Delft University of Technology
Department of Aerospace Engineering

Report LR-339

OUTPUT FEEDBACK REGULATORS FOR AIRCRAFT AUTOMATIC CONTROL SYSTEMS

P. Wilbers, J.A. Hoogstraten

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SUMMARY

In this report a design method for linear proportional and linear dynamic output feedback regulators for discrete time systems is presented. It is shown how this design method may be applied to the design of aircraft automatic control systems.

CONTENTS

0. SYMBOLS AND FRAME OF REFERENCE

0.1. Symbols

A	system matrix
B	control input distribution matrix
$\bar{c}$	mean aerodynamic chord
C	observation matrix
C_m	pitching moment coefficient
C_{m_q}	$= \dfrac{\partial C_m}{\partial \frac{q\bar{c}}{V}}$
C_{m_u}	$= \dfrac{1}{\frac{1}{2}\rho V^2 \bar{c}} \dfrac{\partial M}{\partial \hat{u}}$
C_{m_α}	$= \dfrac{\partial C_m}{\partial \alpha}$
$C_{m_{\alpha_g}}$	$= \dfrac{\partial C_m}{\partial \alpha_g}$
$C_{m_{\dot{\alpha}}}$	$= \dfrac{\partial C_m}{\partial \frac{\dot{\alpha}\bar{c}}{V}}$
$C_{m_{\dot{\alpha}_g}}$	$= \dfrac{\partial C_m}{\partial \frac{\dot{\alpha}_g \bar{c}}{V}}$
C_{vv}, C_{vw}, C_{ww}	covariance matrix
C_X	coefficient of aerodynamic force along the aircraft's X-axis
C_{X_u}	$= \dfrac{1}{\frac{1}{2}\rho V^2 S} \dfrac{\partial X}{\partial \hat{u}}$
C_{X_0}	$= C_X$ in steady flight condition
C_{X_α}	$= \dfrac{\partial C_X}{\partial \alpha}$

$C_{X_{\alpha_g}}$	$= \dfrac{\partial C_X}{\partial \alpha_g}$
$C_{X_{\dot{\alpha}_g}}$	$= \dfrac{\partial C_X}{\partial \dfrac{\dot{\alpha}_g \bar{c}}{V}}$
C_Z	coefficient of aerodynamic force along the aircraft's Z-axis
C_{Z_q}	$= \dfrac{\partial C_Z}{\partial \dfrac{q\bar{c}}{V}}$
C_{Z_u}	$= \dfrac{1}{\frac{1}{2}\rho V^2 S} \cdot \dfrac{\partial Z}{\partial \hat{u}}$
C_{Z_0}	$= C_Z$ in steady flight condition
C_{Z_α}	$= \dfrac{\partial C_Z}{\partial \alpha}$
$C_{Z_{\alpha_g}}$	$= \dfrac{\partial C_Z}{\partial \alpha_g}$
$C_{Z_{\dot{\alpha}}}$	$= \dfrac{\partial C_Z}{\partial \dfrac{\dot{\alpha}\bar{c}}{V}}$
$C_{Z_{\dot{\alpha}_g}}$	$= \dfrac{\partial C_Z}{\partial \dfrac{\dot{\alpha}_g \bar{c}}{V}}$
dq	number of components of controller state vector q_c
du	number of components of input vector u
dv	number of components of observation noise vector v
dw	number of components of system noise vector w
dx	number of components of state vector x
dy	number of components of observation vector y
dz	number of components of vector of controlled variables z
D	output matrix
D_c	dimensionless differential operator

F	gain matrix
h	flight altitude
H	observation noise distribution matrix
J	criterion value
k	discrete time indication
K_e	static gain of elevator servo
K_Y	moment of inertia about the Y-axis
L	criterion value
L_g	integral scale of turbulence
m_q, m_u, m_α, m_{α_g}, $m_{\dot{\alpha}_g}$	stability and gust derivatives in abbreviated notations
M	aerodynamic moment about the Y-axis, system matrix of the discrete-time controller
N	input distribution matrix
p	matrix of Lagrange multipliers
q	angular rate of pitch
q_c	controller state
R_q, R_u, R_z, R_{11}, R_{12}, R_{22}	weighting matrices
S	state covariance matrix, wing area
t	time
u	control input vector, component of the airspeed V along the

	aircraft's X-axis
$\hat{u}$	$= \frac{u}{V}$
u_c	control input vector
$\hat{u}_g$	dimensionless horizontal gust velocity
v	observation noise vector, noise signal
V	airspeed
w	system noise vector, noise signal
W	aircraft weight
x	state vector
x_u, x_α, x_{α_g}, x_θ, x_q	stability and gust derivatives in abbreviated notations
X	aerodynamic force along the aircraft's X-axis
y	observation vector
z	vector of controlled variables
z_q, z_u, z_α, z_{α_g}, z_θ	stability and gust derivatives in abbreviated notations
Z	aerodynamic force along the aircraft's Z-axis
α	angle of attack
α_g	gust angle of attack
Γ	input distribution matrix
δ_e	elevator deflection
δ_k	kronecker delta-function
Δt	sampling interval

θ	angle of pitch
μ_c	dimensionless mass parameter
ρ	air density
σ_h, $\sigma_{\hat{u}}$, σ_{α_g}, σ_θ	standard deviation
τ	time
τ_e	time constant of elevator servo
τ_i	time constant of integrating element
Φ	transition matrix
$\Phi(\omega)$	power spectral density function
ϕ	noise input distribution matrix
ω	circular frequency

0.2. Frame of reference

The frame of reference is a right handed system OXYZ of orthogonal body axes. The origin O lies in the aircraft's centre of gravity. The XOZ plane coincides with the aircraft's plane of symmetry. The positive X-axis is fixed relative to the aircraft. It points forward, parallel to the airspeed V in the steady flight condition. The positive Y-axis points to the right, the positive Z-axis points downward.

1. INTRODUCTION

Due to the rapid development in digital electronics, digital computers become cheaper, smaller and more versatile than their analog predecessors. Furthermore they weigh less and consume less power. Replacing analog computers in aircraft by digital ones offers other advantages too. Some of these are:
1. the amount of electrical wiring is decreased using databus structures.
2. non linear functions can be implemented in a relatively easy way.
3. the computer programs can easily be exchanged and parameter settings altered.
4. the increased computational capacity may be used to perform more complicated computations on board the aircraft.

As a result the next generation of airplanes will exhibit the extensive use of digital computers on board the aircraft.

This report discusses a design method for linear discrete time regulators to be implemented in a small digital computer. Since the regulators feature output feedback and an optimization procedure is used in the design method, this type of regulator is also referred to as 'Linear discrete time optimal output feedback regulator'.

The design method has been embodied in the Control System Analysis and Synthesis Program package for Aerospace Research (CASPAR), developed by the Disciplinary Group for Aircraft Stability and Control at the Department of Aerospace Engineering of Delft University of Technology. This program package is extensively described in ref. 1, some of the subroutines are described in ref. 2.

Chapter two presents a design method for proportional and dynamic output feedback regulators.

Chapter three shows this design method applied to the design of aircraft controllers performing a stabilization task.

2. A DESIGN METHOD FOR OUTPUT FEEDBACK REGULATORS

2.1. Introduction

During the last two decades much attention has been paid to the design of control systems for large multi-variable systems, using 'Modern Control Theory'. A characteristic feature of 'Modern Control Theory' is the simplicity gained when dealing with a more complex, multi-variable structure of the system that is to be controlled.
Nonetheless, even for simple cases the multivariable approach may have its merits, particularly when dealing with severe limitations on input signals, or when dealing with systems where output (sensor) signals are intricately related to state variables and are few in number. The multivariable approach will prevail when dealing with systems or subsystems exhibiting significant interaction in system responses, such as may occur in aircraft having low-frequency elastic modes or for complicated control tasks (see for instance refs. 3 and 4). In these cases the single-input-single-output-approach may prove cumbersome to apply and satisfactory control may be hard to obtain.
In developing a controller, using either a single-input-single-output or a multivariable approach, the primary design objective is to stabilize the system. Furthermore, by implementing a control system, the designer attempts to adjust the characteristics of the system to be controlled, in order to comply with the general design specification. Desired characteristics may thus be indicated as secondary design objectives (as opposed to the mandatory characteristic expressed in the primary design objective). In Modern Control Theory these secondary design objectives are expressed in terms of a mathematically formulated performance criterion. A widely used performance criterion is the so-called 'quadratic cost function' which essentially expresses the compromise between the control effort and the deviations from the desired system state in terms of the variances of input and state variables for the closed-loop system.
Since the behaviour of a system may be qualified in terms of various, often intricately related, characteristics, the designer is usually compelled to seek a compromise, which, in terms of a 'performance criterion', boils down to mutually weighing the importance of the various characteristics included in the criterion. Furthermore, not all desired characteristics can be expressed directly in a mathematical formulation. It will therefore be clear that optimization of a performance criterion will in itself not usually lead directly to a satisfactory solution, but will require some tuning of weighting functions as indicated by further evaluation of the controlled system (note that this essentially also holds for design methods that employ alternative expressions for a 'performance criterion' such as root-locus or Nichols-diagram design).
In the design method presented in this report, a quadratic cost function will be used to obtain the feedback control gains for a number of output variables. Output feedback is applied rather than state feedback since:

1. very often it is not possible to measure the complete state of a system (ref. 5, 6, 7);
2. in general it is desired to control only some state variables, very often only these variables need to be measured;
3. although the performance of a state feedback controller theoretically will yield improved result, the cost of installing and maintaining a large set of observation instruments for directly measuring all state variables can be prohibitive, especially for large-scale systems.

This chapter introduces the system to be controlled. It is represented by a linear model. The performance criterion is given in the form of a quadratic cost function, weighing the deviations from their reference values of the various control input signals, and of the controlled variables. The controlled variables essentially define the control task the controller will perform. They are

represented as linear combinations of state and input variables.
It will be shown that the necessary conditions for minimization of the chosen cost function, when employing an output feedback gain matrix, are given by a set of three recurrent relations. Solution of this set of equations yields the elements of the feedback gain matrix. This basic result is subsequently extended to encompass the design of a dynamic output feedback regulator.

2.2. Design of linear proportional output feedback regulators

Consider the linear, discrete time, time invariant system with state equation (see fig. 1):

$$x(k+1) = \Phi\, x(k) + \Gamma\, u(k) + \Psi\, w(k) \tag{2.1}$$

where $x(k)$ is the dx dimensional state vector,
$u(k)$ is the du dimensional control input vector,
$w(k)$ is the dw dimensional system noise vector,
Φ is the dx * dx dimensional transition matrix,
Γ is the dx * du dimensional control input distribution matrix,
Ψ is the dx * dw dimensional noise input distribution matrix.

The disturbing system noise vector $w(k)$ is a sampled sequence of mutually uncorrelated, zero mean, stochastic vector processes. The dw * dw dimensional diagonal system noise covariance matrix is given by:

$$E\left\{w(i),\, w^T(j)\right\} = C_{ww}\, \delta_K(i-j) \tag{2.2}$$

An example, presenting the equations describing a large jet transport aircraft, is discussed in Chapter 3.

The output equation of the system reads (see fig. 1):

$$y(k) = C\, x(k) + H\, v(k) \tag{2.3}$$

where $y(k)$ is the dy-dimensional observation vector,
$v(k)$ is the dv-dimensional observation noise vector,
C is the dy * dx dimensional state observation matrix,
H is the dy * dv dimensional noise input distribution matrix.

The observation noise vector $v(k)$ is a sampled sequence of mutually uncorrelated zero mean stochastic vector processes, possibly correlated with the state noise vector $w(k)$, but uncorrelated with the state $x(k)$. The dv * dv dimensional diagonal covariance matrix is given by:

$$E\left\{v(i),\, v^T(j)\right\} = C_{vv}\, \delta_k(i-j) \tag{2.4}$$

The correlation between the sequences $w(k)$ and $v(k)$ is given by the covariance matrices:

$$E\{w(i),\ v^T(j)\} = C_{wv}\ \delta_k(i-j) = C_{vw}^T\ \delta_k(i-j) \tag{2.5}$$

The vector of controlled variables z(k) is a linear combination of the state x(k) and the control input u(k).

$$z(k) = D_1\ x(k) + D_2\ u(k) \tag{2.6}$$

where z(k) is the dz-dimensional vector of variables to be controlled
D_1 is a dz * dx dimensional output matrix,
D_2 is a dz * du dimensional output matrix.

The system will be controlled using a proportional output feedback regulator. This type of regulator may be described in the ideal case by the following equation:

$$u(k) = -F_y\ y(k) \tag{2.7}$$

where F_y is a du * dy dimensional gain matrix. In practice, however, it is not possible to process the observations and to generate the control signals instantaneously, so a computational delay of at least one sample will occur. For sufficiently high sampling rate, the effect may be considered negligible, and the delay need not be modelled. In the next section it will be indicated how this delay may be taken into account, if necessary.

The control problem can now be formulated as follows. Given the linear system described by eqs. (2.1), (2.3) and (2.6), where it is assumed that the system is both observable and controllable (see ref. 5), compute the stabilizing time invariant gain matrix F_y, occurring in eq. (2.7):

$$u(k) = -F_y\ y(k) \tag{2.7}$$

which will minimize the criterion:

$$J = \lim_{k_1 \to \infty} \sum_{k=k_o}^{k_1} \{z^T(k)\ R_z\ z(k) + u^T(k)\ R_u\ u(k)\} \tag{2.8}$$

Here R_z and R_u are positive, semi-definite weighting matrices with dimensions dz * dz and du * du respectively. It should be noted from eq. (2.8) that this criterion constitutes a balance between the deviations of the controlled variables from their nominal values and the control energy required to maintain those nominal values. Consequently, as an initial choice for the elements of R_z and R_u, values may be given that are closely related to the maximum variances the designer wishes to allow for the various controlled and input variables (ref. 9, see also refs. 6, 8). As already remarked in the introduction, usually further tuning of the weighting matrices will be necessary to establish satisfactory performance in terms of criteria such as damping, settling time, rise time and indifference to plant parameter variations. Note that, through the use of the controlled variables z, essential limitations to the solution of the optimization problem imposed by the open-loop system characteristics (i.e. the attainable state-space) are included in the problem formulation. Thus, the optimization problem is a constrained one. In the remainder of this section it is shown how this constrained control problem can be solved.

Substitution of eq. (2.3) into (eq. (2.7) yields:

$$u(k) = -F_y\ C\ x(k) - F_y\ H\ v(k) \tag{2.9}$$

Substitution of eq. (2.9) into eq. (2.6) yields:

$$z(k) = D_1\ x(k) - D_2\ F_y\ C\ x(k) - D_2\ F_y\ H\ v(k) \tag{2.10}$$

The following expression for J may be obtained substituting eqs. (2.9) and (2.10) into eq. (2.8):

$$J = \lim_{k_1 \to \infty} \sum_{k=k_o}^{k_1} \{x^T(k)\left(D_1^T R_z D_1 - D_1^T R_z D_2 F_y C - C^T F_y^T D_2^T R_z D_1 + C^T F_y^T D_2^T R_z D_2 F_y C + \right.$$

$$\left. C^T F_y^T R_u F_y C\right)\ x(k) + v^T(k)\left(H^T F_y^T D_2^T R_z D_2 F_y H + H^T F_y^T R_u F_y H\right)\ v(k)\} \tag{2.11}$$

For sake of simplicity in notations, the following matrices are defined:

$$R_{11} \overset{\Delta}{=} D_1^T\ R_z\ D_1 \tag{2.12}$$

$$R_{12} \overset{\Delta}{=} D_1^T\ R_z\ D_2 \tag{2.13}$$

$$R_{22} \overset{\Delta}{=} D_2^T\ R_z\ D_2 + R_u \tag{2.14}$$

By virtue of these definitions and by introducing trace operations (see the Appendix), the criterion J may be rewritten as follows:

$$J = \mathrm{tr}\{\left(R_{11} - R_{12}F_yC - C^T F_y^T R_{12}^T + C^T F_y^T R_{22} F_y C\right) \lim_{k \to \infty} E\{x(k), x^T(k)\}\} +$$

$$\mathrm{tr}\ \{H^T F_y^T R_{22} F_y H C_{vv}\} \tag{2.15}$$

Next the state covariance matrix S(k) is defined as:

$$S(k) \stackrel{\Delta}{=} E\{x(k), x^T(k)\} \tag{2.16}$$

In case the closed loop-controlled system is asymptotically stable, the covariance matrix S(k) will reach a finite steady state value, denoted as $\bar{S}$. Using this definition the criterion J (eq. (2.16)), may be rewritten as:

$$J = tr\ \{(R_{11} - R_{12}F_yC - C^TF_y^TR_{12}^T + C^TF_y^TR_{22}F_yC)\bar{S}\} + tr\ \{H^TF_y^TR_{22}F_yH\ C_{vv}\} \tag{2.17}$$

Substitution of eq. (2.9):

$$u(k) = -F_y\ C\ x(k) - F_y\ H\ v(k) \tag{2.9}$$

into the state equation of the uncontrolled system (eq. (2.1)), yields the state equation of the controlled system:

$$x(k+1) = (\Phi - \Gamma\ F_y\ C)\ x(k) - \Gamma\ F_y\ H\ v(k) + \Psi\ w(k) \tag{2.18}$$

The propagation in time of the covariance matrix S of a system:

$$x'(k+1) = \Phi'\ x'(k) + \psi'\ w'(k)$$

where w'(k) is a sequence of mutually uncorrelated, zero-mean vector-valued stochastic processes with variance matrix $C_{w'w'}$, may be described in terms of the system perturbations w' and the noise input distribution and system matrices ψ' and Φ'. This yields a so-called Lyapunov-equation:

$$S(k+1) = \Phi'\ S(k)\ \Phi'^T + \psi' C_{w'w'}\ \psi'^T, \qquad S(k_o) = S_o \tag{2.16b}$$

(see ref. 5).
Thus, the steady-state covariance matrix $\bar{S}$ of the system represented by eq. (2.18):

$$\bar{S} \overset{\Delta}{=} \lim_{k \to \infty} E \left\{x(k), x^T(k)\right\} = \lim_{k \to \infty} E \left\{x(k+1), x^T(k+1)\right\} \tag{2.16c}$$

or, alternatively, after substitution of eq. (2.18) in eq. (2.16c):

$$\begin{aligned}\bar{S} = \lim_{k \to \infty} E \Big\{&\left[(\Phi - \Gamma F_y C)\, x(k) - \Gamma F_y H\, v(k) + \psi\, w(k)\right], \\ &\left[(\Phi - \Gamma F_y C)\, x(k) - \Gamma F_y H\, v(k) + \psi\, w(k)\right]^T\Big\}\end{aligned} \tag{2.19}$$

may be found from the Lyapunov equation:

$$\begin{aligned}&(\Phi - \Gamma F_y C)\, \bar{S}\, (\Phi - \Gamma F_y C)^T - \bar{S} + \Psi C_{ww} \Psi^T - \Gamma F_y H C_{wv}^T \Psi^T \\ &+ \Psi C_{wv} H^T F_y^T \Gamma^T + \Gamma F_y H C_{vv} H^T F_y^T \Gamma^T = S_o\end{aligned} \tag{2.20}$$

using eqs. (2.2), (2.3), (2.4) and (2.16c).

The above stated equation yields the steady state value $\bar{S}$ of the state covariance matrix S of the controlled system, for given gain matrix F_y and system noise and observation noise covariance matrices C_{ww}, C_{wv} and C_{vv}. For slight perturbations δF_y of the output feedback gain matrix F_y, the value $J+\Delta J$ of the augmented criterion is approximately given by the original value J and its first-order variation δJ:

$$J + \Delta J \cong J + \delta J \tag{2.21a}$$

Denoting:

$$L \overset{\Delta}{=} J + \delta J \tag{2.21b}$$

it is a well-known result that a minimum for J may be found by minimizing, for a certain F_y:

$$L = J + \text{tr}\,[P\,S_o] \tag{2.21c}$$

where P is a matrix of Lagrange multipliers, and S_o follows from eq. (2.20) for the given value of F_y. Thus the constrained optimization problem, laid down in equations (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.8) and the demand for stability of the closed-loop system, is converted into the unconstrained optimization problem of minimizing L.
Substituting eq. (2.17) and eq. (2.20):

$$L = \text{tr}\,\{(R_{11} + C^T F_y^T R_{22} F_y C - C^T F_y^T R_{12}^T - R_{12} F_y C)\bar{S} +$$
$$+ [(\Phi - \Gamma F_y C)\,\bar{S}\,(\Phi - \Gamma F_y C)^T - \bar{S} + \Psi\,C_{ww}\,\Psi^T - \Gamma F_y H C_{wv}^T \Psi^T$$
$$- \Psi C_{wv} H^T F_y^T \Gamma^T + \Gamma F_y H C_{vv} H^T F_y^T \Gamma^T]P\} + \text{tr}\,\{H^T F_y^T R_{22} F_y H C_{vv}\} \tag{2.21d}$$

Since L is expressed in terms of the state covariance matrix S, the matrix of Lagrange multipliers P, and the output feedback gain matrix F_y, the necessary conditions to be placed upon L for J to have a minimum at a certain F_y are:

$$\frac{\delta L}{\delta \bar{S}} = 0 \tag{2.22}$$

$$\frac{\delta L}{\delta P} = 0 \tag{2.23}$$

$$\frac{\delta L}{\delta F_y} = 0 \tag{2.24}$$

Elaborating the rules for matrix trace operations given in the Appendix, the following equations may be derived:

$$\frac{\delta L}{\delta \bar{S}} = (\Phi - \Gamma F_y C)^T\,P(\Phi - \Gamma F_y C) - P + R_{11} - C^T F_y^T R_{12}^T$$
$$- R_{12} F_y C + C^T F_y^T R_{22} F_y C = 0 \tag{2.25}$$

$$\frac{\delta L}{\delta P} = (\Phi - \Gamma F_y C)\ \bar{S}\ (\Phi - \Gamma F_y C)^T - \bar{S} + \Psi C_{ww} \Psi^T - \Gamma F_y H C_{wv}^T \Psi^T$$
$$- \Psi C_{wv} H^T F_y^T \Gamma^T + \Gamma^T F_y^T H^T C_{vv} H F_y \Gamma = 0 \qquad (2.26)$$

$$\frac{\delta L}{\delta F_y} = 2\{(\Gamma^T P \Gamma + R_{22})\ F_y\ (C\bar{S}C^T + HC_{vv}H^T) - R_{12}^T \bar{S}^T C^T - \Gamma^T P \Psi C_{wv} H^T$$
$$- \Gamma^T P \Phi \bar{S} C^T\} = 0 \qquad (2.27)$$

Eq. (2.27) enables the gain matrix F_y to be computed:

$$F_y = (\Gamma^T P \Gamma + R_{22})^{-1} ((R_{12}^T + \Gamma^T P \Phi)\ \bar{S}C^T + \Gamma^T P \Psi C_{wv} H^T)(C\bar{S}C^T + HC_{vv}\ H^T)^{-1} \qquad (2.28)$$

Eqs. (2.25), (2.26) and (2.28) contain the solution of the control problem. The gain matrix F_y following from eq. (2.28), is influenced by the specified system noise- and observation noise covariance matrices through the solution $\bar{S}$ obtained from eq. (2.26), and by the chosen weighting matrices through the solution P obtained from eq. (2.25). The solution of this set of equations may thus be obtained by iterative solution of eqs. (2.25), (2.26) and (2.28). To initiate the iteration process, a stabilizing control law F_{y_s} should be substituted for F_y in eqs. (2.25) and (2.26), and the solution F_y may subsequently be found by updating F_{y_s} with the solution F_y found from eq. (2.28) for each iteration cycle. For practical purposes the iteration process can be started specifying a simple stabilizing control law F_{y_s} found by applying common engineering sense Alternatively one might consider to compute a state feedback control law F_{x_r} first, and to use the solutions $\bar{S}$ and P for this stabilizing state feedback controller to compute the initial value F_{y_s} eq. (2.28). State feedback control is treated extensively in ref. 5.
The conditions stated in eqs. (2.22) through (2.24) are only necessary conditions. Therefore the calculated solution F_y may lead to a local minimum of J which may not be unique. Furthermore, convergence of the iteration process can only be guaranteed if the controlled system has well-damped responses. A tight formulation of necessary conditions for the existence of a solution has not been proposed as yet. Sufficient conditions are discussed in a.o. ref. 10, 11, 12.

2.3. Design of linear dynamic output feedback regulators

In this section a design method for dynamic output feedback regulators is presented. Reasons for investigating this type of regulator are:

1. proportional output feedback regulators usually cannot eliminate completely the effects of constant disturbances (see ref. 5);
2. suitably chosen dynamic output feedback regulators are less sensitive to system parameter variations (see refs. 6, 7);
3. although not all systems may be stabilized by applying proportional output feedback, this may in principle be achieved using dynamic output feedback;
4. reconstructive action is further enhanced in a dynamic output feedback regulator. Sensor requirements may therefore be relinquished further.

A dynamic controller can be described by the following discrete time system equations, see refs. 5, 7 and fig. 2:

$$q_c(k+1) = M\, q_c(k) + N\, y(k) \tag{2.29}$$

$$u(k) = -F_q\, q_c(k) - F_y\, y(k) \tag{2.30}$$

Where $q_c(k)$ is the dq dimensional state vector of the controller,
$y(k)$ is the dy dimensional observation vector,
$u(k)$ is the du dimensional control output vector,
M is the dq * dq dimensional transition matrix of the controller,
N is the dq * dy dimensional observation input distribution matrix,
F_q, F_y are gain matrices of dimensions du * dq and du * dy respectively.

The value of dq, i.e. the dimension of the controller state vector $q_c(k)$ has to be specified by the designer. Although the system matrices M and N of the controller may be included in the optimization process (see ref. 10, 12), the designer may wish to define explicitely the dynamics of the controller. Since usually the designer has some knowledge about the system that is to be controlled, specification of the matrices M and N should pose no undue problems, while yielding benefits in terms of transparency of design process as well as controller implementation. The design process will thus be discussed for given matrices M and N, which will define the differentiating or integrating action, or both, specified for the controller. In chapter 3 the specification of the matrices M and N will be demonstrated. Finally it should be noted that eqs. (2.29) and (2.30) may be used to model the computational delay occurring in the regulator. In that case the matrix F_y will be equal to zero.

Let the uncontrolled open-loop system be given by the following discrete time equations:

$$x(k+1) = \Phi\, x(k) + \Gamma\, u(k) + \Psi\, w(k) \tag{2.1}$$

$$y(k) = C\, x(k) + H\, v(k) \tag{2.3}$$

$$z(k) = D_1\, x(k) + D_2\, u(k) \tag{2.6}$$

The variables in these equations have been defined in section 2.2.
A block diagram of the controlled system is given in fig. 2.

The control law design problem can now be formulated as follows. Given the linear system described by eqs. (2.1), (2.3) and (2.6), compute the linear time invariant dynamic controller described by eqs. (2.29) and (2.30), which will minimize the criterion:

$$J = \lim_{k_1 \to \infty} \sum_{k=k_0}^{k_1} \{z^T(k)\, R_z\, z(k) + u^T(k)\, R_u\, u(k) + q_c^T(k)\, R_q\, q_c(k)\} \qquad (2.31)$$

It should be noted that the controller state $q_c(k)$ has been added to the criterion. The matrices R_z, R_u and R_q are positive, semi-definite weighting matrices with dimensions dz * dz, du * du and dq * dq respectively. Since the matrices M and N are specified by the designer, it remains to compute the gain matrices F_y and F_q.

Substitution of eq. (2.3) into eq. (2.29) yields:

$$q_c(k+1) = M\, q_c(k) + N\, C\, x(k) + N\, H\, v(k) \qquad (2.32)$$

Substitution of eq. (2.3) into eq. (2.30) and substituting the result into eq. (2.1) yields the state equation of the controlled system:

$$x(k+1) = [\Phi - \Gamma\, F_y\, C]\, x(k) - \Gamma\, F_q\, q_c(k) + \Psi\, w(k) - \Gamma\, F_y\, H\, v(k) \qquad (2.33)$$

At this point the augmented state vector $\tilde{x}(k)$, the augmented observation vector $\tilde{y}(k)$ and the augmented noise vector $\tilde{w}(k)$ are introduced, defined as:

$$\tilde{x}(k) \overset{\Delta}{=} \mathrm{col}\,[x(k),\, q_c(k)] \qquad (2.34)$$

$$\tilde{y}(k) \overset{\Delta}{=} \mathrm{col}\,[y(k),\, q_c(k)] \qquad (2.35)$$

$$\tilde{w}(k) \overset{\Delta}{=} \mathrm{col}\,[w(k),\, v(k)] \qquad (2.36)$$

From eqs. (2.2), (2.4) and (2.5) it follows that the covariance matrix $\tilde{C}_{\tilde{w}\tilde{w}}$ is given by:

$$\tilde{C}_{\tilde{w}\tilde{w}} = E\{\tilde{w}(i),\, \tilde{w}^T(j)\} = \begin{bmatrix} C_{ww} & C_{wv} \\ C_{vw} & C_{vv} \end{bmatrix} \delta_K(i-j) \qquad (2.37)$$

$$\tilde{C}_{\tilde{w}v} = E\{\tilde{w}(i), v^T(j)\} = \begin{bmatrix} C_{wv} \\ \\ C_{vv} \end{bmatrix} \delta_K(i-j) \tag{2.38}$$

The augmented state equation is obtained by combining eqs. (2.32) and (2.33):

$$\tilde{x}(k+1) = \begin{bmatrix} \Phi - \Gamma F_y C & - \Gamma F_q \\ \\ N C & M \end{bmatrix} \tilde{x}(k) + \begin{bmatrix} \Psi & - \Gamma F_y H \\ \\ 0 & N H \end{bmatrix} \tilde{w}(k) \tag{2.39}$$

The observation equation reads:

$$\tilde{y}(k) = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \tilde{x}(k) + \begin{bmatrix} H \\ 0 \end{bmatrix} v(k) \tag{2.40}$$

In order to obtain the solution of the optimization problem, leaving the specification of the matrices M and N to the designer, the following partitioned matrices are introduced:

$$\tilde{\Phi} = \begin{bmatrix} \Phi & 0 \\ N C & M \end{bmatrix} \tag{2.41}$$

$$\tilde{\Gamma} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \tag{2.42}$$

$$\tilde{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \tag{2.43}$$

$$\tilde{F}_y = [F_y \quad F_q] \tag{2.44}$$

$$\tilde{\Psi} = \begin{bmatrix} \Psi & 0 \\ 0 & N H \end{bmatrix} \tag{2.45}$$

$$\tilde{H} = \begin{bmatrix} H \\ 0 \end{bmatrix} \tag{2.46}$$

$$\tilde{D}_1 = [D_1 \quad 0] \tag{2.47}$$

Using these matrices, the controlled system may be rewritten to yield the following equations:

$$\tilde{x}(k+1) = [\tilde{\Phi} - \tilde{\Gamma}\,\tilde{F}_y\,\tilde{C}]\,\tilde{x}(k) - \tilde{\Gamma}\,\tilde{F}_y\,\tilde{H}\,v(k) + \tilde{\Psi}\,\tilde{w}(k) \tag{2.48}$$

$$\tilde{y}(k) = \tilde{C}\,\tilde{x}(k) + \tilde{H}\,\tilde{v}(k) \tag{2.49}$$

$$u(k) = -\,\tilde{F}_y\,\tilde{y}(k) \tag{2.50}$$

Similarly the equation of the controlled variables reads:

$$z(k) = \tilde{D}_1\,\tilde{x}(k) + D_2\,u(k) \tag{2.51}$$

Substituting eqs. (2.3), (2.6) and (2.30) into eq. (2.32), and using properties of matrix calculations, the following expression for the criterion J is obtained:

$$J = tr\{[\tilde{R}_{11} - \tilde{R}_{12}\tilde{F}_y\tilde{C} - \tilde{C}^T\tilde{F}_y^T\tilde{R}_{12}^T + \tilde{C}^T\tilde{F}_y^T\tilde{R}_{22}\tilde{F}_y C]\tilde{S}\} + tr\{\tilde{H}^T\tilde{F}_y^T\tilde{R}_{22}\tilde{F}_y\tilde{H}C_{vv}\} \tag{2.52}$$

Where
$$\tilde{R}_{11} = \begin{bmatrix} D_1^T R_2 D_1 & 0 \\ 0 & R_q \end{bmatrix} \tag{2.53}$$

$$\tilde{R}_{12} = \begin{bmatrix} D_1^T R_2 D_2 \\ 0 \end{bmatrix} \tag{2.54}$$

$$\tilde{R}_{22} = D_2^T R_2 D_2 + R_u \tag{2.55}$$

$$\tilde{S} = \lim_{k \to \infty} E\{\tilde{x}(k),\ \tilde{x}^T(k)\} \tag{2.56}$$

Since the control law design problem has been formulated in the same terms as used in the previous section, the solution may be obtained directly from the following expressions, similar to eqs. (2.28), (2.25) and (2.26):

$$\tilde{F}_y = (\tilde{\Gamma}^T\tilde{P}\tilde{\Gamma}+\tilde{R}_{22})^{-1}((\tilde{R}_{12}^T+\tilde{\Gamma}^T\tilde{P}\tilde{\Phi})\tilde{S}\tilde{C}^T+\tilde{\Gamma}^T\tilde{P}\tilde{\Psi}C_{\tilde{w}v}\tilde{H}^T)(\tilde{C}\tilde{S}\tilde{C}^T+\tilde{H}C_{vv}\tilde{H}^T)^{-1} \qquad (2.57)$$

$$(\tilde{\Phi}-\tilde{\Gamma}\tilde{F}_y\tilde{C})\tilde{S}(\tilde{\Phi}-\tilde{\Gamma}\tilde{F}_y\tilde{C})^T-\tilde{S}+\tilde{\Psi}C_{\tilde{w}\tilde{w}}\tilde{\Psi}-\tilde{\Gamma}\tilde{F}_y\tilde{H}C_{\tilde{w}v}^T\tilde{\Psi}^T-\tilde{\Psi}C_{\tilde{w}v}\tilde{H}^T\tilde{F}_y^T\tilde{\Gamma}^T$$

$$+\tilde{\Gamma}^T\tilde{F}_y^T\tilde{H}^TC_{vv}\tilde{H}\tilde{F}_y\tilde{\Gamma} = 0 \qquad (2.58)$$

$$(\tilde{\Phi}-\tilde{\Gamma}\tilde{F}_y\tilde{C})^T\tilde{P}(\tilde{\Phi}-\tilde{\Gamma}\tilde{F}_y\tilde{C})-\tilde{P}+\tilde{R}_{11}-\tilde{C}^T\tilde{F}_y^T\tilde{R}_{12}^T-\tilde{R}_{12}\tilde{F}_y\tilde{C}+\tilde{C}^T\tilde{F}_y^T\tilde{R}_{22}\tilde{F}_y\tilde{C} = 0 \qquad (2.59)$$

Where $\tilde{P}$ is a matrix containing Lagrange multipliers.
The gain matrix $\tilde{F}_y$ can be computed from eqs. (2.57) through (2.59) in the same way as described in section 2.2. Subsequently, the partitions F_y and F_q may be taken from $\tilde{F}_y$.

2.4. Concluding remarks

In the previous sections the design of linear proportional and dynamic output feedback regulators has been discussed. It is shown that the control law can be computed by solving a set of three matrix equations. Since this set of equations only poses necessary conditions for optimal output feedback, uniqueness and existence of stabilizing solutions cannot be guaranteed at this stage. However, for suitably chosen initial stabilizing solutions F_{y_s}, optimization algorithms will usually converge although convergence may be slow (and consequently, computation time may be high).
In the output feedback regulator optimization process, reconstructive and control actions are closely related. However, regulator control and reconstruction properties may be influenced separately by adjusting the weighting matrices R_z, R_u and R_q, respectively the variance matrices C_{vv}, C_{vw} and C_{ww}. The latter are usually chosen on the basis of physical properties of the system to be controlled. The former are usually tuned through further evaluation of the closed-loop controlled system. Increasing the value of the elements of R_z will shift the poles of the closed-loop system to the left, decreasing the system's sensitivity to parameter variations, but increasing the control effort. This effect may also be noticed when decreasing the value of the elements of R_u.
Individual tuning of elements of a weighting matrix will primarily influence the related system variable(-s). Note that a-priori knowledge of system variable interrelations may for instance be acquired through modal decomposition of the system equations.

3. DESIGN OF OUTPUT FEEDBACK REGULATORS FOR FLIGHT CONTROL SYSTEMS

3.1. Introduction

In the previous chapter a design method for proportional and dynamic output feedback controllers, applicable to discrete time systems having stochastic input signals, has been discussed. In this chapter the theory is applied to the design of an aircraft controller.
In section 3.2 a continuous time system model describing the symmetrical motions of an aircraft is presented. In section 3.3 a discrete time version of this system model will be derived. In section 3.4 a proportional and a dynamic regulator will be computed.

3.2. The aircraft model

The linearized differential equations describing the symmetrical motions of an aircraft due to atmospheric turbulence and control input signals read (see refs. 13, 14):

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0 & 0 \\ C_{Z_u} & C_{Z_\alpha} - (2\mu_c - C_{Z_{\dot\alpha}})D_c & -C_{X_0} & 2\mu_c + C_{Z_q} & 0 \\ 0 & 0 & -D_c & 1 & 0 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot\alpha}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c & 0 \\ 0 & -1 & 1 & 0 & D_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \\ \frac{h}{\bar{c}} \end{bmatrix} = \tag{3.1}$$

$$\begin{bmatrix} 0 \\ C_{Z_\delta} \\ 0 \\ C_{m_\delta} \\ 0 \end{bmatrix} \delta_e + \begin{bmatrix} C_{X_{u_g}} & C_{X_{\dot{u}_g}} & C_{X_{\alpha_g}} & C_{X_{\dot\alpha_g}} \\ C_{Z_{u_g}} & C_{Z_{\dot{u}_g}} & C_{Z_{\alpha_g}} & C_{Z_{\dot\alpha_g}} \\ 0 & 0 & 0 & 0 \\ C_{m_{u_g}} & C_{m_{\dot{u}_g}} & C_{m_{\alpha_g}} & C_{m_{\dot\alpha_g}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ D_c \hat{u}_g \\ \alpha_g \\ D_c \alpha_g \end{bmatrix}$$

The numerical values of the matrix coefficients used in this report, pertaining to the BAC SUPER VC 10, are taken from ref. 14 and given in table 1. The above mentioned set of equations can be written in the general form:

$$\dot{x} = A\,x + B\,u + W\,w \tag{3.2}$$

after eliminating the term $C_{m_{\dot{\alpha}}} D_c \alpha$ in eq. (3.1). This can be accomplished by multiplying the second row of eq. (3.1) with $2\mu_c - C_{Z_{\dot{\alpha}}}$, summing the result and the fourth row, and finally dividing the first row by $2\mu_c$, the second row by $(2\mu_c - C_{Z_{\dot{\alpha}}})$ and the fourth row by $2\mu_c K_y^2$. The result is, in abbreviated notation:

$$\frac{\bar{c}}{V}\begin{bmatrix} \dot{\hat{u}} \\ \dot{\alpha} \\ \dot{\theta} \\ \frac{\dot{q}\bar{c}}{V} \\ \frac{\dot{h}}{\bar{c}} \end{bmatrix} = \begin{bmatrix} x_u & x_\alpha & x_\theta & 0 & 0 \\ z_u & z_\alpha & z_\theta & z_q & 0 \\ 0 & 0 & 0 & 1 & 0 \\ m_u & m_\alpha & m_\theta & m_q & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \\ \frac{h}{\bar{c}} \end{bmatrix} + \begin{bmatrix} 0 \\ z_\delta \\ 0 \\ m_\delta \\ 0 \end{bmatrix} \cdot \delta_e +$$

$$\begin{bmatrix} x_{u_g} & x_{\dot{u}_g} & x_{\alpha_g} & x_{\dot{\alpha}_g} \\ z_{u_g} & z_{\dot{u}_g} & z_{\alpha_g} & z_{\dot{\alpha}_g} \\ 0 & 0 & 0 & 0 \\ m_{u_g} & m_{\dot{u}_g} & m_{\alpha_g} & m_{\dot{\alpha}_g} \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_g \\ D_c \hat{u}_g \\ \alpha_g \\ D_c \alpha_g \end{bmatrix} \tag{3.3}$$

The matrix elements written in terms of the elements of eq. (3.1) in this equation are presented in table 2.

The disturbing input signals, the gust velocities u_g and α_g and their first order derivatives, can be modelled as the output signals of linear filters, driven by white noise. These filters may be computed from the power spectral density functions of the gust signals. These spectra are, according to the Dryden model:

horizontal gusts: $$\Phi_{\hat{u}_g \hat{u}_g} = \frac{2}{\pi} \sigma^2_{\hat{u}_g} \frac{L_g}{V} \frac{1}{1 + \left(\frac{\omega L_g}{V}\right)^2} \qquad (3.4)$$

vertical gusts: $$\Phi_{\alpha_g \alpha_g} = \frac{1}{\pi} \sigma^2_{a_g} \frac{L_g}{V} \frac{1 + 3\left(\frac{\omega L_g}{V}\right)^2}{\left(1 + \left(\frac{\omega L_g}{V}\right)^2\right)^2} \qquad (3.5)$$

The following differential equations, describing the turbulence filters, pertaining to these spectra may be derived (ref. 11):

$$\dot{\hat{u}}_g \frac{\bar{c}}{V} = -\frac{\bar{c}}{L_g} \hat{u}_g + \sigma_{\hat{u}_g} \sqrt{\frac{2V}{L_g}}\, w_1 \qquad (3.6)$$

$$\begin{bmatrix} \dot{\alpha}_g \\ \dot{\alpha}^*_g \end{bmatrix} \frac{\bar{c}}{V} = \begin{bmatrix} 0 & 1 \\ -\frac{\bar{c}^2}{L_g^2} & -2\frac{\bar{c}}{L_g} \end{bmatrix} \begin{bmatrix} \alpha_g \\ \alpha^*_g \end{bmatrix} + \begin{bmatrix} \sigma_{\alpha_g} \frac{\bar{c}}{V} \sqrt{\frac{3V}{L_g}} \\ (1-2\sqrt{3}) \frac{\bar{c}}{L_g} \sigma_{\alpha_g} \frac{\bar{c}}{V} \sqrt{\frac{V}{L_g}} \end{bmatrix} \cdot w_2 \qquad (3.7)$$

Here w_1 and w_2 represent white noise signals with intensity 1. Furthermore, the auxiliary variable α^*_g has been introduced:

$$\alpha^*_g = \alpha_g \frac{\bar{c}}{V} - \sigma_{\alpha_g} \frac{\bar{c}}{V} \sqrt{\frac{3V}{L_g}}\, w_2 \qquad (3.8)$$

The elevator servo is modelled as a first order system, described by the following differential equation:

$$\dot{\delta}_e = -\frac{1}{\tau_e} \delta_e + \frac{K_e}{\tau_e} \delta_{e_c} \qquad (3.9)$$

Here δ_{e_c} is the commanded elevator deflection.

Adding the gust turbulence filters and the servo model to the aircraft model, yields the following augmented system model:

$$\dot{x}^+ = A^+ x^+ + B^+ u_c + W^+ w^+ \tag{3.10}$$

Where $x^+ = \text{col}\,[\hat{u},\ \alpha,\ \theta,\ \frac{q\bar{c}}{V},\ \frac{h}{\bar{c}},\ \delta_e,\ \hat{u}_g,\ \alpha_g,\ \alpha_g^*]$

$u_c = \delta_{e_c}$

$w^+ = \text{col}\,[w_1,\ w_2]$

Note that controllability of the pair (A^+, B^+) is guaranteed on the basis of physical considerations.

To demonstrate the applicability of the control law design method presented in the previous chapter to the design of aircraft regulators, a proportional as well as a dynamic altitude regulator will be computed. To facilitate control of the aircraft, the angle of pitch θ and the altitude h will be measured. The vector of controlled variables z contains two variables, the flight altitude h and the rate of pitch q. The latter is considered a controlled variable since, for reasons of comfort and load limitations, pitch rates should be small. Considering q a controlled variable implies the possibility to weigh the deviations of the altitude h and the rate of pitch q versus the control input signal δ_{e_c}.

The observation equation reads:

$$y = C\,x^+ + H\,v \tag{3.11}$$

where:
$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{c} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.12}$$

$$H = \begin{bmatrix} \sigma_\theta & 0 \\ 0 & \sigma_h \end{bmatrix} \tag{3.13}$$

$v = \text{col}\,[v_1,\ v_2]$

Here v_1 and v_2 represent white noise signals with intensity 1.

The equation of controlled variables reads:

$$z = D_1\,x^+ + D_2\,u_c \tag{3.14}$$

Where: $$D_1 = \begin{bmatrix} 0 & 0 & 0 & \bar{c}/V & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{c} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.15)$$

$$D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.16)$$

The matrices A^+, B^+, W^+, C_1, H, D_1 and D_2 are given in table 3.
The eigen values and related response criteria of the uncontrolled aircraft are given in table 4.

3.3. Discretization of the aircraft model

Before applying the control law design methods discussed in chapter 2, an equivalent discrete time system model of the aircraft will be presented, derived from the continuous time model. The discretization process is not treated in depth. Detailed information about the discretization process may be found in ref. 7. Omitting the superscript '+' in eqs. (3.10), (3.11) and (3.14), the aircraft in the continuous time domain is described by the following equations:

$$\dot{x}(t) = A\, x(t) + B\, u_c(t) + W\, w(t) \quad (3.17)$$

$$y(t) = C\, x(t) + H\, v(t) \quad (3.18)$$

$$z(t) = D_1\, x(t) + D_2\, u_c(t) \quad (3.19)$$

According to ref. 7 the state of the system at the sampling instant $t = t_{k+1}$ can be computed from the system state $x(t_k)$, the control input u(t) and the disturbing input signal w(t), using the following equation:

$$x(t_{k+1}) = \Phi(t_{k+1},t_k)\, x(t_k) + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\tau)\, B\, u(\tau)\, d\tau + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\tau)\, W\, w(\tau)\, d\tau \quad (3.20)$$

Assuming the sampling interval constant and considering the system model to be time invariant, the following equation holds:

$$\Phi(t_{k+1},t_k) = \Phi(\Delta t,\ 0) = \Phi = e^{A\,\Delta t} \quad (3.21)$$

Where Δt is the sampling interval.
Furthermore it is assumed that the control signals are computed by a digital computer and passed onto the aircraft using a zero order hold device. Therefore:

$$u(t) = u(t_k) \qquad (t_k \leq t < t_{k+1}) \tag{3.22}$$

Finally the disturbing white noise input signals w(t) are replaced by a sequence of mutually uncorrelated, stochastic vectors $w(t_k)$ with covariance matrix C_{ww}.
Under these assumptions the state equation of the system reads:

$$x(t_{k+1}) = \Phi\, x(t_k) + \Gamma\, u(t_k) + \Psi\, w(t_k) \tag{3.23}$$

Where
$$\Gamma = \int_0^{\Delta t} e^{A\tau}\, B\, d\tau \tag{3.24}$$

$$\Psi = \int_0^{\Delta t} e^{A\tau}\, W\, d\tau \tag{3.25}$$

The discrete time output equation of the system reads:

$$y(t_i) = C\, x(t_i) + H\, v(t_i) \qquad (t_k \leq t_i \leq t_{k+1}) \tag{3.26}$$

Substitution of eq. (20) yields:

$$y(t_i) = C\, \Phi\, (t_i, t_k)\, x(t_k) + C \int_{t_k}^{t_i} \Phi(t_i, \tau)\, B\, u(\tau)\, d\tau$$

$$+ C \int_{t_k}^{t_i} \Phi\, (t_i, \tau)\, W\, u(\tau)\, d\tau + H\, v(t_i) \tag{3.27}$$

From eq. (3.27) it can be seen that if the sampling instances t_i and t_k do not coincide, a direct link between system input and output will be created; even if the continuous time system does not have a direct link.
Assuming that the observation sampling instances t_i, coincide with the instances t_k on which the control signals $u(t_k)$ are passed to the system, eq. (3.27) reduces to:

$$y(t_k) = C\, x(t_k) + H\, v(t_k) \tag{3.28}$$

The continuous time, white noise signals v(t) have been replaced by a sequence of mutually uncorrelated, stochastic vectors $v(t_k)$ with covariance matrix C_{vv}.
In a similar way the equation of controlled variables may be discretized. The result reads:

$$z(t_k) = D_1\ x(t_k) + D_2\ u_c(t_k) \tag{3.29}$$

replacing the time argument t_k by the time interval notation k, the following system equations are obtained:
State equation:

$$x(k + 1) = \Phi\ x(k) + \Gamma\ u(k) + \Psi\ w(k) \tag{3.30}$$

Observation equation:

$$y(k) = C\ x(k) + H\ v(k) \tag{3.31}$$

Equation of controlled variables:

$$z(k) = D_1\ x(k) + D_2\ u(k) \tag{3.32}$$

It will be noted that these last three equations are identical to the ones used in chapter 2.

3.4. Examples

In this section, two experiments that have been carried through will be briefly discussed. In section 3.2 a continuous-time system model of the aircraft + elevator servo + gust turbulence filters has been presented. The system vectors were defined as follows:

$$x^+ = \text{col}\ (\hat{u},\ a,\ \theta,\ \frac{q\bar{c}}{V},\ \frac{h}{\bar{c}},\ \delta_e,\ \hat{u}_g,\ \alpha_g,\ \alpha^*_g)$$

$$u = \delta_{e_c}$$

$$w^+ = \text{col}\ (w_1,\ w_2)$$

$$y = \text{col}\ (\theta_m,\ h_m)$$

$$v = \text{col}\ (v_1, v_2)$$

$$z = \text{col}\ (q, h)$$

with system matrices defined by eqs. (3.2), (3.3), (3.7), (3.9), (3.11), (3.12), (3.13), (3.14), (3.15) and (3.16).
This system model has been discretized as described in section 3.3, for a sample rate of 20 times per second.
Subsequently a regulator has been computed. Table 5 lists the specified weighting matrices and resulting gain matrix which were arrived at after some re-adjustments in the initial specification, according to the principles described in chapter 2. The eigen values and related variables of the controlled aircraft are given in table 6.

The performance of the regulator has been evaluated using a linear simulation computer program. The responses of the uncontrolled and controlled aircraft to a step shaped disturbing input signal $w_2(t)$ are given in fig. 3 and 4. The disturbing input signal is defined by:

$$w_2(t) = 0 \qquad t < 0 \tag{3.33}$$

$$w_2(t) = 0.2 \qquad t \geq 0 \tag{3.34}$$

The responses of the controlled aircraft have been computed, assuming the absence of observation noise. From figures 3 and 4 and tables 4 and 6 it is concluded that the damping of the phugoid has been increased while the damping of the short period oscillation has decreased.
Furthermore it is concluded from fig. 4 that the regulator can not completely eliminate the effect of the disturbing input signal $w_2(t)$ upon the altitude h. For that reason it was decided to add integrating action to the regulator. So a dynamic regulator will be designed.

Before specifying the matrices M and N, consider the following discrete time state equation of an integrating element:

$$x_i(k + 1) = x_i(k) + \frac{\Delta t}{\tau_i} u_i(k) \tag{3.35}$$

Where x_i is the state of the integrator
u_i is the input signal of the integrator
Δt is the sampling time interval
τ_i is the integration time constant

Using this equation, the matrices M and N, describing the dynamics of the regulator can easily be specified:

$$M = [1]$$

$$N = [0 \quad \frac{\Delta t}{\tau_1}]$$

In this experiment Δt equals 0.05 and τ_i was chosen equal to 10. sec. to maintain parameter values for the integrator state variable at magnitudes comparable to the other output signals in the relevant frequency interval (see also table 4). Note that here such a choice is irrelevant for the controller performance since the resulting feedback gain factor is directly related to the magnitude of τ_i and hence optimization will ensure proper scaling (for given weighting factors) in the resulting feedback gain matrix. For the given time interval Δt and integration time constant τ_1:

$$N = [0 \quad 0.005]$$

Subsequently the system matrices were augmented as described in section 3.3 and the gain matrix was computed. The specified weighting matrices and resulting control law are given in table 7. Note that the weighting factors for the proportional feedback signals have remained unaltered, allowing evaluation of the effect of adding integrating actions.
Slight changes have obviously resulted in the feedback gains of the proportional feedback signals to maintain the minimum value of the criterion. The eigen values of the controlled aircraft are given in table 8. The responses of the controlled aircraft to a step shaped disturbing input signal $w_2(t)$ are given in figs. 3 and 4. From these figures and tables 4 and 8 it is concluded that the damping of the phugoid has been increased, while the damping of the short period oscillation has decreased. Both are slightly smaller than in the proportional feedback case. More important, a marked improvement in the stability of the aperiodic (altitude) motion has been achieved. Fig. 4 shows that this regulator completely eliminates the effect of the disturbing input signal w_2 upon the altitude h.

Although this regulator is capable of stabilizing the aircraft at a given altitude h, the design process may be continued and regulator performance may be shaped to conform more closely to the design specification. Adjustment of some characteristic criteria used in classical control theory may be expressed in terms of weighting functions as:

1. the settling time of the controlled aircraft may be decreased by increasing the weighting factor for the integrated deviations of the altitude h, or by decreasing the integration time constant;
2. the damping ratio of the short period oscillation may be increased by adding differentiating action to the controller for the angle of pitch θ.

Such adjustments have not been incorporated in the present example, since no practical implementation of the controller is intended and the above merely serves to illustrate the method and practice of designing automatic aircraft control systems using optimization theory.
For practical purposes a proper design specification should be drafted before engaging in the actual design process. Such a design specification should include all kinds of considerations relating to passenger comfort, structural limitations, safety etc., all of which may be either included in the performance criterion or may be evaluated in the course of the design process. A demonstration of a full-scale design process is however considered outside the scope of this report.

4. CONCLUSIONS

In chapter 2 of this report a design method for linear proportional and linear dynamic regulators for multivariable discrete time systems has been presented. In chapter 3 the design method has been demonstrated by applying it to the design of an altitude regulator of a transport aircraft. Since these examples deal with a conventional control task and a conventional aircraft type, the advantages of a multivariable approach over a classical single loop approach have not been dwelt upon. The example chosen should be regarded as a demonstration tool, which by virtue of its relative simplicity serves to illustrate an innovative concept.

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Table 1: Aircraft data BAC SUPER VC10

W	= 96160 kg	V	= 71.24 m/sec		
S	= 260.68 m^2	μ_c	= 49.315		
$\bar{c}$	= 6.10 m	K_Y^2	= 2.354		
b	= 42.67 m	ρ	= 0.125 kg sec^2/m^4		
$x_{c.g}$	= 0.36 $\bar{c}$				
C_{X_o}	= −0.0507	C_{Z_o}	= −1.163		
C_{X_u}	= −0.370	C_{Z_u}	= −2.326	C_{m_u}	= 0
C_{X_α}	= 0.655	C_{Z_α}	= −5.04	C_{m_α}	= −0.72
		$C_{Z_{\dot{\alpha}}}$	= −0.395	$C_{m_{\dot{\alpha}}}$	= −1.218
		C_{Z_q}	= −4.65	C_{m_q}	= −8.622
		C_{Z_δ}	= −0.342	C_{m_δ}	= −1.055
$C_{X_{\alpha_g}}$	= 0.655	$C_{Z_{\alpha_g}}$	= −5.04	$C_{m_{\alpha_g}}$	= −0.72
$C_{X_{\dot{\alpha}_g}}$	= 0	$C_{Z_{\dot{\alpha}_g}}$	= 4.255	$C_{m_{\dot{\alpha}_g}}$	= 7.400
$C_{X_{u_g}}$	= −0.370	$C_{Z_{u_g}}$	= −2.326	$C_{m_{u_g}}$	= 0
$C_{X_{\dot{u}_g}}$	= 0	$C_{Z_{\dot{u}_g}}$	= −0.957	$C_{m_{\dot{u}_g}}$	= −0.584

Table 2: The matrix elements of equation 3.3.

$$x_u = \frac{C_{X_u}}{2\mu_c} \qquad z_w = \frac{C_{Z_u}}{2\mu_c - C_{z_{\dot\alpha}}} \qquad m_u = \frac{C_{m_u} + C_{Z_u} \cdot \dfrac{C_{m_{\dot\alpha}}}{2\mu_c - C_{Z_{\dot\alpha}}}}{2\mu_c K_Y^2}$$

$$x_\alpha = \frac{C_{X_\alpha}}{2\mu_c} \qquad z_\alpha = \frac{C_{Z_\alpha}}{2\mu_c - C_{Z_{\dot\alpha}}} \qquad m_\alpha = \frac{C_{m_\alpha} + C_{Z_\alpha} \cdot \dfrac{C_{m_{\dot\alpha}}}{2\mu_c - C_{Z_{\dot\alpha}}}}{2\mu_c K_Y^2}$$

$$x_\theta = \frac{C_{z_o}}{2\mu_c} \qquad z_\theta = \frac{-C_{X_o}}{2\mu_c - C_{Z_{\dot\alpha}}} \qquad m_\theta = \frac{-C_{X_o} \cdot \dfrac{C_{m_{\dot\alpha}}}{2\mu_c - C_{Z_{\dot\alpha}}}}{2\mu_c K_Y^2}$$

$$z_q = \frac{2\mu_c + C_{Z_q}}{2\mu_c - C_{Z_{\dot\alpha}}} \qquad m_q = \frac{C_{m_q} + \dfrac{2\mu_c + C_{Z_q}}{2\mu_c - C_{Z_{\dot\alpha}}} \cdot C_{m_{\dot\alpha}}}{2\mu_c K_Y^2}$$

$$x_{u_g} = \frac{C_{X_{u_g}}}{2\mu_c} \qquad z_{u_g} = \frac{C_{Z_{u_g}}}{2\mu_c - C_{Z_{\dot\alpha}}} \qquad m_{u_g} = \frac{C_{m_{u_g}} + C_{Z_{u_g}} \cdot \dfrac{C_{m_{\dot\alpha}}}{2\mu_c - C_{Z_{\dot\alpha}}}}{2\mu_c K_Y^2}$$

$$x_{\dot u_g} = \frac{C_{X_{\dot u_g}}}{2\mu_c} \qquad z_{\dot u_g} = \frac{C_{Z_{\dot u_g}}}{2\mu_c - C_{Z_{\dot\alpha}}} \qquad m_{\dot u_g} = \frac{C_{m_{\dot u_g}} + C_{Z_{\dot u_g}} \cdot \dfrac{C_{m_{\dot\alpha}}}{2\mu_c - C_{Z_{\dot\alpha}}}}{2\mu_c K_Y^2}$$

$$x_{\alpha_g} = \frac{C_{X_{\alpha_g}}}{2\mu_c} \qquad z_{\alpha_g} = \frac{C_{Z_{\alpha_g}}}{2\mu_c - C_{Z_{\dot\alpha}}} \qquad m_{\alpha_g} = \frac{C_{m_{\alpha_g}} + C_{Z_{\alpha_g}} \cdot \dfrac{C_{m_{\dot\alpha}}}{2\mu_c - C_{Z_{\dot\alpha}}}}{2\mu_c K_Y^2}$$

$$x_{\dot\alpha_g} = \frac{C_{X_{\dot\alpha_g}}}{2\mu_c} \qquad z_{\dot\alpha_g} = \frac{C_{Z_{\dot\alpha_g}}}{2\mu_c - C_{Z_{\dot\alpha}}} \qquad m_{\dot\alpha_g} = \frac{C_{m_{\dot\alpha_g}} + C_{Z_{\dot\alpha_g}} \cdot \dfrac{C_{m_{\dot\alpha}}}{2\mu_c - C_{Z_{\dot\alpha}}}}{2\mu_c K_Y^2}$$

$$z_\delta = \frac{C_{Z_\delta}}{2\mu_c - C_{Z_{\dot\alpha}}} \qquad m_\delta = \frac{C_{m_\delta} + C_{Z_\delta} + \dfrac{C_{m_{\dot\alpha}}}{2\mu_c - C_{Z_{\dot\alpha}}}}{2\mu_c K_Y^2}$$

Table 3: The matrices of the continuous time aircraft model

The system matrix A^+ of the augmented linear continuous-time aircraft model:

	1	2	3	4	5
1	-4.3811-02	+7.7557-02	-1.3770-01	-	-
2	-2.7432-01	-5.9440-01	-5.9793-03	+1.1083+01	-
3	-	-	-	+1.1678+01	-
4	+1.4390-03	-3.3098-02	+3.1367-05	-4.9184-01	-
5	-	-1.1678+01	+1.1678+01	-	-
6	-	-	-	-	-
7	-	-	-	-	-
8	-	-	-	-	-
9	-	-	-	-	-

	6	7	8	9
1	-	-4.3811-02	+7.7557-02	-
2	-4.0334-01	-2.7891-01	-5.9440-01	+5.0182-01
3	-	-	-	-
4	-5.2856-02	+2.6096-03	-3.3098-02	+3.6959-01
5	-	-	-1.1678+01	-
6	-9-9999+00	-	-	-
7	-	-4.7493-01	-	-
8	-	-	-	+1.1678+01
9	-	-	-1.9313-02	-9.4986-01

The input distribution matrix B^+ of the augmented linear continous-time aircraft model:

	1
1	-
2	-
3	-
4	-
5	-
6	1.000+01
7	-
8	-
9	-

Table 3 (continued)

The noise input distribution matrix W^+ of the augmented linear continuous-time aircraft model:

	1	2
1	-	-
2	-2.7974-03	1.17132-02
3	-	-
4	+3.5668-05	+1.0652-02
5	-	-
6	-	-
7	+2.8945-01	-
8	-	+3.3660-01
9	-	-1.6675-03

The observation matrix C

	1	2	3	4	5
1	-	-	+1.00000+00	-	-
2	-	-	-	-	+6.0999+00

	6	7	8	9
1	-	-	-	-
2	-	-	-	-

The observation noise input distribution matrix H:

	1	2
1	+8.6999-03	-
2	-	+1.9999-01

Table 3 (continued)

The system matrix D_1 of the equations describing the controlled variables:

	1	2	3	4	5
1	-	-	-	+1.1678+01	-
2	-	-	-	-	+6.0999+00

	6	7	8	9
1	-	-	-	-
2	-	-	-	-

The input distribution matrix D_2 of the equations describing the controlled variables:

	1
1	-
2	-

Table 4: The eigenvalues of the uncontrolled aircraft

Real Part	Imag. Part	Nat. Frequency (rad/sec)	Damping Ratio	Time constant (sec)
-0.5562	± 0.6128	0.8276	0.6721	
-0.0088	± 0.1520	0.1522	0.0581	
0.0000	0.0000			∞

Table 5: The specified weighting matrices and resulting proportional control law

$$R_z = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_u = [200000]$$

$$F_y = [-0.754 \quad -0.0012]$$

Table 6: The eigenvalues of the proportionally controlled aircraft

Real Part	Imag. Part	Nat. Frequency (rad/sec)	Damping Ratio	Time constant (sec)
-0.3830	± 0.8295	0.9137	0.4192	
-0.1485	± 0.2368	0.2796	0.5313	
-0.0069	± 0.0000			143.99

Table 7: The specified weighting matrices and resulting dynamic control law

$$R_z = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_u = [200000]$$

$$R_q = [0.1000]$$

$$F_y = [-0.7446 \quad -0.0014 \quad -0.0005]$$

Table 8: The eigenvalues of the dynamically controlled aircraft

Real Part	Imag. Part	Nat. Frequency (rad/sec)	Damping Ratio	Time Constant (sec)
-0.3885	± 0.8234	0.9194	0.4267	
-0.1341	± 0.2442	0.2786	0.4812	
0.0168	0.000			59.5259

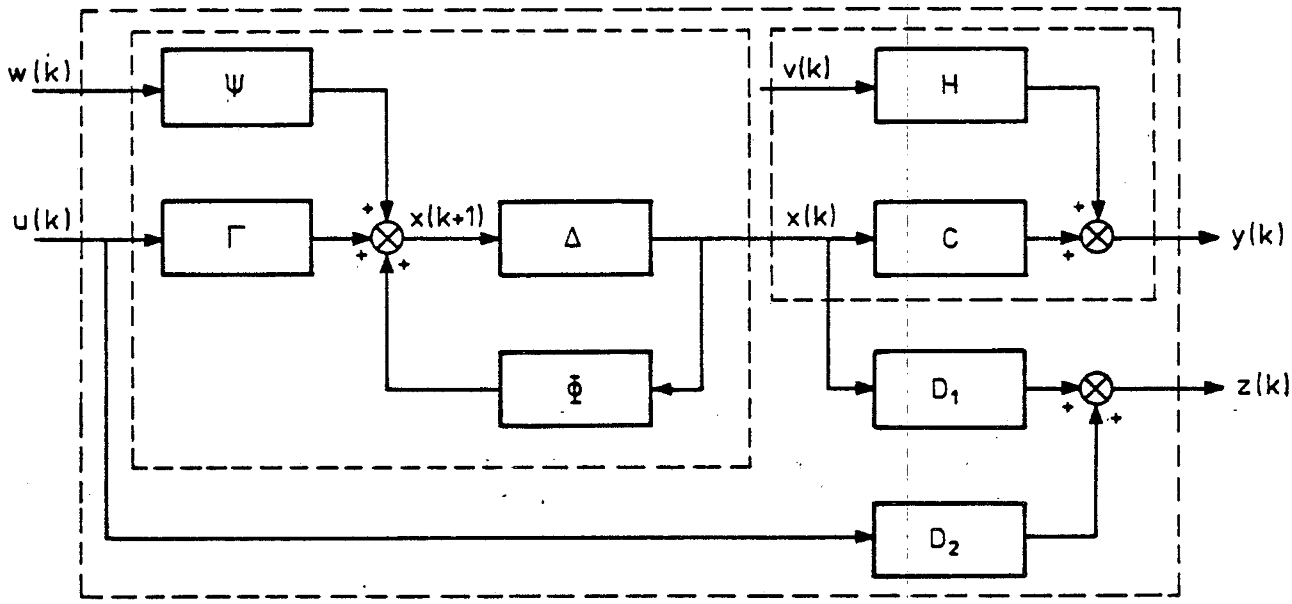


fig. 1 Block diagram of the uncontrolled system

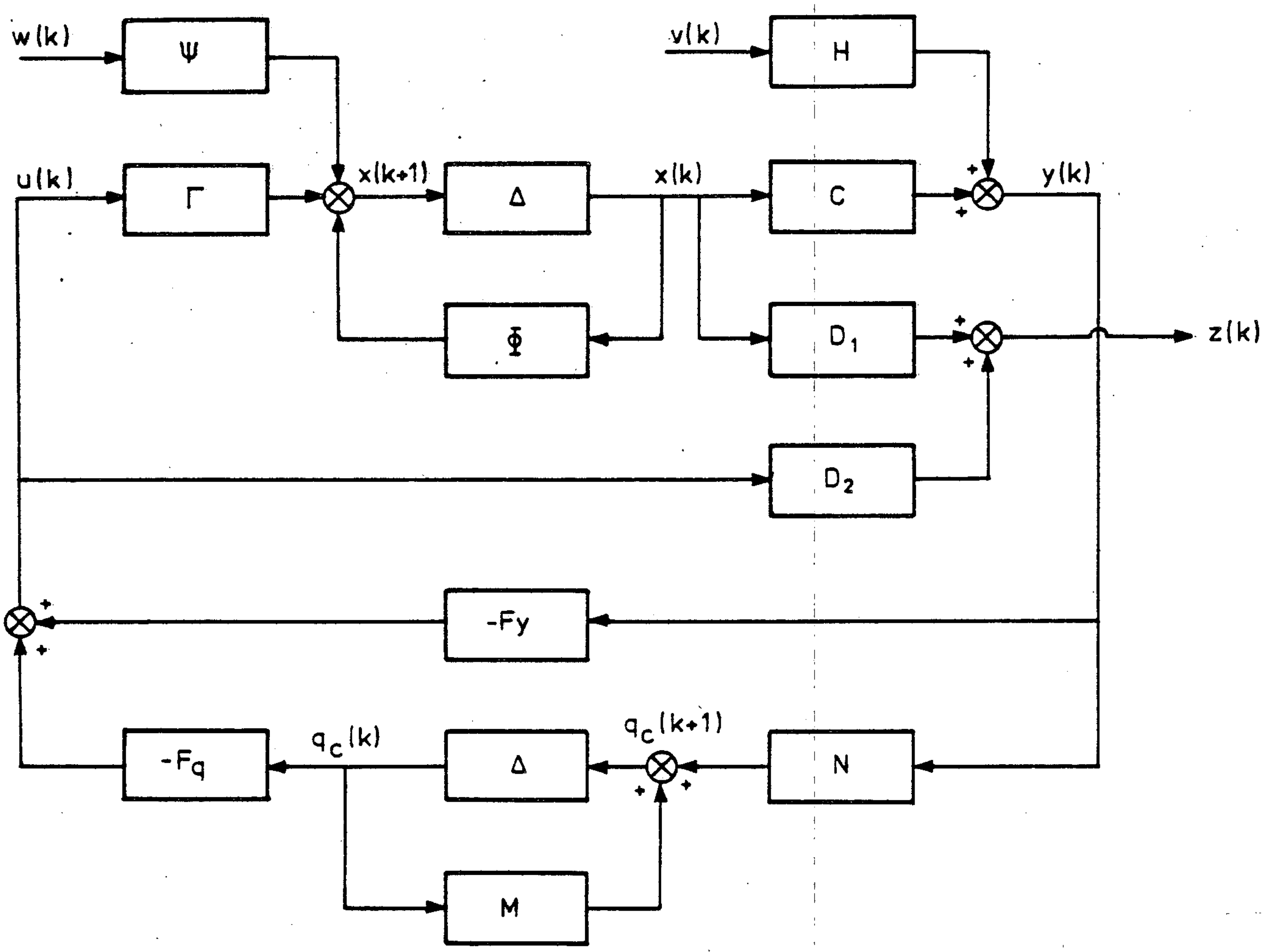
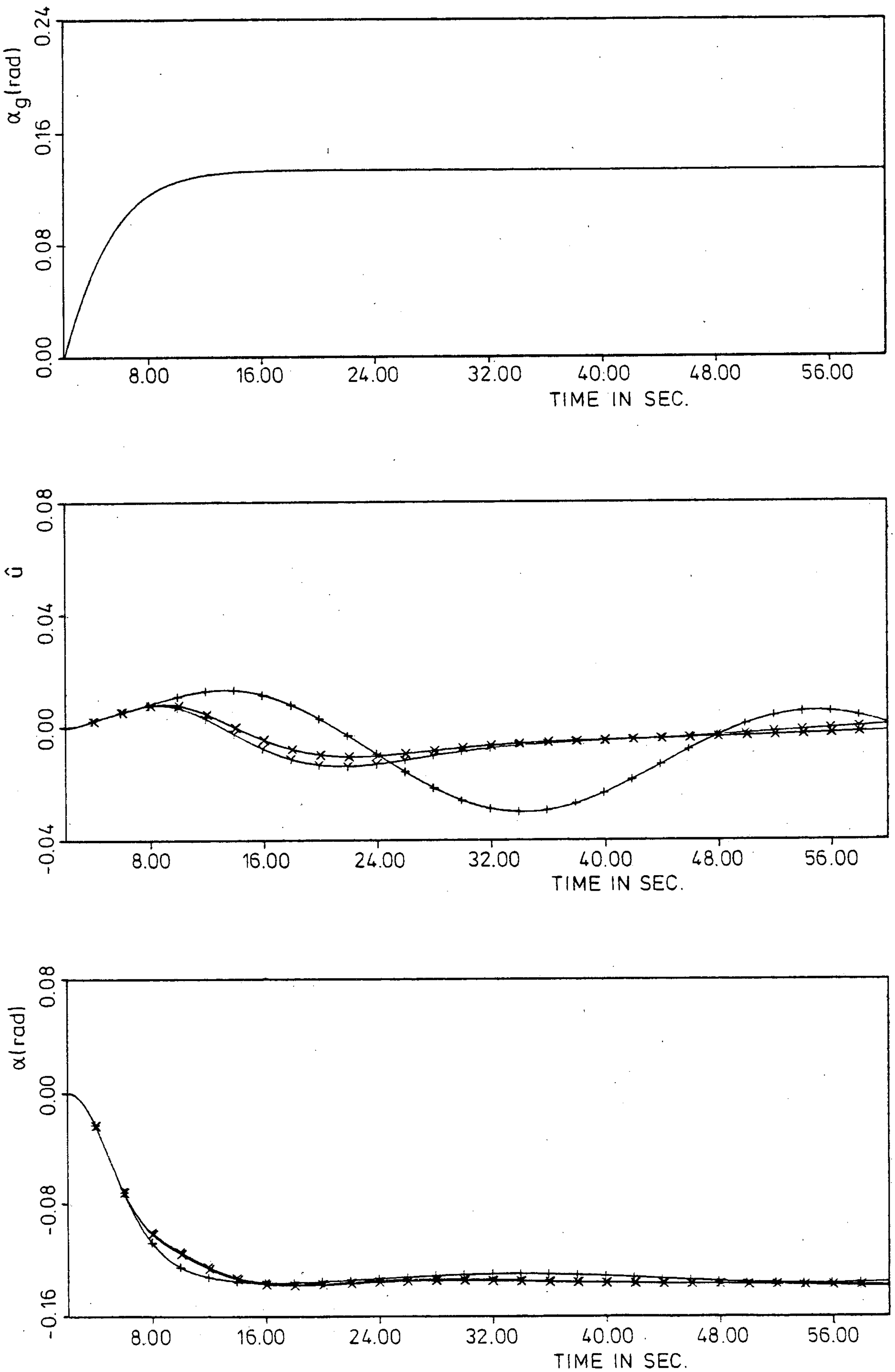
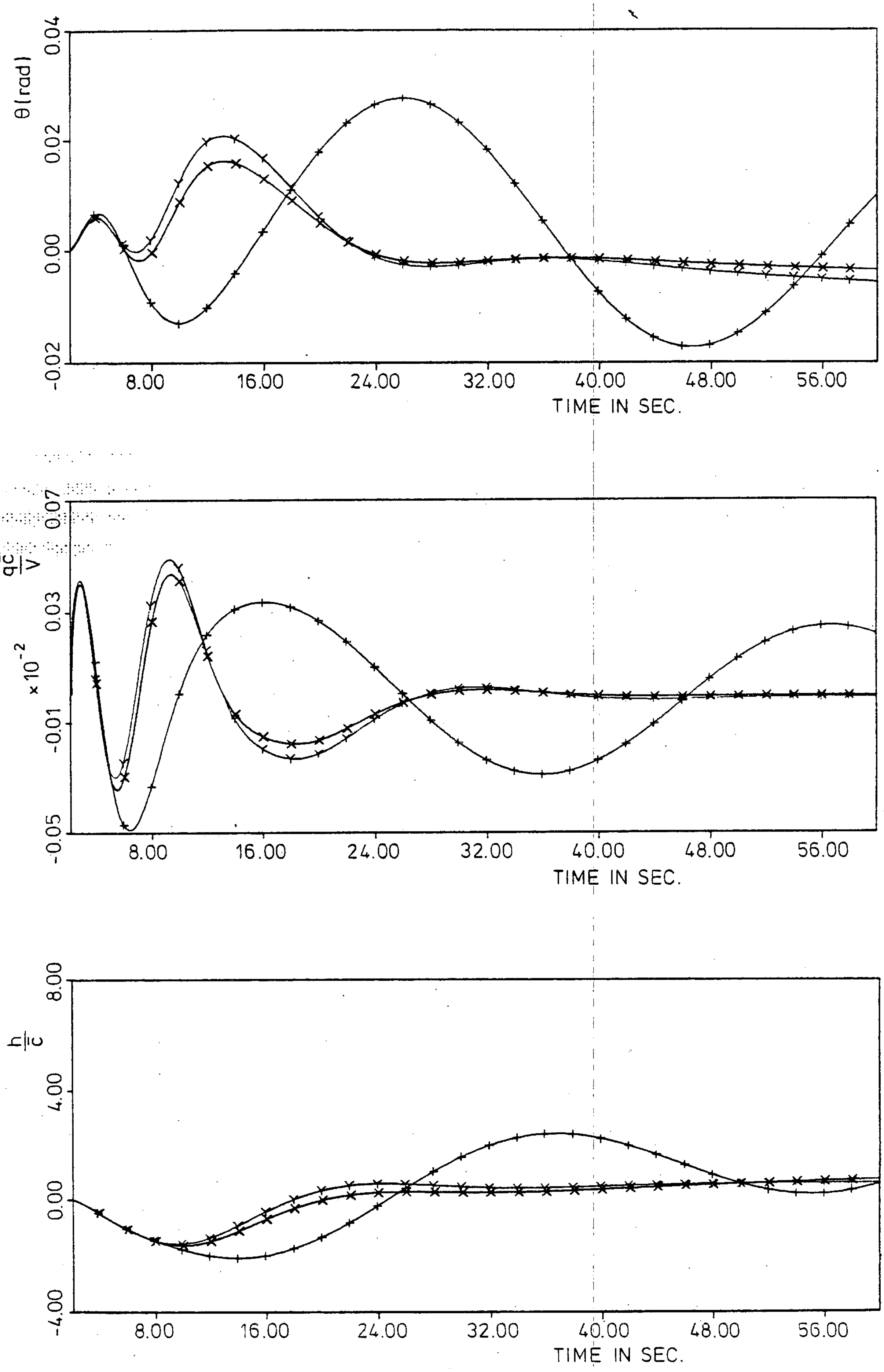


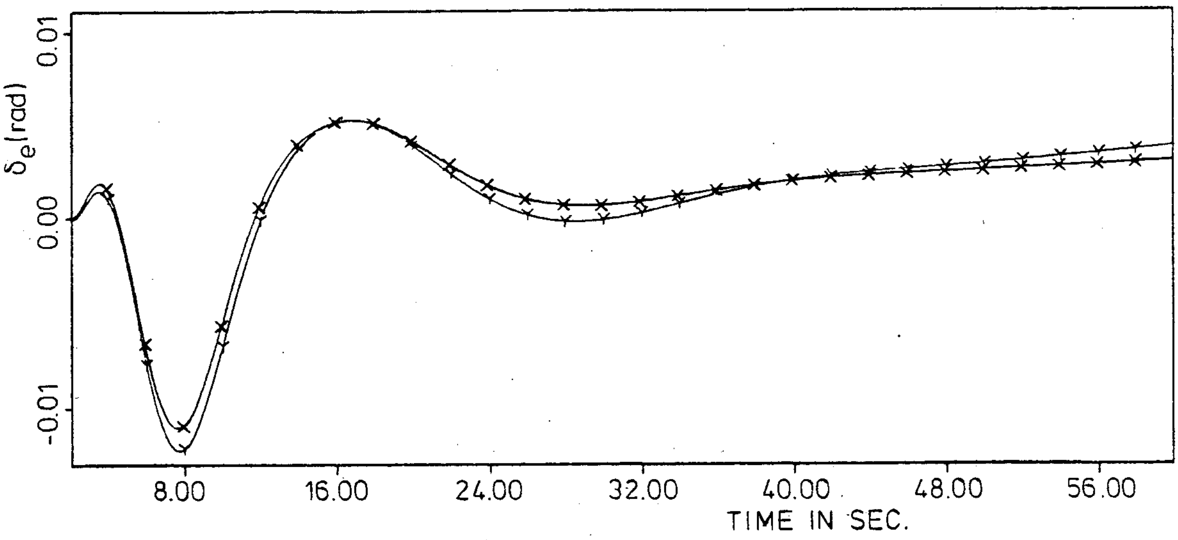
fig. 2 Block diagram of the (dynamically) controlled system



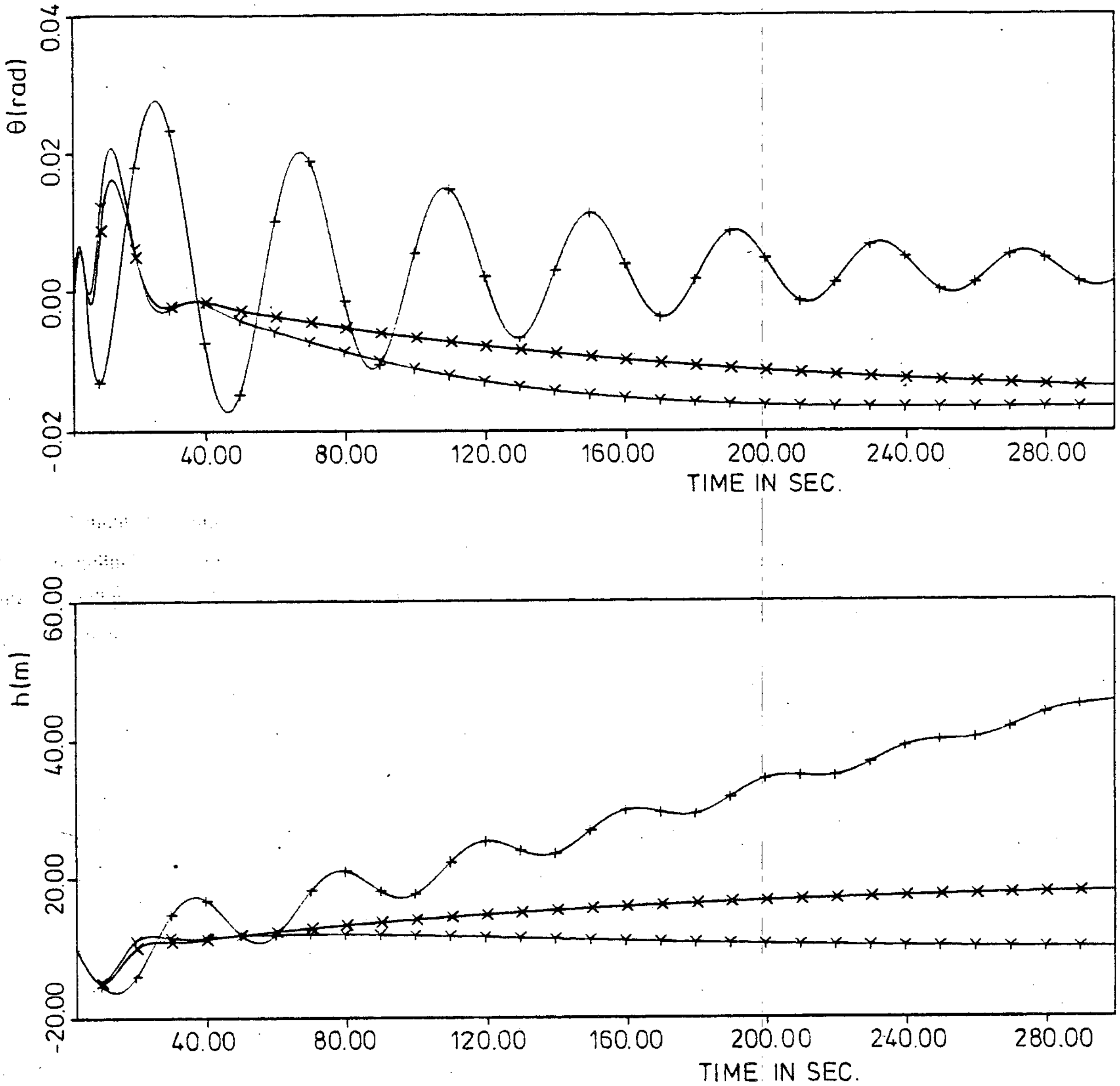
Figuur 3. Aircraft responses of the uncontrolled (+), proportionally controlled (x) and dynamically controlled (y) aircraft to a step-shaped input signal $w_2(t)$.



Figuur 3. Continued.



Figuur 3. Continued.



Figuur 4. Attitude and altitude responses of the uncontrolled (+) proportionally controlled (x) and dynamically controlled (y) aircraft to a step-shaped input signal $w_2(t)$.

APPENDIX 1. DIFFERENTIATION OF MATRICES AND VECTORS

1. Introduction

In Matrix Control Theory advanced mathematics, especially state-space analysis and linear algebra, play a prominent part. Although excellent text books on this subject are available (see for instance ref. 15), this Appendix was included to present the reader with some less well-known results that are of particular interest in the theory of optimal output feedback regulators.

2. Differentials and gradients

Consider the vector x:

$$x \triangleq \text{col}\,[x_1, x_2, \ldots, x_n] \tag{1}$$

The differential of x is defined by:

$$dx \triangleq \text{col}\,[dx_1, dx_2, \ldots, dx_n] \tag{2}$$

Consider the scalar function f(x):

$$f(x) = f(x_1, x_2, \ldots, x_n) \tag{3}$$

The gradient vector of f(x) with respect to x is defined by:

$$\frac{df(x)}{dx} \triangleq \text{col}\,\left[\frac{df(x)}{dx_1}, \frac{df(x)}{dx_2}, \ldots, \frac{df(x)}{dx_n}\right] \tag{4}$$

Consider the n x m matrix A with elements $a_{i,j}$ (i=1,...,n; j=1,...,m). The differential of A is defined by:

$$dA = \begin{bmatrix} da_{11} & da_{12} & \cdots & da_{1m} \\ \vdots & & & \\ da_{n1} & da_{n2} & \cdots & da_{nm} \end{bmatrix} \tag{5}$$

From this definition the following relations can be derived:

$$d(cA) = cdA \qquad \text{(c is a scalar constant)} \tag{6}$$

$$d(A+B) = dA + dB \tag{7}$$

$$d(AB) = (dA)\ B + A\ (dB) \tag{8}$$

$$d(A^{-1}) = -A^{-1}(dA)\ A^{-1} \tag{9}$$

The last relation may be derived by applying relation (8) to the product AA^{-1} = I (I is the unity matrix)

$$d(AA^{-1}) = (dA)\ A^{-1} + A(dA^{-1}) = dI = 0 \tag{10}$$

Consider the real scalar function f(A):

$$f(A) = f(a_{11}, \ldots, a_{1m}, \ldots, a_{n1}, \ldots, a_{nm}) \tag{11}$$

The gradient matrix of f(A) with respect to A is defined by:

$$\frac{df(A)}{dA} = \begin{bmatrix} \frac{df(A)}{da_{11}} & \cdots & \frac{df(A)}{da_{1m}} \\ \vdots & & \\ \frac{df(A)}{da_{n1}} & \cdots & \frac{df(A)}{da_{nm}} \end{bmatrix} \tag{12}$$

The Jacobian:
Differentiation of the vector valued variable z(x) with respect to the vector x is defined by:

$$\frac{dz(x)}{dx} = \begin{bmatrix} \frac{dz_1}{dx_1} & \frac{dz_1}{dx_2} & \cdots & \frac{dz_1}{dx_n} \\ \vdots & & & \\ \frac{dz_m}{dx_1} & \frac{dz_m}{dx_2} & \cdots & \frac{dz_m}{dx_n} \end{bmatrix} \tag{13}$$

3. Differentiation of linear and quadratic expressions

Using the previously given definitions the derivative with respect to x of the linear expression:

$$z(x) = A\ y(x) \tag{14}$$

(both z and y are vectorvalued variables) equals:

$$\frac{dz(x)}{dx} = A\,\frac{dy(x)}{dx} \tag{15}$$

The derivative with respect to the vctor x of the scalar quadratic form:

$$f = y^T(x)\ A\ z(x) \tag{16}$$

(f is a scalar, y = y(x) and z = z(x) are vectors, A is a matrix independent of x) equals:

$$\frac{df}{dx} = \frac{dy^T}{dx}\ A\ z + \frac{dz^T}{dx}\ A^T\ y \tag{17}$$

The derivative of the scalar quadratic form f with respect to A equals:

$$\frac{df}{dA} = y\ z^T \tag{18}$$

$$\frac{df}{dA^T} = z\ y^T \tag{19}$$

4. Derivatives of matrix trace functions

The trace of a square matrix equals the sum of the diagonal elements:

$$\mathrm{tr}\ [A] = \sum_{i=1}^{n} a_{ii} \tag{20}$$

where a_{ii} are the diagonal elements of the matrix A.
The following relations can be derived:

$$\mathrm{tr}\ [A] = \mathrm{tr}\ [A^T] \tag{21}$$

$$\mathrm{tr}\ [A + B] = \mathrm{tr}\ [A] + \mathrm{tr}\ [B] \tag{22}$$

$$\mathrm{tr}\ [A^TB] = \mathrm{tr}\ [B^TA] = \mathrm{tr}\ [AB^T] = \mathrm{tr}\ [BA^T] \tag{23}$$

$$\mathrm{tr}\ [ABC] = \mathrm{tr}\ [BCA] = \mathrm{tr}\ [CAB] \tag{24}$$

Consider the matrix trace function f:

$$f(X) = \text{tr}\ [MX] \tag{25}$$

where M is a m x n matrix and X a n x m matrix.
Since f is a scalar function a gradient matrix can be derived according to relation (12). The gradient matrix can be calculated by direct differentiation or by application of Kleinman's lemma (ref. 16) which will be stated here without further proof.

If the following equation holds:

$$f(X + \varepsilon\Delta X) - f(X) = \varepsilon\ \text{tr}\ [M(X)\ \Delta X] \tag{26}$$

when ε approaches 0, then the gradient matrix equals:

$$\frac{df(X)}{dX} = M^T(X) \tag{27}$$

where it is assumed that the elements of X are mutually independent (see ref. 16).

Without further derivation the gradient matrices of a number of commonly used trace functions will be given:

$$\frac{\partial\ \text{tr}[X]}{\partial X} = I \tag{28}$$

$$\frac{\partial\ \text{tr}\ [AX]}{\partial X} = A^T \tag{29}$$

$$\frac{\partial\ \text{tr}\ [AX^T]}{\partial X} = A \tag{30}$$

$$\frac{\partial\ \text{tr}\ [AXB]}{\partial X} = A^T B^T \tag{31}$$

$$\frac{\partial\ \text{tr}\ [AX^T B]}{\partial X} = BA \tag{32}$$

$$\frac{\partial\ \text{tr}\ [X^T AX]}{\partial X} = [A + A^T]\ X \tag{33}$$

$$\frac{\partial\ \text{tr}\ [XAX^T]}{\partial X} = X\ [A + A^T] \tag{34}$$

$$\frac{\partial\ \text{tr}\ [AXBX]}{\partial X} = A^T X^T B^T + B^T X^T A^T \tag{35}$$

$$\frac{\partial\ \text{tr}\ [AXBX^T]}{\partial X} = AXB + A^T XB^T \tag{36}$$

$$\frac{\partial \text{ tr } [AXX^T B]}{\partial X} = (A^T B^T + BA) X \qquad (37)$$

$$\frac{\partial \text{ tr } [X^n]}{\partial X} = n (X^{n-1})^T \qquad (38)$$

$$\frac{\partial \text{ tr } [X^{-1}]}{\partial X} = - (X^{-1} X^{-1})^T = - (X^{-2})^T \qquad (39)$$

$$\frac{\partial \text{ tr } [e^X]}{\partial X} = [e^X]^T \qquad (40)$$