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Unconstrained Parametrizations of Discrete-Time Linear Input–Output Models: Stability and Dissipativity by Construction

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***Abstract*—It is often required that identified models exhibit certain stability and dissipativity properties, e.g., passivity or ℓ_2-gain. The aim of this article is to develop an unconstrained parametrization of linear parameter-varying (LPV) input–output (IO) discrete-time (DT) models that guarantees stability/dissipativity by construction, i.e., the model is stable/dissipative for any choice of model parameters. To achieve this, it is shown that *any* quadratically stable/dissipative DT-LPV-IO model can be generated by a mapping of transformed coefficient functions that are constrained to the unit ball. The unit ball is reparameterized through a Cayley transformation, resulting in a fully unconstrained parameterization. These results immediately apply to linear time-varying IO models. In the linear time-invariant case, an unconstrained parameterization of all stable/dissipative DT transfer functions is obtained. The unconstrained parametrization enables, among others, the use of neural network coefficient functions in LPV system identification while guaranteeing stability and dissipativity.**

***Index Terms*—Linear parameter-varying (LPV) systems, neural networks, stability, system identification.**

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I. Introduction

It IS often desirable that a model obtained using identification/learning [1], [2] reflects the stability and dissipativity properties of the true system. For example, if the true system is stable, stability of the identified model is crucial for simulation purposes [3]. Next to stability, dissipativity encapsulates a range of input–output (IO) properties, e.g., passivity and an upper bound on the amplification of an input signal (i.e., ℓ_2-gain). If a system is known to satisfy such an IO property, it is desirable that the identified model also has this property. However, even if the underlying system is stable/dissipative, it is not guaranteed that the identified model will also exhibit these properties due to measurement noise, finite-data effects and modeling errors [4]. An important class of models used in system identification are discrete-time (DT) linear IO models, containing both time-invariant [5] and time/parameter-varying (LTV/LPV) IO models [6]. Linear time-invariant (LTI) IO models have been used for identification in both the time [1] and frequency domain [7]. These models are represented in the form of a transfer function, or equivalently, a higher order linear difference equation in terms of previous inputs and outputs, and coefficients describing the relation between these inputs and outputs. In LPV-IO models, this relation remains linear, but the coefficients describing this relation are a function of a time-varying scheduling signal p that is assumed to be measurable online, describing, e.g., a change in operating conditions [8]. Under the correct choice of p, LPV models can even embed certain nonlinear characteristics [9]. Similarly, in LTV-IO models, the coefficients are a direct function of time, which is a special case of LPV-IO models in which p equals the current time step. Consequently, system identification methods based on these LPV/LTV-IO models have been thoroughly developed in the last decades [10], [11], [12], [13], [14], [15], [16].

Currently, techniques to guarantee stability and dissipativity properties possibly deteriorate prediction performance, or require computationally intensive constrained optimization, or cannot be straightforwardly extended to IO models. Specifically, existing literature can be categorized as follows.

1) An LTI-IO model can be projected onto the set of stable models after identification [3], e.g., through mirroring the poles in the unit circle. However, this projection does not

take into account the data, possibly resulting in a loss of model quality.

2) The coefficient functions of an LPV-IO model can be restricted a priori such that the model is stable [17]. However, this restriction is usually done according to a simple, but often too conservative approximation, e.g., through only allowing bounded deviations from a model that is known to be stable, thus limiting the model's flexibility.
3) The stability/dissipativity properties can be enforced using constrained optimization [4], [18], [19], [20]. However, enforcing such a constraint during identification severely increases the computational complexity of the optimization, often preventing their use in practice. This is especially true in the LPV case, in which the constraints that reflect the stability/dissipativity properties are generally formulated in terms of the feasibility of matrix inequalities (MI), see [21] and [22], that need to hold for all possible realizations of the scheduling signal p, representing a semi-infinite constraint [23]. These MI constraints are often verified via semidefinite programming (SDP) that has to be incorporated in the parameter optimization of the identification algorithm. Moreover, only in a select few cases can these infinite MI constraints be reduced to a finite set, e.g., in case of a polytopic scheduling domain and affine dependence of the coefficient functions on p. In the other cases, it is required to grid the scheduling space, which is not only computationally intractable for complex scheduling dependencies but also does not yield exact guarantees as the MI is not validated outside the gridded points in the scheduling space.
4) In the state-space setting, stability, and dissipativity properties can be guaranteed by construction, i.e., the model satisfies these properties for any choice of coefficients [24], [25]. Specifically, in [25], the MI representing the stability/dissipativity constraint is reparametrized through new unconstrained coefficients in a necessary and sufficient manner, such that the MI is satisfied for any choice of these new coefficients. This can be understood as constraint elimination, and enables system identification in these new coefficients using unconstrained gradient-based optimization. These results have been extended to the LPV state-space case in [26] to obtain an unconstrained characterization of coefficient functions for which the model satisfies stability/dissipativity properties. This enables parametrization of these functions by, e.g., neural networks while maintaining stability/IO guarantees by construction.

Although unconstrained parametrizations that guarantee stability/dissipativity have been developed for state-space models, such parametrizations do not exist for linear IO models that are especially relevant in system identification. The main difficulty in obtaining unconstrained parametrizations for IO models is that the corresponding state-space realization used for stability/dissipativity analysis is structured. Consequently, the resulting MI representing stability/dissipativity is also structured, such that not all components can be chosen freely. This is in contrast to state-space models, in which all components can be chosen freely.

The main contribution of this article is an unconstrained parametrization of DT linear IO models that guarantees stability and dissipativity properties by construction. This parametrization includes not only LPV models but also LTV and LTI IO models. This enables system identification using unconstrained gradient-based optimization. In addition, in the LPV case it allows for using any parametrization for the coefficient functions, e.g., neural networks, such that arbitrary functional dependencies can be learned from data up to the approximation capabilities of this parametrization. This is achieved through the following subcontributions.

- C1 An unconstrained characterization of all coefficient functions for which the DT linear IO model is stable with respect to a quadratic p-independent Lyapunov function[1] (Section III).
- C2 An unconstrained characterization of all coefficient functions such that the DT linear IO model is dissipative with respect to a p-independent quadratic storage function and a supply rate that is a generic quadratic function of the input and output, encapsulating dissipativity properties, such as an induced ℓ_2-gain and passivity (Section IV).
- C3 A DT linear IO model class that is stable/dissipative by construction, and an example demonstrating system identification using this model class (Sections V and VI).

Preliminary results related to contribution C1 were reported in [27]. This article extends those results by providing a formal proof, and by considering the multivariable setting.

Notation: $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote, respectively, the set of real and complex matrices of size $n\times m$. For $A\in\mathbb{R}^{n\times m}$ and $B\in\mathbb{C}^{n\times m}$, $A^\top$ and B^H represent the transpose and Hermitian transpose, respectively. $\mathbb{S}^n$ denotes the set of symmetric matrices of size $n\times n$ and $\mathbb{S}^n_{\succ 0}\subset\mathbb{S}^n$ is the set of positive-definite matrices. $\mathbb{U}^n_{>0}$ denotes the set of real square upper triangular matrices of size $n\times n$ with positive diagonal entries and $\mathbb{U}^n_0$ contains those with zeros on the diagonal. $\mathrm{triu}(A)\in\mathbb{U}^n_0$ denotes the strictly upper triangular part of a square matrix A. The Cholesky decomposition of a matrix $S\in\mathbb{S}^n_{\succ 0}$ is denoted by $S=E^\top E$ with $E\in\mathbb{U}^n_{>0}$. For $H\in\mathbb{S}^n$, the inertia of H is given by $\mathrm{In}(H)=(\lambda_+,\lambda_-,\lambda_0)$ with $\lambda_+,\lambda_-,\lambda_0\in\mathbb{N}$ the number of positive, negative, and zero eigenvalues of H, respectively. Last, given a symmetric matrix function $\Xi:\mathbb{P}\to\mathbb{S}^m$ with domain $\mathbb{P}\subseteq\mathbb{R}^n$, the notation $\Xi(p)\succ_{\mathbb{P}} 0$ is shorthand for $\Xi(p)\succ 0\quad\forall p\in\mathbb{P}$.

II. Problem Formulation

To guarantee stability/dissipativity properties for DT linear IO models by construction, a DT-LPV-IO representation is used as a general model class that also covers the LTI and LTV case. Specifically, consider the DT-LPV-IO representation $\mathcal{M}$ where

[1]Note that the choice of a p-independent, quadratic Lyapunov is a common choice for enforcing stability/dissipativity properties, both in constrained [19] and unconstrained [25] approaches, and already represents a significant improvement over the current conservative approximations.

y results from the parameter-varying difference equation

$$\mathcal{M}: \quad y_k = -\sum_{i=1}^{n_a} A_i(p_k) y_{k-i} + \sum_{i=0}^{n_b-1} B_i(p_k) u_{k-i} \tag{1}$$

with discrete time step $k \in \mathbb{Z}$, input $u_k \in \mathbb{R}^{n_u}$, output $y_k \in \mathbb{R}^{n_y}$, and coefficient functions $A_i(\cdot): \mathbb{P} \to \mathbb{R}^{n_y \times n_y}$, $B_i(\cdot): \mathbb{P} \to \mathbb{R}^{n_y \times n_u}$ describing the dependence of the difference equation on the scheduling signal p with $p_k \in \mathbb{P} \subseteq \mathbb{R}^{n_p}$ at time $k \in \mathbb{Z}$. Note that by definition A_i, B_i are bounded over $\mathbb{P}$. The model order satisfies $n_a \geq 0, n_b \geq 1$, with $n_a = 0, n_b > 1$ corresponding to an LPV-FIR setting, $n_a > 0, n_b = 1$ to an inverse LPV-FIR setting, and $n_a = 0, n_b = 1$ to a parameter-varying gain $y_k = B_0(p_k)u_k$. The setting $n_b = 0$, in which (1) reduces to an autonomous system, is not considered.

Remark 1: Representation (1) recovers the LTI case for constant $p_k = \bar{p} \quad \forall k$, in which (1) reduces to the multi-input, multi-output transfer function $G(z) = P(z)^{-1}Q(z)$ with $P(z) = I + \sum_{i=1}^{n_a} A_i(\bar{p}) z^{-i}$ and $Q(z) = \sum_{i=0}^{n_b-1} B_i(\bar{p}) z^{-i}$. For $p_k = k \quad \forall k$, the LTV case is recovered.

To define the set of coefficient functions A_i and B_i for which (1) is stable/dissipative, state-space techniques based on Lyapunov/storage functions are employed. Consequently, (1) is embedded in a state-space description. Here, the state is chosen as a set of delayed inputs and outputs, i.e.,

$$x_k = \begin{bmatrix} \bar{y}_k^\top & \bar{u}_k^\top \end{bmatrix}^\top \in \mathbb{R}^{n_x}$$

$$\bar{y}_k = \begin{bmatrix} y_{k-1}^\top & \dots & y_{k-n_a}^\top \end{bmatrix}^\top \in \mathbb{R}^{\bar{n}_y}$$

$$\bar{u}_k = \begin{bmatrix} u_{k-1}^\top & \dots & u_{k-n_b+1}^\top \end{bmatrix}^\top \in \mathbb{R}^{\bar{n}_u} \tag{3}$$

with $\bar{n}_y = n_y n_a$, $\bar{n}_u = n_u(n_b - 1)$, $n_x = \bar{n}_y + \bar{n}_u$. Given this state definition, (1) can be equivalently represented as the state-space system given in (2), shown at the bottom of this page, or more compactly as

$$\begin{aligned} x_{k+1} &= \mathcal{A}(p_k)x_k + \mathcal{B}(p_k)u_k \\ y_k &= \mathcal{C}(p_k)x_k + \mathcal{D}(p_k)u_k \end{aligned} \tag{4}$$

where

$$\mathcal{A}(\cdot) = \left[\begin{array}{c|c} F_a - G_a A(\cdot) & G_a B(\cdot) \\ \hline 0 & F_b \end{array}\right] \qquad \mathcal{B}(\cdot) = \left[\begin{array}{c} G_a B_0(\cdot) \\ \hline G_b \end{array}\right]$$

$$\mathcal{C}(\cdot) = \left[\begin{array}{c|c} -A(\cdot) & B(\cdot) \end{array}\right] \qquad \mathcal{D}(\cdot) = B_0(\cdot) \tag{5}$$

in which

$$F_a = \begin{bmatrix} 0 & 0 \\ I_{\bar{n}_y - n_y} & 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}_y \times \bar{n}_y} \quad G_a = \begin{bmatrix} I_{n_y} \\ 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}_y \times n_y}$$

$$F_b = \begin{bmatrix} 0 & 0 \\ I_{\bar{n}_u - n_u} & 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}_u \times \bar{n}_u} \quad G_b = \begin{bmatrix} I_{n_u} \\ 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}_u \times n_u}$$

$$A\colon \mathbb{P} \to \mathbb{R}^{n_y \times \bar{n}_y}, A(p_k) = \begin{bmatrix} A_1(p_k) & \dots & A_{n_a}(p_k) \end{bmatrix} \tag{6a}$$

$$B\colon \mathbb{P} \to \mathbb{R}^{n_y \times \bar{n}_u}, B(p_k) = \begin{bmatrix} B_1(p_k) & \dots & B_{n_b-1}(p_k) \end{bmatrix}. \tag{6b}$$

Remark 2: Realization (2) is not minimal. However, it is structurally reachable and completely detectable [6]. More importantly, it ensures that the coefficient functions $A, B,$ and B_0 appear affinely in the state-space matrix functions $\mathcal{A}, \mathcal{B}, \mathcal{C},$ and $\mathcal{D}$. In contrast, methods for obtaining a minimal realizations of (1) result in $\mathcal{A}, \mathcal{B}, \mathcal{C},$ and $\mathcal{D}$ that depend nonlinearly on $A(p_k), B(p_k),$ and $B_0(p_k)$ and potentially also on shifted instances of p_k, i.e., on $A(p_{k-m}), B(p_{k-m}),$ and $B_0(p_{k-m})$.

State-space representation (2) enables determining the stability and dissipativity properties of (1) through the existence of Lyapunov/storage functions [23]. Here, a parameter-invariant quadratic storage function is considered, resulting in the notion of quadratic stability and quadratic dissipativity, as formalized next.

Definition 1 (QS): Given coefficient functions $A, B,$ and B_0, (1) is said to be quadratically stable (QS) if there exists a $\mathcal{P} \in \mathbb{S}_{\succ 0}^{n_x}$ such that $\Xi_{\text{stab}}(A(p_k), B(p_k), \mathcal{P}) \prec 0 \quad \forall p_k \in \mathbb{P}$ where

$$\Xi_{\text{stab}}(A(p_k), B(p_k), \mathcal{P}) = \mathcal{A}^\top(p_k)\mathcal{P}\mathcal{A}(p_k) - \mathcal{P}. \tag{7}$$

When (1) is QS, output y_k asymptotically approaches zero, i.e., $\lim_{k\to\infty} y_k = 0$, for all inputs u for which $u_k = 0 \quad \forall k > \bar{k}$ with $k \in \mathbb{Z}$, all scheduling signals p with $p_k \in \mathbb{P}$, and all initial conditions $y_{k-1}, \dots, y_{k-n_a}, u_{k-1}, \dots, u_{k-n_b+1}$. A similar statement holds in the LTV case with $p_k = k$. In the LTI case, i.e., when $p_k = \bar{p} \quad \forall k \in \mathbb{Z}$, (1) reduces to the transfer function $G(z) = P(z)^{-1}Q(z)$ with $P(z) = I + \sum_{i=1}^{n_a} A_i(\bar{p}) z^{-i}$ and $Q(z) = \sum_{i=0}^{n_b-1} B_i(\bar{p}) z^{-i}$, quadratic stability is equivalent to $G(z)$ having only stable poles.

To also guarantee IO properties, the notion of quadratic dissipativity, or QSR dissipativity [28] is formalized.

$$x_{k+1} = \left[\begin{array}{ccccc|ccccc}
-A_1(p_k) & -A_2(p_k) & \dots & -A_{n_a-1}(p_k) & -A_{n_a}(p_k) & B_1(p_k) & B_2(p_k) & \dots & B_{n_b-2}(p_k) & B_{n_b-1}(p_k) \\
I & & & \emptyset & 0 & 0 & 0 & \dots & 0 & 0 \\
& I & & & 0 & 0 & 0 & \dots & 0 & 0 \\
& & \ddots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\emptyset & & & I & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & I & & & \emptyset & 0 \\
0 & 0 & \dots & 0 & 0 & & I & & & 0 \\
\vdots & \vdots & & \vdots & \vdots & & & \ddots & & \vdots \\
0 & 0 & \dots & 0 & 0 & \emptyset & & & I & 0
\end{array}\right] x_k + \left[\begin{array}{c} B_0(p_k) \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline I \\ 0 \\ 0 \\ \vdots \\ 0 \end{array}\right] u_k, \tag{2a}$$

$$y_k = \left[\begin{array}{ccccc|ccccc} -A_1(p_k) & -A_2(p_k) & \dots & -A_{n_a-1}(p_k) & -A_{n_a}(p_k) & B_1(p_k) & B_2(p_k) & \dots & B_{n_b-2}(p_k) & B_{n_b-1}(p_k) \end{array}\right] x_k + B_0(p_k)u_k. \tag{2b}$$

Definition 2 (QD): Given coefficient functions $A, B,$ and B_0, (1) are said to be quadratically dissipative (QD) with respect to

$$s^{\Pi}(u_k, y_k) = -\begin{bmatrix} y_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix} \begin{bmatrix} y_k \\ u_k \end{bmatrix} = -\begin{bmatrix} y_k \\ u_k \end{bmatrix}^\top \Pi \begin{bmatrix} y_k \\ u_k \end{bmatrix} \quad (8)$$

if there exists a $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ such that $\Xi^{\Pi}_{\text{dis}}(A(p_k), B(p_k), B_0(p_k), \mathcal{P}) \prec 0 \quad \forall p_k \in \mathbb{P}$ where

$$\Xi^{\Pi}_{\text{dis}}(A(p_k), B(p_k), B_0(p_k), \mathcal{P}) = \begin{bmatrix} \mathcal{A}^\top \mathcal{P} \mathcal{A} - \mathcal{P} & \mathcal{A}^\top \mathcal{P} \mathcal{B} \\ \mathcal{B}^\top \mathcal{P} \mathcal{A} & \mathcal{B}^\top \mathcal{P} \mathcal{B} \end{bmatrix} (p_k)$$

$$+ \begin{bmatrix} \mathcal{C}^\top Q \mathcal{C} & \mathcal{C}^\top Q \mathcal{D} + \mathcal{C}^\top L \\ \mathcal{D}^\top Q \mathcal{C} + L^\top \mathcal{C} & \mathcal{D}^\top Q \mathcal{D} + \mathcal{D}^\top L + L^\top \mathcal{D} + R \end{bmatrix} (p_k) \quad (9)$$

with $Q \in \mathbb{S}^{n_y}_{\succeq 0}$, $L \in \mathbb{R}^{n_y \times n_u}$, $R \in \mathbb{S}^{n_u}$ and $\text{In}(\Pi) = (n_y, n_u, 0)$.

Quadratic dissipativity, as in Definition 2, is a means to encapsulate generic IO properties. When $A, B,$ and B_0 are such that (1) is QD, this implies that (1) is QS and for all inputs u, all scheduling signals p with $p_k \in \mathbb{P}$ and zero initial condition, it holds that $\sum_{k=0}^{\infty} s(u_k, y_k) > 0$ [23]. In the LTI case, i.e., $p_k = \bar{p} \; \forall k$, (1) is a transfer function with frequency response $\mathcal{M}(e^{j\omega})$. Then, by the KYP lemma, quadratic dissipativity as in (9) is equivalent to $\mathcal{M}(e^{j\omega})$ satisfying frequency domain inequality $\begin{bmatrix} \mathcal{M}^H(e^{j\omega}) & I \end{bmatrix} \Pi \begin{bmatrix} \mathcal{M}^H(e^{j\omega}) & I \end{bmatrix}^H \preceq 0 \; \forall \omega \in [0, \pi]$ [29]. Two common choices for $Q, R,$ and S are highlighted.

1) $Q = I$, $L = 0$, $R = -\gamma^2 I$: for this choice, (9) implies that $\sum_{k=0}^{\infty} y_k^\top y_k < \sum_{k=0}^{\infty} \gamma^2 u_k^\top u_k$ for all input signals u and scheduling signals p, i.e., (1) has induced ℓ_2-gain smaller than γ. In the LTI case, $\mathcal{M}(e^{j\omega})$ satisfies $\|\mathcal{M}(e^{j\omega})\|_2 < \gamma \; \forall \omega \in [0, \pi]$, i.e., its infinity-norm satisfies $\|M(e^{j\omega})\|_{\mathcal{H}_\infty} < \gamma$.
2) $Q = 0$, $L = -I$, $R = 0$: for this choice, (9) implies that $\sum_{k=1}^{\infty} u_k^T y_k > 0$, i.e., (1) is passive. In the LTI case, $\mathcal{M}(e^{j\omega})$ satisfies $\mathcal{M}^H(e^{j\omega}) + \mathcal{M}(e^{j\omega}) \succ 0 \; \forall \omega \in [0, \pi]$, or with $n_u = n_y = 1$, $\Re(\mathcal{M}(e^{j\omega})) > 0 \; \forall \omega \in [0, \pi]$, i.e., $\mathcal{M}(e^{j\omega})$ is strictly positive real.

Other dissipativity properties, such as strict input/output passivity can be embedded similarly [30] as long as $\text{In}(\Pi) = (n_y, n_u, 0)$.

Remark 3: Note that the stability and dissipativity properties are formulated with respect to a quadratic, p-independent Lyapunov/storage function. In the LTI case, this is a necessary and sufficient condition for guaranteeing stability/dissipativity of (1) since (2) is detectable and reachable. In the LPV case, it is only sufficient, i.e., there may exist $A(p_k)$ for which (1) is stable but for which stability cannot be concluded by the considered class of Lyapunov functions, which could be conservative when compared to a p-dependent Lyapunov function [31].

Given Definitions 1 and 2, define the following sets of coefficient functions:

$$\Omega_{\text{stab}} = \{A, B, B_0 \mid \exists \mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0} \text{ such that } \Xi_{\text{stab}}(A(p_k), B(p_k), \mathcal{P}) \prec 0 \quad \forall p_k \in \mathbb{P}\} \quad (10)$$

$$\Omega^{\Pi}_{\text{dis}} = \{A, B, B_0 \mid \exists \mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0} \text{ such that } \Xi^{\Pi}_{\text{dis}}(A(p_k), B(p_k), B_0(p_k), \mathcal{P}) \prec 0 \quad \forall p_k \in \mathbb{P}\}. \quad (11)$$

The goal of this article is to characterize the sets Ω_{stab} and Ω_{dis} in terms of unconstrained functions and auxiliary variables, i.e., to characterize all coefficient functions $A, B,$ and B_0 for which (1) is, respectively, QS and QD in an unconstrained fashion. Formally, the goal is to find, for $i = \{\text{stab}, \text{dis}\}$, a continuous mapping T_i such that for all $(A, B, B_0) \in \Omega_i$, there exist unconstrained functions $X : \mathbb{P} \to \mathbb{U}^{n_i \times n_i}_{>0}$, $Y : \mathbb{P} \to \mathbb{U}^{n_i \times n_i}_0$, $Z : \mathbb{P} \to \mathbb{R}^{n_i \times m_i}$ with n_i, m_i depending on n_y, n_u and unconstrained auxiliary variables $\phi_i \in \mathbb{R}^{n_{\phi_i}}$ such that

$$[A(p_k) \; B(p_k) \; B_0(p_k)] = T_i(X(p_k), Y(p_k), Z(p_k), \phi_i), \quad (12)$$

that is, T_i is a mapping from a set of unconstrained functions X, Y, Z and unconstrained auxiliary variables ϕ_i onto Ω_i. This mapping T_i consists of a Lyapunov/Riccati equation parametrized by ϕ_i and a Cayley transformation parametrizing all functions contained to the unit ball using unconstrained functions X, Y, Z.

These mappings T_{stab} and T_{dis} are specified in detail in Sections III and IV, respectively. Subsequently, it is shown how to use them in system identification in Sections V and VI to guarantee that the resulting $A, B,$ and B_0 are such that estimated model (1) has the desired properties by construction.

III. Stability

In this section, the set of all $A, B,$ and B_0 for which (1) is QS, i.e., all $(A, B, B_0) \in \Omega_{\text{stab}}$ is characterized, constituting Contribution C1. First the main result is provided, followed by a proof in the subsection after.

A. Main Stability Result

The following theorem describes all $(A, B, B_0) \in \Omega_{\text{stab}}$ in an unconstrained fashion.

Theorem 1: Given coefficient functions $A, B,$ and B_0, it holds that $(A, B, B_0) \in \Omega_{\text{stab}}$ if and only if either $n_a = 0$ or there exist $X : \mathbb{P} \to \mathbb{U}^{n_y}_{>0}$, $Y : \mathbb{P} \to \mathbb{U}^{n_y}_0$, $Z : \mathbb{P} \to \mathbb{R}^{(\bar{n}_y - n_y) \times n_y}$ and $U \in \mathbb{U}^{\bar{n}_y}_{>0}$ such that

$$A(p_k) = E^{-1} M(p_k) U + (G_a^\top P G_a)^{-1} G_a^\top P F_a \quad (13)$$

in which $P \in \mathbb{S}^{\bar{n}_y}_{\succ 0}$ is the solution to

$$F_a^\top P F_a - P - F_a^\top P G_a (G_a^\top P G_a)^{-1} G_a^\top P F_a = -U^\top U \quad (14)$$

and in which $E \in \mathbb{U}^{n_y}_{>0}$ follows from the Cholesky factorization $E^\top E = G_a^\top P G_a$, and $M(p_k)$ is defined by

$$M(p_k) = \begin{bmatrix} (I - N(p_k))(I + N(p_k))^{-1} \\ -2Z(p_k)(I + N(p_k))^{-1} \end{bmatrix}^\top \quad (15a)$$

$$N(p_k) = X^\top(p_k) X(p_k) + Y(p_k) - Y^\top(p_k) + Z^\top(p_k) Z(p_k). \quad (15b)$$

The proof is given in the next subsection. Theorem 1 characterizes any coefficient functions $A, B,$ and B_0 for which (1) is QS in terms of unconstrained transformed coefficient functions X, Y, Z and auxiliary variable U. First, no restrictions are imposed on B and B_0, which can be intuitively understood by noting that these describe the forced behavior of (1) related to u_k, which by linearity of (1) does not influence stability. Thus, only stability of $\bar{y}_k$ has to be considered. Then, if $n_a = 0$, i.e.,

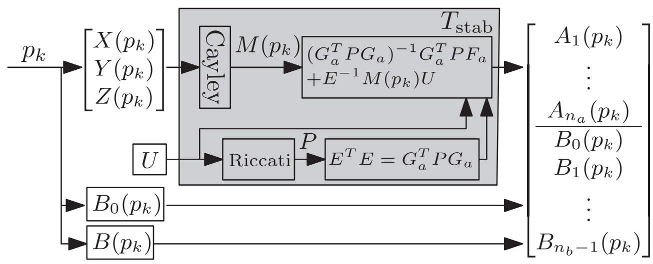

Fig. 1. Construction of the coefficient functions A, B, and B_0 from the transformed coefficient functions X, Y, Z and auxiliary variable U through the mapping T_{stab} consisting of Riccati equation (14) and Cayley transformation (15). This construction guarantees that (1) with A, B, and B_0 is QS.

if (1) does not include dynamics related to $\bar{y}_k$, (1) is trivially stable. In case $n_a > 0$, Theorem 1 states that if A is jointly constructed from U and unconstrained transformed coefficient functions X, Y, Z according to (13)–(15), it is guaranteed that (1) with the resulting A is QS. Conversely, Theorem 1 states that any A for which (1) is QS can be represented through an X, Y, Z, U. Last, the requirement that $X : \mathbb{P} \to \mathbb{U}^{n_y}_{>0}$ has strictly positive diagonal entries can be eliminated through parametrizing its diagonal as the square of a truly unconstrained function, i.e., as $\operatorname{diag}(X(p_k)) = X_d^2(p_k)$ with $X_d : \mathbb{P} \to \mathbb{R}^{n_y}$.

The characterization of all QS functions $A(p_k)$ through T_{stab} and specifically (13)–(15) can be understood as follows.

1) Transformed coefficient functions X, Y, Z characterize a matrix function N in (15b) that has eigenvalues in the right half-plane for all p_k.
2) The Cayley transformation in (15a) maps N to a gain-bounded matrix function M for which it is guaranteed that $M^\top(p_k)M(p_k) \prec I \;\forall p_k \in \mathbb{P}$.
3) Riccati equation (14) characterizes a Lyapunov function candidate $V(\bar{y}) = \bar{y}^\top P \bar{y}$ with $P \in \mathbb{S}^{\bar{n}_y}_{\succ 0}$ the solution of (14) for a given right-hand side $-U^\top U \in \mathbb{S}^{\bar{n}_y}_{\prec 0}$.
4) Given P and M, the construction of A in (13) guarantees that $V(\bar{y})$ is a Lyapunov function for $\bar{y}_k = (F_a - G_a A(\rho_k)\bar{y})$ by construction, such that (1) is QS. This construction can be interpreted as a rotation, scaling and translation of the unit ball $\{M(p_k) \mid \|M(p_k)\|_2 < 1 \;\forall p_k \in \mathbb{P}\}$ based on P.

Theorem 1 defines T_{stab} with $\phi = U$ through (13)–(15); see also Fig. 1.

Remark 4: T_{stab} is continuous with respect to X, Y, Z, U. Specifically,

1) the solution P of Riccati equation (14) is continuous with respect to $U^\top U$ [32];
2) the Cayley transformation in (15) is differentiable since $I + N(p_k)$ is invertible, as proven in Lemma 3;
3) the Cholesky decomposition $E^\top E = G_a^\top P G_a$ is also differentiable [33].

Consequently, all operations in T_{stab} are continuous.

Before proceeding with the proof, Theorem 1 is illustrated through an example.

Example 1: Consider an IO representation in the form of

$$y_k = -\frac{1}{2}\cos(p_k) y_{k-1} - \frac{1}{p_k + 3} y_{k+2} + u_k$$

with $p_k \in \mathbb{P} = [0, 1]$, which corresponds to (1) for $a_1(p) = \frac{1}{2}\cos(p_k)$, $a_2(p_k) = \frac{1}{p_k+3}$ and $b_0(p_k) = 1$, resulting in

$$A(p_k) = \begin{bmatrix} a_1(p_k) & a_2(p_k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cos(p_k) & \frac{1}{p_k+3} \end{bmatrix}.$$

Given the state $\begin{bmatrix} y_{k-1} & y_{k-2} \end{bmatrix}^\top \in \mathbb{R}^2$, for $P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, it holds that

$$Q = \begin{bmatrix} -a_1(p_k) & -a_2(p_k) \\ 1 & 0 \end{bmatrix}^\top P \begin{bmatrix} -a_1(p_k) & -a_2(p_k) \\ 1 & 0 \end{bmatrix} - P$$
$$= \begin{bmatrix} 2a_1^2(p_k) - 1 & 2a_1(p_k)a_2(p_k) \\ 2a_1(p_k)a_2(p_k) & 2a_2^2(p_k) - 1 \end{bmatrix}$$

which has leading principal minors that satisfy

$$2a_1^2(p_k) - 1 = \frac{\cos^2(p_k)}{2} - 1 < 0 \quad \forall p_k \in \mathbb{P}$$
$$\det(Q) = -2a_1^2(p_k) - 2a_2^2(p_k) + 1 > 0 \quad \forall p_k \in \mathbb{P},$$

in which the second inequality can be verified numerically. Consequently, $Q \prec 0 \;\forall p_k \in \mathbb{P}$. Thus, P is a quadratic Lyapunov function for $F_a - G_a A(p_k) = \begin{bmatrix} -a_1(p_k) & -a_2(p_k) \\ 1 & 0 \end{bmatrix}$. Consequently, E, U, X, Y, and Z are constructed by inverting (13)–(15). Specifically, U is computed by evaluating (14) as

$$F_a^\top P F_a - P - F_a^\top P G_a (G_a^\top P G_a)^{-1} G_a^\top P F_a = -I$$

resulting in $U = I$. In addition, $G_a^\top P G_a = 2 = E^\top E$ such that $E = \sqrt{2}$. Consequently, M is computed as

$$M(p_k) = E A(p_k) U^{-1} - (G_a^\top P G_a)^{-1} G_a^\top P F_a$$
$$= \begin{bmatrix} \frac{\sqrt{2}}{2}\cos(p_k) & \frac{\sqrt{2}}{p_k+3} \end{bmatrix} = \begin{bmatrix} M_1(p_k) & M_2(p_k) \end{bmatrix}.$$

It can be verified that for this M, it holds that $\|M(p_k)\|_2 < 1 \;\forall p_k \in \mathbb{P}$. Next, N follows by the relation between M_1 and N in (15b) as

$$N(p_k) = (I - M_1(p_k))(1 + M_1(p_k))^{-1} = -\frac{\sqrt{2}\cos(p_k) - 2}{\sqrt{2}\cos(p_k) + 2}$$

and given N, Z follows as

$$Z(p_k) = -\frac{1}{2} M_2(p_k)(I + N) = -\frac{2\sqrt{2}}{(p_k + 3)(\sqrt{2}\cos(p_k) + 2)}.$$

Last, for $n_y = 1$ with Y strictly upper triangular, it holds that $Y(p_k) = 0$. Thus, X is recovered from (15a) through the Cholesky factorization

$$X^\top(p_k) X(p_k) = N(p_k) - Z^\top(p_k) Z(p_k)$$

yielding

$$X(p_k) =$$
$$\sqrt{-\frac{\sqrt{2}\cos(p_k) - 2}{\sqrt{2}\cos(p_k) + 2} - \frac{8}{\left(\sqrt{2}\cos(p_k)^2 + 2\right)^2 (p_k + 3)^2}}$$

completing the transformation. Note that X is well defined as the components in the square root are both negative for $p_k \in \mathbb{P} = [0, 1]$ since $\cos(p_k) < 1$.

B. Proof of Theorem 1

In this section, Theorem 1 is proven through three lemmas that prove individual claims of Theorem 1. Specifically, the following is shown.

1) First, it is shown that B and B_0 do not influence the stability of (1) and can thus be chosen as any function (Lemma 1).
2) Second, it is shown that A is such that (1) is QS if and only if A can be constructed from an M constrained to the unit ball as in (13) and a U that is related to P as in (14) (Lemma 2).
3) Third, it is shown that any M in the unit ball can be represented by a function X, Y, Z through the Cayley transformation in (15) (Lemma 3).

Last, these steps are combined into a proof of the full result.

1) B and B_0 Can Be Any Function: First, it is shown that (1) is QS if and only $\bar{y}_{k+1} = (F_a - G_a A(p_k))\bar{y}_k$ is QS, i.e., B and B_0 do not influence stability.

Lemma 1: Given $A, B,$ and B_0, there exists a $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ such that $\mathcal{A}^\top(p_k)\mathcal{P}\mathcal{A}(p_k) - \mathcal{P} \prec 0\, \forall p_k \in \mathbb{P}$ if and only if there exists a $P \in \mathbb{S}^{\bar{n}_y}_{\succ 0}$ such that

$$(F_a - G_a A(p_k))^\top P(F_a - G_a A(p_k)) - P \prec 0 \quad \forall p_k \in \mathbb{P}. \tag{16}$$

Proof: First, partition $\mathcal{P}$ as $\mathcal{P} = \begin{bmatrix} P & \mathcal{P}_{12} \\ \mathcal{P}_{12}^\top & \mathcal{P}_{22}\end{bmatrix}$ with $P \in \mathbb{S}^{\bar{n}_y}_{\succ 0}$, $\mathcal{P}_{12} \in \mathbb{R}^{\bar{n}_y \times \bar{n}_u}$ and $\mathcal{P}_{22} \in \mathbb{S}^{\bar{n}_u}_{\succ 0}$. Consequently substitute this partitioned $\mathcal{P}$ into (7) to obtain

$$\mathcal{A}^\top(p_k)\mathcal{P}\mathcal{A}(p_k) - \mathcal{P} = \mathcal{Q}(p_k) = \begin{bmatrix} \mathcal{Q}_{11}(p_k) & \mathcal{Q}_{12}(p_k) \\ \mathcal{Q}_{12}^\top(p_k) & \mathcal{Q}_{22}(p_k)\end{bmatrix} \tag{17}$$

in which

$$\mathcal{Q}_{11}(p_k) = (F_a - G_a A(p_k))^\top P(F_a - GA(p_k)) - P$$
$$\mathcal{Q}_{12}(p_k) = (F_a - G_a A(p_k))^\top (PG_a B(p_k) + \mathcal{P}_{12}F_b) - \mathcal{P}_{12}$$
$$\mathcal{Q}_{22}(p_k) = B^\top(p_k)G_a^\top P G_a B(p_k) + B^\top(p_k)G_a^\top \mathcal{P}_{12}F_b + F_b^\top \mathcal{P}_{12} G_a B(p_k) + F_b^\top \mathcal{P}_{22}F_b - \mathcal{P}_{22}.$$

Recall that $\mathcal{Q}(p_k) \prec_{\mathbb{P}} 0$ is a shorthand notation for $\mathcal{Q}(p_k) \prec 0 \quad \forall p_k \in \mathbb{P}$. It is now proven that $\mathcal{Q}(p_k) \prec_{\mathbb{P}} 0$ if and only if (16) holds.

Necessity: Given A, B, B_0 and $\mathcal{P}$ such that $\mathcal{A}^\top(p_k)\mathcal{P}\mathcal{A}(p_k) - \mathcal{P} \prec_{\mathbb{P}} 0$, then it must also hold that $\mathcal{Q}_{11}(p_k) \prec_{\mathbb{P}} 0$, i.e., (16) necessarily holds.

Sufficiency: Given a $P \in \mathbb{S}^{\bar{n}_y}_{\succ 0}$ such that (16) holds, then by stability of F_b following (5), it is always possible to find a $\mathcal{P}_{22} \in \mathbb{S}^{\bar{n}_u}_{\succ 0}$ for which $\mathcal{P} = \operatorname{blkdiag}(P, \mathcal{P}_{22}) \succ 0$ proves $Q(p_k) \prec_{\mathbb{P}} 0$ by making the term $F_b^\top \mathcal{P}_{22} F_b - \mathcal{P}_{22}$ in $\mathcal{Q}_{22}(p_k)$ as negative definite as desired. Specifically, $\mathcal{Q}(p_k) \prec_{\mathbb{P}} 0$ if and only if its Schur complement $\mathcal{Q}_{22}(p_k) - \mathcal{Q}_{12}^\top(p_k)\mathcal{Q}_{11}^{-1}(p_k)\mathcal{Q}_{12}(p_k) \prec_{\mathbb{P}} 0$ as P satisfies (16), ensuring $\mathcal{Q}_{11}(p_k) \prec_{\mathbb{P}} 0$. Now, set $\mathcal{P}_{12} = 0$ and note that the Schur complement of $\mathcal{Q}(p_k)$ satisfies

$$\mathcal{Q}_{22}(p_k) - \mathcal{Q}_{12}^\top(p_k)\mathcal{Q}_{11}^{-1}(p_k)\mathcal{Q}_{12}(p_k) \preceq_{\mathbb{P}} \lambda I + F_b^\top \mathcal{P}_{22}F_b - \mathcal{P}_{22} - \kappa I \tag{18}$$

in which

$$\lambda = \max_{p_k \in \mathbb{P}} \lambda_{\max}(B^\top(p_k)G_a^\top P G_a B(p_k)) \tag{19a}$$

$$\kappa = \max_{p_k \in \mathbb{P}} \lambda_{\max}(\mathcal{Q}^\top{}_{12}(p_k)\mathcal{Q}_{11}^{-1}(p_k)\mathcal{Q}_{12}(p_k)) \tag{19b}$$

since any matrix $Q \in \mathbb{S}$ satisfies $x^\top Q x \le \lambda_{\max}(Q)x^\top x$. Note that κ and λ are well defined since $A(p_k)$ and $B(p_k)$ are bounded $\forall p_k \in \mathbb{P}$. Given these bounds, any $\mathcal{P}_{22}$ for which it holds that

$$F_b^\top \mathcal{P}_{22} F_b - \mathcal{P}_{22} \prec -(\kappa - \lambda) I \tag{20}$$

implies that $\mathcal{Q}_{22}(p_k) - \mathcal{Q}_{12}^\top(p_k)\mathcal{Q}_{11}^{-1}(p_k)\mathcal{Q}_{12}(p_k) \prec_{\mathbb{P}} 0$, i.e., implies that $\mathcal{Q}(p_k) \prec_{\mathbb{P}} 0$. Since F_b only has eigenvalue inside the unit circle, there exists a $\mathcal{P}_{22} \in \mathbb{S}^{\bar{n}_u}_{\succ 0}$ for which $F_b^\top \mathcal{P}_{22}F_b - \mathcal{P}_{22} = -\alpha I$ for any $\alpha \in \mathbb{R}_{>0}$. Then, for any $\alpha > \kappa - \lambda$, it holds that $Q(p_k) \prec_{\mathbb{P}} 0$, implying $\mathcal{A}^\top(p_k)\mathcal{P}\mathcal{A}(p_k) - \mathcal{P} \prec_{\mathbb{P}} 0$. Lastly, for $\mathcal{P}_{12} = 0$, positive definiteness of $\mathcal{P}_{22} \succ 0$ in combination with $P \succ 0$ guarantees that $\mathcal{P} \succ 0$, completing the proof. □

Intuitively, the above Lemma states that by the block upper triangular structure of $\mathcal{A}(p_k)$, only the diagonal blocks of $\mathcal{A}(p_k)$ determine its stability. Since F_b is stable, $\mathcal{A}(p_k)$ and thus (1) is QS if and only if the dynamics related to $\bar{y}_k$ are QS, i.e., if and only if $y_k = -\sum_{i=1}^{n_a} A_i(p_k)y_{k-i}$ is QS. Consequently, only (16), has to be considered in the remainder. Last, note that the above Lemma also implies that (1) is QS if $n_a = 0$.

2) Construction of A From M: Second, it is shown that if $A(p_k)$ is constructed according to (13) with the P resulting from Riccati equation (14), then A satisfies (16) with exactly that P. Furthermore, it is shown that all $A(p_k)$ for which there exists a P such that (16) holds, can be represented like this.

Lemma 2: Given $A : \mathbb{P} \to \mathbb{R}^{n_y \times \bar{n}_y}$, there exists a $P \in \mathbb{S}^{\bar{n}_y}_{\succ 0}$ such that A, P satisfy (16) if and only if there exists a $M : \mathbb{P} \to \mathbb{R}^{n_y \times \bar{n}_y}$ with $M^\top(p_k)M(p_k) \prec I\, \forall p_k \in \mathbb{P}$ and $U \in \mathbb{U}^{\bar{n}_y}_{>0}$ according to (14) and (13).

Proof: The proof follows by completing the square in (16) to equivalently write

$$\begin{aligned}&(F_a - G_a A(p_k))^\top P(F_a - G_a A(p_k)) - P \\ &= F_a^\top P F_a - P - F_a^\top P G_a (G_a^\top P G_a)^{-1} G_a^\top P F_a \\ &\quad + H^\top(p_k) G_a^\top P G_a H(p_k) \prec 0 \quad \forall p_k \in \mathbb{P}\end{aligned} \tag{21}$$

in which

$$H(p_k) = A(p_k) - (G_a^\top P G_a)^{-1} G_a^\top P F_a. \tag{22}$$

Necessity: Given A, P such that (16) holds, define

$$-S = F_a^\top P F_a - P - F_a^\top P G_a (G_a^\top P G_a)^{-1} G_a^\top P F_a. \tag{23}$$

Then, by (21), it holds that $S \succ 0$ since $H^\top(p_k)G_a^\top P G_a H(p_k) \succeq_{\mathbb{P}} 0$. Given $S \succ 0$, it has Cholesky decomposition $S = U^\top U$ with $U \in \mathbb{U}^{\bar{n}_y}_{>0}$. Similarly, $G_a^\top P G_a \succ 0$ as $P \succ 0$ and G_a is full rank, such that it has decomposition $G_a^\top P G_a = E^\top E$ with $E \in \mathbb{U}^{n_y}_{>0}$. A congruence transformation of (21) with U^{-1} then gives

$$I - U^{-\top} H^\top(p_k) E^\top E H(p_k) U^{-1} \prec 0 \quad \forall p_k \in \mathbb{P} \tag{24}$$

or equivalently $I - M^\top(p_k)M(p_k)$ by defining

$$M(p_k) = E H(p_k) U^{-1} \tag{25}$$

which after substitution of $H(p_k)$ is equivalent to (13).

Sufficiency: Given M and U as stated, set $P \in \mathbb{S}^{\bar{n}_y}_{\succ 0}$ as the positive-definite solution to (14), which exists and is unique as

F, G is controllable and $U^\top U \succ 0$ as U is full rank [34]. Then, constructing A according to (13) and substituting it in (21) yields

$$\begin{aligned}&(F_a - G_a A(p_k))^\top P(F_a - G_a A(p_k)) - P\\ &\quad= F_a^\top P F_a - P - F_a^\top P G_a (G_a^\top P G_a)^{-1} G_a^\top P F_a\\ &\qquad+ U^\top M(p_k)^\top E^{-\top} G_a^\top P G_a E^{-1} M(p_k) U. \qquad (26)\end{aligned}$$

Note that $E \in \mathbb{U}_{>0}^{n_y}$ is defined by $G_a^\top P G_a = E^\top E$, which exists as $P \succ 0$ and G_a is full rank. Then, (26) can be written as

$$-U^\top U + U^\top M(p_k)^\top M(p_k) U \prec 0 \quad \forall p_k \in \mathbb{P} \qquad (27)$$

since P is the solution to (14), and in which negative definiteness follows by $M^\top(p_k)M(p_k) \prec I \quad \forall p_k \in \mathbb{P}$, i.e., the constructed $A(p_k)$ satisfies (16). □

The fact that P has to satisfy Riccati equation (14) can be understood as a consequence of fixing the state coordinates according to (3). Consequently, P also has to be a Lyapunov function in these coordinates, which is reflected by (14). In contrast, in the unconstrained characterization of stable state-space models, the coordinates of the state-space are not fixed and consequently no restrictions are imposed on $P \in \mathbb{S}_{\succ 0}^{n_x}$ [25], [26].

Since the set of functions $\{M : \mathbb{P} \to \mathbb{R}^{n_y \times \bar{n}_y} \mid M(p_k)^\top M(p_k) \prec I \quad \forall p_k \in \mathbb{P}\}$ and the set of triangular matrices $\mathbb{U}_{>0}^{\bar{n}_y}$ are both convex, Lemma 2 provides a convex characterization of all coefficient functions A for which (1) is QS.

3) Reparametrizing the Unit Ball: To obtain an unconstrained characterization of Ω_{stab}, all $M(p_k)$ which are constrained to the unit ball are characterized through a Cayley transformation. Throughout, the dependency of the matrix functions M, X, Y, Z on p_k is left implicit for brevity.

Lemma 3: Given $M : \mathbb{P} \to \mathbb{R}^{n \times m}$ with $n \geq m$, $M^T M \prec I \quad \forall p_k \in \mathbb{P}$ if and only if there exists $X : \mathbb{P} \to \mathbb{U}_{>0}^m$, $Y : \mathbb{P} \to \mathbb{U}_0^m$, $Z : \mathbb{P} \to \mathbb{R}^{n-m \times m}$ related to M as

$$N = X^\top X + Y - Y^\top + Z^\top Z \qquad (28)$$

$$M = \begin{bmatrix} (I - N)(I + N)^{-1} \\ -2Z(I + N)^{-1} \end{bmatrix}. \qquad (29)$$

Proof: The proof is adapted from [26]. Partition M as $M = \begin{bmatrix} M_1^\top & M_2^\top \end{bmatrix}^\top$ with $M_1 : \mathbb{P} \to \mathbb{R}^{m \times m}$ and $M_2 : \mathbb{P} \to \mathbb{R}^{n-m \times m}$.

Necessity: Given M such that $M^\top M \prec I$, then first note that $M_1^\top M_1 + M_2^\top M_2 \prec I \quad \forall p_k \in \mathbb{P}$ implies that $\|M_1\|_2 < 1 \; \forall p_k \in \mathbb{P}$, and thus, $I + M_1$ is invertible for all $p_k \in \mathbb{P}$. Given M, calculate N according to first row of (29) as

$$N = (I + M_1)^{-1}(I - M_1) = (I - M_1)(I + M_1)^{-1}. \qquad (30)$$

The second row of (29) directly gives $Z = -\frac{1}{2} M_2 (I + N)$, i.e., there exists a Z as stated. Next, $I + N$ is simplified using (30) as

$$I + N = (I + M_1)(I + M_1)^{-1} + N = 2(I + M_1)^{-1}. \qquad (31)$$

Then, define $H = -Z^\top Z + \frac{1}{2}(N + N^\top)$ such that

$$\begin{aligned}H =& -(I + M_1)^{-\top} M_2^\top M_2 (I + M_1)^{-1}\\ &+ \frac{1}{2}\left((I - M_1)(I + M_1)^{-1} + (I + M_1)^{-\top}(I - M_1)^\top\right)\\ =& \frac{1}{2}(I + M_1)^{-\top}(I + M_1)^\top(I - M_1)(I + M_1)^{-1})\\ &+ \frac{1}{2}(I + M_1)^{-\top}(I - M_1)^\top(I + M_1)(I + M_1)^{-1}\\ &- (I + M_1)^{-\top} M_2^\top M_2 (I + M_1)^{-1}\\ =& (I + M_1)^{-\top}(I - M_1^\top M_1 - M_2^\top M_2)(I + M_1)^{-1}\\ =& (I + M_1)^{-\top}(I - M^\top M)(I + M_1)^{-1} \succ 0 \quad \forall p_k \in \mathbb{P}\end{aligned} \qquad (32)$$

where the last inequality holds as $M^\top M \prec I \; \forall p_k \in \mathbb{P}$ and $I + M_1$ is full rank. Since $H \succ 0 \; \forall p_k \in \mathbb{P}$, define $X : \mathbb{P} \to \mathbb{U}_{>0}^m$ through $H = X^\top X$. Last, define Y as

$$Y = \frac{1}{2}\text{triu}(N - N^\top) \qquad (33)$$

i.e., $Y : \mathbb{P} \to \mathbb{U}_0^m$ is the skew-symmetric part of N. Then, the constructed X, Y, Z satisfy (28) and (29).

Sufficiency: Given X, Y, Z as stated, define N and M as in (28) and (29), which is well defined as $I + N$ is full rank $\forall p_k \in \mathbb{P}$ since $v^\top(I + N)v = v^\top(I + X^\top X + Z^\top Z)v > 0 \; \forall v \in \mathbb{R}^m \setminus \{0\} \; \forall p_k \in \mathbb{P}$. Then, it holds that

$$\begin{aligned}&(I + N)^\top(I - M^\top M)(I + N)\\ &\quad= (I + N)^\top(I + N) - (I - N)^\top(I - N) - 4Z^\top Z\\ &\quad= 2(N^\top + N) - 4Z^\top Z\\ &\quad= 4(X^\top X + Z^\top Z) - 4Z^\top Z = 4X^\top X \succ 0 \quad \forall p_k \in \mathbb{P} \qquad (34)\end{aligned}$$

where strict positive definiteness follows since $X : \mathbb{P} \to \mathbb{U}_{>0}^m$ is full rank $\forall p_k \in \mathbb{P}$. Since $I + N$ is also full rank $\forall p_k \in \mathbb{P}$, it holds that $I - M^\top M \succ 0 \quad \forall p_k \in \mathbb{P}$. □

Remark 5: Instead of representing $X \in \mathbb{U}_{>0}$ and $Y \in \mathbb{U}_0$, it is also possible to specify them as full matrices as in [26]. However, this introduces extra parameters, namely the lower triangular part of X and Y, and does not allow for a straightforward way to ensure that X is full rank.

Together, Lemmas 1–3 imply Theorem 1.

Proof of Theorem 1. Necessity: given $(A, B, B_0) \in \Omega_{\text{stab}}$, i.e., such that $\exists \mathcal{P} \in \mathbb{S}_{\succ 0}^{n_x}$ for which $\Xi_{\text{stab}}(A(p_k), B(p_k), B_0(p_k), \mathcal{P}) \prec 0 \; \forall p_k \in \mathbb{P}$ with Ξ_{stab} in (7), then by Lemma 1 there exists a $P \in \mathbb{S}_{\succ 0}^{\bar{n}_y}$ such that (16) holds. Subsequently, by Lemma 2 there exists an M with $M^\top(p_k)M(p_k) \prec I \; \forall p_k \in \mathbb{P}$ and $U \in \mathbb{U}_{>0}^{\bar{n}_y}$ related to A and P as (13) and (14). Finally, by Lemma 3, there exist X, Y, Z related to M according to (15a) and (15b).

Sufficiency: given U and functions X, Y, Z as stated, by Lemma 3, the M constructed according to (15a) and (15b) satisfies $M^\top(p_k)M(p_k) \prec I \; \forall p_k \in \mathbb{P}$. Then, by Lemma 2, A and P constructed according to (13) and (14) satisfy (16). Finally, by Lemma 1, there exists a $\mathcal{P}$ such that $\Xi_{\text{stab}}(A(p_k), B(p_k), B_0(p_k), \mathcal{P}) \prec 0 \; \forall p_k \in \mathbb{P}$ for any B, B_0, i.e., $(A, B, B_0) \in \Omega_{\text{stab}}$.

IV. Dissipativity

In this section, the set of all coefficient functions A, B, and B_0 for which (1) is QD with respect to a prespecified supply $s^\Pi(u_k, y_k)$ are characterized, i.e., all $(A, B, B_0) \in$

$\Omega_{\text{dis}}^{\Pi}$, constituting Contribution C2. Based on the choice of s^{Π}, quadratic dissipativity then implies relevant IO properties, such as passivity and an upper bound on the induced ℓ_2-gain of (1), see also Definition 2. First the main result is provided, followed by a proof in the subsection after.

A. Main Dissipativity Result

To state the main result, first define

$$F = \begin{bmatrix} F_a & 0 \\ 0 & F_b \end{bmatrix} \in \mathbb{R}^{n_x \times n_x} \tag{35a}$$

$$G = \begin{bmatrix} G_a \\ 0 \end{bmatrix} \in \mathbb{R}^{n_x \times n_y} \quad \bar{G} = \begin{bmatrix} 0 \\ G_b \end{bmatrix} \in \mathbb{R}^{n_x \times n_u} \tag{35b}$$

such that (2) can be equivalently written as

$$\begin{aligned} x_{k+1} &= (F + GJ(p_k))x_k + (GB_0(p_k) + \bar{G})u_k \\ y_k &= J(p_k)x_k + B_0(p_k)u_k \end{aligned} \tag{36}$$

with $J(p_k) = \begin{bmatrix} -A(p_k) & B(p_k) \end{bmatrix}$. Given these definitions, the following theorem provides an unconstrained characterization of all $(A, B, B_0) \in \Omega_{\text{dis}}^{\Pi}$ in the case that $n_a > 0$, $n_b > 1$. The cases in which $n_a = 0$ and/or $n_b = 1$ are discussed after.

Theorem 2: Given coefficient functions $A, B,$ and B_0 with $n_a > 0, n_b > 1$ and $\Pi = \left[\begin{smallmatrix} Q & L \\ L^\top & R \end{smallmatrix}\right] \in \mathbb{S}^{n_y + n_u}$ with $Q \in \mathbb{S}_{\succeq 0}^{n_y}$ and $\text{In}(\Pi) = (n_y, n_u, 0)$, then it holds that $(A, B, B_0) \in \Omega_{\text{dis}}^{\Pi}$ if and only if there exist $X : \mathbb{P} \to \mathbb{U}_{>0}^{n_y}$, $Y : \mathbb{P} \to \mathbb{U}_0^{n_y}$, $Z : \mathbb{P} \to \mathbb{R}^{n_x + n_u - n_y \times n_y}$, $V \in \mathbb{U}_{>0}^{n_x}$ and $\alpha \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} \begin{bmatrix} -A(p_k) & B(p_k) & \big| & B_0(p_k) \end{bmatrix} &= E^{-1} M(p_k) U \\ &- (G^\top \mathcal{P} G + Q)^{-1} \begin{bmatrix} G^\top \mathcal{P} F & \big| & G^\top \mathcal{P} \bar{G} + L \end{bmatrix} \end{aligned} \tag{37}$$

in which $\mathcal{P} \in \mathbb{S}_{\succ 0}^{n_x}$ follows from

$$\mathcal{P} = (\mathcal{X} + \beta \bar{\mathcal{X}})^{-1} \tag{38a}$$

$$F \mathcal{X} F^\top - \mathcal{X} - \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} = -V^\top V \tag{38b}$$

$$\beta = -\min(\lambda_{\min}(\bar{\mathcal{X}}^{-\frac{1}{2}} \mathcal{X} \bar{\mathcal{X}}^{-\frac{1}{2}}), 0) + \alpha \tag{38c}$$

with $\bar{\mathcal{X}} = \text{blkdiag}(\bar{\mathcal{X}}_a, \bar{\mathcal{X}}_b)$ with $\bar{\mathcal{X}}_a = \text{blkdiag}(I_{n_y}, 2I_{n_y}, \ldots, n_a I_{n_y}) \in \mathbb{D}_{>0}^{\bar{n}_y}$ and $\bar{\mathcal{X}}_b = \text{blkdiag}(I_{n_u}, 2I_{n_u}, \ldots, (n_b - 1) I_{n_u}) \in \mathbb{D}_{>0}^{\bar{n}_u}$, and in which $U \in \mathbb{U}_{>0}^{n_x}$, $E \in \mathbb{U}_{>0}^{n_u}$ follow by Cholesky factorizations $S = U^\top U$, $E^\top E = G^\top \mathcal{P} G + Q$ with

$$\begin{aligned} S = &- \begin{bmatrix} F^\top \\ \bar{G}^\top \end{bmatrix} \mathcal{P} \begin{bmatrix} F & \bar{G} \end{bmatrix} + \begin{bmatrix} \mathcal{P} & 0 \\ 0 & -R \end{bmatrix} \\ &+ \begin{bmatrix} F^\top \mathcal{P} G \\ \bar{G}^\top \mathcal{P} G + L^\top \end{bmatrix} (G^\top \mathcal{P} G + Q)^{-1} \begin{bmatrix} G^\top \mathcal{P} F & G^\top \mathcal{P} \bar{G} + L \end{bmatrix} \end{aligned} \tag{39}$$

and last, in which $M(p_k)$ follows by

$$M(p_k) = \begin{bmatrix} (I - N(p_k))(I + N(p_k))^{-1} \\ -2Z(p_k)(I + N(p_k))^{-1} \end{bmatrix}^\top \tag{40a}$$

$$\begin{aligned} N(p_k) = &\ X^\top(p_k) X(p_k) + Y(p_k) - Y^\top(p_k) \\ &+ Z^\top(p_k) Z(p_k). \end{aligned} \tag{40b}$$

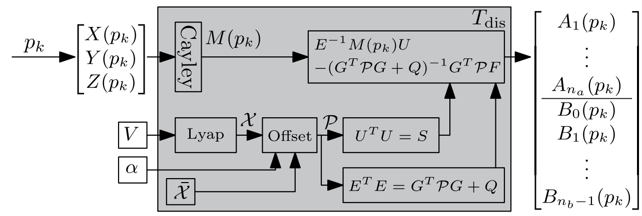

Fig. 2. Coefficient functions $A, B,$ and B_0 constructed from the transformed coefficient functions X, Y, Z and auxiliary variables V, α through the mapping T_{dis} consisting of Lyapunov equation (38) and Cayley transformation (40) with S, as in (39). This construction guarantees that (1) with $A, B,$ and B_0 is QD with respect to the supply s^{Π}, as in Definition 2.

The proof of above theorem is given in the next subsection. Theorem 2 characterizes all $(A, B, B_0) \in \Omega_{\text{dis}}^{\Pi}$ through unconstrained transformed coefficient functions X, Y, Z and unconstrained auxiliary variables V, α. Specifically, it states that if $A, B,$ and B_0 are jointly constructed from M and $\mathcal{P}$ according to (37) with $\mathcal{P}$ constructed as in (38), then $A, B,$ and B_0 satisfy dissipation inequality (9) with exactly this $\mathcal{P}$, i.e., (1) is QD. The other way around, it states that any $A, B,$ and B_0 for which (1) is QD can be represented through some X, Y, Z, V, α. This characterization can be understood as follows.

1) Given V, Lyapunov equality (38b) and offset (38a) define a storage function $x^\top \mathcal{P} x$ for state-space realization (2). Since the state coordinates of (2) are fixed according to (3), $x^\top \mathcal{P} x$ has to be a storage function *in these coordinates*, which is ensured by (38).
2) The Cayley transformation in (40a) constructs a matrix coefficient function M for which $M^\top(p_k) M(p_k) \prec I \; \forall p_k \in \mathbb{P}$ based on N in (40b).
3) Given $\mathcal{P}$ and M, coefficient functions $A, B,$ and B_0 are constructed by scaling, rotating, and shifting the unit ball M, as in (37). This construction is implicitly based on $\mathcal{P}$ through Cholesky factorizations U and E, and ensures that $\Xi_{\text{dis}}^{\Pi}(A, B, B_0, \mathcal{P}) \prec I \; \forall p_k \in \mathbb{P}$ by construction, i.e., that $A, B,$ and B_0 are such that (1) is QD with respect to the storage function defined by $\mathcal{P}$.

Similarly to the stability case, Theorem 2 defines mapping T_{dis} with $\phi = \{V, \alpha\}$ through (37)–(39), which is visualized in Fig. 2.

Remark 6: T_{dis} is continuous with respect to X, Y, Z, V, α. Specifically,

1) Lyapunov equation (38b) represents a square and full rank linear system of equations [35], and thus, its solution $\mathcal{X}$ is differentiable with respect to V;
2) the shift parameter β in (38c) is continuous with respect to $\mathcal{X}$ as $\lambda_{\min}(\mathcal{X})$ is continuous with respect to the entries in $\mathcal{X}$, and differentiable if its multiplicity is 1 [36];
3) $\mathcal{P}$ in (38a) is differentiable as $\mathcal{X} + \beta \bar{\mathcal{X}}$ is full rank;
4) all other operations in (37), (39), and (40) are differentiable.

Consequently, all operations in T_{dis} are continuous.

B. Special Cases

Theorem 2 treats the general case of QD of (1) for which $n_a > 0, n_b > 1$. This result can be narrowed down to the setting that $n_a = 0$ and/or $n_b = 1$ as follows.

1) $n_a > 0$, $n_b = 1$: In this setting, LPV-IO representation (1) only contains dynamics related to y and depends on u only through $B_0 u_k$, i.e., B is empty. Theorem 2 immediately applies to this setting by replacing $\{F, G, \bar{G}, [-A(p_k)\ B(p_k)]\}$ with $\{F_a, G_a, 0, -A(p_k)\}$, respectively, and adjusting the dimensions of Z as $Z : \mathbb{P} \to \mathbb{R}^{(n_y n_a - n_y)\times n_y}$. To see this, note that (1) with $n_b = 1$ can be written as

$$\begin{aligned} x_{k+1} &= (F_a - G_a A(p_k))x_k + G_a B_0(p_k)u_k \\ y_k &= -A(p_k)x_k + B_0(p_k)u_k \end{aligned} \tag{41}$$

with $x_k = [y_{k-1}^\top\ y_{k-2}^\top\ \ldots\ y_{k-n_b}^\top]^\top$, which is equivalent to (36) with $F, G, \bar{G}, J(p_k)$ replaced by $F_a, G_a, 0, -A(p_k)$.

2) $n_a = 0, n_b > 1$: In this setting, (1) only contains dynamics related to u and A is empty, corresponding to an LPV-FIR system. Theorem 2 applies to this setting by replacing $\{F, G, \bar{G}, [-A(p_k)\ B(p_k)]\}$ with $\{F_b, 0, G_b, B(p_k)\}$, adjusting the dimensions of Z as $Z : \mathbb{P} \to \mathbb{R}^{n_u(n_b-1)\times n_y}$, and by requiring $Q \in \mathbb{S}^{n_y}_{\succ 0}$ instead of $Q \in \mathbb{S}^{n_y}_{\succeq 0}$. To see this, note that (1) with $n_a = 0$ can be written as

$$\begin{aligned} x_{k+1} &= F_b x_k + G_b u_k \\ y_k &= B(p_k)x_k + B_0(p_k)u_k \end{aligned} \tag{42}$$

with $x_k = [u_{k-1}^\top\ u_{k-2}^\top\ \ldots\ u_{k-n_b}^\top]^\top$, which is equivalent to (36) with $F, G, \bar{G}, J(p_k)$ replaced by $F_b, 0, G_b, B(p_k)$. $Q \succ 0$ is required to ensure that $E^\top E = G^\top \mathcal{P} G + Q$ is still well defined as in this case $G = 0$.

3) $n_a = 0, n_b = 1$: In this setting, (1) is a parameter-varying gain without any states, i.e., $y_k = B_0(p_k)u_k$. Theorem 2 can be specialized to this setting by zeroing $F, G, \bar{G}, J(p_k)$, deleting the first block of rows/columns, restricting $Q \in \mathbb{S}^{n_y}_{\succ 0}$, and adjusting the dimensions of X, Y, Z as $X : \mathbb{P} \to \mathbb{U}^{n_1}_{>0}$, $Y : \mathbb{P} \to \mathbb{U}^{n_1}_0$, $Z : \mathbb{P} \to \mathbb{R}^{n_2\times n_1}$ with $n_1 = \min(n_u, n_y)$ and $n_2 = |n_y - n_u| - n_1$. Then, B_0 is given by

$$E^{-1}M(p_k)U + Q^{-1}L = \begin{cases} B_0(p_k), & \text{if } n_y \le n_u \\ B_0^\top(p_k), & \text{otherwise} \end{cases} \tag{43}$$

with $E \in \mathbb{U}^{n_y}_{>0}$, $U \in \mathbb{U}^{n_u}_{>0}$ resulting from factorizations $E^\top E = Q$, and $U^\top U = S = -R + L^\top Q^{-1} L$.

C. Proof of Theorem 2

In this section, Theorem 2 is proven through two Lemmas that prove individual claims of Theorem 2. Specifically, the following is shown.

1) First, it is shown that $(A, B, B_0) \in \Omega^{\Pi}_{\text{dis}}$ if and only if these can be constructed from an M in the unit ball, as in (37), given a $\mathcal{P} \succ 0$ for which $S \succ 0$ with S, as in (39) (Lemma 4).
2) Second, it is shown that all $\mathcal{P}$ for which $S \succ 0$ can be generated from V and α, as in (38) (Lemma 5–7).

Last, these steps are combined with a parametrization of M in the unit ball through the Cayley transformation of X, Y, Z, as in (40), to yield a proof of the full result.

1) Construction of (A,B,B$_0$): First, it is shown that if A, B, and B_0 are constructed according to (37) with a $\mathcal{P}$ for which $S \succ 0$, then A, B, and B_0 satisfy $\Xi^{\Pi}_{\text{dis}}(A(p_k), B(p_k), B_0(p_k)) \prec_{\mathbb{P}} 0$ with exactly that $\mathcal{P}$, i.e., (1) is QD. Furthermore, any $(A, B, B_0) \in \Omega^{\Pi}_{\text{dis}}$ can be represented in this way.

Lemma 4: Given A, B, and B_0, then it holds that $(A, B, B_0) \in \Omega^{\Pi}_{\text{dis}}$ if and only if there exists 1) a $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ such that $S \succ 0$ with S in (39); and 2) an $M : \mathbb{P} \to \mathbb{R}^{n_y \times n_x + n_u}$ with $M^\top(p_k)M(p_k) \prec I \quad \forall p_k \in \mathbb{P}$ related to A, B, and B_0 as (37).

Proof: The proof follows by completing the squares. Specifically, $\Xi^{\Pi}_{\text{dis}}(A, B, B_0, \mathcal{P})$ in (9) can be written as

$$\begin{aligned} \Xi^{\Pi}_{\text{dis}} = &\begin{bmatrix} F^\top \\ \bar{G}^\top \end{bmatrix} \mathcal{P} \begin{bmatrix} F & \bar{G} \end{bmatrix} - \begin{bmatrix} \mathcal{P} & 0 \\ 0 & -R \end{bmatrix} + H^\top(p_k) Z H(p_k) \\ &- \begin{bmatrix} F^\top \mathcal{P} G \\ \bar{G}^\top \mathcal{P} G + L^\top \end{bmatrix} (G^\top \mathcal{P} G + I)^{-1} \begin{bmatrix} G^\top \mathcal{P} F & G^\top \mathcal{P} \bar{G} + L \end{bmatrix} \end{aligned} \tag{44}$$

in which $Z = (G^\top \mathcal{P} G + Q)$ and

$$\begin{aligned} H(p_k) = &\left[-A(p_k) \quad B(p_k) \mid B_0(p_k)\right] \\ &+ (G^\top \mathcal{P} G + Q)^{-1} \left[G^\top \mathcal{P} F \mid G^\top \mathcal{P} \bar{G} + L\right]. \end{aligned} \tag{45}$$

Necessity: Given $(A, B, B_0) \in \Omega^{\Pi}_{\text{dis}}$, i.e., an (A, B, B_0) such that $\exists \mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ such that $\Xi^{\Pi}_{\text{dis}}(A(p_k), B(p_k), B_0(p_k), \mathcal{P}) \prec_{\mathbb{P}} 0$, then by (44) it must hold that $S \in \mathbb{S}^{n_x}_{\succ 0}$ since $H^\top(p_k)(G^\top \mathcal{P} G + Q)H(p_k) \succeq_{\mathbb{P}} 0$. Thus S has the Cholesky decomposition $S = U^\top U$ with $U \in \mathbb{U}^{n_x}_{>0}$. Similarly, $G^\top \mathcal{P} G + Q = E^\top E$ with $E \in \mathbb{U}^{n_x}_{>0}$ since $Q \succeq 0$. Then, a congruence transformation of Ξ^{Π}_{dis} with U^{-1} implies

$$-I + U^{-\top} H^\top(p_k) E^\top E H(p_k) U^{-1} \prec 0 \quad \forall p_k \in \mathbb{P} \tag{46}$$

or equivalently $M^\top(p_k)M(p_k) \prec I\ \forall p_k \in \mathbb{P}$ with

$$M(p_k) = E H(p_k) U^{-1} \tag{47}$$

which is equivalent to (37) after substitution of H in (45).

Sufficiency: Given $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ such that $S \in \mathbb{S}^{n_x}_{\succ 0}$ and M such that $M^\top(p_k)M(p_k) \prec_{\mathbb{P}} I$, substitution of A, B, and B_0 constructed as in (37) into (44) and (45) yields

$$\Xi^{\Pi}_{\text{dis}} = -S + U^\top M^\top(p_k) E^{-\top}(G^\top \mathcal{P} G + Q) E^{-1} M(p_k) U \tag{48}$$

by definition of S. Now, $G^\top \mathcal{P} G + Q = E^\top E$ and $S = -U^\top U$ by definition, such that (48) reduces to

$$\Xi^{\Pi}_{\text{dis}} = U^\top(-I + M^\top(p_k)M(p_k))U \prec 0 \quad \forall p_k \in \mathbb{P} \tag{49}$$

where the inequality follows by $M^\top(p_k)M(p_k) \prec_{\mathbb{P}} I$. □

Lemma 4 restricts the set of possible storage functions $\mathcal{P}$ through Riccati-like condition (39). Given such a storage function, the transformed coefficient functions M must be gain-bounded.

2) Characterization of all $\mathcal{P}$: Unlike in the stability case, it is not possible to characterize all $\mathcal{P}$ that satisfy (39) through S. Specifically, given $\mathcal{P}$, the corresponding S can be calculated by (39), but given an S, it is unclear if a solution $\mathcal{P}$ exists and how to compute it. Instead, as an intermediate step, condition (39) is equivalently characterized by a linear matrix inequality (LMI) in $\mathcal{P}^{-1}$.

Lemma 5: $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ satisfies (39) with Π such that $\text{In}(\Pi) = (n_y, n_u, 0)$ if and only if $\mathcal{P}$ satisfies

$$-\mathcal{P}^{-1} + F\mathcal{P}^{-1}F^\top - \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix}^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \prec 0. \tag{50}$$

For the proof of above Lemma, the Haynsworth's Rule of Inertia is used [37], which is stated next for completeness.

Lemma 6: Given $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^\top & Z_{22} \end{bmatrix} \in \mathbb{S}^{n+m}$ with $Z_{11} \in \mathbb{S}^n$, $Z_{22} \in \mathbb{S}^m$, $Z_{12} \in \mathbb{R}^{n\times m}$, then the inertia of Z satisfies

$$\begin{aligned} \text{In}(Z) &= \text{In}(Z_{22}) + \text{In}(Z_{11} - Z_{12}Z_{22}^{-1}Z_{12}^\top) \\ &= \text{In}(Z_{11}) + \text{In}(Z_{22} - Z_{12}^\top Z_{11}^{-1} Z_{12}). \end{aligned} \tag{51}$$

Proof of Lemma 5: The proof is based on first rewriting (39) as a genuine Riccati inequality and an auxiliary condition through similar algebraic steps as in [38] and [39], and second transforming this Riccati equation to (50).

Necessity: Given $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ such that (39) holds, it must hold that $W \prec 0$ with W the (2,2) block of (39) given by

$$\begin{aligned} W &= \bar{G}^\top \mathcal{P}\bar{G} + R \\ &\quad - (\bar{G}^\top \mathcal{P} G + L^\top)(G^\top \mathcal{P} G + Q)^{-1}(G^\top \mathcal{P}\bar{G} + L). \end{aligned} \tag{52}$$

In addition, if (39) holds, the Schur complement over its (2,2) block must also be negative definite, i.e.,

$$\begin{aligned} &F^\top \mathcal{P} F - \mathcal{P} - F^\top \mathcal{P} G (G^\top \mathcal{P} G + Q)^{-1} G^\top \mathcal{P} F \\ &- \left(F^\top \mathcal{P}\bar{G} - F^\top \mathcal{P} G (G^\top \mathcal{P} G + Q)^{-1}(G^\top \mathcal{P}\bar{G} + L)\right) W^{-1} \times \\ &\left(\bar{G}^\top \mathcal{P} F - (\bar{G}^\top \mathcal{P} G + L^\top)(G^\top \mathcal{P} G + Q)^{-1} G^\top \mathcal{P} F\right) \prec 0 \end{aligned}$$

which, by collecting terms, is equivalent to

$$F^\top \mathcal{P} F - \mathcal{P} - F^\top \mathcal{P} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^\top & W^{-1} \end{bmatrix} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \mathcal{P} F \prec 0 \tag{53}$$

with

$$\begin{aligned} W_{11} &= (G^\top \mathcal{P} G + Q)^{-1} + (G^\top \mathcal{P} G + Q)^{-1} \\ &\quad \times (G^\top \mathcal{P}\bar{G} + L) W^{-1} (\bar{G}^\top \mathcal{P} G + L^\top)(G^\top \mathcal{P} G + Q)^{-1} \end{aligned} \tag{54a}$$

$$W_{12} = -(G^\top \mathcal{P} G + Q)^{-1}(G^\top \mathcal{P}\bar{G} + L) W^{-1}. \tag{54b}$$

Then, by the matrix inversion lemma

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{12}^\top & W^{-1} \end{bmatrix} = \begin{bmatrix} G^\top \mathcal{P} G + Q & G^\top \mathcal{P}\bar{G} + L \\ \bar{G}^T \mathcal{P} G + L^\top & \bar{G}^\top \mathcal{P}\bar{G} + R \end{bmatrix}^{-1} \tag{55}$$

such that (53) can be written as

$$\begin{aligned} &-F^\top \mathcal{P} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top \left(\begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \mathcal{P} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top + \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix} \right)^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \mathcal{P} F \\ &+ F^\top \mathcal{P} F - \mathcal{P} \prec 0. \end{aligned} \tag{56}$$

In other words, $\mathcal{P}$ satisfies (39) if and only if it satisfies Riccati inequality (56) and auxiliary condition $W \prec 0$. For $Q = I$, $R = -\gamma^2 I$, these conditions correspond to the conditions derived for the full information $\mathcal{H}_\infty$ control setting [40, Ch. 7].

Next, (52) and (56) are converted to the LMI in (50). Specifically, by the Woodbury Matrix Identity it holds that

$$\begin{aligned} &\mathcal{P} - \mathcal{P} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top \left(\begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \mathcal{P} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top + \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix} \right)^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \mathcal{P} \\ &= \left(\mathcal{P}^{-1} + \begin{bmatrix} G & \bar{G} \end{bmatrix} \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix}^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \right)^{-1} \end{aligned} \tag{57}$$

which after substitution in (53) implies that

$$F^\top \left(\mathcal{P}^{-1} + \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \right)^{-1} F - \mathcal{P} \prec 0. \tag{58}$$

Now, it is claimed that $\mathcal{P}^{-1} + \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G & \bar{G} \end{bmatrix}^\top \succ 0$ if and only if $W \prec 0$, which is proven later. Then, (58) implies that

$$\begin{bmatrix} -\mathcal{P} & F^\top \\ F & -\mathcal{P}^{-1} - \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G & \bar{G} \end{bmatrix}^\top \end{bmatrix} \prec 0 \tag{59}$$

or equivalently (50) by taking the Schur complement.

It remains to prove that $\mathcal{P}^{-1} + \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G & \bar{G} \end{bmatrix}^\top \succ 0$ if and only if $W \prec 0$. By the Haynsworth's Rule of Inertia

$$\begin{aligned} &\text{In} \begin{bmatrix} -\mathcal{P}^{-1} & G & \bar{G} \\ G^\top & Q & L \\ \bar{G}^\top & L^\top & R \end{bmatrix} \\ &= \text{In} \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix} + \text{In} \left(-\mathcal{P}^{-1} - \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix}^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \right) \\ &= \text{In}(-\mathcal{P}^{-1}) + \text{In} \left(\begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix} + \begin{bmatrix} G^\top \mathcal{P} G & G^\top \mathcal{P}\bar{G} \\ \bar{G}^\top \mathcal{P} G & \bar{G}^\top \mathcal{P}\bar{G} \end{bmatrix} \right) \end{aligned} \tag{60}$$

in which by the same rule

$$\text{In} \begin{bmatrix} G^\top \mathcal{P} G + Q & G^\top \mathcal{P}\bar{G} + L \\ \bar{G}^\top \mathcal{P} G + L^\top & \bar{G}^\top \mathcal{P}\bar{G} + R \end{bmatrix} = \text{In}(G^\top \mathcal{P} G + Q) + \text{In}(W).$$

Now, $\text{In}(-\mathcal{P}^{-1}) = (0, n_x, 0)$, $\text{In}(G^\top \mathcal{P} G + Q) = (n_u, 0, 0)$ as $Q \succeq 0$ and G full rank, and $\text{In}(\Pi) = (n_y, n_u, 0)$, see Definition 2. Substituting these inertias in (60) yields

$$\begin{aligned} &(0, n_x, 0) + (n_y, 0, 0) + \text{In}(W) \\ &= (n_y, n_u, 0) + \text{In} \left(-\mathcal{P}^{-1} - \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G & \bar{G} \end{bmatrix}^\top \right). \end{aligned} \tag{61}$$

Thus, $\text{In}(W) = (0, n_u, 0)$ if and only if

$$\text{In} \left(-\mathcal{P}^{-1} - \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G & \bar{G} \end{bmatrix}^\top \right) = (0, n_x, 0). \tag{62}$$

Sufficiency: Follows by reversing the steps in necessity as all of them are also sufficient. Specifically, if $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ satisfies (50), then $-\mathcal{P}^{-1} - \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G & \bar{G} \end{bmatrix}^\top \prec 0$. Then, by (61) it holds that $W \prec 0$. Next, applying the algebraic manipulations as in (55)–(59) reveal that (39) is satisfied. □

Lemma 5 relates the Riccati-like inequality in (39) to the LMI in (50). Consequently, it provides a convex characterization of the set of possible storage functions $\mathcal{P}$. Already, it allows for solving for a $\mathcal{P}$ through SDP techniques.

Remark 7: Condition (50) could have also been obtained by directly applying the Projection Lemma to (9) [23].

To obtain an unconstrained characterization of all possible storage functions, the set of $\mathcal{P}$ that satisfy (50) is reparametrized as follows.

Lemma 7: $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ satisfies (50) if and only if there exists $V \in \mathbb{U}^{n_x}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$ related to $\mathcal{P}$ as

$$F\mathcal{X}F^\top - \mathcal{X} - \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix}^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} = -V^\top V \quad \text{(63a)}$$

$$\beta = -\min(\lambda_{\min}(\bar{\mathcal{X}}^{-\frac{1}{2}}\mathcal{X}\bar{\mathcal{X}}^{-\frac{1}{2}}), 0) + \alpha \quad \text{(63b)}$$

$$\mathcal{P} = (\mathcal{X} + \beta\bar{\mathcal{X}})^{-1} \quad \text{(63c)}$$

in which $\bar{\mathcal{X}} = \text{blkdiag}(\bar{\mathcal{X}}_a, \bar{\mathcal{X}}_b)$ with $\bar{\mathcal{X}} = \text{blkdiag}(\bar{\mathcal{X}}_a, \bar{\mathcal{X}}_b)$ with $\bar{\mathcal{X}}_a = \text{blkdiag}(I_{n_y}, 2I_{n_y}, \ldots, n_a I_{n_y}) \in \mathbb{D}^{\bar{n}_y}_{\succ 0}$ and $\bar{\mathcal{X}}_b = \text{blkdiag}(I_{n_u}, 2I_{n_u}, \ldots, (n_b-1)I_{n_u}) \in \mathbb{D}^{\bar{n}_u}_{\succ 0}$.

Proof: Necessity: Given a $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ that satisfies (50), then by continuity of the eigenvalues and strictness of (50) there exists a $\beta > 0$ that perturbs $\mathcal{P}^{-1}$ such that $\mathcal{P}^{-1} - \beta\bar{\mathcal{X}} \succ 0$ as well as

$$F(\mathcal{P}^{-1} - \beta\bar{\mathcal{X}})F^\top - (\mathcal{P}^{-1} - \beta\bar{\mathcal{X}}) - \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix}^\top \Pi^{-1} \begin{bmatrix} G^\top \\ \bar{G}^\top \end{bmatrix} \prec 0. \quad \text{(64)}$$

Consequently, set $\mathcal{X} = \mathcal{P}^{-1} - \beta\bar{\mathcal{X}} \succ 0$, which is equivalent to (63c). Then, V is defined through the Cholesky factorization of (64), which is equivalent to (63a). Last, since $\mathcal{X} \succ 0$, it holds that $\min(\lambda_{\min}(\bar{\mathcal{X}}^{-\frac{1}{2}}\mathcal{X}\bar{\mathcal{X}}^{-\frac{1}{2}}), 0) = 0$, such that setting $\alpha = \beta > 0$ satisfies (63b).

Sufficiency: Given a $V \in \mathbb{U}^{n_x}_{>0}$, define $\mathcal{X}$ as the solution to (63a). This $\mathcal{X}$ exists and is well defined since $\lambda_i(F) = 0$ for $i = 1, \ldots, n_x$ [35]. Furthermore, since $-\begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G & \bar{G} \end{bmatrix}^\top + V^\top V$ is symmetric, also $\mathcal{X}$ is symmetric. Since $\mathcal{X}$ is not necessarily positive definite, set $\mathcal{P} = (\mathcal{X} + \beta\bar{\mathcal{X}})^{-1}$ with β, as in (63b), to guarantee $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$. To see this, a congruence of $\mathcal{P}^{-1}$ with $\bar{\mathcal{X}}^{-\frac{1}{2}}$ reveals

$$\bar{\mathcal{X}}^{-\frac{1}{2}}\mathcal{P}^{-1}\bar{\mathcal{X}}^{-\frac{1}{2}} = \bar{\mathcal{X}}^{-\frac{1}{2}}\mathcal{X}\bar{\mathcal{X}}^{-\frac{1}{2}} + \beta I \succ \alpha I \quad \text{(65)}$$

where the inequality holds by definition of β in (63b). Matrix $\mathcal{P}$ with this offset also satisfies (50) through the specific choice for $\bar{\mathcal{X}}$. Specifically, evaluating (50) for $\mathcal{P}$ yields

$$\begin{aligned} &-(\mathcal{X} + \beta\bar{\mathcal{X}}) + F(\mathcal{X} + \beta\bar{\mathcal{X}})F^\top - \begin{bmatrix} G & \bar{G} \end{bmatrix} \Pi^{-1} \begin{bmatrix} G & \bar{G} \end{bmatrix}^\top \\ &= -\beta\bar{\mathcal{X}} + \beta F\bar{\mathcal{X}}F^\top - V^\top V \end{aligned} \quad \text{(66)}$$

by definition of $\mathcal{X}$ as the solution to (63a) given V. Now, for $\bar{\mathcal{X}}$ as stated, it holds that $-\bar{\mathcal{X}} + F\bar{\mathcal{X}}F^\top = -I$, such that

$$-\beta\bar{\mathcal{X}} + \beta F\bar{\mathcal{X}}F^\top - V^\top V = -\beta I - V^\top V. \quad \text{(67)}$$

Since $\alpha > 0$, also $\beta > 0$, such that $\mathcal{P}$ satisfies (50). □

Above lemma characterizes all $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ that satisfy (39) in an unconstrained way. The proof is based on solving (63a) for $\mathcal{X}$ given V and guaranteeing that also $\mathcal{P} \succ 0$ through offsetting $\mathcal{X}$ by $\bar{\mathcal{X}} \succ 0$, as in (63c), which is possible as $\bar{\mathcal{X}}$ is a Lyapunov function for F. The specific bound β in (63b) is obtained by simultaneous diagonalization.

With Lemma 5, a $\mathcal{P}$ that satisfies (39) can be obtained by choosing a V and α and constructing $\mathcal{P}$ according to (63a)–(63c). $\mathcal{P}$ is guaranteed to exist for any $V \in \mathbb{U}^{n_x}_{>0}$, and all $\mathcal{P}$ satisfying (39) can be represented by varying V and α. Lemmas 4–7 together with Lemma 3 immediately imply Theorem 2.

Proof of Theorem 2: Necessity: Given $(A, B, B_0) \in \Omega^{\Pi}_{\text{dis}}$, i.e., $\exists \mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ such that $\Xi^{\Pi}_{\text{dis}}(A(p_k), B(p_k), B_0(p_k), \mathcal{P}) \prec_{\mathbb{P}} 0$ with Ξ^{Π}_{dis} in (9), then by Lemma 4, $\mathcal{P}$ satisfies (39) and there exists an M with $M^\top(p_k)M(p_k) \prec I \; \forall p_k \in \mathbb{P}$ related to $A, B,$ and B_0, as in (37). Then, by Lemma 5, $\mathcal{P}$ satisfies (50), and by Lemma 7, there exist V and α related to $\mathcal{P}$ as (38). Finally, by Lemma 3, there exists $X(p_k), Y(p_k),$ and $Z(p_k)$ related to $M(p_k)$, as in (40).

Sufficiency: Given X, Y, Z, by Lemma 3, M constructed, as in (40), satisfies $M^\top(p_k)M(p_k) \prec_{\mathbb{P}} I$. Furthermore, $\mathcal{P} \in \mathbb{S}^{n_x}_{\succ 0}$ constructed, as in (38a) and (38b), satisfies (50) by Lemma 7, and thus, (39) by Lemma 7. Then, by Lemma 4, $A, B,$ and B_0 constructed, as in (37), satisfy $\Xi_{\text{dis}}(A(p_k), B(p_k), B_0(p_k), \mathcal{P}) \prec_{\mathbb{P}} 0$ with this $\mathcal{P}$, i.e., $(A, B, B_0) \in \Omega^{\Pi}_{\text{dis}}$. □

V. Unconstrained Quadratically Dissipative Model Class

In system identification, the equivalence results given in Theorems 1 and 2 allow for identifying models in the form of (1) in terms of X, Y, Z, ϕ such that stability or dissipativity guarantees hold by construction, which is especially useful for identification based on machine-learning methods, such as neural networks. This results in the QS and QD model classes, constituting contribution C3.

Specifically, traditional methods guarantee stability and dissipativity of an identified model in the form of (1) through optimizing $A, B,$ and B_0 according to

$$\begin{aligned} &\arg \min_{A,B,B_0} && \mathcal{L}(u_k, y_k, [A(p_k),\ B(p_k),\ B_0(p_k)]) \\ &\text{subject to} && (A, B, B_0) \in \Omega_i \end{aligned} \quad \text{(68)}$$

with $\mathcal{L}$ an identification criterion, e.g., a simulation error, that fits parametrized coefficient functions $A, B,$ and B_0 to data $\{y_k, u_k, p_k\}_{k=1}^N$ while stability/dissipativity is guaranteed by the SDP-based feasibility constraint $(A, B, B_0) \in \Omega_i$ with $i = \{\text{stab}, \text{dis}\}$. In contrast, Theorems 1 and 2 allow for optimizing

$$\arg \min_{X,Y,Z,\phi} \mathcal{L}(u_k, y_k, T_i(X(p_k), Y(p_k), Z(p_k), \phi)) \quad \text{(69)}$$

where functions X, Y, Z and auxiliary variables $\phi = \{V, \alpha\}$ are converted to $A, B,$ and B_0 for evaluation of (1) through T_i; see Sections III and IV. In (69), Theorems 1 and 2 guarantee that the resulting $A, B,$ and B_0 are such that (1) has the desired quadratic stability or dissipativity properties for any choice of X, Y, Z, ϕ. Moreover, it is guaranteed that all $A, B,$ and B_0 for which (1) has this quadratic property, can be represented by properly picking X, Y, Z, ϕ through optimization.

The functions X, Y, Z can be parametrized in terms of any bounded function depending on parameters ζ. For shorthand notation, define $[\text{vec}_{\text{triu}}(X(p_k))^\top \text{vec}_{\text{triu,s}}(Y(p_k))^\top \text{vec}(Z(p_k))^\top]^\top = g_\zeta(p_k) \in \mathbb{R}^m$ with $\text{vec}_{\text{triu}}(\cdot)$ representing a vectorization of the upper triangular part of a matrix, $\text{vec}_{\text{triu,s}}$ the vectorization of the strictly upper triangular part, and $m = n_y^2 + n_y(n_x + n_u - n_y)$ the total dimension after vectorization. Some common parametrizations are as follows.

Algorithm 1: Quadratically Dissipative LPV-IO Model.

inputs: Coefficient functions $X : \mathbb{P} \to \mathbb{U}^{n_y}_{>0}$, $Y : \mathbb{P} \to \mathbb{U}^{n_y}_{0}$, $Z : \mathbb{P} \to \mathbb{R}^{n_x+n_u-n_y \times n_y}$, auxiliary variables $V \in \mathbb{U}^{n_x}_{>0}$, $\alpha \in \mathbb{R}_{>0}$, a supply function defined by $\Pi = \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix} \in \mathbb{S}^{n_y+n_u}$ with $Q \in \mathbb{S}^{n_y}_{\succeq 0}$ and $\mathrm{In}(\Pi) = (n_y, n_u, 0)$, and input/scheduling trajectories $\{u_k, p_k\}_{k=1}^N$.
set matrices $F, G, \bar{G}$ as in (35) and $\bar{\mathcal{X}}$ as in Theorem 2.
solve Lyapunov equation (38b) to obtain $\mathcal{X}$.
calculate $\mathcal{P}$ as in (38a)–(38c).
calculate S as in (39).
factorize $S = U^\top U$ and $G^\top \mathcal{P} G + Q = E^\top E s$.
initialize $y_k = 0, u_k = 0 \; \forall k < 0$.
for $k = 1, \ldots, N$ **do**
  calculate $M(p_k)$ according to (40).
  calculate $A(p_k), B(p_k), B_0(p_k)$ according to (37).
  calculate y_k according to (1).
end for

1) Affine parametrizations with $g_\zeta(p_k) = W p_k + c$ with parameters $W \in \mathbb{R}^{m \times n_p}$, $c \in \mathbb{R}^m$.
2) Polynomial parametrizations [14] with

$$g_\zeta(p_k) = c + \sum_{i=1}^{d} w_i \prod_{j=1}^{n_p} p_{k,j}^{d_{i,j}}$$

with $p_{k,j}$ the jth element of $p_k \in \mathbb{R}^{n_p}$ and parameters $d_{i,j} \in \mathbb{N}$, $w_i \in \mathbb{R}^m$, $c \in \mathbb{R}^m$.
3) Neural network parametrizations [41] with

$$g_\zeta(p_k) = W_L \sigma(\cdots W_1 \sigma(W_0 p_k + c_0) + c_1 \cdots) + c_L$$

with elementwise nonlinear activation function σ, e.g., $\sigma = \tanh$, and parameters $W_0, \ldots, W_L, c_0, \ldots, c_L$.

Note that the choice of parametrization can limit the set of A, B, B_0 that can be represented, e.g., through using too little monomials or neurons. However, if the parametrization is chosen as a universal functions approximator, all A, B, B_0 can be represented up to arbitrary precision by increasing the degrees of flexibility of the approximator [42].

The mapping T_{dis} together with the choice for X, Y, Z defines an unconstrained QD LPV-IO model. The construction of the coefficient functions a_i and b_i from X, Y, Z and the prediction of output trajectories using this model is summarized in Algorithm 1. The QS LPV-IO model class can be defined analogously, see also [27].

Remark 8: As for any model structure, when using the unconstrained QS/QD model structure for identification, special attention should be paid to the following.

1) The model order and complexity selection of the involved parameterization to ensure a favorable tradeoff between the structural bias and parameter variance.
2) Ensuring that the input is persistently exciting [12], [43] to provide informative data for the estimation process.
3) The choice of the identification criterion in view of the modeling objectives.

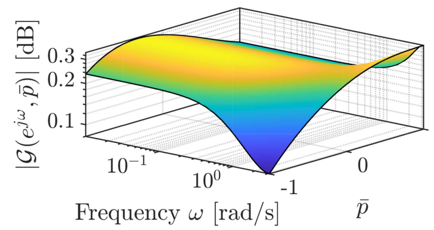

Fig. 3. Bode plot of frozen LTI dynamics of $\mathcal{G}(e^{j\omega}, \bar{p})$ for various constant scheduling values $\bar{p} \in \mathbb{P}$.

The model in Algorithm 1 can be optimized using unconstrained (sub)gradient-based unconstrained optimization methods since all transformations are either continuous or differentiable, see Remark 6.

VI. System Identification Example

In this section, the developed unconstrained QD model class of Theorem 2 is showcased in combination with neural network coefficient functions to identify an LPV-IO model while desired dissipativity properties are enforced by construction. Examples that enforce stability by construction previously appeared in [27] and [44].[2]

A. LPV Output-Error Identification Setup

The considered data-generating LPV system $\mathcal{G}$ is an instance of (1) with $n_a = 1$, $n_b = 1$ and given by

$$\tilde{y}_k = -\tilde{a}_1(p_k)\tilde{y}_{k-1} + \tilde{b}_0(p_k)u_k \tag{70a}$$

$$y_k = \tilde{y}_k + v_k \tag{70b}$$

with input $u_k \in \mathbb{R}$ and scheduling signal $p_k \in \mathbb{P} = \mathbb{R}_{[-1,1]}$. Furthermore, $y_k \in \mathbb{R}$ is a measurement of the true output $\tilde{y}_k \in \mathbb{R}$ perturbed by zero-mean independent and identically distributed white noise $v_k \in \mathbb{R}$ with $\mathbb{E}(v_k^2) = \sigma_v^2$, resulting in an LPV output-error (OE) identification setup [14].[3] Note that by defining $x_k = \tilde{y}_{k-1}$, (70) directly yields a state-space realization by rewriting $y_k = -\tilde{a}_1(p_k)\tilde{y}_{k-1} + \tilde{b}_0(p_k)u_k + v_k$. The coefficient functions are given by

$$\tilde{a}_1(p_k) = -0.1 - 0.5\tanh(-p_k) \tag{71a}$$

$$\tilde{b}_0(p_k) = -0.03 - 0.25e^{-0.7p_k^2} - 0.12\frac{p_k}{1 + e^{p_k}}. \tag{71b}$$

The resulting frozen LTI dynamics of (70a) over $\mathbb{P}$ are shown in Fig. 3 and the coefficient functions are visualized in Fig. 4.

Remark 9: System (70) is chosen as a simple first-order filter for visualization purposes. Specifically, it admits visualizing both coefficient functions a_1, b_0 in a 2-D plane.

For system (70), a dataset $\mathfrak{D} = \{u_k, y_k, p_k\}_{k=1}^N$ of length $N = 1000$ samples is generated from zero initial conditions with multisine $u_k = \sum_{i=1}^{90} \sin(\omega_i k)$ containing normalized frequencies $\omega_i = 0.01\pi \times i$ linearly spaced between 0.01π and

[2]The code for generating the results in this section and verifying Theorem 11 numerically is available at https://gitlab.tue.nl/kon/dissipative_linear_io_identification/-/tree/main. An example demonstrating the unconstrained stable model class of Theorem 1 can be found at https://gitlab.tue.nl/kon/stable-lpv-io-estimation

[3]More generic noise model structures can easily be incorporated, but for ease of notation, an LPV-OE setting is considered.

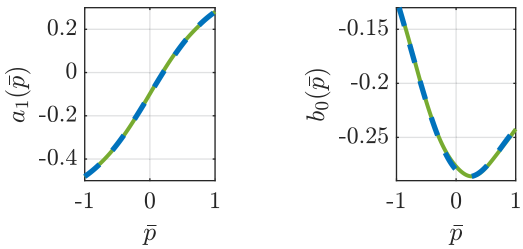

Fig. 4. Estimated neural network coefficient functions of the unconstrained QD model (—) accurately represent the true coefficient functions (- -).

0.9π, and $p_k = 1 - kN^{-1}$ a linear scheduling trajectory. The noise variance is set as $\sigma_v^2 = 0.01$ for signal-to-noise ratio $10\log_{10}(\frac{\sum_{k=1}^{N}\tilde{y}_k^2}{\sum_{k=1}^{N}v_k^2}) = 30$ dB.

B. Unconstrained QD LPV-IO Model

The unconstrained QD LPV-IO model $\mathcal{M}$, see Algorithm 1, is used to identify $\mathcal{G}$ in (70) based on dataset $\mathfrak{D}$. A neural network parametrization for transformed coefficient functions X, Y, Z is considered, and the cost function is chosen as the simulation error, corresponding to the OE identification setting. The model is optimized based on gradient-based optimization.

Specifically, the model order is chosen as $n_a = 1, n_b = 1$, matching the system order. The transformed coefficient functions $X : \mathbb{P} \to \mathbb{U}_{>0}$ and $Z : \mathbb{P} \to \mathbb{R}$ are parametrized as a neural network with two hidden layers of five nodes each and tanh activation function, i.e., according to

$$[X(p_k)\ Z(p_k)] = W_2\sigma(W_1\sigma(W_0 p_k + c_0) + c_1) + c_2 \quad (72)$$

with $\sigma = \tanh$, weight matrices $W_0 \in \mathbb{R}^{5\times 1}$, $W_1 \in \mathbb{R}^{5\times 5}$, and $W_2 \in \mathbb{R}^{2\times 5}$, and bias vectors $c_0 \in \mathbb{R}^5$, $c_1 \in \mathbb{R}^5$, and $c_2 \in \mathbb{R}^2$, for a total parameter set $\zeta = [\mathrm{vec}(W_0)^\top\ \ c_0^\top\ \ \mathrm{vec}(W_1)^\top\ \ c_1^\top\ \ \mathrm{vec}(W_2)^\top\ \ c_2^\top]^\top \in \mathbb{R}^{52}$. Furthermore, with $V \in \mathbb{U}_{>0}^1 = \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}$, the total parameter set is given by $\phi = [\zeta^\top\ V\ \ \alpha]^\top \in \mathbb{R}^{54}$. Note that $Y(p_k) = 0 \quad \forall p_k \in \mathbb{P}$ as $\bar{n}_y = 1$. Last, for this example, the unconstrained parametrization only introduces $n_\phi = 2$ extra parameters compared to directly parametrizing a_1, b_0 by the same neural network.

For the supply function, Π is set as $Q = 0$, $L = -I$, $R = -0.6740$, expressing the desire that the estimated model is dissipative with respect to $s^\Pi(u_k, y_k) = u_k^T y_k + y_k^\top u_k + u_k^\top R u_k$, meaning that the frozen behavior of $\mathcal{M}$ satisfies $\Re(\mathcal{M}(e^{j\omega}, \bar{p})) > R/2 \quad \forall \omega \in [0, 2\pi) \quad \forall \bar{p} \in \mathbb{P}$.

Remark 10: For this simulation example, the true system $\mathcal{G}$ is QD with respect to this s^Π, enabling approximation of $\mathcal{G}$ by $\mathcal{M}$ up to the approximation capabilities of the neural network. The ability of the model to identify the true system can be limited by this choice of supply rate if the true system is not dissipative with respect to the chosen supply rate s^Π.

In the above OE setting with noiseless p_k, the model parameters ϕ are found by minimizing the ℓ_2 loss of the simulation error $V_N(\phi)$ as $\phi^* = \arg\min_\phi V_N(\phi)$ with

$$V_N(\phi) = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(y_k - \hat{y}_{k,\phi})^2} \quad (73)$$

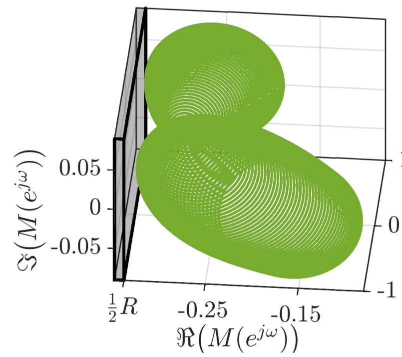

Fig. 5. Frequency response function of frozen LTI transfer functions $\mathcal{M}(e^{j\omega}, \bar{p})$ of the identified model $\mathcal{M}$ for various $\bar{p} \in \mathbb{P}$ (—) are guaranteed to stay outside the region $\Re(\mathcal{M}(e^{j\omega}, \bar{p})) > -0.0337$ (▭) for all $\bar{p} \in \mathbb{P}$ since the estimated model is QD by construction.

where $\hat{y}_{k,\phi}$ is the simulated model response with initial conditions equal to zero.

Criterion (73) is optimized using the Levenberg–Marquardt optimization algorithm with finite differencing for Jacobian estimation, see [45], resulting in a training time of 20 s on a Z-book G5 using an Intel Core i7-8750H CPU.

C. Identification With Dissipativity Guarantees

For the training dataset, the estimated parameter vector ϕ^* achieves $V_N(\phi^*) = 0.1015$, corresponding to the standard deviation of the noise $\sigma_v = 0.1$, implying that the only contribution to $V_N(\phi^*)$ is noise that cannot be predicted. For a validation dataset with the same noise variance and a similar input and scheduling, $V_N(\phi^*) = 0.098$ is obtained, indicating that the model generalizes well.

Fig. 4 shows the true coefficient functions $\tilde{a}_1, \tilde{b}_0$ as well as the estimated coefficient functions a_1, b_0 reconstructed from X, Z, V, α. The optimized neural network weights W_i and biases c_i defining $X(p_k)$ and $Z(p_k)$, as well as the matrix functions $M(p_k)$ and $Z(p_k)$ and the storage function defined by $\mathcal{P}$ can be found in the online repository. It can be concluded that the neural network parametrization of X, Z enables close approximation of $\tilde{a}_1, \tilde{b}_0$. At the same time, it is guaranteed that these neural network coefficient functions are such that (1) is QD. This is illustrated in Fig. 5 through visualizing the FRF of various frozen LTI systems $\mathcal{M}(e^{j\omega}, \bar{p})$ in the complex plane for various values of $\bar{p} \in \mathbb{P}$, which shows that each frozen LTI system satisfies $\Re(\mathcal{M}(e^{j\omega}, \bar{p})) > R/2 = -0.337$, i.e., satisfying the desired IO property.

D. Visualization of Dissipativity Sets

In this section, the set of coefficient functions a_1 and b_0 that can be represented by the unconstrained dissipative model is visualized. Specifically, define the sets

$$\bar{\Omega}_{\mathrm{dis},j}^{\Pi} = \{\bar{a}_1, \bar{b}_0 \mid \Xi_{\mathrm{dis}}^{\Pi}(\bar{a}_1, 0, \bar{b}_0, \mathcal{P}_j) \prec 0\}$$

$$\bar{\Omega}_{\mathrm{dis}}^{\Pi} = \{\bar{a}_1, \bar{b}_0 \mid \exists \mathcal{P} \in \mathbb{S}_{\succ 0}^{n_x} \text{ such that } \Xi_{\mathrm{dis}}^{\Pi}(\bar{a}_1, 0, \bar{b}_0, \mathcal{P}) \prec 0\}$$

where the former represents the coefficient function *values* for which a storage function defined by $\mathcal{P}_j$ proves dissipativity, and the latter represents the coefficient function *values* for

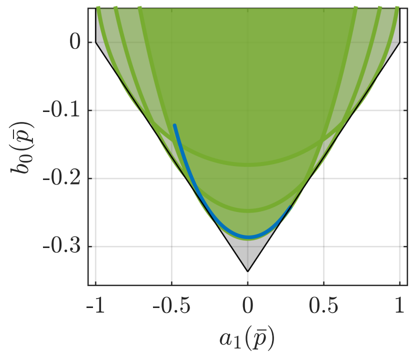

Fig. 6. Coefficient set $\bar{\Omega}^{\Pi}_{\text{dis},j} = \{\bar{a}_1 \in \mathbb{R},\, \bar{b}_0 \in \mathbb{R} \mid \Xi^{\Pi}_{\text{dis}}(\bar{a}_1, 0, \bar{b}_0, \mathcal{P}_j) \prec 0\ \forall \bar{p} \in \mathbb{P}\}$ with supply rate $s^{\Pi}(u_k, y_k) = u_k^T y_k + y_k^\top u_k + u_k^\top R u_k$ with $R = -0.6740$ visualized for $\mathcal{P}_j$ at iteration 3 (▭), 10 (▭), and 30 (▭) of the optimization. $\bar{\Omega}^{\Pi}_{\text{dis},j}$ represents the set in which coefficient functions a_1, b_0 can take on values such that LPV IO-model (1) is QD with respect to a storage function defined by $\mathcal{P}_j$. During optimization, $\mathcal{P}_j$ is adjusted such that the true coefficient functions $\tilde{a}_1, \tilde{b}_0$ (—) are contained within $\bar{\Omega}^{\Pi}_{\text{dis},j}$, corresponding to scaling, rotating, and translating of this set. Any $\bar{\Omega}^{\Pi}_{\text{dis},j}$ is contained to $\bar{\Omega}^{\Pi}_{\text{dis}}$ (▭), i.e., the set of values of $\bar{a}_1, \bar{b}_0$ for which (1) is dissipative with respect to s^{Π} in the LTI case.

which there exists a $\mathcal{P}$ that proves dissipativity. Thus, $\bar{\Omega}^{\Pi}_{\text{dis},j}$ can be interpreted as the region of the coefficient space in which coefficient *functions* a_1 and b_0 can take on values when they are constructed, as in (37), based on $\mathcal{P}_j$; see Theorem 2. The sets $\bar{\Omega}^{\Pi}_{\text{dis},j}$ are visualized for $\mathcal{P}_j$ varying over the optimization iterations in Fig. 6. This figure is interpreted as follows.

1) Mapping (37) corresponds to scaling, rotating, and translating the unit ball $\{M \in \mathbb{R}^{(n_x+n_u)\times n_y} \mid \|M\|_2 < 1\}$, resulting in the ellipsoidal shape of $\bar{\Omega}^{\Pi}_{\text{dis},j}$. Thus, $\mathcal{P}$ can be thought of as describing all possible rotations, translations, and scalings of $A, B,$ and B_0 such that (1) is QD.
2) Unsurprisingly, $\bar{\Omega}^{\Pi}_{\text{dis},j} \subset \bar{\Omega}^{\Pi}_{\text{dis}}$, i.e., the set of $\bar{a}_1$ and $\bar{b}_0 \in \mathbb{R}$ for which $\mathcal{P}_j$ proves dissipativity is necessarily contained within the set of coefficients $\bar{a}_1$ and $\bar{b}_0$ for which there exists a $\mathcal{P}$ that proves dissipativity. Furthermore, by Theorem 2, $\Omega^{\Pi}_{\text{dis}}$ can be fully covered by the union of $\bar{\Omega}^{\Pi}_{\text{dis},j}$ through varying $\mathcal{P}_j$.
3) For an LPV-IO model to be QD, there needs to exist an ellipsoid $\bar{\Omega}^{\Pi}_{\text{dis},j}$ that fully encapsulates the image of its coefficient functions over $\mathbb{P}$. This provides a graphical test for determining dissipativity properties for low-dimensional LPV-IO systems. Then, Fig. 6 illustrates that $\mathcal{G}$ is on the boundary of QD with respect to s^{Π}.
4) Optimizing $\mathcal{P}$ through V, α can be interpreted as transforming $\bar{\Omega}^{\Pi}_{\text{dis},j}$ such that it encapsulates the true coefficient functions.

E. Comparison With Standard LPV-IO Identification

The unconstrained QD LPV-IO structure is compared to a standard LPV-IO model structure, as in (1), in which the coefficient functions a_i and b_i are directly parametrized [14], [46]. For comparison purposes, these coefficient functions are

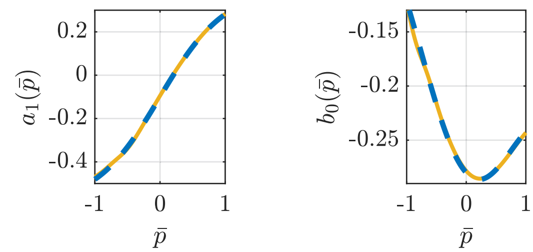

Fig. 7. Estimated neural network coefficient functions of the traditional LPV-IO model (—) accurately represent the true coefficient functions (- -).

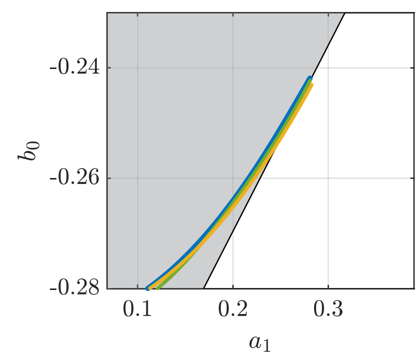

Fig. 8. Even when the coefficient functions $\bar{a}_1(p_k)$ and $\bar{b}_0(p_k)$ of the true system (—) are close to the boundary of the set $\bar{\Omega}^{\Pi}_{\text{dis}}$ (▭), i.e., the set of values of $\bar{a}_1, \bar{b}_0$ for which (1) is dissipative with respect to s^{Π}, the unconstrained QD model class (—) yields coefficient functions that are contained within $\bar{\Omega}^{\Pi}_{\text{dis}}$ by construction. In contrast, due to noisy data and finite approximation capabilities, the coefficient functions of a traditional LPV-IO model (—) are not guaranteed to be contained within this, i.e., do not reflect this dissipativity property of the true system.

parametrized by a neural network with the same structure, as in (72). The weights and biases in the network, i.e., the parameters of this standard LPV-IO model, are identified using an OE noise structure, the same dataset, and a prediction-error criterion, as in (73), resulting in a training time of 18 s. It is observed that during optimization, the estimated parameters regularly yielded an unstable model, necessitating reinitialization of the optimization. Similarly, randomly initializing the neural network often yields an unstable simulated response, complicating optimization.

The estimated model achieves an ℓ_2 loss of the simulation error equal to $V_N = 0.1015$ for the training dataset, and $V_N = 0.098$ for the validation dataset, indicating that the dynamics of the system have been estimated up to the noise floor.

Fig. 7 shows the coefficient functions estimated through the neural network. It is observed that the neural network parametrization of a_1 and b_0 can accurately approximate the true coefficient functions $\bar{a}_1$ and $\bar{b}_0$. Fig. 8 shows the estimated coefficients functions together with $\bar{\Omega}_{\text{dis}^{\Pi}}$, i.e., the set of coefficients *values* for which the model is QD with respect to s^{Π}. Even though the true system is QD with respect to s^{Π}, the estimated model is not: its coefficient functions are not contained in $\bar{\Omega}_{\text{dis}^{\Pi}}$, i.e., there exists a $\bar{p} \in \mathbb{P}$ for which the frozen model is not QD. This is a consequence of the noise in the data and the finite approximation capabilities of the neural network parametrization, combined with the fact that the true system is on the boundary of QD with respect to s^{Π}.

VII. Conclusion

In this article, the class of all QS/dissipative DT linear IO models is reparameterized in terms of unconstrained model parameters. This unconstrained parameterization is achieved through reparameterizing the stability/dissipativity condition in a necessary and sufficient fashion through the solution of a Riccati/Lyapunov equation and a Cayley transformation.

The resulting stable DT-LPV-IO model class enables system identification with a priori guarantees on stability/dissipativity properties of the identified model in the presence of modeling errors and measurement noise. Furthermore, it allows for incorporating arbitrary dependencies of the scheduling coefficients on the scheduling signal p, e.g., a neural network. Since it does not require enforcing an MI condition during optimization, it can be optimized using standard unconstrained optimization routines, significantly decreasing the computational complexity. Moreover, the construction of the Lyapunov/storage function through algebraic equations is numerically efficient such that the model class scales well to systems with high dimensions.

A potential direction for future work is to extend this unconstrained parametrization toward parameter-dependent Lyapunov/Storage functions.

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