\mu) \to L^2(\mu)$ is the multiplication operator $(Mf)(x) = xf(x)$. Moreover, let $p_k = Ue_k$, then the set $\{p_k\}_{k=0}^{\infty}$ is the set of orthonormal polynomials with respect to μ;*

$$\int_{\mathbb{R}} p_k(x)p_l(x)\, \mathrm{d}\mu(x) = \delta_{k,l}.$$

Proof. The map U is defined on the linear subspace $\{J^k e_0 | k \in \mathbb{N}\}$, which is dense in $\ell^2(\mathbb{N})$ by Lemma 3.7. For bounded J the measure μ is compactly supported. Therefore the polynomials are dense in $L^2(\mu)$. So U maps a dense subspace of $\ell(\mathbb{N})$ into a dense subspace of $L^2(\mu)$.
To prove that U is unitary, we can take polynomials P and Q. Then, using the spectral theorem,

$$\begin{aligned}\langle P(J)e_0, Q(J)e_0\rangle &= \left\langle \overline{Q}(J)P(J)e_0, e_0\right\rangle = \int_{\mathbb{R}} \overline{Q}(J)P(J)\,\mathrm{d}\mu(x) \\ &= \langle P, Q\rangle_{L^2(\mu)} = \langle UP(J)e_0, UQ(J)e_0\rangle_{L^2(\mu)},\end{aligned} \tag{3.15}$$

which shows that $\langle u, v\rangle = \langle Uu, Uv\rangle$. Because U is bounded it can uniquely be extended to a unitary operator $\ell^2(\mathbb{N}) \to L^2(\mu)$.
Observe that for $p_k := Ue_k$,

$$\int_{\mathbb{R}} p_k(x)p_l(x)\,\mathrm{d}\mu(x) = \langle p_k, p_l\rangle_{L^2(\mu)} = \langle Ue_k, Ue_l\rangle_{L^2(\mu)} = \langle e_k, e_l\rangle = \delta_{k,l}, \tag{3.16}$$

where we used unitarity of U. The fact that p_k is a polynomial of degree k, together with (3.16), yields a set of orthogonal polynomials $\{p_k\}_{k=0}^{\infty}$ with respect to μ.
We want to prove that $UJe_k = MUe_k$ for all $k \in \mathbb{N}$. Notice that $MUe_k(x) = xp_k(x)$, so it suffices to show that

$$UJe_k = U(a_k e_{k+1} + b_k e_k + a_{k-1}e_{k-1}) = a_k p_{k+1} + b_k p_k + a_{k-1}p_{k-1}$$

corresponds with the three-term recurrence relation. Observe that $a_k = \langle Je_k, e_{k+1}\rangle$ and $b_k = \langle Je_k, e_k\rangle$. Then using the functional calculus and (3.14) yields

$$\begin{aligned}a_k = \langle Je_k, e_{k+1}\rangle &= \int_{\mathbb{R}} x\,\mathrm{d}E_{e_k,e_{k+1}}(x) = \int_{\mathbb{R}} xp_k(x)p_{k+1}(x)\,\mathrm{d}\mu(x), \\ b_k = \langle Je_k, e_k\rangle &= \int_{\mathbb{R}} x\,\mathrm{d}E_{e_k,e_k}(x) = \int_{\mathbb{R}} x\left(p_k(x)\right)^2\,\mathrm{d}\mu(x).\end{aligned}$$

In the proof of Theorem 3.3 we have obtained (3.9) and (3.8), the expressions for a_k and b_k, which coincide with the previous obtained expressions. □

The importance of Theorem 3.9 lies in the fact that it is the reverse of Theorem 3.3. It namely implies that given a_k and b_k, the polynomials generated by (3.5) and (3.6) form an orthonormal set with respect to some unique probability measure μ. As we now have for a given J a unique probability measure μ, we can express the moment generating function in terms of the resolvent operator.

Remark. The moment generating function w for measure μ can be written in terms of the resolvent operator $R(z)$ for the Jacobi operator J:

$$w(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{x-z} = \left\langle (J-z)^{-1}e_0, e_0\right\rangle = \langle R(z)e_0, e_0\rangle, \qquad z \in \mathbb{C}\backslash\mathbb{R}.$$

It turns out that $\sum_{k=0}^{\infty}|p_k(z)|^2$ does not exist for $z \in \mathbb{C}\backslash\mathbb{R}$, as will be shown in Proposition 3.11. In developing an expression for the resolvent operator we want to find a solution of $Jf = zf$ in $\ell^2(\mathbb{N})$.

Definition 3.10. The element $f(z) = \{f_k(z)\}_{k=0}^{\infty}$ is called an **asymptotically free solution** to $Jf(z) = zf(z)$ with $z \in \mathbb{C}\backslash\mathbb{R}$ if it satisfies

1. $(Jf(z))_k = zf_k(z), \qquad$ for $k \geq 1$
2. $\sum_{k=0}^{\infty}|f_k(z)|^2 < \infty$.

Notice that the second requirement can only be fullfilled if $\lim_{k\to\infty} f_k(z) = 0$ with $z \in \mathbb{C}\setminus\mathbb{R}$, see also page 17. On the other hand, an asymptotically free solution does not need to satisfy (3.6) of the three-term recurrence relation.

Proposition 3.11. *Let J be a bounded Jacobi operator. Take $z \in \mathbb{C}\setminus\mathbb{R}$ fixed. Then $f(z) = (J-z)^{-1}e_0$ is an asymptotically free solution for the Jacobi operator J. There exists a unique $w(z) \in \mathbb{C}\setminus\mathbb{R}$ such that $f_k(z) = w(z)p_k(z) + r_k(z)$, with p_k, r_k the polynomials as in Theorem (3.3) and Lemma (3.4). Moreover, w is the Stieltjes transform of the measure μ.*

Proof. Consider w as the Stieltjes transform $w(z) = \int_{\mathbb{R}} (x-z)^{-1}\,\mathrm{d}\mu(x)$. The relation

$$f_k(z) = \langle f(z), e_k\rangle = \langle (J-z)^{-1}e_0, p_k(x)e_0\rangle = \langle p_k(J-z)^{-1}e_0, e_0\rangle \tag{3.17}$$

$$= \int_{\mathbb{R}} \frac{p_k(x)}{x-z}\,\mathrm{d}\mu(x) = w(z)p_k(z) + r_k(z) \tag{3.18}$$

is justified by Corollary 3.5. Also, by the same corollary, we know that $\{w(z)p_k(z) + r_k(z)\}_{k=0}^{\infty} \in \ell^2(\mathbb{N})$, so the second requirement of Definition 3.10 is fullfiled. The first requirement is satisfied due to

$$\begin{aligned} zf_k(z) &= w(z)zp_k(z) + zr_k(z) \\ &= w(z)\left(a_kp_{k+1}(z) + b_kp_k(z) + a_{k-1}p_{k-1}(z)\right) + (a_kr_{k+1}(z) + b_kr_k(z) + a_{k-1}r_{k-1}(z)) \\ &= a_k\left(w(z)p_{k+1}(z) + r_{k+1}(z)\right) + b_k\left(w(z)p_k(z) + r_k(z)\right) + a_{k-1}\left(w(z)p_{k-1}(z) + r_{k-1}(z)\right) \\ &= a_kf_{k+1} + b_kf_k + a_{k-1}f_{k-1} = \left(Jf(z)\right)_k. \end{aligned}$$

So we have found a w that does the job. Now we have to be sure this w is unique. So assume w is not unique, i.e. there is a $\hat{w}$ linearly independent of w, such that $\hat{f}_k(z) = \hat{w}(z)p_k(z) + r_k(z)$ is another solution to $Jf(z) = zf(z)$. But then is $f(z) - \hat{f}(z)$ a solution as well and so

$$\sum_{k=0}^{\infty} |w(z) - \hat{w}(z)|^2 |p_k(z)|^2 = \sum_{k=0}^{\infty} |f(z) - \hat{f}(z)|^2 < \infty$$

implies that $\sum_{k=0}^{\infty} |p_k(z)|^2 < \infty$ for $z \in \mathbb{C}\setminus\mathbb{R}$. Therefore $p(z) = \{p_k(z)\}_{k=0}^{\infty}$, $z \in \mathbb{C}\setminus\mathbb{R}$, is another asymptotically free solution and $Jp(z) = zp(z)$ shows that z is a non-real eigenvalue of J, contradicting the fact that J is symmetric. □

To find the spectral measure for the Jacobi operator J, we need the concept of the following Green kernel for solution p of the three-term recurrence relation and f an asymptotically free solution of J, with $z \in \mathbb{C}\setminus\mathbb{R}$,

$$G_{k,l}(z) := \frac{1}{[f,p]}\begin{cases} f_l(z)p_k(z), & k \le l \\ f_k(z)p_l(z), & k > l, \end{cases} \tag{3.19}$$

using the Wronskian. It is worth mentioning that the Green kernel can be stated easier if we choose the asymptotically free solution f from Proposition 3.11. Then $[f,p] = [wp + r, p] = [r,p] = 1$. Nevertheless, due to the lack of decomposition of the asymptotically free solution in the next chapter, we are forced to define the Green kernel in this way. We consider

$$\begin{aligned} [f,p]^2 \sum_{k=0}^{\infty} \left|G_{k,l}(z)\right|^2 \cdot \sum_{l=0}^{\infty} \left|G_{k,l}(z)\right|^2 = &\left(\left|f_l(z)\right|^2 \sum_{k=0}^{l} \left|p_k(z)\right|^2 + |p_l(z)|^2 \sum_{k=l+1}^{\infty} |f_k(z)|^2\right) \\ &\cdot \left(|f_k(z)|^2 \sum_{l=0}^{k} |p_l(z)|^2 + |p_k(z)|^2 \sum_{l=k+1}^{\infty} |f_l(z)|^2\right). \end{aligned}$$

All terms on the right hand side are finite, in particular because $\sum_{k=0}^{\infty} |f_k(z)|^2 < \infty$ by definition of an asymptotically free solution. Therefore $\{G_{k,l}\}_{k=0}^{\infty}, \{G_{k,l}\}_{l=0}^{\infty} \in \ell^2(\mathbb{N})$. We use the Green kernel to define for $v \in \ell^2(\mathbb{N})$ the map

$$G(z) : v \mapsto (G(z))\,v, \qquad (G(z)\,v)_k := \sum_{l=0}^{\infty} v_l G_{k,l}(z) = \left\langle v, \overline{G_{k,\cdot}(z)} \right\rangle. \tag{3.20}$$

It is clear that this is a well-defined mapping. Notice that the second requirement of Definition 3.10 is needed for this mapping to be well-defined.

Proposition 3.12. *The resolvent of J is given by $(J-z)^{-1} = G(z)$ for $z \in \mathbb{C}\backslash\mathbb{R}$.*

Proof. For $z \in \mathbb{C}\backslash\mathbb{R}$ we know that the inverse $(J-z)^{-1}$ exists and is bounded, because J is symmetric and so $\mathbb{C}\backslash\mathbb{R} \subset \rho(J)$. So if we can show that $(J-z)G(z) = \mathbf{1}$ on the dense subspace $\mathscr{D}(\mathbb{N})$ of $\ell^2(\mathbb{N})$, then by continuity we have proven the proposition.

$$\begin{aligned}
((J-z)G(z)\,v)_k &= \sum_{l=0}^{\infty} v_l \left(a_{k-1}G_{k-1,l}(z) + (b_k - z)G_{k,l}(z) + a_k G_{k+1,l}(z)\right) \\
&= \frac{1}{[f,p]}\Bigg(\sum_{l=0}^{k-1} v_l p_l(z) \left(a_{k-1} f_{k-1}(z) + (b_k - z) f_k(z) + a_k f_{k+1}(z)\right) \\
&+ \sum_{l=k+1}^{\infty} v_l f_l(z) \left(a_{k-1} p_{k-1}(z) + (b_k - z) p_k(z) + a_k p_{k+1}(z)\right) \\
&+ v_k \left(a_{k-1} f_k(z) p_{k-1}(z) + (b_k - z) f_k(z) p_k(z) + a_k f_{k+1}(z) p_k(z)\right)\Bigg) \\
&= \frac{v_k}{[f,p]} \left(a_{k-1} f_k(z) p_{k-1}(z) + (b_k - z) f_k(z) p_k(z) + a_k f_{k+1}(z) p_k(z)\right) \\
&= \frac{v_k}{[f,p]} a_k \left(f_{k+1}(z) p_k(z) - f_k(z) p_{k+1}(z)\right) \\
&= v_k.
\end{aligned}$$

Here we have used that for $k \geq 1$ both f_k and p_k are solutions of $Jf = zf$, and thus satisfy the three-term recurrence relation. □

Proposition 3.12 tells us that $G(z)$ is bounded. Indeed, $z \in \mathbb{C}\backslash\mathbb{R}$ is contained in the resolvent set and so by definition $(J-z)^{-1}$ is bounded. Notice that in the proof we didn't use the fact that the Wronskian equals 1.

Theorem 2.20 gives the inversion formula with respect to the resolvent operator R. By Proposition 3.12 we know that for Jacobi operator J the resolvent is equal to the Green kernel. Therefore, the spectral measure of J is:

$$E_{u,v}((a,b)) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \langle G(x+i\varepsilon)u, v\rangle - \langle G(x-i\varepsilon)u, v\rangle \,\mathrm{d}x. \tag{3.21}$$

To find orthogonality relations we need to express $E_{u,v}$ in terms of p_k. Notice that,

$$\langle G(z)u, v\rangle = \sum_{k=0}^{\infty} (G(z)u)_k \overline{v}_k = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} G_{k,l}(z) u_l \overline{v}_k \tag{3.22}$$

$$= \sum_{k\le l} \frac{f_l(z)p_k(z)}{[f,p]} u_l\overline{v}_k + \sum_{k>l} \frac{f_k(z)p_l(z)}{[f,p]} u_l\overline{v}_k \tag{3.23}$$

$$= \frac{1}{[f,p]}\left(\sum_{k\le l} f_l(z)p_k(z)u_l\overline{v}_k + \sum_{k<l} f_l(z)p_k(z)u_k\overline{v}_l\right) \tag{3.24}$$

$$= \frac{1}{[f,p]}\sum_{k\le l} f_l(z)p_k(z)(u_l\overline{v}_k + u_k\overline{v}_l)(1-\frac{1}{2}\delta_{k,l}), \tag{3.25}$$

by switching letters k and l in the third equality, and using the Dirac delta function to avoid dubbling when $k = l$. Because p_k and r_k are polynomials, $p_k(x+i\varepsilon) \approx p_k(x)$ and $r_k(x+i\varepsilon) \approx r_k(x)$ for small ε. The same results holds for $p_k(x-i\varepsilon)$ and $r_k(x-i\varepsilon)$. Then, for small ε,

$$\begin{aligned}\langle G(x+i\varepsilon)u, v\rangle - \langle G(x-i\varepsilon)u, v\rangle &= \frac{1}{[f,p]}\sum_{k\le l}\left[f_l(x+i\varepsilon)-f_l(x-i\varepsilon)\right]p_k(x-i\varepsilon)(u_l\overline{v}_k + u_k\overline{v}_l)(1-\frac{1}{2}\delta_{k,l})\\ &\approx \frac{[w(x+i\varepsilon)-w(x-i\varepsilon)]}{[f,p]}\sum_{k\le l} p_l(x)p_k(x)(u_l\overline{v}_k + u_k\overline{v}_l)(1-\frac{1}{2}\delta_{k,l}).\end{aligned}$$

where we have used the expression for $f_l(x)$ stated in Proposition 2.11. Using the fact that

$$w(x+i\varepsilon) - w(x-i\varepsilon) = w(x+i\varepsilon) - w(\overline{x+i\varepsilon}) = w(x+i\varepsilon) - \overline{w(x+i\varepsilon)} = 2i\Im(w(x+i\varepsilon)),$$

and combining the previous results into (3.21) yields

$$\begin{aligned}E_{u,v}((a,b)) &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} 2i\Im(w(x+i\varepsilon))\sum_{k\le l} p_l(x)p_k(x)(u_l\overline{v}_k + u_k\overline{v}_l)(1-\frac{1}{2}\delta_{k,l})\,\mathrm{d}x.\\ &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{a+\delta}^{b-\delta} \Im(w(x+i\varepsilon))\sum_{k\le l} p_l(x)p_k(x)(u_l\overline{v}_k + u_k\overline{v}_l)(1-\frac{1}{2}\delta_{k,l})\,\mathrm{d}x.\\ &= \sum_{k\le l}\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{a+\delta}^{b-\delta}\Im(w(x+i\varepsilon)p_l(x)p_k(x))\,\mathrm{d}x\cdot(u_l\overline{v}_k + u_k\overline{v}_l)(1-\frac{1}{2}\delta_{k,l}).\\ &= \sum_{k\le l}\left(\int_{(a,b)} p_l(x)p_k(x)\,\mathrm{d}\mu(x)\right)\cdot(u_l\overline{v}_k + u_k\overline{v}_l)(1-\frac{1}{2}\delta_{k,l})\\ &= \int_{(a,b)}\sum_{k\le l} p_l(x)p_k(x)\cdot(u_l\overline{v}_k + u_k\overline{v}_l)(1-\frac{1}{2}\delta_{k,l})\,\mathrm{d}\mu(x)\\ &= \int_{(a,b)}\sum_{k=0}^{\infty} u_k p_k(x)\sum_{l=0}^{\infty}\overline{v_l p_l(x)}\,\mathrm{d}\mu(x),\end{aligned}$$

where Fubini's theorem justifies the interchange of integral and sum. This becomes

$$E_{u,v}((a,b)) = \int_{(a,b)} (Uu)(x)\overline{(Uv)(x)}\,\mathrm{d}\mu(x),$$

where $U:\ell^2(\mathbb{N})\to L^2(\mu)$ is the unitary operator defined in Favard's Theorem, i.e. $(Uu)(x) = \sum_k u_k p_k(x)$ and $(Uv)(x) = \sum_l v_l p_l(x)$.

In this section we have worked on the next theorem, which can be seen as a summary of the work we have done so far.

Theorem 3.13. *There is one-to-one correspondence between bounded Jacobi operators and probability measures on $\mathbb{R}$ with compact support.*

Proof. Let μ be the probability measure with compact support. Then by (3.2) we see that the moments are finite. Now we are able to construct the corresponding set of orthonormal polynomials by Gram-Schmidt. Theorem 3.3 gives us the coefficients a_k and b_k which results in the Jacobi operator J. Therefore we have the map $\zeta : \mu \mapsto J$. Assume that ζ is not injective. Then there exist different measures μ_1, μ_2 such that they generate the same Jacobi operator J. But that means that they generate the same orthogonal polynomials and thus the same moment generating function w. But then

$$\mu_1((a,b)) = \lim_{\delta \downarrow 0} \int_{a+\delta}^{b-\delta} \Im w(x+i\varepsilon)\,\mathrm{d}x = \mu_2((a,b)).$$

Therefore, ζ is injective. □

3.3. Unbounded Jacobi operators

Up until now we have only spoken about bounded Jacobi operators. The advantage of having a bounded Jacobi operator is that we were able to extend the operator to a symmetric operator on $\ell^2(\mathbb{N})$. In Theorem 3.3 we have seen that if μ is compactly supported, then the coefficients a_k and b_k are bounded. Lemma 3.8 showed that the coefficients are bounded if and only if J is bounded. The following lemma occurs as an example in which the Jacobi operator J is unbounded.

Lemma 3.14. *Let μ be a probability measure with finite moments. Consider the three-term recurrence relation and its corresponding densely defined Jacobi operator J. If $\operatorname{supp}(\mu)$ is unbounded, then J is unbounded.*

Proof. See for [8]. □

We would like to be able to work with the adjoint of an unbounded Jacobi operator J. In (3.13) it was shown that J is a densely defined symmetric operator and for J bounded we could extend the operator to $\ell^2(\mathbb{N})$. For unbounded J we still need to verify for which elements in $\ell^2(\mathbb{N})$ the adjoint exists. This can be done by using symmetry of J, so $Jv = J^* v$, on $\mathscr{D}(\mathbb{N})$. The question becomes for which $v = \sum_{k=0}^{\infty} v_k e_k$ in $\ell^2(\mathbb{N})$ holds

$$J^* v = (a_0 v_1 + b_0 v_0)e_0 + \sum_{k=1}^{\infty} (a_k v_{k+1} + b_k v_k + a_{k-1} v_{k-1})e_k \in \ell^2(\mathbb{N}). \tag{3.26}$$

Proposition 3.15. *Let J be a Jacobi operator on $\mathscr{D}(\mathbb{N})$. The adjoint J^* is given by (3.26) with domain*

$$\mathscr{D}^* = \left\{ v \in \ell^2(\mathbb{N}) \mid J^* v \in \ell^2(\mathbb{N}) \right\}.$$

So we see by Proposition 3.15 that J^* is the extension of J with maximal domain $\mathscr{D}^*$. If J is essentially self-adjoint, then there exists a unique self-adjoint extension. This can only be J^*.

Proof. By Definition 2.24, the domain $\mathscr{D}(J^*)$ of the adjoint J^* consists of all $v \in \ell^2(\mathbb{N})$ such that the linear operator $\phi_v(u) = \langle Ju, v\rangle$, $u \in \mathscr{D}(\mathbb{N})$, is continuous and bounded. So let $v \in \mathscr{D}^*$, then

$$\begin{aligned} |\phi_v(u)| &= \left|\sum_k u_k a_k u_{k+1}\overline{v}_k + b_k u_k \overline{v}_k + a_{k-1} u_{k-1}\overline{v}_k\right| \\ &= \left|\sum_k u_k \left(a_{k-1}\overline{v}_{k-1} + b_k\overline{v}_k + a_k\overline{v}_{k+1}\right)\right| \\ &= \left|\left\langle u, J^* v\right\rangle\right| \leq \|u\| \|J^* v\|, \end{aligned}$$

which shows that ϕ_v is bounded and thus continous, and so $\mathscr{D}^* \subseteq \mathscr{D}(J^*)$.

To prove that $\mathscr{D}(J^*) \subseteq \mathscr{D}^*$, let $v \in \mathscr{D}(J^*)$. Then

$$\left|\sum_k u_k \left(a_{k-1}\overline{v}_{k-1} + b_k\overline{v}_k + a_k\overline{v}_{k+1}\right)\right| = |\phi_v(u)| \leq C\|u\| \tag{3.27}$$

for a constant C, justified by the boundedness of ϕ_v. Now let, for $N > 0$

$$u_k = \begin{cases} (J^* v)_k & 0 \leq k \leq N; \\ 0 & k > N. \end{cases}$$

Then

$$\begin{aligned} \left|\sum_k^N u_k \left(a_{k-1}\overline{v}_{k-1} + b_k\overline{v}_k + a_k\overline{v}_{k+1}\right)\right| &= \left|\sum_k^N (a_{k-1}v_{k-1} + b_k v_k + a_k v_{k+1})\overline{(a_{k-1}v_{k-1} + b_k v_k + a_k v_{k+1})}\right| \\ &= \sum_k^N |a_{k-1}\overline{v}_{k-1} + b_k\overline{v}_k + a_k\overline{v}_{k+1}|^2 = \sum_k |u_k|^2 = \|u\|^2, \end{aligned}$$

which, combined with (3.27), shows that $\|u\|^2 \leq C\|u\|$ and so $\|u\| \leq C$. By letting $N \to \infty$ we see that $\|J^* v\| \leq C$, which proves that $J^* v \in \ell^2(\mathbb{N})$ or $v \in \mathscr{D}^*$. Therefore $\mathscr{D}(J^*) \subseteq \mathscr{D}^*$. □

Observe that, with the convention $v_{-1} = 0$,

$$J^*\overline{v} = \sum_{k=0}^{\infty} (a_k\overline{v_{k+1}} + b_k\overline{v_k} + a_{k-1}\overline{v_{k-1}})e_k = \sum_{k=1}^{\infty} \overline{(a_k v_{k+1} + b_k v_k + a_{k-1}v_{k-1})e_k} = \overline{J^* v},$$

were we used that the coefficients a_k and b_k are real. Therefore, J^* commutes with complex conjugation and by the remark on page 11 the deficiency indices are equal: $n_+ = n_-$. The solution of $J^* f = zf$ is completely determined by $f_0 = \langle f, e_0\rangle$. If no such f exists on $\mathbb{C}\backslash\mathbb{R}$, then $N_z = \emptyset$. Now assume f is a solution of $J^* f = zf$ on $\mathbb{C}\backslash\mathbb{R}$. Let g be another solution, then $g_0 = \langle g, e_0\rangle$ and for $\lambda \in \mathbb{C}$ such that $g_0 = \lambda f_0$ we find that $g = \lambda f$. Therefore N_z is either zero or one dimensional.

In the case that N_z is zero-dimensional, J is essentially self-adjoint by Proposition 2.30. For deficiency indices $(0,0)$ we have a corollary.

Corollary 3.16. *Consider the unbounded Jacobi operator* $(J, \mathscr{D}(\mathbb{N}))$. *Then the following statements are equivalent:*

1. $(J, \mathscr{D}(\mathbb{N}))$ *is essentially self-adjoint.*
2. $\sum_{k=0}^{\infty} |p_k(z)|^2 = \infty$ *for all* $z \in \mathbb{C}\backslash\mathbb{R}$.
3. $\sum_{k=0}^{\infty} |p_k(z)|^2 = \infty$ *for some* $z \in \mathbb{C}\backslash\mathbb{R}$.

Proof. Note that the only possible element in $N_z = \{v \in \mathscr{D}^* | J^* v = zv\}$ is $\{p_k\}_{k=0}^{\infty}$.
(1) ⇔ (2). $(J, \mathscr{D}(\mathbb{N}))$ is essentially self-adjoint if and only if by Proposition 3.15 $N_z = \varnothing$ for all $z \in \mathbb{C}\backslash\mathbb{R}$ if and only if $p(z) \notin \mathscr{D}^*$ for all $z \in \mathbb{C}\backslash\mathbb{R}$ if and only if $p(z) \notin \ell^2(\mathbb{N})$ for all $z \in \mathbb{C}\backslash\mathbb{R}$.
(2) ⇒ (3). Trivial.
(3) ⇒ (2). Let $z \in \mathbb{C}\backslash\mathbb{R}$ be such that $\sum_{k=0}^{\infty} |p_k(z)|^2 = \infty$, then $p(z) \notin \mathscr{D}^*$ and thus $p(z) \notin N_z$. It follows that $\dim N_z = 0$ and by the remark on page 11 $\dim N_z = 0$ for all $z \in \mathbb{C}\backslash\mathbb{R}$. This, in return, implies that $p(z) \notin \ell^2(\mathbb{N})$. □

If J has deficiency indices $(1,1)$, then the self-adjoint extensions, say (J_t, D_t), satisfy

$$(J, \mathscr{D}(\mathbb{N})) \subsetneq (J_t, D_t) \subsetneq (J^*, D^*).$$

Notice that by Proposition 3.15 every self-adjoint extension is a restriction of (J^*, D^*) on the smaller domain D_t. The collection of self-adjoint extensions form a one-parameter family.

In order to say more about the domains of the self-adjoint extensions in the next lemma, we need an expression for the sesquilinear form B described in (2.7). Using the convention $u_{-1} = 0 = v_{-1}$ again yields

$$\begin{aligned} B(u,v) = \langle J^* u, v\rangle - \langle u, J^* v\rangle &= \lim_{N\to\infty}\left(\sum_{k=0}^{N}(J^*u)_k\overline{v}_k - \sum_{k=0}^{N} u_k\overline{(J^*v)_k}\right) \\ &= \lim_{N\to\infty}\left(\sum_{k=0}^{N}(a_k u_{k+1} + b_k u_k + a_{k-1}u_{k-1})\overline{v}_k - u_k(a_k\bar{v}_{k+1} + b_k\bar{v}_k + a_{k-1}\bar{v}_{k-1})\right) \\ &= \lim_{N\to\infty}\left(a_0(u_1\bar{v}_0 - \bar{v}_1 u_0) + \sum_{k=1}^{N}[u,\bar{v}]_k - [u,\bar{v}]_{k-1}\right) = \lim_{N\to\infty}[u,\bar{v}]_N, \end{aligned}$$

by cancellation of the b_k and the cancellation due to the telescoping series. By Cauchy-Schwarz, $u, v \in \mathscr{D}^*$ implies $|\langle J^* u, v\rangle| \le \|J^* u\|\|v\| < \infty$. It follows that $|B(u,v)| < \infty$ and so the limit exist. Expressing B in terms of the Wronskian has the benefit that the Wronskian is much easier to calculate than $B(u,v)$.

With the previous result we can describe the domains of all self-adjoint extensions of J.

Lemma 3.17. *Assume that J has deficiency indices $(1,1)$, then the self-adjoint extensions are in one-to-one correspondence with $(J^*, \mathscr{D}_\theta)$, $\theta \in [0, 2\pi)$, where*

$$\mathscr{D}_\theta = \left\{v \in \mathscr{D}^* \mid \lim_{N\to\infty}[v, e^{i\theta}\xi_i + e^{-i\theta}\xi_{-i}]_N = 0\right\}$$

where $J^*\xi_{\pm i} = \pm i\xi_{\pm i}$, $\overline{(\xi_i)_k} = (\xi_{-i})_k$ and $\|\xi_{\pm i}\| = 1$.

Proof. Because N_{-i} and N_i are one-dimensional, we can consider the basis elements $\xi_{-i} \in N_{-i}$ and $\xi_i \in N_i$, such that $c\xi_{\pm i} \in N_{\pm i}$ for all $c \in \mathbb{C}$. Let U be an isometric bijection as described in Proposition 2.30. It follows that U will rotate the element $c\xi_{-i}$ with a phase factor, say $U_\theta(c\xi_i) = ce^{2i\theta}\xi_{-i}$. Again by Proposition 2.30 we then see that the domains of the self-adjoint extensions are of the form

$$\mathscr{D}(S_\theta) = \left\{u + c(\xi_i + e^{2i\theta}\xi_{-i}) \mid u \in \mathscr{D}(J^{**}), c \in \mathbb{C}\right\}.$$

Note that $B(\xi_i, \xi_i) = 2i$, $B(\xi_{-i}, \xi_{-i}) = -2i$ and $B(\xi_i, \xi_{-i}) = 0$. Then for $v = u + c(\xi_i + e^{2i\theta}\xi_{-i}) \in \mathscr{D}(S_\theta)$ and $u \in \mathscr{D}(J^{**})$,

$$B(v, e^{i\theta}\xi_i + e^{-i\theta}\xi_{-i}) = B\left(u + c(\xi_i + e^{-2i\theta}\xi_{-i}), e^{i\theta}\xi_i + e^{-i\theta}\xi_{-i}\right) = B\left(u, e^{i\theta}\xi_i + e^{-i\theta}\xi_i\right). \tag{3.28}$$

For $v = c\xi_i \in N_i$ we notice that

$$\begin{aligned}
B(u,v) &= \left\langle J^* u, v\right\rangle - \left\langle u, J^* v\right\rangle = \left\langle J^* u, c\xi_i\right\rangle - \left\langle u, J^* c\xi_i\right\rangle \\
&= \left\langle J^* u, -icJ^*\xi_i\right\rangle - \langle u, ic\xi_i\rangle = i\left\langle J^* u, J^*(c\xi_i)\right\rangle + i\langle u, c\xi_i\rangle \\
&= i\langle u, c\xi_i\rangle_{J^*} = i\langle u, v\rangle_{J^*}
\end{aligned}$$

using the graph norm stated in Proposition 2.30. This proposition also tells us that for $u \in \mathcal{D}(J^{**})$ and $v \in N_i$ yields $B(u,v) = i\langle u,v\rangle_{J^*} = 0$. In a similar way, $B(u,v) = -i\langle u,v\rangle_{J^*} = 0$ for $v \in N_{-i}$. Thus, by linearity of B,

$$B\left(u, e^{i\theta}\xi_i + e^{-i\theta}\xi_i\right) = e^{i\theta}B(u,\xi_i) + e^{-i\theta}B(u,\xi_i) = 0,$$

which results in $B(v, e^{i\theta}\xi_i + e^{-i\theta}\xi_{-i}) = 0$ due to (3.28). □

Some previous results will change in the unbounded case. Lemma 3.8 now implies that J is unbounded if at least one sequence $\{a_k\}$ or $\{b_k\}$ is unbounded. Favard's theorem, Theorem 3.9, remains valid for a self-adjoint extension of J, although μ is not compactly supported. Still, μ has finite moments, because $\int_{\mathbb{R}} x^k \,\mathrm{d}\mu(x) = \left\langle J^k e_0, e_0\right\rangle < \infty$. Theorem 3.13 can not be extended in the unbounded case. However, the spectral theorem tells us that for every unbounded self-adjoint operator T there is a unique spectral measure E. Related is the moment problem, explained in the next section.

3.4. Jacobi operators and the moment problem

Consider the Hilbert space $L^2(\mu)$ of square μ-integral functions. We define the **moments** of μ by $m_n := \int_{\mathbb{R}} x^n \,\mathrm{d}\mu(x)$ for all nonnegative integers n. The sequence $(m_n)_{n\in\mathbb{N}}$ is called the **moment sequence** of μ.

Moments can be found in many fields of mathematics and physics. Having a measure immediately gives back the corresponding moments. But of more interest is the following Hamburger moment problem.

The **Hamburger moment problem** consists of two questions:

1. Given a real sequence $(s_n)_{n\in\mathbb{N}}$, does there exist a positive Borel measure μ such that $(s_n)_{n\in\mathbb{N}}$ is the associated moment sequence?
2. If there exist a positive measure, is this measure uniquely determined?

If there exist such a positive Borel measure μ and it is unique, then the moment problem is called **determinate**. If not, the moment problem is called **indeterminate**. Without loss of generality we can assume that $s_0 = 1$. The first question is answered in Hamburger's theorem [10, Theorem 3.8], which involves positive definite sequences. The second question of the Hamburger moment problem will be discussed here.

The moments are uniquely determined by the Jacobi operator, because $\int_{\mathbb{R}} x^n \,\mathrm{d}\mu(x) = \langle J^n e_0, e_0\rangle$. Jacobi operators are very useful in finding solutions to the Hamburger moment problem, i.e. finding corresponding measures to the moment sequence. The next theorem is a powerful tool in answering the second question of the Hamburger moment problem [10, 6.10].

Theorem 3.18. *The moment problem for a sequence s is indeterminate if and only if the corresponding Jacobi operator J is not essentially self-adjoint if and only if $p(z) \in \ell^2(\mathbb{N})$ for some $z \in \mathbb{C}\backslash\mathbb{R}$.*

Note that the last implication follows directly from Corollary 3.16.

4

The q-Meixner polynomials

In this chapter we study a difference operator that is an extension of the Jacobi operator for so-called q-Meixner polynomials. We end up with an orthogonality relation for q-Meixner polynomials.

4.1. q-Meixner polynomials

Now it is time to use the developed theory to find the spectral measure considering q-meixner polynomials. This measure turns out to be the key to an orthogonality relation regarding these polynomials. To define these polynomials we need some basic definitions first. Let $q \in (0,1)$ be fixed. Then for $x \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{\infty\}$ we define the **q-shifted factorials** by

$$\begin{aligned} (x)_0 &:= 1 \\ (x)_n &:= \prod_{k=0}^{n-1} (1 - xq^k) \qquad \text{for } n \geq 1, \text{ including } \infty \\ (x_1, x_2, \ldots, x_k)_n &:= \prod_{j=1}^{k} \left(x_j\right)_n . \end{aligned}$$

Furthermore we define the **basic hypergeometric series** ${}_r\varphi_s$ for $z \in \mathbb{C}$ by

$${}_r\varphi_s\left(\begin{matrix} x_1, x_2 \ldots, x_r \\ y_1, y_2, \ldots, y_s \end{matrix} ; z \right) := \sum_{k=0}^{\infty} \frac{(x_1, x_2 \ldots, x_r)_k}{\left(q, y_1, y_2 \ldots, y_s\right)_k} \left((-1)^k q^{k(k-1)/2}\right)^{1+s-r} z^k,$$

where we assume that $y_i \neq q^{-\mathbb{N}}$. Many identities of basic hypergeometric series are given by Gasper and Rahman in [4]. Let's define $M_n(\gamma)$ by

$$M_n(\gamma; a, b) := {}_2\varphi_1\left(\begin{matrix} q^{-n}, \gamma \\ aq \end{matrix} ; \frac{-q^{n+1}}{b} \right), \tag{4.1}$$

which in many literature is called the q-Meixner polynomials. For convenience we rather define the **q-Meixner polynomials** as

$$m_n(\gamma; a, b) := \frac{1}{\left(q\right)_n} M_n(\gamma; a, b), \tag{4.2}$$

with $a \neq q^{-\mathbb{N}}$, $b \neq 0$ and $\gamma \in \mathbb{C}$.

Remark. To have proper conditions on the Jacobi operator and to have a positive spectral measure we need some conditions on a and b:

1. $a < 0$
2. $q^2 < b < 1$.

Consider the q-interval

$$I := \{-q^{n+1}/b \mid n \in \mathbb{N}\} \cup \{-q^{n+1}/a \mid n \in \mathbb{N}\}. \tag{4.3}$$

The space of complex-valued functions $f : I \to \mathbb{C}$ is denoted by $\mathscr{F}_q$. For the restriction on $-q^{\mathbb{N}+1}/a$ and $-q^{\mathbb{N}+1}/b$ we denote the space by $\mathscr{F}_q^a$ and $\mathscr{F}_q^b$ respectively.

Definition 4.1. Let $f \in \mathscr{F}_q$. Then, for $x \neq -q/b, -q/a$, the **difference operator** L is defined by

$$Lf(x) := A(x)\left[f(qx) - f(x)\right] + B(x)\left[f(x/q) - f(x)\right],$$

with

$$A(x) := \left(a + \frac{1}{x}\right)\left(b + \frac{1}{x}\right)$$
$$B(x) := \frac{q}{x}\left(1 + \frac{1}{x}\right).$$

To define the endpoints of L on the q-interval I we set $B(-q/b) := 0$ and $B(-q/a) := 0$. Later we will see that m_n is an eigenfunction of the operator L. We define the weight function for $x \in I$,

$$w(x) := \frac{(-qx)_\infty |x|}{(-ax, -bx)_\infty}.$$

The weight function is positive on the q-interval I, assuming the conditions on a and b in the remark on page 30. Also notice that w is symmetric in a and b. We will deduce the weight function in section 4.2. Now we define the Hilbert spaces in the weighted $L^2(\mathbb{R}, w)$ space, using the notation $x_n^a = -q^{n+1}/a$ and $x_n^b = -q^{n+1}/b$,

$$\mathscr{H}_a := \left\{ f \in \mathscr{F}_q^a \;\middle|\; \sum_{n=0}^{\infty} |f(x_n^a)|^2 w(x_n^a) < \infty \right\};$$
$$\mathscr{H}_b := \left\{ f \in \mathscr{F}_q^b \;\middle|\; \sum_{n=0}^{\infty} |f(x_n^b)|^2 w(x_n^b) < \infty \right\},$$

with inner products $\langle f, g\rangle_a := \sum_{n=0}^{\infty} w(x_n^a) f(x_n^a)\overline{g(x_n^a)}$ and $\langle f, g\rangle_b := \sum_{n=0}^{\infty} w(x_n^b) f(x_n^b)\overline{g(x_n^b)}$ respectively. This gives rise to the Hilbert space $\mathscr{H} = \mathscr{H}_a \oplus \mathscr{H}_b$ with inner product $\langle f, g\rangle = \langle f_a, g_a\rangle_a + \langle f_b, g_b\rangle_b$ for $f : \mathscr{F}_q^a \oplus \mathscr{F}_q^b \to \mathbb{C}$, $f = f_a + f_b$.

Remark. The operator $L|_{\mathscr{F}_q^a}$ is a three-term operator which is equivalent to the Jacobi operator for the q-Meixner polynomials, as can be seen in section 4.2. So L can be considered as a generalization of a Jacobi operator, where L is a sum of two Jacobi operators.

For the proper conditions on $\mathscr{H}$ for L to be self-adjoint, we define

$$f(0^+) = \lim_{n\to\infty} f_a(x_n), \qquad f(0^-) = \lim_{n\to\infty} f_b(x_n),$$
$$f'(0^+) = \lim_{n\to\infty} a\frac{f_a(x_{n+1}) - f_a(x_n)}{q^{n+1}(1-q)}, \quad f'(0^-) = \lim_{n\to\infty} b\frac{f_b(x_{n+1}) - f_b(x_n)}{q^{n+1}(1-q)},$$

provided the limits exist.

In section 4.3 we are going to prove the next theorem about self-adjointness of L.

Theorem 4.2. *Consider the difference operator L with domain*

$$\mathscr{D} := \{f \in \mathscr{H} \mid Lf \in \mathscr{H}, f(0^+) = f(0^-), f'(0^+) = f'(0^-)\}.$$

Then $(L, \mathscr{D})$ is self-adjoint. Moreover, the spectrum of L consists of the point spectrum

$$\sigma_p(L) = \{-ab(1+\gamma) \mid \gamma \in -q^{-\mathbb{N}-1}\}$$

and there is no continuous spectrum.

The next corollary will be decuded from the theorem, which indirectly gives an orthogonality relation for certain eigenfunctions of L.

Corollary 4.3. *Define the function*

$$\psi_n(x) = \psi_n(x; a, b; q) := \frac{\left(-ax/q^n, -bx\right)_\infty}{\left(-qx\right)_\infty} {}_2\varphi_1\left(\begin{matrix} a/q^{n+1}, q^{-n} \\ -ax/q^n \end{matrix}; q, \frac{q^{n+2}}{b}\right),$$

then for $n, m \in \mathbb{N}$

$$\sum_{x \in I} \psi_n(x)\psi_m(x) w(x) = \delta_{n,m} \frac{\left(a/b, bq/a\right)_\infty \left(q^{-n}\right)_n q^{n+1}}{-a\left(q^{n+2}/b, q^{n+2}/a\right)_\infty} \left(\frac{a}{b}\right)^n.$$

The prove will be given at the end of this chapter. We also give some other related orthogonality relations for both the q-Meixner and the big q-Laguerre polynomials.

4.2. Jacobi operator and its unitary equivalent

An important three-term recurrence relation for M_n, stated in [7, 9.10], is

$$\begin{aligned} q^{2n+1}(1-\gamma)M_n(\gamma) &= b\left(1 - aq^{n+1}\right)M_{n+1}(\gamma) - \left[b\left(1 - aq^{n+1}\right) + q\left(1 - q^n\right)\left(b + q^n\right)\right]M_n(\gamma) \\ &\quad + q\left(1 - q^n\right)\left(b + q^n\right)M_{n-1}(\gamma). \end{aligned}$$

This three-term recurrence relation changes for m_n into

$$\begin{aligned} q^{2n+1}\frac{1}{b}(1-\gamma)m_n(\gamma) &= \left(1 - q^{n+1}\right)\left(1 - aq^{n+1}\right)m_{n+1}(\gamma) - \left[\left(1 - aq^{n+1}\right) + q\left(1 - q^n\right)\left(1 + \frac{q^n}{b}\right)\right]m_n(\gamma) \\ &\quad + q\left(1 + \frac{q^n}{b}\right)m_{n-1}(\gamma), \end{aligned} \tag{4.4}$$

by dividing by $b\left(q\right)_n$ and using the identity $\left(q\right)_{n+1} = \left(1 - q^{n+1}\right)\left(q\right)_n$. Let ϕ_γ be a solution to the eigenvalue equation

$$(Lf)(x) = \mu_\gamma f(x), \qquad x \in I,$$

with eigenvalue $\mu_\gamma = -ab(1+\gamma)$, $\gamma \in \mathbb{C}$. Then

$$-ab(1+\gamma)\phi_\gamma(x) = \frac{1}{x^2}(1+ax)(1+bx)\left(\phi_\gamma(qx) - \phi_\gamma(x)\right) + \frac{q}{x^2}(1+x)\left(\phi_\gamma(x/q) - \phi_\gamma(x)\right)$$

and after rewriting we get

$$-abx^2(1+\gamma)\phi_\gamma(x) = (1+ax)(1+bx)\phi_\gamma(qx) - \left[(1+ax)(1+bx) + q(1+x)\right]\phi_\gamma(x) + q(1+x)\phi_\gamma(x/q).$$

Adding $(abx^2 - ax^2)\phi_\gamma(x)$ to both sides yields

$$-ax^2(1+b\gamma)\phi_\gamma(x) = (1+ax)(1+bx)\phi_\gamma(qx) - \left[(1+bx) + q(1+ax/q)(1+x)\right]\phi_\gamma(x) + q(1+x)\phi_\gamma(x/q).$$

Now consider $x_n^a = -\frac{1}{a} q^{n+1}$. Then

$$-\frac{q}{a} q^{2n+1}(1+b\gamma)\phi_\gamma(x_n^a) = \left(1-q^{n+1}\right)\left(1-\frac{b}{a}q^{n+1}\right)\phi_\gamma(x_{n+1}^a) - \left[\left(1-\frac{b}{a}q^{n+1}\right) + q\left(1-q^{n-1}\right)\left(1-\frac{q^{n+1}}{a}\right)\right]\phi_\gamma(x_n^a)$$
$$+ q\left(1-\frac{q^{n+1}}{a}\right)\phi_\gamma(x_{n-1}^a).$$

Finally, we are able to compare this with the three-term recurrence relation (4.4). If we consider $m_n(\Gamma; A, B)$, then it follows that $A = \frac{b}{a}$, $B = -\frac{a}{q}$ and $\Gamma = -b\gamma$, and so $m_n(-b\gamma; \frac{b}{a}, -\frac{a}{q})$ satisfy the same three-term recurrence relation as $\phi_\gamma(x_n)$. Consequently, m_n is a solution to the same eigenvalue equation.

After a transformation by $g_n(\gamma) := \nu_n \phi_\gamma(x_n)$ we are able to obtain the three-term recurrence relation

$$\gamma g_n(\gamma) = -\frac{(1-q^{n+1})(a-bq^{n+1})}{bq^{2n+2}} \frac{\nu_n}{\nu_{n+1}} g_{n+1}(\gamma) + \left(\frac{(1-q^{n+1})(a-bq^{n+1})}{bq^{2n+2}} - \frac{q(q^{n+1}-a)}{bq^{2n+2}} - \frac{bq^{2n+2}}{bq^{2n+2}}\right) g_n(\gamma)$$
$$+ \frac{q(q^{n+1}-a)}{bq^{2n+2}} \frac{\nu_n}{\nu_{n-1}} g_{n-1}(\gamma). \tag{4.5}$$

To fullfill the requirements of the three-term recurrence relation for a_n it needs to hold that

$$-\frac{(1-q^{n+1})(a-bq^{n+1})}{bq^{2n+2}} \frac{\nu_n}{\nu_{n+1}} = \frac{q(q^{n+2}-a)}{bq^{2n+4}} \frac{\nu_{n+1}}{\nu_n}.$$

With these identity we can form the recurrence

$$\frac{\nu_{n+1}^2}{\nu_n^2} = \frac{q(1-q^{n+1})(a-bq^{n+1})}{(a-q^{n+2})},$$

which yields

$$\nu_n^2 = \frac{\left(q, bq/a\right)_n q^n}{\left(q^2/a\right)_n} \nu_0^2.$$

Together with (4.5) this yields the three-term reccurence relation for g_n:

$$a_n = -\frac{(1-aq^n)(1-bq^n)}{abq^{2n}} \sqrt{\frac{(1-q^{n+1})}{q(1-aq^n)(1-bq^n)}}$$
$$= -\frac{\sqrt{q(1-q^{n+1})(1-aq^n)(1-bq^n)}}{abq^{2n+1}}$$

and

$$b_n = \frac{1-aq^n-bq^n-q(q^n-1)}{abq^{2n}} = \frac{1+q-(a+b+q)q^n}{abq^{2n}}.$$

Without loss of generality we take $\nu_0^2 = -\frac{q}{a}\frac{(q^2/a)_\infty}{(q, bq/a)_\infty}$, which yields

$$g_n(\gamma) = \nu_n \phi_\gamma(x_n), \qquad \nu_n = \sqrt{\frac{\left(-qx_n\right)_\infty x_n}{(-ax_n, -bx_n)_\infty}}$$

for $x_n = -\frac{1}{a} q^{n+1}$. Doing the same calculations for $x_n = -\frac{1}{b} q^{n+1}$ will lead to the same ν_n. Therefore in general the weight function of section 4.2 is defined by $w(x_n) = \nu_n^2$, i.e.

$$w(x) := \frac{\left(-qx\right)_\infty |x|}{(-ax, -bx)_\infty}, \forall x \in I.$$

Now define the Jacobi operator J on $\ell^2(\mathbb{N})$ by

$$Je_n := \begin{cases} a_n e_{n+1} + b_n e_n + a_{n-1} e_{n-1}, & n \geq 1, \\ a_0 e_1 + b_0 e_0, & n = 0. \end{cases}$$

Also define $M : \mathcal{H}_a \to \ell^2(\mathbb{N})$ by $Mf(x_n^a) := \nu_n f(x_n^a)$. Then M is unitary,

$$\langle f, f \rangle_a = \sum_{n=0}^{\infty} |f(x_n^a)|^2 w(x_n^a) \tag{4.6}$$

$$= \sum_{n=0}^{\infty} |\nu_n f(x_n^a)|^2 = \langle Mf, Mf \rangle_{\ell^2}, \tag{4.7}$$

and $MLM^{-1} = J$ as can be seen from (4.5). Therefore spectral analysis of L on $\mathcal{H}_a$ is equivalent to spectral analysis of J on $\ell^2(\mathbb{N})$.

4.3. A suitable domain for L

In this section we are going to prove the first part of Theorem 4.2. If we restrict the domain of L to $\mathcal{H}_a$, then L is not essentially self-adjoint as will be shown in Proposition 4.6. The disandvantage of this situation is that it turns out to be hard to find an explicit description of the spectral measure of a self-adjoint extension of L.

We define the **q-Meixner function** by

$$\phi_\gamma(x; a, b) := {}_2\varphi_2 \left(\begin{matrix} -1/x, -1/\gamma \\ a, b \end{matrix} ; ab\gamma x \right) \qquad x, \gamma \in \mathbb{C} \setminus \{0\}.$$

Then ϕ_γ is an eigenfunction of L. The next two lemma's can be found in [5, 3.7 and 3.8].

Lemma 4.4. *Let ϕ_γ be the q-Meixner function. Then ϕ_γ satisfies $L\phi_\gamma(x) = \mu_\gamma \phi_\gamma(x)$ for $x \in \mathbb{R} \setminus \{0\}$, $\gamma \in \mathbb{C}$ and $\mu_\gamma = -ab(1+\gamma)$.*

Using the Jackson's transformation, [4, Appendix III], we get the ${}_2\varphi_1$ relation

$$\phi_\gamma(x; a, b) := \frac{(-ax)_\infty}{(a)_\infty} {}_2\varphi_1 \left(\begin{matrix} -1/x, -b\gamma \\ b \end{matrix} ; -ax \right).$$

To obtain the spectral measure for L we need an asymptotically free solution, as been discussed in the previous chapter. The proof of the previous and next lemma can be found in [5].

Lemma 4.5. *Let ϕ_γ be the q-Meixner function. Then*

$$\lim_{n\to\infty} \phi_\gamma(x_n) = \lim_{x\to 0} \phi_\gamma(x) = \frac{1}{(a)_\infty} {}_1\varphi_1 \left(\begin{matrix} -b\gamma \\ b \end{matrix} ; a \right) \tag{4.8}$$

The lemma will be used to show that the operator L restricted on $\mathcal{H}_a$ would have not been essentially self-adjoint.

Proposition 4.6. *The unbounded operator L restricted to $\mathcal{H}_a$ is not essentially self-adjoint.*

This proposition implies that the Jacobi operator J is not essentially self-adjoint and so the moment problem for the q-Meixner polynomials is indeterminate.

Proof. The proposition has been proven if we find an eigenfunction of L such that a corresponding eigenvalue is contained in $\mathbb{C}\backslash\mathbb{R}$. So let $\gamma\in\mathbb{C}$ such that eigenvalue μ_γ of eigenfunction ϕ_γ is contained in $\mathbb{C}\backslash\mathbb{R}$. Due to Lemma 4.5,

$$\lim_{n\to\infty}\phi_\gamma(x_n^a)=\lim_{x\to 0}\phi_\gamma(x)=C_\gamma,$$

with constant C_γ defined by the r.h.s of (4.8). Notice that $w(x_n^a)=\mathcal{O}(q^n)$ so we have shown that

$$\sum_{n=0}^{\infty}|v_n\phi_\gamma(x_n^a)|^2=\sum_{n=0}^{\infty}|\phi_\gamma(x_n^a)|^2 w(x_n^a)<\infty. \tag{4.9}$$

By Corollary 3.16 the claim follows. □

We also know that L restricted on $\mathcal{H}_a$ has deficiency indices (1,1) and so the eigenspace N_z is one dimensional. It coincide with the statement in proposition 2.30 that $N_z\neq\emptyset$ with $z\in\mathbb{C}\backslash\mathbb{R}$. From now on we consider $L=J^a+J^b$ with J^a and J^b the Jacobi operators on $\mathcal{H}_a$ and $\mathcal{H}_b$ respectively.

Let's define the truncated inner product for $f,g\in\mathcal{F}_q$ by

$$\langle f,g\rangle_{k,l}:=\sum_{n=0}^{k}f(x_n^a)\overline{g(x_n^a)}w(x_n^a)+\sum_{n=0}^{l}f(x_n^b)\overline{g(x_n^b)}w(x_n^b).$$

Notice that $\lim_{k,l\to\infty}\langle f,g\rangle_{k,l}=\langle f,g\rangle$ for $f,g\in\mathcal{H}$. The Wronskian is defined by

$$\begin{aligned}W(f,g)(x)&:=A(x)w(x)\left[f(x)g(qx)-f(qx)g(x)\right]\\&=\frac{(-qx)_\infty}{(-aqx,-bqx)_\infty|x|}\left[f(x)g(qx)-f(qx)g(x)\right]\\&=\frac{(1-q)(-qx)_\infty}{(-aqx,-bqx)_\infty}\left[f(x)\frac{g(qx)}{(1-q)|x|}-\frac{f(qx)}{(1-q)|x|}g(x)\right].\end{aligned}$$

Lemma 4.7. *For $f,g\in\mathcal{F}_q$ and $k,l\in\mathbb{N}$ we have*

$$\langle Lf,g\rangle_{k,l}-\langle f,Lg\rangle_{k,l}=-W(f,\overline{g})(-q^{k+1}/a)-W(f,\overline{g})(-q^{l+1}/b).$$

Proof. Observe that

$$\langle Lf,g\rangle_{k,l}-\langle f,Lg\rangle_{k,l}=\sum_{n=0}^{k}\left[Lf(x_n^a)\overline{g(x_n^a)}-f(x_n^a)\overline{Lg(x_n^a)}\right]w(x_n^a)+\sum_{n=0}^{l}\left[Lf(x_n^b)\overline{g(x_n^b)}-f(x_n^b)\overline{Lg(x_n^b)}\right]w(x_n^b).$$

For $x\in I$ combining

$$\left[Lf(x)\overline{g(x)}-f(x)\overline{Lg(x)}\right]w(x)=A(x)w(x)\left[f(qx)\overline{g(x)}-\overline{g(qx)}f(x)\right]-B(x)w(x)\left[f(x)\overline{g(q^{-1}x)}-\overline{g(x)}f(q^{-1}x)\right]$$

together with

$$A(x)w(x)=B(qx)w(qx)$$

and $B(x_0^a)=B(x_0^b)=0$ gives the identity

$$\left[Lf(x)\overline{g(x)}-f(x)\overline{Lg(x)}\right]w(x)=\begin{cases}W(f,\overline{g})(q^{-1}x)-W(f,\overline{g})(x), & x\in I\backslash\{x_0^a,x_0^b\}\\-W(f,\overline{g})(x), & x\in\{x_0^a,x_0^b\}\end{cases}.$$

Substitution into the first expression yields the final identity by the telescoping sum. □

Lemma 4.8. *For domain $\mathscr{D}$ as described in Theorem 4.2 we have that $(L,\mathscr{D})$ is a symmetric operator.*

Proof. Let $f,g \in \mathscr{D}$. We use Lemma 4.7 where we let $k,l \to \infty$. For the first limit we have

$$\begin{aligned} W(f,g)(0^-) &= \lim_{k\to\infty} \frac{(1-q)\left(-qx_k^b\right)_\infty}{\left(-aqx_k^b, -bqx_k^b\right)_\infty} \left[f(x_k^b) \frac{bg(x_{k+1}^b)}{(1-q)q^{k+1}} - \frac{bf(x_{k+1}^b)}{(1-q)q^{k+1}} g(x_k^b) \right] \\ &= (1-q)\left[f(0^-)g'(0^-) - f'(0^-)g(0^-) \right]. \end{aligned}$$

Likewise, for the other limit we obtain

$$\begin{aligned} W(f,g)(0^+) &= \lim_{k\to\infty} \frac{(1-q)\left(-qx_k^a\right)_\infty}{\left(-aqx_k^a, -bqx_k^a\right)_\infty} \left[f(x_k^a) \frac{-ag(x_{k+1}^a)}{(1-q)q^{k+1}} - \frac{-af(x_{k+1}^a)}{(1-q)q^{k+1}} g(x_k^a) \right] \\ &= -(1-q)\left[f(0^+)g'(0^+) - f'(0^+)g(0^+) \right]. \end{aligned}$$

Because we required for $f \in \mathscr{D}$ that $f(0^-) = f(0^+)$ and $f'(0^-) = f'(0^+)$, we get

$$W(f,g)(0^-) = -W(f,g)(0^+)$$

and so by Lemma 4.7 it follows that L is symmetric on $\mathscr{D}$. □

The next proposition now proofs the first claim of Theorem 4.2 and will be in line with the proof given in [9, section 2].

Proposition 4.9. *The operator $(L,\mathscr{D})$ is self-adjoint.*

Proof. By Lemma 4.8 we know that $(L,\mathscr{D})$ is symmetric, so $\mathscr{D} \subset \mathscr{D}^*$. Therefore the other inclusion remains to show, $\mathscr{D}^* \subset \mathscr{D}$. So consider $g \in \mathscr{D}^* \subset \mathscr{F}_q$.
Our first claim is that $(L^*g)(x) = (Lg)(x)$. For $f,h \in \mathscr{F}_q$ with f finitely supported, Lemma 4.7 shows that $\langle Lf,h\rangle = \langle f,Lh\rangle$. Therefore, using $h = g$, we obtain

$$\langle f,Lg\rangle = \langle Lf,g\rangle = \langle f,L^*g\rangle.$$

In particular, this holds for function f that is only supported on one $x \in I$ and so the claim follows.

For all $f \in \mathscr{D}$, by Lemma 4.7, we get

$$0 = \langle Lf,g\rangle - \langle f,L^*g\rangle = \langle Lf,g\rangle - \langle f,Lg\rangle = -W(f,g)(0^+) - W(f,g)(0^-).$$

As can be seen in the proof of Lemma 4.8 this can only hold when $g(0^+) = g(0^-)$ and $g'(0^+) = g'(0^-)$. This shows that $g \in \mathscr{D}$ and so the proposition has been proven. □

4.4. Orthogonality relation

It will turn out that finding the spectral measure almost immediately results in the desired orthogonality relation of Corollary 4.3. To get the spectral measure we need the Green kernel first. For this we have to find proper eigenvectors of L that satisfies the right boundary conditions.

4.4.1. Eigenvectors

To define the Green kernel we use the following eigenfunctions of L with eigenvalue μ_γ.

Lemma 4.10. *Define the function ψ_γ by*

$$\psi_\gamma(x) := \psi_\gamma(x;a,b;q) = \frac{\left(qa/b, aq\gamma x, -bx\right)_\infty}{\left(-qx, -q/b\gamma\right)_\infty} {}_2\varphi_2\left(\begin{matrix} -a\gamma, -ax \\ aq\gamma x, qa/b \end{matrix}; q, \frac{q^2}{b}\right),$$

then ψ_γ satisfies $L\psi_\gamma = \mu_\gamma \psi_\gamma$ on I defined in (4.3).

In [5, 3.10] a proof can be found for the function $\left(-q\gamma\right)_\infty^{-1} \psi_\gamma(x)$. Notice that ψ_γ is not symmetric in a and b. The function $\psi_\gamma^\dagger(x) := \psi_\gamma(x;b,a;q)$ is therefore another solution to the eigenvalue equation of L. Furthermore, we can see that $\lim_{x\to 0}\psi_\gamma(x)$ and $\lim_{x\to 0}\psi_\gamma^\dagger(x)$ exist. Equivalent $\lim_{n\to\infty}\psi_\gamma(x_n^a)$ and $\lim_{n\to\infty}\psi_\gamma^\dagger(x_n^b)$ exist. This provides the asymptotically free solutions $\psi_\gamma(x_n^a)v(x_n^a)$ and $\psi_\gamma^\dagger(x_n^b)v(x_n^b)$.

The next lemma provides the poles of $\psi_\gamma(x)$.

Lemma 4.11. *Let $n \in \mathbb{N}$. The mapping $\gamma \mapsto \psi_\gamma(-\frac{1}{a}q^{n+1})$ has simple poles $\gamma = -\frac{1}{b}q^m$, $m \in \mathbb{N}$. Likewise, the mapping $\gamma \mapsto \psi_\gamma^\dagger(-\frac{1}{b}q^{n+1})$ has simple poles $\gamma = -\frac{1}{a}q^m$.*

Proof. Because $\left(aq\gamma x\right)_\infty {}_2\varphi_2(-a\gamma, -ax; aq\gamma x, qa/b)$ is analytic in γ, by definition of ψ_γ, the poles of ψ_γ are the zeros of $\left(-q/b\gamma\right)_\infty$. The same reasoning can be applied for $\psi_\gamma^\dagger$. □

With a transformation formula for q-hypergeometric functions, [4, (III.4)], we obtain

$$\psi_\gamma(x) = \frac{\left(aq\gamma x, -bx\right)_\infty}{\left(-qx\right)_\infty} {}_2\varphi_1\left(\begin{matrix} -a\gamma, -q\gamma \\ aq\gamma x \end{matrix}; q, -\frac{q}{b\gamma}\right), \tag{4.10}$$

which will be helpfull in the analysis.

Now we have to calculate the Green kernel, using ψ_γ and $\psi_\gamma^\dagger$. We need an expansion, say

$$\phi_\gamma(x) = \alpha\psi_\gamma(x) + \beta\psi_\gamma^\dagger(x). \tag{4.11}$$

To obtain α and β we will use the symmetric expression of ϕ_γ and the three-term transformation formula, [4, III.32], with the condition that $|\gamma| > 1$:

$$\begin{aligned}
\phi_\gamma(x;a,b) &= \frac{\left(ab\gamma x,-1/\gamma\right)_\infty}{(a,b)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-a\gamma,-b\gamma\\ ab\gamma x\end{matrix};q,-\frac{1}{\gamma}\right)\\
&= \frac{\left(ab\gamma x,-1/\gamma\right)_\infty}{(a,b)_\infty}\left(\frac{\left(-b\gamma,-bx,a,q/a\right)_\infty}{\left(ab\gamma x,b/a,-1/\gamma,-q\gamma\right)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-a\gamma,-q/bx\\ aq/b\end{matrix};q,-qx\right)+\right.\\
&\left.\frac{\left(-a\gamma,-ax,b,q/b\right)_\infty}{\left(ab\gamma x,a/b,-1/\gamma,-q\gamma\right)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-b\gamma,-q/ax\\ bq/a\end{matrix};q,-qx\right)\right)\\
&= \frac{\left(-b\gamma,-bx,q/a\right)_\infty}{\left(b,b/a,-q\gamma\right)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-a\gamma,-q/bx\\ aq/b\end{matrix};q,-qx\right)+\\
&\frac{\left(-a\gamma,-ax,q/b\right)_\infty}{\left(a,a/b,-q\gamma\right)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-b\gamma,-q/ax\\ bq/a\end{matrix};q,-qx\right)\\
&= \frac{\left(-b\gamma,-bx,q/a\right)_\infty}{\left(b,b/a,-q\gamma\right)_\infty}\cdot\frac{\left(-q/b\gamma,aq\gamma x\right)_\infty}{\left(aq/b,-qx\right)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-a\gamma,-q\gamma\\ aq\gamma x\end{matrix};q,-q/b\gamma\right)+\\
&\frac{\left(-a\gamma,-ax,q/b\right)_\infty}{\left(a,a/b,-q\gamma\right)_\infty}\cdot\frac{\left(-q/a\gamma,bq\gamma x\right)_\infty}{\left(bq/a,-qx\right)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-b\gamma,-q\gamma\\ bq\gamma x\end{matrix};q,-q/a\gamma\right)\\
&= \frac{\left(-b\gamma,q/a,-q/b\gamma\right)_\infty}{\left(b,b/a,aq/b,-q\gamma\right)_\infty}\psi_\gamma(x)+\frac{\left(-a\gamma,q/b,-q/a\gamma\right)_\infty}{\left(a,a/b,bq/a,-q\gamma\right)_\infty}\psi_\gamma^\dagger(x)
\end{aligned}$$

with

$$\alpha=\frac{\left(-b\gamma,q/a,-q/b\gamma\right)_\infty}{\left(b,b/a,aq/b,-q\gamma\right)_\infty},\qquad \beta=\frac{\left(-a\gamma,q/b,-q/a\gamma\right)_\infty}{\left(a,a/b,bq/a,-q\gamma\right)_\infty}.$$

To get the Wronskian $[\psi_\gamma,\psi_\gamma^\dagger]$ in Proposition 4.13 we need to find the Wronskian between ϕ_γ and ψ_γ first. To obtain this we use the fact that we know that $m_n^{a,b}(\gamma):=m_n(\gamma;a,b)$ on $I_{>0}$ is a solution to the eigenvalue problem. It can be easily seen that $m_n^{b,a}(\gamma):=m_n(\gamma;b,a)$ is a solution as well on $I_{<0}$.

Proposition 4.12. *$m_n^{b,a}(\gamma)$ can be written as*

$$m_n^{b,a}(\gamma)=c\psi_\gamma(x_n),$$

with $c=\frac{(-q/b\gamma)_\infty}{(aq/b,q)_\infty}$ and $x_n=-\frac{1}{b}q^{n+1}$.

Proof. Recall that

$$m_n^{b,a}(\gamma)=m_n\left(-a\gamma;a/b,-b/q\right)=\frac{1}{(q)_n}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},-a\gamma\\ aq/b\end{matrix};\frac{q^{n+2}}{b}\right),$$

and

$$\psi_\gamma(x_n)=\frac{\left(-aq^{n+2}\gamma/b,q^{n+1}\right)_\infty}{\left(q^{n+2}/b\right)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-a\gamma,-q\gamma\\ -aq^{n+2}\gamma/b\end{matrix};q,-\frac{q}{b\gamma}\right).$$

By Heine's transformations of ${}_2\varphi_1$ series, [4, III.2], we obtain the relation

$${}_2\varphi_1\left(\begin{matrix}q^{-n},-a\gamma\\ aq/b\end{matrix};\frac{q^{n+2}}{b}\right)=\frac{\left(-q/b\gamma,-aq^{n+2}\gamma/b\right)_\infty}{\left(aq/b,q^{n+2}/b\right)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-a\gamma,-q\gamma\\ -aq^{n+2}\gamma/b\end{matrix};q,-\frac{q}{b\gamma}\right)$$

So

$$m_n^{b,a}(\gamma)=\frac{\left(-q/b\gamma\right)_\infty}{\left(aq/b,q\right)_\infty}\psi_\gamma(x_n).$$

□

We can use this to show that

$$\begin{aligned} m_0^{b,a}(\gamma) &= 1 & \Longrightarrow \psi_\gamma(x_0) &= \tfrac{(aq/b,q)_\infty}{(-q/b\gamma)_\infty} \\ m_1^{b,a}(\gamma) &= \tfrac{b-q^2-aq^2\gamma-aq}{(1-q)(b-aq)} & \Longrightarrow \psi_\gamma(x_1) &= \tfrac{1-q^2/b-aq^2\gamma/b-aq/b}{(1-q)(1-aq/b)}\tfrac{(aq/b,q)_\infty}{(-q/b\gamma)_\infty}. \end{aligned}$$

To find the Wronskian we need the contiguous relation [4, Ex. 1.10 (iii)]

$$q(1-A/C)\varphi(A/q)+(1-A)(1-ABZ/C)\varphi(Aq) = [1+q-A-Aq/C+A^2Z(1-B/A)/C]\varphi(A),$$

where $\varphi(A) := {}_2\varphi_1(A,B;C;q,Z)$. Now using $B=-\gamma^{-1}$, $C=b$, $Z=-a\gamma$ yields the identity

$$q(1-q/b)\varphi(1)+(1-q)\left(1-aq/b\right)\varphi(q^2) = \left[1-q^2/b-aq^2\gamma/b-aq/b\right]\varphi(q).$$

Multiplying by $\frac{(-a\gamma)_\infty}{(a)_\infty}$ and realising that $\varphi(1)=1$ gives us the equality, for $x_n^b = -\frac{1}{b}q^{n+1}$,

$$q\left(1-q/b\right)\frac{(-a\gamma)_\infty}{(a)_\infty} = \left[1-q^2/b-aq^2\gamma/b-aq/b\right]\phi_\gamma(x_0^b)-(1-q)\left(1-aq/b\right)\phi(x_1^b).$$

After some straightforward calculations the Wronskian turns out to be

$$\begin{aligned} [\phi_\gamma,\psi_\gamma] &= \frac{(q^2/b)_\infty}{(aq^2/b,q^2)_\infty q/b}\left(\psi_\gamma(x_1)\phi_\gamma(x_0^b)-\psi_\gamma(x_0)\phi_\gamma(x_1^b)\right) \\ &= \frac{(q^2/b)_\infty}{(aq^2/b,q^2)_\infty q/b}\frac{q(1-q/b)}{(1-q)(1-aq/b)}\frac{(-a\gamma,aq/b,q)_\infty}{(a,-q/b\gamma)_\infty} \\ &= \frac{b\left(q/b,-a\gamma\right)_\infty}{(a,-q/b\gamma,q)_\infty} \end{aligned}$$

Now we are able to calculate the Wronskian between ψ_γ and $\psi_\gamma^\dagger$.

Proposition 4.13.

$$\left[\psi_\gamma,\psi_\gamma^\dagger\right] = \frac{-b\left(a/b,bq/a,-q\gamma\right)_\infty}{(-q/a\gamma,-q/b\gamma,q)_\infty}, \qquad \gamma\notin -\frac{1}{a}q^{-\mathbb{N}-1}\bigcup -\frac{1}{b}q^{-\mathbb{N}-1}.$$

Proof. Substitution of $\psi_\gamma^\dagger(x) = \frac{1}{\beta}\left(\phi_\gamma(x)-\alpha\psi_\gamma(x)\right)$ yields

$$\left[\psi_\gamma^\dagger,\psi_\gamma\right] = \frac{1}{\beta}\left[\phi_\gamma,\psi_\gamma\right] = \frac{\left(a,a/b,bq/a,-q\gamma\right)_\infty}{(-a\gamma,q/b,-q/a\gamma)_\infty}\frac{b\left(q/b,-a\gamma\right)_\infty}{(a,-q/b\gamma,q)_\infty} \tag{4.12}$$

Then by $\left[\psi_\gamma,\psi_\gamma^\dagger\right] = -\left[\psi_\gamma^\dagger,\psi_\gamma\right]$ the claim has been proved. □

We are now in the position to define the Green kernel as in (3.19) by

$$G_\gamma(x,y) := \frac{1}{[\psi_\gamma,\psi_\gamma^\dagger]}\begin{cases} \psi_\gamma(x)\psi_\gamma^\dagger(y), & x\le y \\ \psi_\gamma(y)\psi_\gamma^\dagger(x), & x>y, \end{cases}$$

with $x,y\in I$. Then the resolvent operator $(L-z)^{-1} = G(z)$, as in Propositon 3.12, where

$$(G(z)f)(x) := \left\langle f,\overline{G_z(x,\cdot)}\right\rangle, \qquad z\in\mathbb{C}\setminus\mathbb{R}.$$

Lemma 4.14. *Denote*

$$S_{\text{sing}} := -q^{-\mathbb{N}-1}.$$

For $x, y \in I$, the Green kernel $\gamma \mapsto G_\gamma(x,y)$ has simple poles at S_{sing} and is analytic on $\mathbb{C}\setminus S_{\text{sing}}$.

Proof. The zeros $-q^{-\mathbb{N}-1}$ of the Wronskian are poles of $\gamma \mapsto G_\gamma(x,y)$, because they are not zeros of $\gamma \mapsto \psi_\gamma$ or $\gamma \mapsto \psi_\gamma^\dagger$. The poles of ψ_γ satisfy $\left(-q/b\gamma\right)_\infty = 0$ by Lemma 4.11 and from Proposition 4.13 we see that these will be canceled by the poles of the Wronskian $[\psi_\gamma^\dagger, \psi_\gamma]$. Therefore the poles of the Green kernel is the set S_{sing}. □

If $\gamma \in S_{\text{sing}}$, then $[\psi_\gamma^\dagger, \psi_\gamma] = 0$ and therefore $\psi_\gamma^\dagger$ and ψ_γ are linearly dependent solutions of the eigenvalue equation $Lf = \mu_\gamma f$, which means that $\psi_\gamma = C_\gamma \psi_\gamma^\dagger$ for some constant C_γ. To find C_γ we use the lineair combination (4.11). Then

$$\left(-q\gamma\right)_\infty \phi_\gamma(x) = \hat{\alpha}\psi_\gamma(x) + \hat{\beta}\psi_\gamma^\dagger(x),$$

with $\hat{\alpha} = \left(-q\gamma\right)_\infty \alpha$ and $\hat{\beta} = \left(-q\gamma\right)_\infty \beta$. Now for $\left(-q\gamma\right)_\infty = 0$ we obtain $\hat{\alpha}\psi_\gamma(x) + \hat{\beta}\psi_\gamma^\dagger(x) = 0$ and so for $\gamma_n = -q^{-n-1}$ we obtain

$$\psi_{\gamma_n}(x) = C_{\gamma_n}\psi_{\gamma_n}^\dagger(x), \tag{4.13}$$

with

$$C_{\gamma_n} = -\frac{\left(b, q/b, b/a, aq/b, a/q^{n+1}, q^{n+2}/a\right)_\infty}{\left(a, q/a, a/b, bq/a, b/q^{n+1}, q^{n+2}/b\right)_\infty} = \left(\frac{a}{b}\right)^n.$$

4.4.2. Spectral measure

The resolvent is given as in (3.20) by

$$(G(\mu)f)(x) = \left\langle f, \overline{G_{\gamma_\mu}(x,\cdot)}\right\rangle, \qquad f \in \mathcal{H},\ x \in I,\ \mu \in \mathbb{C}\setminus\mathbb{R}$$

with $\gamma_\mu \in \mathbb{C}$ such that $\mu = -ab(1+\gamma_\mu)$.

To find the spectral measure we need to integrate over μ,

$$E_{f,g}((\mu_1,\mu_2)) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\mu_1+\delta}^{\mu_2-\delta} \langle G(\mu+i\varepsilon)f, g\rangle - \langle G(\mu-i\varepsilon)f, g\rangle \,\mathrm{d}\mu, \tag{4.14}$$

with $\mu_1 < \mu_2$ and $f, g \in \mathcal{H}$. Note that if $(\mu_1,\mu_2)\cap\mu(S_{\text{sing}}) = \emptyset$, then $\lim_{\varepsilon\downarrow 0} G(\mu+i\varepsilon)f = \lim_{\varepsilon\downarrow 0} G(\mu-i\varepsilon)f$ for $\mu \in (\mu_1,\mu_2)$. So then $E_{f,g}((\mu_1,\mu_2)) = 0$. Take $\mu_1 < \mu_2$ such that $(\mu_1,\mu_2)\cap\mu(S_{\text{sing}}) = \{-ab(1+\gamma_n)\}$. To be able to evaluate (4.14) we first notice that

$$\begin{aligned}\langle G(\mu)f, g\rangle &= \sum_{x\in I}\left\langle f, \overline{G_\gamma(x,\cdot)}\right\rangle \overline{g(x)}\,w(x) = \sum_{x\in I}\sum_{y\in I} f(y)G_\gamma(x,y)\overline{g(x)}\,w(x)w(y)\\ &= \sum_{x\le y}\frac{1}{[\psi_\gamma,\psi_\gamma^\dagger]}\psi_\gamma(x)\psi_\gamma^\dagger(y)\left(f(x)\overline{g(y)} + f(y)\overline{g(x)}\right)w(x)w(y)\cdot(1-\frac{1}{2}\delta_{x,y})\end{aligned}$$

Due to the singular points S_{sing}, we need to calculate the residue of $\frac{1}{[\psi_\gamma,\psi_\gamma^\dagger]}$. So by definition

$$\begin{aligned}
\operatorname*{Res}_{\gamma=-q^{-n-1}}\left(\frac{1}{[\psi_\gamma,\psi_\gamma^\dagger]}\right) &= \lim_{\gamma\to -q^{-n-1}}(\gamma+q^{-n-1})\frac{1}{[\psi_\gamma,\psi_\gamma^\dagger]}\\
&= \lim_{\gamma\to -q^{-n-1}}(\gamma+q^{-n-1})\frac{\left(-q/a\gamma,-q/b\gamma,q\right)_\infty}{-b\left(a/b,bq/a,-q\gamma\right)_\infty}\\
&= \frac{(q)_\infty}{-b\left(a/b,bq/a\right)_\infty}\lim_{\gamma\to -q^{-n-1}}\left(-q/a\gamma,-q/b\gamma\right)_\infty\frac{\gamma+q^{-n-1}}{\left(-q\gamma\right)_\infty}\\
&= \frac{\left(q^{n+2}/b,q^{n+2}/a,q\right)_\infty}{-b\left(a/b,bq/a\right)_\infty}\frac{1}{q^{n+1}\left(q^{-n}\right)_n\left(q\right)_\infty}\\
&= \frac{\left(q^{n+2}/b,q^{n+2}/a\right)_\infty}{-bq^{n+1}\left(a/b,bq/a\right)_\infty\left(q^{-n}\right)_n}.
\end{aligned}$$

We used the fact that

$$\begin{aligned}
\lim_{\gamma\to -q^{-n-1}}\frac{\gamma+q^{-n-1}}{\left(-q\gamma\right)_\infty} &= \lim_{\gamma\to -q^{-n-1}}\frac{\gamma+q^{-n-1}}{(1+q\gamma)\cdots(1+q^n\gamma)(1+q^{n+1}\gamma)(1+q^{n+2}\gamma)\cdots}\cdot\frac{q^{n+1}}{q^{n+1}}\\
&= \lim_{\gamma\to -q^{-n-1}}\frac{1}{q^{n+1}(1+q\gamma)\cdots(1+q^n\gamma)(1+q^{n+2}\gamma)\cdots}\\
&= \frac{1}{q^{n+1}(1-q^{-n})\cdots(1-q^{-1})(1-q^1)\cdots} = \frac{1}{q^{n+1}\left(q^{-n}\right)_n\left(q\right)_\infty}.
\end{aligned}$$

Now we can evaluate the spectral measure, assuming that the interval (μ_1,μ_2) contains only one point μ_γ with $\gamma\in S_{\text{sing}}$.

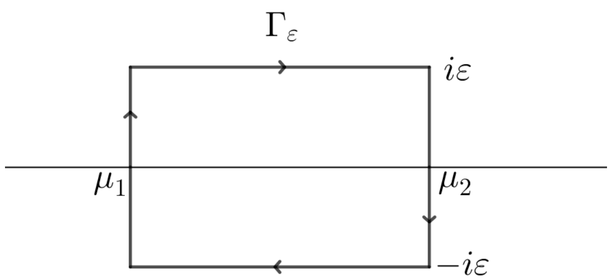

Figure 4.1: Closed curve Γ_ε.

Using the rectangled closed curve Γ_ϵ around the pole, we first observe that

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\varepsilon}^{\varepsilon}\left\langle G(\theta\pm iy)f,g\right\rangle \mathrm{d}y = \lim_{\varepsilon\to 0}\frac{\varepsilon}{\pi i}\max_{y\in[-\varepsilon,\varepsilon]}\left|\left\langle G(\theta\pm iy)f,g\right\rangle\right| = 0$$

for $\theta \in \{\mu_1, \mu_2\}$. Thus now we calculate the spectral measure for $\gamma_n \in S_{\text{sing}}$ by

$$\begin{aligned}
E_{f,g}((\mu_1,\mu_2)) &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\mu_1+\delta}^{\mu_2-\delta}\langle G(\mu+i\varepsilon)f,g\rangle - \langle G(\mu-i\varepsilon)f,g\rangle\,\mathrm{d}\mu \\
&= \lim_{\varepsilon\downarrow 0}\int_{\Gamma_\varepsilon}\left\langle G(\mu)f,g\right\rangle \mathrm{d}\mu = ab\times \operatorname*{Res}_{\gamma_n=-q^{-n-1}}\left(\left\langle G(\gamma_n)f,g\right\rangle\right) \\
&= \frac{-a\left(q^{n+2}/b,q^{n+2}/a\right)_\infty}{q^{n+1}\left(a/b,bq/a\right)_\infty\left(q^{-n}\right)_n}\sum_{x\le y}\psi_{\gamma_n}(x)C_{\gamma_n}^{-1}\psi_{\gamma_n}(y)\left(f(x)\overline{g(y)}+f(y)\overline{g(x)}\right)\cdot(1-\frac{1}{2}\delta_{x,y}) \\
&= \frac{-\left(q^{n+2}/b,q^{n+2}/a\right)_\infty b^n}{\left(a/b,bq/a\right)_\infty\left(q^{-n}\right)_n a^{n-1}q^{n+1}}\left\langle f,\psi_{\gamma_n}\right\rangle\left\langle\psi_{\gamma_n},g\right\rangle
\end{aligned}$$

where we changed parameter $\mu\mapsto -ab(1+\gamma)$ such that the minus sign is canceled by the minus sign of the residue of the clockwise oriented contour Γ_ε. The last equality comes from substitution of $\psi^\dagger_{\gamma_n}(x) = C_{\gamma_n}^{-1}\psi_{\gamma_n}(x)$. Now let $f = g = \psi_{\gamma_n}$, then we obtain

$$\begin{aligned}
\left\langle\psi_{\gamma_n},\psi_{\gamma_n}\right\rangle &= \left\langle E(\mu_1,\mu_2)\psi_{\gamma_n},\psi_{\gamma_n}\right\rangle \\
&= \frac{-\left(q^{n+2}/b,q^{n+2}/a\right)_\infty b^n}{\left(a/b,bq/a\right)_\infty\left(q^{-n}\right)_n a^{n-1}q^{n+1}}\left\langle\psi_{\gamma_n},\psi_{\gamma_n}\right\rangle^2.
\end{aligned}$$

Because ψ_γ are orthogonal eigenfunctions for distinct values of γ we end up with our main result, the orthogonality relation

$$\left\langle\psi_{\gamma_n},\psi_{\gamma_m}\right\rangle = \delta_{n,m}\frac{\left(a/b,bq/a\right)_\infty\left(-q\gamma_n\right)_n}{a\gamma_n\left(-q/b\gamma_n,-q/a\gamma_n\right)_\infty}\left(\frac{a}{b}\right)^n. \tag{4.15}$$

The next corollary proves the last part of Theorem 4.2 about the spectrum of L.

Corollary 4.15. *Let L be the difference operator and S_{sing} the singular points as in Lemma 4.14, then*

$$\sigma_p(L) = -ab(1-q^{-\mathbb{N}-1}).$$

Proof. Notice that $\mu(S_{\text{sing}}) = -ab(1-q^{-\mathbb{N}-1})$. By Lemma 4.10 the operator $\psi_\gamma\in\mathscr{H}$ with $\gamma\in S_{\text{sing}}$, is an eigenfunction for eigenvalue μ_γ. So $\mu(S_{\text{sing}})\subseteq\sigma_p(L)$. By definition, the spectral measure is supported on the spectrum. The only contribution to the spectral measure comes from the poles of the Green kernel $\gamma\mapsto G_\gamma(x,y)$, which is precisely $\mu(S_{\text{sing}})$. Therefore $\sigma_p(L)\subseteq\mu(S_{\text{sing}})$. □

4.4.3. More orthogonality relations

Finally, we get an orthogonality relation for the q-Meixner polynomials.

Corollary 4.16. *Let $m_k^{a,b}$ and $m_k^{b,a}$ be the q-Meixner polynomials defined on $I_{<0}$ and $I_{>0}$ respectively. Then for $\gamma_n, \gamma_m \in S_{\text{sing}}$,*

$$\sum_{k=0}^{\infty} m_k^{b,a}(\gamma_n) m_k^{b,a}(\gamma_m) \frac{\left(aq/b, q\right)_\infty^2}{\left(-q/b\gamma_n, -q/b\gamma_m\right)_\infty} w(x_k^b)$$
$$+ \sum_{k=0}^{\infty} m_k^{a,b}(\gamma_n) m_k^{a,b}(\gamma_m) \frac{\left(bq/a, q\right)_\infty^2}{\left(-q/a\gamma_n, -q/a\gamma_m\right)_\infty} \left(\frac{a}{b}\right)^{m+n} w(x_k^a) = \delta_{n,m} \frac{\left(a/b, bq/a\right)_\infty \left(-q\gamma_n\right)_n}{a\gamma_n \left(-q/b\gamma_n, -q/a\gamma_n\right)_\infty} \left(\frac{a}{b}\right)^n.$$

Proof. Combining Corollary 4.3, Proposition 4.12 and (4.13) yields the desired orthogonality relation.

□

A nice consequence from Corollary 4.3 is that we can deduce an orthogonality relation for the big q-Laguerre polynomials. By Heine's transformation formula, see [4, (III.2)], we obtain

$${}_2\varphi_1\left(\begin{matrix} a/q^{n+1}, q^{-n} \\ -ax/q^n \end{matrix} ; q, \frac{q^{n+2}}{b}\right) = \frac{\left(-ax, q^2/b\right)_\infty}{\left(-axq^{-n}, q^{n+2}/b\right)_\infty} \cdot {}_2\varphi_1\left(\begin{matrix} -q/bx, q^{-n} \\ q^2/b \end{matrix} ; q, -ax\right).$$

The **big q-Laguerre polynomials** are defined by [7]

$$P_n(x; a, b; q) = \frac{1}{\left(b^{-1}q^{-n}\right)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, aqx^{-1} \\ aq \end{matrix} ; q, \frac{x}{b}\right).$$

With change of parameters we get

$$\begin{aligned} P_n(-qx; q/b, q/a; q) &= \frac{1}{\left(aq^{-n-1}\right)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, -q/bx \\ q^2/b \end{matrix} ; q, -ax\right) \\ &= \frac{1}{\left(aq^{-n-1}\right)_n} \frac{\left(-axq^{-n}, q^{n+2}/b\right)_\infty}{\left(-ax, q^2/b\right)_\infty} \frac{\left(-qx\right)_\infty}{\left(-ax/q^n, -bx\right)_\infty} \psi_n(x) \\ &= \frac{\left(q^2/b\right)_n \left(-qx\right)_\infty}{\left(aq^{-n-1}\right)_n (-ax, -bx)_\infty} \psi_n(x). \end{aligned}$$

So the orthogonality relation for $\psi_n(x)$ provides an orthogonality relation for the big q-Laguerre polynomials.

For the last orthogonality we use duality. We write (4.15) by

$$\left\langle \psi_{\gamma_n}, \psi_{\gamma_m} \right\rangle = \delta_{n,m} h_n.$$

Then $\{\psi_{\gamma_n} : n \in \mathbb{N}\}$ is an orthogonal basis for $\mathcal{H}$. Define $\delta_x \in \mathcal{H}$, with $x \in I$, by $\delta_x(y) = \frac{\delta_{x,y}}{w(y)}$. Then $\delta_x = \sum_{n=0}^{\infty} \alpha_n(x) \psi_{\gamma_n}$ for certain $\alpha_n(x)$. Notice that

$$\left\langle \delta_x, \psi_{\gamma_n} \right\rangle = \sum_{y \in I} \delta_x(y) \psi_{\gamma_n}(y) w(y) = \psi_{\gamma_n}(x). \tag{4.16}$$

Also, by substitution, we can write

$$\begin{aligned}\langle \delta_x, \psi_{\gamma_n} \rangle &= \sum_{m=0}^{\infty} \alpha_m(x) \langle \psi_{\gamma_m}, \psi_{\gamma_n} \rangle \\ &= \sum_{m=0}^{\infty} \alpha_m(x) \delta_x h_n \\ &= \alpha_n(x) h_n.\end{aligned}$$

This, together with (4.16), shows that $\psi_{\gamma_n}(x) = \alpha_n(x) h_n$, or equivalently $\alpha_n(x) = \psi_{\gamma_n}(x) h_n^{-1}$. It follows that

$$\sum_{n=0}^{\infty} \psi_{\gamma_n}(x) \psi_{\gamma_n}(y) h_n^{-1} = \delta_{x,y} w(y)^{-1},$$

which gives an orthogonality relation for the q-Meixner polynomials in x and y.

Theorem 4.17. *Let $x, y \in I_{<0}$. The following orthogonality relation for the q-Meixner polynomials $m_k^{b,a}$ holds, writing $x = -\frac{1}{b} q^{k+1}$ and $y = -\frac{1}{b} q^{l+1}$,*

$$\sum_{n=0}^{\infty} m_k^{b,a}(\gamma_n) m_l^{b,a}(\gamma_n) \frac{q\gamma_n \left(aq/b, q\right)_\infty^2 \left(-q/a\gamma_n\right)_\infty}{\left(a/b, bq/a\right)_\infty \left(-q\gamma_n\right)_n} \left(\frac{a}{b}\right)^{1-n} = \frac{\left(aq^{k+1}/b, q^{k+1}\right)_\infty}{\left(q^{k+2}/b\right)_\infty} q^{-k} \delta_{k,l}.$$

As we have seen om page 14, the orthogonality relation is totally determined by the moments. The corresponding measure now yields a solution to the moment problem of the q-Meixner polynomials.

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