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Creation and annihilation operators for Markov processes

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Abstract

In this thesis the theory of Markov processes and creation and annihilation operators will be used to derive the time evolution of a discrete reaction-diffusion system. More specifically, we make use of transition rates to construct the generator of a process. We then transform this generator through suitable quantum mechanical operators to arrive at our result, namely that the Poisson distribution of a reaction-diffusion system stays Poisson distributed in time with varying rate. This applies for the univariate case, with only a simple birth and death process, as well as the multivariate case. Furthermore, simulations support this result while also showing that an interacting particle system with pairwise annihilation does not evolve according to the Poisson distribution.

Contents

1 Introduction

Throughout the early twentieth century many microscopic physical phenomena started to arise that could not be explained using classical physics alone. A new theory had to be formulated, which is now better known as quantum mechanics. An important concept in quantum mechanics is that physical quantities such as position and momentum are represented by observables, or operators. Any pair of observables that satisfy the Heisenberg algebra commutation relation lead to corresponding uncertainty relations.

Around the same time, in 1906, mathematician Andrey Markov published his first paper on the topic of Markov processes, proving that the average outcome of Markov processes would converge to fixed vector values under certain conditions .

The purpose of this thesis is to solve problems of Markov processes via the use of quantum mechanical knowledge, i.e., using appropriate creation and annihilation operators to solve classical probabilistic questions.

We will start by laying the groundwork to construct the generator of some relevant Markov processes. First, in chapter 2 we define the Markov property and describe a process via transition rates, followed by the Markov semigroup, which posesses some properties that will be listed. From the semigroup we can compute the generator of a Markov process that will be used later on. The final part of the chapter will be spent on invariant measures.

In chapter 3 we introduce creation and annihilation operators, with the most well known ones being the ladder operators. Subsequently, we define some operators and their intertwined relation to transform our generator. Moreover, all relevant operators satisfy the Heisenberg algebra commutation relation.

The main derivation for the time evolution of a Markov process is described in chapter 4. Starting with a birth and death process, we use the Feynman-Kac formula to calculate the propagation of Poisson distributions. Furthermore, duality is introduced as a second method to calculate this. Lastly, we expand the problem to a reaction-diffusion system and calculate the time evolution of the distribution. The final results are compiled in chapter 5.

2 Markov Processes

In this section we introduce some basic material for continuous-time Markov processes on countable state spaces Ω.

2.1 Markov property

Let $\mathbf{X} = \{X_t : t \geq 0\}$ be a stochastic process taking values in a countable state space Ω, which for our purpose we will only be considering $\mathbb{N} = \{0, 1, 2, \dots\}$. A Markov process is then such that, given its current state, the future state is independent of its past [1]. Or alternatively, the distribution of future states is only depending on the current state and not on the further past.

Definition 1. A process $\mathbf{X}$ is a Markov process if it satisfies the Markov property

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1}|X_{t_0} = i_0, X_{t_1} = i_{t_2}, ..., X_{t_n} = i_n) = \mathbb{P}(X_{t_{n+1}} = i_{n+1}|X_{t_n} = i_n) \tag{1}$$

for all $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t_{n+1}$ and all $i_0, i_1, ..., i_{n+1} \in \Omega$.

An equivalent formulation of the Markov property is as follows.

$$\mathbb{E}(f(X_t)|X_{t_1}, X_{t_2}, ..., X_{t_n}) = \mathbb{E}(f(X_t)|X_{t_n}) \tag{2}$$

for all $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$ and for all $f : \Omega \to \mathbb{R}$ [2].

Additionally, a Markov process is time-homogeneous if the conditional probability of equation 1 does not depend on the specific time instance of the jump, but only on the time difference between jumps. All of the Markov processes that will be discussed are time-homogeneous.

2.2 Transition rates

On finite or countable space continuous time Markov processes we define a process via so called transition rates $c(x, y)$ that satisfy

$$c(x, y) \geq 0,\ \lambda_x = \sum_y c(x, y) > 0, \text{for all } x \in \Omega.$$

The description of the process with rates $c(x, y)$ is then as follows. Starting from $X_t = x$, the process waits for an exponentially distributed time with parameter λ_x, after which it jumps to y with probability $\pi(x, y) = \frac{c(x,y)}{\lambda_x}$. Then the process repeats the same from y with independent waiting time, and so forth.

Note that $\sum_y \pi(x, y) = 1$. Furthermore, since the exponential distribution is memoryless, this process satisfies the Markov property. This is only possible if the jump times are exponentially distributed and fails with other waiting time distributions.

2.3 Markov semigroup

To define the Markov semigroup $S_t, t \geq 0$ for finite or countable state space continuous time Markov processes, we consider a Markov process $\mathbf{X} = (X_t : t \geq 0)$ [3] [2]. We define

$$S_t f(x) = \mathbb{E}(f(X_t)|X_0 = x) = \sum_y \mathbb{P}(X_t = y|X_0 = x)f(y), \tag{3}$$

for $f : \Omega \to \mathbb{R}$. This semigroup has several properties that will be listed below

Proposition 1. *The semigroup $S_t, t \geq 0$ satisfies the following properties.*
1. Identity at time zero: $S_0 f = f$, for all f,
2. Right continuity: the map $t \to S_t f$ is right continuous,
3. Semigroup property: for all $t, s \geq 0, f : S_{t+s} f = S_t(S_s f) = S_s(S_t f)$,
4. Positivity: $f \geq 0$ implies $s_t f \geq 0$,
5. Normalization: $S_t 1 = 1$,
6. Contraction: $\max_x |S_t f(x)| \leq \max_x |f(x)|$.

For the Markov semigroup there exists a "generator" matrix L such that $S_t = e^{tL}$, where the matrix exponential is defined by the Taylor series

$$e^{tL} = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^n.$$

This generator can be computed via

$$Lf = \lim_{t \to 0} \frac{S_t f - f}{t} = \frac{d}{dt} S_t f|_{t=0}. \tag{4}$$

Additionally, $\psi(t,x) = S_t f(x)$ is the unique solution to the differential equation

$$\frac{\partial \psi(t,x)}{\partial t} = L\psi(t,x)$$

with initial condition $\psi_0 = f$ and where L works on the x variable. This equation is known as the Kolmogorov backwards equation.

2.4 Examples

1 (Poisson process)**.** One example of a continuous time Markov process is the Poisson process.

Definition 2. A Poisson process $\mathbf{N} = (N_t : t \geq 0)$ with rate λ satisfies the following conditions [1]:
1. $N_t \in \mathbb{N}$
2. $N_0 = 0$
3. $N_s \leq N_t$ if $s \leq t$
4. The number of jumps on the time interval $(s,t]$ is independent of jumps prior to time s.
5. There exists a rate λ such that, for small positive h,

$$\mathbb{P}(N_{t+h} = n+1 | N_t = n) = \lambda h + o(h)$$

$$\mathbb{P}(N_{t+h} = n | N_t = n) = 1 - \lambda h + o(h)$$

$$\mathbb{P}(N_{t+h} \geq n+2 | N_t = n) = o(h),$$

where we write $o(h) = \{f(h) : f(h)/h \to 0 \text{ as } h \to 0\}$.

To see that the process N_t indeed satisfies the Markov property, we choose $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$. Using the independence of increments from point 4 of definition 2 we get

$$\mathbb{P}(N_t = k | N_{t_1} = k_1, N_{t_2} = k_2, ..., N_{t_n} = k_n) =$$

$$\mathbb{P}(N_t - N_{t_n} = k - k_n | N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2 - k_1, ...,$$
$$N_{t_n} - N_{t_{n-1}} = k_n - k_{n-1}) = \mathbb{P}(N_t - N_{t_n} = k - k_n).$$

Similarly we have

$$\mathbb{P}(N_t = k | N_{t_n} = k_n) = \mathbb{P}(N_t - N_{t_n} = k - k_n)$$

Hence we indeed find that the Poisson process satisfies the Markov property.

A Poisson process with rate λ has the following generator

$$Lf(n) = \frac{d}{dt}[P(N_t = n+1|N_0 = n)(f(n+1) - f(n))]\Big|_{t=0} = \lambda(f(n+1)-f(n)). \quad (5)$$

2 (Continuous time random walk)**.** A continuous-time random walk on a graph can be constructed as follows. Let $p(x,y)$ denote a set of transition probabilities, i.e. $p(x,y) \geq 0$, $\sum_y p(x,y) = 1$. Then the rate 1 continuous-time random walk based on $p(x,y)$, starting from $X_0 = x$, waits an exponential time of parameter 1 and jumps with probability $p(x,y)$ $x \to y$. In other words, the jumps take place on a rate 1 Poisson process. The transition rates are $c(x,y) = p(x,y)$. The number of jumps N_t is Poisson distributed with parameter 1, i.e.

$$\mathbb{P}(N_t = n) = \frac{t^n}{n!}e^{-t}$$

Then for the continuous time random walk $\mathbf{X} = \{X_t : t \geq 0\}$ the transition probabilities are given by

$$p_t(x,y) = \mathbb{P}(X_t = y|X_0 = x) = \sum_{n=0}^{\infty} \mathbb{P}(X_t = y, N_t = n|X_0 = x) =$$

$$\sum_{n=0}^{\infty} \mathbb{P}(N_t = n)\mathbb{P}(X_t = y|X_0 = x) = \sum_{n=0}^{\infty} \frac{t^n}{n!}e^{-t}P^n_{x,y} = (e^{t(P-I)})_{x,y}$$

Here we define the generator $L = P - I$ which leads to

$$p_t(x,y) = (e^{tL})_{x,y}. \quad (6)$$

As a consequence, the generator acting on a function is given as

$$Lf(x) = \sum_{y \neq x} p(x,y)f(y) - f(x) \quad (7)$$

3 (Birth and death process)**.** Consider a continuous time Markov process on $\mathbb{N}$ moving with death rate λn $n \to n-1$ and birth rate μ $n \to n+1$. In a similar way as discussed earlier in section 2.2, the process starts in state n at time t and waits an exponential time with parameter $\lambda n + \mu$. Then it jumps to $n+1$ with probability $\mu/(\lambda n + \mu)$ or to $n-1$ with probability $\lambda n/(\lambda n + \mu)$. Note that by the choice of transition rates we have a death rate of 0 in state $n = 0$, which makes sense intuitively since we cannot have a negative number of particles.

The generator matrix is then given by

$$L = \begin{pmatrix} -\mu & \mu & 0 & \dots \\ -\lambda & -\lambda-\mu & \mu & \\ 0 & 2\lambda & -2\lambda-\mu & \\ \vdots & & & \ddots \end{pmatrix}.$$

Thus we find for a birth and death process with death rate λn and birth rate μ a generator L acting on a function f that

$$Lf(n) = \lambda n(f(n-1) - f(n)) + \mu(f(n+1) - f(n)). \tag{8}$$

2.5 Invariant measures

The question one might ask is what happens to a process when $t \to \infty$. In other words, what shape does the invariant measure take?

We start by stating the definition of an invariant measure.

Definition 3. A measure μ is invariant if

$$\sum_x S_t f(x)\mu(x) = \sum_x f(x)\mu(x) \tag{9}$$

for all $f : \Omega \to \mathbb{R}$ and for all $t \geq 0$.

Some properties of invariant measures will be listed below.

Proposition 2. *1. μ is invariant if and only if*

$$\sum_x Lf(x)\mu(x) = 0 \tag{10}$$

for all $f : \Omega \to \mathbb{R}$.

2. If the measure μ satisfies the detailed balance equation

$$\mu(n)c(n,m) = \mu(m)c(m,n), \quad \forall n, m \tag{11}$$

with $c(n,m)$ the element on the nth row and mth column of a transition rate matrix L, then μ is invariant.

Proof. 1. Assume μ is invariant. Taking the time derivative evaluated at $t = 0$ of equation 9 yields

$$\sum_x Lf(x)\mu(x) = \frac{d}{dt}\sum_x S_t f(x)\mu(x)\Big|_{t=0} = \frac{d}{dt}\sum_x f(x)\mu(x)\Big|_{t=0} = 0, \tag{12}$$

thus proving the implication from left to right. Now assume Equation 10 holds.

$$\begin{aligned}\sum_x (S_t f(x) - f(x))\mu(x) &= \sum_x \int_0^t LS_s f(x) ds\ \mu(x) \\ &= \int_0^t \sum_x LS_s f(x)\mu(x) ds = 0,\end{aligned} \tag{13}$$

which proves the reverse implication.

2. Assume μ satisfies the detailed balance equation. An equivalent formulation of Equation 11 is given by

$$\mu(x)c(x,y) = \mu(y)c(y,x) \Leftrightarrow \sum_x \mu(x)Lf(x)g(x) = \sum_x \mu(x)f(x)Lg(x),$$

for all $f, g : \Omega \to \mathbb{R}$.

To prove this equivalence we first assume the left side. Then we find

$$\begin{aligned}\sum_x g(x)Lf(x)\mu(x) &= \sum_x g(x)\sum_y c(x,y)(f(y) - f(x))\mu(x) \\ &= \sum_{x,y} g(x)c(x,y)f(y)\mu(x) - \sum_{x,y} g(x)c(x,y)f(x)\mu(x) \\ &= \sum_{x,y} g(x)f(y)c(y,x)\mu(y) - \sum_{x,y} g(y)f(y)c(y,x)\mu(y) \\ &= \sum_{x,y} f(y)c(y,x)(g(x) - g(y))\mu(y) \\ &= \sum_x f(x)Lg(x)\mu(x),\end{aligned}$$

proving the implication '$\Rightarrow$'.

Now assuming the right-hand side, we choose the Kronecker delta function $f(y) = \delta_{y,z}$ and $g(x) = \delta_{x,u}$ where $z \neq u$.

$$\sum_{x,y} \mu(x)c(x,y)\delta_{y,z}\delta_{x,u} = \sum_{x,y} \mu(x)c(x,y)\delta_{y,u}\delta_{x,z}$$
$$\mu(u)c(u,z) = \mu(z)c(z,u)$$

and thus proving the equivalence.

To prove invariance we take $g(x) = 1, \ \forall x \in \mathbb{N}$, so we find that

$$\sum_x \mu(x)Lf(x) = \sum_x \mu(x)f(x)Lg(x)$$
$$= \sum_{x,y} f(y)c(y,x)(g(x) - g(y))\mu(y) = 0,$$

which proves item (2) of the proposition. □

Example (Birth and death process)**.** Consider the invariant measure π of a birth and death process with rates $\lambda n \ n \to n-1$ and $\mu \ n \to n+1$. Using the definition and taking $f(x) = 1$ for all $x \in \mathbb{N}$, we have

$$\pi S_t = \pi. \tag{14}$$

Using the detailed balance equation from Proposition 2 we get

$$\pi(n)c(n,n+1) = \pi(n+1)c(n+1,n)$$
$$\pi(n)\mu = \pi(n+1)\lambda(n+1)$$
$$\frac{\pi(n+1)}{\pi(n)} = \frac{\mu}{\lambda(n+1)}.$$

Substituting in the Poisson distribution $\pi(n) = \frac{c^n}{n!}e^{-c}$ with an unknown parameter c gives

$$\frac{\pi(n+1)}{\pi(n)} = \frac{c^{(n+1)}}{(n+1)!}e^{-c}\frac{1}{\frac{c^n}{n!}e^{-c}} = \frac{c}{n+1}$$

Therefore the invariant measure π is Poisson distributed with parameter $c = \frac{\mu}{\lambda}$.

3 Creation and annihilation operators

Some quantities commonly used in quantum mechanics are the creation and annihilation operators. A well-known example are the raising and lowering operators in the context of the quantum harmonic oscillator [4]. However, these operators will also be useful in connecting reaction-diffusion systems with their corresponding Markov processes.

Creation and annihilation operators function in such away that particles are added or removed from a many-body system. In fact, we are able to rewrite the operator of a many-body system in terms of these operators. In this section we will cover these operators to get a better understanding of their usefulness.

3.1 Ladder operators

To see the usefulness of creation and annihilation operators, consider the time-independent Schrödinger equation for harmonic oscillators

$$\hat{H} |\Psi\rangle = E |\Psi\rangle \tag{15}$$

where E is the energy eigenvalue and $\hat{\Psi}$ denotes the eigenstate of the system. The Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + 2m\omega^2\hat{x}^2$$

where $\hat{p} = -i\hbar\frac{d}{dx}$ is the momentum operator and $\hat{x}$ is the position operator. Note that $\hat{x}$ and $\hat{p}$ do not commute. This can be seen by calculating the commutator acting on a test function:

$$[\hat{x}, \hat{p}]f(x) = (\hat{x}\hat{p} - \hat{p}\hat{x})f(x) = x(-i\hbar)\frac{df}{dx} - (-i\hbar)\frac{d}{dx}(xf) =$$
$$-i\hbar(\frac{df}{dx} - \frac{df}{dx} - f) = i\hbar f(x),$$

and after dropping the test function we find $[\hat{x}, \hat{p}] = i\hbar$.

We can define the raising and lowering operators, a_+ and a_- respectively, in terms of $\hat{x}$ and $\hat{p}$

$$\begin{aligned} a_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p}) \\ a_- &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p}). \end{aligned} \tag{16}$$

These ladder operators will be a useful tool in finding the energy eigenvalues of the harmonic oscillator. It follows that

$$a_-a_+ = \frac{1}{2\hbar m\omega}(m\omega\hat{x} + i\hat{p})(m\omega\hat{x} - i\hat{p}) =$$

$$\frac{1}{2\hbar m\omega}(\hat{p}^2 + m^2\omega^2\hat{x}^2 + im\omega(\hat{x}\hat{p} - \hat{p}\hat{x})) = \frac{1}{\hbar\omega}H + \frac{1}{2}$$

which yields the Hamiltonian

$$\hat{H} = \hbar\omega(a_-a_+ - \frac{1}{2}) \tag{17}$$

Now, using equation 17 and the following relation for the ladder operators acting on the state $|n\rangle$

$$\begin{aligned} a_+ |n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ a_- |n\rangle &= \sqrt{n}\,|n-1\rangle\,, \end{aligned} \tag{18}$$

we can find the energy eigenvalues E_n for state $|n\rangle$:

$$E_n |n\rangle = \hat{H}|n\rangle = \hbar\omega(a_-a_+ - \frac{1}{2})|n\rangle = \hbar\omega(n - \frac{1}{2})|n\rangle\,. \tag{19}$$

3.2 Heisenberg algebra commutation relations

The Heisenberg algebra commutation relation can be formulated into the following definition for any pair of quantities that satisfies it [5].

Definition 4. Any two quantities A and B with corresponding operators $\hat{A}$ and $\hat{B}$ satisfy the Heisenberg algebra commutation relation if

$$[\hat{A}, \hat{B}] = 1. \tag{20}$$

Moreover, A and B also satisfy the conjugate Heisenberg algebra commutation relation

$$[\hat{A}^*, \hat{B}^*] = -1. \tag{21}$$

Example. Earlier we have shown that the commutator of $\hat{x}$ and $\hat{p}$ is given by $[\hat{x}, \hat{p}] = i\hbar$. This is a fundamental relation in quantum mechanics called the canonical commutation relation and is derived from

$$[x, \frac{d}{dx}] = -1. \tag{22}$$

Equation 22 can be easily shown by

$$[x, \frac{d}{dx}]f = x\frac{d}{dx}f - \frac{d}{dx}(xf) = -f.$$

3.3 A second representation

We return to the Markov generator for the birth and death process with death rate λn and spontaneous creation rate μ as described in Equation 8. The goal is to rewrite this equation such that the test function can be dropped. To do this, we define the annihilation operator a and creation operator $a^\dagger$ in a similar fashion to the relation from Equation 18.

Definition 5. Consider the operators a and $a^\dagger$. These operators are defined for functions $f : \mathbb{N} \to \mathbb{R}$ via

$$\begin{aligned} af(n) &= nf(n-1) \\ a^\dagger f(n) &= f(n+1) \end{aligned}$$

These operators are used to increase or decrease the number of particles from a many-body system as indicated by definition 5. Operator a removes a particle from the system and $a^\dagger$ add a particle.

Proposition 3. *1. a and $a^\dagger$ satisfy the Heisenberg algebra commutation relation.*

2. The following relation holds

$$aa^\dagger f(n) = nf(n). \tag{23}$$

Proof. 1. Using a test function gives

$$[a, a^\dagger]f(n) = (aa^\dagger - a^\dagger a)f(n) = nf(n) - (n+1)f(n) = -f(n).$$

Thus we find that the commutator equals the identity matrix $[a, a^\dagger] = -I$, satisfying the commutation relation.
2. From Definition 5 it follows that

$$aa^\dagger f(n) = a(a^\dagger f(n)) = na^\dagger f(n-1) = nf(n).$$

□

As a consequence of Definition 5 and Proposition 3 we can rewrite the generator from equation 8 as

$$Lf(n) = (\lambda(a - aa^\dagger) + \mu(a^\dagger - I))f(n)$$

and after dropping the test function we are left with

$$L = (\lambda(a - aa^\dagger) + \mu(a^\dagger - I)). \tag{24}$$

3.4 Intertwining

To study properties of the process with generator L, we will focus on another representation of the Heisenberg algebra. These generators are related via an intertwining of Markov processes described in the following proposition.

Proposition 4. *a, $\mathcal{A}$ and $a^\dagger$, $\mathcal{A}^\dagger$ are intertwined via the generating function $(\mathcal{G}f)(z) = \sum_{n=0}^{\infty} f(n)\frac{z^n}{n!}$ defined for $f : \mathbb{N} \to \mathbb{R}$. In other words,*

$$(\mathcal{G}af)(z) = \mathcal{A}(\mathcal{G}f)(z)$$

$$(\mathcal{G}a^\dagger f)(z) = \mathcal{A}^\dagger(\mathcal{G}f)(z)$$

with $\mathcal{A}f(z) = zf(z)$ and $\mathcal{A}^\dagger f(z) = \frac{d}{dz}f(z)$ for $f : \mathbb{R} \to \mathbb{R}$.

Proof. From the definition it follows that

$$(\mathcal{G}af)(z) = \sum_{n=0}^{\infty} nf(n-1)\frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{f(n-1)}{(n-1)!}z^n =$$

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!}z^{n+1} = z\sum_{n=0}^{\infty} f(n)\frac{z^n}{(n)!} = \mathcal{A}(\mathcal{G}f)(z).$$

Similarly we have

$$\mathcal{G}(a^\dagger f)(z) = \sum_{n=0}^{\infty} f(n+1)\frac{z^n}{n!} = \sum_{n=1}^{\infty} f(n)\frac{z^{n-1}}{(n-1)!} =$$

$$\sum_{n=0}^{\infty} f(n)\frac{d}{dz}\frac{z^n}{n!} = \frac{d}{dz}\sum_{n=0}^{\infty} f(n)\frac{z^n}{n!} = \mathcal{A}^\dagger(\mathcal{G}f)(z).$$

□

$$(\mathcal{G}aa^\dagger f)(z) = \sum_{n=0}^{\infty} nf(n)\frac{z^n}{n!} = z\frac{d}{dz}\sum_{n=0}^{\infty} f(n)\frac{z^n}{n!} = \mathcal{A}\mathcal{A}^\dagger(\mathcal{G}f)(z).$$

Remarks. 1. The commutator of $\mathcal{A}$ and $\mathcal{A}^\dagger$ leads to $[\mathcal{A}, \mathcal{A}^\dagger] = -I$.
2. Intertwining has the property that it preserves sums and products:

$$\mathcal{G}(a + a^\dagger) = \mathcal{G}a + \mathcal{G}a^\dagger = (\mathcal{A} + \mathcal{A}^\dagger)\mathcal{G}$$

$$\mathcal{G}(aa^\dagger) = (\mathcal{G}a)a^\dagger = \mathcal{A}(\mathcal{G}a^\dagger) = \mathcal{A}\mathcal{A}^\dagger\mathcal{G}$$

As a consequence to Proposition 4 we can find a different representation of the generator in Equation 24, namely

$$\begin{aligned}\mathcal{G}(Lf)(z) &= \mathcal{G}(\lambda(a - aa^\dagger) + \mu(a^\dagger - I))f(z)\\ &= (\lambda(\mathcal{A} - \mathcal{A}\mathcal{A}^\dagger) + \mu(\mathcal{A}^\dagger - I))\mathcal{G}f(z)\\ &= \lambda z\mathcal{G}f(z) - \lambda z\frac{d}{dz}\mathcal{G}f(z) + \mu\frac{d}{dz}\mathcal{G}f(z) - \mu\mathcal{G}f(z)\\ &= \mathcal{L}\mathcal{G}f(z)\end{aligned}$$

with $\mathcal{L}\psi = (\lambda z - \mu)\psi + (\mu - \lambda z)\frac{d}{dz}\psi$.

4 Discrete reaction-diffusion systems

As one might expect, discrete reaction-diffusion systems are stochastic processes which cannot be described by classical mechanics. Instead, they require quantum mechanics to analyse correctly.

In this context there are a few different ways to approach the problem. Notable ones are the Langevin equation, which deals with the degrees of freedom or observables, and the Fokker-Planck equation, describing the probability density.

Our approach concerns writing down a Markov generator of the stochastic processes first. The theory of operators and intertwining introduced in Section 3 allows us to transform this equation and finally the Feynman-Kac formula can be applied from which we can derive the time evolution of the generator.

4.1 Birth and death process

Consider a system in which particles can appear and disappear with certain rates, but where diffusion does not play a role yet. The time evolution of the Markov process is described by the generator as in Section 2.3

$$\frac{d\psi}{dt} = L\psi \tag{25}$$

where L is the same as in Equation 8. We have seen in Section 2.3 that the unique solution is given by the semigroup $S_t f = e^{tL} f$ with $f : \mathbb{N} \to \mathbb{R}$. The stochastic nature of the process prevents us from accurately predicting what the exact analytical solution is. Instead, our main focus of interest will be seeing what happens to this solution "on average" when we let it propagate in time.

After writing the generating function of the solution we can use Section 4 to show that

$$\mathcal{G}(S_t f) = \mathcal{G}(e^{tL} f) = e^{t\mathcal{L}}(\mathcal{G} f) = \mathcal{S}(t)(\mathcal{G} f). \tag{26}$$

So we are left with computing

$$e^{t\mathcal{L}} = e^{t[(\lambda z - \mu) + (\mu - \lambda z)\frac{d}{dz}]} = e^{t[V(z) + b(z)\frac{d}{dz}]}. \tag{27}$$

We will see that Poisson measures are mapped to Poisson measures under the birth and death process. A helpful tool in proving this is the Feynman-Kac formula [6].

4.1.1 The Feynman-Kac formula

Consider a generator of the form $\mathcal{L}\psi = b(z)\frac{d\psi}{dz}$. Let Z_t^ζ be the solution of $\frac{dz}{dt} = b(z)$ with initial condition $Z_0^\zeta = \zeta$. We will show that $\psi(Z_t^\zeta)$ is indeed the solution to $\frac{d\psi}{dt} = \mathcal{L}\psi$.

Starting from the left hand side it follows that

$$\frac{d}{dt}\psi(Z_t^\zeta) = \psi'(Z_t^\zeta)\frac{dZ_t^\zeta}{dt} = \psi'(Z_t^\zeta)b(Z_t^\zeta).$$

The right hand side gives

$$\mathcal{L}\psi(Z_t^\zeta) = b(z)\frac{d}{dz}\psi(Z_t^\zeta) =$$

$$b(z)\psi'(Z_t^\zeta)\frac{dZ_t^\zeta}{dz} \overset{(*)}{=} b(z)\psi'(Z_t^\zeta)\frac{b(Z_t^\zeta)}{b(z)} = \psi'(Z_t^\zeta)b(Z_t^\zeta).$$

To see that $(*)$ holds we want to prove the next equality

$$\frac{dZ_t^\zeta}{dz} = \frac{b(Z_t^\zeta)}{b(z)} \tag{28}$$

Put $Y_t^\zeta = \frac{dZ_t^\zeta}{dz}$. Then it holds that

$$\frac{d}{dt}Y_t^\zeta = \frac{d}{dt}\frac{dZ_t^\zeta}{dz} = \frac{d}{dz}\frac{dZ_t^\zeta}{dt} = \frac{d}{dz}b(Z_t^\zeta) = b'(Z_t^\zeta)Y_t^\zeta.$$

Now if we set $Y_t^\zeta = \frac{b(Z_t^\zeta)}{b(z)}$ it can be verified that this also satisfies the equation. Indeed we have

$$\frac{d}{dt}Y_t^\zeta = \frac{b'(Z_t^\zeta)}{b(z)}\frac{dZ_t^\zeta}{dt} = b'(Z_t^\zeta)\frac{b(Z_t^\zeta)}{b(z)} = b'(Z_t^\zeta)Y_t^\zeta.$$

Thus we have shown that the solution of $\frac{d\psi}{dt} = \mathcal{L}\psi$ is given by $e^{t\mathcal{L}}\psi(z) = \psi(Z_t^\zeta)$. More generally, we have the the following proposition where now the generator is given by $\mathcal{L}\psi = b(z)\frac{d\psi}{dz} + V(z)\psi$ with $V(z)$ the multiplication operator.

Proposition 5. *The solution of the equation* $\frac{d\psi}{dt} = \mathcal{L}\psi$ *is given by the Feynman-Kac formula*

$$\psi(t,\zeta) = e^{t\mathcal{L}}\psi(\zeta) = e^{\int_0^t V(Z_s^\zeta)ds}\psi(Z_t^\zeta) \tag{29}$$

where the generator is of the form $\mathcal{L} = b(z)\frac{d}{dz} + V(z)$ *and* Z_t^ζ *is the solution to* $\frac{dz}{dt} = b(z)$ *with initial condition* $Z_0^\zeta = \zeta$.

Proof. Substitution of Equation 29 yields

$$\begin{aligned}\frac{d}{dt}\psi(t,\zeta)\Big|_{t=0} &= \frac{d}{dt}[e^{\int_0^t V(Z_s^\zeta)ds}\psi(Z_t^\zeta)]\Big|_{t=0}\\ &= [V(Z_t^\zeta)e^{\int_0^t V(Z_s^\zeta)ds}\psi(Z_t^\zeta) + b(\zeta)e^{\int_0^t V(Z_s^\zeta)ds}\frac{d}{d\zeta}\psi(Z_t^\zeta)]\Big|_{t=0}\\ &= V(\zeta)\psi(\zeta) + b(\zeta)\frac{d}{d\zeta}\psi(\zeta) = \mathcal{L}\psi(\zeta).\end{aligned}$$

□

Solving $\frac{dz}{dt} = \mu - \lambda z$ with initial condition $Z_0^\zeta = \zeta$ one finds

$$Z_t^\zeta = \zeta e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t})$$

Plugging this result into Equation 29 we then find

$$e^{t\mathcal{L}}\psi(\zeta) = e^{\int_0^t \mu - \lambda(\zeta e^{-\lambda s} + \frac{\mu}{\lambda}(1-e^{-\lambda s}))ds}\psi(\zeta e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t}))$$

Solving the integral term gives

$$\int_0^t \lambda(\zeta e^{-\lambda s} + \frac{\mu}{\lambda}(1 - e^{-\lambda s})) - \mu\ ds = \int_0^t \lambda\zeta e^{-\lambda s} - \mu e^{-\lambda s}ds =$$
$$\frac{(\lambda\zeta - \mu)(1 - e^{-\lambda t})}{\lambda} = (\zeta - \frac{\mu}{\lambda})(1 - e^{-\lambda t}).$$

Finally, Equation 26 can be rewritten explicitly as

$$e^{t\mathcal{L}}\mathcal{G}f(\zeta) = e^{\zeta}e^{-\zeta e^{-\lambda t} - \frac{\mu}{\lambda}(1-e^{-\lambda t})}\mathcal{G}f(\zeta e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t})). \tag{30}$$

4.1.2 Propagation of Poisson

In section 2.4 we have shown before that the stationary distribution of a birth and death process is Poisson distributed, however Equation 30 shows us more than that. The result of this is formulated into the following theorem.

Theorem 1. *If we start the birth and death process from a distribution* $\nu_0 \sim Poisson(\zeta)$*, the process at time* t *will become* $\nu_t \sim Poisson(Z_t^\zeta)$*, i.e. the Poisson distribution is conserved in time.*

Proof. If we start with $\nu_0 \sim Poisson(\zeta)$

$$\mathbb{E}_{\nu_t}[f] = \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} e^{-\zeta} S_t f(n) = e^{-\zeta} \mathcal{G}(S_t f) = e^{-\zeta} \mathcal{S}_t \mathcal{G}(f)$$

$$e^{-\zeta} e^{t\mathcal{L}} \mathcal{G} f(\zeta) = e^{-\zeta e^{-\lambda t} - \frac{\mu}{\lambda}(1-e^{-\lambda t})} \mathcal{G} f(\zeta e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t})).$$

If we begin with a distribution $\nu_0 \sim Poisson(\zeta)$, then at time t the process has a distribution ν_t which is Poisson with rate $Z_t^\zeta = \zeta e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t})$. □

If we now let $t \to \infty$, we can find that the process has a stationary distribution that is Poisson distributed with rate $\frac{\mu}{\lambda}$. This coincides exactly with the result in Section 2.4.

4.1.3 Simulations

We want to compare the theoretic results with simulated data by programming a Python code according to the birth and death process. To do this, we apply the Euler approximation method [7]. Instead of simulating a Markov process in the way described in Section 2.2, which takes considerable computational time, we partition time into fixed intervals of length τ. This method approximates our process via population changes on τ-intervals that are determined by Poisson random variables with parameters depending on population sizes at the beginning of the intervals.

Figure 1 shows the distribution of a birth and death process with 1000 realizations. One can see that this simulation agrees with Theorem 1 in the sense that the realizations indeed follow the distribution $\nu_t \sim Poisson(Z_t^{\zeta})$ in time.

Figure 2 shows 10 and 100 realizations respectively of a birth and death process starting from a Poisson distribution with parameter 100 and the expected number of particles at time t. This follows from a distribution $\nu_t \sim Poisson(Z_t^{100})$. Furthermore, we find that they converge to 10 as time goes to infinity, which corresponds to the expectation of the stationary distribution.

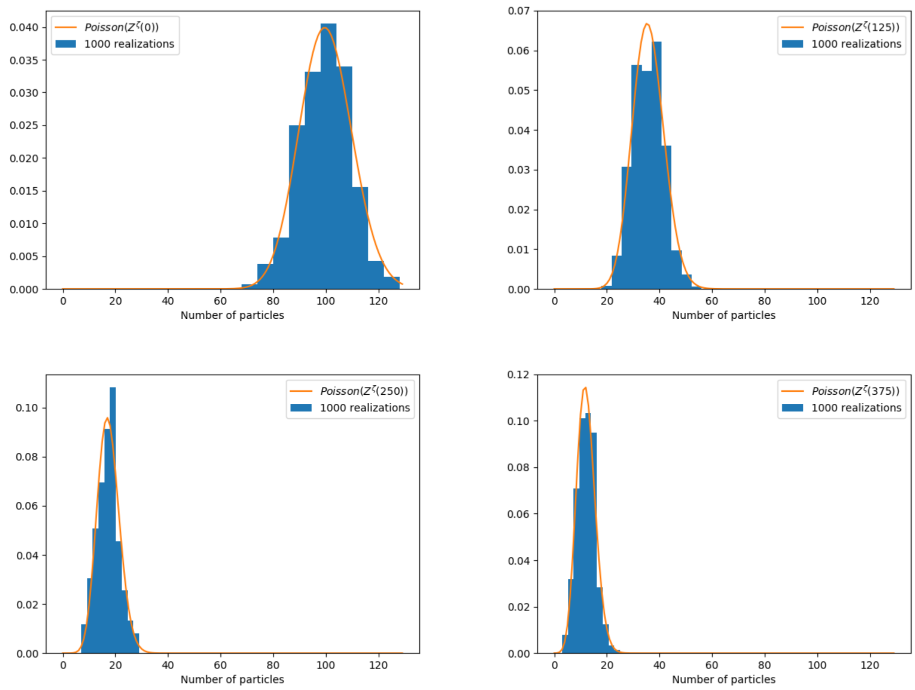
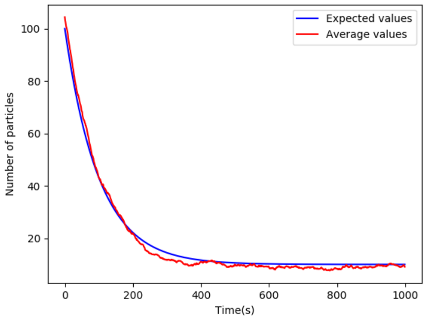
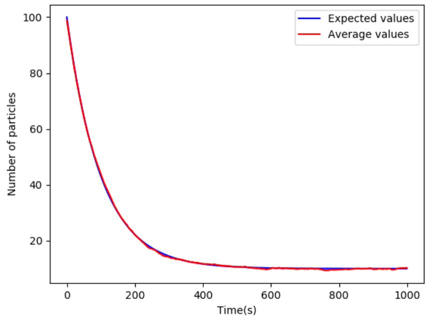


Figure 1: 1000 realizations of a birth and death process. Each point starts with an initial distribution sampled from $Poisson(100)$. The number of particles is plotted at four different times 0, 125, 250 and 375 seconds. The birth and death rates are given by $\mu = 0.1$ and $\lambda = 0.01$.



(a)



(b)

Figure 2: The average of 10 realizations (2a) and 100 realizations (2b) of the birth and death process are plotted with birth rate $\mu = 0.1$, death rate $\lambda = 0.01$ and initial condition $\zeta = 100$. The expected value is given by rate $Z_t^\zeta = \zeta e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t})$.

4.1.4 Duality

Here we give yet another method to prove propagation of Poisson, namely duality. Duality has shown to be very powerful in describing interacting particle systems. Often times duality allows to relate a complicated process to a (usually) simpler process.

Theorem 2. *Take the duality function $D(n, z) = z^n$. Then we have*

$$\begin{aligned} aD(n, z) &= AD(n, z) \\ a^\dagger D(n, z) &= A^\dagger D(n, z) \end{aligned} \tag{31}$$

with $Af(z) = \frac{d}{dz} f(z)$ and $A^\dagger f(z) = zf(z)$. In other words, $a \xrightarrow{D} A$ as well as $a^\dagger \xrightarrow{D} A^\dagger$ are dual to each other with duality function $D(n, z) = z^n$.

Proof. Clearly a and $a^\dagger$ work on the left variable, while A and $A^\dagger$ work on the right variable. It follows that

$$\begin{aligned} AD(n, z) &= \frac{d}{dz} z^n = nz^{n-1} = az^n = aD(n, z) \\ A^\dagger D(n, z) &= z * z^n = z^{n+1} = a^\dagger z^n = a^\dagger D(n, z) \end{aligned}$$

which is indeed Equation 31. □

Remarks. 1. The commutator of A and $A^\dagger$ is given by $[A, A^\dagger] = I$. Note that the sign has changed compared to $\mathcal{A}$ and $\mathcal{A}^\dagger$.
2. As a consequence of Theorem 2

$$aa^\dagger \xrightarrow{D} A^\dagger A. \tag{32}$$

and

$$L \xrightarrow{D} \hat{L} = \lambda(\frac{d}{dz} - z\frac{d}{dz}) - \mu(z-1). \tag{33}$$

3. To see why the the duality function is constructed in the way that it is, we first have to remark that

$$AD(0, z) = 0,$$

because $D(0, z)$ already represents the lowest possible number of particles. This then implies $D(0, z) = 1$. Moreover, the duality function for arbitrary n can now be written as

$$D(n, z) = (A^\dagger)^n D(0, z) = z^n.$$

As a consequence of Equation 33

$$e^{tL} \xrightarrow{D} e^{t\hat{L}}, \tag{34}$$

so we have to compute $e^{t\hat{L}}$.

$$e^{t\hat{L}} f(\zeta) = e^{\int_0^t \mu(Z_s^\zeta - 1)ds} f(Z_t^\zeta) \tag{35}$$

with Z_t^ζ the solution of $\frac{dz}{dt} = \lambda - \lambda z$ and initial condition $Z_0^\zeta = \zeta$. This results in

$$Z_t^\zeta = (\zeta - 1)e^{-\lambda t} + 1 \tag{36}$$

We again solve the integral term to arrive at

$$e^{t\hat{L}} f(\zeta) = e^{\frac{\mu(\zeta-1)}{\lambda}(1-e^{-\lambda t})} f(Z_t^\zeta). \tag{37}$$

We then show that the duality between $S(t)$ and $e^{t\hat{L}}$ again give us that Poisson goes to Poisson. Denoting $e^{t\hat{L}} D(n, z)$, we can write the time dependent generating function

$$\mathbb{E}_n(z^{n(t)}) = e^{\frac{\mu(\zeta-1)}{\lambda}(1-e^{-\lambda t})}((\zeta - 1)e^{-\lambda t} + 1)^n. \tag{38}$$

Letting $t \to \infty$, we find the stationary Poisson distribution $e^{\frac{\mu(z-1)}{\lambda}}$. Furthermore, Poisson averaging over n gives us

$$\sum_n e^{-\rho}\frac{\rho^n}{n!}\mathbb{E}_n(z^{n(t)}) = \sum_n e^{-\rho}\frac{\rho^n}{n!}e^{\frac{\mu(\zeta-1)}{\lambda}(1-e^{-\lambda t})}((\zeta-1)e^{-\lambda t}+1)^n =$$

$$e^{\frac{\mu(\zeta-1)}{\lambda}(1-e^{-\lambda t})}e^{\rho((\zeta-1)e^{-\lambda t}+1)}e^{-\rho} = e^{(\zeta-1)((\rho-\frac{\mu}{\lambda})e^{-\lambda t}+\frac{\mu}{\lambda})}$$

This is exactly the Poisson generating function $e^{\zeta\rho_t-\rho_t}$ with rate $\rho_t = (\rho-\frac{\mu}{\lambda})e^{-\lambda t}+\frac{\mu}{\lambda}$. We can see that this result is indeed consistent with Theorem 4.1.2.

4.2 A reaction-diffusion system

Now that it is clear what happens with a process when only particles appearing and dying off is considered, we consider a graph $G = (V, E)$ with a set of vertices V where birth and death occurs connected by a set of edges E. Additionally, particles can jump from vertex to vertex, i.e. diffusion is also considered [8].

The goal is again to prove that the Poisson distribution is preserved in time, but with a different generator than before. We will deal with each process separately first.

In Section 2.4 we considered random walk with a single particle. This can be easily extended to a multi particle case, because each particle moves independently of the others. On V we consider transition rate $p(i,j) \geq 0$ for $i,j \in V$. We only consider an irreducible random walk, i.e.

$$\forall i,j \in V, \exists n : p^{(n)}(i,j) > 0. \tag{39}$$

Then we have an independent random walk process that moves according to $p(i,j)$, so we write the generator

$$L_1 f(\eta) = \sum_{i,j} p(i,j)[\eta_i[f(\eta-\delta_i+\delta_j)-f(\eta)]+\eta_j[f(\eta-\delta_j+\delta_i)-f(\eta)]]. \tag{40}$$

Here δ_i represents a single particle at vertex i and $\eta = (\eta_i)_{i\in V}$ denotes the number of particles at each vertex i.

At each vertex particles can appear and disappear independently, so the birth and death process with rates $\mu_i \geq 0$ and $\lambda_i \geq 0$, for $i \in V$ has generator

$$L_2 f(\eta) = \sum_i \lambda_i \eta_i [f(\eta - \delta_i) - f(\eta)] + \mu_i [f(\eta + \delta_i) - f(\eta)]. \tag{41}$$

Equations 40 and 41 are also independent, so the total generator is written as

$$\begin{gathered} Lf(\eta) = L_1 f(\eta) + L_2 f(\eta) = \\ \sum_{i,j} p(i,j)[\eta_i [f(\eta - \delta_i + \delta_j) - f(\eta)] + \eta_j [f(\eta - \delta_j + \delta_i) - f(\eta)]] \\ + \sum_i \lambda_i \eta_i [f(\eta - \delta_i) - f(\eta)] + \mu_i [f(\eta + \delta_i) - f(\eta)]. \end{gathered} \tag{42}$$

Similar to the simple birth and death process, we want to rewrite this generator in terms of creation and annihilation operators.

Definition 6. We define the operators a_i and $a_i^\dagger$ for $f : \mathbb{N}^V \to \mathbb{R}$ as follows

$$\begin{aligned} a_i f(\eta) &= \eta_i f(\eta - \delta_i) \\ a_i^\dagger f(\eta) &= f(\eta + \delta_i), \end{aligned} \tag{43}$$

for all $i \in V$. In other words, the annihilation operator a_i removes a particle at vertex i while the creation operator $a_i^\dagger$ adds one.

Remarks. 1. As a consequence

$$a_i [a_i^\dagger f(\eta)] = \eta_i [a_i^\dagger f(\eta - \delta_i)] = \eta_i f(\eta), \text{ for all } i \in V. \tag{44}$$

2. It is important to note that operators acting on different vertices commute. Thus, for all $i \neq j$

$$[a_i, a_j^\dagger] f(\eta) = (a_i a_j^\dagger - a_j^\dagger a_i) f(\eta) = 0. \tag{45}$$

On the other hand, operators acting on the same vertex do not commute.

$$[a_i, a_i^\dagger] f(\eta) = (a_i a_i^\dagger - a_i^\dagger a_i) f(\eta) = -f(\eta). \tag{46}$$

Assuming symmetry, i.e. $p(i,j) = p(j,i)$ for all $i, j \in V$, we can rewrite Equation 42 as

$$\begin{gathered} Lf(\eta) = \sum_{i,j} p(i,j)(a_i - a_j)(a_j^\dagger - a_i^\dagger) f(\eta) + \\ \sum_i \lambda_i (a_i - a_i a_i^\dagger) f(\eta) + \mu_i (a_i^\dagger - I) f(\eta). \end{gathered} \tag{47}$$

We then give an intertwined generator via a new generating function defined for $f : \mathbb{N}^V \to \mathbb{R}$. Unlike the one dimensional case we need to introduce new variables.

Proposition 6. *a_i, $\mathcal{A}_\rangle = z_i$ and $a_i^\dagger$, $\mathcal{A}_i^\dagger = \frac{\partial}{\partial z_i}$ are intertwined via the generating function $(\mathcal{G}f)(z) = \sum_\eta f(\eta)\frac{z^\eta}{\eta!}$ for all $i \in V$ and for $f : \mathbb{N}^V \to \mathbb{R}$. We denote $z = (z_i)_{i\in V}$, $z^\eta = \Pi_{i\in V} z_i^{\eta_i}$ and $\eta! = \Pi_{i\in V}\eta_i!$.*

Proof. The proof is similar to the proof of Proposition 4. □

Intertwining now gives the following relation

$$\begin{aligned}
&\mathcal{G}L = \mathcal{G}(\sum_{i,j} p(i,j)(a_i - a_j)(a_j^\dagger - a_i^\dagger) + \sum_i \lambda_i(a_i - a_i a_i^\dagger) + \mu_i(a_i^\dagger - I)) \\
&= (\sum_{i,j} p(i,j)(\mathcal{A}_\rangle - \mathcal{A}_|)(\mathcal{A}_j^\dagger - \mathcal{A}_i^\dagger) + \sum_i \lambda_i(\mathcal{A}_\rangle - \mathcal{A}_\rangle \mathcal{A}_i^\dagger) + \mu_i(\mathcal{A}_i^\dagger - 1))\mathcal{G} \\
&= \mathcal{L}\mathcal{G}.
\end{aligned} \tag{48}$$

So we find the generator

$$\begin{aligned}
\mathcal{L} &= \sum_{i,j} p(i,j)(z_i - z_j)(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i}) + \sum_i \lambda_i(z_i - z_i\frac{\partial}{\partial z_i}) + \mu_i(\frac{\partial}{\partial z_i} - 1) \\
&= \sum_i \psi_i(z)\frac{\partial}{\partial z_i} + V(z).
\end{aligned} \tag{49}$$

where the multiplication operator is given by

$$V(z) = \sum_i (\lambda_i z_i - \mu_i). \tag{50}$$

Remarks. 1. The generator $\tilde{\mathcal{L}} = \sum_i \psi_i(z)\frac{\partial}{\partial z_i}$ corresponds to the process

$$\begin{cases} \dot{z}_1 = \psi_1(z) \\ \dot{z}_2 = \psi_2(z) \\ \vdots \end{cases} \tag{51}$$

with initial distribution $\zeta = (\zeta_i)_{i\in V}$.

2. As a consequence of Equation 49 we have

$$e^{t\mathcal{L}}f(\zeta) = e^{t(V(z)+\tilde{\mathcal{L}})}f(\zeta) = e^{\int_0^t V(Z^\zeta(s))ds}f(Z^\zeta(t)) \tag{52}$$

with $Z^\zeta(t) = (Z_i^\zeta(t))_{i\in V}$ the solution of Equation 51.

4.2.1 Propagation of Poisson

Starting from a distribution $\nu_\zeta \sim Poisson(\zeta_1) \otimes Poisson(\zeta_2) \otimes \cdots$, we find for $f : \mathbb{N}^V \to \mathbb{R}$

$$\begin{aligned}
\sum_{n=0}^{\infty} \nu_\zeta(n)(S(t)f)(n) =& e^{-\zeta} \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} S(t)f \\
=& e^{-\zeta} \mathcal{G}(S(t)f)(\zeta) \\
=& e^{-\zeta} e^{\int_0^t V(Z^\zeta(s))ds} \mathcal{G} f(Z^\zeta(t)) \\
=& e^{-\zeta} e^{\int_0^t V(Z^\zeta(s))ds} \sum_{n=0}^{\infty} \frac{Z^\zeta(t)^n}{n!} f(n) \\
=& e^{-\zeta} e^{\int_0^t V(Z^\zeta(s))ds} e^{Z^\zeta(t)} \sum_{n=0}^{\infty} \nu_{Z^\zeta(t)}(f)
\end{aligned} \tag{53}$$

Choosing $f = 1$, we find for the exponential term

$$e^{-\zeta} e^{\int_0^t V(Z^\zeta(s))ds} e^{Z^\zeta(t)} = 1$$

and thus the result becomes

$$\sum_{n=0}^{\infty} \nu_\zeta(n)(S(t)f)(n) = \sum_{n=0}^{\infty} \nu_{Z^\zeta(t)}(f). \tag{54}$$

So once again, just like the simple birth and death process, we see in Equation 54 that the Poisson distribution is preserved, despite diffusion being added to the mix. Figure 3 supports this result in the case of a two-point system. At time t the fit is given by $Poisson(Z_t^\zeta)$

However, propagation of Poisson distribution is not always the case, as shown in Figure 4. The death rate is now according to $\lambda_i n_i(n_i - 1)$ $n_i \to n_i - 2$ for $i \in \{1, 2\}$. In other words, the particles are no longer independent since they annihilate in pairs. This results in a collapse of the Poisson distribution.

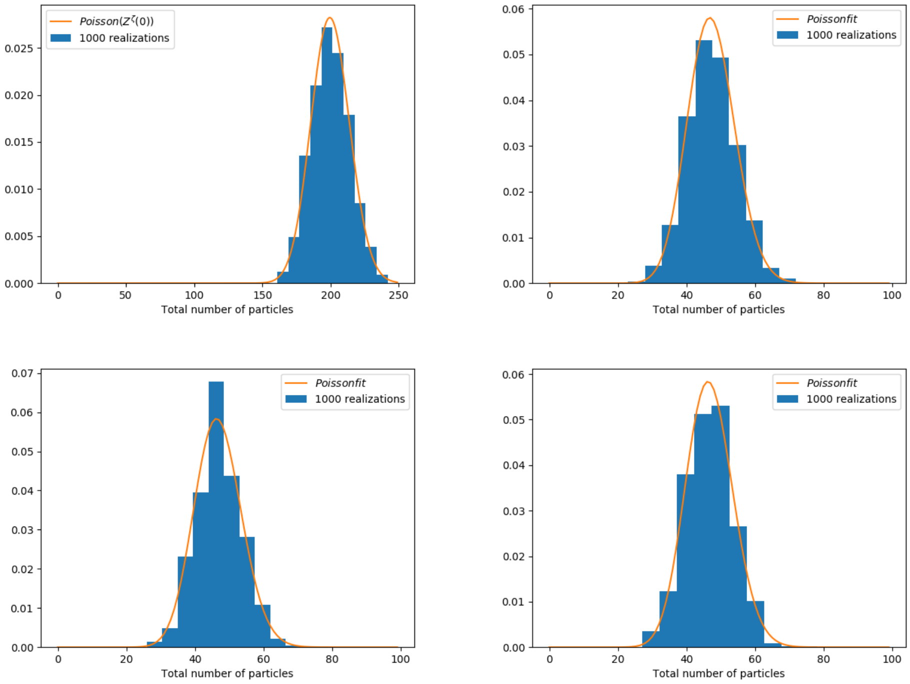


Figure 3: 1000 realizations of a reaction-diffusion system with two points. Each point starts with an initial distribution sampled from $Poisson(100)$. The total number of particles of both points combined is plotted at four different times, namely 0, 125, 250 and 375 seconds. The rates are given by $\mu_1 = \mu_2 = 1, \lambda_1 = 0.01, \lambda_2 = 0.05, p(1,2) = 0.1$ and $p(2,1) = 0.5$.

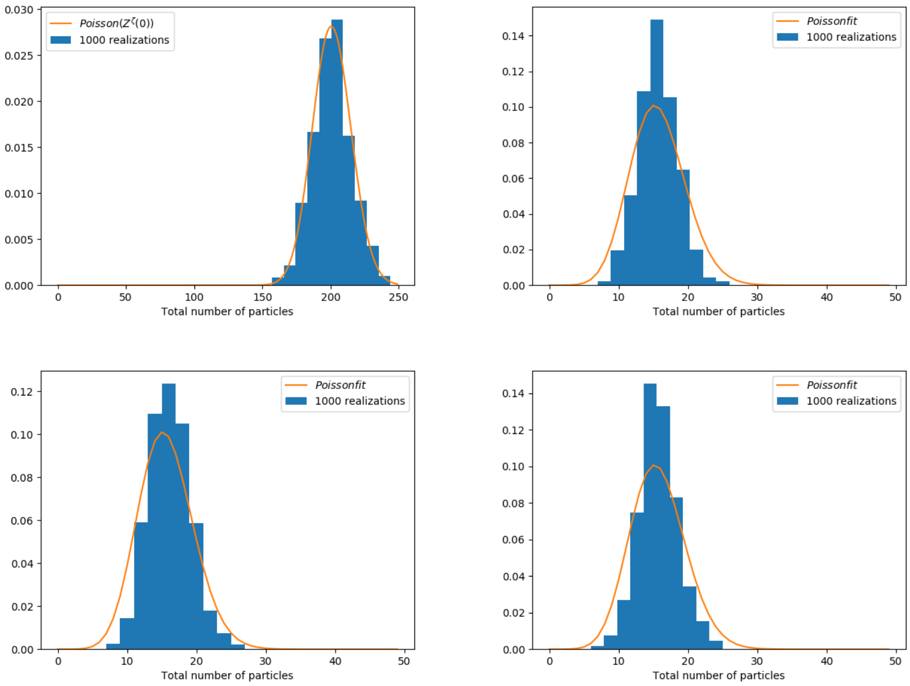


Figure 4: 1000 realizations of a reaction-diffusion system with two points, this time with pairwise annihilation according to rate $\lambda_i n_i(n_i-1)$ $n_i \to n_i - 2$ for $i \in \{1,2\}$. Each point starts with an initial distribution sampled from $Poisson(100)$. The total number of particles of both points combined is plotted at times 0, 125, 250 and 375 seconds. The rates are given by $\mu_1 = \mu_2 = 1, \lambda_1 = 0.01, \lambda_2 = 0.05, p(1,2) = 0.1$ and $p(2,1) = 0.5$.

5 Conclusion

After rewriting the generator of certain Markov processes through a series of creation and annihilation operators and intertwined processes we were able to calculate the time evolution of the corresponding distributions. All relevant operators satisfy the Heisenberg algebra commutation relation.

5.1 Birth and death process

The birth and death was shown to move according to the Poisson distribution in time. Moreover, the Poisson rate Z_t^ζ was calculated explicitly via the Feynman-Kac formula in Section 4.1.2. Starting from a Poisson distribution with rate ζ, the rate at time t becomes

$$Z_t^\zeta = \zeta e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t}) \tag{55}$$

with μ and λ the birth and death rates respectively. As time increases the distribution converges to an invariant measure $\pi \sim Poisson(\frac{\mu}{\lambda})$, which matches the result obtained in Section 2.5.

Duality was provided as an alternative method to prove propagation of Poisson. Poisson averaging the time dependent generating function

$$\mathbb{E}_n(z^{n(t)}) = e^{\frac{\mu(\zeta-1)}{\lambda}(1-e^{-\lambda t})}((\zeta - 1)e^{-\lambda t} + 1)^n \tag{56}$$

resulted in showing the Poisson generating function $e^{\zeta\rho_t - \rho_t}$ with rate $\rho_t = (\rho - \frac{\mu}{\lambda})e^{-\lambda t} + \frac{\mu}{\lambda}$.

5.2 A reaction-diffusion system

Expanding the Markov process to a graph and and adding in diffusive motion did not affect the Poisson distribution. In Section 4.2.1 it was shown that the process remained Poisson distributed in time with the final result being

$$\nu_\zeta(n)(S(t)f)(n) = \nu_{Z^\zeta(t)}. \tag{57}$$

Independence of the particles is necessary to retain Poisson, which is further supported by the simulations. The previously treated birth and death process and reaction-diffusion process have been fitted with Poisson probability mass functions in Figure 2 and Figure 3 respectively.

The same was done in Figure 4 with a process involving pairwise annihilation with rate $\lambda n(n-1)$ $n \to n-2$. This simulation showed that a non-independent process does not necessarily evolve according to the Poisson distribution. T

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