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Accelerated Degradation Data Analysis Based on Gamma Process With Random Effects

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ABSTRACT

The Gamma constant-stress accelerated degradation model is a natural model for monotonous degradation processes. However, unit heterogeneity often exists in practice, necessitating a more realistic model. This study develops a Gamma process with random effects to accurately capture accelerated degradation data for reliability analysis, encompassing both point and interval estimation. First, the Expectation-Maximization (EM) algorithm is developed to obtain point estimates of the proposed model. Since these estimates are sensitive to initial values, potentially impacting the outcomes, an improved EM algorithm is proposed, which iteratively refines the estimation quality by executing two different M-steps, thereby enhancing overall estimation accuracy. Secondly, given the complexity of the model and the constraint of small sample sizes and limited stress levels, a three-step interval estimation method is devised. This method segregates the parameters into three distinct parts and addresses them individually using the generalized pivotal quantity method, which simplifies the parameter interval estimation process and enhances the estimation accuracy. Finally, simulation studies and a real example of O-rings are presented to demonstrate the effectiveness of the proposed method.

1 | Introduction

For many highly reliable products, conducting degradation tests under normal operating conditions requires a significant amount of time to gather sufficient degradation data for reliability analysis [1]. An effective approach for obtaining enough degradation data within a reasonable time duration is to conduct accelerated degradation tests (ADTs) at higher stress levels [2]. ADT can be categorized into various types depending on the application of different stress loadings [3]. When conducting reliability analysis from the accelerated degradation data, the primary focus is on both point estimation and interval estimation of accelerated degradation model parameters.

The stochastic process model can effectively capture random fluctuations in the degradation process [4], numerous studies have employed stochastic processes (i.e., Wiener process, Gamma process, and inverse Gaussian process) to characterize the degradation process [5]. In most cases, the degradation path of product performance characteristics is monotonic, and the Gamma process is an important model for describing this degradation phenomenon. Park and Padgett [6] investigated inferential procedures for accelerated degradation models utilizing geometric Brownian motion and Gamma processes. Pulcini [7] introduced a non-stationary Gamma process incorporating a power-law shape function to model the accumulation of mileage processes. In the field of reliability demonstration, based on the Gamma process,

Zhang et al. [8] discussed the reliability demonstration by ADT for high-reliability products, and Zheng et al. [9] proposed an optimal design method for qualification sampling plan and acceptance sampling plan considering the unit heterogeneity.

In real-world scenarios, there are always uncertainties surrounding factors like environmental conditions, material properties, and usage patterns, among others. Effectively accounting for these uncertainties is crucial for ensuring the credibility of reliability analysis based on degradation laws [10]. There are various random effects models based on the Gamma process. Lawless and Crowder [11] developed a convenient Gamma process model with covariates and random effects for constant stress accelerated degradation test (CSADT). In this model, the rate parameter is treated as a function of the accelerated stress, while the scale parameter follows a Gamma distribution. Luis Alberto Rodríguez-Picón et al. [12] proposed two modifications of the parameters' structure of the Gamma process and derived two Gamma degradation models with random effects, respectively. One modification implies that the random effects only affect the volatility, and the other implies that the random effects only affect the rate. Wang et al. [13] investigated the Gamma degradation model with random effects by assuming the scale parameter follows a Gamma distribution. Duan and Wang [2] gave the theoretical plan of the CSADT design problem based on a Gamma process with random effects, where the relationship between the scale parameter β_k and stress level ζ_k is expressed as $\beta_k = \beta_0 \exp(b\zeta_k)$, and the scale parameter under constant stress levels, β_0, follows a Gamma distribution.

In many cases, particularly when models involve latent variables, it is common to use the maximum likelihood method (MLE) and Expectation-Maximization (EM) algorithm for parameter estimation [14]. These methods typically rely on likelihood functions to estimate parameters by maximizing them [15]. For complex models with latent variables, the EM algorithm is a widely used and effective approach [16]. It iteratively computes maximum likelihood estimates of parameters even in the presence of latent variables. EM is especially suited for exponential family models, as their likelihood functions are often concise, making EM implementation more convenient [17]. Duan and Wang [18] studied the stepped-stress test design problem based on the Gamma model with random effects, and used the EM algorithm to estimate the unknown parameters in the model. However, for the EM algorithm, the choice of initial values often affects the convergence and quality of estimation results [19]. The greater the number of initial values that need to be given, the greater the impact on the estimation results of parameters.

Regarding the interval estimation of model parameters, numerous methods have been developed. For example, Wang [20] obtained confidence intervals (CIs) using the asymptotic normal likelihood methods. Lawless and Crowder employed Bootstrap method to derive CIs for the model parameters. Ling et al. [21] established approximate CIs for the model parameters based on the observed Fisher information matrix. Wang et al. [13] also extended the inference procedures from the degradation case to the accelerated degradation case through the normal distribution approximation. Zhao et al. [22] derived CIs based on large-sample approximations for a stepwise Wiener degradation process with covariates. However, their coverage probabilities often do not meet practical requirements, especially in cases of small sample sizes and measurement numbers. In addition, the constant stress ADTs are often conducted with only 3 or 4 stress levels, and the limited number of stress levels makes it challenging to ensure the estimation accuracy of the accelerated parameters.

The challenge of small sample sizes and the need for accurate inference have prompted researchers to develop more effective interval estimation methods. In this case, the generalized pivotal quantity (GPQ) procedure can be employed to establish CIs for its independence from large sample sizes [13]. In addition, once the GPQs of model parameters are obtained, the invariance property of the GPQ procedure (similar to MLE) allows for the GPQs derivation for reliability indexes through a simple plug-in [23]. Qin et al. [24] applied the GPQ procedure to obtain the generalized prediction interval for the remaining useful life (RUL) of a product under Weibull constant stress accelerated life test with Type II censoring. Jiang et al. [25] proposed an interval estimation method for the inverse Gaussian process with the GPQ procedure. Chen and Ye [26] used the GPQ method to derive the interval estimation for the Gamma process. Chen et al. [27] also developed a comprehensive R package for the Gamma distribution, facilitating the estimation of model parameters using the GPQ method. Zheng et al. [28] proposed a novel two-step interval estimation method for the inverse Gaussian accelerated degradation model with unit heterogeneity based on the GPQ procedure, whose good performances are assessed by Monte Carlo simulation in terms of the coverage percentage and average interval length.

Consequently, this study proposes a parameter estimation method for accelerated degradation data based on the Gamma CSADT model with random effects, including the point and interval estimation. The basic principles of the proposed method are presented in Figure 1. Given the practical occurrence of unit heterogeneity, we employ a Gamma CSADT model with random effects to capture the complexity of accelerated degradation data. Initially, given the complexity of the model and its involvement of latent variables, as well as the susceptibility of the EM algorithm to initial values, an improved EM algorithm is proposed by executing two different M-steps to reduce the effect of initial values. Moreover, considering the imperative for accurate inference and the constraints imposed by small sample sizes and limited stress levels, we propose a three-step interval estimation method. This method divides the parameters into three distinct parts and addresses them separately using the GPQ method. The main contributions of this work are as follows.

1. Considering the unit heterogeneity, a more realistic model based on the nonlinear accelerated model and Gamma process is developed to depict the accelerated degradation data for reliability analysis.
2. Based on the proposed model, an improved EM algorithm is presented by executing two different M-steps to reduce the effect of initial values, improving parameter estimation accuracy.
3. A novel three-step interval estimation method is proposed, wherein the proposed model parameters are divided into three distinct parts and separately addressed using the GPQ

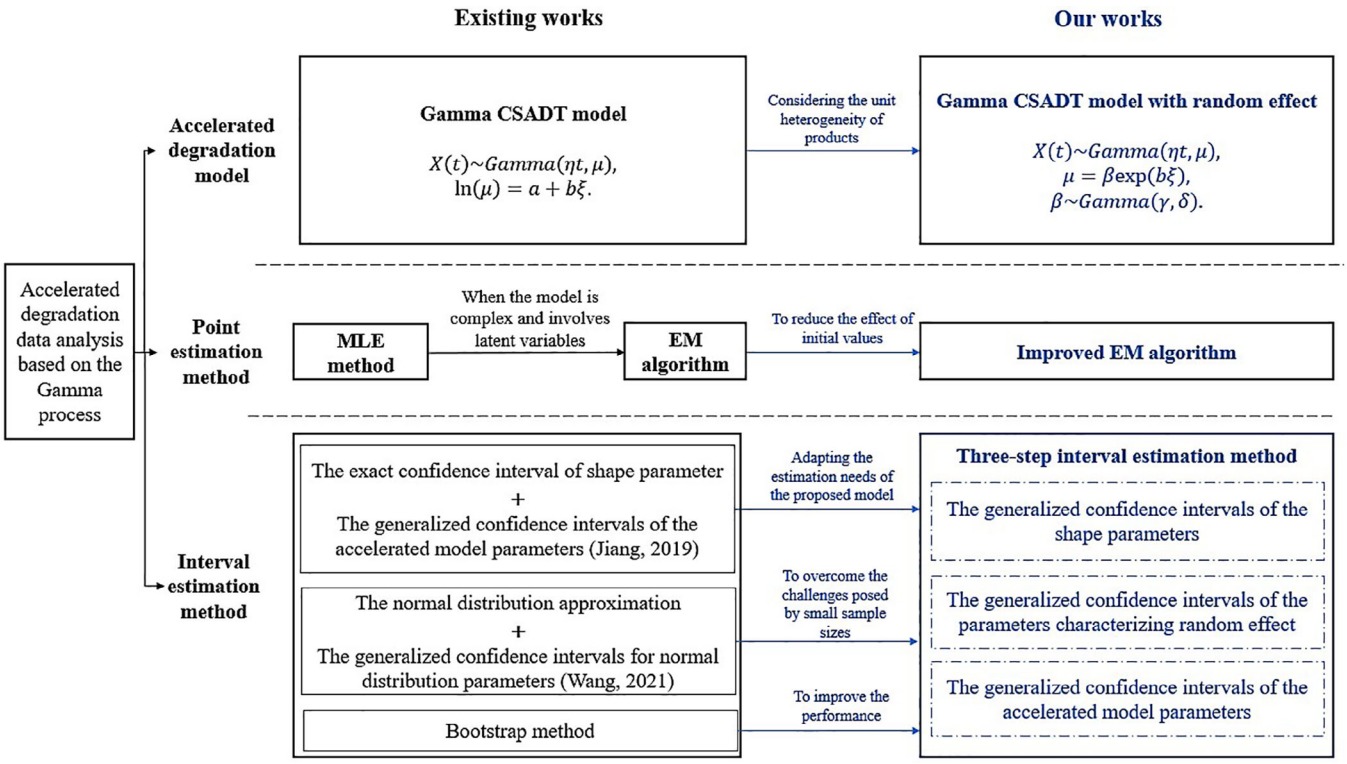

FIGURE 1 | The basic principles for the interval estimation of the Gamma CSADT model with random effects.

method, which simplifies the parameter interval estimation process and enhances the estimation accuracy.

The rest of this study is organized as follows. Section 2 provides an in-depth exploration of the Gamma CSADT model with random effects. Section 3 presents an improved EM algorithm to enhance the accuracy of point estimations for model parameters proposed in this study. In Section 4, a three-step interval estimation method is proposed to derive the generalized confidence interval (GCI) of model parameters and predictive reliability indexes. Section 5 is dedicated to the execution of simulations designed to assess the performance of the proposed method. In Section 6, a practical application involving the sealing of O-rings is presented to demonstrate the real-world implementation of the proposed method. Finally, Section 7 provides a comprehensive conclusion to this work.

2 | The Gamma CSADT Model

Suppose that the CSADT is conducted using K accelerated stress levels $s_1 < s_2 < \cdots < s_K$ and s_0 be the normal operating stress level. Additionally, n_k products are tested under the stress level s_k with m measurements , $k = 1, 2, \ldots, K$. The degradation data of i th product at the time $t_{i,j,k}$ is denoted as $X_{i,j,k}$, and the test interval $\Delta t_{i,j,k} = t_{i,j+1,k} - t_{i,j,k}$ is denoted as $h_{i,k}$, where $i = 1, 2, \ldots, n_k$, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, K$.

Let $\{Xt,\ t \geq 0\}$ denote the degradation process of the product, which is assumed to follow a Gamma process [2]. Specifically, Xt has statistically independent increments and follows a Gamma distribution, $Xt \sim Ga(\mu, \eta t), t > 0$, where $Ga(\mu, \eta t)$ is the Gamma distribution with the shape parameter $\eta t > 0$ and the rate parameter $\mu > 0$.

According to the relevant research [2], the Gamma CSADT model can be described as $Xt \sim Ga(\mu, \eta t)$ with a nonlinear function, that is, $\mu = \beta \exp(b\zeta)$, and ζ presents the standardization of stress levels [2]. Additionally, the random effect in the Gamma process is incorporated by letting the parameter β follows a Gamma distribution with parameter δ and γ, that is, $\beta \sim Ga(\gamma, \delta), \delta > 0, \gamma > 0$. In summary, the accelerated degradation model can be expressed as

$$Xt \sim Ga(\mu, \eta t), \mu = \beta \exp(b\zeta), \beta \sim Ga(\gamma, \delta). \tag{1}$$

Denoting $X_t = X(t)$, and the PDF $f(X_t)$ can be written as follows,

$$f(X_t) = \int_0^{+\infty} f(X_t|\beta) \times f(\beta)\ d\beta$$

$$= \frac{X_t^{\eta t-1}\gamma^{\delta}[\exp(b\zeta)]^{\eta t}\Gamma(\delta+\eta t)}{\Gamma(\eta t)\Gamma(\delta)[\gamma + X_t \exp(b\zeta)]^{\delta+\eta t}}, \tag{2}$$

where $f(X_t|\beta) = \frac{X_t^{\eta t-1}[\beta\exp(b\zeta)]^{\eta t}e^{-X_t\beta\exp(b\zeta)}}{\Gamma(\eta t)}$, $f(\beta) = \frac{\beta^{\delta-1}\gamma^{\delta}e^{-\gamma\beta}}{\Gamma(\delta)}$, and $\Gamma(z) = \int_0^{+\infty} x^{z-1}e^{-x}dx$ is the Gamma function. Note that $X_t|\beta \sim Ga(\beta\exp(b\zeta), \eta t)$, and $\frac{\delta\exp(b\zeta)X_t}{\gamma\eta t}$ follows F-distribution with $2\eta t$ and 2δ degrees of freedom, therefore, the CDF of X_t can be expressed as

$$F(X_t) = P(X_t \leq x) = P\left(\frac{\delta\exp(b\zeta)X_t}{\gamma\eta t} \leq \frac{\delta\exp(b\zeta)x}{\gamma\eta t}\right)$$

$$= \mathrm{F}_{2\eta t, 2\delta}\left(\frac{\delta x\exp(b\zeta)}{\gamma\eta t}\right). \tag{3}$$

The useful life $T = \inf(t|X(t) \geq D)$ be the first passage time of the degradation path to a degradation threshold D, and the CDF of T [11] can be expressed as

$$F_T(t|\eta, b, \gamma, \delta) = P(T \leq t) = P(X(t) > D) = 1 - \mathrm{F}_{2\eta t, 2\delta}\left(\frac{\delta D \exp(b\zeta)}{\gamma \eta t}\right). \quad (4)$$

Similarly, the RUL of the system at time t_ω is defined as $T_\omega = \inf\{t : X(t + t_\omega) > D | X(t_\omega) < D\}$, and the CDF of T_ω can be written as

$$F_{T_\omega}(t|\eta, b, \gamma, \delta, X(t_\omega)) = P(X(t + t_\omega) - X(t_\omega) \geq D - X(t_\omega)) = 1 - \mathrm{F}_{2\eta t, 2\delta}\left(\frac{\delta \exp(b\zeta)(D - X(t_\omega))}{\gamma \eta t}\right), \quad (5)$$

where $X(t_\omega)$ is the degradation data at time t_ω.

3 | Point Estimation

In this study, EM algorithm is used to estimate parameters in model (1), that is, $\Theta = \{\eta, b, \gamma, \delta\}$. Firstly, under the given parameter $\beta_{i,k}$, the likelihood function of the degradation data of i th sample $X_{i,k;1:m_{i,k}} = \{\Delta X_{i,1,k}, \Delta X_{i,2,k}, \ldots, \Delta X_{i,m_{i,k},k}\}$ can be written as

$$P(X_{i,k;1:m_{i,k}}|\beta_{i,k}) = \prod_{j=1}^{m_{i,k}} f\left(\Delta X_{i,j,k}|\beta_{i,k}\right) = \prod_{j=1}^{m_{i,k}} \frac{\Delta X_{i,j,k}^{\eta \Delta t_{i,j,k} - 1}\left(\beta_{i,k}\exp(b\zeta_k)\right)^{\eta \Delta t_{i,j,k}} e^{-\beta_{i,k}\exp(b\zeta_k)\Delta X_{i,j,k}}}{\Gamma\left(\eta \Delta t_{i,j,k}\right)}. \quad (6)$$

Based on Equation (1), the conditional density distribution of $\beta_{i,k}$ can be written as

$$P\left(\beta_{i,k}|X_{i,k;1:m_{i,k}}\right) \propto P\left(X_{i,k;1:m_{i,k}}|\beta_{i,k}\right) f\left(\beta_{i,k}\right) \propto \beta_{i,k}^{\sum_{j=1}^{m_{i,k}} \eta \Delta t_{i,j,k} + \delta - 1} \mathrm{e}^{-\beta_{i,k}\left(\gamma + \exp(b\zeta_k)\sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}\right)}. \quad (7)$$

According to Equation (7), under the given observations of $X_{i,k;1:m_{i,k}}$, the parameter $\beta_{i,k}$ still follows the Gamma distribution, that is, $\beta_{i,k}|X_{i,k;1:m_{i,k}} \sim Ga(\gamma + \exp(b\zeta_k)\sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}, \sum_{j=1}^{m_{i,k}} \eta \Delta t_{i,j,k} + \delta)$. Therefore, based on the properties of the Gamma distribution, the conditional expectations of $\beta_{i,k}$ and $\ln(\beta_{i,k})$ are derived,

$$E_{i,k}^{(1)}(\Theta) = E\left(\beta_{i,k}|X_{i,k;1:m_{i,k}}\right) = \frac{\sum_{j=1}^{m_{i,k}} \eta \Delta t_{i,j,k} + \delta}{\gamma + \exp(b\zeta_k)\sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}}, \quad (8)$$

$$E_{i,k}^{(2)}(\Theta) = E\left(\ln(\beta_{i,k})|X_{i,k;1:m_{i,k}}\right) = \psi\left(\sum_{j=1}^{m_{i,k}} \eta \Delta t_{i,j,k} + \delta\right) - \ln\left(\gamma + \exp(b\zeta_k)\sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}\right), \quad (9)$$

where $\psi(x) = \frac{d\ln(\Gamma(x))}{dx}$ is the digamma function, and its first-partial derivative $\psi'(x) > 1$.

Meanwhile, based on degradation data $\{\Delta X_{i,j,k}, j = 1, 2, \ldots, m_{i,k}; i = 1, 2, \ldots, n_k; k = 1, 2, \ldots, K\}$ and random effects $\beta_{i,k}$, the log-likelihood function $\ln(L_c(\Theta))$ can be written as

$$\begin{aligned}\ln(L_c(\Theta)) = &\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}[(\eta h_{i,k} - 1)\ln(\Delta X_{i,j,k}) + \eta h_{i,k}\ln(\beta_{i,k}) \\ &+ \eta h_{i,k} b\zeta_k - \beta_{i,k}\exp(b\zeta_k)\Delta X_{i,j,k}] \\ &- \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}\ln(\Gamma(\eta h_{i,k})) + \sum_{k=1}^{K}\sum_{i}^{n_k}[(\delta - 1)\ln(\beta_{i,k}) \\ &+ \delta\ln(\gamma) - \gamma\beta_{i,k} - \ln(\Gamma(\delta))]. \end{aligned} \quad (10)$$

Based on Equations (8)–(10), the detailed EM algorithm is derived and presented as follows.

E-step: Assuming that the current value of the model parameter is $\Theta^{(l)} = \{\eta^{(l)}, b^{(l)}, \gamma^{(l)}, \delta^{(l)}\}$, based on Equations (8)–(10), the Q-function $Q(\Theta|\Theta^{(l)}) = E(\ln(L(X, \mu))|\Theta^{(l)})$ can be expressed as

$$\begin{aligned} Q\left(\Theta|\Theta^{(l)}\right) = &\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}\left[(\eta h_{i,k} - 1)\ln(\Delta X_{i,j,k}) + \eta h_{i,k} E_{i,k}^{(2)}(\Theta^{(l)})\right. \\ &\left. + \eta h_{i,k} b\zeta_k - E_{i,k}^{(1)}(\Theta^{(l)})\exp(b\zeta_k)\Delta X_{i,j,k}\right] \\ &- \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}\ln\left(\Gamma\left(\eta h_{i,k}\right)\right) \\ &+ \sum_{k=1}^{K}\sum_{i=1}^{n_k}\left[(\delta - 1)E_{i,k}^{(2)}(\Theta^{(l)}) + \delta\ln(\gamma)\right. \\ &\left. - \gamma E_{i,k}^{(1)}(\Theta^{(l)}) - \ln(\Gamma(\delta))\right]. \end{aligned} \quad (11)$$

M-step: Computes the first derivative of Equation (11) with respect to parameter $\Theta = \{\eta, b, \gamma, \delta\}$, set the results of them to 0, and obtains the system of Equation (12).

$$\begin{cases} \frac{\partial \ln(L_c(\Theta))}{\partial \eta} = \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}\left[\Delta t_{i,j,k}\ln\left(\Delta X_{i,j,k}\right) + h_{i,k}E_{i,k}^{(2)}\left(\Theta^{(l)}\right)\right. \\ \qquad \left. + h_{i,k} b\zeta_k - h_{i,k}\psi\left(\eta \Delta t_{i,j,k}\right)\right] = 0, \\ \frac{\partial \ln(L_c(\Theta))}{\partial b} = \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}\left[\eta h_{i,k}\zeta_k\right. \\ \qquad \left. - E_{i,k}^{(1)}\left(\Theta^{(l)}\right)\exp\left(b\zeta_k\right)\Delta X_{i,j,k}\zeta_k\right] = 0, \\ \frac{\partial \ln(L_c(\Theta))}{\partial \gamma} = \sum_{k=1}^{K}\sum_{i}^{n_k}\left[\frac{\delta}{\gamma} - E_{i,k}^{(1)}\left(\Theta^{(l)}\right)\right] = 0, \\ \frac{\partial \ln(L_c(\Theta))}{\partial \delta} = \sum_{k=1}^{K}\sum_{i}^{n_k}\left[E_{i,k}^{(2)}\left(\Theta^{(l)}\right) + \ln(\gamma) - \psi(\delta)\right] = 0. \end{cases} \quad (12)$$

Solve this system of Equation (12), and the next iteration of parameters $\Theta^{(l+1)} = \{\eta^{(l+1)}, b^{(l+1)}, \gamma^{(l+1)}, \delta^{(l+1)}\}$ can be derived. Repeat the E-step and M-step until $\| \Theta^{(l+1)} - \Theta^{(l)} \|_\infty$ is less than the predetermined threshold ϵ, where $\| \cdot \|_\infty$ is the infinite norm. Then, $\Theta^{(l)}$ can be used as parameter estimators.

The estimators from the EM algorithm are highly sensitive to the choice of initial parameter values [29]. According to Equation (12), four initial parameter values are required, and a nonlinear optimization algorithm is employed to calculate the parameters $\Theta^{(l+1)}$. Consequently, reducing the number of initial parameter values and formulating the explicit expression of parameters can contribute to improving both the parameter point estimation accuracy and convergence speed.

According to model (1), under the given $\beta_{i,k}, i = 1, 2, \ldots, n_k, k = 1, 2, \ldots, K$, the log-likelihood function under given $\beta_{i,k}$ can be written as can be written as

$$\ln(L) = \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{i,k}} [(\eta h_{i,k} - 1) \ln(\Delta X_{i,j,k}) + \eta h_{i,k} \ln(\beta_{i,k} \exp(b\zeta_k))] - \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{i,k}} [\beta_{i,k} \exp(b\zeta_k) \Delta X_{i,j,k} + \ln(\Gamma(\eta h_{i,k}))]. \quad (13)$$

The detailed derivation of Equation (13) in Appendix A. Based on this, the first derivative of Equation (13) on $\beta_{i,k}$ can be written as

$$\frac{\partial \ln(L)}{\partial \beta_{i,k}} = \sum_{j=1}^{m_{i,k}} \left[\frac{\eta h_{i,k}}{\beta_{i,k}} - \exp(b\zeta_k) \Delta X_{i,j,k} \right]. \quad (14)$$

Therefore, the estimators $\hat{\beta}_{i,k}$ can be derived as

$$\hat{\beta}_{i,k} = \frac{m_{i,k} \hat{\eta} h_{i,k}}{\exp(\hat{b}\zeta_k) \sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}}, \quad (15)$$

where $\hat{\eta}$ and $\hat{b}$ are the parameter estimation of η and b, respectively. Furthermore, based on the estimators $\hat{\beta}_{i,k}$ and the MLE method, the estimators $\hat{\delta}$ and $\hat{\gamma}$ can be obtained with the following equations [30].

$$\begin{cases} \hat{\delta} = \dfrac{N \sum_{k=1}^{K} \sum_{i=1}^{n_k} \hat{\beta}_{i,k}}{N \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\hat{\beta}_{i,k} \ln\left(\hat{\beta}_{i,k}\right) \right] - \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\ln\left(\hat{\beta}_{i,k}\right) \sum_{k=1}^{K} \sum_{i=1}^{n_k} \hat{\beta}_{i,k} \right]}, \\ \hat{\gamma} = \dfrac{N^2}{N \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\hat{\beta}_{i,k} \ln\left(\hat{\beta}_{i,k}\right) \right] - \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\ln\left(\hat{\beta}_{i,k}\right) \sum_{k=1}^{K} \sum_{i=1}^{n_k} \hat{\beta}_{i,k} \right]}, \end{cases} \quad (16)$$

where $N = \sum_{k=1}^{K} n_k$.

Therefore, the M-step can be updated, and an improved EM algorithm is introduced to enhance the accuracy of parameter estimation.

M-step-1: Computes the first derivative of Equation (10) with respect to parameter $\{\eta, b\}$, set the results of them to 0, and obtains the system of Equation (17).

$$\begin{cases} \dfrac{\partial \ln(L_c(\Theta))}{\partial \eta} = \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{i,k}} \left[\Delta t_{i,j,k} \ln(\Delta X_{i,j,k}) + h_{i,k} E_{i,k}^{(2)}(\Theta^{(l)}) + h_{i,k} b\zeta_k - h_{i,k} \psi(\eta \Delta t_{i,j,k}) \right] = 0 \\ \dfrac{\partial \ln(L_c(\Theta))}{\partial b} = \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{i,k}} \left[\eta h_{i,k} \zeta_k - E_{i,k}^{(1)}(\Theta^{(l)}) \exp(b\zeta_k) \Delta X_{i,j,k} \zeta_k \right] = 0 \end{cases} \quad (17)$$

Solve the system of Equation (17), the next iteration of parameters $\eta^{(l+1)}, b^{(l+1)}$ can be derived.

M-step-2: With the parameters $\eta^{(l+1)}, b^{(l+1)}$ and Equation (15), the parameters $\delta^{(l+1)}$ and $\gamma^{(l+1)}$ can be derived,

$$\begin{cases} \delta^{(l+1)} = \dfrac{N \sum_{k=1}^{K} \sum_{i=1}^{n_k} \beta_{i,k}^{(l+1)}}{N \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\beta_{i,k}^{(l+1)} \ln\left(\beta_{i,k}^{(l+1)}\right) \right] - \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\ln\left(\beta_{i,k}^{(l+1)}\right) \sum_{k=1}^{K} \sum_{i=1}^{n_k} \beta_{i,k}^{(l+1)} \right]}, \\ \gamma^{(l+1)} = \dfrac{N^2}{N \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\beta_{i,k}^{(l+1)} \ln\left(\beta_{i,k}^{(l+1)}\right) \right] - \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\ln\left(\beta_{i,k}^{(l+1)}\right) \sum_{k=1}^{K} \sum_{i=1}^{n_k} \beta_{i,k}^{(l+1)} \right]}, \end{cases} \quad (18)$$

where $\beta_{i,k}^{(l+1)} = \frac{m_{i,k} \eta^{(l+1)} h_{i,k}}{\exp(b^{(l+1)}\zeta_k) \sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}}$. Repeat the E-step and M-step (M-step-1 and M-step-2) until $\| \Theta^{(l+1)} - \Theta^{(l)} \|_\infty$ is less than the predetermined threshold ϵ. Then, $\Theta^{(l)}$ can be used as parameter estimators.

4 | Interval Estimation

Considering the significance of interval estimation in the field of reliability analysis [31], in this section, a three-step interval estimation method is proposed to derive the GCIs of accelerated degradation model parameters. Then, the generalized prediction intervals under normal stress levels are derived.

4.1 | Three-Step Interval Estimation Method for Model Parameters

Considering the complexity of this Gamma CSADT model with random effects, the GPQ procedure is a good choice for constructing interval estimations for the proposed model parameters.

The main point of this procedure is to find the GPQs for model parameters, which are difficult to derive. According to

the proposed model in Equation (1), the parameters that need to be estimated can be divided into three parts, that is, $\{\eta\}$, $\{b\}$, and $\{\delta, \gamma\}$, which are from the degradation model, accelerated model, and random effects, respectively. According to Zheng et al. [28] and Luo et al. [32], the producer of interval estimation for proposed model parameters can also be divided into three parts and addressed them separately, thereby simplifying the interval estimation procedure.

4.1.1 | The GPQs for δ and γ

Based on Equation (1), $\beta_{i,k}$ follow the Gamma distribution, that is, $\beta_{i,k} \sim Ga(\gamma, \delta), i = 1, 2, \ldots, n_k, k = 1, 2, \ldots, K$. Let $\mathcal{H} = \ln(\frac{(\prod_{k=1}^{K}\prod_{i=1}^{n_k}\hat{\beta}_{i,k})^{\frac{1}{N}}}{\bar{Y}})$, where $\bar{Y} = \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\hat{\beta}_{i,k}}{N}, N = \sum_{k=1}^{K} n_k$. According to S. Weerahandi [33], the random variable $U = F_{\mathcal{H}}$ follows the standard uniform distribution $U(0, 1)$, where $F_{\mathcal{H}}$ is the CDF of the statistic $\mathcal{H}$. Then, leveraging the Cornish-Fisher expansion and pivoting the CDF, as discussed in Chapter 9.2.3 of Casella and Berger [34], for a given $U \sim U(0, 1)$,

$$H = \varkappa_1(\delta) + [\varkappa_2(\delta)]^{\frac{1}{2}} Q_1(\delta, U), \tag{19}$$

where $Q_1(\delta, U)$ in Equation (19) can be expressed as

$$\begin{aligned} Q_1(\delta, U) = {} & z_U + \frac{1}{6}\varkappa_3'(\delta)\left(z_U^2 - 1\right) + \frac{1}{24}\varkappa_4'(\delta)\left(z_U^3 - 3z_U\right) \\ & - \frac{1}{36}\left[\varkappa_3'(\delta)\right]^2\left(2z_U^3 - 5z_U\right) + \frac{1}{120}\varkappa_5'(\delta)\left(z_U^4 - 6z_U^2 + 3\right) \\ & - \frac{1}{24}\varkappa_3'(\delta)\varkappa_4'(\delta)\left(z_U^4 - 5z_U^2 + 2\right) \\ & + \frac{1}{324}\left[\varkappa_3'(\delta)\right]^3\left(12z_U^4 - 53z_U^2 + 17\right), \end{aligned} \tag{20}$$

$\varkappa_i(\cdot)$ is the i th $(i = 1, 2, 3, 4, 5)$ cumulant of $\mathcal{H}$, $\varkappa_i'(\cdot) = \frac{\varkappa_i(\cdot)}{[\varkappa_2(\cdot)]^{\frac{i}{2}}}, i = 3, 4, 5$, and z_U is the U percentile of standard normal distribution $N(0, 1)$. According to Wang and Wu [35], the unique solution $G_\delta = g(\mathcal{H}, U)$ can be derived based on Equation (19) for the value of $\varkappa_1(\delta) + [\varkappa_2(\delta)]^{\frac{1}{2}} Q_1(\delta, U)$ is a strictly increasing function of δ under the given U.

According to Wang and Wu [35], the statistics $(\mathcal{H}, \bar{Y})$ are complete and sufficient, and $2N\gamma\bar{Y} \sim \chi^2(2N\delta)$. Then, for the given U, $V_1 = 2NG_\gamma\bar{Y} \sim \chi^2(2NG_\delta)$, where G_γ represents the GPQ for parameter γ and can be written as

$$G_\gamma = \frac{V_1}{2N\bar{Y}}. \tag{21}$$

4.1.2 | The GPQ for Parameter η

Under the fixed parameter $\beta_{i,k}$, let $Z = \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}} h_{i,k} \ln(\Delta X_{i,j,k}\beta_{i,k}) - \sum_{k=1}^{K} Q_k \ln(X_k)$, the r th cumulant of Z is given by

$$\begin{aligned} \kappa_r(\eta) = {} & \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}} \psi^{r-1}(\eta h_{i,k}) \times \left(h_{i,k}\right)^r \\ & - \sum_{k=1}^{K} \psi^{r-1}(\eta Q_k) Q_k^r, \; r = 1, 2, \ldots \end{aligned} \tag{22}$$

where $\psi^{r-1}(\cdot)$ is the $r-1$ th derivative of $\psi(\cdot)$, $Q_k = \sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}} h_{i,k}$, $X_k = \sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}\beta_{i,k}$, $\Delta X_{i,j,k}\beta_{i,k} \sim Ga(\eta h_{i,k}, \exp(b\zeta_k))$, $X_k \sim Ga(\eta Q_k, \exp(b\zeta_k))$, and its proof can be seen in Appendix B.

Similar to the proof of Lamma 1 in Iliopoulos [36], the CDF $F_Z(z|\eta)$ of Z is a strictly decreasing function of η. Denoting $\varkappa_i'(\cdot) = \frac{\varkappa_i(\cdot)}{[\varkappa_2(\cdot)]^{\frac{i}{2}}}, i = 3, 4, 5$, and z_U is the U percentile of standard normal distribution $N(0, 1)$. According to the procedure proposed by Wang and Wu [35], the GPQ for parameter η (i.e., G_η) can be obtained by solving the following equation:

$$Z = \varkappa_1(G_\eta) + \left[\varkappa_2(G_\eta)\right]^{\frac{1}{2}} Q_2(G_\eta, U), \tag{23}$$

where $Q_2(\eta, U) = Q_1(\eta, U)$.

4.1.3 | The GCIs for parameter b

According to model (1), we have $\ln(\frac{\mu_{i,k}}{\beta_{i,k}}) = b\zeta_k$ with the fixed $\beta_{i,k}$, $i = 1, 2, \ldots, n_k; k = 1, 2, \ldots, K$. Therefore, the variance of $\ln(\frac{\mu_{i,k}}{\beta_{i,k}})$, that is, $D_k = Var(\ln(\mu_{i,k}/\beta_{i,k}))$, can be written as

$$D_k = Var\left(\ln\left(\frac{\mu_{i,k}}{\beta_{i,k}}\right)\right) = \zeta_k^2 Var(b). \tag{24}$$

After calculating the value of D_k, the following weighted sum of squares is considered

$$O(b) = \sum_{k=1}^{K}\sum_{i=1}^{n_k} \frac{\left(\ln(\mu_{i,k}) - \ln(\beta_{i,k}) - b\zeta_k\right)^2}{D_k}. \tag{25}$$

By minimizing Equation (25), we have

$$\begin{aligned} b &= \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\zeta_k(\ln(\mu_{i,k}) - \ln(\beta_{i,k}))}{D_k}}{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\zeta_k^2}{D_k}} \\ &= \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\zeta_k(\ln(\mu_{i,k}) - \ln(\beta_{i,k}))}{\zeta_k^2 Var(b)}}{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\zeta_k^2}{\zeta_k^2 Var(b)}} = \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\ln(\mu_{i,k}) - \ln(\beta_{i,k})}{\zeta_k}}{N}. \end{aligned} \tag{26}$$

Based on this, we proceed to the GPQ for the parameter b. First, let

$$V = \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\ln(X_{i,k}) + \ln(\beta_{i,k})}{\zeta_k}}{N} + b = \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\ln(S_{i,k})}{\zeta_k}}{N}, \tag{27}$$

where $X_{i,k} = \sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}$, $S_{i,k} = \mu_{i,k}\; X_{i,k} \sim Ga(1, \eta B_{i,k})$, $B_{i,k} = m_{i,k}\, h_{i,k}$. Therefore, the GPQ for parameter b can be expressed as

$$W_1 = -\frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\ln(X_{i,k}) + \ln(\beta_{i,k}^*)}{\zeta_k}}{N} + \frac{\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\ln(S_{i,k}^*)}{\zeta_k}}{N}, \tag{28}$$

where $B_{i,k}^* \sim Ga(G_\gamma, G_\delta)$, $S_{i,k}^* \sim Ga(1, G_\eta B_{i,k})$.

With the GPQ for parameters δ, γ, η, and b, the percentiles of them can be obtained by the following Monte Carlo algorithm (i.e., Algorithm 1).

ALGORITHM 1 | GCIs for δ, γ, η, and b.

Input: The degradation data $\{X_{i,j,k}, t_{i,j,k}, \zeta_k\}, j = 1, 2, \ldots, m; i = 1, 2, \ldots, n_k; k = 1, 2, \ldots, K$; Parameter estimation results of the accelerated degradation model, i.e., $\eta, b, \beta_{i,k}$, and denoted as $\hat{\eta}, \hat{b}, \hat{\beta}_{i,k}$; The expected percentile ρ.

Output: GCIs for δ, γ, η, and b.

1. Calculate $\bar{Y} = \sum_{k=1}^{K} \sum_{i=1}^{n_k} \hat{\beta}_{i,k} / N$, and $\mathcal{H} = \ln[(\prod_{k=1}^{K} \prod_{i=1}^{n_k} \hat{\beta}_{i,k})^{\frac{1}{N}} / \bar{Y}]$;
2. Generate $U_1 \sim U(0,1)$, calculate the value of $G_\delta = g(\mathcal{H}, U_1)$ based on Equation (19);
3. Generate $V_1 \sim \chi^2(2Ng(\mathcal{H}, U))$, calculate the value of G_γ based on Equation (21);
4. Generate $\beta^*_{i,k} \sim Ga(G_\gamma, G_\delta)$, calculate $X_{i,k} = \sum_{j=1}^{m_{i,k}} \Delta X_{i,j,k}$;
5. Calculate Z, generate $U_2 \sim U(0,1)$, calculate the value of $G_\eta = g(Z, U_2)$ based on Equation (23);
6. Generate $S^*_{i,k} \sim Ga(1, G_\eta B_{i,k})$, based on Equation (28), calculate the value of W_1;
7. Repeat steps 3 to 8 for B times, then there are B values for W_1, G_γ, G_η, and G_δ respectively;
8. Arrange all $G_\delta, G_\gamma, G_\eta$ and W_1 values in ascending order: $G_{\delta,1} < G_{\delta,2} < \cdots < G_{\delta,B}, G_{\gamma,1} < G_{\gamma,2} < \cdots < G_{\gamma,B}, G_{\eta,1} < G_{\eta,2} < \cdots < G_{\eta,B}, W_{1,1} < W_{1,2} < \cdots < W_{1,B}$;
9. Then, the $100(1-\rho)\%$ GCIs for parameters δ, γ, η, and b can be derived as $[G_{\delta,\frac{B\rho}{2}}, G_{\delta,B(1-\frac{\rho}{2})}]$, $[G_{\gamma,\frac{B\rho}{2}}, G_{\gamma,B(1-\frac{\rho}{2})}]$, $[G_{\eta,\frac{B\rho}{2}}, G_{\eta,B(1-\frac{\rho}{2})}]$, and $[W_{1,\frac{B\rho}{2}}, W_{1,B(1-\frac{\rho}{2})}]$, respectively.

4.2 | Interval Estimation Method for Predictive Reliability Indexes

As described in Section 2, the CSADT is conducted under the accelerated stress level. However, in practical applications, engineers pay great attention to the percentiles of some significant reliability indexes at the normal operating stress level ζ_0, such as the mean of degradation $E(Xt_0)$, the reliability $R(t_0)$ at time t_0, the quantile lifetime t_p, the mean of lifetime $E(T_0)$, and the mean remaining useful life $E(T_\omega)$ at time t_ω. According to Ye et al. [37], they can be expressed as

$$\begin{cases} E\ (Xt_0) = \dfrac{\Gamma(\delta-1)\gamma\eta t_0}{\Gamma(\delta)\exp(b\zeta_0)} \\ R\ (t_0) = 1 - F_T\ (t_0|\eta, b, \gamma, \delta) = \mathrm{F}_{2\eta t_0, 2\delta}\left(\dfrac{\delta D \exp(b\zeta_0)}{\gamma\eta t_0}\right), \\ t_p = \dfrac{\delta D \exp(b\zeta_0)}{\gamma\eta z_{1-p}}, \\ E\ (T_0) = \displaystyle\int_0^{+\infty} [1 - F_T\ (t|\eta, b, \gamma, \delta)]\ dt \\ = \displaystyle\int_0^{+\infty} \mathrm{F}_{2\eta t, 2\delta}\left(\dfrac{\delta D \exp(b\zeta_0)}{\gamma\eta t}\right) dt, \\ E\ (T_\omega) = \displaystyle\int_0^{+\infty} \left[1 - F_{T_\omega}\ (t|X\ (t_\omega))\right]\ dt \\ = \displaystyle\int_0^{+\infty} \mathrm{F}_{2\eta t, 2\delta}\left(\dfrac{\delta \exp(b\zeta_0)\ (D - X\ (t_\omega))}{\gamma\eta t}\right) dt, \end{cases} \tag{29}$$

where z_{1-p} represents the p-th quantile of $\mathrm{F}(2\eta t, 2\delta)$ distribution.

ALGORITHM 2 | GCIs for $E(Xt_0)$, $R(t_0)$, t_p, $E(T_0)$, and $E(T_\omega)$.

Input: The degradation data $\{X_{i,j,k}, t_{i,j,k}, \zeta_k\}, j = 1, 2, \ldots, m; i = 1, 2, \ldots, n_k; k = 1, 2, \ldots, K$; Parameter estimation results of the accelerated degradation model, i.e., $\eta, b, \beta_{i,k}$, and denoted as $\hat{\eta}, \hat{b}, \hat{\beta}_{i,k}$; The expected percentile ρ.

Output: GCIs for $E(Xt_0)$, $R(t_0)$, t_p, $E(T_0)$, and $E(T_\omega)$.

1. Follows steps 1 to 6 from Algorithm 1 to derive the values of W_1, G_γ, G_η, and G_δ;
2. Calculate $W_l, l = 2, 3, 4, 5, 6$ based on Equation (30);
3. Repeat steps 1 and 2 for B times to generate B values for W_l;
4. Arrange all W_l values in ascending order: $W_{l,1} < W_{l,2} < \cdots < W_{l,B}$;
5. Then, the ρ percentiles of $E(Xt_0)$, $R(t_0)$, t_p, $E(T_0)$, and $E(T_\omega)$ can be estimated by $W_{l,B\rho}, l = 2, 3, 4, 5, 6$, respectively.

Then, according to Equation (29) and the substitute method in Chapter 2.4 [33], the GPQs of $E(Xt_0)$, $R(t_0)$, t_p, $E(T_0)$, and $E(T_\omega)$ can be expressed as

$$\begin{cases} W_2 = \dfrac{\Gamma(G_\delta - 1) G_\gamma G_\eta t_0}{\Gamma(G_\delta)\exp(W_1\zeta_0)}, \\ W_3 = \mathrm{F}_{2G_\eta t_0, 2G_\delta}\left(\dfrac{G_\delta D \exp(W_1\zeta_0)}{G_\gamma G_\eta t_0}\right), \\ W_4 = \dfrac{G_\delta D \exp(W_1\zeta_0)}{G_\gamma G_\eta z_{1-p}}, \\ W_5 = \displaystyle\int_0^{+\infty} \mathrm{F}_{2G_\eta t, 2G_\delta}\left(\dfrac{G_\delta D \exp(W_1\zeta_0)}{G_\gamma G_\eta t}\right) dt, \\ W_6 = \displaystyle\int_0^{+\infty} \mathrm{F}_{2G_\eta t, 2G_\delta}\left(\dfrac{G_\delta\ (D - X\ (t_\omega))\exp(W_1\zeta_0)}{G_\gamma G_\eta t}\right) dt. \end{cases} \tag{30}$$

Let $W_{l,\rho}$ represents the ρ percentile of W_l. Then, the $100(1-\rho)\%$ GCIs for $E(Xt_0)$, $R(t_0)$, t_p, $E(T_0)$, and $E(T_\omega)$ are determined by $[W_{l,\frac{\rho}{2}}, W_{l,1-\frac{\rho}{2}}]$, $l = 2, 3, 4, 5, 6$, respectively. Similarly, the percentiles of W_l can be obtained using the following Monte Carlo algorithm (i.e., Algorithm 2).

5 | Simulation Study

In this section, two simulation studies are conducted to demonstrate the performances of the proposed method, that is, the improved point estimation method and interval estimation method, respectively.

5.1 | Point Estimation Simulation Study

In this subsection, we evaluate the performance of the proposed point estimation. Suppose the ADT is conducted under four different conversion-stress levels, $\zeta_1 = 1.5$, $\zeta_2 = 2.0$, $\zeta_3 = 2.5$, $\zeta_4 = 3.0$, and the degradation data under these four levels are collected with the measurement intervals $h_1 = h_2 = h_3 = h_4 = 1$, respectively. The proposed improved EM algorithm is compared with the traditional EM algorithm across 18 distinct parameter settings as outlined in Table 1, sample size n and measurement

TABLE 1 | Parameter setting for point estimation simulation study.

Case	η	b	γ	δ
I	4.0	−1.5	2.0	19.0
II	4.0	−1.0	2.0	19.0
III	4.0	−0.5	2.0	19.0
IV	5.0	−1.5	2.0	19.0
V	5.0	−1.0	2.0	19.0
VI	5.0	−0.5	2.0	19.0
VII	4.0	−1.5	1.0	10.0
VIII	4.0	−1.0	1.0	10.0
IX	4.0	−0.5	1.0	10.0
X	5.0	−1.5	1.0	10.0
XI	5.0	−1.0	1.0	10.0
XII	5.0	−0.5	1.0	10.0
XIII	4.0	−1.5	0.5	5.0
XIV	4.0	−1.0	0.5	5.0
XV	4.0	−0.5	0.5	5.0
XVI	5.0	−1.5	0.5	5.0
XVII	5.0	−1.0	0.5	5.0
XVIII	5.0	−0.5	0.5	5.0

times m are set to 8 and 10, respectively. The predetermined threshold ϵ is set as 10^{-6} and the initialization for the traditional EM is set as $\{\eta, b, \gamma, \delta\} = \{1, -1, 1, 5\}$.

Then, the root mean square error (RMSE) and root mean square percentage error (RMSPE) are adopted to measure the feasibility and effectiveness of the proposed method quantitatively. RMSE is calculated by taking the square root of the mean of squared differences between observed and predicted values. RMSPE represents the average percentage error of predicted values relative to actual values, whose advantage lies in its robustness in terms of percentage error. Generally, a smaller RMSE and RMSPE indicate more accurate parameter estimation results. The expression of RMSE and RMSPE are described as follows:

$$\begin{cases} \text{RMSE} = \sqrt{\dfrac{\sum_{i=1}^{B_{rep}} \left(\hat{\theta}_i - \theta_{true}\right)^2}{B_{rep}}}, \\ \text{RPMSE} = \sqrt{\dfrac{\sum_{i=1}^{B_{rep}} \left[\dfrac{\left(\hat{\theta}_i - \theta_{true}\right)}{\theta_{true}}\right]^2}{B_{rep}}}, \end{cases} \tag{31}$$

where $\hat{\theta}_i, i = 1, 2, \ldots, B_{rep}$ represent the estimator vector for the i th simulation, and θ_{true} represents the true values, respectively.

Based on 5000 replications, Table 2 presents the simulation results. Across all parameter settings, the traditional EM algorithm exhibits larger RMSE and RMSPE values compared to the proposed improved EM algorithm. Hence, we recommend utilizing the proposed method for obtaining point estimators of the proposed model parameters.

TABLE 2 | The point estimation simulation results.

	RMSE		RMSPE	
Case	Traditional EM	The proposed method	Traditional EM	The proposed method
I	659.1584	51.6687	4.5549	0.4098
II	924.0342	51.8752	5.3453	0.4199
III	976.1503	51.8119	6.3304	0.4653
IV	464.5907	42.2726	2.8913	0.3277
V	211.1476	41.7254	1.4478	0.3309
VI	248.5212	41.3628	1.6407	0.3655
VII	27.7100	9.2538	0.7624	0.3159
VIII	30.5530	9.1019	0.8435	0.3260
IX	32.5900	9.3801	0.9305	0.4117
X	24.9581	8.2654	0.6892	0.2642
XI	22.3792	7.9029	0.6326	0.2685
XII	26.0479	8.1924	0.7677	0.3279
XIII	4.3970	1.9071	0.6563	0.3008
XIV	4.6628	1.9525	0.7414	0.3323
XV	4.5838	1.9307	0.7996	0.4408
XVI	3.9827	1.8723	0.6307	0.2803
XVII	4.1575	1.9467	0.6143	0.2951
XVIII	4.0467	1.9153	0.6949	0.3958

Abbreviations: EM, Expectation-Maximization; RMSE, root mean square error; RMSPE, root mean square percentage error.

Additionally, to investigate the impact of different sample sizes n and measurement times m on the results of parameter point estimation, we conducted a sensitivity analysis. The simulation results, corresponding to three combinations, namely $(n, m) \in \{(8, 10), (6, 10), (6, 8)\}$, and the initial six parameter settings from Table 1, are presented in Table 3.

According to Table 3, it is observed that, across all combinations and parameter settings, the RMSE and RMSPE values associated with the traditional EM algorithm are consistently larger than those of the proposed improved EM algorithm. Moreover, with an increase in both sample size n and measurement times m, the RMSE and RMSPE results for the traditional EM algorithm decrease. Notably, the impact of changes in sample size is more pronounced. In contrast, the results for the proposed improved EM algorithm exhibit smaller variation as sample size and measurement times increase. Consequently, we can conclude that the proposed improved EM algorithm in this study demonstrates greater robustness.

5.2 | Interval Estimation Simulation Study

In this subsection, we undertake a Monte Carlo simulation study to showcase the effectiveness of the proposed interval estimation method. The specific parameter configurations are outlined in Table 4, with assumed values for t_0, threshold D, and

TABLE 3 | The point estimation simulation results.

(n, m)	Case	RMSE		RMSPE	
		Traditional EM	The proposed method	Traditional EM	The proposed method
(6,8)	I	7581.7860	66.4239	43.3874	0.4962
	II	9689.0550	65.4983	54.4557	0.4946
	III	12365.7900	64.0458	69.2448	0.5443
	IV	4241.3500	52.8997	23.6704	0.3887
	V	5125.2580	53.1757	29.2484	0.3982
	VI	4310.2460	53.0647	26.4332	0.4385
(6,10)	I	6693.3990	53.7234	36.0521	0.3870
	II	5230.0580	54.0937	29.3180	0.3968
	III	2812.1540	53.1531	16.2314	0.4344
	IV	1460.7460	45.3898	8.3793	0.3232
	V	979.9735	46.2199	6.4027	0.3408
	VI	1580.3480	46.0635	9.9962	0.3722
(8,10)	I	659.1584	51.6687	4.5549	0.4098
	II	924.0342	51.8752	5.3453	0.4199
	III	976.1503	51.8119	6.3304	0.4653
	IV	464.5907	42.2726	2.8913	0.3277
	V	211.1476	41.7254	1.4478	0.3309
	VI	248.5212	41.3628	1.6407	0.3655

Abbreviations: EM, Expectation-Maximization; RMSE, root mean square error; RMSPE, root mean square percentage error.

TABLE 4 | Parameter setting for interval estimation simulation study.

Case	η	b	γ	δ
i	4.0	−1.0	10.0	10.0
ii	4.0	−1.0	10.0	20.0
iii	4.0	−1.0	5.0	10.0
iv	5.0	−1.0	10.0	10.0
v	5.0	−1.5	10.0	10.0
vi	5.0	−0.5	10.0	10.0

degradation data X_{t_ω} set at $t_0 = 120$, $D = 40$, and $X_{t_\omega} = D/10$, respectively.

5.2.1 | The Interval Estimation Results for Model Parameters

For the proposed model parameters, utilizing 5000 replications and the parameter settings outlined in Table 4, along with three combinations of (n, m), we calculate the coverage percentages and average interval lengths (in parentheses) for parameters η, b, δ, and γ at nominal levels 0.9 and 0.95. The results are detailed in Tables 5–8. Generally, the closer the coverage percentage is to the nominal level, the more accurate the reliability estimation result.

To further demonstrate the efficiency of the proposed method, we derive the coverage percentages and average interval lengths for these four parameters at nominal levels 0.9 and 0.95 using the Bootstrap-p method. The Bootstrap-p method primarily involves generating parameter interval estimates through data resampling and subsequent point estimation of model parameters. In essence, the precision of interval estimations using the Bootstrap-p method is directly influenced by the accuracy of point estimations. By combining the point estimation method proposed in this study, the results of interval estimation based on the Bootstrap-p method are also presented in Tables 5–8.

As indicated in Tables 5–8, the coverage percentages for parameters η, b, δ, and γ closely align with the nominal levels, with errors within 1% across all parameter settings. This observation underscores the efficiency of the proposed method. Furthermore, the average interval lengths decrease as anticipated with the increase in both n and m.

Certainly, as evident from Tables 5–8, the coverage obtained from the Bootstrap-p method deviates significantly from the nominal levels, with errors well exceeding 1%. Moreover, the average interval lengths obtained by the Bootstrap-p method are larger than those obtained by the proposed method. Therefore,

TABLE 5 | The coverage percentages and average interval lengths (in parentheses) of parameter η for nominal levels 0.9 and 0.95.

		η (the proposed method)		η (Bootstrap-p method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.8926 (1.0110)	0.9488 (1.2047)	0.8450 (3.0463)	0.9070 (3.8230)
	(6, 10)	0.8992 (1.1700)	0.9472 (1.3942)	0.8378 (4.0982)	0.9056 (5.8485)
	(6, 8)	0.8960 (1.3157)	0.9488 (1.5676)	0.8554 (6.5088)	0.9068 (7.8386)
ii	(8, 10)	0.9036 (1.0114)	0.9526 (1.2049)	0.8440 (2.2896)	0.8910 (2.8390)
	(6, 10)	0.9002 (1.1706)	0.9488 (1.3950)	0.8158 (3.7965)	0.8824 (4.8656)
	(6, 8)	0.9002 (1.3189)	0.9498 (1.5718)	0.8204 (6.2435)	0.8630 (8.1239)
iii	(8, 10)	0.9004 (1.0097)	0.9492 (1.2031)	0.8490 (3.0568)	0.9098 (3.8277)
	(6, 10)	0.9018 (1.1700)	0.9528 (1.3939)	0.8240 (4.9688)	0.8904 (5.6867)
	(6, 8)	0.9022 (1.3142)	0.9488 (1.5661)	0.8456 (5.5451)	0.8960 (7.8580)
iv	(8, 10)	0.9002 (1.2735)	0.9520 (1.5171)	0.8452 (3.8160)	0.9044 (4.8735)
	(6, 10)	0.8978 (1.4807)	0.9504 (1.7636)	0.8554 (4.5088)	0.9086 (5.8380)
	(6, 8)	0.8892 (1.6570)	0.9450 (1.9753)	0.8378 (7.2686)	0.8980 (9.4412)
v	(8, 10)	0.9004 (1.2754)	0.9466 (1.5195)	0.8174 (2.7804)	0.8666 (3.4954)
	(6, 10)	0.9048 (1.4755)	0.9500 (1.7580)	0.8204 (3.2435)	0.8630 (4.1239)
	(6, 8)	0.8926 (1.6567)	0.9452 (1.9746)	0.8316 (7.8832)	0.8892 (10.3264)
vi	(8, 10)	0.8976 (1.2759)	0.9460 (1.5201)	0.8464 (3.8011)	0.9038 (4.8301)
	(6, 10)	0.8948 (1.4820)	0.9452 (1.7654)	0.8456 (4.5451)	0.8960 (5.8580)
	(6, 8)	0.9012 (1.6582)	0.9520 (1.9758)	0.8288 (7.1546)	0.8848 (9.2879)

TABLE 6 | The coverage percentages and average interval lengths of parameter b for nominal levels 0.9 and 0.95.

		b (the proposed method)		b (Bootstrap method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.8970 (0.1293)	0.9466 (0.1541)	0.9002 (0.3745)	0.9514 (0.4461)
	(6, 10)	0.9012 (0.1496)	0.9508 (0.1783)	0.8968 (0.4187)	0.9430 (0.4988)
	(6, 8)	0.8928 (0.1681)	0.9426 (0.2003)	0.8916 (0.4418)	0.9456 (0.5264)
ii	(8, 10)	0.9084 (0.1293)	0.9544 (0.1540)	0.8916 (0.2886)	0.9440 (0.3436)
	(6, 10)	0.9012 (0.1496)	0.9494 (0.1783)	0.8915 (0.3486)	0.9520 (0.4088)
	(6, 8)	0.9080 (0.1682)	0.9554 (0.2004)	0.8942 (0.4136)	0.9484 (0.4993)
iii	(8, 10)	0.9016 (0.1293)	0.9484 (0.1541)	0.9014 (0.3747)	0.9546 (0.4464)
	(6, 10)	0.9054 (0.1496)	0.9512 (0.1783)	0.8802 (0.4181)	0.9400 (0.4980)
	(6, 8)	0.8980 (0.1681)	0.9520 (0.2004)	0.8840 (0.4418)	0.9406 (0.5263)
iv	(8, 10)	0.9066 (0.1282)	0.9488 (0.1528)	0.8872 (0.4299)	0.9470 (0.5122)
	(6, 10)	0.8958 (0.1484)	0.9484 (0.1769)	0.8916 (0.4418)	0.9456 (0.5264)
	(6, 8)	0.8912 (0.1667)	0.9448 (0.1987)	0.8868 (0.4581)	0.9474 (0.5302)
V	(8, 10)	0.9002 (0.1283)	0.9480 (0.1528)	0.8892 (0.3299)	0.9448 (0.3928)
	(6, 10)	0.9040 (0.1484)	0.9508 (0.1768)	0.8942 (0.3436)	0.9484 (0.4092)
	(6, 8)	0.8932 (0.1667)	0.9464 (0.1987)	0.8940 (0.4290)	0.9464 (0.5113)
vi	(8, 10)	0.9016 (0.1282)	0.9516 (0.1528)	0.8868 (0.4301)	0.9438 (0.5124)
	(6, 10)	0.8976 (0.1484)	0.9438 (0.1768)	0.8840 (0.4418)	0.9406 (0.5263)
	(6, 8)	0.9050 (0.1667)	0.9530 (0.1987)	0.8854 (0.4570)	0.9414 (0.5388)

TABLE 7 | The coverage percentages and average interval lengths of parameter γ for nominal levels 0.9 and 0.95.

		γ (the proposed method)		γ (Bootstrap method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.9014 (8.9017)	0.9476 (10.6272)	0.8576 (9.3645)	0.9094 (11.5330)
	(6, 10)	0.8982 (10.5612)	0.9496 (12.6175)	0.8506 (11.3020)	0.9016 (14.2051)
	(6, 8)	0.9030 (10.5698)	0.9502 (12.6275)	0.8594 (11.3706)	0.9094 (14.3203)
ii	(8, 10)	0.8958 (8.8646)	0.9464 (10.5812)	0.8664 (9.1947)	0.9154 (11.3264)
	(6, 10)	0.8994 (10.6523)	0.9492 (12.7251)	0.8506 (11.2804)	0.9036 (14.0838)
	(6, 8)	0.9032 (10.4587)	0.9514 (12.4907)	0.8510 (11.4519)	0.9032 (14.3186)
iii	(8, 10)	0.8898 (4.4499)	0.9422 (5.3111)	0.8590 (4.6714)	0.9076 (5.7513)
	(6, 10)	0.9018 (5.3239)	0.9478 (6.3598)	0.8538 (5.6847)	0.9044 (7.0969)
	(6, 8)	0.8982 (5.3266)	0.9494 (6.3642)	0.8544 (11.4680)	0.9026 (14.3351)
iv	(8,10)	0.8946 (8.9300)	0.9474 (10.6580)	0.8610 (11.3083)	0.9120 (14.1214)
	(6, 10)	0.8970 (10.6266)	0.9464 (12.6896)	0.8594 (11.3706)	0.9094 (14.2103)
	(6, 8)	0.9054 (10.5684)	0.9540 (12.6237)	0.8520 (11.3840)	0.9014 (14.2186)
v	(8, 10)	0.8918 (8.9352)	0.9442 (10.6656)	0.8532 (11.2509)	0.9028 (14.0644)
	(6, 10)	0.8990 (10.5997)	0.9482 (12.6622)	0.8510 (11.2819)	0.9032 (14.0886)
	(6, 8)	0.9018 (10.5771)	0.9498 (12.6361)	0.8512 (11.3284)	0.9028 (14.1520)
vi	(8, 10)	0.8970 (8.9357)	0.9482 (10.6676)	0.8488 (5.6744)	0.9022 (7.0956)
	(6, 10)	0.8994 (10.6143)	0.9508 (12.6782)	0.8544 (5.6847)	0.9026 (7.0969)
	(6, 8)	0.8956 (10.5418)	0.9500 (12.5914)	0.8454 (11.4400)	0.8952 (14.2962)

TABLE 8 | The coverage percentages and average interval lengths of parameter δ for nominal levels 0.9 and 0.95.

		δ (the proposed method)		δ (Bootstrap method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.9062 (8.6813)	0.9472 (10.3612)	0.8646 (9.1008)	0.9124 (11.2040)
	(6, 10)	0.9032 (10.2740)	0.9514 (12.2754)	0.8548 (10.9000)	0.9040 (13.5246)
	(6, 8)	0.8972 (10.3116)	0.9480 (12.3188)	0.8618 (10.9055)	0.9166 (13.6174)
ii	(8, 10)	0.8978 (17.5158)	0.9648 (20.9073)	0.8678 (20.0691)	0.9198 (22.2620)
	(6, 10)	0.8964 (21.0271)	0.9488 (25.1199)	0.8570 (21.0441)	0.9078 (25.7992)
	(6, 8)	0.9036 (20.6193)	0.9514 (24.6215)	0.8578 (22.0187)	0.9094 (27.4853)
iii	(8, 10)	0.8880 (8.6715)	0.9466 (10.3510)	0.8682 (9.0806)	0.9174 (11.1802)
	(6, 10)	0.9000 (10.3803)	0.9450 (12.3393)	0.8630 (11.0674)	0.9118 (13.8320)
	(6, 8)	0.8988 (10.4689)	0.9510 (12.3895)	0.8600 (11.9729)	0.9124 (14.7002)
iv	(8, 10)	0.8980 (8.6813)	0.9436 (10.3629)	0.8650 (9.9968)	0.9134 (13.6113)
	(6, 10)	0.8990 (10.3693)	0.9496 (12.3854)	0.8618 (10.9055)	0.9166 (13.7374)
	(6, 8)	0.9048 (10.2989)	0.9538 (12.3035)	0.8572 (11.0133)	0.9092 (13.7502)
v	(8, 10)	0.8934 (8.7036)	0.9480 (10.3891)	0.8556 (12.1123)	0.9082 (17.6322)
	(6, 10)	0.8986 (10.3248)	0.9474 (12.3334)	0.8578 (12.2187)	0.9094 (17.6853)
	(6, 8)	0.9010 (10.3317)	0.9516 (12.3674)	0.8590 (13.8992)	0.9110 (18.6082)
vi	(8, 10)	0.9012 (8.6982)	0.9488 (10.3848)	0.8572 (10.7971)	0.9050 (13.5454)
	(6, 10)	0.9018 (10.3284)	0.9480 (12.3362)	0.8600 (10.9720)	0.9124 (13.7002)
	(6, 8)	0.8982 (10.2716)	0.9482 (12.2717)	0.8524 (11.0656)	0.9024 (13.8183)

TABLE 9 | The coverage percentages and average interval lengths of index $E(Xt_0)$ for nominal levels 0.9 and 0.95.

		$E(Xt_0)$ (the proposed method)		$E(Xt_0)$ (Bootstrap method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.8928 (379.9209)	0.9476 (459.0710)	0.8650 (1321.0300)	0.9260 (1644.2240)
	(6, 10)	0.9002 (450.2017)	0.9500 (547.6145)	0.8582 (1907.0290)	0.9222 (2440.6070)
	(6, 8)	0.8978 (465.2028)	0.9494 (565.2519)	0.8574 (2507.3890)	0.9198 (3201.4150)
ii	(8, 10)	0.9014 (137.8676)	0.9484 (165.2107)	0.8508 (463.2625)	0.9086 (570.0734)
	(6, 10)	0.9036 (160.5060)	0.9524 (192.7990)	0.8368 (556.4410)	0.8950 (885.2740)
	(6, 8)	0.8972 (171.3945)	0.9452 (205.7613)	0.8250 (642.2855)	0.8796 (887.6425)
iii	(8, 10)	0.9042 (190.4188)	0.9498 (230.1312)	0.8634 (663.5742)	0.9220 (824.2199)
	(6, 10)	0.8920 (224.1197)	0.9418 (272.6827)	0.8318 (961.6478)	0.8986 (888.5330)
	(6, 8)	0.9082 (232.1119)	0.9512 (281.9996)	0.8452 (1481.785)	0.8982 (1224.7310)
iv	(8, 10)	0.9060 (476.4188)	0.9520 (575.9286)	0.8654 (1642.9310)	0.9198 (2074.7410)
	(6, 10)	0.8904 (560.9168)	0.9486 (682.5941)	0.8574 (1907.0290)	0.9198 (2440.6070)
	(6, 8)	0.8928 (580.9434)	0.9484 (705.6332)	0.8422 (2943.2880)	0.9142 (3796.3420)
v	(8, 10)	0.9016 (783.2282)	0.9482 (946.6290)	0.8268 (862.9460)	0.8846 (940.0691)
	(6, 10)	0.9072 (925.8110)	0.9570 (1126.9380)	0.8250 (942.2855)	0.8796 (1207.6425)
	(6, 8)	0.8996 (958.9635)	0.9510 (1165.4000)	0.8286 (1223.7100)	0.8930 (1790.0540)
vi	(8, 10)	0.9000 (288.6317)	0.9504 (348.7957)	0.8510 (824.6184)	0.9056 (1036.0270)
	(6, 10)	0.8994 (341.1171)	0.9448 (415.0613)	0.8452 (961.6478)	0.8982 (1224.7310)
	(6, 8)	0.8964 (353.3636)	0.9502 (429.3448)	0.8214 (1752.5740)	0.8832 (2254.7400)

the method proposed in this study is superior to the traditional Bootstrap-p method.

5.2.2 | The Interval Estimation Results for Predictive Reliability Indexes

As discussed in the Introduction, the percentiles of certain reliability indexes at the normal operating stress level hold significant importance in practical applications. Utilizing the proposed Algorithm 2 and conducting 5000 replications, we estimate the ρ percentiles of $E(Xt_0)$, $R(t_0)$, t_p, $E(T_0)$, and $E(T_\omega)$, respectively. The coverage percentages and average interval lengths (in parentheses) of these reliability indexes at nominal levels 0.9 and 0.95 are detailed in Tables 9–13, and corresponding simulation results obtained from the Bootstrap-p method are also provided in Tables 9–13.

It can be observed that the coverage percentages of the reliability indexes $E(Xt_0)$, $R(t_0)$, $t_{0.1}$, $E(T_0)$, and $E(T_\omega)$ from the proposed method are all close to the nominal levels, with errors within 1% under all the parameter settings. In contrast, the coverage percentages from the Bootstrap-p method deviate significantly from the nominal levels.

In summary, whether in terms of point estimation for model parameters or predictive reliability indexes, the coverage results of the proposed method outperform those of the Bootstrap-p method. Therefore, the proposed method can better guide practical applications.

6 | Case Study

The Gas-Insulated Transmission line (GIL) [38] is a new alternative underground technology developed to meet the necessity of enforcing the electrical transmission network. As an important component of the GIL system, the sealing O-rings play a crucial role in ensuring that the system operates within a well-sealed environment. Its primary function is to prevent gas leakage and mitigate the adverse effects of external environmental factors, thereby maintaining the stability, reliability, and longevity of the system. Therefore, this section takes the sealing O-rings as an example to illustrate the implementation of the proposed method.

To evaluate the reliability of the sealing O-rings, a CSADT was conducted under four different temperatures (i.e., 50°C, 60°C, 70°C, and 80°C). For each temperature, experiments were carried out with sample size $n = 8$, measurements interval $h = 24$ (hour), and the measurement times $m = 10$. The degradation data (i.e., strain level) from the sealing O-rings $\{Y_{i,k}(t), i = 1, 2, \ldots, 8; k = 1, 2, 3, 4\}$, subjected to normalization, are shown in Figure 2.

Let $X_{i,k}(t) = 1 - Y_{i,k}(t), X_{i,k}(t)$ is a random variable that increases with time t. According to model (1), it is necessary to test whether $X_{i,k}(t)$ and the rate parameter μ under different temperatures subject to the Gamma process, respectively. Therefore, we initially employ the Kolmogorov–Smirnov (K–S) test to assess the degradation data $X_{i,k}(t)$. The resulting p values are presented in Table 14 and are found to surpass the specified significance level $\rho = 0.05$. Furthermore, we calculate and document the

TABLE 10 | The coverage percentages and average interval lengths of index $R(t_0)$ for nominal levels 0.9 and 0.95.

		$R(t_0)$ (the proposed method)		$R(t_0)$ (Bootstrap method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.9022 (2.7707×10^{-14})	0.9520 (2.1796×10^{-13})	0.8330 (8.1703×10^{-10})	0.8936 (3.6182×10^{-9})
	(6, 10)	0.8972 (1.2557×10^{-12})	0.9512 (7.6919×10^{-12})	0.8106 (2.9191×10^{-9})	0.8768 (1.2802×10^{-8})
	(6, 8)	0.9014 (3.1674×10^{-12})	0.9492 (2.4230×10^{-11})	0.8318 (4.7801×10^{-9})	0.8932 (1.7841×10^{-8})
ii	(8, 10)	0.8928 (2.1364×10^{-11})	0.9484 (1.1264×10^{-10})	0.8396 (3.3302×10^{-9})	0.8932 (1.1697×10^{-8})
	(6, 10)	0.9000 (1.0353×10^{-9})	0.9486 (4.3626×10^{-9})	0.8016 (5.0302×10^{-9})	0.8708 (1.2804×10^{-8})
	(6, 8)	0.9056 (1.1833×10^{-9})	0.9476 (5.9154×10^{-9})	0.8066 (4.0492×10^{-8})	0.8640 (1.4542×10^{-7})
iii	(8, 10)	0.8914 (2.1520×10^{-7})	0.9444 (5.9819×10^{-7})	0.8362 (1.0502×10^{-5})	0.8994 (2.2310×10^{-5})
	(6, 10)	0.9062 (8.6923×10^{-7})	0.9522 (2.5479×10^{-6})	0.8194 (3.5404×10^{-5})	0.8834 (1.0589×10^{-4})
	(6, 8)	0.8970 (1.51772×10^{-6})	0.9500 (4.1517×10^{-6})	0.8288 (5.7320×10^{-5})	0.8914 (1.0732×10^{-4})
iv	(8, 10)	0.8968 (2.1305×10^{-17})	0.9496 (3.3389×10^{-16})	0.8272 (4.3000×10^{-9})	0.8890 (1.4500×10^{-8})
	(6, 10)	0.8982 (4.4732×10^{-15})	0.9490 (4.1657×10^{-14})	0.8318 (4.7899×10^{-9})	0.8932 (1.7788×10^{-8})
	(6, 8)	0.9004 (5.0513×10^{-15})	0.9544 (6.5382×10^{-14})	0.8178 (2.3853×10^{-8})	0.8796 (7.1542×10^{-8})
v	(8, 10)	0.8910 (2.0044×10^{-17})	0.9422 (4.0880×10^{-16})	0.8108 (4.0301×10^{-8})	0.8744 (1.3911×10^{-7})
	(6, 10)	0.8988 (5.6636×10^{-15})	0.9478 (9.0525×10^{-14})	0.8066 (4.0499×10^{-8})	0.8640 (1.4478×10^{-7})
	(6, 8)	0.8984 (8.1937×10^{-14})	0.9494 (7.6540×10^{-13})	0.8088 (4.9719×10^{-8})	0.8752 (3.0418×10^{-7})
vi	(8, 10)	0.8968 (2.0572×10^{-18})	0.9482 (4.4553×10^{-17})	0.8268 (3.4013×10^{-5})	0.8908 (6.8102×10^{-5})
	(6, 10)	0.8960 (6.5642×10^{-15})	0.9508 (7.6169×10^{-14})	0.8288 (3.7279×10^{-5})	0.8914 (1.4489×10^{-4})
	(6, 8)	0.8996 (1.3876×10^{-14})	0.9516 (1.6793×10^{-13})	0.8098 (5.6234×10^{-5})	0.8774 (6.7523×10^{-4})

TABLE 11 | The coverage percentages and average interval lengths of index t_p for nominal levels 0.9 and 0.95.

		t_p (the proposed method)		t_p (Bootstrap method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.8940 (4.9908)	0.9478 (5.9882)	0.8374 (9.0962)	0.9004 (10.8415)
	(6, 10)	0.9016 (5.8264)	0.9492 (7.0136)	0.8268 (9.3032)	0.8872 (11.0714)
	(6, 8)	0.8970 (6.2701)	0.9494 (7.5411)	0.8406 (11.0398)	0.8992 (13.1522)
ii	(8, 10)	0.8978 (8.0507)	0.9528 (9.6360)	0.8356 (12.8424)	0.8880 (13.3289)
	(6, 10)	0.9060 (9.3728)	0.9520 (11.2418)	0.8084 (13.3479)	0.8732 (15.1259)
	(6, 8)	0.9034 (10.2191)	0.9500 (12.2562)	0.8076 (15.4757)	0.8590 (18.5006)
iii	(8, 10)	0.8976 (9.9350)	0.9474 (11.9252)	0.8434 (18.0028)	0.9032 (20.4576)
	(6, 10)	0.9010 (11.5955)	0.9502 (13.9599)	0.8216 (19.3397)	0.8828 (21.1200)
	(6, 8)	0.9028 (12.4114)	0.9520 (14.9280)	0.8356 (21.8023)	0.8892 (25.9746)
iv	(8, 10)	0.9030 (4.0041)	0.9522 (4.8046)	0.8342 (10.2210)	0.8922 (12.1858)
	(6, 10)	0.9024 (4.6848)	0.9486 (5.6391)	0.8406 (11.0398)	0.8992 (13.1522)
	(6, 8)	0.8920 (5.0404)	0.9494 (6.0634)	0.8202 (19.9514)	0.8850 (21.8226)
v	(8, 10)	0.8946 (4.0046)	0.9462 (4.8056)	0.8088 (14.2669)	0.8650 (17.0567)
	(6, 10)	0.9078 (4.6988)	0.9502 (5.6539)	0.8076 (15.4757)	0.8590 (18.5006)
	(6, 8)	0.8960 (5.0417)	0.9456 (6.0653)	0.8160 (20.0214)	0.8758 (21.9001)
vi	(8, 10)	0.8918 (3.9996)	0.9478 (4.7993)	0.8302 (20.1823)	0.8924 (24.0649)
	(6, 10)	0.8996 (4.6669)	0.9456 (5.6165)	0.8356 (21.8023)	0.8892 (25.9746)
	(6, 8)	0.9028 (5.0422)	0.9526 (6.0663)	0.8154 (22.9424)	0.8792 (26.8243)

TABLE 12 | The coverage percentages and average interval lengths of index $E(T_0)$ for nominal levels 0.9 and 0.95.

		$E(T_0)$ (the proposed method)		$E(T_0)$ (Bootstrap method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.8932 (3.2066)	0.9426 (3.8354)	0.8492 (5.9688)	0.9062 (7.0938)
	(6, 10)	0.8992 (3.7247)	0.9490 (4.4615)	0.8332 (6.0605)	0.8950 (7.1925)
	(6, 8)	0.9020 (4.0397)	0.9490 (4.8399)	0.8504 (7.2259)	0.9002 (8.5870)
ii	(8, 10)	0.9030 (5.7758)	0.9514 (6.9009)	0.8378 (6.3376)	0.8850 (7.1287)
	(6, 10)	0.9031 (6.7100)	0.9564 (8.0251)	0.8128 (9.1028)	0.8778 (11.2444)
	(6, 8)	0.9042 (7.3515)	0.9520 (8.7970)	0.8076 (11.2709)	0.8572 (13.4541)
iii	(8, 10)	0.9040 (6.3945)	0.9504 (7.6474)	0.8460 (6.9433)	0.9084 (7.1971)
	(6, 10)	0.8974 (7.4392)	0.9492 (8.9097)	0.8242 (11.0923)	0.8858 (14.2323)
	(6, 8)	0.9092 (8.0322)	0.9544 (9.6199)	0.8398 (14.4384)	0.8898 (17.1541)
iv	(8, 10)	0.9066 (2.5724)	0.9538 (3.0764)	0.8442 (6.6675)	0.8962 (7.9287)
	(6, 10)	0.8990 (2.9980)	0.9496 (3.5916)	0.8504 (7.2259)	0.9002 (8.5870)
	(6, 8)	0.8906 (3.2489)	0.9460 (3.8928)	0.8280 (16.5146)	0.8896 (17.7165)
v	(8, 10)	0.8996 (2.5743)	0.9462 (3.0784)	0.8106 (10.3358)	0.8612 (12.3409)
	(6, 10)	0.9078 (3.0048)	0.9568 (3.5988)	0.8076 (11.2709)	0.8572 (13.4541)
	(6, 8)	0.8968 (3.2494)	0.9476 (3.8945)	0.8204 (16.5623)	0.8826 (17.7711)
vi	(8, 10)	0.8968 (2.5711)	0.9472 (3.0749)	0.8384 (13.3105)	0.8946 (15.8273)
	(6, 10)	0.8996 (2.9865)	0.9468 (3.5765)	0.8398 (14.4384)	0.8898 (17.1541)
	(6, 8)	0.9014 (3.2490)	0.9528 (3.8926)	0.8220 (16.5257)	0.8792 (17.7355)

TABLE 13 | The coverage percentages and average interval lengths of index $E(T_\omega)$ for nominal levels 0.9 and 0.95.

		$E(T_\omega)$ (the proposed method)		$E(T_\omega)$ (Bootstrap method)	
Case	(n, m)	**0.9**	**0.95**	**0.9**	**0.95**
i	(8, 10)	0.8934 (2.8885)	0.9426 (3.4549)	0.8492 (5.3786)	0.9062 (6.3925)
	(6, 10)	0.8990 (3.3552)	0.9488 (4.0188)	0.8332 (5.4615)	0.8950 (6.4815)
	(6, 8)	0.9020 (3.6392)	0.9490 (4.3600)	0.8504 (6.5116)	0.9002 (7.7381)
ii	(8, 10)	0.9030 (5.2010)	0.9514 (6.2141)	0.8378 (8.4093)	0.8850 (10.0223)
	(6, 10)	0.9031 (6.0422)	0.9564 (7.2265)	0.8130 (9.4996)	0.8778 (10.5283)
	(6, 8)	0.9042 (6.6201)	0.9522 (7.9217)	0.8076 (10.1505)	0.8572 (12.1166)
iii	(8, 10)	0.9040 (5.7576)	0.9504 (6.8857)	0.8460 (10.7558)	0.9084 (12.7854)
	(6, 10)	0.8974 (6.6982)	0.9492 (8.0223)	0.8242 (11.4901)	0.8858 (13.5174)
	(6, 8)	0.9094 (7.2325)	0.9544 (8.6621)	0.8398 (13.0029)	0.8898 (15.4486)
iv	(8, 10)	0.9066 (2.3172)	0.9538 (2.7712)	0.8442 (6.0083)	0.8962 (7.1448)
	(6, 10)	0.8986 (2.7006)	0.9496 (3.2353)	0.8504 (6.5116)	0.9002 (7.7381)
	(6, 8)	0.8904 (2.9268)	0.9458 (3.5068)	0.8282 (6.8707)	0.8896 (7.9538)
v	(8, 10)	0.8998 (2.3189)	0.9462 (2.7730)	0.8106 (9.3083)	0.8612 (11.1140)
	(6, 10)	0.9074 (2.7067)	0.9568 (3.2417)	0.8076 (10.1505)	0.8572 (12.1166)
	(6, 8)	0.8970 (2.9273)	0.9474 (3.5084)	0.8202 (15.9137)	0.8826 (17.0031)
vi	(8, 10)	0.8968 (2.3161)	0.9472 (2.7698)	0.8384 (11.9870)	0.8946 (14.2535)
	(6, 10)	0.8996 (2.6902)	0.9470 (3.2217)	0.8398 (13.0029)	0.8898 (15.4486)
	(6, 8)	0.9014 (2.9269)	0.9528 (3.5067)	0.8220 (15.8808)	0.8794 (16.9710)

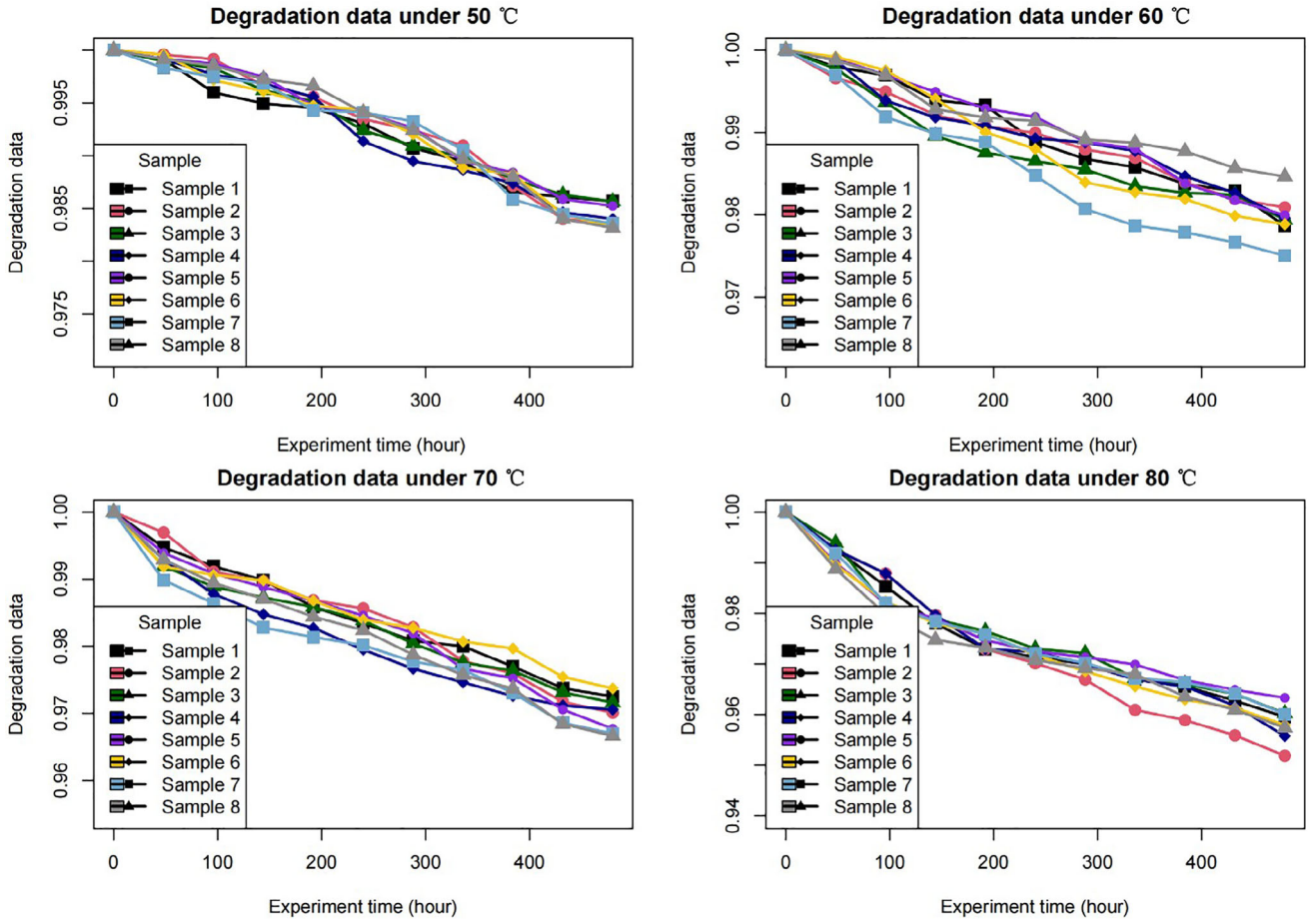

FIGURE 2 | The sealing O-rings degradation data under four different temperatures.

TABLE 14 | The p values and the estimator $\hat{\mu}$ of the 32 samples under four temperatures.

	50°C		60°C		70°C		80°C	
Sample	***p* value**	$\hat{\mu}$	***p* value**	$\hat{\mu}$	***p* value**	$\hat{\mu}$	***p* value**	$\hat{\mu}$
1	0.7791	675.5415	0.8967	1442.7750	0.7624	1316.2810	0.8681	1591.4020
2	0.9092	1105.1280	0.9124	1137.1790	0.9256	2419.4580	0.9269	1324.8120
3	0.994	491.8748	0.8373	3335.8310	0.7817	933.8665	0.7801	1126.0600
4	0.9046	504.7680	0.8554	1897.5560	0.9941	1378.1490	0.9491	948.8634
5	0.7281	583.7786	0.8738	1736.1400	0.6829	2230.5480	0.8357	1587.2310
6	0.7757	823.2154	0.9024	1216.5610	0.9002	1439.8150	0.8795	826.1626
7	0.7396	615.1634	0.8107	1066.2790	0.9079	1237.9580	0.8444	716.7788
8	0.9119	495.6190	0.8036	1672.1320	0.933	1547.7890	0.9441	1631.609

TABLE 15 | The p values of the estimator $\hat{\mu}$ under four temperatures.

Temperatures	**50°C**	**60°C**	**70°C**	**80°C**
p value	0.8737	0.8273	0.7888	0.7318

estimators for $\mu_{i,k}$ across various temperatures in Table 14. Subsequently, Table 15 provides the derived p values for the estimator $\hat{\mu}_{i,k}$, all of which surpass 0.05. Therefore, it is suitable to fit the degradation data of O-rings with the proposed model in Equation (1).

TABLE 16 | The point estimation results for model parameters of O-rings data.

Parameter	$\hat{\eta}(\times 10^{-2})$	$\hat{b}(\times 10^{3})$	$\hat{\gamma}(\times 10^{-12})$	$\hat{\delta}(\times 10^{-1})$
Point estimation	10.4005	−6.1647	8.793069	11.8615

Based on the conclusions drawn in Subsection 5.1, it is advisable to utilize the proposed improved EM algorithm for estimating model parameters. The point estimation results of the O-rings are subsequently derived and presented in Table 16.

TABLE 17 | The interval estimation results for model parameters and predictive reliability indexes.

Parameter	The proposed method		Bootstrap-p method	
	90%	95%	90%	95%
$\eta(\times 10^{-2})$	[9.8601, 12.5887]	[9.6405, 12.8449]	[98.0221, 346.0354]	[96.78659, 356.7403]
$b(\times 10^{3})$	[−6.1929, −6.1000]	[−6.2030, −6.0916]	[−5.5745, −5.1538]	[−5.5887, −5.1489]
$\gamma(\times 10^{-12})$	[5.6553, 14.3152]	[5.1182, 15.3049]	[2.2511, 3.6968]	[2.2158, 3.7099]
$\delta(\times 10^{-1})$	[8.3178, 17.5215]	[7.6488, 18.6040]	[8.5122, 10.5762]	[8.4335, 10.5799]
$R(t_0)(\times 10^{0})$	[0.3798, 0.6069]	[0.3594, 0.6267]	[0.6420, 0.7163]	[0.6379, 0.7307]
$E(T_0)(\times 10^{2})$	[3.0438, 5.2860]	[2.9037, 5.5823]	[6.7510, 9.5561]	[6.6365, 9.6892]
$E(T_\omega)(\times 10^{2})$	[2.5443, 4.4120]	[2.4270, 4.6588]	[5.6266, 7.9637]	[5.5312, 8.0746]
$t_p(\times 10^{2})$	[6.6445, 11.9121]	[6.3686, 12.6585]	[15.4667, 22.4986]	[15.1360, 22.9697]

Moreover, utilizing Algorithm 1 and 2 with B= 5000 replications, we present detailed interval estimations for parameters (η, b, γ, δ) and predictive reliability indexes ($R(t_0)$, $E(T_0)$, $E(T_\omega)$ and t_p) under the normal stress level (i.e., $s_0 = 25°C$) in Table 17, respectively. The specified conditions include with $t_0 = 300$, $D = 0.7$, $X\ (t_\omega) = 0.95$. Additionally, results from the Bootstrap-p method are included in Table 17 for comparison. Significant differences are observed when comparing the outcomes of these two methods (i.e., the proposed method and Bootstrap-p method). Combined with the conclusion derived in Subsection 5.2, we recommend the proposed method.

At the same time, the runtime of both methods under B =5000 is also recorded. The proposed method demonstrates a runtime of 89.34 s, whereas the Bootstrap-p method requires 7180.17 s, markedly surpassing the runtime of the proposed method. The marked disparity in runtimes clearly indicates that the proposed method can enhance the efficiency of reliability analysis. From a practical standpoint, it is recommended to employ the proposed method.

7 | Conclusion

In this study, a more realistic model is developed to depict the accelerated degradation data based on the Gamma process and the nonlinear accelerated model; an improved point estimation method and a novel three-step interval estimation method of model parameters are presented for reliability analysis. First, considering the unit heterogeneity in practice and combining the nonlinear accelerated model, a Gamma CSADT model with random effects is used to depict the accelerated degradation data for reliability analysis. Given the complexity of the model and its involvement of latent variables, as well as the susceptibility of the EM algorithm to initial values, an improved EM algorithm is proposed. In this algorithm, we perform two consecutive different M-steps after each E-step, reducing the number of required initial values from four to two, which not only mitigates the impact of initial values but also enhances parameter estimation accuracy. Then, considering the complexity of the model and the constraint of small sample size and few stress levels, a three-step interval estimation method of model parameters is established by dividing them into three distinct parts and addressing them separately using the GPQ method, which simplifies the parameter interval estimation process and enhances the estimation accuracy. To demonstrate the performance of the proposed method, two simulation studies are conducted. In the point estimation simulation study, under 18 different parameter settings, the RMSE and RMSPE of the traditional EM algorithm are larger than those of the proposed improved EM algorithm, which indicates the effectiveness of the proposed point estimation method. In the interval estimation simulation study, under different parameter settings, the coverage percentages of all parameters and predictive reliability indexes from the proposed method are close to the nominal levels, with errors within 1%. In contrast, the results from the existing method are mostly larger than 1%, further emphasizing the advantage of the proposed interval estimation method. Finally, a real example of sealing O-rings is presented to illustrate the feasibility of the proposed method. According to the estimation results, the interval estimation results of the two methods (i.e., the proposed method and the Bootstrap-p method) are different, and the runtime of the Bootstrap method is far longer than that of the proposed method. Therefore, combined with the conclusion derived from the Simulation study, we recommend the proposed method.

This study provides a parameter estimation method for a Gamma CSADT model with random effects. The Gamma CSADT model has multiple forms, and it is necessary to study other model forms. In further research, exploring different forms of the Gamma CSADT model can be considered to gain a more comprehensive understanding of its applicability and performance. In addition, measurement errors in the observed degradation process may affect the accuracy of reliability analysis, which should also be considered. Finally, reliability analysis of multivariate degradation processes is also an area that deserves attention.

Nomenclature

$s_k, k = 1, 2, \ldots, K$	accelerated stress levels
$\mu > 0, \gamma > 0$	rate parameter for Gamma distribution
$m_{i,k}$	measurement times
s_0	normal operating stress level
n_k	sample size under k th stress level

CDF	cumulative distribution function
$X_{i,j,k}$	degradation data $i = 1, 2, \ldots, n_k, j = 1, 2, \ldots, m_{i,k}$
$\Delta X_{i,j,k}$	degradation increments
$\Delta t_{i,j,k},\ h_{i,k}$	time increments
$X_t = Xt$	degradation data
$E(Xt_0)$	the mean of degradation
$R(t_0)$	the reliability at time t_0
t_p	the quantile lifetime
$E(T_0)$	the mean of lifetime
$E(T_\omega)$	the mean RUL at time t_ω
ζ_k	standardization of stress levels
$\eta > 0, \delta > 0$	shape parameter for Gamma distribution
RUL	remaining useful life
β	random effect parameter
b	accelerated parameter
PDF	probability density function
$G_\eta, W_1, G_\gamma,\ G_\delta$	generalized pivotal quantity for parameters $\{\eta, b, \gamma, \delta\}$
N	the total sample size
$f(X_t)$	PDF of X_t
$F(X_t)$	CDF of X_t
D	the degradation threshold
F_T	CDF of lifetime
F_{T_ω}	CDF of RUL at time t_ω
Θ	parameters set
ϵ	predetermined threshold for EM

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Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A: The detailed derivation of Equation (13)

According to model (1), under the given $\beta_{i,k}, i = 1, 2, \ldots, n_k, k = 1, 2, \ldots, K$, the pdf of degradation increment data $\Delta X = \{\Delta X_{i,j,k}, j = 1, 2, \ldots, m_{i,k}; i = 1, 2, \ldots, n_k; k = 1, 2, \ldots, K\}$

$$f\ (\Delta X) = \prod_{k=1}^{K}\prod_{i=1}^{n_k}\prod_{j=1}^{m_{i,k}} \times \frac{\Delta X_{i,j,k}{}^{\eta h_{i,k}-1}[\beta \exp(b\zeta)]^{\eta h_{i,k}} e^{-\Delta X_{i,j,k}\beta \exp(b\zeta)}}{\Gamma(\eta h_{i,k})}, \tag{A1}$$

Therefore, the log-likelihood function under given $\beta_{i,k}$ can be written as

$$\ln(L) = \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}\left[(\eta h_{i,k}-1)\ln(\Delta X_{i,j,k}) + \eta h_{i,k}\ln(\beta_{i,k}\exp(b\zeta_k))\right] - \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}\left[\beta_{i,k}\exp(b\zeta_k)\Delta X_{i,j,k} + \ln(\Gamma(\eta h_{i,k}))\right].$$

APPENDIX B: Proof of Equation (22)

Let $U_{i,j} = \Delta X_{i,j,k}\beta_{i,k}/X_k$, according to [36], it is easy to verify that $(U_{1,1}, U_{1,2}, \ldots, U_{n_k,m_{i,k}})$ follows the Dirichlet distribution, and the PDF can be written as

$$f\left(u_{1,1}, u_{1,2}, \ldots, u_{n_k,m_{i,k}}\right) = \frac{\Gamma(\eta Q_k)}{\prod_{i=1}^{n_k}\prod_{j=1}^{m_{i,k}}\Gamma(\eta h_{i,k})} u_{1,1}^{\eta h_{1,k}-1} u_{1,2}^{\eta h_{1,k}-1}\cdots u_{n_k,m_{i,k}}^{\eta h_{n_k,k}-1}, \tag{B1}$$

where $u_{i,j} > 0$, $\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}} u_{i,j} = 1$. Notice that $Z = \ln(\prod_{k=1}^{K}\prod_{i=1}^{n_k}\prod_{j=1}^{m_{i,k}} U_{i,j}^{h_{i,k}})$, the moment generating function of Z is thus derived

$$M_Z(t) = E\left(e^{tz}\right) = \prod_{k=1}^{K} E\left(\prod_{i=1}^{n_k}\prod_{j=1}^{m_{i,k}} U_{i,j}^{th_{i,k}}\right)$$

$$= \prod_{k=1}^{K}\frac{\Gamma(\eta Q_k)}{\prod_{i=1}^{n_k}\prod_{j=1}^{m_{i,k}}\Gamma(\eta h_{i,k})}\int\cdots\int\prod_{i=1}^{n_k}\prod_{j=1}^{m_{i,k}} u_{i,j}^{(t+\eta)h_{i,k}-1}\, du_{1,1}\cdots du_{n_k,m_{i,k}}$$

$$= \prod_{k=1}^{K}\frac{\Gamma(\eta Q_k)\prod_{i=1}^{n_k}\prod_{j=1}^{m_{i,k}}\Gamma\left[(t+\eta)h_{i,k}\right]}{\Gamma\left[(t+\eta)Q_k\right]\prod_{i=1}^{n_k}\prod_{j=1}^{m_{i,k}}\Gamma(\eta h_{i,k})}. \tag{B2}$$

Therefore, the r th cumulant of Z can be derived by

$$\kappa_r(\eta) = \frac{d^r \ln(M_Z(t))}{dt^r}\ t = 0 = \sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_{i,k}}\psi^{r-1}(\eta h_{i,k}) \times (h_{i,k})^r - \sum_{k=1}^{K}\psi^{r-1}(\eta Q_k)\, Q_k^r. \tag{B3}$$