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DOI
10.1016/j.orl.2023.10.002

Publication date
2023

Document Version
Final published version

Published in
Operations Research Letters

Operations Research Letters 51 (2023) 601–604

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Minimizing the effective graph resistance by adding links is NP-hard

Robert E. Kooij [a,b,*], Massimo A. Achterberg [a]

[a] Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA, Delft, the Netherlands
[b] Unit ICT, Strategy & Policy, TNO (Netherlands Organisation for Applied Scientific Research), P.O. Box 96800, 2509 JE, The Hague, the Netherlands

A R T I C L E I N F O

Article history:
Received 24 February 2023
Received in revised form 27 September 2023
Accepted 5 October 2023
Available online 12 October 2023

Keywords:
Effective graph resistance
Graph augmentation
NP-hard

A B S T R A C T

The effective graph resistance, also known as the Kirchhoff index, is metric that is used to quantify the robustness of a network. We show that the optimisation problem of minimizing the effective graph resistance of a graph by adding a fixed number of links, is NP-hard.

1. Introduction

Many network metrics have been utilised to quantify the robustness of a network, see for instance [1], [2], [11], [19], [20]. Freitas et al. [6] classify robustness metrics into three types: metrics based on structural properties, such as edge connectivity or diameter; metrics based on the spectrum of the adjacency matrix, such as the spectral radius or spectral gap; and metrics based on the spectrum of the Laplacian matrix, for instance the algebraic connectivity and the effective graph resistance. In this paper we consider the following optimisation problem: how to augment a given graph G by adding at most k links, such that the robustness metric of the augmented network is optimal. As robustness metric we consider the effective graph resistance R_G, also known as the Kirchhoff index, see Ellens et al. [4]. The effective graph resistance not only covers the shortest path between any pair of nodes, but incorporates all paths between any two nodes. Because in addition R_G decreases upon the addition of a link to the graph [9], this makes the effective graph resistance a good metric to evaluate the robustness of a network.

Predari et al. refer to the optimisation problem at hand as k-Graph Robustness Improvement Problem (k-GRIP) [18], in which one has to decide where k links are to be added to a given network G, such that the robustness metric is optimised. Several researchers investigated k-GRIP for specific robustness metrics. For instance, Wang et al. [21] considered 1-GRIP, with as robustness metric the second-smallest eigenvalue of the Laplacian matrix, which was coined the algebraic connectivity by Fiedler [5]. They suggest several strategies to decide which single link to add to the network, in order to increase the algebraic connectivity as much as possible. A nice overview of k-GRIP for the algebraic connectivity is presented in [12]. The NP-hardness of k-GRIP for the algebraic connectivity was proved in [14].

For the effective graph resistance, 1-GRIP was considered by Wang et al. [22]. They investigated different strategies, based upon topological and spectral properties of the graph, to determine the most optimal link to add, and derived a lower bound for R_G after adding a single link. Pizzuti et al. [16], [17] proposed and evaluated several genetic algorithms to find the most optimal link to add, in order to minimize R_G. Clemente et al. [3] studied k-GRIP for the effective graph resistance and gave lower bounds for R_G upon the addition of k links, under some mild conditions for k. For $k = 1$ the lower bound in [3] clearly outperforms the lower bound in [22]. Predari et al. [18] also consider k-GRIP for the effective graph resistance. They focus on heuristics for k-GRIP, based upon sampling and a fast approximation method, to compute the effective graph resistance.

Although for some choices of the robustness metric, k-GRIP is known to be NP-hard, to the best of our knowledge this has not been proved yet for the effective graph resistance. The aim of this paper is to prove that augmenting a given graph G by adding k links, in order to minimize the effective graph resistance, is NP-hard. Note that [9] considered the optimisation problem of the effective graph resistance in the case of weighted links. They pro-

* Corresponding author at: Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA, Delft, the Netherlands.

E-mail addresses: R.E.Kooij@tudelft.nl (R.E. Kooij), M.A.Achterberg@tudelft.nl (M.A. Achterberg).

https://doi.org/10.1016/j.orl.2023.10.002
0167-6377/

vide an efficient (polynomial-time) algorithm under the condition that the sum of the weights is constant. In this paper, however, the graph G is considered unweighted and simple.

2. Definitions and main result

In this paper we consider undirected, connected simple graphs $G = (V, E)$ without self-loops. Here V denotes the set of N nodes, while E is the set of L links connecting node pairs of V. The notation $i \sim j$ indicates that nodes i and j are adjacent in G. We let $G^c = (V, E^c)$ denote the complementary graph of G, where $E^c = \{(u, v)|u, v \in V, u \neq v, (u, v) \notin E\}$. The adjacency matrix A of G is an $N \times N$ symmetric matrix with elements a_{ij} that are either 1 or 0 depending on whether there is a link between nodes i and j or not. The Laplacian matrix Q of G is an $N \times N$ symmetric matrix $Q = \Delta - A$, where $\Delta = diag(d_i)$ is the $N \times N$ diagonal degree matrix with the elements $d_i = \sum_{j=1}^{N} a_{ij}$. The eigenvalues of Q are all real and non-negative and can be ordered as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$.

Interpreting the graph G as an electrical network whose links are resistors of 1Ω, the effective resistance ω_{ij} between node i and j can be computed based on Kirchoff's circuit laws. Then the *effective graph resistance* R_G, also known as the *Kirchhoff index*, is defined as the sum of the resistances over all node pairs [10]

$$R_G(G) = \sum_{1 \leq i < j \leq N} \omega_{ij}. \tag{1}$$

Klein and Randić [10] showed that the effective graph resistance can also be computed using the Laplacian eigenvalues λ_k of the graph G as

$$R_G(G) = N \sum_{k=2}^{N} \frac{1}{\lambda_k}. \tag{2}$$

Ellens et al. [4] argued that the effective graph resistance is an appropriate robustness metric. Note that the smaller the value of R_G the larger the robustness of the network. The smallest value of the effective graph resistance for a graph on N nodes is obtained for the complete graph K_N and satisfies $R_G(K_N) = N - 1$. We will show in this paper that adding a specified number of links to a given graph, in order to minimize the effective graph resistance, is NP-hard. We will now give an explicit description of the considered optimisation problem.

Problem 1 *(Minimum effective graph resistance augmentation problem).* Given an undirected, connected, simple graph $G = (V, E)$, a non-negative integer k and a non-negative threshold t, is there a subset $B \subseteq E^c$ of size $|B| \leq k$ such that the graph $H = (V, E \cup B)$ satisfies $R_G(H) \leq t$?

Problem 1 is clearly in NP, because given a graph G and the set of added links B, the correctness of the given solution can be verified by computing the eigenvalues of the Laplacian matrix, which is an $\mathcal{O}(N^3)$ operation. Then simply computing (2) and comparing the outcome with the given threshold t verifies the solution. Thus the minimum effective graph resistance augmentation problem is in NP.

Problem 1 is the decision version of the following optimisation problem: Given an undirected, connected, simple graph $G = (V, E)$ and a non-negative threshold t, find a set of currently non-existent links of minimum size to add to G such that the effective graph resistance R_G of the augmented graph is at most t. We prove in this work that Problem 1 is NP-hard, which immediately implies that the corresponding optimisation problem is also NP-hard. Thus, the problem of adding a specified number of links to a graph to minimize the effective graph resistance is also NP-hard. We now state the main result of the paper.

Theorem 2. *The minimum effective graph resistance augmentation problem is NP-hard.*

3. Proof of Theorem 2

The proof of Theorem 2 heavily relies on the proof of the NP-hardness of the maximum algebraic connectivity augmentation problem, as given in [14]. The proof is by reduction of our augmentation problem to a problem for which NP-hardness has been proved, namely the 3-colorability problem, see [7]. For our proof we will use a construction and a lemma from [14] and two additional lemmas.

Construction. *[14] Given a graph $G = (V, E)$ with $n > 1$ nodes and m links, a graph $G' = (V', E')$ is constructed which consists of three disjoint copies G_0, G_1 and G_2 of G. This implies that each node $v \in V$ is copied to a node $v_i \in G_i$ and each link $(u, v) \in E$ is copied to $(u_i, v_i) \in G_i$, for $i = 0, 1, 2$. By construction the graph G' has $3n$ nodes and $3m$ links. We now consider the minimum effective graph resistance augmentation problem on G' with $k = 3n^2 - 3m$, such that the augmented graph H has at most $3n^2$ links and $t = \frac{9n-5}{2}$.*

Now, let $K_{n,n,n}$ denote the complete tripartite graph. In order to prove that the minimum effective graph resistance augmentation problem can be reduced to the 3-colorability problem, we will use the following three lemmas.

Lemma 3. *[14] There exists a subset $B \subseteq (E')^c$ of size $|B| \leq k$ such that $H = (V', E' \cup B)$ is (isomorphic to) $K_{n,n,n}$ if and only if G is 3-colorable.*

Lemma 4. *[13] Let G be a simple connected graph with $N \geq 2$ nodes and L links. Then*

$$R_G(G) \geq \frac{N^2(N-1)}{2L} - 1,$$

with equality if and only if $G \cong K_N$, or $G \cong K_{N/2,N/2}$, or $G \in \Gamma_d$.

Here, Γ_d denotes a special class of d-regular graphs defined in [15]. Let $M(i)$ be the set of all neighbours of the node i, that is, $M(i) = \{k|k \in V, k \sim i\}$, where V denotes the set of nodes of the graph. Then for every $1 \leq d \leq n-1$ the set Γ_d denotes the set of all d-regular graphs with diameter 2 and satisfying $|M(i) \cap M(j)| = d$ for every pair of nodes i, j that are not adjacent, i.e. $i \nsim j$.

Lemma 5. *The complete tripartite graph $K_{n,n,n}$ on $3n$ nodes has effective graph resistance $R_G(K_{n,n,n}) = \frac{9n-5}{2}$.*

Proof. We compute the effective graph resistance R_G of the complete tripartite graph $K_{n,n,n}$ using Eq. (1). Gervacio [8] derived the effective resistance between nodes in complete multipartite graphs as:

$$\omega_{ij} = \frac{2}{N - m_i}, \qquad \text{if } i, j \text{ are in the same partition}$$

$$\omega_{ij} = \frac{(N-1)(2N - m_i - m_j)}{N(N - m_i)(N - m_j)}, \qquad \text{otherwise}$$

where m_i and m_j represent the size of the partition of node i and j respectively. In our case, $N = 3n$ and $m_i = m_j = n$. The number of node pairs in the same partition equals $3n(n-1)/2$ and the number of pairs outside of the same partition equals $3n^2$. Then the effective graph resistance of the complete tripartite graph exactly equals $R_G(K_{n,n,n}) = \frac{9n-5}{2}$. □

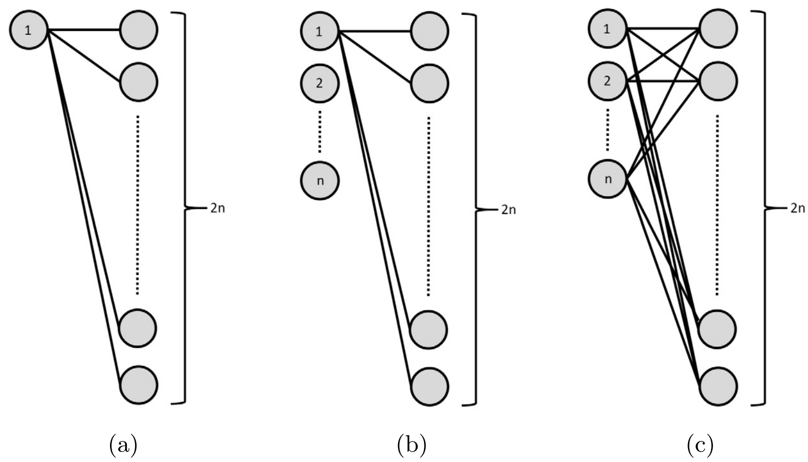

Fig. 1. (a) Node 1 and its $2n$ neighbours. (b) Node 1, its $2n$ neighbours and the $n-1$ remaining nodes. (c) Nodes $\{1, \cdots, n\}$ and their connections to the other $2n$ nodes.

Lemma 6. *A graph $H = (V, E)$ with $N = 3n$ nodes and $L \leq 3n^2$ links for $n > 1$ satisfies $R_G(H) \leq \frac{9n-5}{2}$ if and only if H is (isomorphic to) $K_{n,n,n}$.*

Proof. The backward direction is satisfied by Lemma 5.

To prove the forward direction, using $N = 3n$, $L \leq 3n^2$ and Lemma 4, it follows $R_G(H) \geq \frac{9n^2(3n-1)}{6n^2} - 1 = \frac{9n-5}{2}$. By the condition $R_G(H) \leq \frac{9n-5}{2}$, we deduce that $R_G(H) = \frac{9n-5}{2}$. Also it follows that $L = 3n^2$ because $N = 3n$ and $L < 3n^2$ would imply $R_G(H) > \frac{9n-5}{2}$ according to Lemma 4. Therefore the average degree of H equals $2n$. Since $R_G(H)$ is equal to the lower bound given in Lemma 4, H is either the complete graph K_{3n}, the complete bipartite graph $K_{3n/2,3n/2}$ or it is a $2n$-regular graph belonging to the class Γ_{2n}. First, assume $H \cong K_{3n}$. The number of links of K_{3n} equals $\frac{3n(3n-1)}{2}$ which, for $n > 1$, is larger than $3n^2$, the number of links of H. Therefore $H \ncong K_{3n}$. Next assume $H \cong K_{3n/2,3n/2}$, which can only hold for n even. Then the number of links of $K_{3n/2,3n/2}$ equals $\frac{9n^2}{4}$ which is always smaller than $3n^2$, the number of links of H. Therefore $H \ncong K_{3n/2,3n/2}$. Hence we conclude that the graph H is $2n$-regular and belongs to the class Γ_{2n}.

We will now show that H is isomorphic to $K_{n,n,n}$. We start with an arbitrary node of H and label it as node 1. Because H is $2n$-regular, node 1 has exactly $2n$ neighbours, see Fig. 1a.

The remaining $n-1$ nodes, other than node 1 and its $2n$ neighbours, cannot be adjacent to node 1 because it already has degree $2n$, by construction. We now label these nodes as nodes 2 until n, see Fig. 1b. Now, because H belongs to the class Γ_{2n} and nodes 2 until n are not adjacent to node 1, each of the nodes 2 until n has exactly the same neighbours as node 1, see Fig. 1c.

Next, take an arbitrary node outside the set $\{1, 2, \cdots, n\}$ and label it as $n+1$. To obtain degree $2n$, node $n+1$ needs to be adjacent to n nodes outside the nodes $\{1, 2, \cdots, n\}$. We label this set of n adjacent nodes as $\{2n+1, \cdots, 3n\}$, see Fig. 2a.

Finally, every node not in $\{1, 2, \cdots, n+1\} \cup \{2n+1, \cdots, 3n\}$ needs to share with node $n+1$ its neighbours $\{2n+1, \cdots, 3n\}$, see Fig. 2b.

Denote by S_i the nodes labelled as $\{n(i-1)+1, n(i-1)+2, \cdots, n(i-1)+n\}$, for $i = 1, 2, 3$. Then $|S_i| = n$, every node pair within S_i is not adjacent and for every $i \neq j$ all nodes in S_i are adjacent to all nodes in S_j. This proves that H is a complete tripartite graph $K_{n,n,n}$. $\square$

Finally, Theorem 2 follows from combining Lemma 3 and 6.

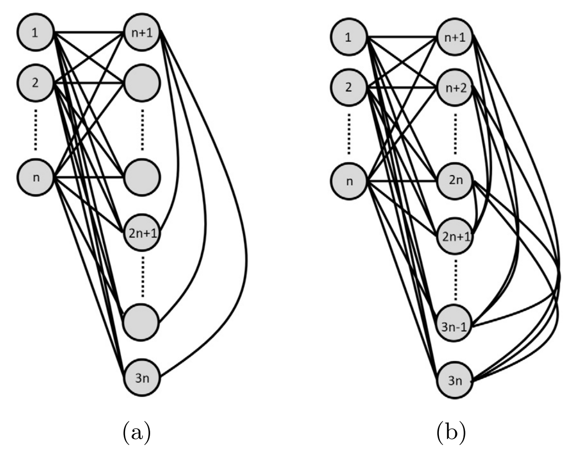

Fig. 2. (a) Nodes $\{1, \cdots, n\}$, their connections to the other $2n$ nodes and the additional n connections of node $n+1$. (b) All connections in graph H.

Funding

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Declaration of competing interest

The authors have no relevant financial or non-financial interests to disclose.

Data availability

No data was used for the research described in the article.

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