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Lokaliseren van een magnetische dipool met behulp van een gradiometer

(Engelse titel: Localizing a magnetic dipole using a gradiometer)

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**“Lokaliseren van een magnetische dipool met behulp van een gradiometer
(Engelse titel: Localizing a magnetic dipole using a gradiometer) ”**

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Abstract

Magnetic anomaly detection (MAD) is a widely used method for detecting ferromagnetic targets, particularly hidden objects. A common way to localize a magnetic target is to look at the maximum of the magnitude of the magneticfield gradient. In this thesis, we consider a three-axis total field gradiometer which measures the gradient of the magnitude of the magnetic field along three orthogonal axes. Based on the measured data from the three-axis total field gradiometer, we decompose the inversion problem into two linear systems. By solving those two systems, we get an approximation for the location and moment that we are looking for. For the considered gradiometer, we can improve the parameters if the detailed geometry of the gradiometer is taken into account.

Acknowledgement

In the past couple of months, I have been working on this thesis as the final part of the Bachelor program of Applied Mathematics at the Delft University of Technology. I would like to my supervisors Aad Vijn (TUD) and Eugène Leplaars (TNO) for allowing me to work on this project and for their pleasant support throughout this project. Furthermore, I would like to thank Prof.dr.ir. A.W. Heemink and Dr.ir. M.B. van Gijzen for their time and willingness to be a committee member.

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Chapter 1

Introduction

1.1 Motivation

As everyone knows, the Earth can be seen as a big magnet. The electron currents due to the motion of molten iron and nickels in the Earth's core, generates a magnetic field. All objects on the Earth that are made of ferromagnetic materials, are magnetized by the Earth magnetic field. After being magnetized, such a ferromagnetic object causes a perturbation to the local Earth magnetic field which forms a ***magnetic anomaly***. The detection, localization and classification of magnetic objects based on the magnetic anomaly is called Magnetic Anomaly Detection (MAD). This method has been used for decades and it is particularly suitable for underwater detection of magnetic objects like sunken ships or containers.

There are two types of sensors that can be used for the localization and classification of a ferromagnetic object:

1. *magnetometers*: There are two basic types for the magnetometer measurements. The *vector magnetometer* measures all components of the magnetic field, while the *total field magnetometer* measures the magnitude of the magnetic field. In this project, we call the magnitude of the magnetic field as the *total magnetic field*.

2. *gradiometers*: the gradiometers are constructed by placing magnetometers along different axes and taking the differences of the measured magnetic field. Just as for the magnetometers, we also have two types of gradiometers: the *total field gradiometers* and the *tensor gradiometers*. The total field gradiometer measures the gradient of the total magnetic field while the tensor gradiometer measures the gradient of the magnetic vector field.

For this project we consider a gradiometer to localize and classify a magnetic object. A common way to localize a magnetic target is to look at the maximum of the magnitude of the magnetic field gradient. The research of Mersch et al [1] introduces a method of how to find the location and the moment for a ferromagnetic object with a small moment in a strong uniformed field by using a three-axis total field gradiometer. They have shown that for an ideal gradiometer, the inversion problem can be separated into two linear systems. Their research has also shown that this method gives accurate results for magnetic objects with moment of $5Am^2$, if the altitude of the gradiometer is between $6m - 10m$.

The aim of this thesis is to answer the question:

'Is this method also suitable for a ferromagnetic object with a larger moment,e.g. $10^3 Am^2$*? If yes, what is the range of the optimal sensor altitude?'*

1.2 Chapter Outlines

This section gives a description of how the remainder of this thesis is organised.

- **Chapter 2 The magnetic field of a magnetic dipole** In this chapter, we derive an expression for the magnetic field of a magnetic point dipole by using the Maxwell's equations.

- **Chapter 3 The forward problem** In this chapter we first describe how the three-axis total field gradiometer that we will use, is composed. Our model consists of two problems: the forward and inverse problem. We first formulate the forward problem and then We explain how we can solve the forward problem.

- **Chapter 4 The inverse problem** In this chapter we will use the definition and theorem of the Euler's homogeneous function to decompose the inverse problem into two linear systems. By solving those two linear systems we get an approximation for the position and moment of the magnetic object. To improve the accuracy of the estimated parameters we also introduce the second step: the nonlinear refinement.

- **Chapter 5 Numerical xperiment without noise** In this chapter we will test our model to see whether this model is accurate for localization and classification of a magnetic object in absence of noise. We will also check whether the altitude of the gradiometer is important for getting acceptable results without nonlinear refinement.

- **Chapter 6 Model with noise** In this chapter we introduce Gaussian noise in our model

- **Chapter 7 Numerical experiment with noise** In this chapter we test whether or not the model with noise still gives acceptable results.

- **Chapter 8 Conclusion and recommendations** In this final chapter we will conclude the results of all the chapters of this thesis. It also gives some idea about the further research with the method we have used.

Chapter 2

The magnetic field of a magnetic dipole

In this chapter, we derive an expression for the magnetic field of a magnetic point dipole by using the Maxwell's equations. This expression is essential for our further research due to the fact that we can treat a magnetic object as a magnetic point dipole when the sensor is located at distance at least three times the dimension of the object [2]. A magnetic point dipole is a point that contains a north pole and a south pole.

Suppose a ferromagnetic object is located at position $\mathbf{d}$ in meters and we detect the magnetic field around it with a sensor at position $\mathbf{r}$ also in meters, see Figure 2.1. Those two positions can be written as vectors in $\mathbb{R}^3$:

$$\mathbf{d} = \begin{bmatrix} d_x & d_y & d_z \end{bmatrix}^T \text{ and } \mathbf{r} = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix}^T.$$

Furthermore, we define $\hat{\mathbf{r}} = \mathbf{r} - \mathbf{d}$, which is the difference vector between $\mathbf{d}$ and $\mathbf{r}$. We assume that this magnetic object has a magnetic moment $\mathbf{m}$ with unit *Ampere square meters* $[Am^2]$ and we define it as:

$$\mathbf{m} = \begin{bmatrix} m_x & m_y & m_z \end{bmatrix}^T.$$

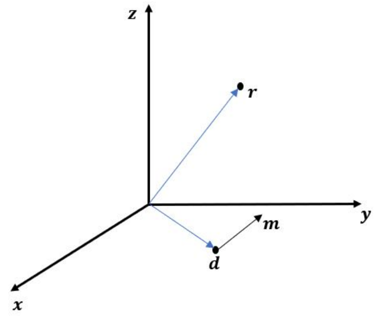

Figure 2.1: Location of sensor and ferromagnetic object.

There is a magnetic field around the magnetic object. We define the magnetic field strength $\mathbf{H}$ in $[A/m]$ and the magnetic induction field $\mathbf{B}$ in $[T]$ at location $\mathbf{r}$ as

$$\mathbf{H} = \mathbf{H}(\mathbf{r}) \text{ and } \mathbf{B} = \mathbf{B}(\mathbf{r}).$$

If **H** is known to us, then we can compute **B** with the following formula:

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{2.1}$$

with $\mu_0 = 4\pi \cdot 10^7 H/m$ is the permeability of free space.

In this chapter, we will first demonstrate equation (2.1) by using the Maxwell's equations and then we will shrink a three-dimensional rod to a magnetic point dipole so that we can find an expression for **H** while keeping the same magnetic moment. We will show that the magnetic field of a magnetic point dipole at location **d** is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} \left(\frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}}{\|\hat{\mathbf{r}}\|^5} - \frac{\mathbf{m}}{\|\hat{\mathbf{r}}\|^3} \right), \tag{2.2}$$

where $\| \cdot \|$ is the Euclidean norm.

2.1 Maxwell's equations

Maxwell's equations are four partial differential equations (PDE's) for describing electric and magnetic fields established by British physicist Maxwell in the 19th century. Those differential equations are shown in (2.3).

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{cases} \tag{2.3}$$

This set of coupled differential equations come from the following four laws:

1. **Gauss' law:** $\nabla \cdot \mathbf{D} = \rho$
   This law states that the electric flux through any closed surface is a measure of the total charge enclosed within the surface. It means that electrical charges produce electric fields. In the equation, **D** is the electric flux density measured in Coulombs per square meters $[C/m^2]$, ρ stands for the electric charge density in Coulombs per cubic meters $[C/m^3]$.

2. **Gauss' law for magnetism:** $\nabla \cdot \mathbf{B} = 0$
   This law states that magnetic monopoles do not exist. **B** is the magnetic induction field just as mentioned before.

3. **Faraday's law:** $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
   This law states that changing magnetic fields induce electric fields. **E** in this equation stands for the intensity of the electric field which is measured in volts per meter $[V/m]$.

4. **Ampère's law:** $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
   This law states that currents across a surface and changing electric fields produce magnetic fields. In this equation, **J** is the electric current density in ampere per squared meter $[A/m^2]$ and **H** is the magnetic field strength just as mentioned before.

When we assume that the electric and magnetic fields do not change in time, then those fields are static. It means that we can split this set of coupled equations in (2.3) into two sets of equations that describe the magnetic field and the electric field separately. Moreover, we assume that there are no currents across the surface which means that $\mathbf{J} \equiv \mathbf{0}$. Then the static magnetic field can be described as:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{0} \end{cases} .$$

2.1.1 The magnetic scalar potential

In this subsection we derive an expression for the magnetic scalar potential that we will use to obtain the magnetic induction field $\mathbf{B}$. The magnetic induction field $\mathbf{B}$ is produced by some magnetization $\mathbf{M} = [M_x\ M_y\ M_z]^T$ in a magnetic object, with $M_x, M_y, M_z \in \mathbb{R}$. Magnetization gives the density of the induced magnetic dipole moment in a magnetic object and its unit is ampere per meter $[A/m]$. So by definition, if $\mathbf{M}$ is known to us, then we can easily calculate the magnetic moment by integrate $\mathbf{M}$ over the volume of the magnetic object

$$\mathbf{m} = \iiint \mathbf{M}\ dV. \tag{2.4}$$

Beside the magnetic moment, we can also use $\mathbf{M}$ to calculate $\mathbf{B}$, since the relation between $\mathbf{M}, \mathbf{B}$ and $\mathbf{H}$ is given by

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{2.5}$$

where $\mu_0 = 4\pi \cdot 10^{-7}\ H/m$ is the magnetic permeability of free space. Outside the magnetic object, the magnetization $\mathbf{M} = \mathbf{0}$. So equation (2.5) becomes $\mathbf{B} = \mu_0\mathbf{H}$, which shows equation (2.1). We want to determine the magnetic induction field $\mathbf{B}$ that is caused by a distribution of $\mathbf{M}$, so we have to solve the following set of PDE's:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 & (2.6a) \\ \nabla \times \mathbf{H} = 0 & (2.6b) \\ \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). & (2.6c) \end{cases}$$

If we substitute (2.6c) into (2.6a), we get $\nabla \cdot (\mu_0(\mathbf{H} + \mathbf{M})) = 0$ and we can rewrite it as $\mu_0(\nabla \cdot \mathbf{H} + \nabla \cdot \mathbf{M}) = 0$. It gives us the following equation:

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}. \tag{2.7}$$

Since $\nabla \times \mathbf{H} = \mathbf{0}$, there exists a magnetic scalar potential φ such that $\mathbf{H} = -\nabla\varphi$. If we substitute it into (2.7), we get

$$\nabla \cdot (-\nabla\varphi) = -\nabla^2\varphi = -\Delta\varphi = -\nabla \cdot \mathbf{M}.$$

Or equivalently,

$$\Delta\varphi = \nabla \cdot \mathbf{M}. \tag{2.8}$$

We can interpret $\nabla \cdot \mathbf{M}$ as the source of $\mathbf{B}$.

An equations of the form,

$$\Delta u = f$$

is called a Poisson equation. We can solve this equation by using Green's function for the Laplace equation in $\mathbb{R}^3$ and it is given by [3]:

$$u(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{-1}{4\pi\|\mathbf{r} - \mathbf{r}'\|} f(\mathbf{r}')\ d\mathbf{r}', \text{with } \mathbf{r}, \mathbf{r}' \in \mathbb{R}^3. \tag{2.9}$$

Note that equation(2.8) is a Poisson equation with $f = \nabla \cdot \mathbf{M}$. By using equation(2.9) we get the magnetic scalar potential

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \nabla' \cdot \mathbf{M}(\mathbf{r}')\ d\mathbf{r}', \text{ with } \nabla' = \left[\frac{\partial}{\partial r_1'}\ \frac{\partial}{\partial r_2'}\ \frac{\partial}{\partial r_3'}\right]^T. \tag{2.10}$$

We will use this expression of φ later to obtain $\mathbf{B}$ of a magnetic point dipole.

2.2 Step-wise derivation of the magnetic induction field

In this section, we shrink a three-dimensional rod step-wise to a magnetic point dipole while keeping the magnetic moment fixed that is caused by the magnetization of the rod. Then we use equation (2.10) to obtain $\mathbf{B}$.

2.2.1 A three-dimensional rod

Suppose we have a rod with length $2l$ and radius a and we use a cylindrical coordinate system (x, ρ, ϕ) with the center point of the rod as origin, see Figure 2.2. In addition, we assume that there exists magnetization $\mathbf{M}$ within the rod only in the x-direction, so $\mathbf{M} = M_x \mathbf{e}_x$ with $\mathbf{e}_x = [1\ 0\ 0]^T$.

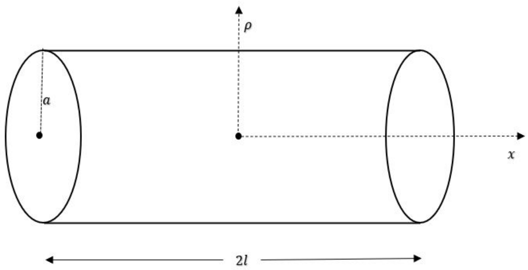

Figure 2.2: Rod with magnetization in x-direction.

As mentioned before, $\mathbf{M} = \mathbf{0}$ outside the rod. So $\mathbf{M} = M_x \mathbf{e}_x$, for $-l < x < l$ and $\rho < a$. By using the Heaviside step function H, we can denote $\mathbf{M}$ as

$$\mathbf{M} = M_x \mathbf{e}_x [H(x+l) - H(x-l)] H(a-\rho).$$

The magnetic moment $\mathbf{m}$ can now be calculated by using equation (2.4) and note $\phi \in [0, 2\pi]$, it gives us that $\mathbf{m}$ is

$$\begin{aligned}
\mathbf{m} = \iiint \mathbf{M}\ dV &= \int_0^{2\pi} \int_0^a \int_{-l}^l M_x \mathbf{e}_x [H(x+l) - H(x-l)] H(a-\rho)\ \rho\ dx\ d\rho\ d\phi \\
&= M_x \mathbf{e}_x \int_{-l}^l H(x+l) - H(x-l)\ dx \cdot \int_0^a H(a-\rho)\rho\ d\rho \cdot \int_0^{2\pi} d\phi \\
&= M_x \mathbf{e}_x \cdot 2l \cdot \left[\frac{1}{2}\rho^2\right]_0^a \cdot 2\pi = 2\pi a^2 l M_x \mathbf{e}_x.
\end{aligned}$$

By letting $m_0 = 2\pi a^2 l M_x$, we get $\mathbf{m} = m_0 \mathbf{e}_x$.

2.2.2 Shrink rod to a line

Now we want to shrink this rod to a line while keeping the same magnetic moment. We let the radius a go to zero, so the magnetization $\mathbf{M}$ of this line is

$$\mathbf{M} = c\ \mathbf{e}_x [H(x+l) - H(x-l)] \delta(y) \delta(z),$$

where c is a constant that we have to compute such that the magnetic moment $\mathbf{m}$ remains the same and δ the Dirac delta function with $\int_{-\infty}^{\infty} \delta(x)\ dx = 1$. Just as for the rod, $\mathbf{m}$ of the line can be determined as follows:

$$\begin{aligned}\mathbf{m} = \iiint_{V_\infty} \mathbf{M}\ dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c\cdot\mathbf{e}_x[H(x+l)-H(x-l)]\delta(y)\delta(z)\ \ dx\ dy\ dz \\ &= c\cdot\mathbf{e}_x\int_{-l}^{l} H(x+l)-H(x-l)\ dx \\ &= c\cdot\mathbf{e}_x\cdot 2l.\end{aligned}$$

We must find a constant c such that $c\cdot\mathbf{e}_x\cdot 2l = m_0\mathbf{e}_x$ which gives us that $c = \frac{m_0}{2l}$. If we substitute c in $\mathbf{M}$, we get

$$\mathbf{M} = \frac{m_0}{2l}\mathbf{e}_x[H(x+l)-H(x-l)]\delta(y)\delta(z). \tag{2.11}$$

Recall the expression of the scalar potential function (2.10). We can use it to determine the potential of the rod that we have shrunk to a line. We let $\mathbf{r} = [x\ y\ z]^T$ and $\mathbf{r}' = [x'\ y'\ z']^T$, together with (2.11), we get

$$\begin{aligned}\nabla'\cdot\mathbf{M}(\mathbf{r}') &= \frac{\partial}{\partial x'}\left[\frac{m_0}{2l}[H(x'+l)-H(x'-l)]\delta(y')\delta(z')\right] \\ &= \frac{m_0}{2l}[\delta(x'+l)-\delta(x'-l)]\delta(y')\delta(z')\end{aligned} \tag{2.12}$$

and

$$\|\mathbf{r}-\mathbf{r}'\| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}. \tag{2.13}$$

Then the scalar potential function is

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\frac{m_0}{2l}[\delta(x'+l)-\delta(x'-l)]\delta(y')\delta(z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\ dx'\ dy'\ dz'. \tag{2.14}$$

The Dirac delta function has the following useful properties that we can use to simplify equation (2.14):

1. $\int_{-\infty}^{\infty} g(x)\delta(x)\ dx = g(0)$ 2. $\int_{-\infty}^{\infty} g(x)\delta(x-t)\ dx = g(t)$.

Let $g(z') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$ and use property 1, then equation (2.14) becomes

$$\begin{aligned}\varphi(\mathbf{r}) &= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{m_0}{2l}[\delta(x'+l)-\delta(x'-l)]\delta(y')g(0)\ dx'\ dy' \\ &= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\frac{m_0}{2l}[\delta(x'+l)-\delta(x'-l)]\delta(y')}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\ dx'\ dy'.\end{aligned}$$

Repeating this step for y', we get

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\frac{m_0}{2l}[\delta(x'+l)-\delta(x'-l)]}{\sqrt{(x-x')^2+y^2+z^2}}\ dx'.$$

Now we can use the second property of the Dirac delta function to obtain the following simplified expression for the scalar potential function for the shrunk rod:

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi}\cdot\frac{m_0}{2l}\left[\frac{1}{\sqrt{(x+l)^2+y^2+z^2}}-\frac{1}{\sqrt{(x-l)^2+y^2+z^2}}\right]. \tag{2.15}$$

2.2.3 Shrink a line to a magnetic point dipole

If we let $l \to 0$ in equation (2.15), we obtain the potential function for the magnetic point dipole. Since $\mathbf{H} = -\nabla\varphi$ and $\mathbf{B} = \mu_0\mathbf{H}$, we can then derive an expression for the magnetic field with the magnetic point dipole located at the origin. To do this, we use the definition of the central difference of a function $f(x)$:

$$f'(x) = \lim_{l\to 0} \frac{f(x+l) - f(x-l)}{2l}.$$

Recall $\mathbf{m} = m_0\mathbf{e}_x$, so $m_x = m_0$. Let $f(x) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, then we can write equation (2.15) as

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi} \cdot m_x \frac{f(x+l) - f(x-l)}{2l}. \tag{2.16}$$

By letting $l \to 0$ and using the central difference formula, we get the scalar potential function

$$\begin{aligned}\varphi(\mathbf{r}) &= -\frac{1}{4\pi} \cdot m_x \cdot \lim_{l\to 0} \frac{f(x+l) - f(x-l)}{2l} \\ &= -\frac{1}{4\pi} \cdot m_x \cdot f'(x) = -\frac{1}{4\pi} \cdot m_x \cdot -x(x^2+y^2+z^2)^{-\frac{3}{2}} \\ &= \frac{1}{4\pi} \cdot m_x \cdot \frac{x}{\|\mathbf{r}\|^3}. \end{aligned} \tag{2.17}$$

Note that we can interpret $m_x \cdot x$ in equation (2.17) as $\mathbf{m} \cdot \mathbf{r}$. So we can generalize this scalar potential function for a magnetic point dipole with $\mathbf{m} = [m_x\ m_y\ m_z]^T$ as :

$$\varphi(\mathbf{r}) = \frac{1}{4\pi} \cdot \frac{\mathbf{m} \cdot \mathbf{r}}{\|\mathbf{r}\|^3}. \tag{2.18}$$

With this equation, we can find

$$\mathbf{H} = -\nabla\varphi = -\nabla\left(\frac{1}{4\pi} \cdot \frac{\mathbf{m} \cdot \mathbf{r}}{\|\mathbf{r}\|^3}\right) = -\frac{1}{4\pi} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \frac{m_x x + m_y y + m_z z}{(x^2+y^2+z^2)^{\frac{3}{2}}}. \tag{2.19}$$

Let $g(x,y,z) = \frac{m_x x + m_y y + m_z z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, then we can write equation (2.19) as

$$\mathbf{H} = -\frac{1}{4\pi} \begin{bmatrix} \frac{\partial g(x,y,z)}{\partial x} \\ \frac{\partial g(x,y,z)}{\partial y} \\ \frac{\partial g(x,y,z)}{\partial z} \end{bmatrix}$$

with

$$\frac{\partial g(x,y,z)}{\partial x} = \frac{m_x}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3x(m_x x + m_y y + m_z z)}{(x^2+y^2+z^2)^{\frac{5}{2}}} = \frac{m_x}{\|\mathbf{r}\|^3} - \frac{3x(\mathbf{m}\cdot\mathbf{r})}{\|\mathbf{r}\|^5}$$

$$\frac{\partial g(x,y,z)}{\partial y} = \frac{m_y}{\|\mathbf{r}\|^3} - \frac{3y(\mathbf{m}\cdot\mathbf{r})}{\|\mathbf{r}\|^5}, \quad \frac{\partial g(x,y,z)}{\partial z} = \frac{m_z}{\|\mathbf{r}\|^3} - \frac{3z(\mathbf{m}\cdot\mathbf{r})}{\|\mathbf{r}\|^5}.$$

Then we obtain

$$\mathbf{H} = \frac{1}{4\pi}\left(\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{\|\mathbf{r}\|^5} - \frac{\mathbf{m}}{\|\mathbf{r}\|^3}\right). \tag{2.20}$$

With $\mathbf{B} = \mu_0 \mathbf{H}$, we get an expression for the magnetic field of a magnetic point dipole which is located at the origin,

$$\mathbf{B} = \frac{\mu_0}{4\pi} \left(\frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}}{\|\mathbf{r}\|^5} - \frac{\mathbf{m}}{\|\mathbf{r}\|^3} \right). \tag{2.21}$$

If we generalize equation (2.21) for a magnetic dipole point at location $\mathbf{d}$ with $\hat{\mathbf{r}} = \mathbf{r} - \mathbf{d}$, then the magnetic field is given by:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \left(\frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}}{\|\hat{\mathbf{r}}\|^5} - \frac{\mathbf{m}}{\|\hat{\mathbf{r}}\|^3} \right). \tag{2.22}$$

Chapter 3

The forward problem

In the previous section we have derived an expression for the magnetic field of a magnetic dipole which is essential to achieve our goal: find the location and the moment of a magnetic object based on the measured data of a gradiometer. In this chapter, we are going to construct the prediction model to achieve our goal. This model consists of two parts: the forward problem and the inverse problem. By solving the forward problem we obtain the data that we will use to solve the inverse problem. Before we start with the formulation of the forward problem, we first explain how the gradiometer that we use in this project, is composed.

3.1 Three-axis total field gradiometer

As mentioned in the introduction, in this project we consider a three-axes total field gradiometer that measures all three components of the gradient of the total magnetic field. This gradiometer is composed of four total-field magnetometers separated from each other along three orthogonal axes., see Figure 3.1. We call the red point in Figure 3.1 as the center point of the gradiometer. When we are talking about the position of the gradiometer, what we mean by this, is actually the position of this center point.

Furthermore, we call the magnetometer that is located along the positive x-axis ***star*** and the one which is located along the negative x-axis, ***port***. The distance between those two magnetometers is defined as $L_{trans} = 0.75$ in meters. The other two magnetometers, ***aft*** and ***down***, are located along the negative y-axis and z-axis respectively. Moreover, we define $L_{long} = 0.55$ and $L_{vert} = 0.25$ in meters as the distance between ***aft*** and ***down*** with the center point of our gradiometer, respectively.

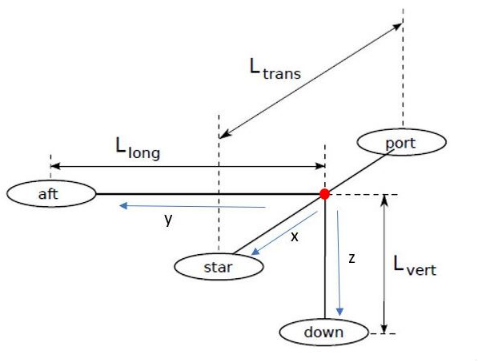

Figure 3.1: Three-axis total field gradiometer.

So if we know that the position of the gradiometer is $\mathbf{r}$, then the position of the four magnetometers are defined as follow:

$$\mathbf{r}_{star} = \mathbf{r} + \frac{1}{2}L_{trans}\mathbf{e}_x, \quad \mathbf{r}_{port} = \mathbf{r} + \frac{-1}{2}L_{trans}\mathbf{e}_x, \quad \mathbf{r}_{aft} = \mathbf{r} - L_{long}\mathbf{e}_y, \quad \mathbf{r}_{aft} = \mathbf{r} - L_{vert}\mathbf{e}_z, \tag{3.1}$$

with $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ the standard basis of $\mathbb{R}^3$. Keep in mind that the magnetometers do not just measure the magnetic anomaly caused by the magnetic object, they measure the total field which has two contributions: the Earth magnetic field $\mathbf{B}_e$ and the magnetic field $\mathbf{B}_d$ due to the magnetic dipole which can be computed with equation (2.22). For example, the data that we receive from the magnetometer at location $\mathbf{r}_{star}$ is:

$$B_{star} = \|\mathbf{B}_{star}\| = \|\mathbf{B}_e + \mathbf{B}_d(\hat{\mathbf{r}}_{star})\|, \text{ with } \hat{\mathbf{r}}_{\mathbf{star}} = \mathbf{r}_{\mathbf{star}} - \mathbf{d}. \tag{3.2}$$

In addition, we denote $B_{port} = \|\mathbf{B}_{\mathbf{port}}\|$, $B_{aft} = \|\mathbf{B}_{\mathbf{aft}}\|$ and $B_{down} = \|\mathbf{B}_{\mathbf{down}}\|$ as the total field that we measure with the magnetometers port, aft, and down. If we combine all of the measurements taken by the four magnetometers, we will get the the following approximation for the gradient of the total magnetic field at location $\mathbf{r}$:

$$\tilde{\mathbf{G}} = \begin{bmatrix} \frac{B_{star}-B_{port}}{L_{trans}} \\ \frac{(B_{star}+B_{port})-2B_{aft}}{2L_{long}} \\ \frac{(B_{star}+B_{port})-2B_{down}}{2L_{vert}} \end{bmatrix}. \tag{3.3}$$

3.2 The forward problem

If we look at the gradiometer that we have constructed, we want to calculate the gradients of the total magnetic field for the locations of the gradiometer during the measurement. From equation (3.2) and (3.3) we can formulate the forward problem.

The forward problem. *Given the Earth magnetic field* $\mathbf{B}_e$ *and a magnetic dipole with moment* $\mathbf{m}$ *which is located at* $\mathbf{d}$, *calculate the gradient of the total magnetic field at each location* $\mathbf{r}_\alpha$ *during the measurement. Suppose we have taken* N *measurements, we want to calculate* $\tilde{\mathbf{G}}_\alpha$ *defined as in equation (3.3) with location of the gradiometer at* $\mathbf{r}_\alpha$, $\alpha = 1, 2, ..., N$.

It is clear that for solving the forward problem we need to know what the Earth magnetic field $\mathbf{B}_e$ is. We assume that $\mathbf{B}_e$ is uniform. Furthermore, the strength of the Earth's magnetic background field ranges between 25 and 65 μT. In this project, we assume that:

$$\mathbf{B}_e = \begin{bmatrix} 19 \\ 1 \\ 50 \end{bmatrix}, \text{in } \mu T \tag{3.4}$$

is always known to us.

If we know the sensor location during the measurement and we also know the position and moment of the magnetic dipole, then we can calculate $\tilde{\mathbf{G}}_\alpha$ in three steps:

Step 1. If we know the position of the gradiometer, then we can compute the positions of the four magnetometers by using the equations in (3.1). We substitute the position of the magnetometers, the position and the moment of the magnetic object in equation (2.22), so that we can obtain $\mathbf{B}_d$ for each magnetometer.

Step 2. We use the calculated $\mathbf{B}_d$ to compute the total fields that are measured by magnetometers, like the example shown in equation (3.2).

Step 3. By substituting the computed total field from the magnetometers in equation(3.3), we will get the gradient of the α-th measurement.

By repeating these three steps for each measurement, the forward problem is executed. The solution of this problem will be used as input data for the inverse problem which we will explain in the next chapter.

Chapter 4

The inverse problem

The inverse problem is to find the location and the moment of the magnetic dipole based on the measured data. In this chapter we will show how to solve this problem by decomposing it into two linear systems. For this step we use the property of the Euler's homogeneous equation. To improve the accuracy of the solution, we also introduce second step with the solutions of the two linear systems as the initial value.

4.1 Euler's homogeneous equation

Definition. ***Euler's homogeneous equation*[4].** *Let $V \subseteq \mathbb{R}^n$ and f: $V \to \mathbb{R}$ is* ***homogeneous of degree d*** *if for all $\lambda > 0$ and $\mathbf{r}, \lambda\mathbf{r} \in V$,*

$$f(\lambda\mathbf{r}) = \lambda^d f(\mathbf{r}).$$

Recall equation (2.22) and let $\lambda > 0$,

$$\mathbf{B}(\lambda\hat{\mathbf{r}}) = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\lambda\hat{\mathbf{r}})\lambda\hat{\mathbf{r}}}{\|\lambda\hat{\mathbf{r}}\|^5} - \frac{\mathbf{m}}{\|\lambda\hat{\mathbf{r}}\|^3}\right) = \frac{\mu_0}{4\pi}\left(\frac{3\lambda^2(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}}{\lambda^5\|\hat{\mathbf{r}}\|^5} - \frac{\mathbf{m}}{\lambda^3\|\hat{\mathbf{r}}\|^3}\right) = \lambda^{-3}\frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}}{\|\hat{\mathbf{r}}\|^5} - \frac{\mathbf{m}}{\|\hat{\mathbf{r}}\|^3}\right) = \lambda^{-3}\mathbf{B}(\hat{\mathbf{r}}).$$

So we can conclude that our function for the magnetic field is homogeneous of degree $d = -3$. Such a positive homogeneous function has a very useful theorem that we will use in the next section to find the position and the moment of our point dipole.

Euler's homogeneous function theorem. *Let $V \subseteq \mathbb{R}^n$ and f: $V \to \mathbb{R}$ is homogeneous of degree d with $f \in C^1$, then $\mathbf{r} \cdot \nabla f(\mathbf{r}) = d \cdot f(\mathbf{r})$.*

Proof. f is homogeneous of degree d, so we have $f(\lambda\mathbf{r}) = \lambda^d f(\mathbf{r})$ for all $\lambda > 0$. If we differentiate $f(\lambda\mathbf{r})$ and $\lambda^d f(\mathbf{r})$ respect to λ using the chain rule, we get $\frac{df(\lambda\mathbf{r})}{d\lambda} = \mathbf{r} \cdot \nabla f(\lambda\mathbf{r})$ and $\lambda^d f(\mathbf{r}) = d \cdot \lambda^{d-1} f(\mathbf{r})$. It gives us that $\mathbf{r} \cdot \nabla f(\lambda\mathbf{r}) = d \cdot \lambda^{d-1} f(\mathbf{r})$. Assume, without the lose of generality, that $\lambda = 1$, we get $\mathbf{r} \cdot \nabla f(\mathbf{r}) = d \cdot f(\mathbf{r})$. □

4.2 Determining position and moment

4.2.1 Expression of magnetic anomaly

Since we will use a three-axes total field gradiometer, we can find a very useful expression for the magnetic anomaly caused by our point dipole. Let $B_T = \|\mathbf{B}_T\|$ be the total magnetic field that we detect at location $\mathbf{r}$, with $\mathbf{B}_T = \mathbf{B}_d + \mathbf{B}_e$. Recall that $\mathbf{B}_d$ is magnetic induction field caused by the magnetic dipole and $\mathbf{B}_e$ is the Earth magnetic field. Moreover, we define the magnetic anomaly T as $T = \|\mathbf{B}_T\| - \|\mathbf{B}_e\|$.

Note that

$$\mathbf{B}_T = \mathbf{B}_e + \mathbf{B}_d \Rightarrow \ \|\mathbf{B}_T\|^2 = (\mathbf{B}_e + \mathbf{B}_d) \cdot (\mathbf{B}_e + \mathbf{B}_d) = \|\mathbf{B}_e\|^2 + \|\mathbf{B}_d\|^2 + 2\mathbf{B}_e \cdot \mathbf{B}_d.$$

The magnetic field due to the magnetic point dipole is very small in comparison with the Earth magnetic field, so $\|\mathbf{B}_d\| \ll \|\mathbf{B}_e\|$. It means

$$\|\mathbf{B}_T\|^2 \cong \|\mathbf{B}_e\|^2 + 2\mathbf{B}_e \cdot \mathbf{B}_d \Rightarrow \|\mathbf{B}_T\| \cong \sqrt{\|\mathbf{B}_e\|^2 + 2\mathbf{B}_e \cdot \mathbf{B}_d} = B_e\sqrt{1 + \tfrac{2\mathbf{B}_e \cdot \mathbf{B}_d}{\|\mathbf{B}_e\|^2}}.$$

Furthermore, $\beta = \frac{\mathbf{B}_e \cdot \mathbf{B}_d}{\|\mathbf{B}_e\|^2} \ll 1$ and by using the Taylor expansion we find $\sqrt{1+2\beta} \cong 1 + \beta$ for $\beta \ll 1$. So if we denote $B_e = \|\mathbf{B}_e\|$, then we get

$$B_e + T = \|\mathbf{B}_T\| \cong B_e(1 + \tfrac{\mathbf{B}_e \cdot \mathbf{B}_d}{B_e^2}) = B_e + \tfrac{\mathbf{B}_e \cdot \mathbf{B}_d}{B_e}.$$

It gives us

$$T \cong \frac{\mathbf{B}_e \cdot \mathbf{B}_d}{B_e} = \mathbf{B}_d \cdot \hat{\mathbf{B}}_e, \text{with } \hat{\mathbf{B}}_e = \frac{\mathbf{B}_e}{B_e}. \tag{4.1}$$

4.2.2 First linear system for finding the position

Since $\mathbf{B}(\hat{\mathbf{r}})$ is homogeneous of degree $d = -3$, it means that $T = \mathbf{B}_d \cdot \hat{\mathbf{B}}_e$ is also homogeneous of degree $d = -3$. By using Euler's homogeneous function theorem, we have $\hat{\mathbf{r}} \cdot \nabla T = -3T$. We have defined the magnetic anomaly as $T = \|\mathbf{B}_T\| - \|\mathbf{B}_e\|$, due to the fact that $\|\mathbf{B}_e\|$ is not dependent on $\hat{\mathbf{r}}$, we have $\nabla T = \nabla\|\mathbf{B}_T\|$. So from Euler's homogeneous function theorem we get the following expression

$$\hat{\mathbf{r}} \cdot \nabla\|\mathbf{B}_T\| = -3(\|\mathbf{B}_T\| - \|\mathbf{B}_e\|), \text{ with } \nabla\|\mathbf{B}_T\| = \begin{bmatrix} \frac{\partial\|\mathbf{B}_T\|}{\partial \hat{r}_1} \\ \frac{\partial\|\mathbf{B}_T\|}{\partial \hat{r}_2} \\ \frac{\partial\|\mathbf{B}_T\|}{\partial \hat{r}_3} \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} = \mathbf{G}. \tag{4.2}$$

By using a three-axes total-field gradiometer which measures the gradients of the total magnetic field, we can find the position of the magnetic point dipole. We denote the measured gradients as $\tilde{G}_i$, for $i = 1, 2, 3$. Since $\hat{\mathbf{r}} = \mathbf{r} - \mathbf{d}$ and we have a measured gradient that approximates the true gradient, we rewrite equation(4.2) as

$$(\mathbf{r} - \mathbf{d}) \cdot \underbrace{\begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ \tilde{G}_3 \end{bmatrix}}_{\tilde{\mathbf{G}}} = 3\|\mathbf{B}_e\| - 3\|\mathbf{B}_T\|.$$

The expression above is equivalent with $\mathbf{r} \cdot \tilde{\mathbf{G}} - \mathbf{d} \cdot \tilde{\mathbf{G}} = 3\|\mathbf{B}_e\| - 3\|\mathbf{B}_T\|$. Since the location of the gradiometer $\mathbf{r}$, the measured gradient $\tilde{\mathbf{G}}$ and the total magnetic field $\|\mathbf{B}_T\|$ at location $\mathbf{r}$ and the Earth magnetic field $\mathbf{B}_e$ are known to us, we can rewrite equation above as:

$$\tilde{\mathbf{G}}^T\mathbf{d} = \mathbf{r} \cdot \tilde{\mathbf{G}} + 3\|\mathbf{B}_T\| - 3\|\mathbf{B}_e\|.$$

The situation above describes when we just take one measurement. If we take N measurements during the survey, then we will get a linear system that can be written in matrix form, $X\mathbf{d}=\mathbf{Y}$,

$$\underbrace{\begin{bmatrix} \tilde{G}_{1,1} & \tilde{G}_{2,1} & \tilde{G}_{3,1} \\ \tilde{G}_{1,2} & \tilde{G}_{2,2} & \tilde{G}_{3,2} \\ \tilde{G}_{1,3} & \tilde{G}_{2,3} & \tilde{G}_{3,3} \\ \vdots & \vdots & \vdots \\ \tilde{G}_{1,N} & \tilde{G}_{2,N} & \tilde{G}_{3,N} \end{bmatrix}}_{X} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} \mathbf{r_1} \cdot \tilde{\mathbf{G}} + 3\|\mathbf{B}_{T,1}\| - 3\|\mathbf{B}_e\| \\ \mathbf{r_2} \cdot \tilde{\mathbf{G}} + 3\|\mathbf{B}_{T,2}\| - 3\|\mathbf{B}_e\| \\ \mathbf{r_3} \cdot \tilde{\mathbf{G}} + 3\|\mathbf{B}_{T,3}\| - 3\|\mathbf{B}_e\| \\ \vdots \\ \mathbf{r_N} \cdot \tilde{\mathbf{G}} + 3\|\mathbf{B}_{T,N}\| - 3\|\mathbf{B}_e\| \end{bmatrix} = \mathbf{Y}. \tag{4.3}$$

For $\alpha = 1, 2, ..., N$, we denote $\mathbf{r}_\alpha$ as the location of the gradiometer and $\mathbf{B}_{T,\alpha}$ the total magnetic field measured at the α-th measurement and $\tilde{G}_{i,\alpha}$ as the i-th component of the measured gradient of the α-th measurement. A linear system is called **overdetermined** if there are more equations than unknowns. In our case we have an overdetermined linear system when $N > 4$. We solve this equation by using the method of least squares.

The method of least squares. *Given a linear system $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix with $m > n$, $\mathbf{x}$ is an $n \times 1$ vector and $\mathbf{b}$ an $m \times 1$ vector. In most cases this linear system does not have a solution. However, we can find an approximation solution $\tilde{\mathbf{x}}$ such that $\|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|$ is minimized and $\tilde{\mathbf{x}}$ is given by*

$$\tilde{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}.$$

By using the method of least squares for equation (4.3), we get $\mathbf{d} = (X^T X)^{-1} X^T \mathbf{Y}$. Now that we have found the position, we can use it to construct the second linear system to find the moment of the point dipole.

4.2.3 Second linear system for finding the moment

Since we have found the position in the previous subsection, we can derive the second linear system from $\mathbf{G} = \tilde{\mathbf{G}}$ which is only dependent on the moment $\mathbf{m}$.
We have defined that $G_i = \frac{\partial \|\mathbf{B}_T\|}{\partial \hat{r}_i}, i = 1, 2, 3$ in equation (4.2) . Furthermore, $\|\mathbf{B}_T\| = \|\mathbf{B}_e\| + T$, it means that $G_i = \frac{\partial T}{\partial \hat{r}_i}$ since $\|\mathbf{B}_e\|$ is not dependent on $\mathbf{r}$. In the previous section we have derived that $T = \mathbf{B}_d \cdot \hat{\mathbf{B}}_e$. We know that $\mathbf{B}_d$ can be obtained by using equation(2.22), the we find

$$T = \mathbf{B}_d \cdot \hat{\mathbf{B}}_e = \frac{\mu_0}{4\pi} \left[\underbrace{\frac{3[(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}] \cdot \hat{\mathbf{B}}_e}{\|\hat{\mathbf{r}}\|^5}}_{a} - \underbrace{\frac{\mathbf{m} \cdot \hat{\mathbf{B}}_\mathbf{e}}{\|\hat{\mathbf{r}}\|^3}}_{b} \right], \text{ with } \hat{\mathbf{r}} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \text{ and } \hat{\mathbf{B}}_e = \begin{bmatrix} \hat{B}_{e,x} \\ \hat{B}_{e,y} \\ \hat{B}_{e,z} \end{bmatrix}.$$

It gives us that G_i can be calculated as

$$G_i = \frac{\partial T}{\partial \hat{r}_i} = \frac{\mu_0}{4\pi} \left(\frac{\partial a}{\partial \hat{r}_i} - \frac{\partial b}{\partial \hat{r}_i} \right), \tag{4.4}$$

with

$$\frac{\partial a}{\partial \hat{r}_i} = \frac{\partial}{\partial \hat{r}_i} \frac{3[(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}] \cdot \hat{\mathbf{B}}_e}{\|\hat{\mathbf{r}}\|^5} = \frac{3\hat{B}_{e,i}(\mathbf{m} \cdot \hat{\mathbf{r}}) + 3m_i(\hat{\mathbf{B}}_\mathbf{e} \cdot \hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^5} - 15\hat{r}_i \frac{\mathbf{m} \cdot [(\hat{\mathbf{B}}_\mathbf{e} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}]}{\|\hat{\mathbf{r}}\|^\mathbf{7}} \tag{4.5a}$$

$$\frac{\partial b}{\partial \hat{r}_i} = \frac{\partial}{\partial \hat{r}_i} \frac{\mathbf{m} \cdot \hat{\mathbf{B}}_\mathbf{e}}{\|\hat{\mathbf{r}}\|^3} = -3\hat{r}_i \frac{\mathbf{m} \cdot \hat{\mathbf{B}}_\mathbf{e}}{\|\hat{\mathbf{r}}\|^5}. \tag{4.5b}$$

The detailed derivation of equation (4.5a) and equation (4.5b) can be found in Appendix A.

If we substitute equation (4.5a) and equation (4.5b) into equation (4.4), it gives us that

$$\frac{\partial T}{\partial \hat{r}_i} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{B}_{e,i}(\mathbf{m}\cdot\hat{\mathbf{r}}) + 3m_i(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}) + 3\hat{r}_i(\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e})}{\|\hat{\mathbf{r}}\|^5} - 15\hat{r}_i\frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{\|\hat{\mathbf{r}}\|^\mathbf{7}}\right] \tag{4.6}$$

So the derivative of T with respect to $\hat{r}_1 = \hat{x}$ is

$$\frac{\partial T}{\partial \hat{x}} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{B}_{e,x}(\mathbf{m}\cdot\hat{\mathbf{r}}) + 3m_x(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}) + 3\hat{x}(\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e})}{\|\hat{\mathbf{r}}\|^5} - 15\hat{x}\frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{\|\hat{\mathbf{r}}\|^\mathbf{7}}\right] \tag{4.7}$$

We know that

$$\hat{B}_{e,x}(\mathbf{m}\cdot\hat{\mathbf{r}}) = \hat{B}_{e,x}(m_x\hat{x} + m_y\hat{y} + m_z\hat{z}) \text{ and } \quad \hat{x}(\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e}) = \hat{x}(m_x\hat{B}_{e,x} + m_y\hat{B}_{e,y} + m_z\hat{B}_{e,z}).$$

Now if we let $f_1 = \frac{3\hat{B}_{e,x}(\mathbf{m}\cdot\hat{\mathbf{r}}) + 3m_x(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}) + 3\hat{x}(\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e})}{\|\hat{\mathbf{r}}\|^\mathbf{5}}$, then we can rewrite f_1 as

$$f_1 = \frac{3m_x(\hat{B}_{e,x}\hat{x} + (\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}) + \hat{B}_{e,x}\hat{x}) + 3m_y(\hat{B}_{e,x}\hat{y} + \hat{x}\hat{B}_{e,y}) + 3m_z(\hat{B}_{e,x}\hat{z} + \hat{x}\hat{B}_{e,z})}{\|\hat{\mathbf{r}}\|^\mathbf{5}}. \tag{4.8}$$

Furthermore, let $f_2 = \frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\hat{x}}{\|\hat{\mathbf{r}}\|^\mathbf{7}}$ then we can write f_2 as:

$$f_2 = \frac{m_x(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}^2 + m_y(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}\hat{y} + m_z(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}\hat{z}}{\|\hat{\mathbf{r}}\|^\mathbf{7}}. \tag{4.9}$$

Since in both equations (4.8) and (4.9) we have terms that are linear in m_x, m_y or m_z, we can also rewrite $\frac{\partial T}{\partial \hat{x}}$ as the following summation:

$$\frac{\partial T}{\partial \hat{x}} = \frac{\mu_0}{4\pi}(f_1 - 15f_2) = \frac{\mu_0}{4\pi}(m_x\cdot t_1 + m_y\cdot t_2 + m_z\cdot t_3), \tag{4.10}$$

where

$$t_1 = \frac{6\hat{B}_{e,x}\hat{x} + (\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^\mathbf{5}} - 15\frac{(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}^2}{\|\hat{\mathbf{r}}\|^\mathbf{7}},\ t_2 = \frac{3\hat{B}_{e,x}\hat{y} + 3\hat{x}\hat{B}_{e,y}}{\|\hat{\mathbf{r}}\|^\mathbf{5}} - 15\frac{(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}\hat{y}}{\|\hat{\mathbf{r}}\|^\mathbf{7}}$$

and

$$t_3 = \frac{3\hat{B}_{e,x}\hat{z} + 3\hat{x}\hat{B}_{e,z}}{\|\hat{\mathbf{r}}\|^\mathbf{5}} - 15\frac{(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}\hat{z}}{\|\hat{\mathbf{r}}\|^\mathbf{7}}.$$

If we repeat these steps for $\frac{\partial T}{\partial \hat{y}}$ and $\frac{\partial T}{\partial \hat{z}}$ (see Appendix B), then we can see that $\nabla T = \mathbf{G}$ can be written in the form of matrix vector multiplication $A\mathbf{m}$, with A a 3×3 matrix where

$$A_{i,j} = \begin{cases} \frac{6\hat{B}_{e,i}\hat{r}_i + 3(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^\mathbf{5}} - 15\hat{r}_i\frac{\hat{r}_i(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^7}, \text{for } i = j \text{ and } i,j = 1,2,3 \\ \\ \frac{3\hat{B}_{e,i}\hat{r}_j + 3\hat{r}_i\hat{B}_{e,j}}{\|\hat{\mathbf{r}}\|^\mathbf{5}} - 15\hat{r}_i\frac{\hat{r}_j(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^\mathbf{7}}, \text{for } i \neq j \end{cases}. \tag{4.11}$$

Since $\tilde{\mathbf{G}}$ is an approximation of the true gradient $\mathbf{G}$, so the gradient of T can be written as a linear system $A\mathbf{m} = \tilde{\mathbf{G}}$. The system described above is just for one measurement. Now if we take

N measurements and since the magnetic moment $\mathbf{m}$ must be the same for all N measurements, we will get $\tilde{A}\mathbf{m} = \hat{\mathbf{G}}$, where $\tilde{A}$ is a $3N \times 3$ matrix and $\tilde{\mathbf{G}}$ is a $3N \times 1$ vector that are given by:

$$\tilde{A} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix}, \qquad \hat{\mathbf{G}} = \begin{bmatrix} \tilde{\mathbf{G}}_\mathbf{1} \\ \tilde{\mathbf{G}}_\mathbf{2} \\ \vdots \\ \tilde{\mathbf{G}}_\mathbf{N} \end{bmatrix}. \tag{4.12}$$

where A_α is a 3×3 matrix defined as (4.4) and $\tilde{\mathbf{G}}_\alpha$ the gradient of the total fieldat the α-th measurement.We can solve this system by using the method of least squares, so $\mathbf{m} = (\tilde{A}^T\tilde{A})^{-1}\tilde{A}^T\tilde{\mathbf{G}}$.

4.3 Nonlinear refinement

For the derivation of those two linear systems described in the previous section, we have assumed that the measured gradient is equal to the true gradient. In practice it is not the case, the measured gradient is just an approximation of the true gradient. It means that the solution that we find by solving the two linear systems are not accurate. To increase the estimation accuracy, we introduce a second step which describes a nonlinear problem. We will use equation(3.3) to formulate this second step.

Nonlinear refinement. *Suppose we have taken N measurements with our gradiometer, which means we have a measured vector* $\hat{\mathbf{G}}$ *defined as in equation (4.12). What we want to do is find the moment* $\mathbf{m}$ *and position* $\mathbf{d}$ *that best satisfies the functional relation* $\mathbf{g} = [\mathbf{g_1}\ \mathbf{g_2}\ \cdots\ \mathbf{g_N}]^T$ *with*

$$\mathbf{g}_\alpha(\mathbf{x}) = \begin{bmatrix} \frac{B_{star,\alpha} - B_{port,\alpha}}{L_{trans}} \\ \frac{(B_{star,\alpha}+B_{port,\alpha})-2B_{aft,\alpha}}{2L_{long}} \\ \frac{(B_{star,\alpha}+B_{port,\alpha})-2B_{down,\alpha}}{2L_{vert}} \end{bmatrix}, with\ \mathbf{x} = \begin{bmatrix} \mathbf{d} \\ \mathbf{m} \end{bmatrix}.$$

We use the Levenberg–Marquardt algorithm to solve this problem. The position and the moment that we have found by solving the two linear systems will be used as initial values for solving this nonlinear refinement.

4.3.1 The Levenberg-Marquardt algorithm

The Levenberg-Marquardt algorithm (LMA) is an iterative method which is used to solve a nonlinear least squares problem. Since LMA can be seen as a "damped version" of the Gauss-Newton method which is also a iterative solver for nonlinear least square problems, we will first derive the Gauss-Newton method.

Nonlinear least squares problem [5]. *Given a vector function* $\mathbf{f} : \mathbb{R}^\mathbf{n} \to \mathbb{R}^\mathbf{m}$ *with* $m \geq n$. *For each* $f_i(\mathbf{x}), i = 1, 2, ..., m,$ *is a nonlinear function defined over* $\mathbb{R}^n$. *Our goal is to find*

$$\mathbf{x}^* = argmin_\mathbf{x}\{F(\mathbf{x})\}\ where\ F(\mathbf{x}) = \tfrac{1}{2}\textstyle\sum_{i=1}^m (f_i(\mathbf{x}))^2 = \tfrac{1}{2}\|\mathbf{f}(\mathbf{x})\|^2 = \tfrac{1}{2}\mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x}).$$

Assume that $\mathbf{f}$ has continuous second partial derivatives, then the Taylor expansion yields:

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) = \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\mathbf{h} + O(\|\mathbf{h}\|^\mathbf{2}), \tag{4.13}$$

with $\mathbf{J}(\mathbf{x})$ the Jacobian matrix, $\mathbf{J}_{i,j} = \frac{\partial f_i}{\partial x_j}$ for $i = 1, ..., m$ and $j = 1, ..., n$. The basis of the Gauss-Newton method is the linear approximation of the components of $\mathbf{f}$ in the neighbourhood of $\mathbf{x}$, so for a small $\|\mathbf{h}\|$, equation (4.13) becomes:

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) \approx \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\mathbf{h}. \tag{4.14}$$

So by definition and let $\mathbf{f} = \mathbf{f}(\mathbf{x})$ and $\mathbf{J} = \mathbf{J}(\mathbf{x})$:

$$\begin{aligned} F(\mathbf{x}+\mathbf{h}) \approx \frac{1}{2}(\mathbf{f}+\mathbf{J}\mathbf{h})^T(\mathbf{f}+\mathbf{J}\mathbf{h}) &= \frac{1}{2}(\mathbf{f}^T+\mathbf{h}^T\mathbf{J}^T)(\mathbf{f}+\mathbf{J}\mathbf{h}) \\ &= \frac{1}{2}\mathbf{f}^T\mathbf{f}+\mathbf{h}^T\mathbf{J}^T\mathbf{f}+\frac{1}{2}\mathbf{h}^T\mathbf{J}^T\mathbf{J}\mathbf{h} \\ &= F(\mathbf{x})+\mathbf{h}^T\mathbf{J}^T\mathbf{f}+\frac{1}{2}\mathbf{h}^T\mathbf{J}^T\mathbf{J}\mathbf{h} = L(\mathbf{h}) \qquad (4.15) \end{aligned}$$

Gauss-Newton is iterative, so given an initial value $\mathbf{x_0}$, this method produces a sequence of vectors $\mathbf{x_1}, \mathbf{x_2}, ...$ that converges to $\mathbf{x}^*$. It means that for each step we need to find $\mathbf{h}$ that minimizes $L(\mathbf{h})$. Taking the gradient of $L(h)$ respect to $\mathbf{h}$ we obtain the following expression:

$$\nabla L(\mathbf{h}) = \mathbf{J}^T\mathbf{f} + \mathbf{J}^T\mathbf{J}\mathbf{h}, \text{and } \nabla L\mathbf{h}) = \mathbf{0} \Rightarrow (\mathbf{J}^T\mathbf{J})\mathbf{h} = -\mathbf{J}^T\mathbf{f} \qquad (4.16)$$

By solving equation (4.16) we will get a minimizer for $L(\mathbf{h})$.

For the LMA we will get the following expression with damping:

$$(\mathbf{J}^T\mathbf{J} + \mu\mathbf{I})\mathbf{h} = -\mathbf{J}^T\mathbf{f}, \qquad (4.17)$$

$\mathbf{I}$ is the identity matrix and $\mu > 0$ is the damping term. This damping term influences both the direction and the size of the step h. Furthermore, the start value for μ has to be related to the elements in $\mathbf{J}(\mathbf{x_0})^T\mathbf{J}(\mathbf{x_0})$. If $L(\mathbf{h})$ is a good approximation to $F(\mathbf{x}+\mathbf{h})$, with $\mathbf{h}$ calculated by solving equation (4.17) and the updated vector $\mathbf{x}+\mathbf{h}$. Then the update is successful and we will repeat this process with a decreased damping term. Otherwise, the damping term will be increased and then iterates the process until we find a $\mathbf{h}$ such that $L(\mathbf{h})$ is a good approximation to $F(\mathbf{x}+\mathbf{h})$. More detailed explanation can be found in [7] .

Our nonlinear refinement is actually a data fitting problem, by letting $\mathbf{f} = \tilde{\mathbf{G}} - \mathbf{g}$. We will solve our problem with the function *LMFnlsq* in MATLAB, created by Miroslav Balda [8].

4.4 Acceptable results

Before we test our model, we first define what is an acceptable result for the location and the moment.

Relative error for the position. *Given the found position* $\mathbf{d_f}$ *by solving the first linear system and the true position* $\mathbf{d}$*, we define the relative error for the position as follow*

$$re_p = \frac{\|\mathbf{d}-\mathbf{d_f}\|}{h-d_z},$$

with h *the* z *coordinate of the gradiometer in meters during the measurement and* d_z *the* z *component in meters where the magnetic object is located.*

For the moment we will use the standard definition.

Relative error for the moment. *Given the found moment* $\mathbf{m_f}$ *by solving the first linear system and the true moment* $\mathbf{m}$*, we define the relative error for the moment as follow*

$$re_m = \frac{\|\mathbf{m}-\mathbf{m_f}\|}{\|\mathbf{m}\|}.$$

By multiplying the relative errors with 100%, we will have some insight about how much the found solution differs from the true values. For the position, we say that a found solution is acceptable if the relative error $re_p \leq 0.1$, or equivalently, $re_p \leq 10\%$. For the moment, we use the suggested maximum for the relative error in [1]. We have an acceptable solution for the moment if $re_m \leq 20\%$.

Chapter 5

Numerical experiment without noise

In this chapter, we will do a numerical experiment in the perfect scenario, i.e., without noise, to see whether the model gives acceptable results. For this experiment we first solve the forward problem and then use the simulated data from the forward problem to solve the inverse problem using the method described in the previous chapter.

For this experiment, we assume that the gradiometer takes measurements along a horizontal survey line at height h in meters, see Figure 5.1. We will first test our model at a fixed height to see whether the model is accurate. After that, we vary the value of h to see whether the altitude of the gradiometer affect the accuracy.

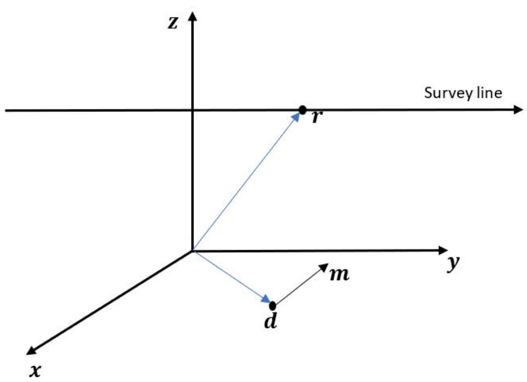

Figure 5.1: Setting for the experiment.

As mentioned before we assume that the Earth magnetic field $\mathbf{B}_e$ is known to us with

$$\mathbf{B}_e = \begin{bmatrix} 19 \\ 1 \\ 50 \end{bmatrix} \text{ in } \mu T.$$

Suppose the we have a survey with range of $400m$ and we take a total of 400 measurements. Furthermore, for this experiment we first solve the forward problem when we have a magnetic object located at

$$\mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} \text{ in meters with } \mathbf{m} = \begin{bmatrix} 1000 \\ 2000 \\ 3000 \end{bmatrix} \text{ in } [Am^2].$$

Recall that $L_{trans} = 0.75m$, $L_{aft} = 0.55m$ and $L_{down} = 0.25m$. If we use those values to simulate measured data in MATLAB and then solve the inverse problem. We see that the found position and moment by solving the two linear systems, gives the following result:

$$\mathbf{d_f} = \begin{bmatrix} -0.2464 \\ 0.1655 \\ -3.1271 \end{bmatrix} \text{ with } \mathbf{m_f} = \begin{bmatrix} 923.85 \\ 1744.94 \\ 2692.97 \end{bmatrix}.$$

If we compute the relative error for the position and moment, we see that for the position we have an relative error of 3.84% and for moment we have 10.86%. We conclude that for this numerical experiment the nonlinear refinement is not even needed to obtain an acceptable result for the position and the moment. To see whether the nonlinear refinement truly improves the result, we use the found values as initial values for the refinement and the found position and moment after refinement become:

$$\mathbf{d_f} = \begin{bmatrix} 0.000382 \\ 0.000123 \\ -3.998 \end{bmatrix} \text{ with } \mathbf{m_f} = \begin{bmatrix} 999.346 \\ 1998.954 \\ 2998.945 \end{bmatrix}.$$

After the refinement, the relative errors for the position and moment are $re_p = 0.009192\%$ and $re_m = 0.0434\%$. To see whether we obtain acceptable values without the nonlinear refinement, we vary the altitude of the gradiometer between $5m - 50m$ while the other values remain the same as above.

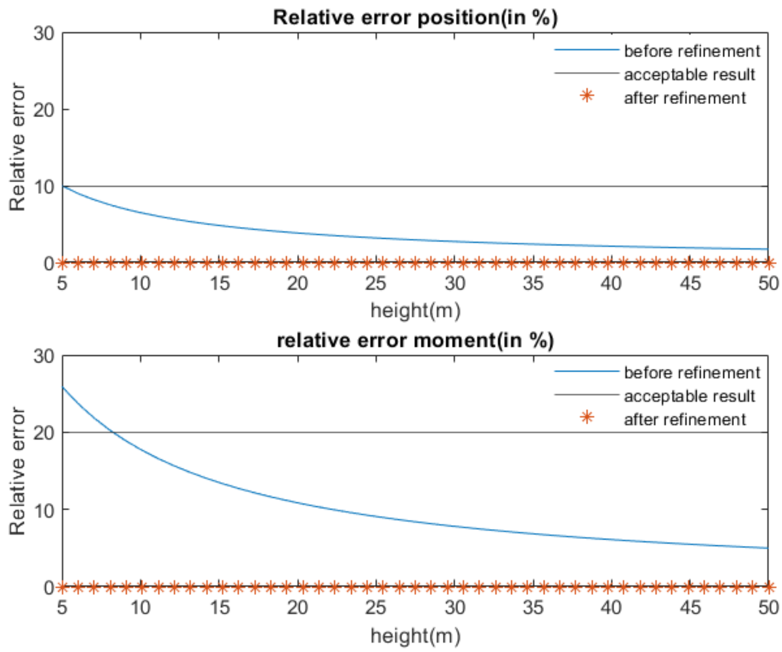

Figure 5.2: Relative errors of position and moment.

In Figure 5.2 we have plotted the relative errors for the situation described above. We observe that for the position we get an acceptable solution when the gradiometer is at a height above $5m$. However, for this altitude, the result for the moment is not acceptable. We must have an altitude at least $9m$ if we want to get acceptable values for both the moment and the position without a nonlinear refinement. Since we have defined the z coordinate of the magnetic dipole is $-4m$, so we get acceptable values if the altitude is at least $13m$. From the figure above we also observe that the nonlinear refinement always improves our results regardless of the altitude.

Chapter 6

Model with Noise

The gradiometer that we use in this project, is composed of four magnetometers. In practise, there is always an intrinsic noise within each magnetometer. It means that the measured data always contains noise. We use the measured data in both linear systems that we have to solve in our inverse problem. In this chapter, we will see how the noise might affect our solutions.

6.1 Gaussian Noise

Suppose the intrinsic noise of the magnetometers are modeled as a uncorrelated Gaussian noise. For each magnetometer, we define the measured total magnetic field that contains noise as $B_m = B_{true} + e = \|\mathbf{B}_{true} + \delta\|$, where $\delta \sim \mathcal{N}(\mu, \sigma^2 I_3)$. The Gaussian distribution has density function:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \mathbb{R}^3 \tag{6.1}$$

As suggested in [1], the Gaussian noise has zero mean and a standard deviation of $0.01nT$. So we have $\mu = 0$ and $\sigma = 0.01 \cdot 10^{-9}T$. With the defined noise in the magnetometers, we get the following expression for our gradiometer for each measurement:

$$\tilde{\mathbf{G}} = \begin{bmatrix} \frac{(B_{true,star}+e_1)-(B_{true,port}+e_2)}{L_{trans}} \\ \frac{(B_{true,star}+e_1)+(B_{true,port}+e_2)-2(B_{true,aft}+e_3)}{2L_{long}} \\ \frac{(B_{true,star}+e_1)+(B_{true,port}+e_2)-2(B_{true,down}+e_4)}{2L_{vert}} \end{bmatrix} = \tilde{\mathbf{G}}_{true} + \mathbf{e}, \tag{6.2}$$

with $\tilde{\mathbf{G}}_{true}$ the gradient without noise.

6.2 Revised model with noise

In our inverse problem, both linear systems use the measured data that contains noise. Recall equation(4.3). We observe that both side of the equation are dependent on the measured data. It means that for this linear system with Gaussian noise, we have to solve the following least squares problem:

$$(X_{true} + E)\mathbf{d} = \mathbf{Y} + \mathbf{F}, \tag{6.3}$$

with E an $N \times 3$ matrix and $\mathbf{F}$ an $N \times 1$ vector for the measurement error. We call such a least squares problem, **the total least squares(TLS)**. For the second linear system, see equation (4.12), we only use the measured data at the right side so this system becomes:

$$\tilde{A}\mathbf{m} = \hat{\mathbf{G}}_{true} + \mathbf{e}. \tag{6.4}$$

The situation of equation (6.4) is more familiar for us. So we will first analysis this problem next section to see why some linear systems in this form might be unstable and how we can solve this problem.

6.3 The second linear system with noise

Consider this linear system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathbb{R}^{m\times n}$,$\mathbf{x} \in \mathbb{R}^n$,$\mathbf{b} \in \mathbb{R}^m$ and $m \geq n$. We know that for every matrix A, we can find the *singular value decomposition* of the form $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are orthogonal matrices. When the $rank(A) = r < n$, we have $\Sigma \in \mathbb{R}^{m\times n}$ of the following form:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \text{where } \Sigma_1 := diag(\sigma_1, ..., \sigma_r) \in \mathbb{R}^{r\times r}. \tag{6.5}$$

For $i \in \{1, 2, ..., r\}, \sigma_i$ is the singular values of matrix A with $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$. As mentioned before, an overdetermined system does not have an solution in most cases. When we have a matrix A defined as above, we can use the *pseudo-inverse* of A to find the best-approximation solution $\mathbf{x}^\dagger$.

Definition (*The pseudo-inverse*)[6]. *The matrix* $A^\dagger = V\Sigma^\dagger U^T$ *with*

$$\Sigma^\dagger = \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \Sigma_1^{-1} := diag\left(\frac{1}{\sigma_1}, ..., \frac{1}{\sigma_r}\right), \tag{6.6}$$

is called the pseudo-inverse of A.

If we know the pseudo-inverse, we can compute $\mathbf{x}^\dagger = A^\dagger\mathbf{b} = V\Sigma^\dagger U^T\mathbf{b}$. We denote $V = [\mathbf{v_1}\ \mathbf{v_2}\ \cdots\ \mathbf{v_n}]$ and $\mathbf{U} = [\mathbf{u_1}\ \mathbf{u_2}\ \cdots\ \mathbf{u_m}]$. Furthermore we know $\Sigma^\dagger$ from equation(6.6). Then we can write $\mathbf{x}^\dagger$ as:

$$\mathbf{x}^\dagger = V\Sigma^\dagger U^T\mathbf{b} = V \begin{bmatrix} \frac{1}{\sigma_1}\mathbf{u_1^T b} \\ \frac{1}{\sigma_2}\mathbf{u_2^T b} \\ \vdots \\ \frac{1}{\sigma_r}\mathbf{u_r^T b} \end{bmatrix} = \sum_{i=1}^{r} \frac{\mathbf{u_i^T b}}{\sigma_i}\mathbf{v_i}. \tag{6.7}$$

Now we have derived an expression to find the best approximation solution, we will analysis the stability of linear systems in the form of equation (6.4), i.e., $A\mathbf{x} = \mathbf{b_{true}} + \mathbf{e}$.

Definition *Condition number*[6]. *Given the singular values with* $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r$ *of matrix* $\mathbf{A}$, *then the condition number is defined as:*

$$K(A) = \frac{\sigma_1}{\sigma_r}. \tag{6.8}$$

Theorem *Condition number*. *Suppose that* $A \in \mathbb{R}^{m\times n}$ *is a rectangular matrix. Let* $\mathbf{b}, \mathbf{e} \in \mathbb{R}^m$ *some nonzero vectors and assume that* $A\mathbf{x}_e = \mathbf{b}$ *and* $A\mathbf{x} = \mathbf{b} + \mathbf{e}$ *holds. Then*

$$\frac{\|\mathbf{x}_e - \mathbf{x}\|}{\|\mathbf{x}_e\|} \leq K(A)\frac{\|\mathbf{e}\|}{\|\mathbf{b}\|}.$$

According to this theorem, the condition number of a matrix gives how sensitive the system is for a small perturbation. So if $\sigma_r \ll \sigma_1$, then we have a large condition number that might tends to infinity. We call a system with a large condition number (when $K(A) \geq 10^6$), ill-conditioned. If a matrix is ill-conditioned, we can conclude from the theorem that the obtained solution can be far away from the true solution. When solving the system $A\mathbf{x} = \mathbf{b}_{true} + \mathbf{e}$ using the pseudo-inverse, we get the following by letting $\mathbf{b} = \mathbf{b}_{true} + \mathbf{e}$ in equation (6.7):

$$\mathbf{x}^{\dagger} = A^{\dagger}(\mathbf{b_{true}} + \mathbf{e}) = \sum_{i=1}^{r} \frac{\mathbf{u_i^T}(\mathbf{b_{true}} + \mathbf{e})}{\sigma_i}\mathbf{v_i} = \sum_{i=1}^{r} \frac{\mathbf{u_i^T b_{true}}}{\sigma_i}\mathbf{v_i} + \sum_{i=1}^{r} \frac{\mathbf{u_i^T e}}{\sigma_i}\mathbf{v_i} \tag{6.9}$$

We can conclude from equation above that the solution of an ill-condition problem with presence of noise is not reliable for us.

6.3.1 Regularization

For an ill-conditioned system, regularization is needed. The idea behind the regularization methods is to reduce the influence of noise on the best-approximate solution. One of the regularization methods is Tikhonov regularization. For the Tikhonov approach, we need to solve the following problem

$$arg\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|^2 + \lambda^2\|L\mathbf{x}\|^2, \tag{6.10}$$

with $\lambda > 0$ is the regularization parameter and L is some matrix that consists of a-priori information about the solution(s) of the system of equations $A\mathbf{x} = \mathbf{b}$. When we take L as the identity matrix I, then the we get the Tikhonov regularization with the following form:

$$arg\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|^2 + \lambda^2\|\mathbf{x}\|^2. \tag{6.11}$$

It follows that the solution of equation (6.11) is

$$\mathbf{x}_{\lambda} = (A^T A + \lambda I)^{-1} A^T \mathbf{b}.$$

It is essential to find the correct value for λ such that the solution $\mathbf{x}_{\lambda}$ is most optimal. One way of finding the best λ is by using the "L-curve" method. A detailed explanation of this method can be found in [9]. In this project, if regularization is needed, then we will use the *Regularization Tools* in MATLAB which is created by Per Christian Hansen [10].

6.4 The first linear system with noise

Definition *The Total Least Squares(TLS)*[11]. *Given an overdetermined linear system* $A\mathbf{x} = \mathbf{b}$*, if both sides of this linear system are perturbed, then we will get a total least squares of the form*

$$(A + \Delta A)\mathbf{x} = (\mathbf{b} + \Delta\mathbf{b}). \tag{6.12}$$

This problem seeks to

$$\min_{\Delta A \in \mathbb{C}^{m\times n},\ \Delta\mathbf{b} \in \mathbb{C}^m} \|[\Delta A\ \Delta\mathbf{b}]\|_F,\ \textit{subject to}\ (\mathbf{b} + \Delta\mathbf{b}) \in R(A + \Delta A). \tag{6.13}$$

$||A||_F$ *is the Frobenius norm of an arbitrary matrix* A*, which is defined as* $\sqrt{trace(A^*A)}$*, where* A^* *stands for the conjugate transpose of* A *and* $trace(A^*A)$ *is the sum of the elements on the diagonal of* A^*A*.* $R(A)$ *denotes the column space of matrix* A*. Once a minimizing* ΔA *and* $\Delta\mathbf{b}$ *are found, then any* $\mathbf{x}$ *satisfying* $(A + \Delta A)\mathbf{x} = (\mathbf{b} + \Delta\mathbf{b})$ *is called a TLS solution.*

The TLS problem can be analysed by using the singular value decomposition.
First, we rewrite $A\mathbf{x} = \mathbf{b}$ in the following form

$$[A\ \mathbf{b}] \begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix} = \mathbf{0} \tag{6.14}$$

Let the singular value decomposition of $C = [A\ \mathbf{b}] = U\Sigma V^T$, with $\Sigma := diag(\sigma_1, ..., \sigma_{n+1})$. If $\sigma_{n+1} \neq 0$, then C is of rank $n+1$. It implies that $\mathbf{b}$ is not in the column space of A, so no solution exists. To obtain a solution, the rank of C must be reduced to n. We can do this by using the Eckart-Young-Mirsky theorem [11]. This theorem states that the best rank n approximation of C which satisfies the TLS problem, can be found by using the n largest singular values and their corresponding singular vectors. Therefore, the best rank n approximation of C, denote it as $[\hat{A}\ \hat{\mathbf{b}}]$, is given by $\sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$. Then according to [11], the solution of TLS $\mathbf{x}_{TLS}$ is given by

$$\mathbf{x}_{TLS} = -\frac{1}{v_{n+1,n+1}} \begin{bmatrix} v_{1,n+1} \\ v_{2,n+1} \\ \vdots \\ v_{n+1,n+1} \end{bmatrix},$$

if $v_{n+1,n+1} \neq 0$. For finding the solution of TLS, we will use the function *tls* in MATLAB which is created by Ivo Houtzager [13].

Chapter 7

Numerical experiment with noise

Recall the experiment that we have done in Chapter 5. We used a horizontal survey with a range of 400 m and we took a total of 400 measurements. In this chapter, we repeat this experiment with the same set up, but this time the measured data contains Gaussian noise defined in the previous chapter. Recall the location and the moment of the magnetic object that we have used:

$$\mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} \text{ in } m \text{ with } \mathbf{m} = \begin{bmatrix} 1000 \\ 2000 \\ 3000 \end{bmatrix} \text{ in } Am^2.$$

If we use those values to simulate data which contains noise and then solve the inverse problem. The found position and moment are:

$$\mathbf{d_f} = \begin{bmatrix} 0.146 \\ 0.158 \\ -2.866 \end{bmatrix} \text{ in } m \text{ with } \mathbf{m_f} = \begin{bmatrix} 724.43 \\ 1693.85 \\ 2620.31 \end{bmatrix} \text{ in } Am^2.$$

The relative errors of the found values are $re_p = 4.81\%$ and $re_m = 14,97\%$. We see that just as the situation without noise, both results are acceptable for us. To find out whether the altitude of the gradiometer influences the accuracy of the found results, we vary the value of h from $5m - 50m$, see Figure 7.1.

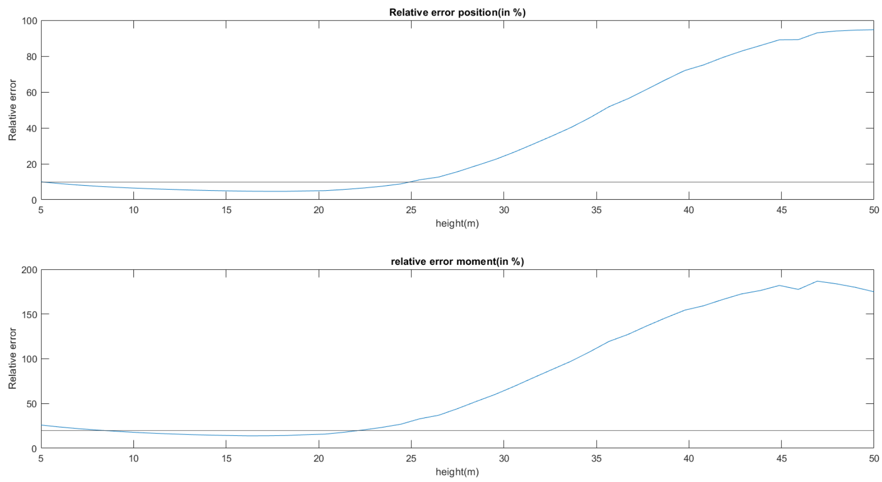

Figure 7.1: Relative errors of position and moment.

The curves presented in the figure above, are the mean errors of the relative errors calculated on 200 runs. We see for the position we must have an altitude between a range of $5m - 24m$ while for the moment we must have a range between $8m - 22m$. We can conclude that there exists an optimal altitude range for this experiment. Since in this situation we have the dipole at depth of $-4m$, so we might suggest that the true optimal altitude of the gradiometer must be $12m - 27m$. To check whether this intuition is correct, we repeat this experiment with **d** located at the origin.

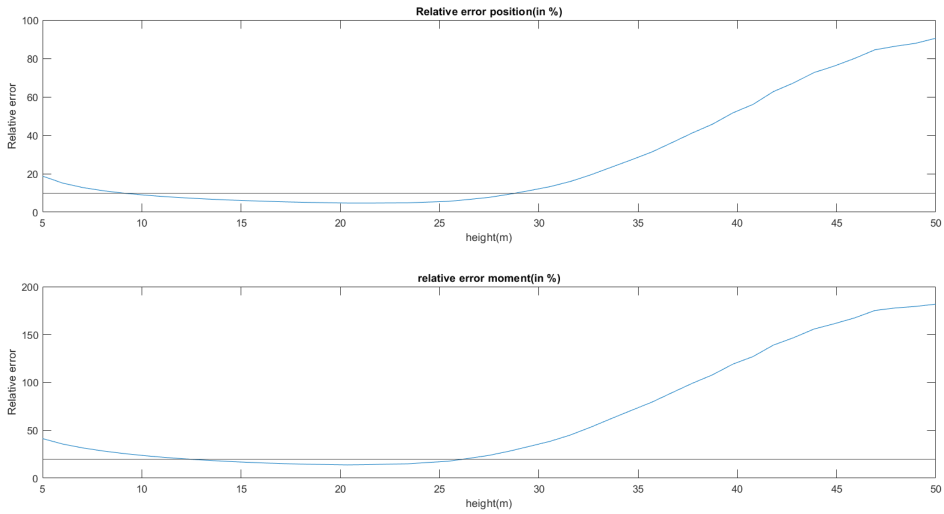

Figure 7.2: Relative errors of position and moment

From Figure 7.2 we can conclude that the optimal range of the gradiometer altitude must between $12m$ and $27m$. So even without the nonlinear refinement, we can find acceptable results if the altitude of the gradiometer is between 12 m and 27 m. However, this method is limited. Since we do not know the depth of the magnetic object in practise, it is hard to choose the correct height of the gradiometer to receive acceptable results. For more experiments with different **d** and **m**, see Appendix C.

Chapter 8

Conclusion and recommendations

8.1 Conclusion

The magnetic fields produced by a ferromagnetic object causes a magnetic anomaly to the local Earth magnetic field. A method for localization and classification of a magnetic object with a small magnetic moment, is shown in [1] by using a three-axis total field gradiometer. In this project, we want to answer the following questions:

1. Is the method in [1] also useful for magnetic objects like sunken ships or containers with the moment of $10^3 Am^2$?
2. Does the altitude of the gradiometer influence the result? If the answer is yes, what is then the optimal range for the gradiometer altitude?

A magnetic object can be seen as a magnetic point dipole, when the distance between the magnetic target and the sensor is large. To answering those two questions, we have first derived an expression for the magnetic field of a magnetic point dipole. After that we have constructed a three-axis total field gradiometer that is composed of four magnetometers, see Figure 3.1. By combining the measurements taken by the magnetometers, an approximation of the gradient is obtained, see equation(3.3)). Afterwards, we have formulated the forward problem where we calculate the gradients of the magnetic field during N measurements along a flown path. We can solve this forward problem in three steps:

1. calculate the magnetic field caused by the target at the position of each magnetometer.
2. use results of step 1 to calculate the total field that we will measure with the four magnetometers.
3. calculate the gradient using equation(3.3).

The solution of the forward problem is used as the input data for the inverse problem. The inverse problem consists of two steps. In the first step, we used the definition and theorem of the Euler's homogeneous functions to get two linear systems. By solving those two linear systems, we get position and moment of the magnetic object that we are looking for. To improve the accuracy of the obtained results from the first step, we have introduced the second step where the gradiometer geometry is taken into account. For this step we used the found position and moment of the first step as the initial values. We say that the results are acceptable, when the relative error of the position is smaller or equal to 10% and the relative error of the moment is smaller or equal to 20%. In the experiment without noise that we have set up in Chapter 5, we found that the altitude of the gradiometer must be at least $23m$ to get acceptable values for both the position and moment without refinement. Furthermore, we saw that the second step significantly improves the results, see Figure 5.2.

To analyse the situation in practice, we have added an uncorrelated Gaussian noise with zero mean and $0.01nT$ standard deviation to the magnetometers. Due to the fact that both sides of the first linear system are dependent on the measured data, we get instead of a linear least squares problem, a total least squares problem. For solving this problem, we can use the singular value decomposition. For the second linear system, we only have the right side that is perturbed by noise. We have analysed that the solution of the system is very sensitive for noise, when the matrix is ill-conditioned. To make the calculation less sensitive to noise, we can use a regularization method. The regularization parameters can be found by using the L-curve method. We have set up an experiment with the same situation described as in Chapter 5, but this time the measured data contains noise. We have found that the optimal range of the altitude of the gradiometer without refinement, is $12m - 27m$.

In conclusion, the method used in [1] is also useful for a magnetic object with a larger moment and the optimal range for the altitude of the gradiometer is $12m - 27m$ for all three considered experiments. Nevertheless, this method is limited since we do not know the depth of the magnetic object. Therefore, it is hard to choose the correct altitude of the gradiometer.

8.2 Recommendations

Even though the goal for this project was achieved, there is more to further investigate. This section explains only a few short ideas that could be considered as further research.

1. In this project we have simulated the total field at the position of the gradiometer. For further research, one could assume that the total field at the position of the gradiometer is the weighted average of the total field measured by the four magnetometers.

2. In this project, we have assumed that the earth magnetic field is known to us. For further research, one could assume that the earth magnetic field is unknown and then check whether the method of Euler deconvolution still works for finding the position and the moment by using the three-axis total field gradiometer.

3. In this project, we consider a three-axis total field gradiometer, one can use for example a tensor gradiometer to see whether similar algorithms exist for finding the location and the moment.

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Appendix A

Derivation of $\frac{\partial T_\alpha}{\partial \hat{r}_{i,\alpha}}$

$$T_\alpha = \mathbf{B}_{s,\alpha} \cdot \hat{\mathbf{B}}_e = \frac{\mu_0}{4\pi}\left(\underbrace{\frac{3[(\mathbf{m}\cdot\hat{\mathbf{r}}_\alpha)\hat{\mathbf{r}}_\alpha]\cdot\hat{\mathbf{B}}_e}{\|\hat{\mathbf{r}}_\alpha\|^5}}_{a} - \underbrace{\frac{\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e}}{\|\hat{\mathbf{r}}_\alpha\|^3}}_{b}\right) \Rightarrow \frac{\partial T_\alpha}{\partial \hat{r}_{i,\alpha}} = \frac{\mu_0}{4\pi}\left(\frac{\partial a}{\partial \hat{r}_{i,\alpha}} - \frac{\partial b}{\partial \hat{r}_{i,\alpha}}\right) \tag{A.1}$$

$a = 3[\hat{B}_{e,x}(m_x x_\alpha + m_y y_\alpha + m_z z_\alpha)x_\alpha + \hat{B}_{e,y}(m_x x_\alpha + m_y y_\alpha + m_z z_\alpha)y_\alpha + \hat{B}_{e,z}(m_x x_\alpha + m_y y_\alpha + m_z z_\alpha)z_\alpha](x_\alpha^2 + y_\alpha^2 + z_\alpha^2)^{-\frac{5}{2}}$

$\frac{\partial a}{\partial x_\alpha} = [\hat{B}_{e,x}(\mathbf{m}\cdot\hat{\mathbf{r}}_\alpha) + \hat{B}_{e,x}m_x x_\alpha + \hat{B}_{e,y}m_x y_\alpha + \hat{B}_{e,z}m_x z_\alpha](x_\alpha^2+y_\alpha^2+z_\alpha^2)^{-\frac{5}{2}} - 3\cdot5\cdot x_\alpha\frac{3[(\mathbf{m}\cdot\hat{\mathbf{r}}_\alpha)\hat{\mathbf{r}}_\alpha]\cdot\hat{\mathbf{B}}_e}{\|\hat{\mathbf{r}}_\alpha\|^7}$
$= \frac{3\hat{B}_{e,x}(\mathbf{m}\cdot\hat{\mathbf{r}}_\alpha)+3m_x(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}_\alpha)}{\|\hat{\mathbf{r}}_\alpha\|^5} - 15x_\alpha\frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}_\alpha)\hat{\mathbf{r}}_\alpha]}{\|\hat{\mathbf{r}}_\alpha\|^\mathbf{7}}$.
The derivative of a respect to y_α and z_α are given by :

$$\frac{\partial a}{\partial y_\alpha} = \frac{3\hat{B}_{e,y}(\mathbf{m}\cdot\hat{\mathbf{r}}_\alpha)+3m_y(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}_\alpha)}{\|\hat{\mathbf{r}}_\alpha\|^5} - 15y_\alpha\frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}_\alpha)\hat{\mathbf{r}}_\alpha]}{\|\hat{\mathbf{r}}_\alpha\|^\mathbf{7}}$$
$$\frac{\partial a}{\partial z_\alpha} = \frac{3\hat{B}_{e,z}(\mathbf{m}\cdot\hat{\mathbf{r}}_\alpha)+3m_z(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}_\alpha)}{\|\hat{\mathbf{r}}_\alpha\|^5} - 15z_\alpha\frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}_\alpha)\hat{\mathbf{r}}_\alpha]}{\|\hat{\mathbf{r}}_\alpha\|^\mathbf{7}}$$

We can generalize the derivative of a respect to $\hat{r}_{i,\alpha}$ with $i = 1, 2, 3$ as:

$$\frac{\partial a}{\partial \hat{r}_{i,\alpha}} = \frac{3\hat{B}_{e,i}(\mathbf{m}\cdot\hat{\mathbf{r}}_\alpha) + 3m_i(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}_\alpha)}{\|\hat{\mathbf{r}}_\alpha\|^5} - 15\hat{r}_{i,\alpha}\frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}_\alpha)\hat{\mathbf{r}}_\alpha]}{\|\hat{\mathbf{r}}_\alpha\|^\mathbf{7}} \tag{A.2}$$

$b = \frac{\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e}}{\|\hat{\mathbf{r}}_\alpha\|^3}$, since $\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e}$ is not dependent on $\hat{\mathbf{r}}_\alpha$, we can write $b = (\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e})\cdot(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)^{-\frac{3}{2}}$.

The derivative of b respect to $\hat{r}_{i,\alpha}$ with $i = 1, 2, 3$ is :

$$\frac{\partial b}{\partial \hat{r}_{i,\alpha}} = -3\hat{r}_{i,\alpha}\frac{\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e}}{\|\hat{\mathbf{r}}_\alpha\|^5} \tag{A.3}$$

Appendix B

Derivation from equation(4.3)

We have derived that for all i the derivative of the magnetic anomaly T respect to $\hat{r}_i$ is:

$$\frac{\partial T}{\partial \hat{r}_i} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{B}_{e,i}(\mathbf{m}\cdot\hat{\mathbf{r}}) + 3m_i(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}) + 3\hat{r}_i(\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e})}{\|\hat{\mathbf{r}}\|^5} - 15\hat{r}_i\frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{\|\hat{\mathbf{r}}\|^\mathbf{7}}\right],\ \hat{\mathbf{r}} = \begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} \tag{B.1}$$

So the derivative of T respect to $\hat{r}_x = \hat{x}$ is

$$\frac{\partial T}{\partial \hat{x}} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{B}_{e,x}(\mathbf{m}\cdot\hat{\mathbf{r}}) + 3m_x(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}) + 3\hat{x}(\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e})}{\|\hat{\mathbf{r}}\|^5} - 15\hat{x}\frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{\|\hat{\mathbf{r}}\|^\mathbf{7}}\right] \tag{B.2}$$

We know that

$$\hat{B}_{e,x}(\mathbf{m}\cdot\hat{\mathbf{r}}) = \hat{B}_{e,x}(m_x\hat{x} + m_y\hat{y} + m_z\hat{z}) \text{ and } \quad \hat{x}(\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e}) = \hat{x}(m_x\hat{B}_{e,x} + m_y\hat{B}_{e,y} + m_z\hat{B}_{e,z}).$$

Now if we let $f_1 = \frac{3\hat{B}_{e,x}(\mathbf{m}\cdot\hat{\mathbf{r}}) + 3m_x(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}) + 3\hat{x}(\mathbf{m}\cdot\hat{\mathbf{B}}_\mathbf{e})}{\|\hat{\mathbf{r}}\|^\mathbf{5}}$, then we can rewrite f_1 as following:

$$f_1 = \frac{3m_x(\hat{B}_{e,x}\hat{x} + (\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}}) + \hat{B}_{e,x}\hat{x}) + 3m_y(\hat{B}_{e,x}\hat{y} + \hat{x}\hat{B}_{e,y}) + 3m_z(\hat{B}_{e,x}\hat{z} + \hat{x}\hat{B}_{e,z})}{\|\hat{\mathbf{r}}\|^\mathbf{5}}. \tag{B.3}$$

Furthermore, let $f_2 = \frac{\mathbf{m}\cdot[(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\hat{x}}{\|\hat{\mathbf{r}}\|^\mathbf{7}}$ then we can write f_2 as:

$$f_2 = \frac{m_x(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}^2 + m_y(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}\hat{y} + m_z(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}\hat{z}}{\|\hat{\mathbf{r}}\|^\mathbf{7}}. \tag{B.4}$$

Since in both equations (B.3) and (B.4) we have terms that are only dependent on m_x, m_yor m_z, so we can also rewrite $\frac{\partial T}{\partial \hat{x}}$ as the following summation:

$$\frac{\partial T}{\partial \hat{x}} = \frac{\mu_0}{4\pi}(f_1 - 15f_2) = \frac{\mu_0}{4\pi}(m_x\cdot t_1 + m_y\cdot t_2 + m_z\cdot t_3), \tag{B.5}$$

where

$$t_1 = \frac{6\hat{B}_{e,x}\hat{x} + (\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^\mathbf{5}} - 15\frac{(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}^2}{\|\hat{\mathbf{r}}\|^\mathbf{7}}, t_2 = \frac{3\hat{B}_{e,x}\hat{x} + 3\hat{x}\hat{B}_{e,y}}{\|\hat{\mathbf{r}}\|^\mathbf{5}} - 15\frac{(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}\hat{y}}{\|\hat{\mathbf{r}}\|^\mathbf{7}} \text{ and}$$

$$t_3 = \frac{3\hat{B}_{e,x}\hat{x} + 3\hat{x}\hat{B}_{e,z}}{\|\hat{\mathbf{r}}\|^\mathbf{5}} - 15\frac{(\hat{\mathbf{B}}_\mathbf{e}\cdot\hat{\mathbf{r}})\hat{x}\hat{z}}{\|\hat{\mathbf{r}}\|^\mathbf{7}}.$$

If derive the derivative of T respect to $\hat{r}_y$ and $\hat{r}_z$ at the same way, we will get the following expressions for $\frac{\partial T}{\partial \hat{y}}$ and $\frac{\partial T}{\partial \hat{z}}$:

$$\frac{\partial T}{\partial \hat{y}} = \frac{\mu_0}{4\pi}(m_x \cdot t_4 + m_y \cdot t_5 + m_z \cdot t_6) \tag{B.6}$$

$$t_4 = \frac{3\hat{B}_{e,y}\hat{x} + 3\hat{y}\hat{B}_{e,x}}{\|\hat{\mathbf{r}}\|^{\mathbf{5}}} - 15\frac{(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})\hat{x}\hat{y}}{\|\hat{\mathbf{r}}\|^{\mathbf{7}}}, t_5 = \frac{6\hat{B}_{e,y}\hat{y} + (\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^{\mathbf{5}}} - 15\frac{(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})\hat{y}^2}{\|\hat{\mathbf{r}}\|^{\mathbf{7}}} \text{ and}$$

$$t_6 = \frac{3\hat{B}_{e,y}\hat{z} + 3\hat{y}\hat{B}_{e,z}}{\|\hat{\mathbf{r}}\|^{\mathbf{5}}} - 15\frac{(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})\hat{y}\hat{z}}{\|\hat{\mathbf{r}}\|^{\mathbf{7}}}.$$

$$\frac{\partial T}{\partial \hat{z}} = \frac{\mu_0}{4\pi}(m_x \cdot t_7 + m_y \cdot t_8 + m_z \cdot t_9) \tag{B.7}$$

$$t_7 = \frac{3\hat{B}_{e,x}\hat{x} + 3\hat{x}\hat{B}_{e,x}}{\|\hat{\mathbf{r}}\|^{\mathbf{5}}} - 15\frac{(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})\hat{x}\hat{z}}{\|\hat{\mathbf{r}}\|^{\mathbf{7}}}, t_8 = \frac{3\hat{B}_{e,y}\hat{z} + 3\hat{y}\hat{B}_{e,z}}{\|\hat{\mathbf{r}}\|^{\mathbf{5}}} - 15\frac{(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})\hat{y}\hat{z}}{\|\hat{\mathbf{r}}\|^{\mathbf{7}}}.$$

$$t_9 = \frac{6\hat{B}_{e,z}\hat{z} + (\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^{\mathbf{5}}} - 15\frac{(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})\hat{z}^2}{\|\hat{\mathbf{r}}\|^{\mathbf{7}}}.$$

So we can write ∇T in the form of matrix vector multiplication:

$$\nabla T = \underbrace{\begin{bmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \tag{B.8}$$

For matrix $\mathbf{A}$, the elements of the diagonal are:

$$\frac{6\hat{B}_{e,i}\hat{r}_i + 3(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^{\mathbf{5}}} - 15\hat{r}_i\frac{\hat{r}_i(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^7}, \text{for } i = 1, 2, 3$$

and the elements off diagonal:

$$\frac{3\hat{B}_{e,i}\hat{r}_j + 3\hat{r}_i\hat{B}_{e,j}}{\|\hat{\mathbf{r}}\|^{\mathbf{5}}} - 15\hat{r}_i\frac{\hat{r}_j(\hat{\mathbf{B}}_{\mathbf{e}} \cdot \hat{\mathbf{r}})}{\|\hat{\mathbf{r}}\|^{\mathbf{7}}}$$

for $i = 1, 2, 3$ for the rows and $j = 1, 2, 3$ for the columns.

Appendix C

Numerical experiments

C.1 Experiment 1

If we have magnetic object with:

$$\mathbf{d} = \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix} \text{ in meters with } \mathbf{m} = \begin{bmatrix} 3000 \\ 2000 \\ 1000 \end{bmatrix} \text{ in } [Am^2].$$

The found results are by solving the two linear systems:

$$\mathbf{d}_f = \begin{bmatrix} 1.244 \\ 5.371 - 4.576 \end{bmatrix} \text{ in meters with } \mathbf{m}_f = \begin{bmatrix} 2440.900 \\ 1657.727 \\ 907.552 \end{bmatrix} \text{ in } [Am^2].$$

The relative errors are $re_p = 5.735\%$ and 17.694%. For both we have acceptable results. If we vary the altitude from $5m - 50m$, we see that the optimal range of the altitude is $5m - 21m$. By adding the depth of the magnetic dipole to the range, the optimal range is $11m - 27m$.

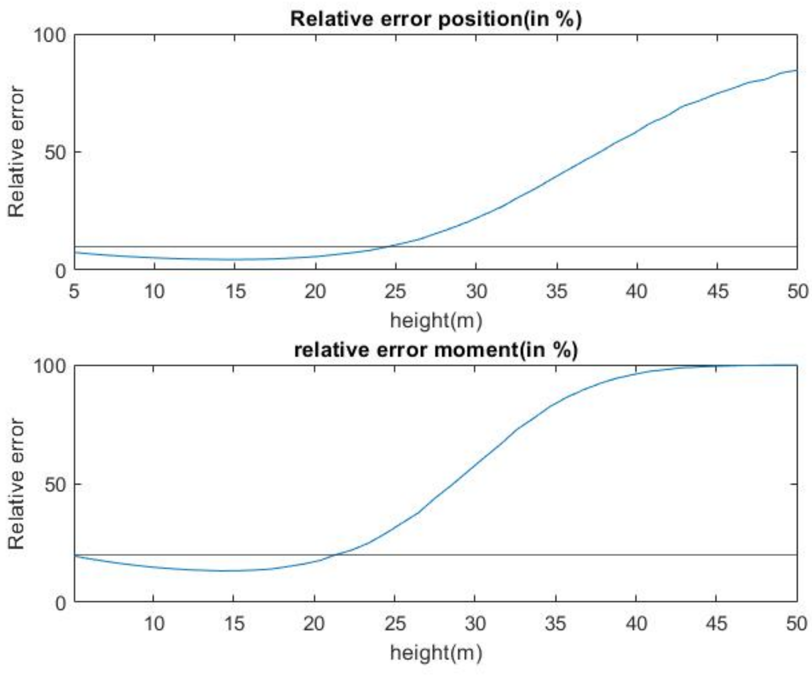

Figure C.1: Relative errors of position and moment

C.2 Experiment 2

If we have magnetic object with:

$$\mathbf{d} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} \text{ in meters with } \mathbf{m} = \begin{bmatrix} 3000 \\ 1000 \\ 5000 \end{bmatrix} \text{ in } [Am^2].$$

The found results are by solving the two linear systems:

$$\mathbf{d}_f = \begin{bmatrix} 5.497 \\ -0.100 \\ -4.591 \end{bmatrix} \text{ in meters with } \mathbf{m}_f = \begin{bmatrix} 2089.672 \\ 935.464 \\ 4335.310 \end{bmatrix} \text{ in } [Am^2].$$

And the relative errors are $re_p = 5.760\%$ and $re_m = 19.084\%$. So for this experiment we get acceptable results. If we vary the altitude from $5m - 50m$, we see that the optimal range of the altitude is $11m - 26m$. Since we have the dipole at depth $-1m$, we can conclude the optimal range must be $12m - 27m$.

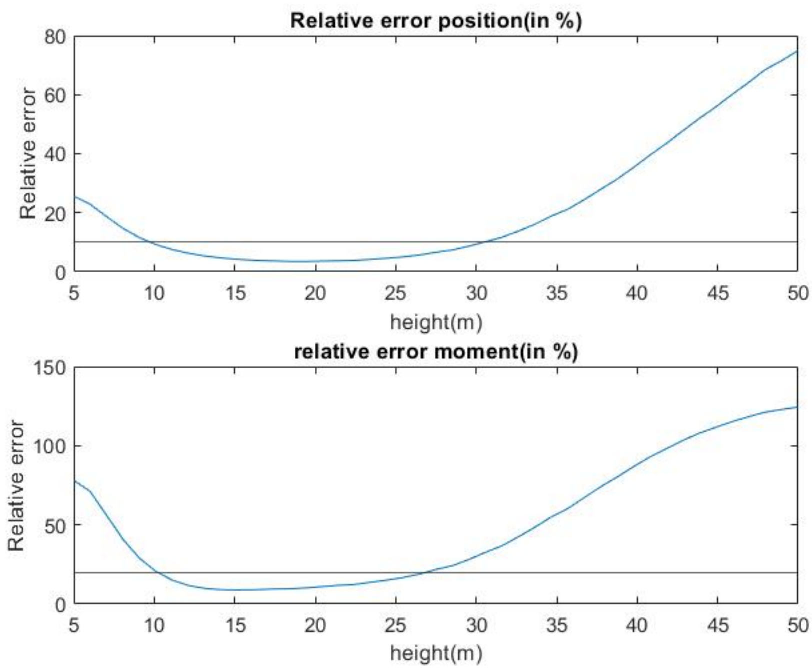

Figure C.2: Relative errors of position and moment