On the primitive equations and the hydrostatic Stokes operator in L^2

by

G. Muguruza Lasa

to obtain the degree of Master of Science in Applied Mathematics
at Delft University of Technology,
to be defended publicly on Thursday February 18, 2021 at 13:00 PM.

Student number:	4923782	
Future correspondence:	garazimugu@gmail.com	
Responsible full professor:	Prof. Dr. M. C. Veraar,	TU Delft, Analysis
Rest of thesis committee:	Prof. Dr. H. M. Schuttelaars,	TU Delft, Mathematical Physics
	Dr. J. Sauer,	TU Delft, Analysis

TU Delft Delft University of Technology

Abstract

In 2016 Hieber and Kashiwabara showed that the three dimensional primitive equations admit a unique, global, strong solution for all initial data in a closed subspace of the Bessel space $H^{2/p,p}(\Omega)$ provided $p \geqslant 6/5$, being this the first result in the general L^p-setting. Their approach consisted in studying the properties of the hydrostatic Stokes operator A_p defined on the solenoidal subspace $L^p_{\overline{\sigma}}(\Omega)$ of $L^p(\Omega)$. In 2017 Giga et. al. further proved that the hydrostatic Stokes operator A_p admits a bounded H^∞-calculus, obtaining maximal $L^q - L^p$ regularity estimates for the linearized primitive equations in a much simpler way.

In this work we will study Giga et. al.'s and Hieber and Kashiwabara's works particularized for the L^2-case as well as all the necessary literature to replicate the proofs. The goal of the thesis is to present an extended version of Giga et. al.'s proof to make it more accessible. Although the L^p-case is not studied for lack of time, we differentiate between the Sobolev-Slobodeckij, Bessel potential and Besov spaces to accentuate how we could extend the proofs to the L^p-setting.

Preface

With this thesis concludes my journey as a master student at the Analysis department at Delft University of Technology.

I decided to come here by chance, as most decisions are made in life, because a friend recomended me the university. Two weeks in Jan van Neerven's Functional Analysis class were enough to know it was my place; thank you for sharing your enthusiasm for math, your difficult exercises and being a constant support by our long coffees. I would also like to thank my supervisor Mark Veraar for his guidance through the thesis, for all the help in a rather strange year and for sharing with me the wonders of semigroups, interpolation, multipliers, and the rest of new worlds I discovered and tried to fit into this small work.

I would not have finished this thesis without the love and care of Uri (my partner); Gustavo, Monique and Cecilia (my second family); Ankita, Irem, Jessie, Rijk, Sam and Simone (mijn geliefde huisgenoten); Reuben, Aniket and Jinwan (wonderful friends).

Izan zirelako gara, eta garelako izango dira. Gurasoei, bihotzez.

Garazi
Delft, January 2021

Contents

Symbols and notation

$\mathbb{N} = \{0, 1, 2...\}$ - non-negative integers
$\mathbb{Z}$ - integers
$\mathbb{R}$ - real numbers
$\mathbb{R}_+$ - positive real numbers $\mathbb{R}_+ = (0, \infty)$
$\mathbb{C}$ - complex numbers
$\mathbb{K}$ - scalar field ($\mathbb{R}$ or $\mathbb{C}$)
$x \times y$ - cross product of $x, y \in \mathbb{R}^3$
$x << y$ - x is *much less* than y for $x, y \in \mathbb{R}$
$x \cdot y$ - inner product of $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ given by

$$x \cdot y := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$|x|$ - euclidean norm of $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ given by

$$|x| := (x_1^2 + x_2^2 + \cdots x_n^2)^{1/2}$$

$O(g(x))$ - for $g(x)$ real-valued, $f(x) = O(g(x))$ if there exist $M > 0$, $x_0 \in \mathbb{R}$, such that

$$|f(x)| \leqslant M|g(x)| \quad \text{for all } x \geqslant x_0$$

$\partial\Omega$ - boundary of $\Omega \subseteq \mathbb{R}^n$
$\|\cdot\|_X$ - norm in Banach space X
$\|\cdot\|_p$ - L^p-norm
$(\cdot|\cdot)_H$ - inner product in Hilbert space H
$(\cdot|\cdot)_\Omega$ - inner product in $L^2(\Omega)$
$\hookrightarrow$ - continuous embedding
$\Re$ - real part
$\mathcal{L}(X, Y)$ - space of bounded linear operators from X to Y
$\mathcal{L}(X)$ - space of bounded linear operators from X to $\mathbb{K}$
C^∞ - space of smooth functions
C_b - space of bounded continuous functions
$\mathcal{S}$ - space of Schwartz functions
X^* - dual Banach space
$\langle \cdot, \cdot \rangle$ - duality, for $x \in X$ and $x^* \in X^*$ given by

$$\langle x, x^* \rangle := x^*(x)$$

x^α - for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and a multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ we set

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$\partial^\alpha f$ - for some function f on $\mathbb{R}^n$ and a multiindex α we set

$$\partial^\alpha f := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$$

$\lfloor x \rfloor$ - greatest integer less than or equal to x
$\complement$ - complement

Chapter 1

Introduction

The primitive equations of the ocean, derived from the Navier-Stokes equations by assuming hydrostatic balance, constitute the fundamental model for geophysical flow. In this work we consider the simplified model consisting of the momentum and continuity equations, explicitly given by

$$\begin{aligned} \partial_t v + \mathbf{v} \cdot \nabla v + \nabla_H \pi - \Delta v &= f \quad \text{on } \Omega \times (0,T), \\ \partial_z \pi &= 0 \quad \text{on } \Omega \times (0,T), \\ \operatorname{div} \mathbf{v} &= 0 \quad \text{on } \Omega \times (0,T), \\ v(0) &= v_0 \quad \text{on } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega := G \times (a,b)$ is a cylindrical domain with $G := (0,1)^2$ and $a < b$. The velocity of the fluid is $\mathbf{v} = (v,w)$ with $v = (v_1, v_2)$ and w horizontal and vertical components of $\mathbf{v}$ respectively, π denotes the pressure of the fluid and f the external force.

The mathematical analysis of primitive equations was initiated in 1992 by Lions, Teman and Wang [25, 24], who proved that given initial data in L^2, there exists a global weak solution to the problem. Note that uniqueness of solutions in three dimensions is still an open problem. In 1995 Ziane [36, 37] proposed studying the linearized problem instead and showed H^2-regularity for the solution of the resolvent problem. Based on this result, Guillén-González, Masmoudi and Rodríguez-Bellido [11] proved in 2001 the existence of a local, strong solution for initial data in H^1.

In 2007 Cao and Titi [3] took a big step forward and proved the existence of a unique, global, strong solution for arbitrary initial data in H^1. In later works they included modifications on the viscosity and diffusion, establishing global well-posedness for initial data belonging to H^2.

These results were extended to the L^p-setting in 2016 by Hieber and Kashiwabara [15], assuming Neumann and Dirichlet boundary conditions on the upper and bottom layers of the cylinder respectively. They studied the problem from the point of view of evolution equations, seeing the linearized primitive equations as semi-linear parabolic evolution equations in certain solenoidal subspaces of L^p. Their proof starts with the claim that the solution of the linearized equations is governed by an analytic semigroup $(T_p(t))_{t \geqslant 0}$ on the hydrostatic solenoidal subspace $L^p_{\overline{\sigma}}(\Omega)$ of $L^p(\Omega)$, see (4.20). Following Sohr's approach for the classical Navier-Stokes equations [31], they introduced the hydrostatic Helmholtz projection, denoted P_p, which eliminates the horizontal pressure gradient $\nabla_H \pi_s$ and has range precisely $L^p_{\overline{\sigma}}(\Omega)$. The generator of the $(T_p(t))_{t \geqslant 0}$ semigroup, $A_p = P_p \Delta$, is called the hydrostatic Stokes operator and they proved that $-A_p$

is a sectorial operator of spectral angle 0, generating an exponentially decaying analytic semigroup. Finally, adapting the Fujita-Kato approach for the Navier-Stokes equation [20, 6], they constructed a unique, global strong solution to the nonlinear primitive equations for arbitrary initial data in the complex interpolation space $[L^p_{\overline{\sigma}}(\Omega), D(A_p)]_{1/p}$ for $p \in [6/5, \infty)$. Since $[L^p_{\overline{\sigma}}(\Omega), D(A_p)]_{1/p} \hookrightarrow H^{2/p,p}(\Omega)$, for $p \geqslant 6/5$ large the result extends to initial data having less differentiability properties than H^1.

In this work we are going to study the recent article by Giga et. al. [7] where it is further proved that the hydrostatic Stokes operator admits a bounded H^∞-calculus. By rewriting the hydrostatic Stokes operator A_p as a perturbation of the Laplacian

$$A_p v = \Delta v + Bv, \quad Bv = -\nabla_H \Delta_H^{-1} \operatorname{div}_H \partial_z v\big|_D,$$

they use perturbation theorems and the fact that Laplace operator Δ admits a bounded H^∞-calculus to show sectoriality of A_p and that it generates and analytic semigroup on $L^p_{\overline{\sigma}}(\Omega)$ in a much shorter way than Hieber and Kashiwabara. Moreover, multiple corollaries are obtained immediately, such as maximal $L^q - L^p$-regularity and a characterization of domains of fractional powers. As a consequence, in another recent article by the same authors [8] they give a new proof of a unique, strong global solution for the primitive equations for initial data in the real interpolation space $(L^p_{\overline{\sigma}}(\Omega), D(A_p))_{1/q',q}$.

In particular, we will restrict the proof to the L^2-setting. Although the perturbation argument is the same in both cases, it relies on the hydrostatic Stokes operator A_p being invertible and sectorial with spectral angle 0 and the L^p-realization of the Laplacian admitting a bounded H^∞-calculus. Both proofs can be found in Hieber and Kashiwabara's previous work [15] and Nau's dissertation [27] respectively, but contain bigger mathematical difficulties. However, being the goal of this work to understand Giga et. al.'s article, we decided the L^2-approach to be a good start.

1.1 Outline

The thesis is divided into three parts. In chapter 2 we introduce the relevant equations to construct a model of the large-scale ocean and derive the primitive equations of the ocean through the Boussinesq and hydrostatic approximations. We conclude the chapter by stating the simplified linearization of the primitive equations, also called the hydrostatic Stokes equations.

In chapter 3 we introduce the relevant preliminaries to understand Giga et. al.'s [7] proof. Finally, in chapter 4 we show that the hydrostatic Stokes operator admits a bounded H^∞-calculus. To this end, in section 4.1 we establish the weak solvability of the Poisson problem

$$\Delta_H \pi = \operatorname{div}_H \overline{f} \text{ on } G, \quad \pi \text{ periodic on } \partial G,$$

which will allow us to define the hydrostatic Helmholtz projection

$$P_2 v = v - \nabla_H \pi \quad \text{with} \quad \operatorname{Ran} P_2 = L^2_{\overline{\sigma}}(\Omega).$$

We also include a useful characterization of the hydrostatic solenoidal subspace $L^2_{\overline{\sigma}}(\Omega)$ of $L^2(\Omega)$. In section 4.2 we define the hydrostatic Stokes operator

$$A_2 v = P_2 \Delta v, \quad D(A_2) = \{v \in W^{2,2}_{\text{per}}(\Omega)^2 \ : \ v\big|_{\Gamma_D} = 0,\ \partial_z v\big|_{\Gamma_N} = 0\} \cap L^2_{\overline{\sigma}}(\Omega),$$

and show that it generates an exponentially stable analytic semigroup following Hieber et. al. [15] for the $p = 2$ case. A construction for the L^p-case can also be found in the referenced article. The proof consists of two steps, first we show the resolvent estimate (4.36) by finding a unique solution of the weak formulation of the resolvent problem

$$\begin{cases} \lambda v - \Delta v + \nabla_H \pi_s = f & \text{on } \Omega, \\ \operatorname{div}_H \overline{v} = 0 & \text{on } G, \end{cases} \tag{1.2}$$

for every $\lambda \in \Sigma_{\pi-\varepsilon} \cup \{0\}$ through the Ladyzhenskaya-Babuška-Brezzi theorem and applying difference quotients. Second, we use the resolvent estimate to show that the hydrostatic Stokes operator is sectorial and invertible, and apply the Laplace transform representation to obtain the exponentially decaying bound. Finally, in section 4.3 we show that $-A_2$ admits a bounded H^∞-calculus through perturbation arguments by rewriting it as

$$A_p v = \Delta v + Bv, \quad Bv = -\nabla_H \Delta_H^{-1} \operatorname{div}_H \partial_z v\big|_D.$$

We start by establishing H^∞-boundedness of the Laplacian through reflection arguments and then apply the perturbation theorems of section 3.7 to obtain the desired result. Although the assertion for the general L^p-setting relies on $-A_p$ generating an exponentially decaying analytic semigroup, the H^∞-boundedness of the L^p-Laplacian is an easy generalization of the L^p-case and can be found on [27]. We finish the work including two immediate corollaries of the H^∞-boundedness of the hydrostatic Stokes operator, an explicit characterization of the domains of fraction powers $(-A_2)^\theta$ through Bessel potential spaces and maximal L^p-regularity estimates for the linearized primitive equations.

Our work ends at chapter 5 with a summary of the proof and possible future directions.

Chapter 2

Primitive equations

The primitive equations of the ocean are the standard model for the study of geophysical flows. They are derived from the Navier-Stokes equations assuming hydrostatic balance for the pressure in the vertical direction, justified by the difference of scale between the depth of the ocean ($\sim$ 11km) and the width (10^3 to 10^4km). In this chapter we provide a brief explanation of the construction of the hydrostatic Stokes equations studied by Giga et. al. in [7]. We will start by introducing the relevant equations to construct a model of the large-scale ocean, while defining the basic concepts of oceanic fluid dynamics. In sections 2.1 and 2.2 we explain the Boussinesq and hydrostatic approximation respectively, continuing with several simplifications. Finally, in section 2.3 we state the final form of the hydrostatic Stokes equations we will study in the rest of the work. This chapter is a compilation of several works [13, 12, 29]. The general theory on fluid dynamics and the respective definitions are obtained from [35].

In general, the ocean is modeled as a slightly compressible fluid with Coriolis force. The full set of equations of the large-scale ocean are: the momentum equation, the continuity equation, the diffusion-transport equations for the temperature and salinity and the equation of state. In what continues we will explain each of these equations.

We will start with the conservation of momentum. Newton's second law of motion for inertial frames of reference states that the acceleration of a body is equal to the net force acting on it. Since we are interested in describing the flow relative to Earth's surface, we have to translate this law to a rotating frame of reference. Moreover, in case of fluids, individual molecules cannot be followed and we instead consider an approximation by a continuum, this is, the momentum per unit volume is $\rho\mathbf{v}$, where $\rho : \mathbb{R}^3 \to \mathbb{R}$ is the density of the fluid and $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ the velocity. The resulting equation is the momentum equation in a rotating frame, which describes how the velocity of a fluid $\mathbf{v}$ responds to inertial and imposed forces, assumed to be the pressure and gravity:

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla\pi - \rho\mathbf{g} - 2\rho\omega \times \mathbf{v} + D\mathbf{v}, \tag{2.1}$$

Here, $\pi : \mathbb{R}^3 \to \mathbb{R}$ is the pressure, $\mathbf{g} = (0, 0, g)$ the constant gravity vector, ω the angular momentum, D is the molecular dissipation modeling the viscosity part of the velocity changes by the anisotropic Laplacian

$$D = \mu_1\Delta_H + \mu_2\partial_z^2, \quad \mu_1, \mu_2 > 0, \tag{2.2}$$

and the Coriolis force $2\rho\omega \times \mathbf{v}$ is a fictitious force that arises from considering a non-inertial coordinate system. Note that the acceleration is not simply the partial derivative

$\frac{\partial \mathbf{v}}{\partial t}$ but the material derivative, which describes the change of momentum of the fluid subject to space-time dependent velocity field and is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t} := \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \tag{2.3}$$

where ∇ is the covariant derivative.

Next is the conservation of mass. Although in classical mechanics mass is absolutely conserved, in fluid-mechanics the fluid flows into and away from regions. Therefore, the continuity equation describes the relation between the rate at which the mass enters and leaves the system, taking into account the accumulation of masses within the system:

$$\rho \text{ div } \mathbf{v} + \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0. \tag{2.4}$$

The temperature and salinity of the ocean are modeled by diffusion transport equations

$$\frac{\mathrm{d}T}{\mathrm{d}t} = D_T T + Q_T \quad \text{and} \quad \frac{\mathrm{d}S}{\mathrm{d}t} = D_S S + Q_S, \tag{2.5}$$

where D_T and D_S are the heat and salinity diffusivities associated with the anisotropic Laplacian (2.2) and Q_T and Q_S are sources of temperature and salinity respectively.

Finally, the equation of state. Note that the momentum and continuity equations in three dimensions provide two equations but three unknowns (density ρ, pressure π and velocity $\mathbf{v}$). Consequently, in order to obtain a solution another equation is needed, an equation of state. For seawater liquid such a model is not easily derivable, we usually rely on semi-empirical equations. In our case, a priori we will just assume the density of the ocean to be a general function of pressure, temperature and salinity

$$\rho = f(\pi, T, S). \tag{2.6}$$

The combination of equations (2.1)-(2.6) constitutes the set of equations governing the dynamics of the large-scale ocean:

$$\begin{aligned} \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= -\nabla\pi - \rho\mathbf{g} - 2\rho\omega \times \mathbf{v} + D\mathbf{v}, \\ \rho \text{ div } \mathbf{v} + \frac{\mathrm{d}\rho}{\mathrm{d}t} &= 0, \\ \frac{\mathrm{d}T}{\mathrm{d}t} &= D_T T + Q_T, \\ \frac{\mathrm{d}S}{\mathrm{d}t} &= D_S S + Q_S, \\ \rho &= f(\pi, T, S). \end{aligned} \tag{2.7}$$

Nevertheless, these equations are still mathematically complicated to study and specially computationally hard, so several simplifications are commonly considered which will be studied in the following sections.

2.1 Boussinesq approximation

The variations of density in the ocean are due to three effects: the compression of water by pressure ($\Delta_\pi \rho$), the thermal expansion of water by temperature changes ($\Delta_T \rho$) and the haline contraction by salinity changes ($\Delta_S\rho$). However, these variations are relatively little ($\sim 5\%$) around the ocean in comparison to the mean density and, accordingly, the *Boussinesq approximation* exploits this feature of the ocean to obtain a simpler equation of motion. Here we will give a brief explanation of its consequences, but for a rigorous justification of the approximation we refer the reader to [35].

Fixing ρ_0 as a reference value for the density, we may write

$$\rho = \rho_0 + \rho'(t; x, y, x) \quad \text{with} \quad \rho' << \rho_0, \tag{2.8}$$

The appropriate equation of state that approximately evaluates the change in pressure, temperature and salinity is the linear one

$$\rho = \rho_0(1 - \beta_T(T - T_0) + \beta_S(S - S_0)),$$

where β_T, β_S are the expansion coefficients and T_0, S_0 the reference values of temperature and respectively. Substituting (2.8) in (2.1) and dividing by ρ_0 the momentum equation can be written as

$$\left(1 + \frac{\rho'}{\rho_0}\right)\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho_0}\nabla\pi - \frac{\rho}{\rho_0}\mathbf{g} - \left(1 + \frac{\rho'}{\rho_0}\right)2\omega \times \mathbf{v} + \nu_1\Delta_H\mathbf{v} + \nu_2\partial_z^2\mathbf{v},$$

where $\nu_j = \mu_j/\rho_0$ is the kinematic viscosity. If $\rho' << \rho_0$, we can neglect the ρ'/ρ_0 term on the left and in the Coriolis force. However, the gravity term g is relatively big and there is no reason to disregard it. Consequently, the equation of motion after Boussinesq approximation takes the form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho_0}\nabla\pi - \frac{\rho}{\rho_0}\mathbf{g} - 2\omega \times \mathbf{v} + \nu_1\Delta_H\mathbf{v} + \nu_2\partial_z^2\mathbf{v}.$$

The same substitution (2.8) applied to the continuity equation (2.4) gives

$$\frac{\partial\rho'}{\partial t} + \mathbf{v}\cdot\nabla\rho' + \rho_0 \operatorname{div}\mathbf{v} + \rho' \operatorname{div}\mathbf{v} = 0,$$

where we already developed the material derivative (2.3). Dividing by ρ_0 again, taking $\rho' << \rho$ results in

$$\operatorname{div}\mathbf{v} = 0.$$

Finally, separating the velocity $\mathbf{v} = (v, w) : \mathbb{R}^3 \to \mathbb{R}^3$ into its horizontal $v = (v_1, v_2)$ and vertical w coordinates we obtain the *Boussinesq equations of the ocean*:

$$(BEs)\begin{cases} \partial_t v + v\cdot\nabla_H v + w\cdot\partial_z v = -\dfrac{1}{\rho_0}\nabla_H\pi - 2\omega\times v + \nu_1\Delta_H v + \nu_2\partial_z^2 v, \\ \partial_t w + v\cdot\nabla_H w + w\cdot\partial_z w = -\dfrac{1}{\rho_0}\partial_z\pi - \dfrac{\rho}{\rho_0}g + \nu_1\Delta_H w + \nu_2\partial_z^2 w, \\ \operatorname{div}_H v + \partial_z w = 0, \\ \partial_t T + v\cdot\nabla_H T + w\cdot\partial_z T - D_T T = Q_T, \\ \partial_t S + v\cdot\nabla_H S + w\cdot\partial_z S - D_S S = Q_S, \\ \rho = \rho_0(1 - \beta_T(T - T_0) + \beta_S(S - S_0)). \end{cases} \tag{2.9}$$

In absence of subscript we will assume all the operators (e.g. ∇ and Δ) to be 3-dimensional, and we will denote by the subscript H (e.g. Δ_H) the 2-dimensional operators that refer only to horizontal coordinates.

2.2 Hydrostatic approximation

As mentioned earlier, the difference of horizontal and vertical scale of the ocean is such that the momentum equation of the vertical motion can be substituted by the hydrostatic equation. A careful study of scales, see [29, Section II.3], shows that the aspect ratio (reason between the vertical Z and horizontal L characteristic lengths) is small:

$$\varepsilon = \frac{Z}{L} \approx 10^{-3}$$

and related to this aspect ratio, the a priori viscosities

$$\nu_1 = O(1), \quad \text{and} \quad \nu_2 = O(\varepsilon^2),$$

lead to primitive equations with full viscosity, while lower orders would lead to only partial viscosity. We define the scaled Boussinesq equations of the ocean, omitting temperature, salinity and the Coriolis force for simplicity, by

$$\begin{aligned} \partial_t v + v \cdot \nabla_H v + w \cdot \partial_z v - \Delta_H v - \partial_z^2 v + \nabla_H \pi &= 0, \\ \varepsilon^2(\partial_t w + v \cdot \nabla_H w + w \cdot \partial_z w - \Delta_H w - \partial_z^2 w) + \partial_z \pi &= -\rho g, \\ \operatorname{div}_H v + \partial_z w &= 0. \end{aligned} \tag{2.10}$$

Letting $\varepsilon \to 0^+$ formally gives the *hydrostatic approximation* for the vertical velocity by

$$\frac{\partial \pi}{\partial z} = -\rho g, \tag{2.11}$$

For a rigorous justification of the hydrostatic balance see [13, Section 1.14].

Replacing the moment equation for the vertical velocity by the hydrostatic approximation in the Boussinesq equations, we obtain the so called *primitive equations of the ocean*:

$$(PEs) \begin{cases} \partial_t v + v \cdot \nabla_H v + w \cdot \partial_z v + \dfrac{1}{\rho_0} \nabla_H \pi + 2\omega \times v - \nu_1 \Delta_H v - \nu_2 \partial_z^2 v = 0, \\ \partial_z \pi = -\rho g, \\ \operatorname{div}_H v + \partial_z w = 0, \\ \partial_t T + v \cdot \nabla_H T + w \cdot \partial_z T - D_T T = Q_T, \\ \partial_t S + v \cdot \nabla_H S + w \cdot \partial_z S - D_S S = Q_S, \\ \rho = \rho_0(1 - \beta_T(T - T_0) + \beta_S(S - S_0)). \end{cases} \tag{2.12}$$

2.3 Model situation

Temperature and salinity do not impose any added complexity to the problem, but rather just complicate the mathematical analysis. Therefore, we will assume a simpler isothermal scenario in our model, i.e. temperature and salinity are assumed to be constant. By the same argument, we will neglect the Coriolis and gravity term from our analysis and all constants are normalized to one. The resulting set of equations is

$$\begin{aligned} \partial_t v + v \cdot \nabla_H v + w \cdot \partial_z v &= -\nabla_H \pi + \Delta v, \\ \partial_z \pi &= 0, \\ \operatorname{div}_H v + \partial_z w &= 0. \end{aligned} \tag{2.13}$$

Moreover, we will assume the simplest domain possible for the ocean, the cylindrical domain. Given $a, b \in \mathbb{R}$, we set the domain to be

$$\Omega = G \times (a,b) \subset \mathbb{R}^3 \quad \text{with} \quad G = (0,1) \times (0,1),$$

and we denote the bottom, upper and lateral parts of the boundary of Ω respectively by

$$\Gamma_a = G \times \{a\}, \quad \Gamma_b = G \times \{b\} \quad \text{and} \quad \Gamma_l = \partial G \times (a,b).$$

Note that the boundary $\partial\Omega$ is not smooth, but we will overcome this difficulty by assuming lateral periodicity. The primitive equations of the ocean (2.13) can be supplemented by diverse boundary conditions, following Giga et. al.'s approach [7], we will focus on the mixed case

$$v,\ w,\ \pi \ \text{ are periodic on } \ \Gamma_l \times (0,\infty), \tag{2.14}$$
$$w = 0 \ \text{ on } \ (\Gamma_a \cup \Gamma_b) \times (0,\infty), \tag{2.15}$$
$$v = 0 \ \text{ on } \ \Gamma_D \times (0,\infty) \quad \text{and} \quad \partial_z v = 0 \ \text{ on } \ \Gamma_N \times (0,\infty), \tag{2.16}$$
$$v(0;\cdot,\cdot,\cdot) = v_0 \ \text{ on } \ \Omega, \tag{2.17}$$

where Dirichlet and Neumann boundary conditions are given by the notation

$$\Gamma_D \in \{\varnothing, \Gamma_a, \Gamma_b, \Gamma_a \cup \Gamma_b\} \quad \text{and} \quad \Gamma_N = (\Gamma_a \cup \Gamma_b) \backslash \Gamma_D.$$

In particular, in the new set of equations (2.13) the full pressure $\pi : \Omega \to \mathbb{R}$ can be determined from the surface pressure $\pi_s : G \to \mathbb{R}$. On the other hand, when substituting the continuity equation for the vertical velocity by the hydrostatic approximation, we lose an evolution equation for the vertical velocity. One of Lions et. al.'s [25] biggest contribution, in their own words, was overcoming the difficulties caused by the absence of an evolution equation for the vertical velocity w. Indeed, integrating the continuity equation with respect to the vertical variable over (a, z) we obtain

$$w(t;x,y,z) - w(t;x,y,a) = -\int_a^z \operatorname{div}_H v(t;x,y,s)\,\mathrm{d}s \tag{2.18}$$

and substituting the bottom boundary condition $w\big|_{\Gamma_a} = 0$, cf. (2.15), we arrive at

$$w(t;x,y,z) = w(v)(t;x,y,z) := -\int_a^z \operatorname{div}_H v(t;x,y,s)\,\mathrm{d}s. \tag{2.19}$$

which gives the vertical component of the velocity in terms of the horizontal one. Analogously, integrating over the whole vertical interval (a, b) and substituting both the bottom and upper boundary conditions for the vertical velocity we obtain that

$$\operatorname{div}_H \overline{v} = 0, \quad \text{with} \quad \overline{v}(t;x,y) := \frac{1}{b-a}\int_a^b v(t;x,y,s)\,\mathrm{d}s. \tag{2.20}$$

The simplified set of equations modeling the large-scale ocean is

$$\begin{aligned} \partial_t v + v \cdot \nabla_H v + w(v)\partial_z v + \nabla_H \pi_s - \Delta v = f \quad &\text{on } \ \Omega \times (0,T), \\ \operatorname{div}_H \overline{v} = 0 \quad &\text{on } \ G \times (0,T), \\ v(0) = v_0 \quad &\text{on } \ \Omega, \end{aligned} \tag{2.21}$$

where $f : \Omega \to \mathbb{R}$ is a given external force and $v_0 \in \mathbb{R}^3$ the initial horizontal velocity. The linearized version of the above set of equations is what we refer to as the *hydrostatic Stokes equations*:

$$(HSEs)\begin{cases} \partial_t v + \nabla_H \pi_s - \Delta v = f & \text{on } \Omega \times (0,T), \\ \operatorname{div}_H \overline{v} = 0 & \text{on } G \times (0,T), \\ v(0) = v_0 & \text{on } \Omega. \end{cases} \tag{2.22}$$

Chapter 3

Preliminaries

3.1 Vector-valued Fourier transform

This section is divided into two main parts. In the first subsection we have compiled some notions about integration in Banach spaces and we introduce the vector-valued Fourier transform. In the second section we summarize without proof basic facts about R-boundedness and state the vector-valued Mikhlin multiplier theorem. We will only touch a few basic aspects of the theory necessary to get a general understanding of the H^∞-functional calculus in section 3.6 and maximal regularity in section 3.8. However, for the interested reader a thorough study of the field is carried out in [18, 19], a concise introduction to R-boundedness and operator-valued Fourier multipliers can be found in [22].

3.1.1 Introduction to the vector-valued setting

In order to work with vector-valued function spaces, we need to extend the notion of integrability to vector-valued functions. This is done analogous to the scalar case, by approximation of simple functions. Fix a measure space $(\Omega, \mathcal{F}, \mu)$ and a Banach space X. Let $f : \Omega \to \mathbb{K}$ and $x \in X$, we define $f \otimes x : \Omega \to \mathbb{K}$ by

$$(f \otimes x)(\omega) := f(w)x.$$

Definition 3.1.1. A function $g : \Omega \to X$ is called *simple* if there exist a finite number of elements of the Banach space $\{x_j\}_{j=1}^n \subset X$ and disjoint sets of finite measure $\{F_j\}_{j=1}^n \subset \mathcal{F}$, such that g is a linear combination of the form

$$g = \sum_{j=1}^{n} \mathbf{1}_{F_j} \otimes x_j. \tag{3.1}$$

A function $f : \Omega \to X$ is called *strongly measurable* if it is a pointwise limit of a sequence of simple functions $(g_n)_{n\geqslant 1}$, i.e.

$$f(\omega) = \lim_{n\to\infty} g_n(\omega), \quad \omega \in \Omega.$$

We can define the integral of simple functions as

$$\int_\Omega g \, \mathrm{d}\mu := \sum_{j=1}^{n} \mu(F_j)x_j,$$

and the triangle inequality implies

$$\left\| \int_\Omega g \, \mathrm{d}\mu \right\|_X \leqslant \int_\Omega \|g\|_X \, \mathrm{d}\mu.$$

Pointwise limit of simple functions is what makes the following definition of integration in Banach spaces allowable.

Definition 3.1.2. A strongly measurable function $f : \Omega \to X$ is said to be *Bochner integrable* with respect to μ if there exists a sequence $(g_n)_{n\geqslant 1}$ of simple functions $g_n : \Omega \to X$ such that

$$\lim_{n\to\infty} \int_\Omega \|f - g_n\|_X \, \mathrm{d}\mu = 0.$$

In that case we define the Bochner integral of f by

$$\int_\Omega f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega g_n \, \mathrm{d}\mu.$$

Perhaps a simpler characterization is the following.

Proposition 3.1.3 ([18, Proposition 1.2.2])**.** *A strongly measurable function $f : \Omega \to X$ is Bochner integrable with respect to μ if and only if*

$$\int_\Omega \|f\|_X \, \mathrm{d}\mu < \infty.$$

In this case we have

$$\left\| \int_\Omega f \, \mathrm{d}\mu \right\|_X \leqslant \int_\Omega \|f\|_X \, \mathrm{d}\mu.$$

Proof. To prove the sufficient condition let f be a strongly measurable function such that $\int_\Omega \|f\|_X < \infty$, then by definition there exists a sequence of simple functions $(g_n)_{n\geqslant 1}$ such that $g_n \to f$ pointwise. We can define a new sequence of simple functions by

$$f_n := \mathbf{1}_{\{\|g_n\|\leqslant 2\|f\|\}} g_n,$$

which converge to f pointwise as well. Since $\|f_n\| \leqslant 2\|f\|$ and each f_n is simple, by the dominated convergence theorem we get that

$$\lim_{n\to\infty} \int_\Omega \|f - f_n\|_X \, \mathrm{d}\mu = 0.$$

For the necessary condition let $(g_n)_{n\geqslant 1}$ be a sequence of simple functions as in definition 3.1.2. Then for n large enough we get

$$\int_\Omega \|f\| \, \mathrm{d}\mu \leqslant \int_\Omega \|f - g_n\| \, \mathrm{d}\mu + \int_\Omega \|g_n\| \, \mathrm{d}\mu < \infty.$$

□

The Bochner integral allows us to easily generalize many classical function spaces. For example, the vector-valued L^p-spaces, where we say that two strongly measurable functions $f : \Omega \to X$ and $g : \Omega \to X$ are in the same equivalence class if $f(\omega) = g(\omega)$ for μ-almost all $\omega \in \Omega$.

Definition 3.1.4. For $1 \leqslant p < \infty$, we denote by $L^p(\Omega; X)$ the space of all strongly measurable functions $f : \Omega \to X$ such that

$$\int_\Omega \|f\|_X^p \, \mathrm{d}\mu < \infty.$$

For $p = \infty$, we denote by $L^\infty(\Omega; X)$ the space of all strongly measurable functions $f : \Omega \to X$ for which there exists $r \geqslant 0$ such that $\mu(\|f\| \geqslant r) = 0$.

Endowed with the norms

$$\|f\|_{L^p(\Omega;X)} := \left(\int_\Omega \|f\|_X^p \, \mathrm{d}\mu\right)^{1/p} \quad \text{and} \quad \|f\|_{L^\infty(\Omega;X)} := \operatorname*{ess\,sup}_{\omega\in\Omega} \|f(\omega)\|_X,$$

the spaces $L^p(\Omega; X)$, $1 \leqslant p \leqslant \infty$ are Banach spaces.

Before we define the vector-valued Fourier transform, we will introduce the vector-valued Schwartz class and tempered distributions. All these notions are straightforward modifications of their scalar counterparts, but we gather them here for the sake of clarity.

Definition 3.1.5. The *Schwartz class* of X-valued functions on $\mathbb{R}^n$ is the space

$$\mathcal{S}(\mathbb{R}^n; X) := \left\{ f \in C^\infty(\mathbb{R}^n; X) \ : \ \|f\|_{\alpha,\beta} := \|x \mapsto x^\beta \partial^\alpha f(x)\|_{L^\infty(\mathbb{R}^n;X)} < \infty, \ \forall \alpha, \beta \in \mathbb{N}^d \right\}.$$

The space of X-valued *tempered distributions* is defined by

$$\mathcal{S}'(\mathbb{R}^n; X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n); X).$$

Denoting by $\mathcal{D}(\mathbb{R}^n; X)$ the space of X-valued smooth function with compact support on $\mathbb{R}^n$, and the space of X-valued distributions $\mathcal{D}'(\mathbb{R}^n; X)$ analogous to the tempered distributions, we have the usual inclusions

$$\mathcal{D}(\mathbb{R}^N; X) \hookrightarrow \mathcal{S}(\mathbb{R}^n; X) \quad \text{and} \quad S'(\mathbb{R}^n; X) \hookrightarrow \mathcal{D}'(\mathbb{R}^n; X).$$

Definition 3.1.6. An X-valued tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n; X)$ is called *regular* if there exists a strongly measurable function $f : \mathbb{R}^n \to X$ such that

$$u(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x) \, \mathrm{d}x, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Note that, in particular, every Bochner integrable function $f \in L^1(\mathbb{R}^n; X)$ defines a regular distribution $u_f \in \mathcal{S}'(\mathbb{R}^n; X)$ by the identification

$$u_f(\varphi) := \int_{\mathbb{R}^n} f(x)\varphi(x) \, \mathrm{d}x, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

By abuse of notation, we continue to write f to denote the regular distribution u_f.

We now have set the framework to define the vector-valued Fourier transform.

Definition 3.1.7. Let X be a complex Banach space. Given $f \in L^1(\mathbb{R}^n; X)$, the *Fourier transform* of f, $\mathcal{F} : L^1(\mathbb{R}^n; X) \to L^\infty(\mathbb{R}^n; X)$, is defined by

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x, \quad \xi \in \mathbb{R}^n.$$

Restricted to the Schwartz class $\mathcal{S}(\mathbb{R}^n; X)$, the Fourier transform is an isomorphism whose inverse is given by

$$\mathcal{F}^{-1}(f)(x) := \check{f}(x) := \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} \, \mathrm{d}\xi, \quad x \in \mathbb{R}^n.$$

And this property extends to the space of X-valued tempered distributions when we define the Fourier transform here via duality. Given $u \in \mathcal{S}'(\mathbb{R}^n; X)$ we define the Fourier transform of u by

$$\mathcal{F}(u)(f) := u(\mathcal{F}(f)), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Many of the L^1-properties of the scalar-valued Fourier transform continue to be true in the vector-valued case, but in general the L^2-results fail unless X is a Hilbert space. For example, Plancherel's theorem is essential in order to extend isometrically the scalar-valued Fourier-Plancherel transform from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. However, the same construction cannot be applied in the general vector-valued case since Plancherel's theorem only holds for Hilbert-valued functions. Furthermore, the following theorem is true.

Theorem 3.1.8 ([18, Theorem 2.1.18], Kwapień)**.** *For a Banach space X the following assertions are equivalent:*

i) the Fourier-Plancherel transform extends to a bounded operator on $L^2(\mathbb{R}^n; X)$;

ii) X is isomorphic to a Hilbert space.

3.1.2 R-boundedness and operator-valued Fourier multipliers

When trying to extend the Mikhlin multiplier theorem to the vector-valued setting we encounter a similar problem.

Definition 3.1.9. Let $1 \leqslant p < \infty$ and let X and Y be complex Banach spaces. A function $m \in L^\infty(\mathbb{R}^n; \mathcal{L}(X, Y))$ is called an *operator-valued L^p-Fourier multiplier* if the *Fourier multiplier operator*

$$T_m(f) := \mathcal{F}^{-1}(m\hat{f}), \quad f \in \mathcal{S}(\mathbb{R}^n; X),$$

extends uniquely to a bounded operator on $L^p(\mathbb{R}^n; X)$. In other words, $T_m(f) \in L^p(\mathbb{R}^n; Y)$ and there exists $C > 0$ such that

$$\|T_m(f)\|_{L^p(\mathbb{R}^n; Y)} \leqslant C \|f\|_{L^p(\mathbb{R}^n; X)}, \quad f \in \mathcal{S}(\mathbb{R}^n; X).$$

Note that the operator $T_m : \mathcal{S}(\mathbb{R}^n; X) \to \mathcal{S}'(\mathbb{R}^n; Y)$ is well defined because for $f \in \mathcal{S}(\mathbb{R}^n; X)$ the multiplication $m\hat{f} \in L^\infty(\mathbb{R}^n; Y)$ defines a regular distribution in $\mathcal{S}'(\mathbb{R}^n; Y)$.

In particular, if $X = Y$ is a Hilbert space J. Schwartz proved the following theorem.

Theorem 3.1.10 ([22, Theorem 1.6])**.** *Let X be a Hilbert space. Assume that for the function $m \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X))$ the sets*

$$\{m(u) \, : \, u \in \mathbb{R}\backslash\{0\}\} \quad \textit{and} \quad \{um'(u) \, : \, u \in \mathbb{R}\backslash\{0\}\}$$

are bounded in $\mathcal{L}(X)$. Then the Fourier multiplier operator T_m extends to a bounded operator on $L^p(\mathbb{R}; X)$ for $1 < p < \infty$.

Unfortunately, as in the case of the Fourier-Plancherel transform, G. Pisier proved that if this theorem holds for a Banach space X, then X is isomorphic to a Hilbert space. Consequently, if we want a more general multiplier theorem for a wider class of multipliers and Banach spaces we need to change some assumptions. We will devote the rest of this section to introduce the framework to state Weis' operator-valued Mikhlim multiplier theorem.

Definition 3.1.11. A Banach space X is called a *UMD-space* if the Hilbert transform

$$H(f)(t) := \lim_{\substack{\varepsilon\downarrow 0\\ R\to\infty}} \frac{1}{\pi}\int_{\varepsilon<|x-y|<R} \frac{f(y)}{x-y}\,\mathrm{d}y, \quad f\in\mathcal{S}(\mathbb{R};X),$$

extends to a bounded operator on $L^p(\mathbb{R};X)$ for $1<p<\infty$.

Definition 3.1.12. Let X and Y be Banach spaces. A family of operators $\mathcal{F}\subseteq\mathcal{L}(X,Y)$ is called *R-bounded* if for some $p\in[1,\infty)$ there exists a constant $C\geqslant 0$ independent of $N\geqslant 1$ such that we have

$$\left\|\sum_{n=1}^N \varepsilon_n T_n x_n\right\|_{L^p(\Omega;Y)} \leqslant C\left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^p(\Omega;X)}$$

for any finite choice of $x_1,...,x_N\in X$ and $T_1,...,T_N\in\mathcal{F}$, where $\{\varepsilon_n\}_{n=1}^N$ is a sequence of independent, symmetric $\{-1,1\}$-valued random variables on Ω.

The definition of R-boundedness is independent of $p\in[1,\infty)$, i.e. if $\mathcal{F}$ is R-bounded for some $p\in[1,\infty)$, then it is for all $p\in[1,\infty)$, this follows from the Kahane-Khintchine inequality below.

Proposition 3.1.13 ([19, Theorem 6.2.4])**.** *Let $(\varepsilon_n)_{n\geqslant 1}$ be a sequence of independent, symmetric $\{-1,1\}$-valued random variables on Ω. Then for all $0<p<q<\infty$ and all finite sequences $\{x_n\}_{n=1}^N$ in any Banach space X, we have*

$$\left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^q(\Omega;X)} \leqslant \kappa_{q,p}\left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^p(\Omega;X)}.$$

for some constant $\kappa_{q,p}>0$.

Although the constant C in definition 3.1.12 depends on p, for most purposes there is no need to distinguish the p-dependence. The next two propositions are included to get a general feeling of the relation between R-boundedness and Hilbert spaces.

Proposition 3.1.14. *If $\mathcal{F}\subseteq\mathcal{L}(X,Y)$ is a R-bounded family of operators, then $\mathcal{F}$ is uniformly bounded.*

Proof. It is a simple matter of checking the definition 3.1.12 for $N=1$. If ε is a symmetric $\{-1,1\}$-valued random variable on Ω, then

$$\|\varepsilon\|_{L^p(\Omega)} = (\mathbb{E}|\varepsilon|^p)^{1/p} = (|1|^p P(\varepsilon=-1)+|-1|^p P(\varepsilon=1))^{1/p} = 1,$$

and therefore for every $x\in X$ and $T\in\mathcal{F}$ we have the uniform bound

$$\|Tx\|_Y = \|\varepsilon Tx\|_{L^p(\Omega;Y)} \leqslant \|\varepsilon x\|_{L^p(\Omega;X)} = \|x\|_X.$$

$\square$

The converse is true if X and Y are Hilbert spaces.

Proposition 3.1.15. *Let X and Y be Hilbert spaces. Then $\mathcal{F} \subseteq \mathcal{L}(X, Y)$ is R-bounded if and only if $\mathcal{F}$ is uniformly bounded.*

Proof. We already proved the sufficient condition. For the necessary, let $p = 2$. Recalling that the norm in a Hilbert space comes from an inner product we obtain that

$$\begin{aligned}
\left\| \sum_{n=1}^{N} \varepsilon_n T_n x_n \right\|_{L^2(\Omega;Y)}^2 &= \mathbb{E}_\Omega \left\| \sum_{n=1}^{N} \varepsilon_n T_n x_n \right\|_Y^2 = \mathbb{E}_\Omega \Big(\sum_{n=1}^{N} \varepsilon_n T_n x_n \Big| \sum_{n=1}^{N} \varepsilon_n T_n x_n \Big)_Y \\
&= \sum_{i,j=1}^{N} \mathbb{E}_\Omega \varepsilon_i \varepsilon_j (T_i x_i | T_j x_j)_Y = \sum_{n=1}^{N} \mathbb{E}_\Omega \| T_n x_n \|_Y^2 \\
&\leqslant C \sum_{n=1}^{N} \mathbb{E}_\Omega \| x_n \|_X^2 = C \sum_{i,j=1}^{N} \mathbb{E}_\Omega \varepsilon_i \varepsilon_j (x_i | x_j)_X \\
&= C \mathbb{E}_\Omega \Big(\sum_{n=1}^{N} \varepsilon_n x_n \Big| \sum_{n=1}^{N} \varepsilon_n x_n \Big)_X = C \mathbb{E}_\Omega \left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|_X^2 = C \left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|_{L^2(\Omega;X)}^2,
\end{aligned}$$

where we used that since the random variables $\{\varepsilon_n\}_{n=1}^N$ are independent,

$$\mathbb{E}_\Omega \varepsilon_i \varepsilon_j = \begin{cases} \mathbb{E}_\Omega \varepsilon_i \mathbb{E}_\Omega \varepsilon_j = 0 & \text{if } i \neq j \\ \mathbb{E}_\Omega \varepsilon_i^2 = 1 & \text{if } i = j. \end{cases}$$

□

We can finally state the main theorem of this section, L. Weis' celebrated vector-valued multiplier theorem.

Theorem 3.1.16 ([22, Theorem 1.10])**.** *Let X and Y be UMD spaces. Assume that for $m \in C^1(\mathbb{R} \backslash \{0\}, \mathcal{L}(X, Y))$ the sets*

$$\{m(u) \ : \ u \in \mathbb{R} \backslash \{0\}\} \quad \text{and} \quad \{u m'(u) \ : \ u \in \mathbb{R} \backslash \{0\}\}$$

are R-bounded. Then the Fourier multiplier operator

$$T_m(f) = \mathcal{F}^{-1}(m \widehat{f}), \quad f \in \mathcal{S}(\mathbb{R}; X)$$

extends to a bounded operator $T_m : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; Y)$ for all $p \in (1, \infty)$.

3.2 Interpolation theory

In this section we introduce interpolation couples, and look more closely to real and complex interpolation. However, we do not intend to cover more than the essential properties for the self-containment of this work. For a more extensive study of Interpolation theory we refer the reader to [26], [33] and [18], which constitute the basis of this introduction.

Definition 3.2.1. A couple of Banach spaces (X, Y) is said to be an *interpolation couple* if both X and Y are continuously embedded in a Hausdorff topological vector space $\mathcal{V}$.

Proposition 3.2.2. *If (X, Y) is an interpolation couple, then the spaces*

$$\begin{aligned} X \cap Y :=& \{v \in \mathcal{V} \,:\, v \in X \quad \textit{and} \quad v \in Y\}, \\ X + Y :=& \{v \in \mathcal{V} \,:\, v = x + y \quad \textit{with} \quad x \in X,\, y \in Y\}. \end{aligned}$$

are Banach spaces with the respective norms

$$\|v\|_{X\cap Y} := \max\{\|v\|_X,\ \|v\|_Y\}, \tag{3.2}$$

$$\|v\|_{X+Y} := \inf_{\substack{x+y=v \\ x\in X,\, y\in Y}} \{\|x\|_X + \|y\|_Y\}. \tag{3.3}$$

Proof. We start with $X \cap Y$. It is evident that $\|\cdot\|_{X\cap Y}$ defines a norm, and we only need to show completeness. For this purpose let $(v_n)_{n\geqslant 1} \subset X \cap Y$ be a Cauchy sequence. From the definition of the norm it follows $(v_n)_{n\geqslant 1}$ is a Cauchy sequence in X and Y as well, therefore there exist limits in X and Y respectively. However, since X and Y are continuously embedded in a Hausdorff space $\mathcal{V}$, the limits coincide, where we will denote the common limit by v. We thus have that $v_n \to v$ both in X and Y, and consequently also in the intersection $X \cap Y$.

For $\|\cdot\|_{X+Y}$ the only non-trivial norm property is point-separation. Assume $\|v_n\|_{X+Y} = 0$, since the norm is taken as an infimum there exist sequences $(x_n)_{n\geqslant 1} \subset X$ and $(y_n)_{n\geqslant 1} \subset Y$ such that $v = x_n + y_n$ for all $n \in \mathbb{N}$ and $\|x_n\|_X + \|y_n\|_Y \to 0$. However, X and Y are continuously embedded in the Hausdorff space $\mathcal{V}$, $v = x_n + y_n \to 0$ in $\mathcal{V}$, thus $v = 0$. Recall that completeness is equivalent to every absolutely convergent series in $X + Y$ converging in $X + Y$. Let $(v_n)_{n\geqslant 1} \subseteq X + Y$ be absolutely convergent $\sum_{n\geqslant 1} \|v_n\|_{X+Y} < \infty$. By the infimum in the definition of the norm, we can find sequences $(x_n)_{n\geqslant 1} \subseteq X$ and $(y_n)_{n\geqslant 1} \subseteq Y$ such that

$$v_n = x_n + y_n \quad \text{and} \quad \|x_n\|_X + \|y_n\|_Y < 2^{-n} + \|v_n\|_{X+Y}, \quad n \in \mathbb{N}.$$

This gives that both $\sum_{n\geqslant 1} \|x_n\|_X$ and $\sum_{n\geqslant 1} \|y_n\|_Y$ are absolutely convergent sequences of the Banach spaces X and Y respectively, thus the limits $x = \sum_{n\geqslant 1} x_n$ and $y = \sum_{n\geqslant 1} y_n$ exist. Defining now $v_n = x + y$, we get that

$$\left\| v - \sum_{n=1}^{N} v_n \right\|_{X+Y} = \left\| \sum_{n\geqslant N+1} x_n + y_n \right\|_{X+Y} \leqslant 2^{-N} + \sum_{n\geqslant N+1} \|v_n\|_{X+Y},$$

therefore letting $N \to \infty$ we conclude that $v = \sum_{n\geqslant 1} v_n$ in $X + Y$ as desired. □

Definition 3.2.3. Let (X, Y) be an interpolation couple. An *intermediate space* is any Banach space E such that

$$X \cap Y \hookrightarrow E \hookrightarrow X + Y.$$

Interpolation theory is the study of spaces that are in certain sense intermediate to X and Y. To begin with, we would like to guarantee that continuous functions in X and Y are also continuous in any intermediate space E. The precise meaning of this is contained in the following definition.

Definition 3.2.4. Let (X, Y) be an interpolation couple. An *interpolation space* between X and Y is any intermediate space E such that for every linear operator $T : X + Y \to X + Y$ whose restriction to X belongs to $\mathcal{L}(X)$ and whose restriction to Y belongs to $\mathcal{L}(Y)$, then $T\big|_E \in \mathcal{L}(E)$.

3.2.1 Real interpolation

Given an interpolation couple (X, Y) we will define real interpolation spaces by the K-method which assigns to each $\theta \in (0,1)$ and $1 \leqslant p \leqslant \infty$ an interpolation space $X_{\theta,p}$.

For $t > 0$ and $v \in X + Y$ we define the *K-functional*:

$$K(t, v; X, Y) := \inf_{\substack{v=x+y \\ x\in X,\, y\in Y}} \{\|x\|_X + t\|y\|_Y\}.$$

If it does not lead to confusion we will simply write $K(t, x)$ instead. Note that $K(1, \cdot) = \|\cdot\|_{X+Y}$ and $K(t, \cdot)$ is a norm in $X + Y$ for every $t > 0$, equivalent to $\|\cdot\|_{X+Y}$.

Definition 3.2.5. Let $\theta \in (0,1)$, $1 \leqslant p \leqslant \infty$. We define *real interpolation spaces* as

$$(X, Y)_{\theta,p} := \{v \in X + Y : t \mapsto t^{-\theta} K(t, v) \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})\},$$

endowed with the norm

$$\|v\|_{(X,Y)_{\theta,p}} := \|t^{-\theta} K(t, v)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})}. \tag{3.4}$$

These are indeed interpolation spaces between X and Y. Given $v \in X+Y$ and $t > 0$, from the inequality

$$\min\{1, t\}\|v\|_{X+Y} \leqslant K(t, v) \leqslant \min\{1, t\}\|v\|_{X\cap Y}$$

and the independence of the norms $\|\cdot\|_{X\cap Y}$ and $\|\cdot\|_{X+Y}$ from t it is inferred that $(X, Y)_{\theta,p}$ is an intermediate space

$$X \cap Y \hookrightarrow (X, Y)_{\theta,p} \hookrightarrow X + Y. \tag{3.5}$$

Consider now a linear operator $T : X + Y \to X + Y$ such that $T\big|_X \in \mathcal{L}(X)$ and $T\big|_Y \in \mathcal{L}(Y)$ with norms

$$\|T\|_{\mathcal{L}(X)} = A_X \quad \text{and} \quad \|T\|_{\mathcal{L}(Y)} = A_Y.$$

Without loss of generality we can assume $A_X \neq 0$. Fix $v \in (X, Y)_{\theta,p}$ and take $x \in X$, $y \in Y$ such that $v = x + y$. If $t > 0$, then

$$\|Tx\|_X + t\|Ty\|_Y \leqslant A_X \left(\|x\|_X + \frac{A_Y}{A_X} t\|y\|_Y\right),$$

or equivalently

$$K(t, Tv) \leqslant A_X K\left(\frac{A_Y}{A_X} t, v\right).$$

Using this inequality we can show that $T\big|_{(X,Y)_{\theta,p}}$ is continuous

$$\begin{aligned}\|Tv\|^p_{(X,Y)_{\theta,p}} &= \int_0^\infty [t^{-\theta} K(t, Tv)]^p \frac{\mathrm{d}t}{t} \leqslant \int_0^\infty \left[t^{-\theta} A_X K\left(t\frac{A_Y}{A_X}, v\right)\right]^p \frac{\mathrm{d}t}{t} \\ &= A_X \left(\frac{A_Y}{A_X}\right)^{\theta p} \int_0^\infty [s^{-\theta} K(s, v)]^p \frac{\mathrm{d}s}{s} = A_X^{(1-\theta)p} A_Y^{\theta p} \|v\|^p_{(X,Y)_{\theta,p}}.\end{aligned}$$

Where we used the change of variable $s = tA_Y/A_X$. In conclusion, $T\big|_{(X,Y)_{\theta,p}} \in \mathcal{L}((X, Y)_{\theta,p})$ with norm

$$\|T\|_{\mathcal{L}((X,Y)_{\theta,p})} \leqslant A_X^{(1-\theta)} A_Y^{\theta}. \tag{3.6}$$

Remark 1. In particular, let $y \in X \cap Y$ and take the operator $T : \mathbb{K}+\mathbb{K} \to X+Y$ such that $T : \lambda \mapsto \lambda y$ in (3.6). If we follow the same prove by choosing $A_X = \|T\|_{\mathcal{L}(\mathbb{K},Y)} = \|y\|_X$ and $A_Y = \|T\|_{\mathcal{L}(\mathbb{K},Y)} = \|y\|_Y$ instead we get the inequality

$$\|y\|_{(X,Y)_{\theta,1}} = \|T\|_{\mathcal{L}(\mathbb{K},(X,Y)_{\theta,1})} \leqslant C(\theta)\|y\|_X^{1-\theta}\|y\|_Y^{\theta}, \quad y \in X \cap Y.$$

Proposition 3.2.6. *Let $\theta \in (0,1)$ and $1 \leqslant p \leqslant \infty$. The real interpolation space $(X,Y)_{\theta,p}$ is a Banach space endowed with the norm* (3.4).

Proof. Take a Cauchy sequence $(v_n)_{n\geqslant 1}$ in $(X,Y)_{\theta,p}$, we can denote the limit of $(v_n)_{n\geqslant 1}$ in $X+Y$ as v. Given $\varepsilon > 0$, choose $N \geqslant 1$ such that $\|v_n - v_m\|_{(X,Y)_{\theta,p}} < \varepsilon$ for all $n, m \geqslant N$. If $m \geqslant n \geqslant N$ we can apply the triangle inequality

$$\begin{aligned}\left(\int_0^\infty [t^{-\theta}K(t, v - v_n)]^p \frac{\mathrm{d}t}{t}\right)^{1/p} &\leqslant \varepsilon + \left(\int_0^\infty [t^{-\theta}K(t, v - v_m)]^p \frac{\mathrm{d}t}{t}\right)^{1/p} \\ &\leqslant \varepsilon + \|v - v_m\|_{X+Y}\left(\int_0^\infty t^{-\theta p}\frac{\mathrm{d}t}{t}\right)^{1/p},\end{aligned}$$

where in the last inequality we applied the property $K(t,y) \leqslant \max\{1,t\}\|y\|_{Y+X}$. Since $v_m \to v$ in $X+Y$ when $m \to \infty$ we get that

$$\|v - v_n\|_{(X,Y)_{\theta,p}} < \varepsilon \quad \text{for all} \quad n \geqslant N,$$

with N sufficiently large, thus $(v_n)_{n\geqslant 1}$ converges to $x \in (X,Y)_{\theta,p}$ in $(X,Y)_{\theta,p}$ as desired. $\square$

Since both X^* and Y^* are continuously embedded in $(X\cap Y)^*$, taking as the ambient Hausdorff space $\mathcal{V} = (X\cap Y)^*$ we see that (X^*,Y^*) is an interpolation couple. Therefore, the following characterization of duality of interpolation spaces is rather expected.

Theorem 3.2.7 ([26, Theorem 1.18]). *Let $1 \leqslant p < \infty$. If $X \cap Y$ is dense in X and Y, then for each $\theta \in (0,1)$ we have*

$$((X,Y)_{\theta,p})^* = (X^*,Y^*)_{\theta,p'}, \quad \textit{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

3.2.2 Complex interpolation

The construction of complex interpolation spaces is less intuitive than real interpolation, but nevertheless they will be very useful to treat fractional exponents of operators. Take the open strip S in the complex plane

$$S := \{z \in \mathbb{C} \,:\, 0 < \Re z < 1\},$$

and let $\overline{S}$ be its closure.

Definition 3.2.8. We denote by $\mathcal{F}(X,Y)$ the complex vector space of all functions $f : \overline{S} \to X+Y$ such that

(i) $f \in \mathcal{H}(S; X+Y) \cap C_b(\overline{S}; X+Y)$;
(ii) $t \mapsto f(it) \in C_b(\mathbb{R}; X)$, $t \mapsto f(1+it) \in C_b(\mathbb{R}; Y)$;

(iii)
$$\|f\|_{\mathcal{F}(X,Y)} := \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X,\ \sup_{t\in\mathbb{R}} \|f(1+it)\|_Y\} < \infty. \tag{3.7}$$

As expected, $\mathcal{F}(X,Y)$ is a Banach space endowed with the norm (3.7). However for its proof we will need the maximum principle for strips in general Banach spaces X. Remember that the scalar-valued maximum principle stated that if f is a nonzero holomorphic function on a bounded connected open subset Ω of the complex plane $\mathbb{C}$, continuous up to the boundary of Ω, taking complex values then $|f|$ attains its minimum value on the boundary of Ω. We state the generalization of the theorem for general complex Banach spaces in the following lemma.

Lemma 3.2.9 (Maximum modulus principle for strips)**.** *Let $f : \overline{S} \to X$ be holomorphic in S and continuous in $\overline{S}$. Then*

$$\|f(\xi)\|_X \leqslant \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_X\} \quad \textit{for every} \quad \xi \in \overline{S}.$$

Proof. We will start by proving it for a general bounded open connected subset Ω of the complex plane $\mathbb{C}$. The proof relies on writing the theorem in terms of a general scalar product. From the Riesz theorem we know that there exists $x^* \in X^*$ such that for every $\xi \in \overline{\Omega}$ we can write $\|f(\xi)\|_X = \langle f(\xi), x^*\rangle$ with $\|x^*\|_{X^*} = 1$. The complex function $z \mapsto \langle f(z), x^*\rangle$ is nonzero, holomorphic on Ω and continuous on $\overline{\Omega}$, we can thus apply the maximum principle to get that

$$\|f(\xi)\|_X = |\langle f(\xi), x^*\rangle| \leqslant \max_{z\in\partial\Omega} |\langle f(z), x^*\rangle| \leqslant \max_{z\in\partial\Omega} \|f(z)\|_X, \quad \xi \in \overline{\Omega}.$$

The proof for unbounded strips S follows analogously applying the Hadamard three-lines theorem instead. □

Proposition 3.2.10. *$\mathcal{F}(X,Y)$ is a Banach space endowed with the norm* (3.7)*.*

Proof. Take $(f_n)_{n\geqslant 1}$ a Cauchy sequence in $\mathcal{F}(X,Y)$. Let $\xi \in \overline{S}$, by the maximum modulus principle for strips 3.2.9 we have the bound

$$\begin{aligned}
\|f_n(\xi) - f_m(\xi)\|_{X+Y} &\leqslant \max\left\{\sup_{t\in\mathbb{R}} \|(f_n - f_m)(it)\|_{X+Y},\ \sup_{t\in\mathbb{R}} \|(f_n - f_m)(1+it)\|_{X+Y}\right\} \\
&\leqslant \max\left\{\sup_{t\in\mathbb{R}} \|(f_n - f_m)(it)\|_{X},\ \sup_{t\in\mathbb{R}} \|(f_n - f_m)(1+it)\|_{Y}\right\} \\
&= \|f_n(\xi) - f_m(\xi)\|_{\mathcal{F}(X,Y)}
\end{aligned}$$

Thus for every $\xi \in S$ there exists $f(\xi) = \lim_{n\to\infty} f_n(\xi)$ in $X+Y$. Since the limit is uniform $f \in \mathcal{H}(S; X+Y) \cap C_b(\overline{S}; X+Y)$ and consequently $f \in \mathcal{F}(X,Y)$. Moreover, $t \mapsto f_n(it)$ and $t \mapsto f_n(1+it)$ converge in $C_b(\mathbb{R}; X)$ and $C_b(\mathbb{R}; Y)$ respectively, so $f_n \to f$ in $\mathcal{F}(X,Y)$ as desired. □

Definition 3.2.11. Let $\theta \in [0,1]$. We define *complex interpolation spaces* as

$$[X,Y]_\theta := \{f(\theta) :\ f \in \mathcal{F}(X,Y)\},$$

endowed with the norm

$$\|x\|_{[X,Y]_\theta} := \inf_{\substack{f\in\mathcal{F}(X,Y)\\ f(\theta)=x}} \|f\|_{\mathcal{F}(X,Y)}. \tag{3.8}$$

They are indeed interpolation spaces between X and Y and the proof is analogous to the real case. We will start by proving that they are intermediate spaces.

Lemma 3.2.12. *Let $\theta \in (0,1)$, then*

$$X \cap Y \hookrightarrow [X,Y]_\theta \hookrightarrow X + Y.$$

Proof. For the first embedding consider $x \in X \cap Y$, the constant function $f(z) := x$ is obviously holomorphic in S and bounded in $\overline{S}$ with $t \mapsto f(it) \in C_b(\mathbb{R};X)$ and $t \mapsto f(1+it) \in C_b(\mathbb{R};Y)$, thus $f \in \mathcal{F}(X,Y)$ and $f(\theta) = x \in [X,Y]_\theta$ with

$$\|x\|_{[X,Y]_\theta} \leqslant \|f\|_{\mathcal{F}(X,Y)} \leqslant \max\{\|x\|_X, \|x\|_Y\} = \|x\|_{X\cap Y}.$$

For the second embedding take $x \in [X,Y]_\theta$, then by definition there exists $f \in \mathcal{F}(X,Y)$ such that $f(\theta) = x$. By the maximum modulus principle for strips 3.2.9:

$$\begin{aligned}\|x\|_{X+Y} \leqslant \|f(\theta)\|_{X+Y} &\leqslant \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X+Y}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X+Y}\}\\ &\leqslant \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_Y\} = \|f\|_{\mathcal{F}(X,Y)},\end{aligned}$$

taking te infimum of all admissible $f \in \mathcal{F}(X,Y)$ gives the continuous embedding $X+Y \subset [X,Y]_\theta$. □

Theorem 3.2.13. *Let $\theta \in (0,1)$. The spaces $[X,Y]_\theta$ are interpolation spaces.*

Proof. Consider a linear operator $T : X+Y \to X+Y$ such that $T\big|_X \in \mathcal{L}(X)$ and $T\big|_Y \in \mathcal{L}(Y)$ with norms

$$\|T\|_{\mathcal{L}(X)} = A_X \quad \text{and} \quad \|T\|_{\mathcal{L}(Y)} = A_Y.$$

Without loss of generality we can assume $A_X, A_Y \neq 0$. If $x \in [X,Y]_\theta$, then there exists $f \in \mathcal{F}(X,Y)$ such that $f(\theta) = x$. We are going to define $g : \overline{S} \to X+Y$ by

$$g(z) = \left(\frac{A_X}{A_Y}\right)^{z-\theta} Tf(z), \quad z \in \overline{S}.$$

Defined this way g is bounded in the boundary of the strip S:

$$\begin{aligned}\|g(it)\|_X &\leqslant \left\|\left(\frac{A_X}{A_Y}\right)^{it-\theta} Tf(it)\right\|_X \leqslant A_X^{-\theta}A_Y^{\theta}A_X\|f(it)\|_X = A_X^{1-\theta}A_Y^{\theta}\|f\|_X\\ \|g(1+it)\|_Y &\leqslant \left\|\left(\frac{A_X}{A_Y}\right)^{1+it-\theta} Tf(1+it)\right\|_Y \leqslant A_X^{1-\theta}A_Y^{\theta-1}A_Y\|f(1+it)\|_Y\\ &= A_X^{1-\theta}A_Y^{\theta}\|f(1+it)\|_Y\end{aligned} \tag{3.9}$$

Since $f \in \mathcal{F}(X,Y)$ and T is continuous, $g \in \mathcal{F}(X,Y)$. And the above norms imply that

$$\begin{aligned}\|g\|_{\mathcal{F}(X,Y)} &= \max\{\sup_{t\in\mathbb{R}} \|g(it)\|_X, \sup_{t\in\mathbb{R}} \|g(1+it)\|_Y\}\\ &\leqslant A_X^{1-\theta}A_Y^{\theta} \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_Y\} = A_X^{1-\theta}A_Y^{\theta}\|f\|_{\mathcal{F}(X,Y)}.\end{aligned}$$

Moreover, $g(\theta) = Tf(\theta) = Tx$, so $Tx \in [X,Y]_\theta$ with norm

$$\|Tx\|_{[X,Y]_\theta} = \inf_{\substack{f\in\mathcal{F}(X,Y)\\ f(\theta)=x}} \|f\|_{\mathcal{F}(X,Y)} \leqslant \|g\|_{\mathcal{F}(X,Y)} \leqslant A_X^{1-\theta} A_Y^{\theta} \|f\|_{\mathcal{F}(X,Y)}.$$

Taking the infimum over all $f \in \mathcal{F}(X,Y)$ we conclude that $T\big|_{[X,Y]_\theta} \in \mathcal{L}([X,Y]_\theta)$, with norm

$$\|T\|_{\mathcal{L}([X,Y]_\theta)} \leqslant A_X^{1-\theta} A_Y^{\theta}.$$

□

We can generalize the above proof by considering the domain and range of the bounded linear operator T belonging to two different interpolation spaces to obtain the following stronger statement.

Proposition 3.2.14. *Let (X_0, X_1) and (Y_0, Y_1) be two complex interpolation couples. If a linear operator $T : X_0+X_1 \to Y_0+Y_0$ belong to $\mathcal{L}(X_j, Y_j)$ for $j \in \{0,1\}$, then the restriction of T to the complex interpolation space $[X_0, X_1]_\theta$ belongs to $\mathcal{L}([X_0,X_1]_\theta, [Y_0,Y_1]_\theta)$ for every $\theta \in (0,1)$. Moreover,*

$$\|T\|_{\mathcal{L}([X_0,X_1]_\theta,[Y_0,Y_1]_\theta)} \leqslant (\|T\|_{\mathcal{L}(X_0,Y_0)})^{1-\theta} (\|T\|_{\mathcal{L}(X_1,Y_1)})^{\theta}.$$

Proof. Follow the proof of theorem 3.2.13 taking $A_X = \|T\|_{\mathcal{L}(X_0,Y_0)}$ and $A_Y = \|T\|_{\mathcal{L}(X_1,Y_1)}$ instead. □

We can now prove that complex interpolation spaces are Banach spaces.

Proposition 3.2.15. *Let $\theta \in (0,1)$. The complex interpolation space $[X,Y]_\theta$ is a Banach space endowed with the norm (3.8).*

Proof. Take an absolutely convergent sequence $(x_n)_{n\geqslant 1}$ in $[X,Y]_\theta$. For every x_n we can find $g_n \in \mathcal{F}(X,Y)$ such that $g_n(\theta) = x_n$ and $\|g_n\|_{\mathcal{F}(X,Y)} \leqslant \|x\|_{[X,Y]_\theta} + 2^{-n}$. Then $(g_n)_{n\geqslant 1}$ is absolutely convergent in $\mathcal{F}(X,Y)$ and let $g = \sum_{n=0}^\infty g_n$ be the limit. If we define $x := g(\theta)$, we have that

$$\left\| x - \sum_{n=0}^{\infty} x_n \right\|_{[X,Y]_\theta} \leqslant \left\| \sum_{n\geqslant N+1}^{\infty} g_n \right\|_{\mathcal{F}(X,Y)} \leqslant 2^{-N} + \sum_{n\geqslant N+1}^{\infty} \|x_n\|_{[X,Y]_\theta},$$

which from absolute convergence of $(x_n)_{n\geqslant 1}$ converges to 0 when $N \to \infty$. We conclude that $x = \sum_{n=0}^\infty x_n$ in $[X,Y]_\theta$. □

The following simple properties are a direct consequence of the definition of complex interpolation spaces and give us a flavor of how they work.

Proposition 3.2.16. *Let $\theta \in (0,1)$ and $0 < \theta_1 < \theta_2 < 1$, then*

(i) $[X,Y]_\theta = [Y,X]_{1-\theta}$;

(ii) *if $X = Y$, then* $[X,X]_\theta = X$;

(iii) *if $Y \subset X$, then* $[X,Y]_{\theta_2} \hookrightarrow [X,Y]_{\theta_1}$.

Proof. For (i) note that $f \in \mathcal{F}(X,Y)$ if and only if $f(1-\cdot) \in \mathcal{F}(Y,X)$, with equal norms

$$\begin{aligned}\|x\|_{[X,Y]_\theta} &= \inf_{\substack{f\in\mathcal{F}(X,Y)\\ f(\theta)=x}} \max\left\{\sup_{t\in\mathbb{R}}\|f(it)\|_X, \sup_{t\in\mathbb{R}}\|f(1+it)\|_Y\right\}\\ &= \inf_{\substack{f\in\mathcal{F}(Y,X)\\ f(1-\theta)=x}} \max\left\{\sup_{t\in\mathbb{R}}\|f(1-it)\|_X, \sup_{t\in\mathbb{R}}\|f(-it)\|_Y\right\} = \|x\|_{[Y,X]_{1-\theta}}.\end{aligned}$$

For (ii) we use the maximum modulus principle for strips 3.2.9. Take $x \in [X,X]_\theta$, then theres exists $f \in \mathcal{F}(X,X)$ such that $f(\theta) = x$ and

$$\|x\|_X = \|f(\theta)\|_X \leqslant \max\{\sup_{t\in\mathbb{R}}\|f(it)\|_X, \sup_{t\in\mathbb{R}}\|f(1+it)\|_X\} = \|f\|_{\mathcal{F}(X,X)},$$

taking the infimum of all such $f \in \mathcal{F}(X,X)$ we obtain $\|x\|_X \leqslant \|x\|_{[X,X]_\theta}$. For the reverse take $x \in X$, then the constant function $g : z \mapsto x$ with $z \in S$ is in $\mathcal{F}(X,X)$ with $g(\theta) = x$ so

$$\|x\|_{[X,X]_\theta} \leqslant \|g\|_{\mathcal{F}(X,X)} = \|x\|_X.$$

For (iii) we claim that if $X \subset Y$ then $[X,Y]_{\theta_1} \hookrightarrow [X,Y]_{\theta_2}$. Take $x \in [X,Y]_{\theta_1}$, then we can choose $f \in \mathcal{F}(X,Y)$ such that $f(\theta_1) = x$ and

$$\|f\|_{\mathcal{F}(X,Y)} \leqslant \|x\|_{[X,Y]_{\theta_1}} + \varepsilon.$$

Let $\lambda := \theta_1/\theta_2 < 1$, then $\theta_1 = \lambda\theta_2$ and we define the function

$$g(z) := f(\theta_2 z)e^{\varepsilon(z^2-\lambda^2)}.$$

We have that $g \in \mathcal{F}(X,[X,Y]_{\theta_2})$ since

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\|g(it)\|_X = \sup_{t\in\mathbb{R}}\|f(it)e^{\varepsilon(-t^2-\lambda^2)}\|_X \leqslant e^\varepsilon\|f\|_{\mathcal{F}(X,Y)},\\ &\sup_{t\in\mathbb{R}}\|g(1+it)\|_{[X,Y]_{\theta_2}} = \sup_{t\in\mathbb{R}}\|f(\theta_2+it)e^{\varepsilon((1+it)^2-\lambda^2)}\|_{[X,Y]_{\theta_2}} \leqslant e^\varepsilon\|f\|_{\mathcal{F}(X,Y)},\end{aligned}$$

thus

$$\|g\|_{\mathcal{F}(X,[X,Y]_{\theta_2})} \leqslant e^\varepsilon(\|x\|_{[X,Y]_{\theta_1}} + \varepsilon).$$

Note that if $X \subset Y$, then $X \hookrightarrow [X,Y]_{\theta_2}$ and consequently

$$[X,[X,Y]_{\theta_2}]_\lambda \hookrightarrow [[X,Y]_{\theta_2},[X,Y]_{\theta_2}]_\lambda = [X,Y]_{\theta_2}$$

where the last equality follows from (ii). Now since $g(\lambda) = f(\theta_1) = x$, the claim follows

$$\|x\|_{[X,Y]_{\theta_2}} \leqslant \|x\|_{[X,[X,Y]_{\theta_2}]_\lambda} \leqslant \|g\|_{\mathcal{F}(X,[X,Y]_{\theta_2})} \leqslant \|x\|_{[X,Y]_{\theta_1}}.$$

Finally, if $Y \subset X$ we can use the claim and (i) to prove the proposition

$$[X,Y]_{\theta_2} = [Y,X]_{1-\theta_2} \hookrightarrow [Y,X]_{1-\theta_1} = [X,Y]_{\theta_2}.$$

□

3.2.3 Interpolation with domains

To finish the section on interpolation we will include some tools to construct interpolation spaces with domains. They are included in a rather abstract setting following [23] in order to apply it for the construction of Bessel and Besov spaces in section 3.3, as well as to characterize the domains of the fractional Stokes operator, corollary 4.3.6, by retraction and coretraction arguments.

Definition 3.2.17. Let X and Y be two Banach spaces. An operator $R \in \mathcal{L}(X, Y)$ is said to be a *retraction* if there exists an operator $S \in \mathcal{L}(Y, X)$ such that

$$RS = I \quad \text{in} \quad \mathcal{L}(Y, Y),$$

holds. In this case, S is said to be a *coretraction.*

The following theorem will allow us to characterize unknown interpolation spaces in terms of known ones via retraction/coretraction arguments.

Theorem 3.2.18 ([23, Lemma 5.3])**.** *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples and let $X_\theta = [X_0, X_1]_\theta$ and $Y_\theta = [Y_0, Y_1]_\theta$ for a given $\theta \in (0,1)$. Let*

$$S \in \mathcal{L}((Y_0, Y_1), (X_0, X_1)) \quad \textit{and} \quad R \in \mathcal{L}((X_0, X_1), (Y_0, Y_1)),$$

be operators such that the restrictions $S \in \mathcal{L}(Y_j, X_j)$ are coretractions, with corresponding retractions $R \in \mathcal{L}(X_j, Y_j)$ and $RS = I$ on Y_j for $j \in \{0, 1\}$. Then SR defines a projection on X_θ and R is an isomorphism from $SR(X_\theta)$ onto Y_θ with inverse S. Moreover, the following estimates hold

$$C_S^{-1}\|Sy\|_{X_\theta} \leqslant \|y\|_{Y_\theta} \leqslant C_R\|Sy\|_{X_\theta}, \quad y \in Y_\theta,$$
$$C_R^{-1}\|Rx\|_{Y_\theta} \leqslant \|x\|_{X_\theta} \leqslant C_S\|Rx\|_{Y_\theta}, \quad x \in X_\theta$$

where $C_R = \max_{j\in\{0,1\}} \|R\|_{\mathcal{L}(X_j,Y_j)}$ and $C_S = \max_{j\in\{0,1\}} \|S\|_{\mathcal{L}(X_j,Y_j)}$.

Proof. Since $RS = I$ in both Y_0 and Y_1, from the definition of complex interpolation it is clear that $RS = I$ in Y_θ as well, and consequently $(SR)^2 = S(RS)R = SR$ is a projection. The upper bounds follow from proposition 3.2.14. To finish the proof it suffices to show the lower bounds, which for the coretraction S is an easy calculation

$$\|y\|_{Y_\theta} = \|RSy\|_{Y_\theta} \leqslant C_R\|Sy\|_{X_\theta}, \quad y \in Y_\theta,$$

and for the retraction R note that if $x := SRu \in SR(X_\theta)$, then

$$\|x\|_{X_\theta} = \|SRSRu\|_{X_\theta} \leqslant C_S\|RSRu\|_{Y_\theta} = C_S\|Rx\|_{Y_\theta}.$$

□

In particular, theorem 3.2.18 is very useful to treat interpolation spaces with boundary. We will start with some notation.

Definition 3.2.19. Let $F \hookrightarrow \mathcal{D}'(\mathbb{R}^n; X)$ be a Banach space. We define the *factor space* to an open set $\Omega \subseteq \mathbb{R}^n$ as

$$F(\Omega) := \{f \in \mathcal{D}'(\mathbb{R}^n; X) \ : \ \exists g \in F, \ f = g\big|_\Omega\},$$

and the norm

$$\|f\|_{F(\Omega)} := \inf\{\|g\|_F \ : \ g\big|_\Omega = f\}.$$

Definition 3.2.20. We say that E is an *extension operator for* $F(\Omega)$ if
i) for all $f \in F(\Omega)$, $(Ef)\big|_\Omega = f$;
ii) $E : F(\Omega) \to F$ is bounded.

We can now prove the main theorem of this section.

Theorem 3.2.21 ([23, Lemma 5.4])**.** *Let $F^0, F^1 \hookrightarrow \mathcal{D}'(\mathbb{R}^n; X)$ be two Banach spaces. For $\theta \in (0,1)$ let*

$$F^\theta = [F^0, F^1]_\theta.$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and define the factor $F^\theta(\Omega)$ as in definition 3.2.19. If there exists an extension operator E for $F^s(\Omega)$ for $s \in \{0,1\}$, then

$$[F^0(\Omega), F^1(\Omega)]_\theta = F^\theta(\Omega)$$

and there exists C only depending on the norms of the extension operator such that

$$C^{-1}\|f\|_{F^\theta(\Omega)} \leqslant \|f\|_{[F^0(\Omega),F^1(\Omega)]_\theta} \leqslant \|f\|_{F^\theta(\Omega)}.$$

Moreover, E is an extension operator for $F^\theta(\Omega)$.

Proof. The exists an extension operator E for $F^s(\Omega)$ for $s \in \{0,1\}$ by hypothesis, hence we can define the retraction $R : F^s \to F^s(\Omega)$ by $Rf = f\big|_\Omega$ and corresponding coretraction $S : F^s(\Omega) \to F^s$ by $S = E$. Both operators are bounded by construction with $\|R\| \leqslant 1$ and $\|S\| \leqslant C$, where C is the boundedness constant of the extension operator. From theorem 3.2.18 we get that if $f \in [F^0(\Omega), F^1(\Omega)]_\theta$, then

$$C^{-1}\|f\|_{F^\theta(\Omega)} \leqslant C^{-1}\|Ef\|_{F^\theta} \leqslant \|f\|_{[F^0(\Omega),F^1(\Omega)]_\theta}.$$

Conversely, if $f \in F^\theta(\Omega)$ we can choose $g \in F^\theta$ such that $Rg = g\big|_\theta = f$ and we get

$$\|f\|_{[F^0(\Omega),F^1(\Omega)]_\theta} \leqslant \|g\|_{[F^0,F^1]_\theta} \leqslant \|g\|_{F^\theta},$$

taking the infimum over all g the inequality follows. To prove the assertion note that the extension operator $E : F^\theta(\Omega) \to F^\theta$ is bounded by the above inequalities. Moreover, for $f \in F^0(\Omega) \cap F^1(\Omega)$ the extension operator $(Ef)\big|_\Omega = f$ by hypothesis, and this extends to all $F^\theta(\Omega)$ by density [33, Theorem 1.9.3]. ☐

3.3 Function spaces

We assume the reader is familiar with distributions and Fourier transform results on the scalar-valued setting, for a general reference consult [9].

Recall that the derivative of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, with respect to a multiindex $\alpha \in \mathbb{N}^n$, is defined via duality

$$\langle \varphi, \partial^\alpha f\rangle = (-1)^{|\alpha|}\langle \partial^\alpha \varphi, f\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

A tempered distribution f is called *regular* if there exists a measurable function g such that

$$\langle \varphi, f\rangle = \int_{\mathbb{R}^n} g(x)\varphi(x)\,\mathrm{d}x, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

In particular, every $f \in L^1(\mathbb{R}^n)$ defines a regular distribution u_f setting

$$\langle \varphi, u_f \rangle := \int_{\mathbb{R}^n} f(x)\varphi(x)\,\mathrm{d}x.$$

If there is no room for confusion we will denote u_f by f. Finally, if the distributional derivative of a tempered distribution is regular, we call $\partial^\alpha f$ the *weak derivative* of f.

Definition 3.3.1. Let $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. We define the *Sobolev space* of m-times weakly differentiable, L^p-integrable functions in $\mathbb{R}^n$ as

$$W^{m,p}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) \,:\, \partial^\alpha f \in L^p(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}^n \,:\, |\alpha| \leqslant m\}.$$

It is a Banach space equipped with the norm

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} = \sum_{|\alpha| \leqslant m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

We can define Sobolev spaces in smooth domains $\Omega \subseteq \mathbb{R}^n$ directly by substituting the tempered distributions for general distributions

$$W^{m,p}(\Omega) := \{f \in \mathcal{D}'(\Omega) \,:\, \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^n \,:\, |\alpha| \leqslant m\}. \tag{3.10}$$

However, in the hydrostatic Stokes equation we will be treating a non-smooth domain where we are interested in requiring boundary conditions to our functions. With this in mind, it is simpler to define our function spaces in domains through extension/restriction operators.

Definition 3.3.2. Let $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define the Sobolev space of m-times weakly differentiable, L^p-integrable functions in Ω as

$$W^{m,p}(\Omega) := \{f \in \mathcal{D}'(\Omega) \,:\, \exists g \in W^{m,p}(\mathbb{R}^n) \text{ with } g\big|_\Omega = f\},$$

where the extension is taken in the distributional sense.

Although for bounded C^∞-domains both definitions coincide for Sobolev spaces, for more complicated spaces we cannot expect an inner description of the type (3.10) to exist [34, Section 3.1.2]. Every smooth compactly supported function has clearly an extension to the real space, and we have the embeddings

$$\mathcal{D}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

The completion of smooth functions $C_c^\infty(\Omega)$ on the $\|\cdot\|_{W^{m,p}(\Omega)}$-norm provides us with the space of Sobolev functions with trace zero

$$W_0^{m,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}},$$

which is in general different from $W^{m,p}(\Omega)$. This notation allows us to define negative order Sobolev spaces via duality in the following way.

Definition 3.3.3. Let $m \in \mathbb{N}$ and $1 \leqslant p < \infty$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define the Sobolev space of order $-m$ and integrability p' on Ω as

$$W^{-m,p'} := (W_0^{m,p}(\Omega))^*, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

Definition 3.3.4. Let $m \in \mathbb{N}$, $s \in (m, m+1)$ and $1 \leqslant p < \infty$. We define the fractional Sobolev space or the *Sobolev-Slobodeckij* space $W^{s,p}$ on $\mathbb{R}^n$ as the real interpolation space

$$W^{s,p}(\mathbb{R}^n) := (W^{m,p}(\mathbb{R}^n), W^{m+1,p}(\mathbb{R}^n))_{\theta,p} \quad \text{with } \theta = s - m.$$

We can define Sobolev-Slobodeckij spaces in domains through factors as in theorem 3.2.21 as long an extension exists for $W^{m,p}(\mathbb{R}^n)$ and $W^{m+1,p}(\mathbb{R}^n)$. Note that this characterization coincides with definition 3.3.2. We can take a complex interpolation instead of a real one and define the following space.

Definition 3.3.5. Let $m \in \mathbb{N}$, $\theta \in (0,1)$ and $1 \leqslant p \leqslant \infty$. We define the *Bessel potential space* of order $s = m + \theta$ and integrability p in $\mathbb{R}^n$ as the complex interpolation space

$$H^{s,p}(\mathbb{R}^n) := [W^{m,p}(\mathbb{R}^n), W^{m+1,p}(\mathbb{R}^n)]_\theta.$$

As for Sobolev-Slobodeckij functions, we can define Bessel potential spaces in domains through extension operators. Moreover, if the domain is *good enough* we can impose boundary conditions.

Definition 3.3.6. Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞-domain. Let $\{B_j\}_{j=0}^k$ be a finite family of $k \in \mathbb{N}$ differential operators on $\partial\Omega$ such that

$$B_j f(x) = \sum_{|\alpha| \leqslant j} b_{j,\alpha}(x)(\partial^\alpha f)\big|_{\partial\Omega}, \quad \text{for} \quad j = 0, 1, ...k, \tag{3.11}$$

where $b_{j,\alpha} : \partial\Omega \to \mathbb{R}$. For $s > 0$ and $1 < p < \infty$ the *Bessel potential spaces with boundary conditions* (3.11) are defined as

$$H^{s,p}_{\{B_j\}}(\Omega) := \left\{ f \in H^{s,p}(\Omega) \ : \ B_j f\big|_{\partial\Omega} = 0 \quad \text{for} \quad j + \frac{1}{p} < s \right\}. \tag{3.12}$$

For the proper meaning of trace in Bessel spaces we refer the reader to section 3.4.

Following Hieber's notation [15], a smooth function $f : [0,1]^n \to \mathbb{R}$ is said to be *periodic of order* m if

$$\frac{\partial^\alpha f}{\partial x^\alpha}(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) = \frac{\partial^\alpha f}{\partial x^\alpha}(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_n),$$

for every $x_i' = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in [0,1]^{n-1}$, all $|\alpha| \leqslant m$ and $1 \leqslant i \leqslant n$. However, with the above notation we can extend periodicity to Bessel functions by choosing an adequate family $\{B_j\}_{j=0}^k$ of differential operators on $\partial\Omega$.

Definition 3.3.7. A Bessel function $f \in H^{s,p}([0,1]^n)$ is said to be *periodic of order* $\lfloor s \rfloor$ if $f \in H^{s,p}_B([0,1]^n)$ where

$$B = \{(\partial^\alpha f)\big|_{\{x_i=0\}} - (\partial^\alpha f)\big|_{\{x_i=1\}}, \ i = 1, 2, \ldots, n, \ |\alpha| < 2s\}.$$

Finally, the complex interpolation of Bessel functions on domains with boundary conditions is again a Bessel function as expected.

Theorem 3.3.8 ([33, Section 4.3.3])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞-domain. Let the family $\{B_j\}_{j=1}^k$ be as in definition 3.3.6. Let $m > k$, $1 < p < \infty$ and $\theta \in (0,1)$. Then*

$$[L^p(\Omega), H^{m,p}_{\{B_j\}}(\Omega)]_\theta = H^{\theta m,p}_{\{B_j\}}(\Omega), \quad j + \frac{1}{p} < m\theta.$$

Definition 3.3.9. Let $m \in \mathbb{N}$, $s \in (m, m+1)$ and $1 \leqslant p, q < \infty$. We define the *Besov space* $B^s_{p,q}$ on $\mathbb{R}^n$ as the real interpolation space

$$B^s_{p,q}(\mathbb{R}^n) := (W^{m,p}(\mathbb{R}^n), W^{m+1,p}(\mathbb{R}^n))_{\theta,q} \quad \text{with } \theta = s - m.$$

Moreover, we can define them for negative exponents as

$$B^{-s}_{p',q'}(\mathbb{R}^n) := (B^s_{p,q}(\mathbb{R}^n))^*.$$

Attending to theorem 3.2.7, we can rewrite the dual spaces of real interpolation spaces by developing

$$\begin{aligned} B^{-s}_{p',q'}(\mathbb{R}^n) &= \Big[(W^{m,p'}(\mathbb{R}^n), W^{m+1,p'}(\mathbb{R}^n))_{\theta,q'}\Big]^* \\ &= \Big((W^{m,p'}(\mathbb{R}^n))^*, (W^{m+1,p'}(\mathbb{R}^n))^*\Big)_{\theta,q} \\ &= (W^{-m,p}(\mathbb{R}^n), W^{-m-1,p}(\mathbb{R}^n))_{\theta,q} \end{aligned} \tag{3.13}$$

Therefore, for $m \in \mathbb{N}$, $s \in (m, m+1)$ and $1 \leqslant p, q < \infty$, the Besov space of order $-s < 0$ in $\Omega \subseteq \mathbb{R}^n$ is

$$B^{-s}_{p',q'}(\Omega) = (W^{-m,p}(\Omega), W^{-m-1,p}(\Omega))_{\theta,q} \quad \text{with } \theta = s - m,$$

as long as extension operators exist for $W^{-m,p}(\Omega)$ and $W^{-m-1,p}(\Omega)$.

3.4 Traces

Given a bounded Lipschitz domain Ω, boundary values of smooth functions can be pointwise defined as continuous functions in $\partial\Omega$. However, for L^p-functions the restriction to the boundary does not make sense in the usual way. In the following chapter we are going to introduce a short overview of how functional analytical methods can be employed to overcome the problems of defining traces of Sobolev spaces. Although the text is largely based in Sohr's book [31] difficult proofs of important theorems, avoided in the text for simplicity, are cited properly through the chapter.

We start by extending the concept of trace for continuous functions to Sobolev functions.

Theorem 3.4.1 ([1, Theorem 7.39])**.** *Let $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a bounded Lipschitz domain. Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then there exists a bounded and surjective operator*

$$\begin{array}{rccc} \Gamma: & W^{m,p}(\Omega) & \to & B^{m-\frac{1}{p}}_{p,p}(\partial\Omega) \\ & u & \mapsto & \Gamma u \end{array} \tag{3.14}$$

such that

$$\Gamma u = u\big|_{\partial\Omega} \quad \text{for all } u \in C^\infty(\overline{\Omega}). \tag{3.15}$$

We call this operator the trace operator.

Notation (3.15) is used for all $u \in W^{m,p}(\Omega)$ as long as it does not lead to confusion. In particular, surjectivity of the trace operator implies that for every $g \in B^{m-\frac{1}{p}}_{p,p}(\partial\Omega)$

there exists at least one $u \in W^{m,p}(\Omega)$ such that $g = u\big|_{\partial\Omega}$. Furthermore, $u \in W^{m,p}(\Omega)$ can be chosen in a way such that the map

$$\begin{array}{rccl} E: & B^{m-\frac{1}{p}}_{p,p}(\partial\Omega) & \to & W^{m,p}(\Omega) \\ & g & \mapsto & u \qquad \text{with } g = u\big|_{\partial\Omega} \end{array} \tag{3.16}$$

is bounded. This map is called *extension operator* E, which by definition has the property

$$\Gamma E g = g.$$

Note that this is consistent with definition 3.2.20, and consequently the interpolation spaces in domains from section 3.3 are well defined.

Remark 2. The trace operator allows us to generalize Green's theorem initially given for $u \in C^\infty(\overline{\Omega})$ and $v \in C^\infty(\overline{\Omega})^n$:

$$(u|\operatorname{div} v)_\Omega = (u|\nu_{\partial\Omega} \cdot v)_{\partial\Omega} - (\nabla u|v)_\Omega, \tag{3.17}$$

where $\nu_{\partial\Omega}$ is the outer unit normal on $\partial\Omega$. Using the density of smooth functions and the continuity of the trace operator, we can extend the equation to $u \in W^{m,p}(\Omega)$ and $v \in W^{m,p'}(\Omega)^n$, with $p' = \frac{p}{p-1}$, so that $(u|\nu_{\partial\Omega} \cdot v)_{\partial\Omega}$ is still well defined as a surface integral with

$$u\big|_{\partial\Omega} \in B^{m-\frac{1}{p}}_{p,p}(\partial\Omega) \quad \text{and} \quad \nu \cdot v_{\partial\Omega}\big|_{\partial\Omega} \in B^{m-\frac{1}{p'}}_{p',p'}(\partial\Omega). \tag{3.18}$$

The trace theorem generalizes to Besov spaces following the same technique as for Sobolev spaces.

Theorem 3.4.2 ([1, Theorem 7.43])**.** *Let $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a bounded Lipschitz domain. Let $1 < p, q < \infty$ and $s > 1/p$. Then there exists a bounded and surjective operator*

$$\begin{array}{rccc} \Gamma: & B^s_{p,q}(\Omega) & \to & B^{s-\frac{1}{p}}_{p,q}(\partial\Omega) \\ & u & \mapsto & \Gamma u \end{array} \tag{3.19}$$

such that

$$\Gamma u = u\big|_{\partial\Omega} \quad \textit{for all } u \in C^\infty(\overline{\Omega}). \tag{3.20}$$

However, when treating the hydrostatic Stokes operator we are going to deal with regular distributions, meaning that we are interested in extending the Green's theorem further. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and define

$$E_p(\Omega) := \{v \in L^p(\Omega)^n \ : \ \operatorname{div} v \in L^p(\Omega)\},$$

where the divergence is taken in the distributional sense. $E_p(\Omega)$ is a Banach space equipped with the norm

$$\|v\|_{E_p(\Omega)} := \left(\|v\|_p^p + \|\operatorname{div} v\|_p^p\right)^{1/p},$$

and clearly $W^{1,p}(\Omega)^n \subseteq E_p(\Omega)$. We can define a generalized trace for functions in $E_p(\Omega)$. Let $u = Eg \in W^{1,p'}(\Omega)$ such that $u\big|_{\partial\Omega} = g \in B^{1-\frac{1}{p'}}_{p',p'}(\partial\Omega)$ and $v \in W^{1,p}(\Omega)^n$. Substituting

u and v in (3.17), using Hölder's inequality and the continuity of the extension operator we get that

$$\begin{aligned} |(u|\nu_{\partial\Omega} \cdot v)_{\partial\Omega}| &\leqslant \|Eg\|_{p'}\| \operatorname{div} v\|_p + \|\nabla Eg\|_{p'}\|v\|_p \leqslant \|Eg\|_{W^{1,p'}(\Omega)}(\|v\|_p + \| \operatorname{div} v\|_p) \\ &\leqslant C\|g\|_{B^{1-\frac{1}{p'}}_{p',p'}(\partial\Omega)} \|v\|_{E_p(\Omega)}. \end{aligned} \tag{3.21}$$

We can therefore see $u \mapsto (u|\nu_{\partial\Omega} \cdot v)_{\partial\Omega}$ as a continuous functional in $B^{1-\frac{1}{p'}}_{p',p'}(\partial\Omega)$ for every $v \in W^{1,p}(\Omega)$. In other words, $(\cdot|\nu_{\partial\Omega} \cdot v)_{\partial\Omega} \in B^{-1/p}_{p,p}(\partial\Omega)$ for every $v \in W^{1,p}(\Omega)$. Moreover, from equation (3.21) we can deduce that

$$\begin{array}{ccc} W^{1,p}(\Omega) & \to & B^{-1/p}_{p,p}(\partial\Omega) \\ v & \mapsto & (\cdot|\nu_{\partial\Omega} \cdot v)_{\partial\Omega} \end{array} \tag{3.22}$$

is continuous in the $\| \cdot \|_{E_p(\Omega)}$ norm.

Since smooth functions are dense in $E_p(\Omega)$ we can extend the map (3.22) to conclude that there exists a generalized trace operator

$$\Gamma_\nu : E_p(\Omega) \to B^{-1/p}_{p,p}(\partial\Omega)$$

such that

$$\Gamma_\nu v = (\cdot|\nu_{\partial\Omega} \cdot v)_{\partial\Omega} \quad \text{for } v \in C^\infty(\overline{\Omega})^n.$$

We will once again make use of the notation $\Gamma_\nu v = \nu_{\partial\Omega} \cdot v\big|_{\partial\Omega}$ for every $v \in E_p(\Omega)$.

Theorem 3.4.3 ([31, Lemma 1.2.3], Green's generalized theorem)**.** *Let $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a bounded Lipschitz domain with boundary $\partial\Omega$. Let $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Then for all $u \in W^{1,p}(\Omega)$ and $v \in E_{p'}(\Omega)$,*

$$(u| \operatorname{div} v)_\Omega = (u|\nu_{\partial\Omega} \cdot v)_{\partial\Omega} - (\nabla u|v)_\Omega, \tag{3.23}$$

where $(u|\nu_{\partial\Omega} \cdot v)_{\partial\Omega}$ is well defined in the sense of generalized trace with

$$\nu_{\partial\Omega} \cdot v\big|_{\partial\Omega} \in B^{-1/p'}_{p',p'}(\partial\Omega) \quad \text{and} \quad u\big|_{\partial\Omega} \in B^{1-\frac{1}{p}}_{p,p}(\partial\Omega).$$

3.5 Operator semigroups

Given a Banach space X and an unbounded linear operator $A : D(A) \subseteq X \to X$, we are interested in solving the abstract Cauchy problem

$$(ACP) \begin{cases} u'(t) = Au(t), & t \in [0,T], \\ u(0) = x & , \end{cases} \tag{3.24}$$

where $x \in X$ is the initial value and $u : [0,T] \to X$ the unknown solution. In this section, we will build intuition for the framework of the solutions starting from the scalar field. The presented notes are a summary of the classical work by Engel and Nagel [5] combined with [4] and [19].

The problem (3.24) has an easy answer in the scalar field $X = \mathbb{C}$ if the associated operator A is a matrix $A \in \mathcal{M}_n(\mathbb{C})$. Here the unique solution is given by the matrix exponential

$$e^{tA} := \sum_{k=1}^{N} \frac{t^k A^k}{k!},$$

which is well-defined because the truncated sums of the series form a Cauchy sequence. Moreover, the map $t \to e^{tA}$ has some interesting properties:

i) for $t \geqslant 0$, $t \to e^{tA}$ is continuous;

ii) for $t, s \geqslant 0$, $t \to e^{tA}$ satisfies the *semigroup properties*, i.e.

$$e^{0A} = I \quad \text{and} \quad e^{(t+s)A} = e^{tA} e^{sA}. \tag{3.25}$$

Attending to the second property, the family of bounded operators $(e^{tA})_{t \geqslant 0}$ is called the *semigroup generated by the matrix A*. Let us now study the behavior of the solution at infinity. A continuous semigroup $(e^{tA})_{t \geqslant 0}$ is called *stable* if

$$\lim_{t \to \infty} \|e^{tA}\| = 0.$$

And Liapunov's theorem [5, Theorem 2.10] states that for continuous semigroups generated by matrices $A \in \mathcal{M}_n(\mathbb{C})$, stability is equivalent to all eigenvalues of A having negative real part. In other words, we can characterize the stability of a semigroup through the spectral properties of its generator.

Consequently, it is not surprising that spectral theory plays a big role in the study of solutions of abstract Cauchy problems in general Banach spaces X. We will start by recalling some notions.

Definition 3.5.1. We call *resolvent set* of A, denoted $\rho(A)$, to the set of complex scalars $\lambda \in \mathbb{C}$ such that $\lambda - A$ has a bounded two-sided inverse, i.e. there exists a bounded linear operator B on X such that $Bx \in D(A)$ for all $x \in X$ and

$$(\lambda - A)B = B(\lambda - A) = I.$$

In this case, we call B the *resolvent operator* associated with A and we write

$$R(\lambda, A) := B = (\lambda - A)^{-1}.$$

The *spectrum* of A, denoted $\sigma(A)$, is defined as $\sigma(A) := \mathbb{C} \backslash \rho(A)$.

The following basic properties of spectrums and resolvents of unbounded operators can be found in any introductory book to Functional Analysis, see for instance [30, Chapters 12-13]. We gather them here without proof for the sake of clarity.

Proposition 3.5.2. *Let A be an unbounded operator in X.*

i) If the resolvent set $\rho(A)$ is nonempty, then A is closed. Recall that an unbounded operator is closed if whenever $x_n \to x$ in X and $Ax_n \to y$ in Y, then $x \in D(A)$ and $Ax = y$. In particular, every bounded operator A is closed.

ii) The resolvent set $\rho(A)$ is open.

iii) The resolvent identity holds, i.e. if $\lambda, \mu \in \rho(A)$, then

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A).$$

iv) If A is bounded, its spectrum is contained in the open ball $B(0, \|A\|)$.

For general Banach spaces X and linear bounded operators A, we can still define exponentials through the Dunford functional-calculus for holomorphic functions. More precisely, by the last property of proposition 3.5.2 there exists a bounded open set $\Omega \subseteq \mathbb{C}$ such that the spectrum of A is contained in Ω, $\sigma(A) \subseteq \Omega$, and a suitable contour Γ in Ω with winding number one around every point of the spectrum $\sigma(A)$, such that the integral

$$e^{tA} := \frac{1}{2\pi i} \int_\Gamma e^{tz} R(z, A)\, \mathrm{d}z \tag{3.26}$$

converges. Moreover, it defines a bounded operator on X and its definition does not depend on the particular choice of contour Γ. Recall that Dunford functional-calculus is actually well-defined for every holomorphic function in the open set Ω, $f \in H(\Omega)$, through

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, \mathrm{d}z.$$

It is an easy calculation to see that the matrix exponential defined this way is a solution of the abstract Cauchy problem (3.24).

Proposition 3.5.3 ([5, Proposition 3.5])**.** *Let A be a bounded operator on a Banach space X. Let $(e^{tA})_{t\geqslant 0}$ be a a family of operators defined by equation* (3.26)*. Then*

i) the family $(e^{tA})_{t\geqslant 0}$ is a uniformly continuous semigroup, i.e. has the semigroup properties (3.25) *and $t \to e^{tA}$ is continuous with respect to the operator topology $(\mathcal{L}(X), \|\cdot\|)$;*

ii) the mapping $t \to e^{tA}$ is differentiable and satisfies the differential equation

$$\partial_t e^{tA} = A e^{tA}, \quad t \geqslant 0.$$

Furthermore, the converse is also true. Every uniformly continuous semigroup is of the form $(e^{tA})_{t\geqslant 0}$ for some bounded operator $A \in \mathcal{L}(X)$, determined uniquely by the derivative of the semigroup at $t = 0$. In conclusion, if A is a bounded, the mapping $t \to e^{tA}$ is actually the *unique solution* to the abstract Cauchy problem 3.24.

Stability in general Banach spaces is characterized through the following notion.

Definition 3.5.4. A semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X is called *uniformly exponentially stable* is there exists constants $\varepsilon > 0$, $M \geqslant 1$ such that

$$\|T(t)\| \leqslant M e^{-\varepsilon t}, \quad t \geqslant 0.$$

For uniformly continuous semigroups, uniform exponential stability is equivalent to

$$\lim_{t\to\infty} \|T(t)\| = 0.$$

Finally, we would like to generalize solutions for unbounded operators A. Since uniformly continuous semigroups uniquely characterize solutions to the abstract Cauchy problem (3.24) with bounded generators A, we need a weaker notion of semigroup for our purpose.

Definition 3.5.5. A family $(T(t))_{t\geqslant 0}$ of bounded linear operators on a Banach space X is called a *strongly continuous semigroup* or *C_0-semigroup* if it satisfies the following properties:

i) $T(0) = I$;
ii) $T(t)T(s) = T(t+s)$ for all $t, s \geqslant 0$;
iii) $\lim_{t\to 0} \|T(t)x - x\| = 0$ for all $x \in X$.

The generator of the family $(T(t))_{t\geqslant 0}$ is the linear operator $(A, D(A))$ given by

$$D(A) := \{x \in X \ : \ \lim_{t\to 0} \frac{1}{t}(T(t)x - x) \text{ exists in } X\}, \tag{3.27}$$

$$Ax := \lim_{t\to 0} \frac{1}{t}(T(t)x - x), \quad x \in D(A). \tag{3.28}$$

It turns out that the generator $(A, D(A))$ is a closed and densely defined operator uniquely determined by the C_0-semigroup. Moreover, strong continuity implies continuously differentiability of the orbits $t \mapsto T(t)x$ if $x \in D(A)$, with

$$T(t)x \in D(A) \quad \text{and} \quad \partial_t T(t)x = AT(t)x = T(t)Ax, \quad t \geqslant 0.$$

Consequently, for initial values $x \in D(A)$ the function

$$u(t) := T(t)x, \quad t \geqslant 0,$$

solves the abstract Cauchy problem (3.24).

There is a special class of unbounded operators for which the respective C_0-semigroup can be characterized as a general form of the Dunford integral (3.26). Given $\omega \in (0, \pi)$, a *sector* of angle ω is given by

$$\Sigma_\omega := \{x \in \mathbb{C}\backslash\{0\} \ : \ |\arg(z)| < \omega\},$$

where the argument is taken in $(-\pi, \pi)$.

Definition 3.5.6. A linear operator $(A, D(A))$ is called *sectorial of angle* ω if $\sigma(A) \subseteq \overline{\Sigma_\omega}$ and

$$M_\omega := \sup_{z \in \mathbb{C}\overline{\Sigma_\omega}} \|zR(z, A)\| < \infty.$$

We will also use the notation

$$C_\omega := \sup_{z \in \mathbb{C}\overline{\Sigma_\omega}} \|AR(z, A)\|.$$

We call A *sectorial* if it is sectorial for some angle $\omega \in (0, \pi)$. The infimum ω for which A is ω-sectorial is called the *angle of sectoriality* of A and denoted by $\omega(A)$.

Now the exponential function can be properly defined for the ω-sectorial operator A by the Cauchy integral

$$e^{zA} := \frac{1}{2\pi i} \int_\Gamma e^{z\lambda} R(\lambda, A) \,\mathrm{d}\lambda, \tag{3.29}$$

where the contour Γ is taken as the boundary of a sector Σ_ν for some $\nu \in (\omega, \pi)$.

Definition 3.5.7. A C_0-semigroup $(T(t))_{t\geqslant 0}$ of bounded linear operators on a Banach space X is called *analytic on the sector* Σ_ω if for all $x \in X$, the function $t \mapsto T(t)x$ extends analytically to Σ_ω and satisfies

$$\lim_{\substack{z \in \Sigma_\omega \\ z \to 0}} T(z)x = x.$$

We call $(T(t))_{t\geqslant 0}$ an *analytic C_0-semigroup* if $(T(t))_{t\geqslant 0}$ is analytic on Σ_ω for some $\omega \in (0, \pi)$. Moreover, if the family $(T(t))_{t\geqslant 0}$ is uniformly bounded we call it a *bounded analytic C_0-semigroup*.

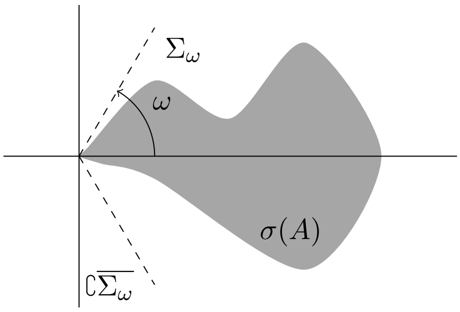


Figure 3.1: Spectrum of a sectorial operator.

It is a simple matter of checking that the exponentials defined by equation (3.29) form a bounded analytic C_0-semigroup $(e^{tA})_{t\geqslant 0}$, see [5, Proposition 4.3].

We may now state the main theorem of this chapter, it gives a characterization of bounded analytic C_0-semigroups fundamental to solve the hydrostatic Stokes equation.

Theorem 3.5.8 ([19, Theorem G.5.2]). *For a closed and densely defined operator A on a Banach space X the following are equivalent:*
(1) A is sectorial of angle $\omega \in (0, \pi/2)$;
(2) $-A$ generates a bounded analytic C_0-semigroup $(T(t))_{t\geqslant 0}$ on Σ_ω.

3.6 R-boundedness and bounded H^∞-calculus

In this section we are going to develop a functional calculus for sectorial operators taking the Dunford functional calculus as inspiration. The main problem is that the Cauchy integral defined for the matrix exponential in (3.29) does not necessarily give a bounded operator if we take an arbitrary bounded holomorphic function $f \in H^\infty(\Sigma_\omega)$ instead of the exponential. Furthermore, the space of functions such that $f(A)$ is a bounded linear operator does not admit an explicit characterization. Here we are going to limit ourselves to the definitions of the prerequisites to understand the H^∞-calculus and some immediate properties, but the full construction is carried out in [19] and [22].

In order to ensure the convergence of the Dunford integral, the first step is to restrict ourselves to a smaller class of functions that have certain decay properties on zero and infinity.

Definition 3.6.1. Let $1 \leqslant p \leqslant \infty$ and $\sigma \in (0, \pi)$. We define the *Hardy space of order p* as:

$$H^p(\Sigma_\sigma) := \{f : \Sigma_\sigma \to \mathbb{C} \ : \ \|f\|_{H^p(\Sigma_\sigma)} = \sup_{|\nu|<\sigma} \|t \mapsto f(e^{i\nu}t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} < \infty\}.$$

For functions $f \in H^1(\Sigma_\sigma)$ is easy to see that the operator $f(A)$ defined by the generalized Dunford integral

$$f(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(z)R(z,A)\,\mathrm{d}z, \quad \nu \in (\omega(A), \sigma),$$

is bounded. Take $\nu \in (\omega(A), \sigma)$ arbitrary and consider the contour $\partial\Sigma_\nu$ oriented "downwards" such that the spectrum of A is on the left-hand side, then we can bound the integral by

$$\|f(A)\| \leqslant \frac{M_\nu}{2\pi}\int_{\partial\Sigma_\nu} |f(z)|\frac{|dz|}{|z|} \leqslant \frac{M_\nu}{2\pi}\|f\|_{H^1(\Sigma_\sigma)}.$$

Taking the infimum over all $\nu \in (\omega(A), \sigma)$ we get that the operator $f(A)$ is bounded

$$\|f(A)\| \leqslant \frac{M_\sigma}{2\pi}\|f\|_{H^1(\Sigma_\sigma)}.$$

The next step is to restrict ourselves to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and extend the calculus to $H^\infty(\Sigma_\sigma)$ by approximation arguments, if this extension is possible depends on the properties of the sectorial operator A.

Definition 3.6.2. Let A be a sectorial operator on X and $\sigma \in \omega(A), \pi)$. Then A is said to admit a *bounded $H^\infty(\Sigma_\sigma)$-calculus* if there exists a constant $C \geqslant 0$ such that

$$\|f(A)\| \leqslant C\|f\|_\infty, \quad f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma).$$

We define the angle of H^∞-boundedness as

$$\omega_{H^\infty}(A) := \inf\{\sigma \in (\omega(A), \pi) : \text{ A has bounded } H^\infty(\Sigma_\sigma)\text{-calculus}\}.$$

Finally, we say that A admits a *bounded H^∞-calculus* if there exists $\sigma \in (\omega(A), \pi)$ such that A admits a bounded $H^\infty(\Sigma_\sigma)$-calculus.

The following couple of examples are going to help us understand the H^∞-functional calculus and provide the basis to prove that the Laplacian in chapter 4 admits a bounded H^∞-calculus.

Proposition 3.6.3 ([19, Proposition 10.2.22])**.** *Let $\sigma \in (0, \pi)$ and $1 \leqslant p < \infty$. Let $(X, \mathscr{A}, \mu)$ be a finite measure space and $m : X \to \mathbb{C}$ a measurable function taking values in Σ_σ μ-almost everywhere. Consider the pointwise multiplication operator M_m on $L^p(X)$ given by*

$$M_m(\phi)(x) := m(x)\phi(x), \quad \phi \in L^p(X),\ x \in X \tag{3.30}$$

$$D(M_m) := \{\phi \in L^p(X) : m\phi \in L^p(X)\}. \tag{3.31}$$

Then M_m admits a bounded $H^\infty(\Sigma_\sigma)$-calculus on $L^p(X)$.

Proof. Let $z \in \complement\overline{\Sigma_\sigma}$, then $z \in \rho(M_m)$ and the resolvent

$$R(z, M_m)\phi(x) = (z - m(x))^{-1}\phi(x).$$

Take $\nu \in (\omega(M_m), \sigma)$. For $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $\phi \in L^p(X)$ Dunford's integral is well-defined with

$$\begin{aligned} f(M_m)\phi(x) &= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(z)R(z, M_m)\phi(x)\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(z)(z - m(x))^{-1}\phi(x)\,\mathrm{d}z = f(m(x))\phi(x) \end{aligned} \tag{3.32}$$

for μ-almost all $x \in X$. Thus $\|f(M_m)\| \leqslant \|f\|_\infty$ and M_m admits a bounded $H^\infty(\Sigma_\sigma)$-calculus on $L^p(X)$. □

Proposition 3.6.4 ([19, Proposition 10.2.13]). *Let A be a densely defined, positive, self-adjoint operator on a complex Hilbert space H. Then A admits a bounded H^∞-calculus of angle $\omega_{H^\infty}(A) = 0$.*

Proof. By the spectral theorem for positive, self-adjoint operators [28, Theorem VIII.4], there exists a finite measure space $(X, \mathscr{A}, \mu)$, a measurable function $m : X \to [0, \infty)$ and a unitary transformation $U : H \to L^2(X)$ such that

$$A = U^{-1} M_m U,$$

where M_m is as in equation (3.30) for $p = 2$. Since A is injective, m is strictly positive μ-almost surely. By proposition 3.6.3, M_m admits a bounded H^∞-calculus of angle 0 on $L^2(S)$. Finally, since bounded H^∞-calculus is preserved under similarity transforms because

$$(z - U^{-1} M_m U)^{-1} = (U^{-1}Uz - U^{-1} M_m U)^{-1} = U^{-1}(z - M_m)^{-1} U,$$

we conclude that A admits a bounded H^∞-calculus with angle $\omega_{H^\infty}(A) = 0$. □

In particular, the bounded H^∞-calculus allows us to define the fractional powers of sectorial operators.

Theorem 3.6.5 ([4, Theorem 2.5]). *If A is admits a bounded H^∞-calculus on X, then*

$$D(A^\alpha) = [X, D(A)]_\alpha \quad \textit{for all} \quad \alpha \in (0, 1).$$

Similar to the case of the vector-valued Mikhlin multiplier theorem 3.1.16, in order to derive properties of vector-valued operator families uniform boundedness is not enough and we need to make use of R-boundedness. We will finish this section introducing R-sectoriality and stating the relation between R-boundedness of the H^∞-functional calculus.

Definition 3.6.6. A sectorial operator A is called *R-sectorial* if there exists $\sigma \in (\omega(A), \pi)$ such that the family of resolvents

$$\{zR(z, A) \;:\; z \in \complement\overline{\Sigma_\sigma}\}$$

is R-boundend.

Definition 3.6.7. Let A be a sectorial operator admitting a bounded H^∞-functional calculus. We say that A admits a R-bounded H^∞-calculus if the functional calculus

$$\{f(A) \;:\; f \in H^\infty(\Sigma_\sigma),\ \|f\|_\infty \leqslant 1\}$$

is R-bounded. The set of operators A admitting a R-bounded H^∞-calculus on X is denoted by $RH^\infty(X)$.

Theorem 3.6.8 ([19, Theorem 10.3.4]). *If a Banach space X is good enough and A admits a bounded H^∞-calculus, then*
 i) A is R-sectorial and;
 ii) the full H^∞-calculus of A is R-bounded.

For the proper definition of *good enough* in this context see the referred literature. For all practical purposes in the following chapters, $L^p(\Omega)$ as a L^p space with values in a Hilbert space is indeed *good enough.*

3.7 Perturbation theorems

In this section we study permanence properties of sectorial operators under additive perturbations. We will touch only a few aspects of the theory relevant for the proof of the main theorems in chapter 4. For a deeper discussion of perturbation results of the H^∞-calculus we refer the reader to [22].

Definition 3.7.1. Let $A : D(A) \subseteq X \to X$ be a generator of a C_0-semigroup and consider a second operator $B : D(B) \subseteq X \to X$. Then the sum $A+B$ is defined as

$$(A+B)x := Ax + Bx \quad \text{in} \quad D(A+B) := D(A) \cap D(B).$$

If the sum $A+B$ generates a C_0-semigroup again, we say that B is a *perturbation* of A.

Note that the domain $D(A+B)$ might be trivial in general. Even in the simplest case $B = -A$, the sum $A+B$ is the zero operator. Therefore, we need to add more requirements to the perturbation term in order to obtain properties such as sectoriality or H^∞-boundedness of the functional calculus. The following theorem ensures at least the sectoriality of the sum.

Theorem 3.7.2 ([4, Theorem 1.5])**.** *Let $A : D(A) \subseteq X \to X$ be a sectorial operator and let $B : D(B) \subseteq X \to X$ be a perturbation subordinated to A, i.e., $D(A) \subset D(B)$ and*

$$\|Bx\| \leqslant b\|Ax\|, \quad x \in D(A), \tag{3.33}$$

for some constant $b \geqslant 0$. Then $b < 1$ implies that the sum $A+B$ is closed, densely defined and $N(A+B) = \{0\}$. Moreover, $A+B$ is sectorial with spectral angle

$$\omega(A+B) \leqslant \inf\{\sigma > \omega(A) \;:\; bC_\sigma(A) < 1\}.$$

Proof. We first show closedness. Let $x_n \to x$ and $(A+B)x_n \to z$, we want to prove that $x \in D(A+B) = D(A)$ and $z = (A+B)x$. From the definition of subordinate perturbation (3.33) it is easy to check that $\{x_n\}_n$ defines a Cauchy sequence

$$\begin{aligned}\|A(x_n - x_m)\| &\leqslant \|(A+B)(x_n - x_m)\| + \|B(x_n - x_m)\| \\ &\leqslant \|(A+B)(x_n - x_m)\| + b\|x_n - x_m\| \to 0.\end{aligned}$$

Since A is closed, this implies that $x \in D(A)$ and $Ax_n = x$, thus the sequence $\{x_n\}_n$ is convergent in $D(A)$. We conclude that $D(A)$ is a Banach space with respect to the graph norm $\|\cdot\|_{D(A)}$, hence $A+B$ is closed. To prove injectivity assume that $(A+B)x = 0$, it is immediate from the bound (3.33) that $Ax = 0$, which by injectivity of A implies that $x = 0$. Finally, to show sectoriality take $\nu \in (\omega(A), \pi)$ and let $\lambda \in \complement\overline{\Sigma_\nu}$. Then $\lambda \in \rho(A)$ and we have

$$\lambda - (A+B) = (1 - B(\lambda - A)^{-1})(\lambda - A),$$

hence $\lambda - (A+B)$ is invertible whenever $\|B(\lambda - A)^{-1}\| < 1$, with inverse

$$(\lambda - (A+B))^{-1} = (\lambda - A)^{-1}(1 - B(\lambda - A)^{-1})^{-1}. \tag{3.34}$$

Consider $\nu > \omega(A)$ such that $bC_\nu(A) < 1$, then using the above characterization for $\lambda \in \complement\overline{\Sigma_\nu}$ we obtain the bound

$$\|\lambda(\lambda - (A+B))^{-1}\| \leqslant \frac{\|\lambda(\lambda - A)^{-1}\|}{\|1 - B(\lambda - A)^{-1}\|} \leqslant \frac{M_\nu(A)}{1 - bC_\nu(A)},$$

where the lower bound of the denominator is evident from

$$\|B(\lambda - A)^{-1}\| \leqslant b\|A(\lambda - A)^{-1}\| \leqslant bC_\nu(A) < 1.$$

□

An immediate consequence of the above theorem is the following corollary, where assuming a more general perturbation we prove sectoriality of the right shift $\mu + A + B$ for some $\mu \geqslant 0$.

Corollary 3.7.3 ([4, Corollary 1.6])**.** *Let $A : D(A) \subseteq X \to X$ be a sectorial operator and let $B : D(B) \subseteq X \to X$ be a relative perturbation of A, i.e., $D(A) \subset D(B)$ and*

$$\|Bx\| \leqslant a\|x\| + b\|Ax\|, \quad x \in D(A), \tag{3.35}$$

for some constants $a, b \geqslant 0$. Then there exists $b_0 > 0$ such that the sum $\mu + A + B$ is sectorial whenever $b < b_0$ and $\mu \geqslant 0$ is large enough.

Proof. Consider $\sigma > \omega(A)$ and $\lambda \in \complement\overline{\Sigma_\sigma} \subseteq \rho(A)$. Repeating the argument (3.34), from the hypothesis (3.35) we get the bound

$$\|B(\lambda - A)^{-1}\| \leqslant a\|(\lambda - A)^{-1}\| + b\|A(\lambda - A)^{-1}\| \leqslant a\frac{M_\sigma(A)}{|\lambda|} + bC_\sigma(A),$$

hence $\lambda - (A + B)$ is invertible provided

$$bC_\sigma(A) < 1 \quad \text{and} \quad |\lambda| > \mu_0 := a\frac{M_\sigma(A)}{1 - bC_\sigma(A)}.$$

In other words, $\lambda - (\mu_0 + A + B)$ is invertible for all $\lambda \in \complement\overline{\Sigma_\sigma}$. □

Note that (3.35) is equivalent to the graph norm $\|\cdot\|_{D(A)}$ if $a, b > 0$. Since $(X, D(A))_{\alpha,1} \hookrightarrow D(A^\alpha)$ for $\alpha \in (0, 1)$, see [26, Proposition 4.7], we could further ask if sectoriality holds for perturbations of *lower order type* like

$$\|Bx\| \leqslant a\|x\| + b\|A^\alpha x\|, \quad x \in D(A^\alpha). \tag{3.36}$$

Turns out that the answer is affirmative whenever $\alpha \in [0, 1)$, even without applying any restriction to $a, b \geqslant 0$.

Theorem 3.7.4. *Let $A : D(A) \subseteq X \to X$ be a sectorial operator and let $B : D(B) \subseteq X \to X$ be a lower order perturbation of A, i.e., $D(A) \subset D(B)$ and inequality* (3.36) *holds for arbitrary constants $a, b \geqslant 0$. Then the sum $\mu + A + B$ is sectorial whenever $\alpha \in [0, 1)$ and $\mu \geqslant 0$ is large enough.*

Proof. Let $\alpha \in (0, 1)$ be such that inequality (3.36) holds. Since $(X, D(A))_{\alpha,1} \hookrightarrow D(A^\alpha)$ there exists a constant $C \geqslant 0$ such that we can rewrite (3.36) equivalently as

$$\|Bx\| \leqslant (a + b)\|x\|_{D(A^\alpha)} \leqslant C(a + b)\|x\|_{(X,D(A))_{\alpha,1}}.$$

We may now bound the norm in the real interpolation space following the same reasoning as in remark 1 by

$$\|Bx\| \leqslant C(a + b)\|x\|_X^{1-\alpha}\|x\|_{D(A)}^\alpha.$$

Further, multiplying/dividing by ε and applying Young's inequality we get

$$\|Bx\| \leqslant C(a+b)\left(\varepsilon^{-\alpha/(1-\alpha)}(1-\alpha)\|x\|_X + \varepsilon\alpha\|x\|_{D(A)}\right) = C_{\varepsilon,\alpha}\|x\|_X + C(a+b)\varepsilon\alpha\|Ax\|_X,$$

for some constant $C_{\varepsilon,\theta} > 0$. As ε is arbitrary small, taking $\varepsilon < 1/(C(a+b)\alpha)$ we see that B is a relative perturbation of A and the assertion follows from corollary 3.7.3. □

Moreover, for perturbations of lower order type the following permanence of functional calculus for sectorial operators is true.

Theorem 3.7.5 ([22, Proposition 13.1])**.** *Let $A : D(A) \subseteq X \to X$ be an invertible, sectorial operator admitting a bounded $H^\infty(\Sigma_\sigma)$-calculus on X. Let $\delta \in (0,1)$ and suppose that $B : D(B) \subseteq X \to X$ is a lower order perturbation of A of the type*

$$\|Bx\| \leqslant C\|A^{1-\delta}x\|,$$

where $C > 0$. Then the sum $\mu + A + B$ admits a bounded $H^\infty(\Sigma_\sigma)$-calculus on X for $\mu \geqslant 0$ sufficiently large.

Proof. From theorem 3.7.4 we can assume, possibly by a right shift, that $A + B$ is sectorial. For $\lambda \in \complement\overline{\Sigma_{\omega(A+B)}}$ we can rewrite the resolvent as

$$\begin{aligned} R(\lambda, A+B) &= R(\lambda, A) + R(\lambda, A+B)BR(\lambda, A) \\ &= R(\lambda, A) + R(\lambda, A+B)BA^{\delta-1}A^{1-\delta}R(\lambda, A) \\ &= R(\lambda, A) + M(\lambda). \end{aligned} \tag{3.37}$$

Let $\sigma > \omega(A+B)$ and $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. For every $\nu \in (\omega(A+B), \sigma)$, by the above characterization, we can write

$$f(A+B) = f(A) + \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(\lambda)M(\lambda)\,\mathrm{d}\lambda,$$

where $f(A)$ is bounded because A admits a bounded H^∞-calculus. Moreover, we can estimate the integrand by

$$\|f(\lambda)M(\lambda)\| \leqslant \|f\|_\infty \frac{M_{\nu,A+B}}{|\lambda|}\|BA^{\delta-1}\|C'\frac{M^\delta_{\nu,A}}{|\lambda|^\delta}C^{1-\delta}_{\nu,A} \leqslant C\|f\|_\infty M_{\nu,A+B}\frac{M^\delta_{\nu,A}C^{1-\delta}_{\nu,A}}{|\lambda|^{1+\delta}},$$

where we applied remark 1 in the second inequality, with $x = R(\lambda, A)$, to get

$$\|A^{1-\delta}R(\lambda, A)\| \leqslant C'\|R(\lambda, A)\|^\delta\|AR(\lambda, A)\|^{1-\delta} \leqslant C'\frac{M^\delta_{\nu,A}}{|\lambda|^\delta}C^{1-\delta}_{\nu,A}.$$

Taking the infimum over all $\nu \in (\omega(A+B, \sigma)$ the integral converges absolutely and its norm is bounded by $\lesssim \|f\|_\infty$. □

One last result about lower order perturbations is needed in order to prove the main theorem of the work. The next proposition characterizes how much we can modify the sum of two operators $A + B$ without losing the functional calculus.

Proposition 3.7.6 ([4, Proposition 2.7]). *Let $A : D(A) \subseteq X \to X$ be a sectorial operator and $B : D(B) \subseteq X \to X$ a lower order perturbation of A of the type*

$$\|Bx\| \leqslant a\|x\| + b\|A^{\alpha}x\|, \quad x \in D(A),$$

for arbitrary constants $a, b > 0$ and $\alpha \in [0, 1)$. Assume that $A + B$ is sectorial and invertible. Then $h(A) \in \mathcal{L}(X)$ implies $h(A + B) \in \mathcal{L}(X)$ for any $h \in H^{\infty}(\Sigma_{\sigma})$, where $\sigma > \omega(A), \omega(A + B)$.

Proof. Let $f = \psi h$ with $\psi(\lambda) = \lambda(1 + \lambda)^{-2}$. For special properties of $\psi \in H^1(\Sigma_{\sigma}) \cap H^{\infty}(\Sigma_{\sigma})$ see [4, Section 2.1]. Then

$$\begin{aligned} h(A+B) &= \psi^{-1}(A+B)f(A+B) = (2 + A + B + (A+B)^{-1})f(A+B) \\ &= (2 + A + B(I+A)^{-1}(I+A) + (A+B)^{-1})f(A+B) \\ &= (2 + B(I+A)^{-1} + [I + B(I+A)^{-1}]A + (A+B)^{-1})f(A+B). \end{aligned}$$

From invertibility of $A + B$ and sectoriality of A, both $(A + B)^{-1}$ and $(I + A)^{-1}$ are bounded. Since $f \in H^1(\Sigma_{\sigma}) \cap H^{\infty}(\Sigma_{\sigma})$, the only point remaining concern is the boundedness of $Af(A + B)$. Let $\nu \in (\omega(A + B), \sigma)$, from the resolvent equation (3.37) we get that

$$Af(A+B) = Af(A) + \frac{1}{2\pi i}\int_{\partial\Sigma_{\nu}} f(\lambda)A(\lambda - A)^{-1}B(\lambda - A - B)^{-1}\,\mathrm{d}\lambda.$$

The first term $Af(A) = A\psi(A)h(A)$ is bounded because $h(A)$ is bounded by hypothesis, and the same computation as in theorem 3.7.5 proves that the integral is absolutely convergent. □

3.8 Maximal regularity

As mentioned in the introduction, in order to analyze the full nonlinear problem it is enough to consider its linearized part. Although it is not our purpose to study the exact construction of solutions to the full nonlinear problem -which is done via fixed point arguments from the solutions to the linearized problem-, this construction is tightly linked to the property of maximal regularity. In this section we will state some basic properties of maximal L^q-regularity and give an intuition behind the relation between maximal L^q-regularity and the bounded H^{∞}-calculus of an operator. However, the proofs of these theorems exceed the scope of this work and we refer the interested reader to [22, 17].

Let X be a Banach space and A a closed operator in X. For $1 \leqslant q \leqslant \infty$ and $0 < T \leqslant \infty$, consider the abstract Cauchy problem

$$\begin{aligned} u'(t) + Au(t) &= f(t), \quad t \in (0, T) \\ u(0) &= 0, \end{aligned} \tag{3.38}$$

If $f \in L^q(0, T; X)$, from $f = u' + Au$ it is clear that the solution u and Au cannot be more regular than the external force f. Therefore, maximal regularity refers to the best scenario, when u and Au have the same regularity as f.

Definition 3.8.1. Let $q \in [1, \infty]$ and $0 < T \leqslant \infty$. A closed linear operator A has *maximal L^q-regularity* in $(0, T)$ if for each $f \in L^q(0, T; X)$, the equation (3.38) admits a unique solution u satisfying

$$u \in L^q(0, T; D(A)) \cap W^{1,q}(0, T; X),$$

and there exists a constant $C \geqslant 0$ such that

$$\|u\|_{L^q(0,T;D(A)) \cap W^{1,q}(0,T;X)} + \|Au\|_{L^q(0,T;X)} \leqslant C\|f\|_{L^q(0,T;X)}.$$

We will often write "maximal L^q-regularity (for all q)", this is because for a closed and densely defined operator A in X, A having maximal L^{q_0}-regularity for some $q_0 \in [1, \infty]$ implies A having maximal L^q-regularity for all $q \in (1, \infty)$.

The next bundle of classical theorems uncovers the relation between maximal L^q-regularity and semigroup theory. For the remainder of the section, A is assumed to be closed and densely defined in X.

1. **Dore's theorem:** If A has maximal L^q-regularity in a bounded interval $(0, T)$, with $0 < T < \infty$, then $-A$ generates an analytic C_0-semigroup on X. On the other hand, if A has maximal L^q-regularity in the positive real line $\mathbb{R}_+$, then $-A$ generates a bounded analytic C_0-semigroup on X.
2. **De Simone's theorem:** In particular, if X is a Hilbert space the reverse also holds true.
3. **Kalton-Lancien:** If X has an unconditional basis and every generator of a (bounded) analytic C_0-semigroup on X has maximal L^q-regularity in $(0, T)$ (respectively $\mathbb{R}_+$), then X is isomorphic to a Hilbert space.

As one would imagine, maximal L^q-regularity in the positive real line implies maximal L^q-regularity in every bounded interval $(0, T)$ with $0 < T < \infty$. However, the converse is nontrivial and contained in the following theorem.

Theorem 3.8.2 (Dore-Kato)**.** *Let $q \in [1, \infty]$ and suppose that A is a densely defined closed operator which has maximal L^p-regularity in a finite interval $(0, T)$ with $0 < T < \infty$. If $-A$ generates a uniformly exponentially stable semigroup $\{S(t)\}_{t \geqslant 0}$, i.e.*

$$\|S(t)\| \leqslant Me^{-\omega t}, \quad \text{with} \quad M \geqslant 1,\ \omega > 0,$$

then A has maximal L^q-regularity in $\mathbb{R}_+$.

One final remark regarding the initial data in the Cauchy problem (3.38). We can of course instead consider the inhomogeneous Cauchy problem

$$\begin{aligned} u'(t) + Au(t) &= f(t), \quad t \in (0, T) \\ u(0) &= u_0, \end{aligned} \tag{3.39}$$

as long as the initial data satisfies

$$u_0 \in (X, D(A))_{1/q',q}, \quad \text{where } \frac{1}{q'} + \frac{1}{q} = 1.$$

Although chapter 4 focuses solely in the L^2 case, we will be following Giga et. al.'s proof for the general L^p-case [7], thus it is of interest to ask whether a form of De Simone's result extends for general Banach spaces. Turns out that R-boundedness 3.1.12 offers a powerful tools to deal with the present situation. The main theorem of this section is a consequence of the operator-valued Fourier multiplier theorem 3.1.16, the following provides a necessary and sufficient condition for maximal regularity in UMD spaces.

Theorem 3.8.3 ([22, Theorem 1.11]). *Let A be a generator of a bounded analytic semi-group on a UMD-space X. Then A has maximal L^q-regularity in $\mathbb{R}_+$ for one (all) $q \in (1, \infty)$ on X if and only if,*

$$\{\lambda R(\lambda, A) \,:\, \lambda \in \Sigma_\sigma\}$$

is R-bounded for some $\sigma < \pi/2$.

Corollary 3.8.4. *If A admits a bounded H^∞-calculus of angle $\sigma < \pi/2$ on a UMD space X, then A has maximal L^q-regularity in $\mathbb{R}_+$ on X for all $q \in (1, \infty)$.*

Chapter 4

Hydrostatic Stokes operator in L^2

In order to construct a unique, global strong solution for the non-linear equations modeling the large-scale ocean (2.21) in the L^p-setting, the study of the linearized problem is crucial. Recall that the hydrostatic Stokes equations are given by

$$(HSE)\begin{cases} \partial_t v + \nabla_H \pi_s - \Delta v = f & \text{in } \Omega \times (0,T), \\ \operatorname{div}_H \overline{v} = 0 & \text{in } G \times (0,T) \\ v(0) = v_0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where v denotes the horizontal velocity of the fluid, π_s the surface pressure, f the external force and v_0 the initial horizontal velocity. We consider the cylindrical domain for $a, b \in \mathbb{R}$ with $a < b$,

$$\Omega = G \times (a,b) \subset \mathbb{R}^3 \quad \text{with} \quad G = (0,1) \times (0,1),$$

where the bottom, upper and lateral part of the boundary $\delta\Omega$ are denoted by

$$\Gamma_a = G \times \{a\}, \quad \Gamma_b = G \times \{b\} \quad \text{and} \quad \Gamma_l = \partial G \times (a,b).$$

Here and subsequently, $(x,y) \in G$ stand for horizontal variables and $z \in (a,b)$ for the vertical variable, with this notation,

$$\nabla_H = (\partial_x, \partial_y)^T, \quad \operatorname{div}_H v = \partial_x v_1 + \partial_y v_2 \quad \text{and} \quad \overline{v} := \frac{1}{b-a}\int_a^b v(\cdot,\cdot,s)\,\mathrm{d}s$$

whereas Δ denotes the three dimensional Laplacian. Recall that the vertical velocity of the fluid w is determined by the horizontal velocity v via the relation

$$w(t;x,y,z) = w(v)(t;x,y,z) := -\int_a^z \operatorname{div}_H v(t;x,y,s)\,\mathrm{d}s,$$

and that due to the hydrostatic approximation (2.11), $\partial_z \pi = 0$, the full pressure π is actually determined only by the surface pressure π_s. The equations (4.1) are supplemented by the boundary conditions

$$\begin{aligned} v,\ \pi_s \text{ are periodic on } \Gamma_l \times (0,T), \\ v = 0 \text{ on } \Gamma_D \times (0,T), \\ \partial_z v = 0 \text{ on } \Gamma_N \times (0,T), \end{aligned} \tag{4.2}$$

where Dirichlet, Neumann, and mixed boundary conditions are given by

$$\Gamma_D \in \{\varnothing, \Gamma_a, \Gamma_b, \Gamma_a \cup \Gamma_b\}, \quad \text{and} \quad \Gamma_N \in (\Gamma_a \cup \Gamma_b) \backslash \Gamma_D.$$

In this chapter we are going to study the article by Giga et. al. [7], where they prove that the hydrostatic Stokes operator admits a bounded H^∞-calculus. The chapter is structured in four sections. In the first two sections, we follow Hieber et. al.'s [15] construction of the hydrostatic Helmholtz and Stokes operators respectively. In particular, we show that the negative hydrostatic Stokes operator $-A_p$ is sectorial of spectral angle 0 and generates an exponentially decaying analytic semigroup. Based on that, we continue with Giga et. al.'s [7] approach by rewriting the hydrostatic Stokes operator as a perturbation of the Laplacian of the form

$$A_p = \Delta_p + B_p, \quad \text{with} \quad B_p v := -\nabla_H \Delta_H^{-1} \operatorname{div}_H \left(\partial_z v\big|_{\Gamma_b} - \partial_z v\big|_{\Gamma_a} \right).$$

Note that the boundary terms play an important role here since pure Neumann boundary conditions yield $A_p = \Delta_2$, i.e. the Laplacian and the hydrostatic Stokes projection P_p commute. Finally, the third section 4.3 contains the main theorems of this work, we prove that the hydrostatic Stokes operator admits a bounded H^∞-calculus and mention some of the immediate corollaries.

Throughout this chapter, we follow the notation of [15] to model horizontally periodic function spaces. Let $m \in \mathbb{N}$ and $\nu \in \{0,1\}^2$, a smooth function $f : \overline{\Omega} \to \mathbb{R}$ is called *ν-periodic of order m on $\Gamma_l = \partial G \times (a, b)$* if

$$\frac{\partial^\alpha f}{\partial x^\alpha}(0, y, z) = (-1)^{\nu_1} \frac{\partial^\alpha f}{\partial x^\alpha}(1, y, z) \quad \text{and} \quad \frac{\partial^\alpha f}{\partial y^\alpha}(x, 0, z) = (-1)^{\nu_2} \frac{\partial^\alpha f}{\partial y^\alpha}(x, 1, z),$$

for all $\alpha = 0, ..., m$. In particular if $\nu = (0,0)$, then f is called periodic and if instead $\nu = (1,1)$, then f is called anti-periodic. (Anti)-periodicity on ∂G is defined in the same way for smooth functions in $\overline{G}$. Note that we only require periodicity in the horizontal directions, and no assumption is made over the vertical axis. Using this notion, the Bessel potential spaces of functions with periodic boundary conditions in the horizontal directions are

$$H^{s,p}_{\text{per}}(\Omega) := \{f \in H^{s,p}(\Omega) \mid f \text{ is periodic of arbitrary order on } \Gamma_l\}, \tag{4.3}$$
$$H^{s,p}_{\text{per}}(G) := \{f \in H^{s,p}(G) \mid f \text{ is periodic of arbitrary order on } \partial G\}. \tag{4.4}$$

For $s = m$ natural, the spaces $H^{m,p}_{\text{per}}(\Omega)$ and $H^{m,p}_{\text{per}}(G)$ respectively coincide with the Sobolev spaces

$$W^{m,p}_{\text{per}}(\Omega) := \{f \in W^{m,p}(\Omega) \mid f \text{ periodic of order } m-1 \text{ on } \partial\Gamma_l\}, \tag{4.5}$$
$$W^{m,p}_{\text{per}}(G) := \{f \in W^{m,p}(G) \mid f \text{ periodic of order } m-1 \text{ on } \partial G\}. \tag{4.6}$$

By theorem [33, Theorem 46.2], the spaces of smooth functions with periodic boundary conditions in the horizontal directions

$$C^\infty_{\text{per}}(\overline{\Omega}) := \{f \in C^\infty(\overline{\Omega}) \mid f \text{ is periodic of arbitrary order on } \Gamma_l\}, \tag{4.7}$$
$$C^\infty_{\text{per}}(\overline{G}) := \{f \in C^\infty(\overline{G}) \mid f \text{ is periodic of arbitrary order on } \partial G\}, \tag{4.8}$$

and dense in the respective Bessel potential spaces

$$H^{s,p}_{\text{per}}(\Omega) := \overline{C^\infty_{\text{per}}(\overline{\Omega})}^{\|\cdot\|_{H^{s,p}(\Omega)}} \quad \text{and} \quad H^{s,p}_{\text{per}}(G) := \overline{C^\infty_{\text{per}}(\overline{G})}^{\|\cdot\|_{H^{s,p}(G)}}. \tag{4.9}$$

Indeed this can be made precise using periodic extensions (4.59), mollifying the extensions, and restricting back.

Finally, for an open set $M \subset \mathbb{R}^n$, we define the closed subspace

$$L^p_0(M) := \{u \in L^p(M) \,:\, \int_M u \,\mathrm{d}x = 0\} \subseteq L^p(M). \tag{4.10}$$

Remark 3. Since it is a closed subspace of a Hilbert space, $L^2_0(M)$ is a Hilbert space.

4.1 The hydrostatic Helmholtz projection

As mentioned previously, the basic idea of the proof is to adapt the construction of the classical Helmholtz projection for the solenoidal subspace $L^2_\sigma(\Omega)$ of $L^2(\Omega)$. This is, we intend to find a projection that will eliminate the pressure gradient

$$P_2 v := v - \nabla_H \pi, \quad v \in L^2(\Omega)^2, \tag{4.11}$$

in other words, the existence of the hydrostatic Helmholtz projection depends on finding a solution $\nabla_H \pi$ of the Poisson problem

$$\Delta_H \pi = \operatorname{div}_H \overline{f} \;\text{ on }\; G, \quad \pi \text{ periodic on } \partial G, \tag{4.12}$$

in the distributional sense. The solution operator $\overline{f} \mapsto \nabla_H \pi$ is the closure operator in $L^2(G)$ of

$$\nabla_H \pi = -\nabla_H (-\Delta_H)^{-1} \operatorname{div}_H \overline{f},$$

where Δ_H denotes the two-dimensional Laplacian defined on $H^{2,2}_{\text{per}}(G)^2$ with inverse in $L^2_0(G)$. Note that although we have dropped the subindex of surface pressure in order to simplify notation, as a result of the hydrostatic approximation the pressure is a function of only two variables.

Proposition 4.1.1 ([15], Weak solvability of the Poisson problem)**.** *Let $f \in L^2(G)^2$. Then there exists a unique $\pi \in W^{1,2}_{per}(G) \cap L^2_0(G)$ satisfying*

$$\langle f, \nabla_H \phi \rangle_{L^2(G)} = \langle \nabla_H \pi, \nabla_H \phi \rangle_{L^2(G)}, \quad \phi \in W^{1,2}_{per}(G) \cap L^2_0(G). \tag{4.13}$$

Furthermore, there exists a constant $C > 0$ such that

$$\|\pi\|_{W^{1,2}(G)} \leqslant C \|f\|_{L^2(G)}, \quad f \in L^2(G)^2. \tag{4.14}$$

Remark 4. Let us first observe that the theorem is actually equivalent to the unique solvability of the Poisson problem in the distributional sense. Equation (4.12) is equivalent to

$$\int_G \left(\frac{\partial^2 \pi}{\partial x_1^2} + \frac{\partial^2 \pi}{\partial x_2^2} \right) \phi \,\mathrm{d}x = \int_G \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \phi \,\mathrm{d}x, \quad \phi \in C^\infty_{\text{per}}(\overline{G}) \tag{4.15}$$

which from the definition of distributional derivative can be rewritten as

$$-\int_G \frac{\partial \pi}{\partial x_1} \frac{\partial \phi}{\partial x_1} + \frac{\partial \pi}{\partial x_2} \frac{\partial \phi}{\partial x_2} \,\mathrm{d}x = -\int_G f_1 \frac{\partial \phi}{\partial x_1} + f_2 \frac{\partial \phi}{\partial x_2} \,\mathrm{d}x, \quad \phi \in C^\infty_{\text{per}}(\overline{G}), \tag{4.16}$$

or in other words

$$\langle \nabla_H \pi, \nabla_H \phi \rangle_{L^2(G)} = \langle f, \nabla_H \phi \rangle_{L^2(G)}, \quad \phi \in C^\infty_{\text{per}}(\overline{G}). \tag{4.17}$$

Proof. (*Proposition 4.1.1*) Let $H := W^{1,2}_{\text{per}}(G) \cap L^2_0(G)$ be our Hilbert space, with norm $\|\nabla_H v\|_{L^2(G)}$. Let $f \in L^2(G)^2$, the functional

$$F : \phi \mapsto \langle f, \nabla_H \phi \rangle_{L^2(G)}, \quad \phi \in W^{1,2}_{\text{per}}(G) \cap L^2_0(G),$$

is continuous in $\|\nabla_H \phi\|_{L^2(G)}$ since

$$|F(\phi)| = |\langle f, \nabla_H \phi \rangle_{L^2(G)}| \leqslant \|f\|_{L^2(G)} \|\nabla_H \phi\|_{L^2(G)}.$$

By Riesz' representation theorem, there exists a unique $\pi \in W^{1,2}_{\text{per}}(G) \cap L^2_0(G)$ satisfying

$$\langle f, \nabla_H \phi \rangle_{L^2(G)} = (\nabla_H \pi | \nabla_H \phi)_{L^2(G)}, \quad \phi \in W^{1,2}_{\text{per}}(G) \cap L^2_0(G). \tag{4.18}$$

Setting $\phi = \pi$ yields

$$\|\nabla_H \pi\|^2_{L^2(G)} \leqslant \|f\|_{L^2(G)} \|\nabla_H \pi\|_{L^2(G)},$$

and thus $\|\nabla_H \pi\|_{L^2(G)} \leqslant \|f\|_{L^2(G)}$ as desired. □

This proposition allows us to clarify the precise meaning of (4.11).

Definition 4.1.2. Given $v \in L^2(\Omega)^2$, let $\pi \in W^{1,2}_{\text{per}}(G) \cap L^2_0(G)$ be the unique solution of equation (4.13) with $f = \overline{v}$. The hydrostatic Helmholtz projection P_2 is defined by

$$P_2 v := v - \nabla_H \pi. \tag{4.19}$$

To see that it is indeed a continuous projection take $v \in L^2(\Omega)^2$ and let $\pi \in W^{1,2}_{\text{per}}(G) \cap L^2_0(G)$ be the unique solution of (4.13) with $f = \overline{v}$, thus $P_2 v = v - \nabla_H \pi$. We have $v - \nabla_H \pi \in L^2(\Omega)^2$, and since π is independent of the vertical axis, $\overline{v - \nabla_H \pi} = \overline{v} - \nabla_H \pi$ follows. Consider now $f = \overline{v} - \nabla_H \pi$, the unique solution $\pi' \in W^{1,2}_{\text{per}}(G) \cap L^2_0(G)$ of equation (4.13) satisfies

$$0 = \langle \overline{v} - \nabla_H \pi, \nabla_H \phi \rangle_{L^2(G)} = \langle \nabla_H \pi', \nabla_H \phi \rangle_{L^2(G)}, \quad \phi \in C^\infty_{\text{per}}(G),$$

this is, $\pi' = 0$ necessarily. We conclude that

$$P_2^2 v = P_2(v - \nabla_H \pi) = P_2 v,$$

which proves that P_2 is a projection on $L^2(\Omega)^2$ as desired.

The range of the hydrostatic Helmholtz projection

$$L^2_{\overline{\sigma}}(\Omega)^2 := \operatorname{Ran} P_2, \tag{4.20}$$

will play an analogous role in the study of primitive equations to the solenoidal subspace $L^2_\sigma(\Omega)$ of $L^2(\Omega)$ for the Stokes equations. We begin by giving some useful equivalent characterizations of $L^2_{\overline{\sigma}}(\Omega)^2$. We denote by $\nu_{\partial G}$ the exterior normal vector field at ∂G.

Proposition 4.1.3 ([15], Proposition 4.3)**.** *The range of the hydrostatic Helmholtz projection $L^2_{\overline{\sigma}}(\Omega)^2$ coincides with the following subsets of $L^2(\Omega)^2$:*

a) $X_1 := \{v \in L^2(\Omega)^2 \ : \ \langle \overline{v}, \nabla_H \phi \rangle_{L^2(G)} = 0 \quad$ *for all* $\phi \in W^{1,2}_{per}(G)\}$

b) $X_2 := \{v \in L^2(\Omega)^2 \ : \ \operatorname{div}_H \overline{v} = 0, \quad \overline{v} \cdot \nu_{\partial G}$ *is anti-periodic of order* 0 *on* $\partial G\}$

c) $X_3 := \overline{V}^{\|\cdot\|_{L^2(\Omega)}}$*, where*

$$V := \{v \in C^\infty_{per}(\overline{\Omega})^2 \ : \ \operatorname{div}_H \overline{v} = 0 \ \textit{in } G, \quad \operatorname{supp} v \subset \overline{G} \times (a, b)\} \tag{4.21}$$

Proof. ($L^2_{\overline{\sigma}}(\Omega)^2 = X_1$) Take $v \in X_1$. By proposition 4.1.1 there exists a unique $\pi \in W^{1,2}_{\text{per}}(G) \cap L^2_0(G)$ such that equation (4.13) holds for $f = \overline{v}$. Since

$$0 = \langle \overline{v}, \nabla_H \phi \rangle_{L^2(G)} = \langle \nabla_H \pi, \nabla_H \phi \rangle_{L^2(G)}, \quad \phi \in C^\infty_{\text{per}}(\overline{G}),$$

then $\pi = 0$ and v is invariant under the projection $P_2 v = v$, consequently $v \in L^2_{\overline{\sigma}}(\Omega)^2$. Conversely, if $v \in L^2_{\overline{\sigma}}(\Omega)^2$ there must exist $w \in L^2(\Omega)^2$ and $\pi \in W^{1,2}_{\text{per}}(G) \cap L^2_0(G)$ solution of (4.13) with $f = \overline{w}$ such that $v = w - \nabla_H \pi$. But π is independent of the vertical axis z, thus $\overline{v} = \overline{w - \nabla_H \pi} = \overline{w} - \nabla_H \pi$. Substituting in (4.13) we get

$$\langle \overline{v}, \nabla_H \phi \rangle_{L^2(G)} = \langle \overline{w} - \nabla_H \pi, \nabla_H \phi \rangle_{L^2(G)} = 0, \quad \phi \in W^{1,2}_{\text{per}}(G),$$

concluding that $v \in X_1$.

($X_1 = X_2$) Recall that we introduced the notion of trace for $\overline{v} \cdot \nu_{\partial G}$ in theorem 3.4.3, i.e. $\overline{v} \cdot \nu_{\partial G}$ is a well defined element of $B^{1/2}_{2,2}(\partial G)^*$ via the relation

$$\langle \operatorname{div}_H \overline{v}, \phi \rangle_{L^2(G)} = \langle \overline{v} \cdot \nu_{\partial G}, \phi \rangle_{B^{1/2}_{2,2}(\partial G)} - \langle \overline{v}, \nabla_H \phi \rangle_{L^2(G)}, \quad \phi \in W^{1,2}_{\text{per}}(G). \tag{4.22}$$

Let now $v \in X_1$. Since in particular $C^\infty_{\text{per}}(\overline{G}) \subset W^{1,2}_{\text{per}}(G)$, it is easy to check that $\operatorname{div}_H \overline{v} = 0$ in the distributional sense

$$\begin{aligned} 0 = \int_G \overline{v} \cdot \nabla_H \phi \, \mathrm{d}x &= \int_G \overline{v}_1 \frac{\partial \phi}{\partial x_1} + \overline{v}_2 \frac{\partial \phi}{\partial x_2} \, \mathrm{d}x \\ &= -\int_G \frac{\partial \overline{v}_1}{\partial x_1} \phi + \frac{\partial \overline{v}_2}{\partial x_2} \phi \, \mathrm{d}x = \int_G \operatorname{div}_H(\overline{v}) \phi \, \mathrm{d}x, \quad \phi \in C^\infty_{\text{per}}(\overline{G}). \end{aligned} \tag{4.23}$$

In order to prove anti-periodicity in each direction we first substitute $\operatorname{div}_H \overline{v} = 0$ in (4.22) to obtain the useful relation

$$\langle \overline{v} \cdot \nu_{\partial G}, \phi \rangle_{B^{1/2}_{2,2}(\partial G)} = 0, \quad \phi \in W^{1,2}_{\text{per}}(G). \tag{4.24}$$

Let us split the boundary by defining $G_i = G \cap \{x_i = 0\}$ for $i = 1, 2$, where x_1 and x_2 are x and y respectively. The idea of the proof is to extend one-dimensional functions defined

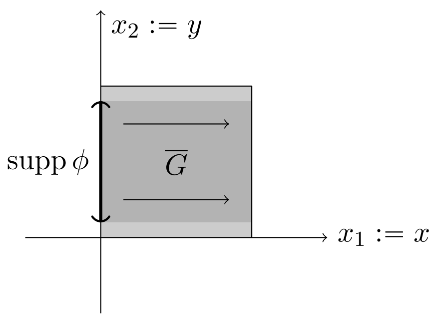


Figure 4.1: Extension of $\phi \in C^\infty_c(G_1)$ to $\overline{G}$.

on G_i to $\overline{G}$, where we have the property (4.24). Choosing an arbitrary $\phi \in C^\infty_c(G_i)$, we can extend it constantly along the direction x_i and by abuse of notation, regard it as $\phi \in C^\infty_{\text{per}}(\overline{G})$. Note that since ϕ is compactly supported in G_i it will vanish on the boundary of the opposite direction $\partial G \backslash G_i$. Now substituting ϕ in equation (4.24) we get

$$\langle \overline{v} \cdot \nu_{\partial G}\big|_{\{x_i=0\}}, \phi \rangle_{C^\infty_c(G_i)} + \langle \overline{v} \cdot \nu_{\partial G}\big|_{\{x_i=1\}}, \phi \rangle_{C^\infty_c(G_i)} = 0, \tag{4.25}$$

where we used that $\phi\big|_{G\cap\{x_i=1\}} = \phi\big|_{G_i}$. We have thus proven that

$$\overline{v}\cdot\nu_{\partial G}\big|_{\{x_i=0\}} = -\overline{v}\cdot\nu_{\partial G}\big|_{\{x_i=1\}},$$

i.e. $\overline{v}\cdot\nu_{\partial G}$ is anti-periodic of order 0 on ∂G as desired. Conversely, let $v \in X_2$. From anti-periodicity of $\overline{v}\cdot\nu_{\partial G}$ and periodicity of ϕ, we have that in each direction

$$\langle \overline{v}\cdot\nu_{\partial G}, \phi\rangle_{B^{1/2}_{2,2}(G\cap\{x_i=0\})} + \langle \overline{v}\cdot\nu_{\partial G}, \phi\rangle_{B^{1/2}_{2,2}(G\cap\{x_i=1\})} = 0, \quad \phi \in W^{1,2}_{\mathrm{per}}(G).$$

Since $\mathrm{div}_H\,\overline{v} = 0$, substituting in (4.22) the proof is simple

$$\begin{aligned}\langle \overline{v}, \nabla_H\phi\rangle_{W^{1,2}_{\mathrm{per}}(G)} &= \langle \overline{v}\cdot\nu_{\partial G}, \phi\rangle_{B^{1/2}_{2,2}(\partial G)}\\ &= \sum_{i=1}^{2}\langle \overline{v}\cdot\nu_{\partial G}, \phi\rangle_{B^{1/2}_{2,2}(G\cap\{x_i=0\})} + \langle \overline{v}\cdot\nu_{\partial G}, \phi\rangle_{B^{1/2}_{2}(G\cap\{x_i=1\})}\\ &= 0, \quad \phi \in W^{1,2}_{\mathrm{per}}(G).\end{aligned} \tag{4.26}$$

($X_1 = X_3$) Since $V \subset X_2 = X_1$, it is clear that $X_3 \subseteq X_1$. In order to get equality, we first claim that $X_1^* = X_1$. Indeed, it is immediate from the canonical inclusion that

$$X_1 \subset L^2(\Omega) = L^2(\Omega)^* \subset X_1^*,$$

thus it suffices to show $X_1^* \subseteq X_1$. Given $F \in X_1^*$, by the Hahn-Banach theorem it has an extension to a functional on $(L^2(\Omega)^2)^*$, which we will represent by $f \in L^2(\Omega)^2$. Moreover, from the weak solvability of the Poisson problem, proposition 4.1.1, the projection $P_2 f$ onto X_1 is determined independently of the way F is extended, proving our claim.

Suppose now that $X_3 \neq X_1$, i.e. X_3 is a proper closed subspace of X_1. By the Hahn-Banach theorem again, there exists a non-zero functional $F \in X_1^*$ vanishing on X_3, which by the above claim can be represented as $f \in X_1$. Summarizing, there exists a non-zero functional $f \in L^2(\Omega)^2$ such that

$$\begin{aligned}\langle f, v\rangle_{L^2(\Omega)} &= 0, \quad \text{for all } v \in V,\\ \langle \overline{f}, \nabla_H\phi\rangle_{L^2(G)} &= 0, \quad \text{for all } \phi \in W^{1,2}_{\mathrm{per}}(G).\end{aligned} \tag{4.27}$$

The main idea of the proof is to apply the Helmholtz decomposition [31, Section II.2.5], i.e.

$$L^2(G)^2 = L^2_\sigma(G) \oplus H(G) \quad \text{with} \quad H(G) := \{f \in L^2(G)^2\,;\, \exists\pi \in L^2(G)\,:\, f = \nabla_H\pi\},$$

and to show that the respective pressure gradient $\nabla_H\pi$ is zero, reaching a contradiction.

Note that $f \in L^2(G)^2$ is independent of the z direction, which can be seen by taking $\partial_z v \in V$ with $v \in V$ as a test function in $(4.27)_1$. We can thus regard $f = \overline{f} \in L^2(G)^2$ and will first check that it is orthogonal to the solenoidal subspace $L^2_\sigma(G)$. In order to make use of (4.27) starting with $\phi \in C^\infty_{\mathrm{per},\sigma}(\overline{G})$, where $C^\infty_{\mathrm{per},\sigma}(\overline{G}) := \{\phi \in C^\infty_{\mathrm{per}}(\overline{G}) : \mathrm{div}_H\,\phi = 0\}$, we want to construct a function in $V = \{v \in C^\infty_{\mathrm{per}}(\overline{\Omega})^2 \,:\, \mathrm{div}_H\,\overline{v} = 0 \text{ in } G, \quad \mathrm{supp}\, v \subset \overline{G}\times(a,b)\}$. Let χ be a cut-off function on the vertical interval, $\chi \in C^\infty_c(a,b)$, such that

$$\int_a^b \chi(z)\,\mathrm{d}z = b - a. \tag{4.28}$$

We can now define $v \in V$ as

$$v(x, y, z) = \chi(z)\phi(x, y).$$

Indeed, clearly $v \in C^\infty_{\text{per}}(\overline{\Omega})^2$ with $\operatorname{supp} v \subset \overline{G} \times (a, b)$ and

$$\operatorname{div}_H \overline{v} = \operatorname{div}_H \left(\int_a^b \chi(z)\phi(x, y) \,\mathrm{d}z \right) = (b - a) \operatorname{div}_H \phi = 0.$$

Recall that f vanished in X_3 $(4.27)_1$, in particular

$$0 = \langle f, v \rangle_{L^2(\Omega)} = \langle f, \phi \rangle_{L^2(G)}, \quad \phi \in C^\infty_{\text{per},\sigma}(\overline{G}). \tag{4.29}$$

The Helmholtz decomposition in $L^2(G)^2$ now implies that there exists $\pi \in W^{1,2}(G)$ such that $\nabla_H \pi = f$.

Our next goal is to prove that π is actually the unique solution of the weak Poisson problem (4.13). For $\pi \in W^{1,2}(G)$, according to the generalized Green's theorem 3.4.3, we have

$$\langle \pi, \phi \cdot \nu_{\partial G} \rangle_{L^2(\partial G)} = \langle \nabla_H \pi, \phi \rangle_{L^2(G)}, \quad \phi \in C^\infty_{\text{per},\sigma}(\overline{G}). \tag{4.30}$$

Moreover, since $\nabla_H \pi = f$, from (4.29) we obtain that

$$\begin{aligned} 0 = \langle \pi, \phi \cdot \nu_{\partial G} \rangle_{L^2(\partial G)} &= \langle \pi, \phi \cdot (-1, 0) \rangle_{L^2(G \cap \{x_1 = 0\})} + \langle \pi, \phi \cdot (1, 0) \rangle_{L^2(G \cap \{x_1 = 1\})} \\ &\quad + \langle \pi, \phi \cdot (0, -1) \rangle_{L^2(G \cap \{x_2 = 0\})} + \langle \pi, \phi \cdot (0, 1) \rangle_{L^2(G \cap \{x_2 = 1\})} \\ &= \sum_{i=1}^{2} \langle \pi, \phi \rangle_{L^2(G \cap \{x_i = 0\})} - \langle \pi, \phi \rangle_{L^2(G \cap \{x_i = 1\})}, \quad \phi \in C^\infty_{\text{per},\sigma}(\overline{G}). \end{aligned} \tag{4.31}$$

It follows that π is periodic on ∂G.

Note that we have actually proved that there exists $\pi \in W^{1,2}_{\text{per}}(G)$ such that

$$\langle \nabla_H \pi, \nabla_H \phi \rangle_{L^2(G)} = \langle f, \nabla_H \phi \rangle_{L^2(G)} = 0, \quad \phi \in W^{1,2}_{\text{per}}(G).$$

In conclusion, π must be the unique solution of proposition 4.1.1, therefore $\pi = 0$, which in turn gives $f = 0$. This contradicts the assumption $F \neq 0$, hence proving the proposition. □

4.2 The hydrostatic Stokes operator

The hydrostatic Helmholtz projection allows us to define the hydrostatic Stokes operator, analogous to the classical one. By applying the projection P_2 to the hydrostatic Stokes equations 4.1 we obtain the equivalent Cauchy problem

$$\partial_t v - A_2 v = P_2 f, \quad v(0) = v_0, \tag{4.32}$$

where $A_2 = P_2 \Delta$. The operator A_2 is called the *hydrostatic Stokes operator* and its $L^2_{\overline{\sigma}}(\Omega)$-realization is defined as

$$A_2 v := P_2 \Delta v, \quad D(A_2) := \{ v \in W^{2,2}_{\text{per}}(\Omega)^2 \,:\, v\big|_{\Gamma_D} = 0,\ \partial_z v\big|_{\Gamma_N} = 0 \} \cap L^2_{\overline{\sigma}}(\Omega). \tag{4.33}$$

Giga et. al. established in their article [7] that the hydrostatic Stokes operator admits a bounded H^∞-calculus. In order to prove this property, they refer to Hieber and Kashiwara's work [15], where it is shown that $-A_2$ generates a strongly continuous, exponentially stable, analytic semigroup of angle 0. In this section we are going to focus on the latter result and leave the bounded H^∞-calculus for the next section.

Before we begin with the proofs, note that our domain differs from Hieber and Kashiwabara's [15], who only considered $\Gamma_D = \Gamma_a$ and $\Gamma_N = \Gamma_b$, but this does not carry any further complexity and the same proof holds as long as $\Gamma_D \neq \varnothing$. However, in the pure Neumann case we just have $A_2 v = \Delta v$ for $v \in D(A_2)$, see theorem 4.3.5. In particular, zero is an eigenvalue of A_2. Since injectivity is required for sectoriality, we will study the spectral properties of $-A_2 + \mu$ instead, for some $\mu > 0$.

Let $f \in L^2(\Omega)^2$ and $\lambda \in \Sigma_{\pi-\varepsilon} := \{\lambda \in \mathbb{C} \ : \ |\arg(\lambda)| < \pi - \varepsilon\}$ for some $\varepsilon \in (0, \pi/2)$. Consider the resolvent problem

$$\begin{cases} \lambda v - \Delta v + \nabla_H \pi_s = f & \text{on } \Omega, \\ \operatorname{div}_H \overline{v} = 0 & \text{on } G, \end{cases} \tag{4.34}$$

with boundary conditions

$$\begin{gathered} v, \ \pi \ \text{ are periodic on } \Gamma_l, \\ v\big|_{\Gamma_D} = 0 \quad \text{and} \quad \partial_z v\big|_{\Gamma_N} = 0. \end{gathered} \tag{4.35}$$

The proof of $-A_2$ generating an exponentially stable analytic semigroup will be divided into two steps. We first have to prove the following resolvent estimate.

Theorem 4.2.1 ([15, Theorem 3.1])**.** *Assume $\Gamma_D \neq \varnothing$. Let $f \in L^2(\Omega)^2$ and $\lambda \in \Sigma_{\pi-\varepsilon} \cup \{0\}$ for $\varepsilon \in (0, \pi/2)$. Equations (4.34) and (4.35) admit a unique solution $(v, \pi) \in W^{2,2}_{per}(\Omega)^2 \times W^{1,2}_{per}(G) \cap L^2_0(G)$. Moreover, there exists a constant $C > 0$, depending only on ε, such that*

$$|\lambda| \|v\|_{L^2(\Omega)} + \|v\|_{W^{2,2}(\Omega)} + \|\pi\|_{W^{1,2}(G)} \leqslant C \|f\|_{L^2(\Omega)}. \tag{4.36}$$

The basic idea is to find a unique solution of the weak formulation of the problem and then applying difference quotients to obtain the $H^2 - H^1$ estimate. Nevertheless, let us start by proving a useful lemma to treat the weak formulation of the problem.

Lemma 4.2.2 (Ladyzhenskaya-Babuška-Brezzi theorem (LBB), [2, Section 12.2])**.** *Let V and W be two Hilbert spaces. Assume $\mathfrak{a} : V \times V \to \mathbb{C}$ and $\mathfrak{b} : V \times W \to \mathbb{C}$ to be bounded sesquilinear forms. Suppose $\mathfrak{a}$ is coercive and $\mathfrak{b}$ verifies the Babuška-Brezzi condition, i.e.*

$$\Re \mathfrak{a}(\varphi, \varphi) \geqslant \alpha \|\varphi\|_V^2, \quad \varphi \in V \quad \textit{and} \quad \sup_{\varphi \in V} \frac{|b(\varphi, \phi)|}{\|\varphi\|_V} \geqslant \beta \|\phi\|_W, \quad \phi \in W, \tag{4.37}$$

for some constants $\alpha > 0$ and $\beta > 0$. Then for $f \in V^*$ the variational problem

$$\begin{cases} \mathfrak{a}(v, \varphi) + \mathfrak{b}(\varphi, \pi) = \langle f, \varphi \rangle, & \varphi \in V \\ \mathfrak{b}(v, \phi) = 0, & \phi \in W. \end{cases} \tag{4.38}$$

admits a unique solution $(v, \pi) \in V \times W$. Furthermore, for some $C > 0$, the solution satisfies the estimate

$$\|v\|_V + \|\pi\|_W \leqslant C \|f\|_{V^*}. \tag{4.39}$$

Proof. Let Z denote the subspace of V defined by

$$Z = \{\varphi \in V \ : \ \mathfrak{b}(\varphi, \phi) = 0, \quad \phi \in W\}.$$

It is simple to check that Z is closed, therefore Z is a Hilbert space with the inner product $(\cdot|\cdot)_V$. Take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset Z$ converging to φ and an arbitrary $\phi \in W$, since the form $\mathfrak{b}$ is bounded, then $\mathfrak{b}(\varphi, \phi) = \lim_{n\to\infty} \mathfrak{b}(\varphi_n, \phi) = 0$, thus $\varphi \in Z$. As a consequence, $V = Z \oplus Z^\perp$ and the solution $v \in V$ can be determined by testing only against $\varphi \in Z$. In other words, it suffices to show that there exists a unique $v \in Z$ such that

$$\mathfrak{a}(v, \varphi) = \langle f, \varphi\rangle, \quad \varphi \in Z, \tag{4.40}$$

but since $\mathfrak{a}$ is coercive, this is exactly the Lax-Milgram theorem for the continuous functional $f \in V^*$. It remains to find $\pi \in W$, which is uniquely determined by v as the solution of the equation

$$\mathfrak{b}(\varphi, \pi) = -\mathfrak{a}(v, \varphi) + \langle f, \varphi\rangle, \quad \varphi \in V. \tag{4.41}$$

Since the behaviour of $b(\cdot,\cdot)$ is trivial on Z we can rewrite the problem as

$$\mathfrak{b}(\varphi, \pi) = \langle g, \varphi\rangle, \quad \varphi \in Z^\perp, \tag{4.42}$$

where $g \in (Z^\perp)^*$ is a continuous functional

$$|\langle g, \varphi\rangle| \leqslant |\mathfrak{a}(v, \varphi)| + |\langle f, \varphi\rangle| \leqslant (C_{\mathfrak{a}}\|v\|_V + \|f\|_{V^*})\|\varphi\|_V, \quad \varphi \in Z^\perp$$

with $C_{\mathfrak{a}}$ the continuity constant of $\mathfrak{a}$. We will start by proving the existence of solutions. Take $\phi \in W$, the functional $\varphi \mapsto \mathfrak{b}(\varphi, \phi)$ is continuous on $Z^\perp$. Then by the Riesz representation theorem, there exists a unique $T\phi \in Z^\perp$ such that

$$\mathfrak{b}(\varphi, \phi) = (T\phi|\varphi)_V, \quad \varphi \in Z^\perp. \tag{4.43}$$

The mapping $T : \phi \mapsto T\phi$ is obviously linear and continuous

$$\|T\phi\|_V = \|b(\cdot, \phi)\|_{V^*} = \sup_{\varphi\in V} \frac{|b(\varphi, \phi)|}{\|\varphi\|_V} \leqslant C_{\mathfrak{b}}\|\phi\|_W,$$

where the last inequality follows from the boundedness of $\mathfrak{b}$. Moreover, we claim that the range of T is the whole space $R(T) = Z^\perp$. Since $g \in (Z^\perp)^*$ is a continuous functional, once again by the Riesz representation theorem there exists a unique $w \in Z^\perp$ such that

$$\langle g, \varphi\rangle = (w|\varphi)_V, \quad \varphi \in Z^\perp.$$

In combination with $T : W \to Z^\perp$ being surjective and (4.43), there exists $\pi \in W$ such that $T\pi = w \in Z^\perp$ and

$$\langle g, \varphi\rangle = (T\pi|\varphi) = \mathfrak{b}(\varphi, \pi), \quad \varphi \in Z^\perp.$$

To prove the claim we will show the closedness of T and derive a contradiction if $R(T) \neq Z^\perp$. Take a sequence $\{T\phi_n\}_{n\in\mathbb{N}}$ converging to w in $Z^\perp$, then $\{T\phi_n\}_n$ is a Cauchy sequence and consequently $\{\phi_n\}_n$ is a Cauchy sequence as well since

$$\begin{aligned} \|\phi_k - \phi_m\|_W &\leqslant \frac{1}{\beta} \sup_{\varphi\in Z^\perp} \frac{|\mathfrak{b}(\varphi, \phi_k - \phi_m)|}{\|\varphi\|_V} = \frac{1}{\beta} \sup_{\varphi\in Z^\perp} \frac{(T\phi_k - T\phi_m|\varphi)_V}{\|\varphi\|_V} \\ &= \frac{\|T\phi_k - T\phi_m\|_V}{\beta} \to 0, \quad \text{when } k, m \to \infty. \end{aligned} \tag{4.44}$$

Since $Z^\perp$ is complete, there exists $\phi \in Z^\perp$ such that $\varphi_n \to \phi$, but T is continuous, thus $T\phi = w$. If $R(T) \neq Z^\perp$, there would exist a nonzero element $v \in R(T)^\perp$ such that

$$0 = (T\phi|v)_V = \mathfrak{b}(v, \phi), \quad \phi \in W,$$

but then $v \in Z$, a contradiction. The solution is easily seen to be unique by the Babuška-Brezzi condition. Suppose π_1 and π_2 are two solutions of equation (4.42), then

$$\begin{aligned} \|\pi_1 - \pi_2\|_W &\leqslant \frac{1}{\beta} \sup_{\varphi \in V} \frac{|b(\pi_1 - \pi_2, \varphi)|}{\|\varphi\|_V} = \frac{1}{\beta} \sup_{\varphi \in V} \frac{|b(\pi_1, \varphi) - \mathfrak{b}(\pi_2, \varphi)|}{\|\varphi\|_V} \\ &= \frac{1}{\beta} \sup_{\varphi \in V} \frac{|g(\varphi) - g(\varphi)|}{\|\varphi\|_V} = 0, \end{aligned} \tag{4.45}$$

thus $\pi_1 = \pi_2$ necessarily. Finally, the proof of inequality (4.39) is straightforward from the construction of the solution

$$\|v\|_V^2 \leqslant \frac{1}{\alpha} \Re \mathfrak{a}(v, v) \leqslant \frac{1}{\alpha} |\mathfrak{a}(v, v)| = \frac{1}{\alpha} |\langle f, v \rangle| \leqslant \frac{1}{\alpha} \|f\|_{V*} \|v\|_V, \tag{4.46}$$

$$\|\pi\|_W \leqslant \frac{1}{\beta} \sup_{\varphi \in V} \frac{|\mathfrak{b}(\varphi, \pi)|}{\|\varphi\|_V} \leqslant \frac{1}{\beta} \sup_{\varphi \in V} \frac{|\mathfrak{a}(v, \varphi)| + |\langle f, \varphi \rangle|}{\|\varphi\|_V} \leqslant \frac{2}{\beta} \|f\|_{V*}. \tag{4.47}$$

□

As mentioned previously the first step is to study a weak formulation of the problem (4.34). Consider the spaces

$$V := \{\varphi \in W^{1,2}_{\mathrm{per}}(\Omega)^2 \,:\, \varphi = 0 \text{ in } \Gamma_D\} \quad \text{and} \quad W := L^2_0(G), \tag{4.48}$$

which are closed subspaces of the Hilbert spaces $W^{1,2}_{\mathrm{per}}(\Omega)^2$ and $L^2(G)$ respectively, thus Hilbert spaces with respect to the inherited inner product. If (v, π) is a classical solution of (4.34), multiplying by $(\varphi, \phi) \in V \times W$ and integrating over Ω we obtain

$$\begin{cases} \lambda(v|\varphi)_{L^2(\Omega)} + (\nabla v|\nabla \varphi)_{L^2(\Omega)} - (\pi|\operatorname{div}_H \overline{\varphi})_{L^2(G)} = (f|\varphi)_{L^2(\Omega)}, & \varphi \in V, \\ \qquad\qquad -(\phi|\operatorname{div}_H \overline{v})_{L^2(G)} = 0, & \phi \in W, \end{cases} \tag{4.49}$$

where we applied Green's identity 3.4.3 and the fact that

$$\begin{aligned} (\nabla v \cdot \nu_{\partial\Omega}|\varphi)_{B^{1/2}_{2,2}(\partial\Omega)} =& (\partial_z v|\varphi)_{B^{1/2}_{2,2}(\partial\Omega \cap \Gamma_b)} - (\partial_z v|\varphi)_{B^{1/2}_{2,2}(\partial\Omega \cap \Gamma_a)} \\ &+ (\nabla_H v \cdot \nu_{\partial G}|\varphi)_{B^{1/2}_{2,2}(\partial\Omega \cap \Gamma_l)} = 0. \end{aligned} \tag{4.50}$$

Conversely, if (v, π) is smooth satisfying (4.49), then it defines a classical solution of (4.34). Note that if we test the second equation against constant functions $\phi \equiv c$ we have $\int_G c \operatorname{div}_H \overline{v} \, dx = c \int_{\partial G} \overline{v} \cdot \nu_{\partial G} = 0$ by the divergence theorem, independently of ϕ, hence it is enough to test against functions in $L^2_0(G)$.

Proposition 4.2.3. *Assume $\Gamma_D \neq \varnothing$. Let V and W be defined as in* (4.48)*. If $f \in V^*$, then there exists a unique solution $(v, \pi) \in V \times W$ to the weak resolvent problem* (4.49)*. Moreover, there exists a constant $C > 0$ such that*

$$\|v\|_V + \|\pi\|_W \leqslant C \|f\|_{V*}.$$

Proof. We can rephrase the weak resolvent problem (4.49) as a mixed variational problem

$$\begin{cases} \mathfrak{a}_\lambda(v,\varphi) + \mathfrak{b}(\varphi,\pi) = \langle f,\varphi\rangle, & \varphi \in V \\ \mathfrak{b}(v,\phi) = 0, & \phi \in W. \end{cases} \tag{4.51}$$

where $\mathfrak{a}_\lambda(\varphi,\psi) = \lambda(\varphi|\psi)_{L^2(\Omega)} + (\nabla\varphi|\nabla\psi)_{L^2(\Omega)}$ and $\mathfrak{b}(\varphi,\phi) = -(\phi|\operatorname{div}_H\overline{\varphi})$, for $\varphi,\psi \in V$ and $\phi \in W$. Both forms are easily seen to be sesquilinear and bounded

$$\begin{aligned} |\mathfrak{a}_\lambda(\varphi,\psi)| &\leqslant |\lambda|\|\varphi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} \\ &\leqslant (1+|\lambda|)\|\varphi\|_{W^{1,2}(\Omega)}\|\varphi\|_{W^{1,2}(\Omega)}, \end{aligned} \tag{4.52}$$

$$|\mathfrak{b}(\varphi,\phi)| \leqslant \|\phi\|_{L^2(G)}\|\operatorname{div}_H\overline{\varphi}\|_{L^2(G)} \leqslant \|\phi\|_{L^2(G)}\|\varphi\|_{W^{1,2}(G)}, \tag{4.53}$$

where the boundedness of the divergence operator in G follows from

$$\begin{aligned} \|\operatorname{div}_H\overline{\varphi}\|_{L^2(G)} &= \left(\int_G \left|\frac{\partial\overline{\varphi}_1}{\partial x_1} + \frac{\partial\overline{\varphi}_2}{\partial x_2}\right|^2 \mathrm{d}x\right)^{1/2} \leqslant \sqrt{2}\left(\sum_{i=1}^{2}\int_G \left|\frac{\partial\overline{\varphi}_i}{\partial x_i}\right|^2 \mathrm{d}x\right)^{1/2} \\ &\leqslant \left(\sum_{i,j=1}^{2}\int_G \left|\frac{\partial\overline{\varphi}_i}{\partial x_j}\right|^2 \mathrm{d}x\right)^{1/2} = \|\overline{\varphi}\|_{W^{1,2}(G)} \leqslant \|\varphi\|_{W^{1,2}(\Omega)}. \end{aligned} \tag{4.54}$$

By the LBB theorem 4.2.2, to prove the assertion it suffices to show that $\mathfrak{a}_\lambda$ and $\mathfrak{b}$ are respectively coercive and complying with the Babuška-Brezzi condition. Coercivity of $\mathfrak{a}_\lambda$ follows from Poincaré's inequality and the estimate $|s\lambda+t| \geqslant C_\varepsilon(s|\lambda|+t)$ for $s,t \geqslant 0$ and some constant $C_\varepsilon > 0$. Indeed take $\varphi \in V$, then there exists a constant $C > 0$ such that

$$\begin{aligned} |\mathfrak{a}_\lambda(\varphi,\varphi)| &= \left|\lambda\|\varphi\|^2_{L^2(\Omega)} + \|\nabla_H\varphi\|^2_{L^2(\Omega)}\right| \geqslant C(|\lambda|\|\varphi\|^2_{L^2(\Omega)} + \|\nabla_H\varphi\|^2_{L^2(\Omega)}) \\ &\geqslant C(|\lambda|\|\varphi\|^2_{L^2(\Omega)} + \|\nabla_H\varphi\|^2_{W^{1,2}(\Omega)}). \end{aligned} \tag{4.55}$$

The estimate can be shown to hold true by writing λ in polar coordinates $\lambda = re^{i\alpha}$ for $|\alpha| \leqslant \pi - \varepsilon$ and developing

$$\begin{aligned} |\lambda+t|^2 &= (r\cos\alpha + t)^2 + (r\sin\alpha)^2 = r^2\cos^2\alpha + t^2 + 2tr\cos\alpha + r^2\sin^2\alpha \\ &= r^2 + t^2 + 2tr\cos\alpha, \end{aligned} \tag{4.56}$$

- if $|\alpha| < \pi/2$, then $\cos\alpha \geqslant 0$ and

$$r^2 + t^2 + 2tr\cos\alpha \geqslant r^2 + t^2 \geqslant \frac{1}{2}(r^2+t^2) = \frac{1}{2}(|\lambda|^2 + t^2);$$

- if $|\alpha| > \pi/2$, since $\cos\alpha \leqslant 0$ and $2rt \leqslant r^2 + t^2$, we get $2rt\cos\alpha \geqslant (r^2+t^2)\cos\alpha$ and consequently

$$r^2+t^2+2tr\cos\alpha \geqslant (r^2+t^2)(1+\cos\alpha) \geqslant \frac{1}{2}(r^2+t^2)(1+\cos(\pi-\varepsilon)) \geqslant (|\lambda|^2+t^2)\sin^2\left(\frac{\varepsilon}{2}\right).$$

To prove that the Babuška-Brezzi condition holds we will apply a similar technique to proposition 4.1.3, proving it first for functions in G and extending it to Ω afterwards. Let

$\phi \in W = L^2_0(G)$, then there exists $\psi \in W^{1,2}_0(G)^2$ such that $\operatorname{div}_H \psi = \phi$ and $\|\psi\|_{W^{1,2}(G)} \leqslant C(G)\|\phi\|_{L^2(G)}$, see [31, Section II.2]. In particular, the inequality

$$\frac{|(\phi|\operatorname{div}_H \psi)_{L^2(G)}|}{\|\psi\|_{W^{1,2}(G)}} = \frac{\|\phi\|^2_{L^2(G)}}{\|\psi\|_{W^{1,2}(G)}} \geqslant \frac{1}{C(G)}\|\phi\|_{L^2(G)}, \tag{4.57}$$

holds. We can now define a function defined in the whole space Ω as

$$\varphi(x,y,z) = \chi(z)\psi(x,y),$$

where χ is a cut-off function $0 \leqslant \chi \leqslant 1$ in (a,b) taken as in (4.28). It is clear that $\varphi \in V$, because we are actually imposing Dirichlet conditions in the whole boundary $\partial\Omega$. We can estimate the norm of φ by

$$\begin{aligned}
\|\varphi\|_V &= \int_\Omega \left(|\varphi|^2 \,\mathrm{d}x + \sum_{i=1}^3 \int_\Omega \left|\frac{\partial\varphi}{\partial x_i}\right|^2 \,\mathrm{d}x \right)^{1/2} \\
&\leqslant \left(\int_\Omega \chi|\psi|^2 \,\mathrm{d}x + \sum_{i=1}^2 \int_\Omega \chi \left|\frac{\partial\psi}{\partial x_i}\right|^2 \,\mathrm{d}x + 2\int_\Omega \frac{\partial\chi}{\partial z}|\psi|^2 \,\mathrm{d}x \right)^{1/2} \\
&= (b-a)^{1/2} \left(\int_G |\psi| \,\mathrm{d}x + \sum_{i=1}^2 \int_G \left|\frac{\partial\psi}{\partial x_i}\right| \,\mathrm{d}x \right)^{1/2} = C(\Omega)\|\psi\|_{W^{1,2}(G)}.
\end{aligned} \tag{4.58}$$

Moreover,

$$\overline{\varphi} = \frac{1}{b-a}\int_a^b \varphi(\cdot,\cdot,z)\,\mathrm{d}z = \frac{1}{b-a}\int_a^b \chi(z)\psi(\cdot,\cdot)\,\mathrm{d}z = \psi,$$

thus $\operatorname{div}_H \overline{\varphi} = \operatorname{div}_H \psi$. In conclusion, given $\phi \in W$ we use the previous construction to obtain the bound

$$\frac{1}{C(G)}\|\phi\|_W \leqslant \frac{|(\phi|\operatorname{div}_H \psi)_{L^2(G)}|}{\|\psi\|_{W^{1,2}(G)}} \leqslant C(\Omega)\frac{|(\phi|\operatorname{div}_H \overline{\varphi})|_{L^2(G)}}{\|\varphi\|_V} = C(\Omega)\frac{|\mathfrak{b}(\varphi,\phi)|}{\|\varphi\|_V},$$

for some $\varphi \in V$. In fact, this is a lower bound in the supremum and the Babuška-Brezzi condition holds, completing the proof. $\square$

What is left to show are the $H^2 - H^1$ estimates for the solution (v,π). Let us first outline some properties of difference quotients. Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R}\backslash\{0\}$, the ith difference quotient of size $|h|$ is the function $D^h_i f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$D^h_i f(x) = \frac{f(x+he_i) - f(x)}{h},$$

where e_i is the unit vector in the ith direction. From now on, M stands for either Ω or G. In order to treat difference quotients of functions $f : \overline{M} \to \mathbb{R}$ we will make use of periodical extensions Ef. Let $\overline{\Omega}_1 := G_1 \times (a,b)$ with $G_1 := (-1/2, 3/2)^2$, we can extend $f : \overline{\Omega} \to \mathbb{R}$ to $\overline{\Omega}_1$ by

$$Ef(x + j/2, y + k/2, z) := f(x,y,z), \quad (x,y,z) \in \Omega, \quad j,k \in \{-1,0,1\}. \tag{4.59}$$

The same definition applies to $f : \overline{G} \to \mathbb{R}$, which by abuse of notation we will denote Ef as well. Note that if $f \in W^{1,2}_{\mathrm{per}}(M)$, then $Ef \in W^{1,2}(M_1)$ and $\|Ef\|_{W^{1,2}(M_1)} = 2^2\|f\|_{W^{1,2}(M)}$. Moreover, this extension is independent of vertical averaging, i.e. $E\overline{f} = \overline{Ef}$.

Proposition 4.2.4 ([16, Appendix 4.C.]). *Let $i \in \{1,2\}$ and $|h| < 1/2$. The difference quotient has the following properties:*

1. *Commutativity with weak derivatives: if $f, \partial_i f \in L^1_{loc}(M_1)$, then*

$$\partial_i D_j^h f = D_j^h \partial_i f.$$

2. *Integration by parts: if $f, g \in L^2(M)$, then*

$$(f|D_i^{-h}(Eg))_{L^2(M)} = (D_i^h(Ef)|g)_{L^2(M)}.$$

3. *Boundedness: if $\partial_i f \in L^2(M_1)$, then*

$$\|D_i^h(f)\|_{L^2(M)} \leqslant \|\partial_i f\|_{L^2(M)}.$$

4. *Uniform boundedness: if $f \in L^2(M_1)$ and $\|D_i^h f\|_{L^2(M)} \leqslant C$ for all $|h| < 1/2$ and some $C > 0$, then $f \in W^{1,2}(M)$ and $\|\partial_i f\|_{L^2(M)} \leqslant C$.*

Proof. (1) Is an immediate consequence of the linearity of weak derivatives

$$\partial_i D_j^h f = \partial_i \left(\frac{f(x+he_j) - f(x)}{h} \right) = \frac{\partial_i f(x+he_j) - \partial_i f(x)}{h} = D_j^h \partial_i f.$$

(2) Is an easy computation as well

$$\begin{aligned}(f|D_i^{-h}(Eg))_{L^2(M)} &= \int_M f D_i^{-h}(Eg)\,\mathrm{d}x = \frac{1}{h}\int_M f(x)(Eg(x-he_i) - Eg(x))\,\mathrm{d}x \\ &= \frac{1}{h}\int_M Ef(x'+he_i)g(x')\,\mathrm{d}x' - \frac{1}{h}\int_M f(x)g(x)\,\mathrm{d}x \\ &= \frac{1}{h}\int_M (Ef(x+he_i) - Ef(x))g(x)\,\mathrm{d}x = \int_M D_i^h(Ef)g\,\mathrm{d}x \\ &= (D_i^h(Ef)|g)_{L^2(M)}.\end{aligned} \tag{4.60}$$

(3) By an approximation argument we can assume f to be smooth. Then

$$f(x+he_i) - f(x) = h\int_0^h \partial_i f(x+te_i)\,\mathrm{d}t.$$

Applying Jensen's theorem for convex functions to $x \mapsto |x|^2$ we obtain the inequality

$$|f(x+he_i) - f(x)|^2 = \left| h\int_0^h \partial_i f(x+te_i)\,\mathrm{d}t \right|^2 \leqslant |h|^2 \int_0^h |\partial_i f(x+te_i)|^2\,\mathrm{d}t.$$

Now, integrating over M and noting that if $x \in M$, then $x + te_i \in M_1$ for all $|t| \leqslant h$, we get

$$\int_M \left| \frac{f(x+he_i) - f(x)}{h} \right|^2 \mathrm{d}x \leqslant \int_{M_1} |\partial_i f(x)|^2\,\mathrm{d}x,$$

where we applied Fubini's theorem. In conclusion, $\|D_i^h f\|_{L^2(M)} \leqslant \|\partial_i f\|_{L^2(M_1)}$ as desired.
(4) Fix i. Since the set $\{D_i^{-h} f \;:\; 0 < |h| < 1/2\}$ is bounded in $L^2(M_1)$, by Banach-Alaouglu's theorem there exists a subsequence $(h_k)_{k \geqslant 0}$ converging to 0 and a function

$g_i \in L^2(M_1)$ such that $D_i^{h_k} f \to g_i$ in $L^2(M_1)$ when $k \to \infty$. Take now an arbitrary $\phi \in C_c^\infty(M_1)$. For h_k small enough we have

$$\int_{M_1} f D_i^{-h_k} \phi \, \mathrm{d}x = \int_{M_1} \left(D_i^{h_k} f \right) \phi \, \mathrm{d}x,$$

hence letting $k \to \infty$, since ϕ is smooth $D_i^{-h_k}\phi$ converges uniformly to $\partial_i \phi$ and we get

$$\int_{M_1} f \partial_i \phi \, \mathrm{d}x = \int_{M_1} g_i \phi \, \mathrm{d}x.$$

We conclude that f is weakly differentiable with weak derivative $\partial_i f = g_i \in L^2(M_1)$ as desired. □

We can now proceed with the proof of theorem 4.2.1.

Proof. (*Theorem 4.2.1*) By proposition 4.2.3 there exists a unique solution $(v, \pi) \in V \times W$ to the weak resolvent problem (4.49). Let $i \in \{1, 2\}$ and choose $D_i^{-h}(E\varphi)$ as a test function in the variational problem 4.51, i.e. the first equation becomes

$$\mathfrak{a}_\lambda(v, D_i^{-h}(E\varphi)) + \mathfrak{b}(D_i^{-h}(E\varphi), \pi) = (f | D_i^{-h}(E\varphi))_{L^2(\Omega)}, \quad \varphi \in V.$$

By the integration by parts formula 4.2.4(2), we can rewrite it as

$$\mathfrak{a}_\lambda(D_i^h(Ev), \varphi) + \mathfrak{b}(\varphi, D_i^h(E\pi)) = (f | D_i^{-h}(E\varphi))_{L^2(\Omega)}, \quad \varphi \in V. \tag{4.61}$$

In particular, we can take $\varphi = D_i^h(Ev)$. Observe that substituting φ in the second equation in the variational problem 4.51, we necessarily have

$$\mathfrak{b}(D_i^h(Ev), D_i^h(E\pi)) = \mathfrak{b}(v, D_i^{-h} D_i^h(E\pi)) = 0.$$

Combining these results with the coercivity of $\mathfrak{a}_\lambda$, proven in proposition 4.2.3, we get the inequality

$$\begin{aligned} \alpha \| D_i^h(Ev) \|_V^2 &\leqslant |\mathfrak{a}_\lambda(D_i^h(Ev), D_i^h(Ev))| \leqslant \|f\|_{L^2(\Omega)} \| D_i^{-h} D_i^h(Ev) \|_{L^2(\Omega)} \\ &\leqslant \|f\|_{L^2(\Omega)} \| D_i^h(Ev) \|_V . \end{aligned} \tag{4.62}$$

where in the last equation we used boundedness of the difference quotients 4.2.4(3). Thus the difference quotient of v is bounded by $\|D_i^h(Ev)\|_{W^{1,2}(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}$. We can deduce the inequality on π by a similar approach, taking $\phi = D_i^h(E\pi)$. From the Babuska-Brezzi condition for $\mathfrak{b}$ and equation 4.61 we get

$$\begin{aligned} \beta \| D_i^h(E\pi) \|_W &\leqslant \sup_{\varphi \in V} \frac{|\mathfrak{b}(\varphi, D_i^h(E\pi))|}{\|\varphi\|_V} = \sup_{\varphi \in V} \frac{|(f | D_i^{-h}(E\varphi))_{L^2(\Omega)} - \mathfrak{a}_\lambda(D_i^h(Ev), \varphi)|}{\|\varphi\|_V} \\ &\leqslant \sup_{\varphi \in V} \frac{\|f\|_2 \|D_i^{-h}(E\varphi)\|_2 + |\lambda| \|D_i^h(Ev)\|_2 \|\varphi\|_2 + \|\nabla D_i^h(Ev)\|_2 \|\nabla \varphi\|_2}{\|\varphi\|_V} \\ &\leqslant \sup_{\varphi \in V} \frac{\|f\|_2 \|\partial_i \varphi\|_2 + |\lambda| C \|f\|_2 \|\varphi\|_2 + C\|f\|_2 \|\nabla\varphi\|_2}{\|\varphi\|_V} \\ &\leqslant \sup_{\varphi \in V} \frac{\|f\|_2 \|\varphi\|_V (1 + C|\lambda| + C)}{\|\varphi\|_V} = C' \|f\|_{L^2(\Omega)}, \end{aligned} \tag{4.63}$$

for some constant $C' > 0$, where we constantly applied properties of difference quotients and the boundedness of the difference quotient of v. We have thus proven that the difference quotients are uniformly bounded in h for $i \in \{1,2\}$, which leads to $\nabla_H v \in W^{1,2}_{\mathrm{per}}(\Omega)^2$ and $\nabla_H \pi \in L^2(G)^2$. Moreover, the resolvent problem (4.34) implies that $-\partial_z^2 v = f - \lambda v + \Delta_H v - \nabla_H \pi \in L^2(\Omega)^2$, hence $v \in W^{2,2}_{\mathrm{per}}(\Omega)^2$ and $\pi \in W^{1,2}_{\mathrm{per}}(G)$ as desired. Finally, the inequality (4.36) is obtained taking $\varphi = v$ in the variational problem (4.51), note that $\mathfrak{b}(\pi, v) = 0$, and applying the coercivity of $\mathfrak{a}_\lambda$ as in (4.55), this is

$$C|\lambda|\|v\|^2_{L^2(\Omega)} \leqslant |\mathfrak{a}_\lambda(v,v)| \leqslant \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)},$$

thus $|\lambda|\|v\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}$. □

We can now prove the main result of this subsection, that the hydrostatic Stokes operator A generates an exponentially stable analytic semigroup in $L^2_{\overline{\sigma}}(\Omega)$, which was first shown by Hieber et. al. [15] and will be very helpful in the next section to prove that A admits a bounded H^∞-calculus.

Theorem 4.2.5 ([15], Proposition 4.4)**.** *Assume $\Gamma_D \neq \emptyset$. The hydrostatic Stokes operator A_2 is invertible and generates a bounded analytic C_0-semigroup $(T_2(t))_{t\geqslant 0}$ on $L^2_{\overline{\sigma}}(\Omega)$. Moreover, there exist constants $C, \beta > 0$ such that*

$$\|T_2(t)f\|_{L^2_{\overline{\sigma}}(\Omega)} \leqslant Ce^{-\beta t}\|f\|_{L^2_{\overline{\sigma}}(\Omega)}, \quad t > 0. \tag{4.64}$$

Proof. The hydrostatic Stokes operator A_2 is clearly densely defined since $V \subset D(A_2)$ and according to proposition 4.1.3 the completion of V in the L^2-norm is the whole space $L^2_{\overline{\sigma}}(\Omega)$. Let now $\lambda \in \Sigma_{\pi-\varepsilon} \cup \{0\}$ for some $\varepsilon \in (0, \pi/2)$ and $f \in L^2_{\overline{\sigma}}(\Omega)$. Then there exists $v \in D(A_2)$ such that $(\lambda - A_2)v = f$ if and only if the resolvent problem (4.34)-(4.35) admits a unique solution $(v, \pi) \in W^{2,2}_{\mathrm{per}}(\Omega)^2 \times W^{1,2}_{\mathrm{per}}(G) \cap L^2_0(G)$. However, the latter is true by theorem 4.2.1, thus $\Sigma_{\pi-\varepsilon} \cup \{0\} \subseteq \rho(A_2)$. Moreover, the bound (4.36) implies

$$\begin{aligned}\sup_{\lambda\in\Sigma_{\pi-\varepsilon}} \|\lambda R(\lambda, A_2)\|_{\mathcal{L}(L^2_{\overline{\sigma}}(\Omega))} &= \sup_{\lambda\in\Sigma_{\pi-\varepsilon}} \sup_{\substack{f\in L^2_{\overline{\sigma}}\\ \|f\|\leqslant 1}} \|\lambda(\lambda - A_2)^{-1}f\|_{L^2_{\overline{\sigma}}(\Omega)} \\ &= \sup_{\lambda\in\Sigma_{\pi-\varepsilon}} \sup_{\substack{f\in L^2_{\overline{\sigma}}\\ \|f\|\leqslant 1\\ v=(\lambda-A_2)^{-1}f}} \|\lambda v\|_{L^2_{\overline{\sigma}}(\Omega)} \leqslant C.\end{aligned}$$

Note that A_2 is closed since $0 \in \rho(A_2)$. By the generation theorem for analytic semigroups 3.5.8, we have that A_2 generates a bounded analytic semigroup of angle $\pi/2$ and we have the inverse Laplace transform representation

$$T_2(t)f = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} e^{zt}R(z, A_2)f\,\mathrm{d}z, \quad t > 0,\ f \in L^2_{\overline{\sigma}}(\Omega),$$

where $\partial\Sigma_\nu$ is the upwards oriented contour line for any $\nu \in (0, \pi/2)$. Since $0 \in \rho(A_2)$ and the resolvent is open, there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subset \rho(A_2)$. As a consequence, the open sector of angle $\nu' \in (0, \nu)$ and center $-\beta$ with $\beta \in (0, \varepsilon)$ is contained in the resolvent set of A_2. In particular,

$$\Sigma_{\nu'} \subset \rho(A_2 + \beta) \quad \text{and} \quad \sup_{\lambda\in\Sigma_{\nu'}} \|\lambda R(\lambda, A_2 + \beta)\| = \sup_{\lambda\in\Sigma_{\nu'}} \|\lambda R(\lambda - \beta, A_2)\| \leqslant C.$$

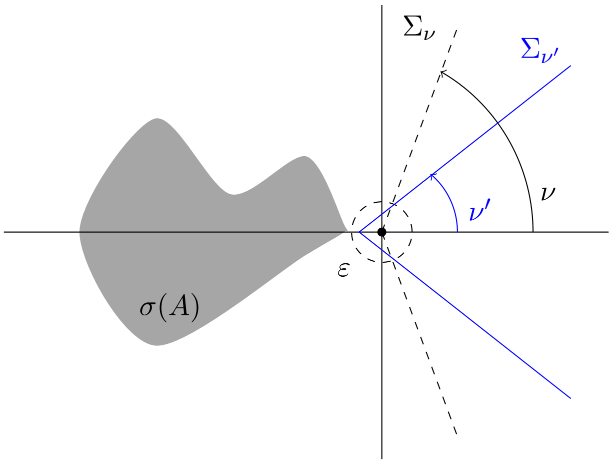


Figure 4.2: Spectrum of a sectorial operator.

Therefore, $A_2 + \beta$ generates a bounded analytic semigroup $\{T_\beta(t)\}_{t\geqslant 0}$. The two semigroups are closely related

$$T_\beta(t) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\nu'}} e^{zt}R(z, A_2+\beta)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} e^{(z+\beta)t}R(z, A_2)\,\mathrm{d}z = e^{\beta t}T_2(t),$$

which proportionates the desired exponential stability

$$\|T_2(t)f\|_{L^2_{\overline{\sigma}}(\Omega)} = \|e^{-\beta t}T_\beta(t)f\|_{L^2_{\overline{\sigma}}(\Omega)} \leqslant Ce^{-\beta t}\|f\|_{L^2_{\overline{\sigma}}(\Omega)}$$

□

4.3 H^∞-calculus of the hydrostatic Stokes operator

Giga et. al. [7] further showed that $-A_2$ actually admits a bounded H^∞-calculus making use of perturbation techniques studied in section 3.7. This section is devoted to the proof of this result and some immediate corollaries.

We begin with a restatement of the Cauchy problem (4.32). Averaging the resolvent problem (4.34) vertically yields

$$\begin{aligned} \lambda\overline{v} - \Delta_H\overline{v} - \left(\partial_z v\big|_{\Gamma_b} - \partial_z v\big|_{\Gamma_a}\right) + \nabla_H\pi_s &= \overline{f} \qquad \text{on } G,\\ \mathrm{div}_H\,\overline{v} &= 0 \qquad \text{on } G, \end{aligned} \tag{4.65}$$

where we applied Leibniz's integral rule for $\Delta_H v$ and the fundamental theorem of calculus for $\partial_z^2 v$. Taking horizontal divergence we can rewrite it as the weak problem

$$\Delta_H\pi_s = \mathrm{div}_H\,\overline{f} + \mathrm{div}_H\,\partial_z v\big|_{\Gamma_D},$$

where we introduce the simplified notation for the boundary term since $\partial_z v\big|_{\Gamma_N} = 0$. Solving this for $\nabla_H\pi_s$ we obtain the characterization

$$\nabla_H\pi_s = \nabla_H\Delta_H^{-1}\,\mathrm{div}_H\,\overline{f} + \nabla_H\Delta_H^{-1}\,\mathrm{div}_H\,\partial_z v\big|_{\Gamma_D}, \tag{4.66}$$

which inserted in the hydrostatic Stokes equations (4.1) provides

$$\begin{cases} \partial_t v - \Delta v + \nabla_H \Delta_H^{-1} \operatorname{div}_H \partial_z v\big|_D = f - \nabla_H \Delta_H^{-1} \operatorname{div}_H \overline{f} & \text{on } \Omega, \\ \operatorname{div}_H \overline{v} = 0 & \text{on } G. \end{cases} \tag{4.67}$$

Applying the hydrostatic Helmholtz projection we obtain the equivalent problem

$$\partial_t v - (\Delta + B)v = P_2 f, \quad \operatorname{div}_H \overline{v} = 0, \quad v(0) = v_0,$$

where

$$Bv := -\nabla_H \Delta_H^{-1} \operatorname{div}_H \partial_z v\big|_D. \tag{4.68}$$

Here and subsequently, we will study the resolvent problem

$$\lambda v - (\Delta_2 + B_2)v = P_2 f$$

where Δ_2 denotes the $L^2(\Omega)$-realization of the Laplacian with general boundary conditions

$$\Delta_2 v := \Delta v \quad \text{with} \quad D(\Delta_2) := \{v \in H^{2,2}_{\text{per}}(\Omega)^2 : \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, \tag{4.69}$$

and B_2 is defined for some $\delta \in (0, 1/2)$ as

$$B_2 v := Bv, \quad \text{with} \quad D(B_2) = H^{1+1/2+\delta,2}(\Omega)^2.$$

Before we state the main theorem let us first establish the H^∞-boundedness of the $L^2(\Omega)$-realization of the Laplacian, which is not only essential for the later perturbation techniques but also of interest on its own. However, we shall first prove some important reflection arguments, whose construction is adapted from Nau's dissertation [27, Proposition 7.16] and Krylov's book [21, Lemma 8.2.1].

Lemma 4.3.1. *Let a function* $u \in H^{2,2}_{b.c.}(0, 1/2)$ *with*

$$H^{2,2}_{b.c.}(0, 1/2) := \{u \in H^{2,2}(0, 1/2) : u\big|_{\Gamma_D} = \partial u\big|_{\Gamma_N} = 0\}.$$

On the one hand, define $\overline{u}$ *via the odd extension to* $(0, 1)$ *given by*

$$\overline{u}(x) := \begin{cases} u(x) & \text{if } x \leqslant 1/2 \\ -u(1-x) & \text{if } x > 1/2 \end{cases} \tag{4.70}$$

if we impose

i) Dirichlet conditions $u\big|_{\{x=0\}} = u\big|_{\{x=1/2\}} = 0$ *or;*

ii) Neumann-Dirichlet conditions $\partial u\big|_{\{x=0\}} = u\big|_{\{x=1/2\}} = 0$ *in the trace sense.*

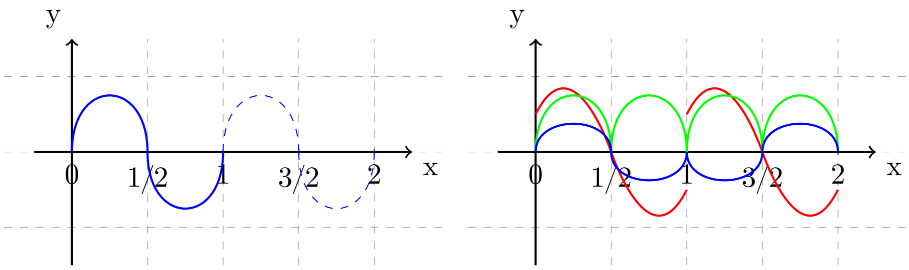
On the other hand, define $\overline{u}$ *via the even extension to* $(0, 1)$ *given by*

$$\overline{u}(x) := \begin{cases} u(x) & \text{if } x \leqslant 1/2 \\ u(1-x) & \text{if } x > 1/2 \end{cases} \tag{4.71}$$

if we impose

iv) Neumann conditions $\partial u\big|_{\{x=0\}} = \partial u\big|_{\{x=1/2\}} = 0$ *or;*

v) Dirichlet-Neumann conditions $u\big|_{\{x=0\}} = \partial u\big|_{\{x=1/2\}} = 0$ *in the trace sense.*



(a) Dirichlet boundary, odd-periodic extension.

(b) Other boundary values, even or antiperiodic extensions.

Figure 4.3

Then $u \in H^{2,2}_{b.c.}(0,1/2)$ *if and only if, we have* $\overline{u} \in H^{2,2}_{\mathrm{per}}(0,1)$ *for pure boundary conditions and* $\overline{u} \in H^{2,2}_{\mathrm{antiper}}(0,1)$ *for mixed boundary conditions.*

Proof. (*i*) We first prove the necessary condition for the Dirichlet case in detail. For a visual understanding of why an odd periodic extension is required for Dirichlet boundary conditions, we refer the reader to figure (4.3).

Let $\overline{u} \in H^{2,2}_{\mathrm{per}}(0,1)$, then by density of smooth periodic functions there exists a sequence $\{v_n\}_n \subset C^\infty_{\mathrm{per}}[0,1]$ such that v_n converges to $\overline{u}$ in $\|\cdot\|_{H^{2,2}(0,1)}$. Since $\overline{u}$ is odd with respect to $1/2$ by definition, it further holds that

$$-v_n(1-x) \to -\overline{u}(1-x) = \overline{u}(x).$$

Therefore, we can define

$$\overline{v}_n = \frac{v_n(x) - v_n(1-x)}{2},$$

which trivially converges to the extension $\overline{u}$. These periodic smooth functions can be evaluated on the boundary, in particular, we see that when restricted to the interval $(0,1/2)$ they comply with Dirichlet boundary conditions

$$\overline{v}_n(0) = \frac{v_n(0) - v_n(1)}{2} = 0 \quad \text{and} \quad \overline{v}_n(1/2) = \frac{v_n(1/2) - v_n(1/2)}{2} = 0.$$

We only need to show that $\overline{v}_n \to u$, which is a simple matter of checking

$$\|\overline{v}_n - u\|_{(0,1/2)} = \|\overline{v}_n - \overline{u}\|_{(0,1/2)} \leqslant \|\overline{v}_n - \overline{u}\|_{(0,1)} \to 0.$$

Finally, since $\{\partial\overline{v}_n\}_n$ and $\{\partial^2\overline{v}_n\}_n$ are Cauchy sequences

$$\|\partial\overline{v}_n - \partial\overline{v}_m\|_{(0,1/2)} = \|\partial\overline{v}_n - \partial\overline{v}_m\|_{(0,1)} \to 0,$$
$$\|\partial^2\overline{v}_n - \partial^2\overline{v}_m\|_{(0,1/2)} = \|\partial^2\overline{v}_n - \partial^2\overline{v}_m\|_{(0,1)} \to 0,$$

when $n, m \to \infty$, we have that $u \in H^{2,2}_{b.c.}(0,1/2)$ as desired.

Next we prove the sufficient condition. Let $u \in H^{2,2}_{b.c.}(0,1/2)$ and take a defining sequence $\{w_n\}_n \subset C^\infty_{b.c.}[0,1/2]$. We can extend the functions $\{w_n\}_n$ oddly as in (4.70), which leads to the following definition of the derivative

$$\partial\overline{w}_n(x) := \begin{cases} \partial w_n(x) & \text{if } x \leqslant 1/2 \\ (\partial w_n)(1-x) & \text{if } x > 1/2. \end{cases} \tag{4.72}$$

Periodicity of order 1 is then an immediate consequence

$$\overline{w}_n(0) = w_n(0) = 0 = -w_n(1) = \overline{w}_n(1),$$
$$\partial\overline{w}_n(0) = \partial w_n(0) = \partial\overline{w}_n(1).$$

Furthermore, equation (4.72) shows that $\{\partial\overline{w}_n\}_n$ is a Cauchy sequence and obviously $\overline{w}_n \to \overline{u}$, thus $\overline{u} \in H^{1,2}_{\text{per}}(0,1)$ and its derivative is given by

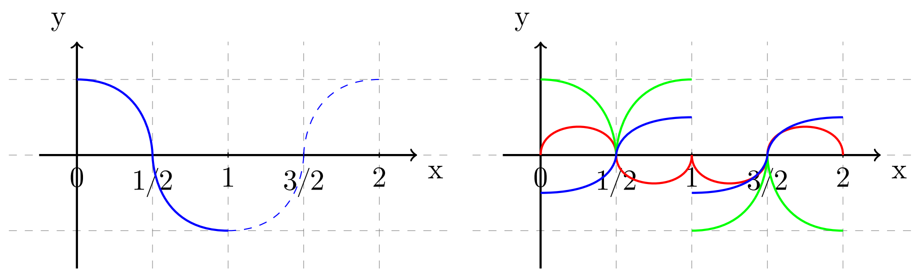
$$\partial\overline{u}(x) := \begin{cases} \partial u(x) & \text{if } x \leqslant 1/2 \\ (\partial u)(1-x) & \text{if } x > 1/2. \end{cases} \tag{4.73}$$

We can now deduce that the second derivative

$$\partial^2\overline{u}(x) := \begin{cases} \partial^2 u(x) & \text{if } x \leqslant 1/2 \\ -(\partial^2 u)(1-x) & \text{if } x > 1/2, \end{cases} \tag{4.74}$$

is clearly in $L^2(0,1)$. We have therefore proved that $\overline{u} \in H^{2,2}_{\text{per}}(0,1)$.

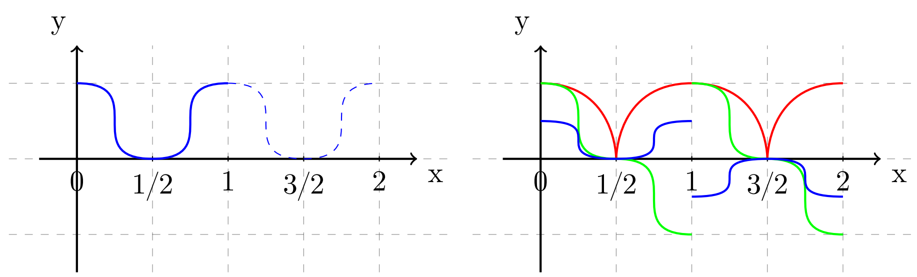
(ii) The construction of the mixed Neumann-Dirichlet case is completely analogous, for a visual understanding of why the extension $\overline{u}$ is odd and antiperiodic see figure (4.4).



(a) Neumann-Dirichlet boundary, odd-antiperiodic extension. (b) Other boundary values, even or periodic extensions.

Figure 4.4

(iii) For the sake of clarity we will include a brief summary of the case of pure Neumann boundary values, see figure (4.5). Let $\overline{u} \in H^{2,2}_{\text{per}}(0,1)$ and take a defining



(a) Neumann boundary, even-periodic extension. (b) Other boundary values, odd or antiperiodic extensions.

Figure 4.5

sequence $\{v_n\}_n \subset C^\infty_{\text{per}}[0,1]$ such that $v_n \to \overline{u}$. Since $\overline{u}$ is even, it holds that

$$\overline{v}_n(x) := \frac{v_n(x) + v_n(1-x)}{2} \to \overline{u}, \quad \partial\overline{v}_n = \frac{\partial v_n(x) - (\partial v_n)(1-x)}{2}, \tag{4.75}$$

and it complies with Neumann boundary values

$$\partial\overline{v}_n(0) = \frac{\partial v_n(0) - \partial v_n(1)}{2} = 0 = \frac{\partial v_n(1/2) - \partial v_n(1/2)}{2} = \partial\overline{v}_n(1/2).$$

Following the same argumentation as in (i) we see that $\overline{v}_n \to u$ and $u \in H^{2,2}_{b.c.}(0,1/2)$, thus the necessary condition is proven. For the sufficient condition let $u \in H^{2,2}_{b.c.}(0,1/2)$ and take a defining sequence $\{w_n\}_n \subset C^\infty_{b.c.}[0,1/2]$. Extend the functions $\{w_n\}_n$ evenly as in (4.71) leads to the derivatives

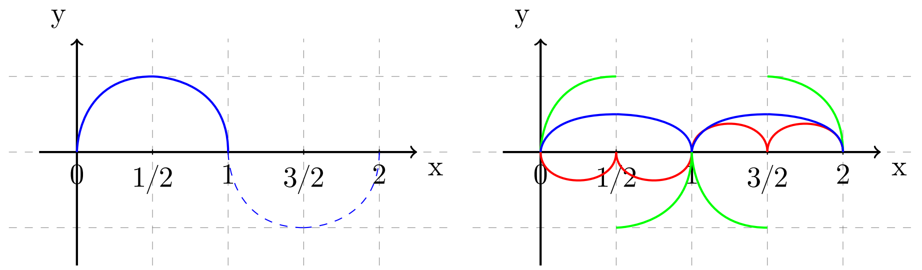
$$\partial\overline{w}_n(x) := \begin{cases} \partial w_n(x) & \text{if } x \leqslant 1/2 \\ -(\partial w_n)(1-x) & \text{if } x > 1/2. \end{cases} \tag{4.76}$$

which comply with Neumann boundary conditions and show periodicity of order 1 of the sequence

$$\overline{w}_n(0) = w_n(0) = 0 = \overline{w}_n(1)$$
$$\partial\overline{w}_n(0) = \partial w_n(0) = 0 = \partial\overline{w}_n(1).$$

Once again by the same argument as in (i) we conclude that $\overline{u} \in H^{2,2}_{\text{per}}(0,1)$.

(iv) The construction of the mixed Dirichlet-Neumann case is completely analogous, for a visual understanding of why the extension $\overline{u}$ is even and antiperiodic see figure (4.6).



(a) Dirichlet-Neumann boundary, even-antiperiodic extension. (b) Other boundary values, odd or periodic extensions.

Figure 4.6

□

Lemma 4.3.2. *Given $f \in L^2([0,1]^3)$, there exists a unique solution $u \in H^{2,2}([0,1]^3)$ of the partial differential equation*

$$(I - \Delta)u = f \quad in \quad [0,1]^3 \tag{4.77}$$
$$u \quad (anti)periodic\ in\ x_j\ for\ j = 1,2,3. \tag{4.78}$$

Proof. By a periodic extension and a linear transformation of coordinates, we can regard antiperiodic function as periodic functions and the problem reduces to finding $u \in H^{2,2}(\mathbb{T}^3)$ solving :

$$(I - \Delta)u = f, \quad \text{for } f \in L^2(\mathbb{T}^3). \tag{4.79}$$

Since the Fourier transform on the torus $\mathcal{F} : L^2(\mathbb{T}^3) \to l^2(\mathbb{Z}^3)$ given by

$$\mathcal{F}(f)(k) := \widehat{f}(k) := \int_{\mathbb{T}^3} f(x)e^{-2\pi i x\cdot k}\,\mathrm{d}x, \quad k \in \mathbb{Z}^3$$

defines an isometry, we can apply the Fourier transform to the problem (4.79) and find solutions there, which will uniquely define our functions in $L^2(\mathbb{T}^3)$. Recall that the Fourier transform of a derivative obeys

$$\widehat{(\partial_j f)}(k) = \int_{\mathbb{T}^3} (\partial_j f)(x)e^{-2\pi i x\cdot k}\,\mathrm{d}x = (-2\pi i k)\int_{\mathbb{T}^3} f(x)e^{-2\pi i x\cdot k}\,\mathrm{d}x = (-2\pi i k)\widehat{f}(k),$$

thus the problem (4.79) becomes finding $\widehat{u} \in l^2(\mathbb{Z}^3)$ solving

$$(1 + (2\pi|k|)^2)\widehat{u}(k) = \widehat{f}(k), \quad \text{for } \widehat{f} \in l^2(\mathbb{Z}^3).$$

This is a trivial task since we can clear the coefficients of $\widehat{u}$ by

$$\widehat{u}(k) = \frac{1}{1 + 4\pi^2|k|^2}\widehat{f}(k),$$

and recover $u \in L^2(\mathbb{T}^3)$ uniquely from its Fourier series

$$u(\xi) = \sum_{k\in\mathbb{Z}^3} \frac{1}{1 + 4\pi^2|k|^2}\widehat{f}e^{2\pi i k\cdot\xi}, \quad \xi \in \mathbb{T}^3,$$

which converges because $f \in l^2(\mathbb{T}^3)$. To see that $u \in H^{2,2}(\mathbb{T}^3)$ it suffices to show that its mixed partial derivatives

$$\widehat{\partial_j\partial_r u}(k) = -4\pi^2 k_j k_r \widehat{u}(k)$$

also define a bounded series

$$u(\xi) = \sum_{k\in\mathbb{Z}^3} \frac{-4\pi^2 k_j k_r}{1 + 4\pi^2|k|^2}\widehat{f}e^{2\pi i k\cdot\xi}, \quad \xi \in \mathbb{T}^3.$$

□

Proposition 4.3.3. *Let $\nu \geqslant 0$. Then the operator $-\Delta_2 + \nu$ admits a H^∞-calculus on $L^2(\Omega)$ of angle 0 provided $\nu > 0$. If $\Gamma_D \neq \emptyset$, then the above assertion holds true even for $\nu = 0$.*

Proof. By proposition 3.6.4 it suffices to show that the negative Laplacian $-\Delta_2$ is a densely defined, positive, self- adjoint operator on the Hilbert space $L^2(\Omega)$.

Density of the domain $D(-\Delta_2)$ in $L^2(\Omega)$ follows immediately from the density of

$$V := \{v \in C^\infty_{\text{per}}(\overline{\Omega}) \ : \ z \mapsto v(\cdot,\cdot,z) \text{ is compactly supported in } (a,b)\}$$

in $L^2(\Omega)$ and the chain of inclusions $V \subset D(-\Delta_2) \subset L^2(\Omega)$. Positivity holds trivially as well since given $u \in D(-\Delta_2)$, we have

$$\begin{aligned}(-\Delta_2 u|u) &= -\int_\Omega \sum_{j=1}^3 \frac{\partial^2 u}{\partial x_j^2}\widetilde{u}\,\mathrm{d}x = -\left[\sum_{j=1}^3 \frac{\partial u}{\partial x_j}\widetilde{u}\right]_{\partial\Omega} + \int_\Omega \sum_{j=1}^3 \frac{\partial u}{\partial x_j}\frac{\partial \widetilde{u}}{\partial x_j}\,\mathrm{d}x \\ &= \int_\Omega \sum_{j=1}^3 \left|\frac{\partial u}{\partial x_j}\right|^2 \mathrm{d}x \geqslant 0,\end{aligned} \tag{4.80}$$

where the vanishing of boundary terms in the last equality is possible by periodicity of u in the lateral boundary Γ_l and Dirichlet-Neumann conditions in the vertical one, i.e. either $u = 0$ or $\partial_z u = 0$ on $\partial\Omega\backslash\Gamma_l$.

What is left to show is the self-adjointness of the negative Laplacian in $L^2(\Omega)$. Let $H_D^{1,2}(\Omega)$ denote the space of functions vanishing on Γ_D,

$$H_D^{1,2}(\Omega) := \{v \in H_{\mathrm{per}}^{1,2}(\Omega)^2 \;:\; v\big|_{\Gamma_D} = 0\}.$$

Note that $H_D^{1,2}(\Omega)$ is closed in $H^{1,2}(\Omega)$, thus it is a Hilbert space with respect to the inner product $\|\cdot\|_{H^{1,2}(\Omega)}$. Let the mapping S be defined by

$$\langle Su, v\rangle = (u|v) + (\partial u|\partial v) \tag{4.81}$$

from $H_D^{1,2}(\Omega)$ to its dual $H_D^{1,2}(\Omega)^*$. The proof will be divided into two steps: first we prove the self-adjointness of S^{-1} restricted to $L^2(\Omega)$ following Taylor's trick in [32, Chapter 8.2] and then show that the domains of its inverse and the L^2-Laplacian Δ_2 coincide by applying reflection arguments (see [21, Chapter 8]).

Let us first prove that S is bijective. Injectivity follows from its definition (4.81) taking $v = u$, then

$$\|Su\|_{H^{1,2}(\Omega)^*}\|u\|_{H^{1,2}(\Omega)} \geqslant |\langle Su, u\rangle| = \|u\|_{H^{1,2}(\Omega)}^2,$$

hence if $Su = 0$, then $u = 0$. Suppose that S is not surjective, namely that $R(S)^\perp \neq \varnothing$ in $H_D^{1,2}(\Omega)^*$. Then there exists a nonzero element $v \in H_D^{1,2}(\Omega)$ such that

$$\langle Su, v\rangle = 0 \quad \text{for all } u \in H_D^{1,2}(\Omega),$$

in particular, for $u = v$ we have that $\langle Sv, v\rangle = \|v\|_{H^{1,2}(\Omega)} = 0$, thus $v = 0$, contradicting our assumption. The uniquely determined inverse of S, denoted by

$$T := S^{-1} : H_D^{1,2}(\Omega)^* \to H_D^{1,2}(\Omega),$$

is self-adjoint when restricted to $L^2(\Omega)$ because S is symmetric. Indeed for simplicity of notation we use the same letter T for the restriction $T\big|_{L^2(\Omega)}$. Take $\varphi, \psi \in L^2(\Omega) \subset H_D^{1,2}(\Omega)^*$. Then there exist $u, v \in H_D^{1,2}(\Omega)$ such that $\varphi = Su$ and $\psi = Sv$, and consequently

$$\begin{aligned}\langle T\varphi, \psi\rangle &= \langle TSu, Sv\rangle = \langle u, Sv\rangle \\ &= (u|v) + (\partial u|\partial v) \\ &= \langle Su, v\rangle = \langle Su, TSv\rangle \\ &= \langle \varphi, T\psi\rangle.\end{aligned} \tag{4.82}$$

Since the inverse of an injective self-adjoint operator on a Hilbert space is also self-adjoint with dense domain, see for instance [30, Theorem 13.11], it follows that T^{-1} is self-adjoint with dense domain $R(T)$.

Since $T^{-1}u = (I - \Delta_2)u$ if $u \in D(\Delta_2)$, it only remains to prove that the domains of T^{-1} and the Laplacian Δ_2 as defined in (4.69) coincide. By the definition of range $R(T) = \{Tf \ : \ f \in L^2(\Omega)\}$, if we show that given $f \in L^2(\Omega)$ there exists a unique $u \in D(\Delta_2)$ such that $u = Tf$, i.e. $T^{-1}u = f$, the theorem follows. The proof relies on lemmas 4.3.1 and 4.3.2.

Note that in the current setting $D(\Delta_2)$ imposes periodic boundary conditions in the horizontal directions and Dirichlet-Neumann ones in the vertical one, i.e. given $f \in L^2(\Omega)$ we want to obtain a unique solution $u \in H^{2,2}(\Omega)$ to the problem

$$\begin{aligned} T^{-1}u = f \qquad & \text{in} \quad \Omega, \\ u\big|_{\{x_j=0\}} = u\big|_{\{x_j=1\}} \qquad & \text{for } \ j = 1, 2, \\ u\big|_{\Gamma_D} = \partial_3 u\big|_{\Gamma_N} = 0. & \end{aligned} \tag{4.83}$$

Thus it suffices to transform the vertical variable to the periodic/antiperiodic case, for which we will use the reflection arguments in lemma 4.3.1. We can assume, by a linear transformation of coordinates if necessary, that $(a, b) = (0, 1/2)$.

We will start with pure Dirichlet boundary conditions. Let $f \in L^2(\Omega)$ be arbitrary, we can extend f oddly to $[0, 1]^3$ in the following way

$$\bar{f}(x) := \begin{cases} f(x) & \text{if } x_3 \leqslant 1/2 \\ -f(x_1, x_2, 1 - x_3) & \text{if } x_3 > 1/2. \end{cases} \tag{4.84}$$

Since $\bar{f} \in L^2([0, 1]^3)$, there exists a unique solution $\bar{u} \in H^{2,2}([0, 1]^3)$ to the problem (4.77) with periodic boundary conditions in all three directions. Moreover, from the properties of $I - \Delta_2$ we deduce that $-\bar{u}(x_1, x_2, 1 - x_3)$ is also a solution

$$(I-\Delta)(-\overline{u}(x_1, x_2, 1-x_3)) = -[\overline{u}(x_1, x_2, 1-x_3)-(\Delta\overline{u})(x_1, x_2, 1-x_3)] = -f(x_2, x_2, 1-x_3),$$

which by uniqueness yields that $\bar{u}\big|_{x_3} \in H^{2,2}_{\text{per}}(0, 1)$. Now, by lemma 4.3.1 the solution $u\big|_{x_3} \in H^{2,2}_{b.c.}(0, 1/2)$ complies with Dirichlet boundary conditions

$$u\big|_{x_3=0} = u\big|_{x_3=1/2} = 0,$$

therefore $u \in D(\Delta_2)$. It remains to prove uniqueness of solutions in $[0, 1/2]$. Let $v \in H^{2,2}(\Omega)$ be another solution of (4.77), if we extend v and f oddly to $[0, 1]^3$ by the same method as (4.84), then the uniqueness of solution in $[0, 1]^3$ implies uniqueness of u as well, and the assertion follows.

The same conclusion can be drawn for pure Neumann boundary conditions, where given $f \in L^2(\Omega)$ the extension to $[0, 1]^3$ is constructed evenly instead

$$\bar{f}(x) := \begin{cases} f(x) & \text{if } x_3 \leqslant 1/2 \\ f(x_1, x_2, 1 - x_3) & \text{if } x_3 > 1/2 \end{cases} \tag{4.85}$$

In this case, the solution restricted to the vertical variable is $\bar{u}\big|_{x_3} \in H^{2,2}_{\text{per}}(0, 1)$, thus lemma 4.3.1 yields that the restriction to $[0, 1/2]$, $u \in H^{2,2}_{b.c.}(0, 1/2)$ complies with Neumann boundary conditions

$$\partial_3 u\big|_{x_3=0} = \partial_3 u\big|_{x_3=1/2} = 0.$$

The rest of the proof runs as before.

The only remaining concern is the case of mixed Dirichlet-Neumann boundary conditions. It can be proved in much the same way, the only difference being that instead of transforming the problem to periodic boundary conditions in all three variables, we use the result of existence and uniqueness of solutions in lemma 4.3.2 with periodic boundary conditions in horizontal variables x_1 and x_2, and antiperiodic in the vertical one x_3. On the one hand if $\Gamma_D = \Gamma_a$ and $\Gamma_N = \Gamma_b$, then we extend f evenly to $[0,1]^3$. Restricted to x_3 the solution is $\overline{u}\big|_{x_3} \in H^{2,2}_{\text{antiper}}(0,1)$, thus by lemma 4.3.1 we have

$$u\big|_{x_3=0} = \partial_3 u\big|_{x_3=1/2} = 0.$$

On the other hand, if $\Gamma_D = \Gamma_b$ and $\Gamma_N = \Gamma_a$, then we extend f oddly to $[0,1]^3$. Restricted to x_3 the solution is $\overline{u}\big|_{x_3} \in H^{2,2}_{\text{antiper}}(0,1)$, thus by lemma 4.3.1 we have

$$\partial_3 u\big|_{x_3=0} = u\big|_{x_3=1/2} = 0.$$

In summary, we have shown that S is a self-adjoint operator such that $D(S) = D(\Delta_2)$ and $Su = (I - \Delta_2)u$ for every $u \in D(\Delta_2)$, or in other words, that Δ_2 is a self-adjoint operator in $L^2(\Omega)$ as desired. $\square$

It follows that the domains of fractional powers of the Laplacian can be computed using complex interpolation arguments.

Corollary 4.3.4 ([14, Proposition 4.1])**.** *Let $\theta \in [0,1]$ with $2\theta \notin \{1/2, 3/2\}$. Then*

$$D((-\Delta_2)^\theta) = \begin{cases} \{v \in H^{2\theta,2}_{per}(\Omega)^2 \,:\, \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, & 3/2 < 2\theta \leqslant 2, \\ \{v \in H^{2\theta,2}_{per}(\Omega)^2 \,:\, v\big|_{\Gamma_D} = 0\}, & 1/2 < 2\theta < 3/2, \\ \{v \in H^{2\theta,2}_{per}(\Omega)^2\}, & 2\theta < 1/2. \end{cases} \tag{4.86}$$

Proof. We start with a simple characterization of the domain. Since $-\Delta_2$ admits a bounded H^∞-calculus on $L^2(\Omega)$, by theorem 3.6.5 we can express the domain of fractional powers as a complex interpolation space

$$D((-\Delta_2)^\theta) = [L^2(\Omega), D(-\Delta_2)]_\theta.$$

Now, the result is known for C^∞-boundaries by theorem 3.3.8. Therefore, it suffices to construct a C^∞-domain $\widetilde{\Omega}$ extending Ω such that $\Gamma_a \subset \widetilde{\Gamma}_a$, $\Gamma_b \subset \widetilde{\Gamma}_b$ and $\widetilde{\Gamma}_a, \widetilde{\Gamma}_b \subset \widetilde{\Omega}$. Such $\widetilde{\Omega}$ is depicted in figure (4.7), which construction is due to Hieber et. al. [14]. Since Ω is

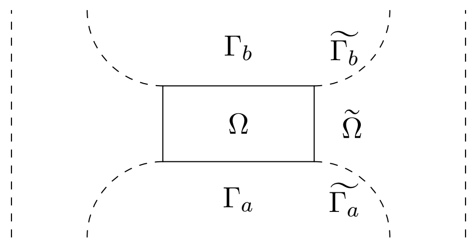


Figure 4.7: Extension of Ω to $\widetilde{\Omega}$.

compact in the topology induced by periodicity, i.e. identifying the lateral boundaries of G, Ω is compact with the topology of $S^1 \times S^1 \times (a, b)$. Consequently, there exists a finite cover $\{U_j\}_{j=1}^k$ and a smooth partition of unity $\{\varphi_j\}_{j=1}^k$, $\varphi_j : \Omega \to [0, 1]$, such that

$$\operatorname{supp} \varphi_j \subset U_j, \quad \text{and} \quad \sum_{j=1}^{k} \varphi_j \equiv 1.$$

Let $j \in \{1, \dots, k\}$, denoting by $\widetilde{\Omega}_j$ a copy of $\widetilde{\Omega}$, taking U_j small enough we can identify it with an open subset $\widetilde{U}_j$ of $\widetilde{\Omega}_j$. We can now define a retraction/correctraction for $s \in [0, \infty)$ by

$$\begin{array}{rccccrccc} S: & H^{s,2}_{\mathrm{per},b.c.}(\Omega) & \to & \bigoplus_{j=1}^k H^{s,2}_{b.c.}(\widetilde{\Omega}_j), & \qquad & R: & \bigoplus_{j=1}^k H^{s,2}_{b.c.}(\widetilde{\Omega}_j) & \to & H^{s,2}_{\mathrm{per},b.c.}(\Omega) \\ & v & \mapsto & \{\sqrt{\varphi_j} v\}_{j=1}^k & & & u & \mapsto & \sum_{j=1}^k \sqrt{\varphi_j} u_j, \end{array}$$

where $b.c.$ refers to boundary conditions as in (4.86). It follows easily that $RSv = R(\{\sqrt{\varphi_j} v\}_j) = \sum_j \varphi_j v = v$. By theorem 3.2.18 we have the following relation between interpolation spaces

$$\begin{aligned} [L^2(\Omega), D(-\Delta_2)]_\theta &= [R(\bigoplus_{j=1}^k L^2(\widetilde{\Omega}_j)), R(\bigoplus_{j=1}^k H^{2,2}_{b.c.}(\widetilde{\Omega}_j))]_\theta \\ &= R(\bigoplus_{j=1}^k [L^2(\widetilde{\Omega}_j), H^{2,2}_{b.c.}(\widetilde{\Omega}_j)]_\theta) \end{aligned}$$

Finally, since the complex interpolation in the right hand side is defined for C^∞-domains, by theorem 3.3.8 we conclude that

$$\begin{aligned} D((-\Delta_2)^\theta) &= R\left(\bigoplus_{j=1}^k \begin{cases} \{v \in H^{2\theta,2}(\widetilde{\Omega}_j)^2 \ : \ \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, & 3/2 < 2\theta \leqslant 2, \\ \{v \in H^{2\theta,2}(\widetilde{\Omega}_j)^2 \ : \ v\big|_{\Gamma_D} = 0\}, & 1/2 < 2\theta < 3/2, \\ \{v \in H^{2\theta,2}(\widetilde{\Omega}_j)^2\}, & 2\theta < 1/2. \end{cases} \right) \\ &= \begin{cases} \{v \in H^{2\theta,2}_{\mathrm{per}}(\Omega)^2 \ : \ \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, & 3/2 < 2\theta \leqslant 2, \\ \{v \in H^{2\theta,2}_{\mathrm{per}}(\Omega)^2 \ : \ v\big|_{\Gamma_D} = 0\}, & 1/2 < 2\theta < 3/2, \\ \{v \in H^{2\theta,2}_{\mathrm{per}}(\Omega)^2\}, & 2\theta < 1/2. \end{cases} \end{aligned}$$

□

We can now formulate the main theorem of this work.

Theorem 4.3.5 ([7], Theorem 3.1)**.** *Let $\nu \geqslant 0$. Then the operator $-A_2 + \nu$ admits a bounded H^∞-calculus on $L^2_{\overline{\sigma}}(\Omega)$ of angle 0 provided $\nu > 0$. If $\Gamma_D \neq \emptyset$, then the above assertion holds true even for $\nu = 0$.*

Proof. Let us first assume that $\Gamma_D \neq \emptyset$. Making use of perturbation techniques for the H^∞-calculus the proof falls naturally into two parts. Firstly, by theorem 3.7.5 we will show that $\nu - \Delta_2 - B_2$ admits a bounded H^∞-calculus for $\nu \geqslant 0$ sufficiently large. Secondly, from the sectoriality of $-A_2$ and theorem 3.7.6 we will conclude that the assertion holds true even for $\nu = 0$.

From proposition 4.3.3 we already know that $-\Delta_2$ admits a bounded H^∞-calculus. Since $D(\Delta_2) \subset D(B_2)$, boundedness of $B_2 : D(\Delta_2) \to L^2(\Omega)$ in $D(\Delta_2)$ can be seen via the following diagram

$$D(B_2) \xrightarrow{\partial_z} H^{1/2+\delta,2}(\Omega)^2 \xrightarrow{\cdot|_{\Gamma_D}} B^\delta_{2,2}(G)^2 \hookrightarrow L^2(G)^2 \xrightarrow{-\nabla_H \Delta_H^{-1} \operatorname{div}_H} L^2(G)^2 \hookrightarrow L^2(\Omega)^2.$$

Let $v \in H^{1+1/2+\delta,2}(\Omega)^2$, then there exists a periodic extension $Ev \in H^{1+1/2+\delta,2}(\mathbb{T}^3)^2$ and we can use the Fourier representation of Bessel potential spaces to obtain that the derivative $\partial_z Ev$ is bounded by

$$\begin{aligned}\|\partial_z Ev\|_{H^{1/2+\delta,2}(\mathbb{T}^3)^2} &= \left\|\mathcal{F}^{-1}\left[(1+|k|^2)^{(1/2+\delta)/2}\mathcal{F}(\partial_z Ev)\right]\right\|_{L^2(\mathbb{T}^3)} \\ &= 2\pi\left\|(1+|k|^2)^{(1/2+\delta)/2} 2\pi k_z \mathcal{F}(Ev)\right\|_{l^2(\mathbb{Z}^3)} \\ &\leqslant 2\pi\left\|\mathcal{F}^{-1}\left[(1+|k|^2)^{(1+1/2+\delta)/2}\mathcal{F}(v)\right]\right\|_{L^2(\mathbb{T}^3)} = 2\pi\|Ev\|_{H^{1+1/2+\delta,2}(\mathbb{T}^3)^2},\end{aligned} \tag{4.87}$$

where we applied Plancherel's theorem and the inequality $|k| \leqslant (1+|k|^2)^{1/2}$. Boundedness of the trace operator follows from theorem 3.4.2. As a consequence, it is easy to see that B_2 is a lower order perturbation of $-\Delta_2$. Indeed note that by corollary 4.3.4 the domain of the fractional Laplacian actually satisfies

$$D((-\Delta_2)^{1-\theta}) \hookrightarrow H^{2(1-\theta),2}_{\mathrm{per}}(\Omega)^2 \hookrightarrow H^{1+1/2+\delta,2}(\Omega)^2, \tag{4.88}$$

whenever the second inclusion holds, this is, $2(1-\theta) > 1 + 1/2 + \delta$, thus $\theta < 1/4 - \delta/2$. Combining (4.88) with the bounded invertibility of the Laplacian Δ_2 and the boundedness of the perturbation term B_2, we obtain that B_2 is a lower order perturbation of the Laplacian with

$$\|B_2 v\|_{L^2(\Omega)} \leqslant C_0 \|v\|_{H^{1+1/2+\delta}(\Omega)} \leqslant C_1 \|(-\Delta_2)^{1-\theta} v\|_{L^2(\Omega)}, \quad v \in D(-\Delta_2), \tag{4.89}$$

for some constant $C_1 > 0$ and $\theta \in (0, 1/4 - \delta/2)$, recall that $\delta \in (0, 1/2)$. Theorem 3.7.5 yields that $\nu - \Delta_2 - B_2$ admits a bounded H^∞-calculus on $L^2(\Omega)$ of angle

$$\omega_{H^\infty}(\nu - \Delta_2 - B_2) = 0$$

for $\nu \geqslant 0$ sufficiently large.

Having disposed of this preliminary step, we can now return to the restriction $\nu - A_2$. Note that through this work we have constructed $\Delta_2 + B_2$ as an extension of A_2 from the closed subspace $L^2_{\overline{\sigma}}(\Omega)$ to $L^2(\Omega)^2$, and the same conclusion can be drawn for the resolvent. Let $\lambda \in \rho(\Delta_2 + B_2)$ and take $w \in L^2_{\overline{\sigma}}(\Omega)$, i.e. $w = P_2 f$ for some $f \in L^2(\Omega)^2$. Substituting the pressure gradient (4.66) in the resolvent problem (4.34):

$$\begin{cases} \lambda v - \Delta v - Bv = P_2 f & \text{on } \Omega, \\ \operatorname{div}_H \overline{v} = 0 & \text{on } G, \end{cases} \tag{4.90}$$

we have that $v := (\lambda - \Delta_2 - B_2)^{-1} P_2 f$ is the unique solution to the resolvent problem

$$\begin{cases} \lambda v - \Delta v + \nabla_H \pi_s = f & \text{on } \Omega, \\ \operatorname{div}_H \overline{v} = 0 & \text{on } G, \end{cases} \tag{4.91}$$

In other words, $(\lambda - \Delta_2 - B_2)^{-1}$ leaves the solenoidal subspace $L^2_\sigma(\Omega)$ invariant with

$$\rho(\Delta_2 + B_2) \subset \rho(A_2) \quad \text{and} \quad (\lambda - \Delta_2 - B_2)^{-1}\big|_{L^2_\sigma(\Omega)} = (\lambda - A_2)^{-1}. \tag{4.92}$$

The property of H^∞-calculus is preserved through invariant subspaces since

$$\begin{aligned}\|f(\nu - A_2)\|_{\mathcal{L}(L^2_\sigma)} &= \|f\left((\nu - \Delta_2 - B_2)\big|_{L^2_\sigma}\right)\|_{\mathcal{L}(L^2_\sigma)} = \|f\left(\nu - \Delta_2 - B_2\right)\big|_{L^2_\sigma}\|_{\mathcal{L}(L^2_\sigma)} \\ &\leqslant \|f(\nu - \Delta_2 - B_2)\|_{L^2(\Omega)}, \quad f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)\end{aligned}$$

where we used (4.92) in the second equality. Consequently, $-A_2 + \nu$ admits a bounded H^∞-calculus on $L^2_\sigma(\Omega)$ of angle

$$\omega_{H^\infty}(-A_2 + \nu) \leqslant \omega_{H^\infty}(\nu - \Delta_2 - B_2) = 0,$$

for $\nu \geqslant 0$ sufficiently large.

We now proceed with the second step, proving that $\nu \geqslant 0$ may be zero. This poses no problem because we proved in theorem 4.2.5 that $-A_2$ is invertible and sectorial. The constant operator $-\nu$ is linear and bounded, hence $D(\nu - A_2) \subset D(-\nu) = L^2_\sigma(\Omega)$ and it is a relative perturbation of $-A_2$. Since the addition

$$(\nu - A_2) - \nu = -A_2,$$

is invertible and sectorial, by theorem 3.7.6 we obtain that $-A_2$ admits a bounded H^∞-calculus.

We now turn our attention to the case $\Gamma_D = \varnothing$. The definition 4.68 clearly forces $B_2 v = 0$. Moreover, it is known that $\nu - \Delta_2$ admits a bounded H^∞-calculus on angle 0 on $L^2(\Omega)^2$ by proposition 4.3.3. Consequently, the above construction applies to this case as well, which proves the theorem. □

To conclude this work, we include to important corollaries that the bounded H^∞-calculus of the hydrostatic Stokes operator A_2 implies.

Corollary 4.3.6. *Let $\theta \in [0, 1]$ with $2\theta \notin \{1/2, 3/2\}$. Then*

$$D((\nu - A_2)^\theta) = \begin{cases} \{v \in H^{2\theta,2}_{per}(\Omega)^2 \,:\, \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\} \cap L^2_\sigma(\Omega), & 3/2 < 2\theta \leqslant 2, \\ \{v \in H^{2\theta,2}_{per}(\Omega)^2 \,:\, v\big|_{\Gamma_D} = 0\} \cap L^2_\sigma(\Omega), & 1/2 < 2\theta < 3/2, \\ \{v \in H^{2\theta,2}_{per}(\Omega)^2\} \cap L^2_\sigma(\Omega), & 2\theta < 1/2. \end{cases} \tag{4.93}$$

for $\nu > 0$. If $\Gamma \neq \varnothing$ it holds even for $\nu = 0$.

Proof. Since $\nu - A_2$ admits a bounded H^∞-calculus, by theorem 3.6.5 we can express its fractional powers as a complex interpolation space

$$D((\nu - A_2)^\theta) = [L^2_\sigma(\Omega), D(\nu - A_2)]_\theta,$$

and the rest of the proof is completely analogous to 4.3.4. □

Corollary 4.3.7. *For $\nu > 0$ the operator $\nu - A_2$ has maximal L^q-regularity. If $\Gamma_D \neq \varnothing$ then it holds true even for $\nu = 0$.*

Proof. Since $L^2_\sigma(\Omega)$ is a Hilbert space, the assertion follows from corollary 3.8.4. □

Chapter 5

Conclusions and future directions

The main goal of this thesis was to reproduce the proof of the hydrostatic Stokes operator $-A_2$ admitting a bounded H^∞-calculus, which was the topic of chapter 4. Although the main proof is short and concise, it relies on deep functional analytical theory which we introduced in chapter 3.

In particular, we have been exposed to perturbation theorems for the H^∞-calculus, section 3.7, and interpolation theory, section 3.2.3. These notions were taken for granted in the main article but required an study of operator semigroups, section 3.5 and the construction of interpolation spaces in order to make the work self-contained. Moreover, we had to make precise the notion of trace for distributions, section 3.4, and interpolation with domains, subsection 3.2.3, which allowed a neat introduction of the function spaces treated throughout the work, section 3.3. Finally, we also included the notion of maximal regularity, section 3.8, to understand the main corollary of the work, which is a conclusion of the vector-valued Fourier multiplier theorem, section 3.1.2, and R-sectoriality, section 3.6.

Once introduced all the preliminaries in the L^p-setting, in chapter 4 we went through Giga et. al.'s proof [7] of bounded H^∞-calculus for the hydrostatic Stokes operator $-A_p$. The definition of the hydrostatic Helmholtz projection and hydrostatic Stokes operator, as well as the characterization of the hydrostatic solenoidal subspace $L^2_{\overline{\sigma}}(\Omega)$ and the hydrostatic Stokes operator being invertible and generating a bounded analytic C_0-semigroup on $L^2_{\overline{\sigma}}(\Omega)$, are an extension of Hieber and Kashiwabara's proof [15]. Giga et. al. take these notions for granted and hence are able to present a concise proof of the bounded H^∞-calculus for the L^p-case. However, the aforementioned properties of the solenoidal subspace $L^p_{\overline{\sigma}}(\Omega)$ and the hydrostatic Stokes operator A_p differ from the L^2-case to L^p. Consequently, now that intuition is built on the L^2-case, the next logical step of this work would be to follow Hieber and Kashiwabara's proof for the general L^p-setting.

Finally, Giga et. al.'s proof is contained in section 4.3.5. Here we start by rewriting the hydrostatic Stokes equation as a perturbation of the L^2-Laplacian and proving H^∞-boundedness of the Laplacian. This is done through reflection arguments adapting Taylor's [32], Krylov's [21] and Nau's [27] works. Although we only prove it for the L^2-case, Nau's dissertation contains a generalization to the L^p-setting. Moreover, the H^∞-calculus of the Laplacian provides a characterization of fractional powers of the Laplacian, which also works in the general L^p-case, for this we followed [14]. The main idea of the proof is show that B_2 is a lower order perturbation of the Laplacian, which admits a bounded H^∞-calculus, and thus $\nu - \Delta_2 - B_2$ admits a bounded H^∞-calculus for $\nu \geqslant 0$ large enough. We conclude the proof by applying that $-A_2$ is invertible and sectorial, hence by perturbation theorems again, $\nu \geqslant 0$ may be zero. Once H^∞-boundedness of the L^p-Laplacian and the invertibility of the L^p-hydrostatic Stokes operator are proven, the proof of admitting a bounded H^∞-calculus should be a simple generalization. We finish the work by showing that the hydrostatic Stokes operator $-A_2$ has maximal L^q-regularity.

Once maximal L^q-regularity is shown, we would still need to show the well-posedness of the full nonlinear primitive equations. In this direction, in 2020 Giga et. al. published a new article [8] simplifying Hieber and Kashiwabara's original proof. We would also like to recommend the interested reader Gries' dissertation [10] on the works published jointly with Giga et. al.

Bibliography

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[3] C. Cao and E. S. Titi. Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. *Ann. of Math. (2)*, 166(1):245–267, 2007.

[4] R. Denk, M. Hieber, and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788):viii+114, 2003.

[5] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[6] H. Fujita and T. Kato. On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.*, 16:269–315, 1964.

[7] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara. Bounded H^∞-calculus for the hydrostatic Stokes operator on L^p-spaces and applications. *Proc. Amer. Math. Soc.*, 145(9):3865–3876, 2017.

[8] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara. Analyticity of solutions to the primitive equations. *Math. Nachr.*, 293(2):284–304, 2020.

[9] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[10] M. Gries. *On the primitive equations and the hydrostatic Stokes operator*. 2018. PhD dissertation, available at: `https://tuprints.ulb.tu-darmstadt.de/id/eprint/7675`.

[11] F. Guillén-González, N. Masmoudi, and M. A. Rodríguez-Bellido. Anisotropic estimates and strong solutions of the primitive equations. *Differential Integral Equations*, 14(11):1381–1408, 2001.

[12] F. Guillén-González and M. A. Rodríguez-Bellido. A review on the improved regularity for the primitive equations. In *Regularity and other aspects of the Navier-Stokes equations*, volume 70 of *Banach Center Publ.*, pages 85–103. Polish Acad. Sci. Inst. Math., Warsaw, 2005.

[13] M. Hieber and A. Hussein. An Approach to the Primitive Equations for Oceanic and Atmospheric Dynamics by Evolution Equations. In T. Bodnár, G. P. Galdi, and S. Necasová, editors, *Fluids Under Pressure*, pages 1–109. Birkhäuser, 2020.

[14] M. Hieber, A. Hussein, and T. Kashiwabara. Global strong L^p well-posedness of the 3D primitive equations with heat and salinity diffusion. *J. Differential Equations*, 261(12):6950–6981, 2016.

[15] M. Hieber and T. Kashiwabara. Global strong well-posedness of the three dimensional primitive equations in L^p-spaces. *Arch. Ration. Mech. Anal.*, 221(3):1077–1115, 2016.

[16] J. K. Hunter. Notes on partial differential equations. *Lecture Notes, https://www. math. ucdavis. edu/hunter/pdes/pdes. html, Department of Mathematics, University of California*, 2014.

[17] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach spaces. Vol. III.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham. In preparation.

[18] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer, Cham, 2016.

[19] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach spaces. Vol. II*, volume 67 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer, Cham, 2017. Probabilistic methods and operator theory.

[20] T. Kato and H. Fujita. On the nonstationary Navier-Stokes system. *Rend. Sem. Mat. Univ. Padova*, 32:243–260, 1962.

[21] N. V. Krylov. *Lectures on elliptic and parabolic equations in Sobolev spaces*, volume 96 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[22] P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.

[23] N. Lindemulder, M. Meyries, and M. Veraar. Complex interpolation with Dirichlet boundary conditions on the half line. *Math. Nachr.*, 291(16):2435–2456, 2018.

[24] J.-L. Lions, R. Temam, and S. H. Wang. New formulations of the primitive equations of atmosphere and applications. *Nonlinearity*, 5(2):237–288, 1992.

[25] J.-L. Lions, R. Temam, and S. H. Wang. On the equations of the large-scale ocean. *Nonlinearity*, 5(5):1007–1053, 1992.

[26] A. Lunardi. *Interpolation theory*, volume 16 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2018. Third edition.

[27] T. Nau. *Lp-Theory of Cylindrical Boundary Value Problems: An Operator-Valued Fourier Multiplier and Functional Calculus Approach.* Springer Science & Business Media, 2012.

[28] M. Reed and B. Simon. *Methods of modern mathematical physics. I.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.

[29] M. A. Rodríguez-Bellido. *Análisis matemático de algunos sistemas de tipo Navier-Stokes: fluidos quasi-newtonianos y ecuaciones primitivas del océano*. Universidad de Sevilla, 2001.

[30] W. Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[31] H. Sohr. *The Navier-Stokes equations.* Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2001. An elementary functional analytic approach.

[32] M. E. Taylor. *Partial differential equations II. Qualitative studies of linear equations*, volume 116 of *Applied Mathematical Sciences.* Springer, New York, second edition, 2011.

[33] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam-New York, 1978.

[34] H. Triebel. *Theory of function spaces.* Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2010. Reprint of 1983 edition.

[35] G. K. Vallis. *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation.* Cambridge University Press, Cambridge, U.K., 2nd edition, 2017.

[36] M. Ziane. Regularity results for Stokes type systems. *Appl. Anal.*, 58(3-4):263–292, 1995.

[37] M. Ziane. Regularity results for the stationary primitive equations of the atmosphere and the ocean. *Nonlinear Anal.*, 28(2):289–313, 1997.