

# DUAL-BASED A-POSTERIORI ERROR ESTIMATION FOR FLUID-STRUCTURE INTERACTION BY THE EMBEDDED DOMAIN METHOD

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**Key words:** Fluid-Structure Interaction, Free-Boundary Problems, A-Posteriori Error Estimation, Goal-Oriented Error Estimation, Dual-Based Error Estimation, Adaptive Finite Element Method

**Abstract.** *Numerical simulations of fluid-structure interaction typically require vast computational resources. Finite-element techniques employing goal-oriented hp-adaptation strategies could offer a substantial improvement in the efficiency of such simulations. These strategies rely on dual-based a-posteriori error estimates for quantities of interest. However, the free-boundary character of fluid-structure-interaction problems forms a fundamental complication, as it yields the underlying domain unknown a-priori. Instead, the domain comprises part of the solution. Consequently, the well-established generic framework for dual-based error estimation is not applicable.*

*In this work we develop a framework for dual-based a-posteriori error estimation for free-boundary problems such as fluid-structure interaction. The framework is based on the embedded-domain approach and an extension operator which enables the comparison of approximate solutions on distinct domains. Given an approximate fluid and structure solution, we present a dual problem on the corresponding approximate fluid domain. Finally, we employ the dual solution to present an exact error representation formula.*

## 1 INTRODUCTION

The numerical solution of fluid-structure-interaction problems is a challenging endeavor. Typically, in numerical procedures for fluid-structure interaction, vast computational resources are consumed by the subsystem that is of least practical interest, viz., the fluid. Finite-element techniques employing *goal-oriented hp-adaptation* strategies could offer a substantial improvement in the efficiency of such simulations. These adaptation

strategies rely on dual-based a-posteriori error estimates for quantities of interest. Considerable work on a-posteriori error estimation and adaptation for *generic boundary value problems* and the corresponding *canonical variational formulations* has been performed by Oden and Prudhomme, Becker and Rannacher, a.o.; see Refs [2, 8].

The aforementioned framework for a-posteriori error estimation and adaptation does however not encompass fluid-structure interaction owing to the free-boundary character of the interface between the fluid and the structure. The position of the interface and, hence, the shape of the fluid domain are unknown a-priori. Instead, they form part of the solution. Consequently, fluid-structure-interaction problems lead to so-called *shape-dependent variational formulations*, which are noncanonical. This forms a significant complication for the dual-based error estimation.

In the present work we develop a framework for dual-based a-posteriori error estimation for fluid-structure interaction. To enable a comparison, the discrepancy in the underlying domains of distinct approximate solutions is accommodated by embedding all domains in a sufficiently-large fixed hold-all domain and by introducing an extension operator which extends functions to this hold-all domain. Then, given an approximate fluid and structure solution, we propose a dual problem on the corresponding approximate fluid domain and, as usual, the primal residual functional evaluated at the dual solution yields the error estimate. We furthermore employ the dual solution to obtain an exact error representation formula.

The contents of this paper are arranged as follows: Section 2 presents the fluid-structure-interaction model problem in shape-dependent variational form and introduces several prototypical quantities of interest. In section 3 we review dual-based error estimation for canonical variational formulations. Furthermore, we elucidate the complication of shape-dependent variational formulations with respect to canonical forms. Using the embedded-domain approach, we develop in section 4 the dual-based error estimation for fluid-structure interaction. Numerical experiments for a simple application are presented in section 5. Finally, section 6 contains concluding remarks.

## 2 PROBLEM STATEMENT

For simplicity, we restrict ourselves to a simple model problem which possesses the prominent features of fluid-structure interaction. We assume that the fluid and the structure separately are governed by linear differential equations. It is to be noted, however, that the aggregated fluid-structure-interaction problem can be nonlinear on account of geometric nonlinearity induced by the coupling at the fluid-structure interface. This nonlinearity originates from the variations in the fluid domain and from the external force by the fluid on the structure in the displaced structure configuration. Moreover, we assume that it is appropriate to select identical test and trial spaces for the variational formulation of the fluid problem.

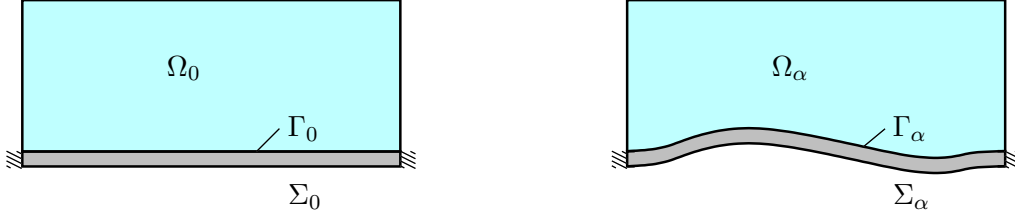


Figure 1: Definition of fluid and structure domains and interface for the 2D panel flow problem. Reference structure configuration ( $\alpha = 0$ ) [left] and current structure configuration due to structure displacement  $\alpha$  [right].

## 2.1 Fluid-structure-interaction problem

To formulate the variational problem associated with the fluid, we consider in conjunction with each (admissible) structural displacement  $\alpha$  the open bounded fluid domain  $\Omega_\alpha \subset \mathbb{R}^N$  ( $N = 1, 2, 3$ ). The boundary  $\partial\Omega_\alpha$  of the fluid domain consists of the *interface*  $\Gamma_\alpha$  between the fluid and structure and the fixed boundary  $\partial\Omega_\alpha \setminus \Gamma_\alpha$ . See Fig. 1 for an illustration of the domains for the familiar 2D panel flow problem [6].

On the domain  $\Omega_\alpha$  the fluid is subject to a boundary-value problem which we condense into the abstract variational statement:

$$\begin{aligned} \text{Find } u \in U(\Omega_\alpha) : \\ F_{\Omega_\alpha}(u, v) = f_{\Omega_\alpha}(v) , \quad \forall v \in U(\Omega_\alpha) , \end{aligned} \tag{1a}$$

where the fluid solution  $u$  and test function  $v$  are elements of an appropriate Hilbert (or Banach) space  $U(\Omega_\alpha)$  on the variable fluid domain  $\Omega_\alpha$ . Furthermore,  $F_{\Omega_\alpha} : U(\Omega_\alpha) \times U(\Omega_\alpha) \rightarrow \mathbb{R}$  and  $f_{\Omega_\alpha} : U(\Omega_\alpha) \rightarrow \mathbb{R}$  are a bilinear and linear functional, respectively. We use the subscript  $\Omega_\alpha$  on  $F$  and  $f$  to indicate that these are *shape* functionals which depend on the fluid domain  $\Omega_\alpha$ .

To formulate the variational problem associated with the structure, we consider the structural reference configuration  $\Sigma_0 \subset \mathbb{R}^N$  and we associate with each structural displacement field  $\alpha$  in the Hilbert (or Banach) space  $A(\Sigma_0)$  the deformed structure configuration  $\Sigma_\alpha \supset \Gamma_\alpha$ . We condense the boundary-value problem for the structure into the abstract variational statement:

$$\begin{aligned} \text{Find } \alpha \in A(\Sigma_0) : \\ S(\alpha, \beta) = s_{\alpha, u}(\beta) , \quad \forall \beta \in A(\Sigma_0) , \end{aligned} \tag{1b}$$

where  $S : A(\Sigma_0) \times A(\Sigma_0) \rightarrow \mathbb{R}$  is a bilinear form corresponding to the standard internal work and  $s_{\alpha, u} : A(\Sigma_0) \rightarrow \mathbb{R}$  is a linear form corresponding to the external work. The latter contains the loads exerted by the fluid on the structure in the deformed configuration. Therefore, it bears a (nonlinear) dependence on  $\alpha$  and  $u$ . This is indicated by the subscript.

We combine the two separate variational equations into one aggregated equation by introducing the semilinear form (nonlinear in its first entry)

$$\begin{aligned} ((\alpha, u), (\beta, v)) \mapsto C((\alpha, u); (\beta, v)) &:= F_{\Omega_\alpha}(u, v) - f_{\Omega_\alpha}(v) + S(\alpha, \beta) - s_{\alpha, u}(\beta) : \\ (A(\Sigma_0) \times U(\Omega_\alpha)) \times (A(\Sigma_0) \times U(\Omega_\alpha)) &\rightarrow \mathbb{R} \end{aligned} \quad (2)$$

The coupled fluid-structure system is then described by

Find  $(\alpha, u) \in A(\Sigma_0) \times U(\Omega_\alpha) :$   
 $C((\alpha, u); (\beta, v)) = 0 \ , \quad \forall (\beta, v) \in A(\Sigma_0) \times U(\Omega_\alpha) \ .$

(3)

As the structure and the fluid are involved simultaneously, (3) constitutes a nonlinear mixed variational problem. However, (3) is a noncanonical variational problem in that  $u$  and  $v$  reside in the shape-dependent function space  $U(\Omega_\alpha)$  which depends on the unknown  $\alpha$ . We therefore refer to (3) as a *shape-dependent variational formulation*. Throughout, we assume that (3) has a (locally unique) solution  $(\alpha, u)$ .

## 2.2 Errors in quantities of interest

In particular, our interest will be restricted to specific quantities  $q(\alpha, u) \in \mathbb{R}$  (*objective functionals*) of the solution  $(\alpha, u)$  of (3), rather than the solution itself. It is to be noted that the functional  $q(\cdot, \cdot)$  can be domain dependent via  $\alpha$ , i.e.,

$$(\alpha, u) \mapsto q(\alpha, u) : A(\Sigma_0) \times U(\Omega_\alpha) \rightarrow \mathbb{R} . \quad (4)$$

A quantity of interest can for instance be the weighted-average of the fluid solution  $u$  on  $\Omega_\alpha$  (assuming  $u$  to be scalar):

$$q^{\text{ave}}(\alpha, u) := \int_{\Omega_\alpha} \varphi u \, d\Omega ,$$

where the weight  $\varphi : \Omega_\alpha \rightarrow \mathbb{R}$  is a prescribed smooth function. Another example of a quantity of interest pertains to properties of the fluid solution at the interface:

$$q^{\text{fb}}(\alpha, u) := \int_{\Gamma_\alpha} \varphi \gamma(u) \, d\Gamma ,$$

where again the weight  $\varphi : \Gamma_\alpha \rightarrow \mathbb{R}$  is a given smooth function and where  $\gamma$  denotes a trace operator.

Let  $(\alpha^h, u^h) \in A(\Sigma_0) \times U(\Omega_{\alpha^h})$  be an approximation, obtained by the Galerkin finite-element method for example. The corresponding approximate value of the quantity of interest is  $q(\alpha^h, u^h)$ . Given our restricted interest, the primary measure for the accuracy of the approximation is the target-quantity error

$$\mathcal{E}_q := q(\alpha, u) - q(\alpha^h, u^h) .$$

Our objective is to develop for (3) a framework for dual-based a-posteriori estimation of the error in the quantity of interest,  $\mathcal{E}_q$ , analogous to the existing framework for canonical variational problems. In the next section we review relevant theory regarding a-posteriori error estimation for canonical variational problems and, subsequently, we develop an analogous framework accommodating shape-dependent variational formulations such as (3).

### 3 DUAL-BASED ERROR ESTIMATION FOR CANONICAL VARIATIONAL FORMULATIONS

A general paradigm for the a-posteriori error estimation of quantities of interest (objective functionals) has been established for canonical variational formulations in [2, 8]. Essentially, a computable error estimate is obtained by evaluating the primal residual functional at the solution of an appropriately defined dual problem. This section presents a brief summary of the theory.

#### 3.1 Canonical setting

Consider the canonical variational problem

$$\boxed{\begin{array}{l} \text{Find } u \in U : \\ B(u; v) = b(v) , \quad \forall v \in U , \end{array}} \quad (5)$$

where  $B(\cdot; \cdot)$  is a semilinear form (nonlinear in the first entry) and  $b(\cdot)$  a linear functional on the Banach space  $U$  with norm  $\|\cdot\|$ . The quantity of interest is the value of the (possibly nonlinear) functional

$$q : U \rightarrow \mathbb{R} \quad (6)$$

for the solution  $u$  of (5). Let  $u^h \in U$  be any approximation to the solution  $u$  of (5). The purpose of a-posteriori error estimation is to obtain a computable estimate of the target quantity error

$$\mathcal{E}_q := q(u) - q(u^h) .$$

#### 3.2 Dual-based error representation

A dual-based approach to this estimation solves the linear dual (or adjoint) problem

$$\boxed{\begin{array}{l} \text{Find } z \in U : \\ B'(u^h; z)(\delta u) = q'(u^h)(\delta u) , \quad \forall \delta u \in U , \end{array}} \quad (7)$$

where  $(\cdot)'$  indicates the Gâteaux derivative of  $(\cdot)$  with respect to its arguments up to the semi-colon “;”. That is,  $B'(u^h; z)$  and  $q'(u^h)$  are linear functionals on  $U$  such that

$$\begin{aligned} B'(u^h; z)(\delta u) &= \lim_{t \rightarrow 0} \frac{B(u^h + t \delta u; z) - B(u^h; z)}{t} , \\ q'(u^h)(\delta u) &= \lim_{t \rightarrow 0} \frac{q(u^h + t \delta u) - q(u^h)}{t} , \end{aligned}$$

for all  $\delta u \in U$ .

Let us define the residual functional at  $u^h$  corresponding to the primal problem (5) as

$$R(u^h; \cdot) := b(\cdot) - B(u^h; \cdot) .$$

Furthermore, we set the error  $e := u - u^h$ . The dual solution  $z$  is the key element in linking the primal problem with the error in the quantity of interest. This is expressed in the following a-posteriori error representation formula:

**Theorem 1 (Becker and Rannacher [2])** *Given any approximation  $u^h \in U$  of the solution  $u$  of (5), let  $z \in U$  be the solution of the dual problem (7). It holds that*

$$\boxed{\mathcal{E}_q := q(u) - q(u^h) = R(u^h; z) + \mathcal{R}} , \quad (8)$$

with remainder

$$\mathcal{R} := \int_0^1 \left( q''(u^h + te)(e)(e) - B''(u^h + te, z)(e)(e) \right) (1 - t) dt .$$

**Proof** The proof is based on standard Taylor-series formulas such as

$$q(u) - q(u^h) = q'(u^h)(e) + \int_0^1 q''(u^h + te)(e)(e)(1 - t) dt ,$$

and can be found in [2]. ■

Note that the remainder term  $\mathcal{R}$  in (8) is quadratic in the error  $e$ . Hence, the residual evaluated with the dual solution,  $R(u^h; z)$ , provides an error estimate for  $\mathcal{E}_q$  which is second-order accurate. It is exact if  $B(\cdot; \cdot)$  and  $q(\cdot)$  are linear functionals.

### 3.3 Approximate dual problem

The dual problem (7) cannot generally be solved exactly and we will have to content ourselves with approximations instead. Let  $z^h \in U$  be an approximation to the solution  $z$  of (7). Furthermore, we set  $\bar{e} := z - z^h$ . We clearly have the representation formula

$$\mathcal{E}_q = R(u^h; z^h) + R(u^h; \bar{e}) + \mathcal{R} . \quad (9)$$

Eq. (9) is the basis for many a-posteriori error estimates and adaptive finite-element techniques. We refer to [4, 5] for additional details and supplementary examples.

### 3.4 Complication for shape-dependent variational formulations

The above general paradigm for error estimation is unsuitable for the shape-dependent variational formulation (3), as (3) does not conform to the canonical variational form (5). In particular, the space  $U(\Omega_\alpha)$  which accomodates the solution to the fluid problem depends on the unknown  $\alpha$ .

To elucidate this complication, let us consider an approximation  $(\alpha^h, u^h) \in A(\Sigma_0) \times U(\Omega_{\alpha^h})$  of the solution  $(\alpha, u) \in A(\Sigma_0) \times U(\Omega_\alpha)$  of (3) and let us try to obtain an error representation formula as in Theorem 1. We need a Taylor-series formula of the form:

$$q(\alpha, u) - q(\alpha^h, u^h) = q'(\alpha^h, u^h)(e_\alpha, e_u) + \text{higher order terms} ,$$

where  $e_\alpha := \alpha - \alpha^h \in A(\Sigma_0)$ . However, at this point it is not clear what  $e_u$  should be. Interpreting the derivative  $q'(\alpha^h, u^h)$  as a linear functional on  $A(\Sigma_0) \times U(\Omega_{\alpha^h})$ , it follows that  $e_u \in V(\Omega_{\alpha^h})$ . However, it cannot be equal to  $u - u^h$ ; this does not make sense since  $u \in V(\Omega_\alpha)$  and  $u^h \in V(\Omega_{\alpha^h})$ , i.e., the actual solution  $u$  and its approximation  $u^h$  are defined on different domains. A proper interpretation of  $e_u$  is presented below.

#### 4 DUAL-BASED ERROR ESTIMATION FOR SHAPE-DEPENDENT VARIATIONAL FORMULATIONS

In this section we describe a framework for the dual-based a-posteriori error estimation of the shape-dependent variational formulation (3). It is based on a conception of the variable fluid domains as subsets embedded in a sufficiently-large fixed hold-all domain. The domain-dependent functions can then be extended onto the fixed hold-all domain. In this manner, the shape-dependent variational form can be recast into canonical form, which in turn allows us to formulate the dual-problem appropriately. As usual, evaluating the primal residual functional with the dual solution yields the error estimate. In addition, we also present an exact error representation formula.

##### 4.1 Embedded-domain approach

Consider a large-enough fixed hold-all open domain  $\Omega_E$ , such that for each admissible  $\alpha$  the corresponding fluid domain  $\Omega_\alpha$  is a subset of  $\Omega_E$ . See Fig. 2 for an illustration of the fixed hold-all domain for the 2D panel flow problem. If the fluid domains  $\Omega_\alpha$  are Lipschitzian, then there generally exists a continuous linear *extension operator*  $E_\alpha : U(\Omega_\alpha) \rightarrow U(\Omega_E)$  which extends functions from  $\Omega_\alpha$  onto  $\Omega_E$ , i.e.,

$$(E_\alpha u)|_{\Omega_\alpha} = u , \quad \forall u \in U(\Omega_\alpha) .$$

See, e.g., [1], page 83.

Let  $u_E$  and  $v_E$  denote functions in  $U(\Omega_E)$ . We define the extended form  $C_E : (A(\Sigma_0) \times U(\Omega_E)) \times (A(\Sigma_0) \times U(\Omega_E)) \rightarrow \mathbb{R}$  of the semilinear functional  $C$  given in (2) as

$$C_E((\alpha, u_E); (\beta, v_E)) := C((\alpha, u_E|_{\Omega_\alpha}); (\beta, v_E|_{\Omega_\alpha})) . \quad (10)$$

Clearly, the extended solution  $(\alpha, E_\alpha(u))$  of our shape-dependent variational formulation (3) is a solution of the extended problem

Find  $(\alpha, u_E) \in A(\Sigma_0) \times U(\Omega_E) :$   
 $C_E((\alpha, u_E); (\beta, v_E)) = 0 , \quad \forall (\beta, v_E) \in A(\Sigma_0) \times U(\Omega_E) .$

(11)

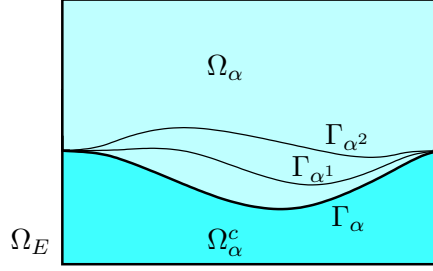


Figure 2: Illustration of the fixed hold-all domain  $\Omega_E = \Omega_\alpha \cup \Gamma_\alpha \cup \Omega_\alpha^c$  used in the embedded-domain approach for the 2-D panel flow problem.

The extended formulation (11) has a unique solution  $u_E$  up to its value on the complement  $\Omega_\alpha^c := \Omega_E \setminus (\Omega_\alpha \cup \Gamma_\alpha)$ . The extended form of the quantity of interest is straightforwardly defined as the functional  $q_E : A(\Sigma_0) \times U(\Omega_E) \rightarrow \mathbb{R}$ ,

$$q_E(\alpha, u_E) := q(\alpha, u_E|_{\Omega_\alpha}) . \quad (12)$$

Note that the variational problem (11) and the quantity of interest (12) are in the canonical form according to (5) and (6), respectively. This allows us to employ the framework for dual-based error estimation described in section 3.

## 4.2 Dual-based error representation

Let  $(\alpha^h, u^h) \in A(\Sigma_0) \times U(\Omega_{\alpha^h})$  be an approximation to the fluid-structure-interaction model problem (3). The extension of  $u^h$  is  $E_{\alpha^h} u^h \in U(\Omega_E)$ . Following the approach of section 3, we obtain the dual problem by linearizing the extended forms (11) and (12) about  $(\alpha^h, E_{\alpha^h} u^h)$ :

$$\begin{aligned} \text{Find } (\zeta, z_E) \in A(\Sigma_0) \times U(\Omega_E) : \\ C'_E((\alpha^h, E_{\alpha^h} u^h); (\zeta, z_E))(\delta\alpha, \delta u_E) &= q'_E(\alpha^h, E_{\alpha^h} u^h)(\delta\alpha, \delta u_E) , \\ \forall (\delta\alpha, \delta u_E) \in A(\Sigma_0) \times U(\Omega_E) . \end{aligned} \quad (13)$$

Similar to the extended problem (11), we assume that the dual problem (13) has a unique dual solution  $z_E$  up to its value on the complement  $\Omega_{\alpha^h}^c = \Omega_E \setminus (\Omega_{\alpha^h} \cup \Gamma_{\alpha^h})$ . To resolve the nonuniqueness, note that

$$C'_E((\alpha^h, E_{\alpha^h} u^h); (\zeta, z_E))(\delta\alpha, \delta u_E) = C'((\alpha^h, u^h); (\zeta, z_E|_{\Omega_{\alpha^h}}))(\delta\alpha, \delta u_E|_{\Omega_{\alpha^h}}) , \quad (14)$$

$$q'_E(\alpha^h, E_{\alpha^h} u^h)(\delta\alpha, \delta u_E) = q'(\alpha^h, u^h)(\delta\alpha, \delta u_E|_{\Omega_{\alpha^h}}) , \quad (15)$$

where the Gâteaux derivatives  $C'((\alpha^h, u^h); (\zeta, z_E|_{\Omega_{\alpha^h}}))$  and  $q'(\alpha^h, u^h)$  are linear functionals on  $A(\Sigma_0) \times U(\Omega_{\alpha^h})$ . In conclusion, we can define the dual problem as the following



variational problem on the approximate domain  $\Omega_{\alpha^h}$ :

$$\boxed{\begin{aligned} \text{Find } (\zeta, z) \in A(\Sigma_0) \times U(\Omega_{\alpha^h}) : \\ C'((\alpha^h, u^h); (\zeta, z))(\delta\alpha, \delta u) = q'(\alpha^h, u^h)(\delta\alpha, \delta u) , \\ \forall (\delta\alpha, \delta u) \in A(\Sigma_0) \times U(\Omega_{\alpha^h}) . \end{aligned}} \quad (16)$$

That the solution  $(\zeta, z)$  of the dual problem (16) is indeed appropriate for linking the primal problem (3) with the error in the quantity of interest is expressed by the following theorem:

**Theorem 2 (Error representation)** *Given any approximation  $(\alpha^h, u^h) \in A(\Sigma_0) \times U(\Omega_{\alpha^h})$  of the solution  $(\alpha, u) \in A(\Sigma_0) \times U(\Omega_\alpha)$  of (3), let  $(\zeta, z) \in A(\Sigma_0) \times U(\Omega_{\alpha^h})$  be the solution of the dual problem (16). It holds that*

$$\boxed{\mathcal{E}_q := q(\alpha, u) - q(\alpha^h, u^h) = -C((\alpha^h, u^h); (\zeta, z)) + \mathcal{R}} \quad (17)$$

with remainder  $\mathcal{R} = \mathcal{R}_q + \mathcal{R}_C$ , where

$$\begin{aligned} \mathcal{R}_q &:= \int_0^1 q_E''(\alpha^h + te_\alpha, E_{\alpha^h}u^h + te_u)(e_\alpha, e_u)(e_\alpha, e_u)(1-t) dt , \\ \mathcal{R}_C &:= \int_0^1 -C_E''((\alpha^h + te_\alpha, E_{\alpha^h}u^h + te_u); (\zeta, E_{\alpha^h}z))(e_\alpha, e_u)(e_\alpha, e_u)(1-t) dt , \end{aligned}$$

and where  $e_\alpha := \alpha - \alpha^h$  and  $e_u := E_\alpha u - E_{\alpha^h}u^h$ .

The error representation formula (17) for shape-dependent variational formulations is the analogue of (8) for canonical variational problems. Formula (17) has a remainder which is quadratic in the errors  $e_\alpha$  and  $e_u$ . Therefore,  $-C((\alpha^h, u^h); (\zeta, z))$  provides an effective error estimate. Furthermore, (17) can be used as the basis for other a-posteriori error estimates and adaptive finite-element techniques.

Although shape-dependent variational formulations are inherently nonlinear, there is an interesting case for which the error estimate  $-C((\alpha^h, u^h); (\zeta, z))$  coincides with the actual error. This occurs if the error in the geometry vanishes, i.e.,  $e_\alpha = 0$ , and if the functionals are linearly-dependent on the fluid solution, i.e., the functionals  $C((\alpha, u); (\beta, v))$  and  $q(\alpha, u)$  are linear with respect to  $u$ .

**Proof of Theorem 2** The extended forms  $C_E$  and  $q_E$  of  $C$  and  $q$ , respectively, imply an appropriate Taylor-series formula. Let  $p_E(\cdot, \cdot)$  denote the functional  $C_E((\cdot, \cdot); (\beta, v))$  or  $q_E(\cdot, \cdot)$ . We clearly have the following Taylor-series formula for the extended forms:

$$\begin{aligned} p_E(\alpha, u_E) - p_E(\alpha^h, u_E^h) &= p_E'(\alpha^h, u_E^h)(e_\alpha, e_{E,u}) \\ &+ \int_0^1 p_E''(\alpha^h + se_\alpha, u_E^h + se_{E,u})(e_\alpha, e_{E,u})(e_\alpha, e_{E,u})(1-s) ds , \end{aligned} \quad (18)$$

where  $e_\alpha = \alpha - \alpha^h \in A(\Sigma_0)$  and  $e_{E,u} = u_E - u_E^h \in U(\Omega_E)$ . Hence, if we consider the target quantity error  $\mathcal{E}_q = q(\alpha, u) - q(\alpha^h, u^h)$  and make use of the extension operator to transform to the extended form, we can write

$$\begin{aligned}\mathcal{E}_q &= q_E(\alpha, E_\alpha u) - q_E(\alpha^h, E_{\alpha^h} u^h) \\ &= q'_E(\alpha^h, E_{\alpha^h} u^h)(e_\alpha, e_u) + \mathcal{R}_q \\ &= q'(\alpha^h, u^h)(e_\alpha, e_u|_{\Omega_{\alpha^h}}) + \mathcal{R}_q ,\end{aligned}$$

where we used (15) in the last step. Next, we use the dual problem (16) to obtain

$$\mathcal{E}_q = C'((\alpha^h, u^h); (\zeta, z))(e_\alpha, e_u|_{\Omega_{\alpha^h}}) + \mathcal{R}_q .$$

It then follows from (14), the Taylor-series formula (18) and (10) that

$$\begin{aligned}\mathcal{E}_q &= C'_E((\alpha^h, E_{\alpha^h} u^h); (\zeta, E_{\alpha^h} z))(e_\alpha, e_u) + \mathcal{R}_q \\ &= C_E((\alpha, E_\alpha u); (\zeta, E_{\alpha^h} z)) - C_E((\alpha^h, E_{\alpha^h} u^h); (\zeta, E_{\alpha^h} z)) + \mathcal{R}_q + \mathcal{R}_C \\ &= C((\alpha, u); (\zeta, (E_{\alpha^h} z)|_{\Omega_\alpha})) - C((\alpha^h, u^h); (\zeta, z)) + \mathcal{R}_q + \mathcal{R}_C .\end{aligned}$$

Finally, we obtain the proof by noting that  $C((\alpha, u); (\zeta, (E_{\alpha^h} z)|_{\Omega_\alpha})) = 0$  according to our primal problem (3). ■

## 5 APPLICATION

Our goal is to apply the above framework for dual-based error estimation to fluid-structure-interaction problems. However, the framework also encompasses other types of free-boundary problems. To exemplify the essential attributes of the framework, in this section we consider numerical experiments for a simple one-dimensional free-boundary problem. For this simple model problem, we derive the shape-dependent variational form, we formulate the dual problem and we present illustrative numerical results. In particular, we demonstrate by numerical computation that the dual-based error estimate indeed represents a second-order-accurate approximation of the true error, in accordance with Theorem 2.

### 5.1 One-dimensional free-boundary problem

In the one-dimensional setting, we characterize the variable domain as  $\Omega_\alpha = (0, \alpha) \subset \mathbb{R}$ . The free-boundary corresponds to a single point,  $\Gamma_\alpha = \{\alpha\}$ . The model free-boundary problem that we consider is to find the suitably smooth function  $u : \Omega_\alpha \rightarrow \mathbb{R}$  and the free-boundary position  $\alpha \in A(\Gamma_0) = \mathbb{R}$  subject to the following equations:

$$-u_{xx} = f , \quad \text{in } \Omega_\alpha , \tag{19a}$$

$$u = 0 , \quad \text{on } \partial\Omega_\alpha \setminus \Gamma_\alpha = \{0\} , \tag{19b}$$

$$u_x = 0 , \quad \text{on } \Gamma_\alpha , \tag{19c}$$

$$u = g , \quad \text{on } \Gamma_\alpha , \tag{19d}$$

where we indicate a spatial derivative using the subscript  $(\cdot)_x = \frac{d}{dx}(\cdot)$ . The data  $f$  and  $g$  are smooth and have support  $[0, \infty)$ .

We will be interested in the following quantities:

$$\begin{aligned} q^{\text{ave}}(\alpha, u) &= \int_{\Omega_\alpha} u(x) \, dx , \\ q^{\text{fb}}(\alpha, u) &= u(\alpha) . \end{aligned}$$

## 5.2 Shape-dependent variational formulation

We recast this problem into the shape-dependent variational form (3) by interpreting Eqs. (19a–c) as an ordinary elliptic boundary value problem with a homogeneous Dirichlet and Neumann condition, and (19d) as the free-boundary condition. Introducing the domain-dependent function space

$$U(\Omega_\alpha) = \{u \in H^1(\Omega_\alpha) \mid u(0) = 0\} ,$$

we condense (19a–c) into the standard variational form conforming to (1a)

$$\begin{aligned} \text{Find } u \in U(\Omega_\alpha) : \\ \int_{\Omega_\alpha} u_x v_x \, dx = \int_{\Omega_\alpha} f v \, dx , \quad \forall v \in U(\Omega_\alpha) , \end{aligned} \tag{20a}$$

The equivalent of Eq. (19d) conforming to (1b) is

$$\begin{aligned} \text{Find } \alpha \in \mathbb{R} : \\ g(\alpha)\beta = u(\alpha)\beta , \quad \forall \beta \in \mathbb{R} , \end{aligned} \tag{20b}$$

Adding (20a) and (20b), we obtain the final shape-dependent variational formulation of (19) conforming to (3) with

$$C((\alpha, u); (\beta, v)) = \int_{\Omega_\alpha} (u_x v_x - f v) \, dx + (g(\alpha) - u(\alpha))\beta .$$

As a digression, we mention that the existence of (possibly nonunique) solutions  $(\alpha, u)$  to (20) can be established using the Green's function  $\phi_\alpha^y \in U(\Omega_\alpha)$  for (20a), that is,

$$u(y) = \int_{\Omega_\alpha} \phi_\alpha^y(x) f(x) \, dx . \tag{21}$$

Substituting this in (20b) shows that an admissible solution for  $\alpha$  exists if and only if the map

$$\alpha \mapsto \int_{\Omega_\alpha} \phi_\alpha^\alpha(x) f(x) \, dx - g(\alpha)$$

has roots in  $(0, \infty)$ . Each root is a solution for  $\alpha$ . The corresponding solution for  $u$  is determined by Eq. (21).

### 5.3 Dual problem

Given an approximation  $(\alpha^h, u^h) \in \mathbb{R} \times U(\Omega_{\alpha^h})$ , the dual problem is given by (16), where in this case  $(\zeta, z) \in \mathbb{R} \times U(\Omega_{\alpha^h})$  and the derivative of  $C$  is given by

$$\begin{aligned} C'((\alpha^h, u^h); (\zeta, z))(\delta\alpha, \delta u) &= \int_{\Omega_{\alpha^h}} z_x \delta u_x \, dx - \zeta \delta u(\alpha^h) \\ &\quad + \left( (u_x^h z_x - fz)(\alpha^h) + (g_x - u_x^h)(\alpha^h) \zeta \right) \delta\alpha, \quad \forall (\delta\alpha, \delta u) \in \mathbb{R} \times U(\Omega_{\alpha^h}) \end{aligned}$$

Note that the first two terms originate from the differentiation at  $u^h$  in direction  $\delta u$ . The other terms are induced by the shape-derivative at  $\alpha^h$  in direction  $\delta\alpha$ . These terms only contain values at the free boundary  $\Gamma_{\alpha^h} = \{\alpha^h\}$ . We refer to [3, 7] for more details regarding shape derivatives.

The derivatives of the quantities of interest are

$$\begin{aligned} q^{\text{ave}'}(\alpha^h, u^h)(\delta\alpha, \delta u) &= \int_{\Omega_{\alpha^h}} \delta u \, dx + u^h(\alpha^h) \delta\alpha \\ q^{\text{fb}'}(\alpha^h, u^h)(\delta\alpha, \delta u) &= \delta u(\alpha^h) + u_x^h(\alpha^h) \delta\alpha. \end{aligned}$$

In this case, the dual-based error estimate  $-C((\alpha^h, u^h); (\zeta, z))$  can be divided in three contributions:

$$-C((\alpha^h, u^h); (\zeta, z)) = \int_{\Omega_{\alpha^h}} \underbrace{(f + u_{xx}^h)}_{r^{\text{int}}} z \, dx - \underbrace{u_x^h(\alpha^h)}_{r^{\text{Neu}}} z(\alpha^h) - \underbrace{(g(\alpha^h) - u^h(\alpha^h))}_{r^{\text{fbc}}} \zeta.$$

The first contribution can be attributed to the interior residual  $r^{\text{int}}$ , weighted by the dual solution  $z$ . The second contribution is due to the Neumann-boundary-condition residual  $r^{\text{Neu}}$ , weighted by the dual solution at the free boundary,  $z(\alpha^h)$ . The third contribution emanates from the residual pertaining to the free-boundary condition,  $r^{\text{fbc}}$ , weighted by the dual solution  $\zeta$ .

### 5.4 Numerical experiments

In the numerical experiments, we specify the data as:

$$f(x) = \frac{16}{9}, \quad g(x) = \frac{7}{2} - x,$$

which gives the solution

$$(\alpha, u(x)) = \left( \frac{3}{2}, \frac{8}{9}x(3-x) \right). \quad (22)$$

The exact values of the quantities of interest are

$$q^{\text{ave}}(\alpha, u) = q^{\text{fb}}(\alpha, u) = 2.$$

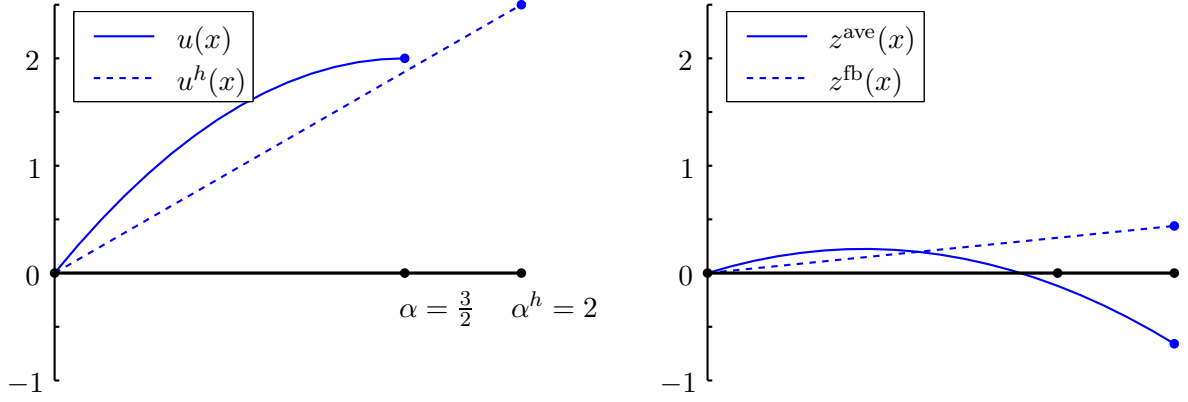


Figure 3: Exact solution  $(\alpha, u(x))$  and approximation  $(\alpha^h, u^h(x))$  of the free-boundary problem (20) [left]. Dual solutions  $z^{\text{ave}}(x)$  and  $z^{\text{fb}}(x)$  of the dual problem corresponding with the quantity of interest  $q^{\text{ave}}$  and  $q^{\text{fb}}$ , respectively [right].

**Dual-based error estimates** To show some typical results, we set the approximation

$$(\alpha^h, u^h(x)) = (2, \frac{5}{4}x) .$$

Figure 3 [left] plots  $u^h(x)$  and  $u(x)$ . The approximation gives the following approximate values for the quantities of interest:

$$q^{\text{ave}}(\alpha^h, u^h) = q^{\text{fb}}(\alpha^h, u^h) = \frac{5}{2} .$$

For the quantity of interest  $q^{\text{ave}}$ , it can be verified that the dual solution is given by

$$(\zeta^{\text{ave}}, z^{\text{ave}}(x)) = (\frac{109}{82}, x(\frac{55}{82} - \frac{1}{2}x)) .$$

See Fig. 3 [right] for a plot of  $z^{\text{ave}}(x)$ . The true value of the target-quantity error,  $\mathcal{E}_{q^{\text{ave}}}$ , and the corresponding dual-based estimate,  $\text{Est}_{q^{\text{ave}}}$ , are

$$\begin{aligned} \mathcal{E}_{q^{\text{ave}}} &= q^{\text{ave}}(\alpha, u) - q^{\text{ave}}(\alpha^h, u^h) = -\frac{1}{2} , \\ \text{Est}_{q^{\text{ave}}} &= -C((\alpha^h, u^h); (\zeta^{\text{ave}}, z^{\text{ave}})) = -\frac{2177}{4428} \approx -0.4916 . \end{aligned}$$

For the quantity of interest  $q^{\text{fb}}$ , the dual solution is given by

$$(\zeta^{\text{fb}}, z^{\text{fb}}(x)) = (\frac{32}{41}, \frac{9}{41}x) .$$

See Fig. 3 [right] for a plot of  $z^{\text{fb}}(x)$ . In this case, the true value of the error in the quantity of interest,  $\mathcal{E}_{q^{\text{fb}}}$ , and the corresponding dual-based estimate,  $\text{Est}_{q^{\text{fb}}}$ , are

$$\begin{aligned} \mathcal{E}_{q^{\text{fb}}} &= q^{\text{ave}}(\alpha, u) - q^{\text{ave}}(\alpha^h, u^h) = -\frac{1}{2} , \\ \text{Est}_{q^{\text{fb}}} &= -C((\alpha^h, u^h); (\zeta^{\text{ave}}, z^{\text{ave}})) = -\frac{45}{82} \approx -0.5488 . \end{aligned}$$

The above exposition clearly demonstrates the applicability of the established dual-based error-estimation framework for shape-dependent variational formulations.

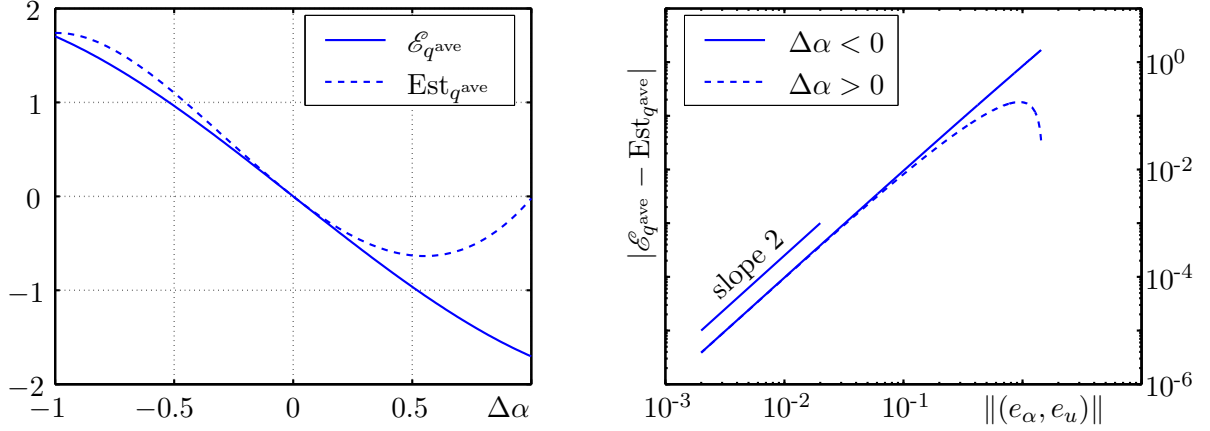


Figure 4: True value of the target quantity error  $\mathcal{E}_{q^{\text{ave}}}$  and the dual-based error estimate  $\text{Est}_{q^{\text{ave}}}$  for the  $\Delta\alpha$ -family of approximations  $(\alpha^h, u^h)$  given in (23) [left]. Convergence of the error estimate  $\text{Est}_{q^{\text{ave}}}$  towards the true error  $\mathcal{E}_{q^{\text{ave}}}$  with respect to the norm  $\|(e_\alpha, e_u)\|$  given in (24) [right].

**Convergence of error estimates** Error estimates for which we have an exact geometry ( $\alpha^h = \alpha$ ) are equal to the true error as our 1-D problem and functionals of interest are linear with respect to  $u$ ; see also the discussion following Theorem 2. Hence, it is interesting to investigate the convergence of the error estimate for the following  $\Delta\alpha$ -family of approximate solutions:

$$(\alpha^h, u^h(x)) = \left(\frac{3}{2} + \Delta\alpha, \frac{8}{9}x(3-x)\right). \quad (23)$$

This family converges to the exact solution (22) as  $\Delta\alpha \rightarrow 0$ . Note that although  $u^h$  has the same expression as the exact  $u$ , their domains,  $(0, \frac{3}{2} + \Delta\alpha)$  and  $(0, \frac{3}{2})$ , respectively, are different for  $\Delta\alpha \neq 0$ .

For the quantity of interest corresponding to the average,  $q^{\text{ave}}$ , Figure 4 [left] plots the true value of the target quantity error  $\mathcal{E}_{q^{\text{ave}}}$  and the dual-based error estimate  $\text{Est}_{q^{\text{ave}}}$  with respect to  $\Delta\alpha$ . It can be seen that the estimate approaches the exact error as  $\Delta\alpha \rightarrow 0$ . Moreover, the curve associated with the estimate is tangent to the curve for the exact error at  $\Delta\alpha = 0$ . This corroborates that the estimate is second-order accurate.

To further elucidate the convergence behaviour, we first recall from Theorem 2 that the error estimate  $\text{Est}_{q^{\text{ave}}}$  is an  $O(\|(e_\alpha, e_u)\|^2)$  approximation of the true error. Here, the norm corresponds to

$$\|(e_\alpha, e_u)\|^2 = |\alpha - \alpha^h|^2 + \|E_\alpha u - E_{\alpha^h} u^h\|_{U(\Omega_E)}^2. \quad (24)$$

In this case, the fixed hold-all domain can be set as  $\Omega_E = (0, \alpha_E)$ , where  $\alpha_E$  is a sufficiently large positive constant. Furthermore, for the extension operator  $E_\alpha : U(\Omega_\alpha) \rightarrow U(\Omega_E) =$

$\{u \in H^1(\Omega_E) \mid u(0) = 0\}$ , we can take the *constant* extension defined by:

$$(E_\alpha u)|_{\Omega_\alpha^c} = u(\alpha) .$$

We associate the following norm with  $U(\Omega_E)$ :

$$\|u\|_{U(\Omega_E)} = \int_{\Omega_E} |u_x|^2 \, dx .$$

Figure 4 [right] plots the error in the estimate,  $|\mathcal{E}_{q^{\text{ave}}} - \text{Est}_{q^{\text{ave}}}|$ , versus the norm of the solution error  $\|(e_\alpha, e_u)\|$  in a log-log plot. The figure clearly illustrates that the convergence of the estimate is of second-order, in compliance with Theorem 2.

## 6 CONCLUSION

As fluid-structure-interaction problems are free-boundary problems, the generic framework for dual-based a-posteriori error estimation for boundary value problems is not trivially applicable. We developed a framework for dual-based error estimation which encompasses fluid-structure-interaction problems. The framework is based on the embedded-domain approach and an extension operator, which enable a comparison of solutions on distinct underlying fluid domains. Given an approximate solution of the fluid and structure equations, we proposed a dual problem on the corresponding approximate fluid domain. Finally, employing the dual solution, we presented an exact error representation formula.

We assessed the developed framework on the basis of a 1D free-boundary problem. The results confirmed the convergence behavior of the dual-based error estimate as predicted by the theory.

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