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Maximal regularity for parabolic equations with measurable dependence on time and applications

Maximal regularity for parabolic equations with measurable dependence on time and applications

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Introduction

1

INTRODUCTION

The subject of this thesis is the maximal L^p-regularity of the Cauchy problem:

$$\begin{aligned} u'(t) + A(t)u(t) &= f(t),\ t \in (0,T) \\ u(0) &= x. \end{aligned} \tag{1.0.1}$$

Here $(A(t))_{t\in(0,T)}$ is a family of closed operators on a Banach space X_0, and the dependence on time is assumed to be measurable. We assume the operators have a constant domain $D(A(t)) = X_1$ for $t \in [0,T]$. Maximal L^p-regularity means that for all $f \in L^p(0,T;X_0)$, the solution to the evolution problem (1.0.1) has the "maximal" regularity in the sense that u', Au are both in $L^p(0,T;X_0)$.

The main goal of this thesis is two-fold:

- introduce a new abstract approach to maximal L^p-regularity in the case where the dependence on time is merely measurable, based on functional calculus and $\mathscr{R}$-boundedness techniques [1];
- apply the abstract approach to evolution equations and systems to obtain mixed $L^p(L^q)$-estimates, for every $p,q \in (1,\infty)$.

The last part of this thesis is devoted to the study of higher-order parabolic equations on the upper half space with general boundary conditions. Here, the leading coefficients of the operators involved are assumed to have vanishing mean oscillation in both the time and space variables. This is based on a different approach and results in $L^p(L^q)$-estimates, for every $p,q \in (1,\infty)$, for this case.

1.1. THE ABSTRACT APPROACH

In the last decades there has been much interest in maximal regularity techniques and their application to nonlinear partial differential equations (PDEs). Maximal regularity is a very useful tool, as it allows to obtain *a priori* estimates which give global existence results. For example, using maximal regularity it is possible to solve quasilinear and fully nonlinear PDEs by elegant linearization techniques combined with the contraction mapping principle [8, 10, 28, 29, 113, 127]. Various approaches to problems from mathematical physics, such as fluid dynamics,

[1] The $\mathscr{R}$-boundedness is a random boundedness condition on a family of operators which is a strengthening of uniform boundedness (see Definition 2.7.1).

reaction-diffusion equations, material science, etc., can be found for example in [1, 28, 39, 66, 78, 106, 116, 118, 125, 127, 130, 137, 158]. Maximal regularity can also be defined for other classes of function spaces instead of L^p-spaces, in particular for Hölder spaces. Hölder maximal regularity is in fact easier to establish and it is known to hold under rather broad assumptions, both in the autonomous and non-autonomous case [3, 113, 157]. In comparison, maximal L^p-regularity has the advantage that it usually requires the least regularity of the data in the PDEs. On the other hand, it is far more difficult to establish.

An important step in the theory of maximal L^p-regularity was the discovery of an operator-theoretic characterization in terms of $\mathscr{R}$-boundedness properties of the differential operator A, due to Weis in [152, 153]. This characterization was proved for the class of Banach spaces X with the UMD property In such spaces, the boundedness of the Hilbert transform can be translated into $\mathscr{R}$-boundedness of certain operator families.[2]

The case in which the operator A is time-dependent is not as well understood. If $t \mapsto A(t)$ is (piecewise) continuous, one can study maximal L^p-regularity using perturbation arguments (see [9, 13, 128]). In particular, Prüss and Schnaubelt in [128] showed that maximal L^p-regularity of (1.0.1) is equivalent to the maximal L^p-regularity for each operator $A(t_0)$ for $t_0 \in [0, T]$ fixed. This, combined with the characterization of [153], yields a very precise condition for maximal L^p-regularity. The disadvantage is that continuity in time is not a natural assumption in the L^p-setting. In fact, in many real-life models, the differential operator A has time-dependent coefficients, and the dependence on time can be rather rough (e.g. the coefficient could be a stochastic process). If this is the case, the operator-theoretic characterization of maximal regularity just mentioned does not apply or leads to unwanted restrictions.

In this thesis we develop a functional analytic approach to maximal L^p-regularity in the case where $t \mapsto A(t)$ is only measurable (see Chapter 3, in particular Theorem 3.3.8). Using a mild formulation, one sees that to prove maximal L^p-regularity one needs to bound a singular integral with operator-valued kernel $Ae^{-(t-s)A}$. With this motivation, our approach is based on the L^p-boundedness of a new class of vector-valued singular integrals of non-convolution type (see Theorem 3.2.4). It is important to note that we do not assume any Hörmander conditions on the kernel in the time variable. For discussion and references on (vector-valued) singular integrals we refer the reader to Section 3.2.

When the time dependence is just measurable, an operator-theoretic condition for maximal L^p-regularity is known only in the Hilbert space setting for $p = 2$ (see [110, 111] and [143, Section 5.5]). The assumption here is that A arises from a coercive form $a(t, \cdot, \cdot) : V \times V \to \mathbb{C}$ and $V \hookrightarrow X_0 \hookrightarrow V'$. Unfortunately, this only yields a theory of maximal L^2-regularity on V' in general (see [59] for a counterexample).

[2] i.e. the Hilbert transform is bounded in $L^p(\mathbb{R}; X)$ for every $p \in (1, \infty)$ (see Definition 2.7.8).

In many situations one would like to have maximal L^p-regularity on X_0 and also for any $p \in (1,\infty)$. Results of this type have been obtained in [14, 43, 44, 73] using regularity conditions on the form in the time variable.

The case where the domains $D(A(t))$ vary in time will not be considered in this thesis. In that setting maximal L^p-regularity results can be obtained under certain Hölder regularity assumptions in the time variable (see [126] and references therein).

An overview of our main result is given in the following theorem, where we assume the problem (1.0.1) to have zero initial value. The corresponding version with non-zero initial value can be treated via an application of related trace theorems. Details about the initial value problem will be given in Section 3.3.4.

Theorem 1.1.1. *Let $T \in (0,\infty)$ and let X_0, X_1 be Banach spaces, and assume that X_0 has finite cotype and that X_1 is densely and continuously embedded in X_0. Assume $A\colon (0,T) \to \mathscr{L}(X_1, X_0)$ is such that for all $x \in X_1$, $t \mapsto A(t)x$ is measurable and*

$$c_1\|x\|_{X_1} \le \|x\|_{X_0} + \|A(t)x\|_{X_0} \le c_2\|x\|_{X_1}, \quad t \in (0,T),\ x \in X_1.$$

Assume there is an operator A_0 on X_0 with $D(A_0) = X_1$ such that

- *A_0 has a bounded H^∞-functional calculus of angle $\theta \in (0, \frac{\pi}{2})$,*
- *$(A(t) - A_0)_{t\in(0,T)}$ generates an evolution family $(T(t,s))_{0\le s\le t\le T}$ on X_0 which commutes with $(e^{-rA_0})_{r\ge 0}$:*

$$e^{-rA_0}T(t,s) = T(t,s)e^{-rA_0}, \quad 0 \le s \le t \le T, \quad r \ge 0.$$

Moreover, let $\mathscr{K}$ be the set of all functions $k : \mathbb{R} \to \mathbb{C}$ such that $|k * f| \le Mf$, where M denotes the Hardy-Littlewood maximal function. For $k \in \mathscr{K}$ and $f : (0,T) \to X_0$ define now the operator I_k on $L^p((0,T), X_0)$ by

$$I_k f(t) = \int_0^T k(t-s)T(t,s)f(s)ds.$$

Assume that the family $\{I_k : k \in \mathscr{K}\}$ is $\mathscr{R}$-bounded.

Then A has maximal L^p-regularity for every $p \in (1,\infty)$, i.e. for every $f \in L^p((0,T), X_0)$ there exists a unique solution $u \in W^{1,p}((0,T), X_0) \cap L^p((0,T), X_1)$ of the problem (1.0.1) *and there is a constant C independent of f such that*

$$\|u'\|_{L^p((0,T),X_0)} + \|u\|_{L^p((0,T),X_1)} \le C\|f\|_{L^p((0,T),X_0)}.$$

This result is derived as a consequence of Theorem 3.3.8, where the more general case of maximal L^p-regularity on the whole real line is considered and where Muckenhoupt weights are included.

The condition on $A(t) - A_0$ can be seen as an abstract ellipticity condition. The assumption that the operators commute holds for instance if $A(t)$ and A_0 are differential operators with coefficients independent of the space variable on $\mathbb{R}^d$. In our proof, we show that the space dependence can be added later on by perturbation arguments. The family $T(t,s) \in \mathscr{L}(X)$ is a two-parameter evolution family (see Sections 2.4 and 3.3.1 for details). The $\mathscr{R}$-boundedness of the family $\{I_k : k \in \mathscr{K}\}$ plays a central role here. Details will be given in Section 3.2. A sufficient condition for this $\mathscr{R}$-boundedness condition in the case $X_0 = L^q$ will be discussed later on in this introduction.

As a consequence of Theorem 3.3.8, we also obtain a characterization of maximal L^p-regularity when X_0 is a Hilbert space (see Theorem 3.3.20).

1.2. APPLICATIONS

Many concrete parabolic PDEs can be formulated in terms of the abstract Cauchy problem (1.0.1). For applications to quasilinear and nonlinear parabolic problems, it is useful to look for minimal smoothness assumptions on the coefficients of the differential operators involved.

As applications of our abstract approach, we consider higher-order parabolic equations and systems in which the operators A under consideration are assumed to have leading coefficients measurable in the time variable and continuous in the space variable. In particular, we prove maximal L^p-regularity for the following class of parabolic PDEs:

$$u'(t,x) + A(t)u(t,x) = f(t,x), \ t \in (0,T), \ x \in \mathbb{R}^d, \tag{1.2.1}$$

with and without initial value, where A is given by

$$A(t)u(t,x) = \sum_{|\alpha| \le 2m} a_\alpha(t,x) D^\alpha u(t,x). \tag{1.2.2}$$

with $a_\alpha : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^{N \times N}$, α multiindex such that $|\alpha| = \alpha_1 + ... + \alpha_d$ for $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$ and $D^\alpha = D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d}$, where $D_j := -i \frac{\partial}{\partial_j}$.

This will be treated in Section 3.4 in the case $N = 1$ and in Chapter 5 in the case of systems.

For such concrete equations with coefficients which depend on time in a measurable way, maximal L^p-regularity results have been derived using PDE techniques. This approach has been developed in a series of papers by Krylov, Dong and Kim in which several $L^p(L^q)$-regularity results for (1.2.1) are derived under the assumption of coefficients measurable in time and have vanishing mean oscillation in space (see the monograph [102] and [51] and references therein). In [102, Theorem 4.3.8] the case $N = 1$, $p = q$ and $m = 2$ has been considered. Extensions to the case $1 < q \le p < \infty$ have been given in [101] and [96]. The case $p = q$ for higher-order equations and systems under the same assumptions for the coefficients was considered by Dong and Kim in [51].

Our results enable us to give an alternative approach to several of these problems. Moreover, we are the first to obtain a full $L^p(\mathbb{R}; L^q(\mathbb{R}^d))$-theory, whereas previous articles usually only give results for $p = q$ or $q \le p$. Very recently, Dong and Kim in [52] generalized the approach of [51] to the full range of $p, q \in (1, \infty)$, with Muckenhoupt weights and small bounded mean oscillation assumptions on the space variable, also in the case of systems.

An overview of the first of our applications is given in the next result, where we consider $N = 1$. We will use condition (C) on A which will be introduced in Section 3.4.2. It basically says that A is uniformly elliptic and the highest order coefficients are continuous in space, but only measurable in time.

Theorem 1.2.1. *Let $T \in (0, \infty)$. Assume that family of operators $(A(t))_{t\in(0,T)}$ given by* (1.2.2) *satisfy condition (C). Let $p, q \in (1, \infty)$. Then the operator A has maximal L^p-regularity on $(0, T)$, i.e. for every $f \in L^p(0, T; L^q(\mathbb{R}^d))$ there exists a unique*

$$u \in W^{1,p}(0, T; L^q(\mathbb{R}^d)) \cap L^p(0, T; W^{2m,q}(\mathbb{R}^d))$$

such that (1.2.1) *holds a.e. and there is a $C > 0$ independent of f such that*

$$\|u\|_{L^p(0,T;W^{2m,q}(\mathbb{R}^d))} + \|u\|_{W^{1,p}(0,T;L^q(\mathbb{R}^d))} \le C\|f\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))}.$$

The above result is derived in Section 3.4 as a consequence of Theorem 3.4.5, where we consider the more general case of maximal L^p-regularity with $t \in \mathbb{R}$ and where we also include Muckenhoupt weights in time and space. In the case of systems of operators, the corresponding maximal L^p-regularity result is stated in Theorem 5.1.3, where we assume a Legendre–Hadamard ellipticity condition on the operators involved, and in Theorem 5.1.4 for divergence form operators. The proofs are an application of the operator-theoretic method introduced in Theorem 1.1.1, combined with PDE techniques as the localization procedure and the method of continuity. As a consequence, via an application of related trace theorems we also obtain maximal L^p-regularity with non-zero initial value (see Theorems 3.4.8 and 5.3.2).

In order to apply our abstract approach to concrete PDEs, we need to construct the evolution family $(S(t,s))_{s\le t}$ generated by $A(t)$ on $X_0 = L^q$ in the case where the coefficients of the operator are space-independent. The main difficulty in obtaining the evolution family is that the operators $A(t)$ and $A(s)$ do not commute in general. While in the case $N = 1$ an explicit formula for the evolution family exists and is well-known, see e.g. Example 3.3.3, in the case of systems as far as we know the existence and uniqueness of the evolution family was unknown even in the case $q = 2$. In this thesis we explicitly construct the evolution family generated by systems of higher-order differential operators, for every $q \in (1, \infty)$. This will be treated in Section 5.2. This "generation" result is interesting on its own and can be found in Theorem 5.2.4. Its proof is based on Fourier multiplier theory (see Section 2.6). Since we are dealing with systems, the symbol is not explicitly known and only given as the solution to an ordinary differential equation. In order to provide estimates for the symbol, we use the implicit function theorem.

As a further application of our abstract approach we consider boundary value problems on the upper half space with homogeneous general boundary conditions. When the coefficients are measurable functions of t only, a weighted mixed $L^p(L^q)$-regularity result for second-order parabolic equations on the half space was proved by Krylov [99] and Kozlov-Nazarov [98], with power type weights. Their proofs rely on Gaussian bounds for the derivatives of the Green's kernel of the fundamental solution of the parabolic equation under consideration. In this thesis, we prove maximal L^p-regularity for systems of higher-order differential operators with coefficients measurable in the time variable and continuous in the space variable, provided that the operator under consideration generates an evolution family which is bounded on weighted L^q-spaces. This is stated in Theorem 6.2.2. The proof is developed as an application of the abstract approach introduced in Theorem 1.1.1 combined with PDEs techniques. In Section 6.4 we will show as an example that a second-order elliptic differential operator with x-independent coefficients generates an evolution family which is bounded on weighted L^q-spaces. In particular, we will see that Gaussian estimates for the evolution family play an important role here as well (see Lemma 6.4.2). In the case of second-order divergence form operators, Gaussian estimates for the fundamental solutions of the equations under consideration were proven by Aronson [16] in the autonomous case and by Sturm [142] in the non-autonomous case with measurable dependence on time. In the case of non-divergence form operators, Ladyženskaja-Solonnikov-Ural'ceva in [105] proved Gaussian estimates for the fundamental solution of the non-autonomous boundary value problem under the assumption that the dependence on time is Hölder continuous. It is still unclear how to prove these results for higher-order operators and systems, and it is the subject of further investigation.

1.2.1. Sufficient conditions

In the characterization of maximal L^p-regularity state in Theorem 1.1.1, a central role is played by the $\mathscr{R}$-boundedness of the family of integral operators $\{I_k : k \in \mathscr{K}\} \subseteq L^p(\mathbb{R}; X)$, with I_k given by

$$I_k f(t) = \int_{\mathbb{R}} k(t-s)T(t,s)f(s)ds. \tag{1.2.3}$$

Details will be given in Section 3.2. Therefore, in order to apply our abstract approach to show $L^p(L^q)$-regularity for concrete PDEs, one needs sufficient conditions for the $\mathscr{R}$-boundedness of such families.

In this thesis we show that in the scalar case $X = L^q$ with $N = 1$, the following is a sufficient condition for the $\mathscr{R}$-boundedness of such families (see Theorem 3.1.4).

Theorem 1.2.2. *Let $\mathscr{O} \subseteq \mathbb{R}^d$ be open. Let $q_0 \in (1,\infty)$ and let $\{T(t,s) : s,t \in \mathbb{R}\}$ be a family of bounded operators on $L^{q_0}(\mathscr{O})$. Assume that for all A_{q_0}-weights w,*

$$\|T(t,s)\|_{\mathscr{L}(L^{q_0}(\mathscr{O},w))} \leq C, \tag{1.2.4}$$

where C depends on w and is independent of $t,s \in \mathbb{R}$. Then the family of integral operators $\mathscr{I} = \{I_{kT} : k \in \mathscr{K}\} \subseteq \mathscr{L}(L^p(\mathbb{R},v;L^q(\mathscr{O},w)))$ as defined in (1.2.3) *is $\mathscr{R}$-bounded for all $p,q \in$* $(1,\infty)$ *and all $v \in A_p$ and $w \in A_q$. Moreover, in this case the $\mathscr{R}$-bounds $\mathscr{R}(\mathscr{I})$ depend on the weights v and w.*

This allows us to apply Theorem 1.1.1 in the proof of Theorem 1.2.1. It is valid for general families of operators $\{T(t,s) : -\infty < s \leq t < \infty\} \subseteq \mathscr{L}(L^q(\Omega,w))$, and we do not use any regularity conditions for $(t,s) \mapsto T(t,s)$. In the setting where $T(t,s) = e^{-(t-s)A}$ with A as in (1.0.1), the condition (1.2.4) also appears in [61] and [77, 81] in order to obtain $\mathscr{R}$-sectoriality of A. There (1.2.4) is checked by using Calderón-Zygmund and Fourier multiplier theory. Examples of such results for two-parameter evolution families will be given in Section 3.3.1.[3]

The idea behind Theorem 1.2.2 is as follows. As a consequence of the Kahane-Khintchine inequality, in standard spaces such as L^p-spaces, $\mathscr{R}$-boundedness is equivalent to the so-called ℓ^2-boundedness (see Section 2.7). The latter is a special case of ℓ^s-boundedness property (see Section 4.1). In L^p-spaces this boils down to classical $L^p(\ell^s)$-estimates from harmonic analysis (see [70, 71], [65, Chapter V] and [32, Chapter 3]). It follows from the work of Rubio de Francia (see [132–134] and [65]) that $L^p(\ell^s)$-estimates are strongly connected to estimates in weighted L^p-spaces. Details will be given in Chapter 4.

Even if it is a sufficient condition for the scalar case, Theorem 1.2.2 is not enough for systems of operators. For this case, we need to generalize Theorem 1.2.2 to the setting of operators with values in a Hilbert space H, i.e. $X(H) = L^q(\mathbb{R}^d;H)$. In the case H has finite dimension N, one could apply Theorem 1.2.2 coordinate-wise, but this only yields estimates with N-dependent constants.

To avoid this, in this thesis (Chapter 4) we directly consider H-valued operators and we introduce the notion of ℓ^s_H-boundedness, which is an extension of ℓ^s-boundedness to this setting. We then give a class of examples for which we can prove the ℓ^s_H-boundedness of the family $\{I_k : k \in \mathscr{K}\}$. The main result is stated in Theorem 4.3.12 and it gives a sufficient condition for the ℓ^s_H-boundedness of such a family. For $H = \mathbb{C}^N$, this will be sufficient for our purpose. Theorem 1.2.2 is then shown as a special case for $H = \mathbb{C}$.

To prove Theorem 1.2.2 we apply weighted techniques of Rubio de Francia. Without additional effort we actually prove the more general Theorem 4.3.5, which

[3] $\mathscr{R}$-sectoriality stands for $\mathscr{R}$-boundedness of a family of resolvents on a sector, see Section 2.7

states that the family of integral operators on $L^p(v, L^q(w;H))$ is ℓ^s_H-bounded for all $p,q,s \in (1,\infty)$ and for arbitrary A_p-weights v and A_q-weights w. Both the modern extrapolation methods with A_q-weights (as explained in the book of Cruz-Uribe, Martell and Pérez [32]) and the factorization techniques of Rubio de Francia (see [65, Theorem VI.5.2] or [71, Theorem 9.5.8]), play a crucial role in our work. It is unclear how to apply the extrapolation techniques of [32] to the inner space L^q directly, but it does play a role in our proofs for the outer space L^p. The factorization methods of Rubio de Francia enable us to deal with the inner spaces (see the proof of Proposition 4.3.8).

In the literature there are many more $\mathcal{R}$-boundedness results for integral operators (e.g. [38, Section 6], [40, Proposition 3.3 and Theorem 4.12], [69], [75, Section 3], [87, Section 4], [104, Chapter 2]). However, it seems they are of a different nature and cannot be used to prove our results Theorem 4.3.5, Corollary 4.3.9 and Theorem 4.3.12.

1.3. $L^p(L^q)$-ESTIMATES FOR PARABOLIC PROBLEMS WITH VMO ASSUMPTIONS AND GENERAL BOUNDARY CONDITIONS

In the last chapter of this thesis, we investigate $L^p(L^q)$-estimates for parabolic equations with general boundary conditions. This will be done using a different approach, based on PDE techniques, which allows us to consider operators whose leading coefficients have vanishing mean oscillation both in the time and in the space variables. The interest in these problems comes from their application to quasilinear and nonlinear PDEs (see e.g. [42, 117]).

In particular, in Chapter 7 we establish $L^p(L^q)$-estimates with $p,q \in (1,\infty)$ for the higher-order parabolic equations

$$\begin{cases} u_t + (\lambda + A)u = f & \text{on} \quad \mathbb{R}\times\mathbb{R}^d_+ \\ \mathrm{tr}_{\mathbb{R}^{d-1}} B_j u = g_j & \text{on} \quad \mathbb{R}\times\mathbb{R}^{d-1},\ j=1,\dots,m, \end{cases} \tag{1.3.1}$$

where $\mathrm{tr}_{\mathbb{R}^{d-1}}$ denotes the trace operator, A is an elliptic differential operator of order $2m$, and (B_j) is a family of differential operators of order $m_j < 2m$ for $j = 1,\dots,m$. The coefficients of A are assumed to have vanishing mean oscillation (VMO) both in the time and space variable, while the leading coefficients of B_j are assumed to be constant in time and space. On the boundary, we assume the Lopatinskii–Shapiro condition to hold. This condition was first introduced by Lopatinskii [112] and Shapiro [150]. See also the work of Agmon–Dougalis–Nirenberg [6]. Roughly speaking, this is an algebraic condition involving the symbols of the principal part of the operators A and B_j with fixed coefficients, which is equivalent to the solvability of certain ODE systems.

Research on $L^p(L^q)$-regularity for these kind of equations has been developed in the last decades by mainly two different approaches.

On the one hand, for parabolic problems on the half space with Dirichlet boundary conditions, a PDE approach has been developed mainly by Krylov, Dong, and Kim (see [51, 102] and references therein). For the case of the whole space, Krylov in [101] showed $L^p(L^q)$-regularity for second-order operators with coefficients merely measurable in time and VMO in space, with the restriction $q \le p$. This approach is based on *mean oscillation estimates* of solutions to the equation. To give a general idea, if the system under consideration is elliptic of the form $Au = f$ with constant coefficients in the whole space, then by the mean oscillation estimate of $D^{2m}u$ we mean a pointwise estimate of the form

$$\begin{aligned}
&\fint_{B_r(x_0)} \Big| D^{2m}u - \fint_{B_r(x_0)} D^{2m}u\,dy \Big| dx \\
&\qquad \le C\kappa^{-1}\Big(\fint_{B_{\kappa r}(x_0)} |D^{2m}u|^2\,dx\Big)^{1/2} + C\kappa^{d/2}\Big(\fint_{B_{\kappa r}(x_0)} |f|^2\,dx\Big)^{1/2},
\end{aligned}$$

for all $x_0 \in \mathbb{R}^d$, $r \in (0,\infty)$ and $\kappa \in [\kappa_0,\infty)$ and where $B_r(x_0)$ is a ball with center x_0 and radius r.

The methodology of Krylov was then extended by Dong and Kim in [49] and [51] to higher-order systems with the same class of coefficients. In particular, in [51] a new technique was developed to produce mean oscillation estimates for higher-order equations and systems in the whole and half spaces with Dirichlet boundary conditions, for $p = q$. This technique had been extended recently by the same authors in [52] to mixed $L^p(L^q)$-spaces with Muckenhoupt weights and small bounded mean oscillations assumptions on the space variable, for any $p, q \in (1,\infty)$.

On the other hand, there is the operator-theoretic approach in which $L^p(L^q)$-regularity is shown as an application of maximal L^p-regularity. With coefficients in the VMO class, higher-order systems in the whole space have been investigated in several papers, for example in [77, 81] where the leading coefficients are VMO with respect to the space variable and independent of the time variable, by using Muckenhoupt weights and estimates of integral operators of Calderon-Zygmund type.

Concerning $L^p(L^q)$-regularity for equations on the half space with boundary conditions satisfying the Lopatinskii–Shapiro condition, a breakthrough result was obtained by Denk, Hieber, and Prüss in [40] in the case of autonomous initial-boundary value problems with homogeneous boundary conditions and operator-valued constant coefficients. They combined operator sum methods with tools from vector-valued harmonic analysis to show $L^p(L^q)$-regularity, for any $p, q \in (1,\infty)$, for parabolic problems with general boundary conditions of homogeneous type, in which the leading coefficients are assumed to be bounded and uniformly continuous. Later, in [41], the same authors characterized optimal $L^p(L^q)$-regularity for non-autonomous, operator-valued parabolic initial-boundary value problems with inhomogeneous boundary data, where the dependence on time is assumed to be continuous. It is worth noting that in the special case where $m = 1$, complex-valued coefficients and $q \le p$, a similar result was obtained by Weidemaier [151]. The results of [41] have been generalized by Meyries and Schnaubelt in [120] to the weighted time-dependent setting, where the weights considered are Muckenhoupt power-type weights. See also [117]. Very recently, Lindemulder in [108] generalized these results to vector-valued parabolic initial boundary value problems with Muckenhoupt power-type weights both in time and space.

In this thesis, we relax the assumptions on the coefficients of the operators involved. We obtain weighted $L^p(L^q)$-estimates for parameter-elliptic operators on the half space with VMO coefficients in the time and space variables, and with boundary operators having constant leading coefficients and satisfying the Lopatin-skii–Shapiro condition. An overview of our main result is given in the following theorem.

Theorem 1.3.1. *Let $p, q \in (1,\infty)$. Then there exists $\lambda_0 \ge 0$ such that for every $\lambda \ge \lambda_0$, there exists a constant $C > 0$ such that the following holds. For any $u \in W_p^1(\mathbb{R}; L_q(\mathbb{R}^d_+)) \cap L_p(\mathbb{R}; W_q^{2m}(\mathbb{R}^d_+))$ satisfying* (1.3.1)*, where*

$$f \in L_p(\mathbb{R}; L_q(\mathbb{R}^d_+)) \quad \text{and} \quad g_j \in F^{k_j}_{p,q}(\mathbb{R}; L_q(\mathbb{R}^{d-1})) \cap L_p(\mathbb{R}; B^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))$$

with $k_j = 1 - m_j/(2m) - 1/(2mq)$, we have

$$\|u_t\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))}$$
$$\le C\|f\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + C\|g_j\|_{F^{k_j}_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R};B^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))}.$$

This is stated in Theorem 7.2.4, where we also consider Muckenhoupt weights, and in the elliptic setting in Theorem 7.2.5. The proofs are based on the results in [40] combined with an extension of the techniques developed in [50–52, 100, 102]. In particular, in the main result of Section 7.3, Lemma 7.3.5, we prove mean oscillation estimates for equations on the half space with the Lopatinskii–Shapiro condition. A key ingredient of the proof is a Poincaré type inequality for solutions to equations satisfying the Lopatinskii–Shapiro condition.

1.4. FURTHER COMMENTS

Most results of this thesis will be presented in the setting of weighted L^p-spaces, with Muckenhoupt weights. For instance in Theorem 3.4.5 we will present a weighted $L^p(L^q)$-maximal regularity result in the case A is a $2m$-th order elliptic operator, assuming only measurability in the time variable and continuity in the space variable. Weighted results can be important for several reasons. Maximal L^p-regularity with a Muckenhoupt power weight t^α in time (e.g. see [97, 118]) allows one to consider rather rough initial values. It can also be used to prove compactness properties which in turn can be used to obtain global existence of solutions. Another advantage of using weights comes from a harmonic analytic point of view. The theory of Rubio de Francia (see [32] and references therein) enables one to extrapolate from weighted L^p-estimates for a single $p \in (1,\infty)$, to any $p \in (1,\infty)$. In Section 3.4, A_p-weights in space will be used to check $\mathscr{R}$-boundedness of certain integral operators. We refer to Theorem 3.1.4 and Step 1 of the proof of Theorem 3.4.5 for details. Weights in time will be used for extrapolation arguments more directly, as for instance in step 4 of the proof of Theorem 3.4.5.

Moreover, in the main results we will study maximal L^p-regularity on $\mathbb{R}$, instead on a bounded time interval. The reason is that this case is more general, and allows us to avoid the technicalities caused by the non-zero initial value, as we will see in Section 3.3 (and in particular Proposition 3.3.18). Therefore, in this thesis we will focus on $t \in \mathbb{R}$ and we will derive the initial-valued results via related trace theorems.

1.5. OUTLINE OF THE THESIS

In Chapter 2, we introduce the background results and notation that will be used throughout this thesis. In particular, we will introduce the notion of a solution to a non-homogeneous evolution equation and we will introduce elliptic differential operators. Preliminaries on functional calculus, weights, Fourier multipliers and $\mathscr{R}$-boundedness will also be given.

Chapter 3 is the core of this thesis. There, we show our new abstract approach to maximal L^p-regularity, and we consider applications to $2m$-th order differential operators and quasilinear equations. The chapter is organized as follows. In Section 3.1 we discuss the $\mathscr{R}$-boundedness of a particular class of integral operators, which will be used in Section 3.2 to prove the L^p-boundedness of a new class of singular integrals. The main result on maximal L^p-regularity is presented in Section 3.3. In Section 3.4 we show how to use our new approach to derive maximal L^p-regularity for (1.2.1). Finally in Section 3.5 we extend the results of [28] and [127] on quasilinear equations to the time-dependent setting.

In Chapter 4 we prove an ℓ^s_H-boundedness result for integral operators with operator-valued kernel. In Section 4.2 we discuss weighted extrapolation in $L^p(L^q)$-spaces, while the main result is stated and proved in Section 4.3. Besides its intrinsic interest, the main result of this chapter has relevant applications in both Chapter 3 and 5.

In Chapter 5 we further apply our abstract approach to the case of systems of differential operators. The main results are stated in Section 5.1. In Section 5.2, we prove that in the case of x-independent coefficients, the operator $A(t)$ generates an evolution family on weighted L^q-spaces, which is the main novelty of this chapter. In Section 5.3 we present the proofs of the main theorems and we show how to deduce maximal regularity results for the initial value problem as well.

As a natural last application of our abstract approach, we study in Chapter 6 maximal L^p-regularity for systems of differential operators in the upper half space with homogeneous general boundary conditions. The generation of the evolution family will play an important role here as well, and it will be shown in full details in Section 6.4, in which we consider the example of a second-order differential operator.

Chapter 7 is the last chapter of this thesis. There, we show mixed $L^p(L^q)$-estimates in the case the coefficients of the operator A are VMO both in the time and the space variable, and we consider general boundary conditions. The chapter is organized as follows. In Section 7.1 we give some preliminary results and introduce some notation. In Section 7.2 we list the main assumptions on the operators and we state the main result, Theorem 7.2.4. In Section 7.3 we prove the mean oscillation estimates needed for the proof of Theorem 7.2.4, which is given in Section 7.4. Finally, in Section 7.5 we prove a solvability result by using the *a priori* estimates in the previous sections.

2

PRELIMINARIES

Before presenting the background results that will be used throughout this thesis, we introduce some basic notation. We denote the set of natural numbers as $\mathbb{N} = \{1,2,3,\dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote the half-line $\mathbb{R}_+ = [0,\infty)$ and the upper half-space $\mathbb{R}^d_+ = \{x = (x_1, x') \in \mathbb{R}^d : \, x_1 > 0, \, x' \in \mathbb{R}^{d-1}\}$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$ we denote $|\alpha| = \alpha_1 + ... + \alpha_d$ and we consider the standard multi-index notation $D^\alpha = D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d}$, where $D_j := -i \frac{\partial}{\partial_j}$ denotes $-i$ times the partial derivative in the $j-$th coordinate direction.

The letters X and Y are used to denote Banach spaces, and we write X^* for the dual of X. We denote as $\mathscr{L}(X,Y)$ the space of all bounded linear operators, with norm $\|\cdot\|_{\mathscr{L}(X,Y)}$.

A function $f : \mathbb{R}^d \to X$ is called *strongly measurable* if it is the a.e. limit of a sequence of simple functions, and it is called *strongly continuous* if it is continuous in the strong operator topology.

In the next sections, we introduce and motivate definitions that will be relevant for this thesis. In Section 2.1 we introduce the function spaces that will be used. In Section 2.2, we present Muckenhoupt weights and classical extrapolation results. In Section 2.3 we introduce some basic results on functional calculus, with particular attention to analytic semigroups and their generators, and H^∞-calculus. In Section 2.4 we define the mild solutions of a non-homogeneous evolution equation via a functional analytic point of view. In Section 2.5 we will introduce elliptic differential operators. In Sections 2.6 and 2.7, we define Fourier multipliers and $\mathscr{R}$-boundedness. Finally, in Section 2.8 we define maximal L^p-regularity for autonomous problems and we introduce the characterization of maximal-L^p regularity due to Weis in [152].

2.1. FUNCTION SPACES

Let X be a Banach space. For $p \in [1,\infty]$, $L^p(\mathbb{R}^d;X)$ is the space of all strongly measurable functions $f : \mathbb{R}^d \to X$ such that

$$\|f\|_{L^p(\mathbb{R}^d;X)} = \left(\int_{\mathbb{R}^d} \|f\|_X^p \, dx \right)^{\frac{1}{p}} < \infty \text{ if } p \in [1,\infty),$$

and $\|f\|_{L^\infty(\mathbb{R}^d;X)} = \text{ess.}\sup_{x\in\mathbb{R}^d}\|f(x)\|$. For $p\in[1,\infty]$ and $k\in\mathbb{N}_0$ we define the Sobolev space

$$W^{k,p}(\mathbb{R}^d;X) = \{u\in L^p(\mathbb{R}^d;X):\ D^\alpha u\in L^p(\mathbb{R}^d;X),\ \forall|\alpha|\le k\}.$$

For $\mathbb{K}=\mathbb{R},\mathbb{C}$ we write $L^p(\mathbb{R}^d;\mathbb{K}) = L^p(\mathbb{R}^d)$. For $p\in[1,\infty]$ we let $p'\in[1,\infty]$ be the Hölder conjugate of p, defined by $\frac{1}{p}+\frac{1}{p'}=1$.

We denote as $\mathscr{S}(\mathbb{R}^d;X)$ the Schwartz class of rapidly decreasing smooth functions from $\mathbb{R}^d$ into X. The Fourier transform $\mathscr{F}:\mathscr{S}(\mathbb{R}^d;X)\to\mathscr{S}(\mathbb{R}^d;X)$ is defined by

$$(\mathscr{F}f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x)\,dx,$$

and it is a bijection whose inverse is given by

$$(\mathscr{F}^{-1}f)(x) := \check{f}(x) := \int_{\mathbb{R}^d} e^{2\pi i x\xi}\widehat{f}(\xi)\,d\xi,$$

where $f\in\mathscr{S}(\mathbb{R}^d;X)$ and $x,\xi\in\mathbb{R}^d$.

Motived by the study of the regularity of initial value problems, as for instance in Section 3.4, we introduce in what follows the so called *Besov spaces*. A complete characterization can be found in [147] and [148].

Let $\Phi(\mathbb{R}^d)$ be the set of all sequences $(\varphi_k)_{k\ge0}\subset\mathscr{S}(\mathbb{R}^d)$ such that

$$\widehat{\varphi}_0=\widehat{\varphi},\quad \widehat{\varphi}_1(\xi)=\widehat{\varphi}(\xi/2)-\widehat{\varphi}(\xi),\quad \widehat{\varphi}_k(\xi)=\widehat{\varphi}_1(2^{-k+1}\xi),$$

for $k\ge2$, $\xi\in\mathbb{R}^d$, and where the Fourier transform $\widehat{\varphi}$ of the generating function $\varphi\in\mathscr{S}(\mathbb{R}^d)$ satisfies $0\le\widehat{\varphi}(\xi)\le1$ for $\xi\in\mathbb{R}^d$ and

$$\widehat{\varphi}(\xi)=1 \quad \text{if } |\xi|\le1,\quad \widehat{\varphi}(\xi)=0\quad \text{if } |\xi|\ge\frac{3}{2}.$$

Definition 2.1.1. Given $(\varphi_k)_{k\ge0}\in\Phi(\mathbb{R}^d)$ we define the *Besov space*

$$\mathscr{B}^r_{p,q}(\mathbb{R}^d) = \{f\in\mathscr{S}'(\mathbb{R}^d):\ \|f\|_{\mathscr{B}^r_{p,q}(\mathbb{R}^d)} := \|(2^{kr}\mathscr{F}^{-1}(\widehat{\varphi}_k\hat{f}))_{k\ge0}\|_{\ell_q(L_p(\mathbb{R}^d))}<\infty\},$$

The following representation of Besov spaces will be needed. We refer the reader to [147, Theorem 1.6.4] for the proof.

Theorem 2.1.2. *Let $p\in(1,\infty)$, $q\in[1,\infty)$, $k\in\mathbb{N}$ and $\theta\in(0,1)$. Then the Besov space $\mathscr{B}^s_{p,q}(\mathbb{R}^d)$ has the following representation via real interpolation*

$$\mathscr{B}^s_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), W^k_p(\mathbb{R}^d))_{\theta,q},$$

where $s=k\theta$.

We will not deal with interpolation spaces here. The interested reader can find an exhaustive description in [114, 148].

2.2. A_p-WEIGHTS AND EXTRAPOLATION

In this section, we introduce Muckenhoupt A_p-weights and we state some of their properties. Details can be found in [71, Chapter 9] and [141, Chapter V].

A *weight* is a locally integrable function on $\mathbb{R}^d$ with $w(x) \in (0,\infty)$ for a.e. $x \in \mathbb{R}^d$. For a Banach space X and $p \in [1,\infty]$, $L^p(\mathbb{R}^d, w; X)$ is the space of all strongly measurable functions $f : \mathbb{R}^d \to X$ such that

$$\|f\|_{L^p(\mathbb{R}^d,w;X)} = \Big(\int_{\mathbb{R}^d} \|f\|^p \, w \, dx\Big)^{\frac{1}{p}} < \infty \quad \text{if } p \in [1,\infty),$$

and $\|f\|_{L^\infty(\mathbb{R}^d,w;X)} = \text{ess.}\sup_{x\in\mathbb{R}^d} \|f(x)\|$.

For $p \in (1,\infty)$ a weight w is said to be an A_p*-weight* if

$$[w]_{A_p} = \sup_Q \fint_Q w(x)\,dx \Big(\fint_Q w(x)^{-\frac{1}{p-1}}\,dx\Big)^{p-1} < \infty.$$

Here the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$ with axes parallel to the coordinate axes and $\fint_Q = \frac{1}{|Q|}\int_Q$. The extended real number $[w]_{A_p}$ is called the A_p*-constant*. The Hardy-Littlewood maximal operator is defined as

$$M(f)(x) = \sup_{Q\ni x} \fint_Q |f(y)|\,dy, \quad f \in L^p(\mathbb{R}^d, w),$$

while the sharp maximal function is defined as

$$f^\sharp(x) = \sup_{Q\ni x} \fint_Q |f(y) - (f)_Q|\, dyds,$$

with $Q \subseteq \mathbb{R}^d$ cubes as before. Recall that $w \in A_p$ if and only if the Hardy-Littlewood maximal operator M is bounded on $L^p(\mathbb{R}^d, w)$. In the case of the half-space $\mathbb{R}^d_+$, we obtain an equivalent definition by replacing the the cubes Q with $Q \cap \mathbb{R}^d_+ =: Q^+$ with center in $\overline{\mathbb{R}^d_+}$.

Next we will summarize a few basic properties of weights which we will need. The proofs can be found in [71, Theorems 9.1.9 and 9.2.5], [71, Theorem 9.2.5 and Exercise 9.2.4], [71, Proposition 9.1.5].

Proposition 2.2.1. *Let $w \in A_p$ for some $p \in [1,\infty)$. Then we have*

1. *If $p \in (1,\infty)$ then $w^{-\frac{1}{p-1}} \in A_{p'}$ with $[w^{-\frac{1}{p-1}}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}$.*
2. *For every $p \in (1,\infty)$ and $\kappa > 1$ there is a constant $\sigma = \sigma_{p,\kappa,d} \in (1,p)$ and a constant $C_{p,d,\kappa} > 1$ such that $[w]_{A_{\frac{p}{\sigma}}} \le C_{p,\kappa,d}$ whenever $[w]_{A_p} \le \kappa$. Moreover, $\kappa \mapsto \sigma_{p,\kappa,d}$ and $\kappa \mapsto C_{p,\kappa,d}$ can be chosen to be decreasing and increasing, respectively.*
3. *$A_p \subseteq A_q$ and $[w]_{A_q} \le [w]_{A_p}$ if $q > p$.*

4. *For $p \in (1,\infty)$, there exists a constant $C_{p,d}$ such that*

$$\|M\|_{L^p(\mathbb{R}^d,w)\to L^p(\mathbb{R}^d,w)} \le C_{p,d}\cdot [w]_{A_p}^{\frac{1}{p-1}}.$$

Example 2.2.2 (power weights)**.** Let $\alpha \in (-d,\infty)$ and let $w(x) := |x|^\alpha$. For $p \in (1,\infty)$, it holds that $w \in A_p(\mathbb{R}^d)$ if and only if $a \in (-d, d(p-1))$ (See [141] for details). Power weights will play an important rôle in the study of maximal L^p-regularity of problems with non-zero initial values (see Section 3.3.4)

Let w be a weight. The duality relation $L^p(\mathbb{R}^d, w)^* = L^{p'}(\mathbb{R}^d, w')$ holds with

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)g(x)\,dx, \quad f \in L^p(\mathbb{R}^d, w), \;\; g \in L^{p'}(\mathbb{R}^d, w').$$

On the other hand, $L^p(\mathbb{R}^d, w)^* = L^{p'}(\mathbb{R}^d, w)$ if we let

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)g(x)w(x)\,dx, \quad f \in L^p(\mathbb{R}^d, w), \;\; g \in L^{p'}(\mathbb{R}^d).$$

The celebrated result of Rubio de Francia (see [132–134], [65, Chapter IV]) allows one to extrapolate from weighted L^p-estimates for a single p to weighted L^q-estimates for all q. As a remarkable consequence, one gets that $L^p(\ell^s)$-estimates are strongly connected to estimates in weighted L^p-spaces. This will play an important rôle in Chapter 4. The proofs and statements have been considerably simplified and clarified in [32] and can be formulated as follows (see [32, Theorem 3.9] and [32, Corollary 3.12]).

Theorem 2.2.3. *Let $f, g : \mathbb{R}^d \to \mathbb{R}_+$ be a pair of nonnegative, measurable functions and suppose that for some $p_0 \in (1,\infty)$ there exists an increasing function α on $\mathbb{R}_+$ such that for all $w_0 \in A_{p_0}$*

$$\|f\|_{L^{p_0}(\mathbb{R}^d,w_0)} \le \alpha([w_0]_{A_{p_0}})\|g\|_{L^{p_0}(\mathbb{R}^d,w_0)}.$$

Then for all $p \in (1,\infty)$ there is a constant $c_{p,d}$ s.t. for all $w \in A_p$,

$$\|f\|_{L^p(\mathbb{R}^d,w)} \le 4\alpha\Big(c_{p,d}[w]_{A_p}^{\frac{p_0-1}{p-1}+1}\Big)\|g\|_{L^p(\mathbb{R}^d,w)}.$$

Corollary 2.2.4. *Let (f_i, g_i) be a family of pairs of non-negative, measurable functions $f_i, g_i : \mathbb{R}^d \to \mathbb{R}_+$ and suppose that for some $p_0 \in (1,\infty)$ and every $w_0 \in A_{p_0}$ there exists a constant $C = C([w_0]_{A_{p_0}})$ such that*

$$\|f_i\|_{L^{p_0}(\mathbb{R}^d,w_0)} \le C\|g_i\|_{L^{p_0}(\mathbb{R}^d,w_0)}.$$

Then, for all p and s, $1 < p, s < \infty$, $w \in A_p$ there is a constant $c = c(p,d,[w]_{A_p})$ s.t.

$$\|(\sum_i |f_i|^s)^{1/s}\|_{L^p(\mathbb{R}^d,w)} \le c\|(\sum_i |g_i|^s)^{1/s}\|_{L^p(\mathbb{R}^d,w)}.$$

The following simple extension of Theorem 2.2.3 will be needed.

Theorem 2.2.5. *For every $\lambda \geq 0$, let $f_\lambda, g_\lambda : \mathbb{R}^d \to \mathbb{R}_+$ be a pair of nonnegative, measurable functions and suppose that for some $p_0 \in (1,\infty)$ there exist increasing functions α_{p_0}, β_{p_0} on $\mathbb{R}_+$ such that for all $w_0 \in A_{p_0}$ and all $\lambda \geq \beta_{p_0}([w_0]_{A_{p_0}})$,*

$$\|f_\lambda\|_{L^{p_0}(\mathbb{R}^d,w_0)} \leq \alpha_{p_0}([w_0]_{A_{p_0}})\|g_\lambda\|_{L^{p_0}(\mathbb{R}^d,w_0)}. \tag{2.2.1}$$

Then for all $p \in (1,\infty)$ there is a constant $c_{p,d} \geq 1$ such that for all $w \in A_p$, and all $\lambda \geq \beta_{p_0}(\phi([w]_{A_p}))$

$$\|f_\lambda\|_{L^p(\mathbb{R}^d,w)} \leq 4\alpha_{p_0}(\phi([w]_{A_p}))\|g_\lambda\|_{L^p(\mathbb{R}^d,w)},$$

where $\phi(x) = c_{p,d}x^{\frac{p_0-1}{p-1}+1}$.

Note that Theorem 2.2.3 corresponds to the case that f_λ and g_λ are constant in λ. To obtain the above extension one can check that in the proof [32, Theorem 3.9] for given p and $w \in A_p$, the A_{p_0}-weight w_0 which is constructed satisfies $[w_0]_{A_{p_0}} \leq \phi([w]_{A_p})$. This clarifies the restriction on the λ's.

Estimates of the form (2.2.1) with increasing function α_{p_0} will appear frequently in this thesis. In this situation we say there is an A_{p_0}-*consistent* constant C such that

$$\|f\|_{L^{p_0}(\mathbb{R}^d,w_0)} \leq C\|g\|_{L^{p_0}(\mathbb{R}^d,w_0)}.$$

Note that the L^p-estimate obtained in Theorem 2.2.3 is again A_p-consistent for all $p \in (1,\infty)$.

The following observation will be often applied. For a bounded Borel set $A \subset \mathbb{R}^d$ and for every $f \in L^p(\mathbb{R}^d, w; X)$ one has $\mathbf{1}_A f \in L^1(\mathbb{R}^d; X)$ and by Hölder's inequality

$$\|\mathbf{1}_A f\|_{L^1(\mathbb{R}^d;X)} \leq C_{w,A}\|f\|_{L^p(\mathbb{R}^d,w;X)}.$$

A linear subspace $Y \subseteq X^*$ is said to be *norming* for X if for all $x \in X$, $\|x\| = \sup\{|\langle x, x^*\rangle| : x^* \in Y, \|x^*\| \leq 1\}$. The following simple duality lemma will be needed.

Lemma 2.2.6. *Let $p, p' \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let v be a weight and let $v' = v^{-\frac{1}{p-1}}$. Let $Y \subseteq X^*$ be a subspace which is norming for X. Then setting*

$$\langle f, g\rangle = \int_{\mathbb{R}} \langle f(t), g(t)\rangle\, dt, \quad f \in L^p(\mathbb{R}, v; X), \quad g \in L^{p'}(\mathbb{R}, v'; X^*),$$

the space $L^{p'}(\mathbb{R}, v'; X^*)$ can be isometrically identified with a closed subspace of $L^p(\mathbb{R}, v; X)^*$. Moreover, $L^{p'}(\mathbb{R}, v'; Y)$ is norming for $L^p(\mathbb{R}, v; X)$.

2.3. FUNCTIONAL CALCULUS

In this section, we focus our attention to sectorial operators and generators of analytic semigroups. These notions will be used in the next section to introduce the semigroup approach to evolution equations. Furthermore, we recall the H^∞-calculus that was developed by McIntosh and collaborators (see e.g. [7, 17, 31, 115]). We refer to [57, 76, 104] for an extensive treatment of these subjects.

2.3.1. *Sectorial operators and analytic semigroups*

Let X be a Banach space. We denote as $D(A) \subseteq X$ the domain of an operator A on X. If A is closed and unbounded, then $D(A)$ is a Banach space when endowed with the norm $\|x\|_{D(A)} := \|x\| + \|Ax\|,\ x \in D(A)$. Let $\sigma(A)$ be the spectrum of A and $\rho(A) := \mathbb{C} \backslash \sigma(A)$ be the resolvent set. The identity operator on X is denoted by I.

For $\sigma \in [0, \pi]$ we set

$$\Sigma_\sigma = \begin{cases} \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \sigma\} & \text{if } \sigma \in (0, \pi] \\ (0, \infty) & \text{if } \sigma = 0 \end{cases}$$

where $\arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$.

Definition 2.3.1. A closed densely defined linear operator $(A, D(A))$ on X is said to be *sectorial of type* $\sigma \in (0, \pi)$ if

(i) it is injective and has dense range,

(ii) its spectrum is contained in $\overline{\Sigma_\sigma}$,

(iii) for all $\sigma' \in (\sigma, \pi)$ the set

$$\left\{z(z + A)^{-1} : \ z \in \mathbb{C} \setminus \{0\},\ |\arg(z)| > \sigma'\right\}$$

is uniformly bounded by some constant C_A.

The infimum of all $\sigma \in (0, \pi)$ such that A is sectorial of type σ is called the *sectoriality angle* of A.

Definition 2.3.2. A mapping $T : [0, \infty) \to \mathscr{L}(X)$ is said to be a *semigroup* if $T(0) = I$ and it possesses the *semigroup property* $T(t)T(s) = T(t + s),\ t, s > 0$. The semigroup T is called *bounded* if $\sup_{0<t<\infty} \|T(t)\| > \infty$.

A semigroup T is called *strongly continuous* (or C_0-semigroup) if $\lim_{t \downarrow 0} T(t)x = x$, for every $x \in X$. If there exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, then the semigroup is called *exponentially bounded*. Moreover, it is said to be *exponentially stable* if $\omega_0(T) := \inf\{\omega \in \mathbb{R} : \exists\ M \text{ such that } \|T(t)\| \leq Me^{\omega t},\ t \geq 0\} < 0$.

Definition 2.3.3. Let $\theta \in (0, \pi/2]$. A mapping $T : \Sigma_\theta \to \mathscr{L}(X)$ is said to be a *bounded analytic semigroup* if it has the following properties:

(i) $T(0) = I$ and the semigroup law $T(r)T(s) = T(r + s)$ holds for all $r, s \in \Sigma_\theta$,

(ii) the mapping $T : \Sigma_\theta \to \mathscr{L}(X)$ is analytic, and

(iii) the mapping T satisfies $\sup_{z \in \Sigma_\varphi} \|T(z)\| < \infty$ for each $\varphi \in (0, \theta)$.

The following proposition states that if A is sectorial of angle $\sigma < \pi/2$, then the mapping T given by

$$T(z) = e^{-zA}, \quad \text{for } |\arg(z)| < \frac{\pi}{2} - \sigma,$$

is a bounded analytic semigroup of angle $\theta = \frac{\pi}{2} - \sigma$ and

$$\|T(z)\| \le C_A C_\sigma. \tag{2.3.1}$$

In this case, the family $(T(z)) = (e^{-zA})$ with $|\arg(z)| < \pi/2 - \sigma$ is called the analytic semigroup *generated* by A, and the operator A is said to be the *generator* of an analytic semigroup. We refer to [76, Proposition 3.4.1] for the proof (see also [57, 113]).

Proposition 2.3.4. *Let A be a sectorial operator of angle $\sigma < \pi/2$. Then the following assertions hold.*

(i) $e^{-rA}e^{-sA} = e^{-(r+s)A}$ *for all* $r, s \in \Sigma_{\pi/2-\sigma}$.

(ii) The mapping $\Sigma_{\pi/2-\sigma} \ni z \mapsto e^{-zA} \in \mathscr{L}(X)$ *is analytic.*

(iii) If $x \in \overline{D(A)}$ *then* $\lim\limits_{z\to 0,\ |\arg z| \le \varphi} e^{-zA}x = x$ *for each* $\varphi \in (0, \pi/2 - \sigma)$.

(iv) Let $\varphi \in (0, \pi/2 - \sigma)$ *Then, for each choice of* $\varphi' \in (\sigma, \pi/2 - \varphi)$ *there exists a constant* $C_{\varphi'}$ *such that* $\|e^{-zA}\| \le C_{\varphi'} C_A$ *for all* $|\arg(z)| \le \varphi$.

For further details on semigroups and their generators, we refer to [76, Appendix A.8]. We list some examples of generators of analytic semigroups that will play a prominent role in what follows. They are taken from [104, Example 1.2].

Example 2.3.5. Let $X = L^p(\mathbb{R}^d)$, $p \in (1, \infty)$.

(1) Consider the *Laplace operator* $\Delta = \sum_{j=1}^d \frac{\partial}{\partial x_j}$, with $D(\Delta) = W^{2,p}(\mathbb{R}^d)$, $p \in (1, \infty)$. Then $-\Delta$ has spectral angle 0 and it generates the Gaussian semigroup

$$(G(t)f)(x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y)\, dy.$$

A proof can be found in [76, Proposition 8.3.1].

(2) Elliptic differential operators (see the precise definition in Section 2.5). Details can be found in [104, Example 1.2.b and Section 6].

2.3.2. H^∞-*calculus*

We now consider the H^∞-calculus for sectorial operators.

Let $\theta \in (0, \pi)$ and let $H^\infty(\Sigma_\theta)$ be the the set of all bounded complex-valued holomorphic functions defined on Σ_θ. This is a Banach space endowed with the norm

$$\|f\|_{H^\infty(\Sigma_\theta)} = \sup\{f(z)| : z \in \Sigma_\theta\}.$$

Let now $H_0^\infty(\Sigma_\theta)$ denote the linear subspace of all $f \in H^\infty(\Sigma_\theta)$ for which there exists $\varepsilon > 0$ and $C \geq 0$ such that

$$|f(z)| \leq \frac{C|z|^\varepsilon}{(1+|z|)^{2\varepsilon}}, \quad z \in \Sigma_\theta.$$

If A is sectorial of type $\sigma_0 \in (0,\pi)$, then for all $\sigma \in (\sigma_0, \pi)$ and $f \in H_0^\infty(\Sigma_\sigma)$ we define the bounded operator $f(A)$ by

$$f(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} f(z)(z+A)^{-1}\,dz.$$

A sectorial operator A of type $\sigma_0 \in (0,\pi)$ is said to have a *bounded* $H^\infty(\Sigma_\sigma)$-*calculus* for $\sigma \in (\sigma_0,\pi)$ if there exists a $C \geq 0$ such that

$$\|f(A)\| \leq C\|f\|_{H^\infty(\Sigma_\sigma)}, \quad f \in H_0^\infty(\Sigma_\sigma).$$

If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus, then the mapping $f \mapsto f(A)$ extends to a bounded algebra homomorphism from $H^\infty(\Sigma_\sigma)$ to $\mathscr{L}(X)$ of norm $\leq C$.

Many differential operators on L^q-spaces with $q \in (1,\infty)$ are known to have a bounded H^∞-calculus (see [40, 104] and the survey [154]). For instance, it includes all sectorial operators A of angle $< \pi/2$ for which e^{-tA} is a positive contraction (see [94]).

Example 2.3.6. The operator $A = -\Delta$ on $L^p(\mathbb{R}^d, w)$ has a bounded H^∞-calculus of arbitrary small angle $\sigma \in (0,\pi)$ for every $w \in A_p$ and $p \in (1,\infty)$. This easily follows from the weighted version of Mihlin's multiplier theorem (see [104, Example 10.2] and [65, Theorem IV.3.9]). Details about Fourier multipliers will be given in Section 2.6.

2.4. EVOLUTION EQUATIONS AND MILD SOLUTION

Throughout this thesis we use the semigroup approach to evolution equations. In this approach, the solution of a non-homogeneous parabolic Cauchy problem is defined in terms of the sectoriality properties of the operator A.

In this section, we recall this abstract approach both in the autonomous and the non-autonomous case, and we refer the reader to [113, 124] and [143] for more details.

2.4.1. Autonomous problems

Let X be a Banach space, $A : D(A) \subset X \to X$ with $D(A)$ dense in X and let $T > 0$. Consider the problem

$$\begin{cases} u'(t) + Au(t) = f(t) & 0 < t < T \\ u(0) = x \end{cases} \tag{2.4.1}$$

where $f : (0,T) \to X$ and $x \in X$. The following definition is taken from [124].

Definition 2.4.1. A function u is said to be a strong solution of of the problem (2.4.1) if $u \in W^{1,1}(0,T;X) \cap L^1(0,T;D(A)) \cap C([0,T],X_0)$ if $u(0) = x$ and $u'(t) + Au(t) = f(t)$ for almost all $t \in (0,T)$.

The spectral properties of A allows one to define the solution of the non-homogeneous problem (2.4.1) in terms of the analytic semigroup generated by A. More precisely, if A is sectorial of angle $< \pi/2$, then the solution of (2.4.1) may be represented by the *variation of constants formula*

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-s)A} f(s)ds, \quad 0 \le t \le T, \tag{2.4.2}$$

where e^{-tA} is the analytic semigroup generated by A.

Definition 2.4.2. The function $u \in C([0,T];X)$ defined in (2.4.2) is said to be the *mild solution* of (2.4.1) on $[0,T]$.

A proof of the following result can be find in [124, Corollary 4.3.3] and [113, Proposition 4.1.2].

Proposition 2.4.3. *Let $f \in L^1(0,T;X)$ and let $x \in X$. Let A be the generator of an analytic semigroup e^{-tA}. If* (2.4.1) *has a strong solution, then it is given by* (2.4.2).

Further details on mild solutions can be found in [124, Chapter 4], [143, Chapter 3] and [113, Chapter 4].

2.4.2. Non-autonomous problems

Let X be a Banach space, $A(t) : D(A(t)) \subset X \to X$ and let $T > 0$. Consider the problem

$$\begin{cases} u'(t) + A(t)u(t) = f(t) & 0 < t \le T \\ u(0) = x \end{cases} \tag{2.4.3}$$

where $f(t)$ is an X-valued function and $x \in X$. We consider only the simple case in which the operators $A(t)$ have common domains $D(A(t)) = D(A)$ and $t \mapsto A(t)$ is continuous in the uniform operator topology. We assume $D(A)$ to be dense in X.

The strong solution of (2.4.3) is defined as in Definition 2.4.1.

The role of the analytic semigroup e^{-tA} is played now by the evolution family $S(t,s)$.

Definition 2.4.4. Let $(A(t))_t$ be as in (2.4.3). A two parameter family of bounded linear operators

$$\{S(t,s) : 0 \le s \le t \le T\} \subset \mathscr{L}(X)$$

is called an *evolution family* for A if

(i) $S(t,s)S(s,r) = S(t,r),\ S(s,s) = I,\ 0 \le r \le s \le t \le T.$

(ii) $(t,s) \to S(t,s)$ is strongly continous for $0 \le s < t \le T$.

(iii) $t \mapsto S(t,s)$ is differentiable in $(s,T]$ with values in $\mathscr{L}(X)$, and

$$\frac{\partial}{\partial t} S(t,s) = -A(t)S(t,s), 0 \le s < t \le T.$$

(iv) $s \mapsto S(t,s)$ is differentiable in $[0,t)$ with values in $\mathscr{L}(X)$, and

$$\frac{\partial}{\partial s} S(t,s) = S(t,s)A(s), 0 \le s < t \le T.$$

The construction of the evolution family associated with the initial value problem is quite technical and will not be done here. For an exhaustive treatment of this subject, we refer to and [124, Chapter 5], [143, Chapters 4 and 5] and [113, Chapter 6].

Example 2.4.5. If $A(t) = A$ is independent on t and sectorial of angle strictly less then $\pi/2$, then $S(t,s) = e^{-(t-s)A}$ and the two parameter family of operators reduces to the one parameter family $(e^{-tA})_{0\le t\le T}$ which is the semigroup generated by A.

However, if there exists an evolution family $S(t,s)$ such that properties (i)-(iv) are satisfied, then one can show that under reasonable assumptions on $f : [s,T] \subset [0,T] \to X$, the solution of the problem

$$\begin{cases} u'(t) + A(t)u(t) = f(t) & 0 \le s < t \le T, \\ u(s) = x, \end{cases} \tag{2.4.4}$$

can be represented by the variation of constants formula

$$u(t) = S(t,s)x + \int_s^t S(t,r)f(r)dr, \ s \le t \le T. \tag{2.4.5}$$

In fact, the analogue of Proposition 2.4.3 holds in the non-autonomous case (see [113, Corollary 6.2.4] and also [124, Theorem 5.7.1]).

Proposition 2.4.6. *Let $f \in L^1(s,T;X)$ and let $x \in X$. Let $S(t,s)$ be the evolution family generated by $A(t)$. If the problem* (2.4.4) *has a strong solution, then it is given by the formula* (2.4.5).

Evolution equations and evolution families are extensively studied in the literature (see [3, 57, 113, 114, 124, 138, 143, 144, 156]). In Section 3.3.1 we will discuss in details evolution families in the case in which $t \mapsto A(t)$ is only measurable, and we will give an example. In Section 5.2 we will construct the evolution family in the case of systems of equations.

2.5. ELLIPTIC DIFFERENTIAL OPERATORS

An example that will play a prominent rôle in this thesis is that of elliptic partial differential operators. Below we consider only the simpler case in which the operator A has time-independent, scalar-valued coefficients, as we want to give a general idea. The more general cases of non-autonomous equations and systems will be considered in Sections 3.4 and 5.1.1.

Let $m \geq 1$ be an integer and α be a multiindex such that $|\alpha| = \alpha_1 + ... + \alpha_d$ for $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$. For $a_\alpha \in \mathbb{C}$, we define the differential operator A of order $2m$ as

$$A = \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha,$$

with the multi-index notation $D^\alpha = D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d}$, where $D_j = -i\frac{\partial}{\partial_j}$. Let $D(A) = W^{2m,p}(\mathbb{R}^d)$ and $X = L^p(\mathbb{R}^d)$, $p \in (1, \infty)$.

In the following we denote the *principal part* of a differential operator A as

$$A_\sharp := \sum_{|\alpha|=2m} a_\alpha D^\alpha.$$

Let $A_\sharp(\xi)$ be the *principal symbol* of A, which is defined by

$$A_\sharp(\xi) := \sum_{|\alpha|=2m} a_\alpha \xi^\alpha.$$

We have, formally,

$$\sum_{|\alpha|=2m} a_\alpha D^\alpha u(x) = \mathscr{F}_\xi^{-1}(\xi \mapsto A_\sharp(\xi)\mathscr{F}u(\xi))(x),$$

which underline the importance of the principal symbol in connection to Fourier multiplier theory (see Section 2.6).

Example 2.5.1. The principal symbol of $A = -\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is given by $A_\sharp(\xi) = -|\xi|^2$.

Definition 2.5.2. We say that A is *uniformly elliptic* of angle $\theta \in (0, \pi)$ if there exists a constant $\kappa \in (0, 1)$ such that

$$A_\sharp(\xi) \subset \Sigma_\theta \text{ and } |A_\sharp(\xi)| \geq \kappa, \quad \xi \in \mathbb{R}^d, \ |\xi| = 1.$$

If additionally there is a constant K such that $|a_\alpha| \leq K$ for all $|\alpha| \leq 2m$, then we write $A \in \mathrm{Ell}(\theta, \kappa, \mathrm{K})$.

In the case $m = 1$ and real-valued coefficients, the above ellipticity condition is equivalent to the following (see [102]).

Definition 2.5.3. Let a_{ij}, b_i, $c \in \mathbb{R}$, for $i, j = 1, \ldots, d$. Assume $a_{ij} = a_{ji}$. Then the second-order differential operator

$$A = \sum_{i,j=1}^{d} a_{ij} D_i D_j + \sum_{i=1}^{d} b_i D_i + c, \tag{2.5.1}$$

is said to be uniformly elliptic if there exists costants $K > 1, \kappa \in (0,1)$ such that for all $\xi \in \mathbb{R}^d$ we have the ellipticity condition

$$\kappa|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij} \xi^i \xi^j \leq K|\xi|^2. \tag{2.5.2}$$

2.6. FOURIER MULTIPLIERS

Here, we recall briefly definitions and some properties of vector-valued Fourier multipliers that will be used thorough this thesis. For details, we refer to [70, 104].

Definition 2.6.1. Let X and Y be Banach spaces. Consider a bounded measurable function $m : \mathbb{R}^d \to \mathscr{L}(X, Y)$. It induces a map $T_m : \mathscr{S}(\mathbb{R}^d; X) \to L^\infty(\mathbb{R}^d; Y)$, $p \in (1, \infty)$ where

$$f \mapsto T_m f = \mathscr{F}^{-1}(m(\cdot)[\widehat{f}(\cdot)]).$$

We call m a L^p*-Fourier multiplier* if there exists a constant C_p such that

$$\|T_m f\|_{L^p(\mathbb{R}^d;Y)} \leq C_p \|f\|_{L^p(\mathbb{R}^d;X)}, \quad \forall\, f \in \mathscr{S}(\mathbb{R}^d; X).$$

The map T_m extends uniquely to an operator $T_m \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$, which is called the L^p*-Fourier multiplier operator* corresponding to m and whose operator norm is the smallest constant C_p for which the above estimate holds.

Let $M_p(X, Y) = \{m : \mathbb{R}^d \to \mathscr{L}(X; Y) : \ m \text{ is a Fourier multiplier}\}$. If $m \in M_p(X, Y)$, then we define

$$\|m\|_{M_p(X,Y)} := \|m\|_p := \|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))}.$$

Consider the scalar case $X = Y = \mathbb{C}$. The following Mihlin's multiplier theorem gives a sufficient condition for m to be a L^p-Fourier multiplier. Details on the proof can be found in [70, Theorem 5.2.7]. A weighted version of the Mihlin's multiplier theorem can be found in [65, Theorem IV.3.9].

Theorem 2.6.2 (Mihlin's Multiplier Theorem). *Let $m(\xi)$ be a complex-valued bounded function on $\mathbb{R}^d \backslash \{0\}$ that satisfies the Mihlin's condition*

$$|\xi|^{|\alpha|} |\partial_\xi^\alpha m(\xi)| \leq C_d$$

for all multiindices $|\alpha| \leq [d/2] + 1 \in \mathbb{N}$. Then, for all $p \in (1, \infty)$, m is a L^p-Fourier multiplier, i.e. $m \in M_p(\mathbb{R}^d)$ and the following estimate is valid

$$\|m\|_{M_p} \leq C_d \max(p, (p-1)^{-1})(C_d + \|m\|_{L^\infty}).$$

2.7. $\mathscr{R}$-BOUNDEDNESS

The concept of $\mathscr{R}$-boundedness plays a major rôle in the characterization of maximal regularity that will be shown in Chapter 3. We introduce below the definition of $\mathscr{R}$-boundedness and we state some of its most important properties. Details can be found in [27, 40, 104] and [85].

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space and let X be a Banach space. An *X-valued random variable* is a strongly measurable function $\xi : \Omega \to X$. Let I be an index set. A collection $\{\xi_i\}_{i\in I}$ of X-valued random variables is called *independent* if for all choices of distinct indices $i_1, \dots, i_N \in I$ and all Borel sets $B_1, \dots, B_N \subseteq X$ we have

$$\mathbb{P}(\xi_{i_1} \in B_1, \dots, \xi_{i_N} \in B_N) = \Pi_{n=1}^{N} \mathbb{P}(\xi_n \in B_n).$$

A sequence of independent $\mathbb{R}$-valued random variables $(r_n)_{n\geq 1}$ on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ is called a *Rademacher sequence* if $\mathbb{P}(r_n = 1) = \mathbb{P}(r_n = -1) = \frac{1}{2}$.

Definition 2.7.1. Let X and Y be Banach spaces. A family of operators $\mathscr{T} \subseteq \mathscr{L}(X, Y)$ is said to be *$\mathscr{R}$-bounded* if there exists a constant C such that for all $N \in \mathbb{N}$, all sequences $(T_n)_{n=1}^{N}$ in $\mathscr{T}$ and $(x_n)_{n=1}^{N}$ in X,

$$\Big\| \sum_{n=1}^{N} r_n T_n x_n \Big\|_{L^2(\Omega;Y)} \leq C \Big\| \sum_{n=1}^{N} r_n x_n \Big\|_{L^2(\Omega;X)}. \tag{2.7.1}$$

The least possible constant C is called the *$\mathscr{R}$-bound* of $\mathscr{T}$ and is denoted by $\mathscr{R}(\mathscr{T})$.

The $L^2(\Omega; X)$-norms in (4.3.8) can be replaced by $L^p(\Omega; X)$, for any $p \in [1, \infty)$, to obtain an equivalent definition up to a constant depending on p. In this case, we denote the $\mathscr{R}$-bound as $\mathscr{R}_p(T)$. This is a consequence of the Kahane-Khintchine inequality (see [45, 11.1]).

Theorem 2.7.2 (Kahane-Khintchine)**.** *For every $p, q \in [1, \infty)$, there exists a $C_{p,q} > 0$ such that for all $N \in \mathbb{N}$ and $(x_n)_{n=1}^{N} \in X$,*

$$\Big\| \sum_{n=1}^{N} r_n x_n \Big\|_{L^p(\Omega;X)} \leq C_{p,q} \Big\| \sum_{n=1}^{N} r_n x_n \Big\|_{L^q(\Omega;X)}. \tag{2.7.2}$$

Example 2.7.3.

(1) Let $T \in \mathscr{L}(X, Y)$. Then, $\mathscr{R}_p(T) = \|T\|$.

(2) Let $X = L^p(\mathbb{R})$, $p \in (0, \infty)$, $p \neq 2$. The set of translation operators $T_j f(\cdot) = f(\cdot - j)$ for $j \in \mathbb{N} \cup \{0\}$ is not $\mathscr{R}$-bounded on X. (see [104, Example 2.12] for the proof).

Every $\mathscr{R}$-bounded family of operators $\mathscr{T} \subseteq \mathscr{L}(X, Y)$ is uniformly bounded, with $\sup\{\|T\| : T \in \mathscr{T}\} \leq \mathscr{R}_p(\mathscr{T})$. A converse holds for Hilbert spaces X and Y: every uniform bounded family of operators is automatically $\mathscr{R}$-bounded.

Let now $q \in (1,\infty)$. If $X = Y = L^q(\mathbb{R}^d)$, then $\mathscr{T} \subseteq \mathscr{L}(L^q(\mathbb{R}^d))$ is $\mathscr{R}$-bounded if and only if there exists a constant C such that for all $N \in \mathbb{N}$, all sequences $(T_n)_{n=1}^N$ in $\mathscr{T}$ and $(f_n)_{n=1}^N$ in $L^q(\mathbb{R}^d)$, the square function estimate

$$\Big\| \Big(\sum_{n=1}^N |T_n f_n|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathbb{R}^d)} \leq C \Big\| \Big(\sum_{n=1}^N |f_n|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathbb{R}^d)}.$$

holds. This is a consequence of Khintchine's inequality.

Proposition 2.7.4 (Khintchine). *For every $p \in [1,\infty)$, there exists a $C_p > 0$ such that for all $N \in \mathbb{N}$, and $(x_n)_{n=1}^N \in \mathbb{C}$,*

$$C_p^{-1} \Big\| \sum_{n=1}^N r_n x_n \Big\|_{L^p(\Omega)} \leq \Big(\sum_{n=1}^N |x_n|^2 \Big)^{1/2} \leq C_p \Big\| \sum_{n=1}^N r_n x_n \Big\|_{L^p(\Omega)}. \tag{2.7.3}$$

This reformulation allows one to connect with the square function estimates known in harmonic analysis (see Chapter 4), and allows for special results not available in the general Banach space setting.

With the definition of $\mathscr{R}$-boundedness in mind, we now introduce functional analytic notions in terms of $\mathscr{R}$-boundedness.

Definition 2.7.5. A closed operator A on a Banach space X with dense domain is said to be *$\mathscr{R}$-sectorial* of angle $\sigma < \pi/2$ if it is sectorial of angle σ and for all $\sigma' \in (\sigma, \pi)$ the set $\{z(z+A)^{-1} : z \in \mathbb{C}\backslash\{0\}, |\arg(z)| > \sigma'\}$ is $\mathscr{R}$-bounded.

The following theorem shows the equivalence of various $\mathscr{R}$-boundedness conditions. We refer to [104, Theorem 2.20] for the proof.

Theorem 2.7.6. *Let A be the generator of a bounded analytic semigroup $(T(t))_{t\geq 0}$ in a Banach space X. Then the following are equivalent:*

(i) A is $\mathscr{R}$-sectorial of angle $\sigma < \pi/2$;

(ii) for some $n \in \mathbb{N}$, $\{(it)^n(it+A)^n : t \in \mathbb{R}\backslash\{0\}\}$ is $\mathscr{R}$-bounded;

The concept of $\mathscr{R}$-boundedness has been used first implicitly by Paley, Marcinkiewicz and Zygmund in the '30s in terms of square functions, and later in the '80s by Rubio de Francia, who connected $L^p(\ell^s)$-estimates to estimates in weighted L^p-settings for every $p, s \in (1,\infty)$ (see [65]). In its randomized form, $\mathscr{R}$-boundedness was introduced in 1986 by Bourgain [22] to prove vector-valued multiplier theorems, and later by Clement, de Pagter, Sukochev and Witvliet [27], in 2000, to prove boundedness of certain operators, multipliers theorems, etc. The connection between $\mathscr{R}$-boundedness and maximal L^p-regularity was first introduced in 2001 by Weis [152]. It is based on a Fourier multiplier theorem under $\mathscr{R}$-boundedness assumptions, proved by the same author in [153]. For completeness, we recall it below.

Theorem 2.7.7 (Weis). *Let X,Y be UMD spaces, $m \in C^1(\mathbb{R}\backslash\{0\}, \mathscr{L}(X,Y))$ be such that the sets*

$$\{m(\xi) : \xi \in \mathbb{R}\backslash\{0\}\} \text{ and } \{\xi m'(\xi) : \xi \in \mathbb{R}\backslash\{0\}\}$$

are $\mathscr{R}$-bounded. Then the operator $T_m f = \mathscr{F}^{-1}(m(\xi)\widehat{f}(\xi)$, $f \in S(\mathbb{R}, X)$, extends to a bounded operator $T : L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, Y)$ for all $p \in (1,\infty)$.

An exhaustive explanation on UMD spaces can be found in [65, 86] and will not be treated here. Briefly speaking, in the case of Bochner spaces $L^p(\mathbb{R}; X)$ it make sense to consider only those Banach spaces X for which the simplest multiplier function $m(\cdot) = \operatorname{sign}(\cdot) I_X$ is a Fourier multiplier, which is equivalent to the boundedness of the Hilbert transform on $L^p(\mathbb{R}; X)$, $p \in (1,\infty)$. Such Banach spaces X are called UMD spaces. Examples of UMD spaces are $L^q(\mathbb{R}^d)$ for $q \in (1,\infty)$, and $L^q(\mathbb{R}, X)$ is UMD if X is UMD. On the other hand, the space $L^1(\mathbb{R}^d)$ does not have the UMD property. Below we give the precise definition for completeness.

Definition 2.7.8. X is UMD if the Hilbert transform

$$Hf(x) := \lim_{\substack{\varepsilon \downarrow 0 \\ R \to \infty}} \left(\frac{1}{\pi} \int_{\varepsilon < |x-y| < R} \frac{f(y)}{x-y} ds \right),$$

extends to a bounded operator on $L^p(\mathbb{R}, X)$ for every $p \in (1,\infty)$. Equivalently, X is UMD if and only if $m : \mathbb{R}\backslash\{0\} \to \mathscr{L}(X,X)$, $m(t) = \operatorname{sign}(t) I_X$ is a Fourier multiplier on $L^p(\mathbb{R}, X)$ for every $p \in (1,\infty)$.

2.8. MAXIMAL L^p-REGULARITY

We are finally in the position to introduce maximal L^p-regularity, for any $p \in (1,\infty)$. In what follows we will consider autonomous problems with zero initial values. The non-autonomous case will be introduced in Definition 3.3.10.

Let X_0 and X_1 be Banach spaces, X_1 densely and continuously embedded in X_0, and let $A : D(A) = X_1 \to X_0$ be a closed (bounded) linear operator on X_0. Consider the problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in (0,T) \\ u(0) = 0. \end{cases} \tag{2.8.1}$$

Definition 2.8.1. Let $p \in (1,\infty)$. The operator A has maximal L^p-regularity if there exists a constant $C \geq 0$ such that for every $f \in L^p(\mathbb{R}_+; X_0)$ there exists a unique solution $u \in W^{1,p}(\mathbb{R}_+; X_0) \cap L^p(\mathbb{R}_+; X_1)$ of (2.8.1) that satisfies the *a priori* estimate

$$\|u\|_{W^{1,p}(\mathbb{R}_+;X_0)} + \|u\|_{L^p(\mathbb{R}_+;X_1)} \leq C\|f\|_{L^p(\mathbb{R}_+;X_0)}. \tag{2.8.2}$$

This implies that for each $f \in L^p(\mathbb{R}_+; X_0)$, both u', Au are in $L^p(\mathbb{R}_+; X_0)$. The space $W^{1,p}(\mathbb{R}_+; X_0) \cap L^p(\mathbb{R}_+; X_1)$ is the *maximal regularity space*, and we denote it by $MR^p(\mathbb{R}_+)$. In particular, this is a Banach space for the norm

$$\|u\|_{MR^p(\mathbb{R}_+)} := \|u\|_{W^{1,p}(\mathbb{R}_+;X_0)} + \|u\|_{L^p(\mathbb{R}_+;X_1)}.$$

Example 2.8.2. Let $p \in (1,\infty)$. Elliptic differential operators on $L^p(\mathbb{R}^d_+)$ of the form

$$Au = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha u, \quad x \in \mathbb{R}^d,$$

where the coefficients $a_\alpha : \mathbb{R}^d_+ \to \mathbb{C}$ are bounded uniformly continuous for $|\alpha| = 2m$ and $a_\alpha \in L^\infty(R^d_+, \mathbb{C})$ for $|\alpha| < 2m$, with general boundary conditions have the maximal L^p-regularity property(see [104, Sections 6 and 7]).

On the other hand, the operator $A = -\Delta$ on $L^1(\mathbb{R}^d)$ does not enjoy the maximal L^p-regularity property (see [104, Counterexample 1.15]).

The maximal regularity estimate (2.8.2) has a long tradition. The first positive result on maximal L^p-regularity was obtained in the '60s by Ladyžhenskaya, Solonnikov and Ural'tseva [105], where $X = L^p(\Omega)$, $\Omega \subset \mathbb{R}^d$ being a bounded smooth domain, and A is a $2nd$-order elliptic differential operator.

The first abstract results were obtained by Sobolevskii [139], de Simon [37], and da Prato-Grisvard [33], where they considered the problem in the framework of generators of analytic semigroups on Banach spaces. Da Prato and Grisvard initiated a new approach via the operator sum method, that was extended by Dore and Venni [54], and Kalton and Weis [94].

In [37] De Simon showed that every generator of a bounded analytic semigroup on a Hilbert space has maximal L^p-regularity. A natural question was posed by Brezis in the '80s, about whether every generator of an analytic semigroup on L^q, $q \in (1,\infty)$ have maximal L^p-regularity. A counterexample was found by Kalton and Lancien [93] in 2000, showing basically that Hilbert spaces are the only ones in which the question posed by Brezis has a positive answer. Their example is not a differential operator, and their result was revisited by Fackler in [58].

In 2001, Weis [152, 153] discovered an operator-theoretic characterization of maximal L^p-regularity in terms of $\mathcal{R}$-boundedness of the resolvent of A and the Mihlin's theorem for operator valued Fourier multipliers. This leads the way to numerous results about maximal L^p-regularity, as discussed in the introduction.

2.8.1. Weis' characterization of maximal regularity

In the following we recall Weis' characterization of maximal L^p-regularity from [104]. We only give an idea of the proof. Details can be found in [104, 152, 153].

Theorem 2.8.3 (Weis)**.** *Let A be the generator of a bounded analytic semigroup on a UMD space X and let $0 \in \rho(A)$. Then, A has maximal L^p-regularity for $p \in (1,\infty)$ if and only if A is $\mathcal{R}$-sectorial.*

Proof. It is well-known that if A is the generator of an analytic semigroup, the solution to problem (2.8.1) is given by the variation of constant formula (2.4.2), that we recall below:

$$u(t) = \int_0^\infty e^{-(t-s)A} f(s)\,ds.$$

so that

$$Au(t) = \int_0^\infty Ae^{-(t-s)A} f(s)ds.$$

Consider the integral operator defined by

$$Kf(t) := \int_{\mathbb{R}} Ae^{-(t-s)A} f(s)ds, \tag{2.8.3}$$

where we extended f by zero on $(-\infty, 0)$ and $k(t) = Ae^{-tA}$ if $t > 0$ or equals zero if $t < 0$. This implies that A has maximal L^p-regularity if K extends to a bounded operator $K : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)$. Observe that K is a convolution operator with an operator-valued kernel, singular at zero, i.e. $\|Ae^{-tA}\| \sim \frac{1}{t}$.

Apply now the Fourier transform to (2.8.3) to get

$$\mathscr{F}(Kf)(\xi) = \mathscr{F}(Ae^{-tA})(\xi)[\mathscr{F}(f)(\xi)], \ \xi \in \mathbb{R}.$$

So, we can identify $\mathscr{F}(Ae^{-tA})(\xi)$ as a Fourier multiplier of the form

$$m(\xi) := \mathscr{F}(Ae^{-tA})(\xi) = AR(i\xi, A) = I - i\xi R(i\xi, A),$$

with

$$\xi m'(\xi) = -i\xi AR(i\xi, A)^2 = i\xi R(i\xi, A)[i\xi R(i\xi, A) - I].$$

where $R(i\xi, A) = \int_0^\infty e^{-i\xi t} T(t)dt$ and $T(t)$ denotes the analytic semigroup generated by A. From this it follows that in order to show the L^p-boundedness of K, we need to prove that m is a Fourier multiplier on $L^p(\mathbb{R}; X)$. For this, it is enough to show that $m : \xi \mapsto i\xi R(i\xi, A)$ is a Fourier multiplier. By the $\mathscr{R}$-sectoriality of A and equivalence $(i) \Leftrightarrow (ii)$ of Theorem 2.7.6, we get the $\mathscr{R}$-boundedness of the sets $\{m(\xi)\}$ and $\{\xi m'(\xi)\}$. This, together with Theorem 2.7.7, implies that m is a Fourier multiplier. Therefore, $K \in \mathscr{L}(L^p(\mathbb{R}; X))$ and A has maximal L^p-regularity. □

This implies that A has maximal L^p-regularity if one of the equivalent conditions of Theorem 2.7.6 are satisfied.

Remark 2.8.4. The need for UMD condition becomes natural from the following result of Couhlon and Lamberton [30]: the generator of the Poisson semigroup on $L^2(\mathbb{R}, X)$ has maximal L^p-regularity if and only if X is a UMD space.

II

A new approach to maximal L^p-regularity for parabolic PDEs

3

Maximal regularity for parabolic equations with measurable dependence on time

In this chapter we study maximal L^p-regularity for evolution equations of the form (1.0.1) with time-dependent operators A, where we merely assume a measurable dependence on time. In the first part, we present a new sufficient condition for the L^p-boundedness of a class of vector-valued singular integrals which does not rely on Hörmander conditions in the time variable. This is then used to develop an abstract operator-theoretic approach to maximal regularity (Section 3.3).

The results are applied to the case of m-th order elliptic operators A with time and space-dependent coefficients, where the highest order coefficients are assumed to be measurable in time and continuous in the space variable (Section 3.4). This results in an $L^p(L^q)$-theory for such equations for $p, q \in (1, \infty)$.

In the final section we extend a well-posedness result for quasilinear equations to the time-dependent setting. Here we give an example of a nonlinear parabolic PDE to which the result can be applied.

This chapter is the heart of this thesis. The results of Section 3.3 will be applied in the second part of this work, where we will consider maximal L^p- regularity for systems of parabolic equations in the whole and half-space (Chapters 5 and 6). We remark that in the estimates below, C can denote a constant which varies from line to line. The results here presented are based on [64].

3.1. Preliminaries: $\mathscr{R}$-boundedness of integral operators

In this section we introduce a certain family of integral operators that will be needed later on and we give sufficient conditions for the L^p-boundedness and the $\mathscr{R}$-boundedness of such a family.

Let $\mathscr{K}$ be the class of kernels

$$\mathscr{K} = \{k \in L^1(\mathbb{R}^d) : \text{for all simple } f : \mathbb{R}^d \to \mathbb{R}_+ \text{ one has } |k * f| \leq Mf \text{ a.e.}\}, \tag{3.1.1}$$

where M denotes the Hardy-Littlewood maximal operator. There are many examples of classes of functions k with this property (see [70, Chapter 2] and [123, Proposition 4.5 and 4.6]). It follows from [123, Lemma 4.3] that every $k \in \mathcal{K}$ satisfies $\|k\|_{L^1(\mathbb{R}^d)} \le 1$.

The next example gives an important class of kernels which are in $\mathcal{K}$.

Example 3.1.1. Let $k : (0,\infty) \times \mathbb{R} \to \mathbb{C}$, be such that $|k(u,t)| \le h(\frac{|t|}{u})\frac{1}{u}$, $u > 0$, where $h \in L^1(\mathbb{R}_+) \cap C_b(\mathbb{R}_+)$, h has a maximum in $x_0 \in [0,\infty)$ and h is radially decreasing on $[x_0,\infty)$. Then,

$$\int_0^\infty \sup_{|t|\ge x} |k(u,t)|\, dx \le \int_0^\infty \sup_{t\ge x} |h(\tfrac{t}{u})| \frac{dx}{u} = \int_0^\infty \sup_{s\ge \frac{x}{u}} |h(s)| \frac{dx}{u} = \int_0^\infty \sup_{s\ge y} |h(s)|\, dy$$
$$= \int_0^{x_0} \sup_{s\ge y} |h(s)|\, dy + \int_{x_0}^\infty |h(y)|\, dy = x_0|h(x_0)| + \|h\|_{L^1(x_0,\infty)}.$$

Now by [123, Proposition 4.5] we find $\{\frac{k(u,\cdot)}{C} : u > 0\} \subseteq \mathcal{K}$ with $C = x_0|h(x_0)| + \|h\|_{L^1(x_0,\infty)}$.

Suppose $T : \{(t,s) \in \mathbb{R}^2 : t \ne s\} \to \mathcal{L}(X)$ is such that for all $x \in X$, $(t,s) \mapsto T(t,s)x$ is measurable. For $k \in \mathcal{K}$ let

$$I_{kT} f(t) = \int_{\mathbb{R}} k(t-s)T(t,s)f(s)\, ds. \tag{3.1.2}$$

Consider the family of integral operators $\mathcal{I} := \{I_{kT} : k \in \mathcal{K}\} \subseteq \mathcal{L}(L^p(\mathbb{R};X))$. The $\mathcal{R}$-boundedness of such families $\mathcal{I}$ of operators will play an important rôle in this chapter. In fact, as we will see in Section 3.2, it constitutes a sufficient condition for the L^p-boundedness of a class of vector-valued singular integrals. In the case $X = L^q$, $T(t,s)$ in (3.1.2) denotes the evolution family generated by a higher-order differential operator. This will be treated in Section 3.4.

Proposition 3.1.2. *If $\{T(t,s) : s,t \in \mathbb{R}\}$ is uniformly bounded on X, then $\mathcal{I}$ is uniformly bounded on $L^p(\mathbb{R},v;X)$ for every $p \in (1,\infty)$ and $v \in A_p$. Moreover, it is also uniformly bounded on $L^1(\mathbb{R};X)$.*

Proof. For any $p \in (1,\infty)$, note that

$$\|I_{kT} f(t)\|_X \le \int_{\mathbb{R}} |k(t-s)| \|T(t,s)f(s)\|_X\, ds$$
$$\le C \int_{\mathbb{R}} |k(t-s)| \|f(s)\|_X\, ds \le CM(\|f\|_X)(t).$$

for a.e. $t \in \mathbb{R}$. Therefore the uniform boundedness of I_{kT} follows from the boundedness of the maximal operator. The case $v \equiv 1$ and $p = 1$ follows from Fubini's theorem and the fact that $\|k\|_{L^1(\mathbb{R})} \le 1$ (see [123, Lemma 4.3]). □

The $\mathcal{R}$-boundedness of (3.1.2) has the following simple extrapolation property:

Proposition 3.1.3. *Let $p_0 \in (1,\infty)$. If for all $v \in A_{p_0}$, $\mathcal{I} \subseteq \mathcal{L}(L^{p_0}(\mathbb{R},v;X))$ is $\mathcal{R}$-bounded by a constant which is A_{p_0}-consistent, then for every $p \in (1,\infty)$ and $v \in A_p$, $\mathcal{I} \subseteq \mathcal{L}(L^p(\mathbb{R},v;X))$ is $\mathcal{R}$-bounded by a constant which is A_p-consistent.*

Proof. The special structure of $\mathscr{I}$ will not be used in this proof. Let $I_1,\dots,I_N \in \mathscr{I}$, $f_1,\dots,f_N \in L^p(\mathbb{R},v;X)$ and let

$$F_p(t) = \Big\| \sum_{n=1}^{N} r_n I_n f_n(t) \Big\|_{L^p(\Omega;X)} \quad \text{and} \quad G_p(t) = \Big\| \sum_{n=1}^{N} r_n f_n(t) \Big\|_{L^p(\Omega;X)}.$$

Then the assumption combined with Fubini's theorem yields that for all $v \in A_{p_0}$,

$$\|F_{p_0}\|_{L^{p_0}(\mathbb{R},v)} \le C\|G_{p_0}\|_{L^{p_0}(\mathbb{R},v)},$$

where C is a constant which is A_{p_0}-consistent. Therefore, by Theorem 2.2.5 we find that for each $p \in (1,\infty)$, there is an A_p-consistent constant C' (depending only on C) such that

$$\|F_{p_0}\|_{L^p(\mathbb{R},v)} \le C'\|G_{p_0}\|_{L^p(\mathbb{R},v)}. \tag{3.1.3}$$

Now by (2.7.3), $F_p \le \kappa_{p,p_0} F_{p_0}$, $G_{p_0} \le \kappa_{p_0,p} G_p$, and the result follows from (3.1.3) and another application of Fubini's theorem. □

In the case $X = L^q$, the following is a simple sufficient condition for the $\mathscr{R}$-boundedness of such families.

Theorem 3.1.4. *Let $\mathscr{O} \subseteq \mathbb{R}^d$ be open. Let $q_0 \in (1,\infty)$ and let $\{T(t,s) : s,t \in \mathbb{R}\}$ be a family of bounded operators on $L^{q_0}(\mathscr{O})$. Assume that for all A_{q_0}-weights w,*

$$\|T(t,s)\|_{\mathscr{L}(L^{q_0}(\mathscr{O},w))} \le C, \tag{3.1.4}$$

where C is A_{q_0}-consistent and independent of $t,s \in \mathbb{R}$. Then the family of integral operators $\mathscr{I} = \{I_{kT} : k \in \mathscr{K}\} \subseteq \mathscr{L}(L^p(\mathbb{R},v;L^q(\mathscr{O},w)))$ as defined in (3.1.2) *is $\mathscr{R}$-bounded for all $p,q \in (1,\infty)$ and all $v \in A_p$ and $w \in A_q$. Moreover, in this case the $\mathscr{R}$-bounds $\mathscr{R}(\mathscr{I})$ are A_p- and A_q-consistent.*

The proof of this result will be given in the end of Chapter 4 as a special case of Corollary 4.3.9, and it is is based on extrapolation techniques of Rubio de Francia. As for fixed $t,s \in \mathbb{R}$, $T(t,s)$ on $L^q(\mathscr{O})$ is usually defined by a singular integral of convolution type in $\mathbb{R}^d$, one can often apply Calderón-Zygmund theory and multiplier theory to verify (3.1.4). In this case it is usually not more difficult to prove the boundedness for all A_q-weights, than just $w = 1$. The reason for this is that for large classes of operators, boundedness implies weighted boundedness (see [65, Theorem IV.3.9], [71, Theorem 9.4.6] and [79, Corollary 2.10]). Another situation where weights are used to obtain $\mathscr{R}$-boundedness can be found in [61, 77].

Example 3.1.5. For a bounded measurable function $\theta : \mathbb{R}^2 \to \mathscr{K}$ let $T(t,s)f = \theta(t,s)f$, $f \in L^{q_0}(\mathbb{R}^d, w)$. Then (3.1.4) holds and hence Theorem 3.1.4 implies that $\mathscr{I} \subseteq \mathscr{L}(L^p(\mathbb{R},v;L^q(\mathbb{R}^d,w)))$ is $\mathscr{R}$-bounded for all $p,q \in (1,\infty)$ and all $v \in A_p$ and $w \in A_q$.

3.2. A CLASS OF SINGULAR INTEGRALS WITH OPERATOR-VALUED KERNEL

Let X be a Banach space. In this section, we present a new sufficient condition for the L^p-boundedness of a class of vector-valued singular integrals of the form

$$I_K f(t) = \int_{\mathbb{R}} K(t,s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{3.2.1}$$

where $K : \{(t,s) : t \neq s\} \to \mathscr{L}(X)$ is an operator-valued kernel.

There is a natural generalization of the theory of singular integrals of *convolution type* to the vector-valued setting (see [89]). In the case the singular integral is of *non-convolution type*, the situation is much more complicated. An extensive treatment can be found in [83, 84, 88], where $T1$-theorems [34] and Tb-theorems [35] have been obtained in an infinite dimensional setting. Checking the conditions of these theorems can be hard. For instance, from [145] it follows that the typical BMO conditions one needs to check, have a different behavior in infinite dimensions. Our motivation comes from the application to maximal L^p-regularity of (1.2.1). At the moment we do not know whether the $T1$-theorem and Tb-theorem can be applied to study maximal L^p-regularity for the time dependent problems we consider. Below we study a special class of singular integrals with operator-valued kernel for which we prove L^p-boundedness. The assumptions on K are formulated in such a way that they are suitable for proving maximal L^p-regularity of (1.2.1) later on.

3.2.1. *Assumptions*

The assumptions in the main result of this section are as follows.

(H1) Let X be a Banach space and let $p \in [1,\infty)$ and[1] $v \in A_p$.

(H2) The kernel K factorizes as

$$K(t,s) = \frac{\phi_0(|t-s|A_0)T(t,s)\phi_1(|t-s|A_1)}{t-s}, \quad (t,s) \in \mathbb{R}^2, t \neq s. \tag{3.2.2}$$

Here A_0 and A_1 are sectorial operators on X of angle $< \sigma_0$ and $< \sigma_1$ respectively, and $\phi_j \in H_0^\infty(\Sigma_{\sigma'_j})$ and $\sigma'_j \in (\sigma_j, \pi)$ for $j = 0,1$. Moreover, we assume $T : \{(t,s) : t \neq s\} \to \mathscr{L}(X)$ is uniformly bounded and for all $x \in X$, $\{(t,s) : t \neq s\} \mapsto T(t,s)x$ is strongly measurable.

(H3) Assume X has finite cotype. Assume A_j has a bounded $H^\infty(\Sigma_{\sigma_j})$-calculus with $\sigma_j \in [0,\pi)$ for $j = 0,1$.

[1]For the case $p = 1$, the convention will be that $v \equiv 1$.

(H4) Assume the family of integral operators $\mathscr{I} := \{I_{kT} : k \in \mathscr{K}\} \subseteq \mathscr{L}(L^p(\mathbb{R}, v; X))$ is $\mathscr{R}$-bounded.

The class of kernels $\mathscr{K}$ is as in Definition 3.1.1. Recall from (3.1.2) that

$$I_{kT} f(t) = \int_{\mathbb{R}} k(t-s) T(t,s) f(s)\, ds.$$

Since T is uniformly bounded, the operator I_{kT} is bounded on $L^p(\mathbb{R}, v; X)$ by Proposition 3.1.2.

Remark 3.2.1.

1. The class of Banach spaces with finite cotype is rather large. It contains all L^p-spaces, Sobolev, Besov and Hardy spaces as long as the integrability exponents are in the range $[1,\infty)$. The spaces c_0 and L^∞ do not have finite cotype. The cotype of X will be used in order to have estimates for certain continuous square functions (see (3.2.6)). We have collected some general facts about type and cotype in Appendix 3.5.3.

2. In the theory of singular integrals in a vector-valued setting one usually assumes X is a UMD space. Note that every UMD has finite cotype and non-trivial type by the Maurey-Pisier theorem (see [45]).

3. A sufficient condition for the $\mathscr{R}$-boundedness condition in the case $X = L^q$ can be deduced from Theorem 3.1.4.

4. In (H2), $\phi_j(|t-s|A_j)$ could be replaced by $\phi_j((t-s)A_j)$ if the A_j's are bisectorial operators. On the other hand, one can also consider $T(t,s)\mathbf{1}_{\{s<t\}}$ and $T(t,s)\mathbf{1}_{\{t<s\}}$ separately. Indeed, the hypothesis (H1)–(H4) holds for these operators as well whenever they hold for $T(t,s)$.

Example 3.2.2. Typical examples of functions ϕ_j which one can take are $\phi_j(z) = z^\alpha e^{-z}$ for $j = 0, 1$. If $T(t,s) = I\mathbf{1}_{\{s<t\}}$, then for $A = A_0 = A_1$ one would have

$$K(t,s) = (t-s)^{2\alpha-1} A^{2\alpha} e^{-2(t-s)A} \mathbf{1}_{\{s<t\}}.$$

This kernel satisfies $\|K(t,s)\| \sim (t-s)^{-1}$ for t close to s. If one takes $T(t,s)$ varying in t and s one might view it as a multiplicative perturbation of the above kernel.

The following simple observation shows that I_K as given in (3.2.1) can be defined on $L^p(\mathbb{R}; D(A_1) \cap R(A_1))$, where $D(A_1)$ denotes the domain of A_1 and $R(A_1)$ the range of A_1.

Lemma 3.2.3. *Under the assumptions (H1) and (H2), I_K from* (3.2.1) *is bounded as an operator from $L^p(\mathbb{R}, v; D(A_1) \cap R(A_1))$ into $L^p(\mathbb{R}, v; X)$.*

Proof. As $\phi_1 \in H_0^\infty(\Sigma_{\sigma_1'})$, we can find a constant C and $\varepsilon \in (0,1)$ such that $|\phi_1(z)| \leq C|z|^\varepsilon|1+z|^{-2\varepsilon}$. We claim that for all $x \in D(A_1) \cap R(A_1)$,

$$\|\phi_1(tA_1)x\| \leq C\min\{t^\varepsilon, t^{-\varepsilon}\}(\|x\| + \|A_1x\| + \|A_1^{-1}x\|), \quad t > 0. \tag{3.2.3}$$

Before proving the claim, we show how to deduce the assertion of the lemma. From the claim and the assumptions that $\phi_0 \in H_0^\infty(\Sigma_{\sigma_0'})$ and $\|T(t,s)\|$ is uniformly bounded we obtain

$$\begin{aligned}|t-s|\,\|K(t,s)x\| &\leq \|\phi_0(|t-s|A_0)\|\,\|T(t,s)\|\,\|\phi_1(|t-s|A_1)x\|\\ &\leq C\min\{|t-s|^\varepsilon, |t-s|^{-\varepsilon}\}(\|x\| + \|A_1x\| + \|A_1^{-1}x\|).\end{aligned}$$

Therefore, $K:\{(t,s): t\neq s\} \to \mathcal{L}(R(A_1)\cap D(A_1), X)$ is essentially nonsingular, and the assertion of the lemma easily follows. Indeed, for $f \in L^p(\mathbb{R}, v; D(A_1)\cap R(A_1))$, we find

$$\|I_Kf(t)\| \leq C\int_{\mathbb{R}} \min\{|t-s|^{\varepsilon-1}, |t-s|^{-\varepsilon-1}\}\|f(s)\|_{D(A_1)\cap R(A_1)}\,ds.$$

As $k(s) = \min\{|s|^{\varepsilon-1}, |s|^{-\varepsilon-1}\}$ is radially decreasing and integrable it follows from [70, Theorem 2.1.10] that

$$\|I_Kf(t)\| \leq \|k\|_{L^1(\mathbb{R})}Mg(t),$$

where M is the Hardy-Littlewood maximal operator and $g(s) = \|f(s)\|_{D(A_1)\cap R(A_1)}$. Therefore, the boundedness follows from the fact that M is bounded on $L^p(\mathbb{R}, v)$, for $p \in (1,\infty)$. The case $p = 1$ follows from Young's inequality.

To prove (3.2.3), let $g(A_1) = A_1(1+A_1)^{-2}$ and note that

$$g(A_1)^{-1} = 2 + A_1 + A_1^{-1} \;\text{ and }\; \phi_1(zA_1) = (\phi_1(z\cdot)g)(A_1)g(A_1)^{-1}$$

(see [104, Appendix B]). For $y = g(A_1)^{-1}x$ one has $\|y\| \leq 2\|x\| + \|A_1x\| + \|A_1^{-1}x\|$ and it remains to show

$$\|(\phi_1(t\cdot)g)(A_1)\| \leq C\min\{t^\varepsilon, t^{-\varepsilon}\}, \text{ for } t > 0.$$

For this, let $\Gamma = \partial\Sigma_\sigma$ with $\sigma \in (\sigma_1, \sigma_1')$. By [104, Theorem 9.2 and Appendix B]

$$\begin{aligned}\|(\phi_1(t\cdot)g)(A_1)\| &\leq \frac{1}{2\pi}\int_\Gamma |\phi_1(t\lambda)||\lambda||1+\lambda|^{-2}\|(\lambda+A_1)^{-1}\|\,|d\lambda|\\ &\leq C\int_\Gamma \frac{|\lambda t|^\varepsilon}{|1+\lambda t|^{2\varepsilon}}|1+\lambda|^{-2}\,|d\lambda|\\ &\leq C|t|^\varepsilon\int_\Gamma \frac{|\lambda|^\varepsilon}{|1+\lambda|^2}\,|d\lambda|\\ &\leq C'|t|^\varepsilon.\end{aligned}$$

This prove the required estimate for $0 < t < 1$. To prove an estimate for $t > 1$, note that $\frac{1}{|1+\mu|} \le \frac{C}{1+|\mu|} \le \frac{C}{|\mu|}$ for $\mu \in \Sigma_\sigma$. Therefore,

$$\begin{aligned}\|(\phi_1(t\cdot)g)(A_1)\| &\le C\int_\Gamma \frac{|\lambda t|^\varepsilon}{|1+\lambda t|^{2\varepsilon}}|1+\lambda|^{-2}|d\lambda| \\ &\le C\int_\Gamma \frac{|\lambda t|^{-\varepsilon}}{|1+\lambda|^2}|d\lambda| \le C|t|^{-\varepsilon}\int_\Gamma \frac{|\lambda|^{-\varepsilon}}{|1+\lambda|^2}|d\lambda| \le C'|t|^{-\varepsilon}.\end{aligned}$$

□

3.2.2. *Main result on singular integrals*

Theorem 3.2.4. *Assume (H1)-(H4). Then I_K defined by* (3.2.1) *extends to a bounded operator on $L^p(\mathbb{R}, v; X)$.*

The proof is inspired by the solution to the stochastic maximal L^p-regularity problem given in [122].

Before we turn to the proof, we have some preliminary results and remarks.

Example 3.2.5. Assume (H2) and (H3). If $T(t,s)$ is as in Example 3.1.5 then (H4) holds. Therefore, I_K is bounded by Theorem 3.2.4. Surprisingly, we do not need any smoothness of the mapping $(t,s) \mapsto K(t,s)$ in this result. In particular we do not need any regularity conditions for $K(t,s)$ (such as Hörmander's condition) in (t,s).

Recall the following Poisson representation formula (see [122, Lemma 4.1]).

Lemma 3.2.6. *Let $\alpha \in (0,\pi)$ and $\alpha' \in (\alpha,\pi]$ be given, let X be a Banach space and let $f : \Sigma_{\alpha'} \to X$ be a bounded analytic function. Then, for all $s > 0$ we have*

$$f(s) = \sum_{j\in\{-1,1\}} \frac{j}{2}\int_0^\infty k_\alpha(u,s) f(ue^{ij\alpha})\,du,$$

where $k_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$k_\alpha(u,t) = \frac{(t/u)^{\frac{\pi}{2\alpha}}}{(t/u)^{\frac{\pi}{\alpha}}+1}\frac{1}{\alpha u}. \tag{3.2.4}$$

Proof. Let $g : \Sigma_{\frac{1}{2\pi}+\varepsilon} \to X$ be analytic and bounded for some $\varepsilon > 0$. Then, by the Poisson formula on the half space (see [82, Chapter 8])

$$g(t) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{t}{t^2+v^2} g(iv)\,dv.$$

For small $\varepsilon > 0$ let $\varphi : \Sigma_{\frac{1}{2\pi}+\varepsilon} \to \alpha'$ be defined by $\varphi(t) := t^{2\alpha/\pi}$. Then φ is analytic, and taking $g = f\circ g$ gives

$$f(t^{2\alpha/\pi}) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{t}{t^2+v^2}$$

Then taking $s = t^{2\alpha/\pi}$ and $u = |v|^{2\alpha/\pi}$ we get the required result. □

Remark 3.2.7. In the proof below we will use the probabilistic notions of γ-boundedness and γ-radonifying operators, i.e $T \in \gamma(\frac{du}{u}; X)$, with $u \in \mathbb{R}_+$ and X Banach space. Detailed definitions will be given in Appendix 3.A.2. As a general idea, in the special case $X = L^q(S)$ with $q \in (1,\infty)$, we present some identification of spaces which can be used to simplify the proof below. This might be of use to readers who are only interested in L^q-spaces. In this case one can take

$$\begin{aligned}\gamma(\tfrac{du}{u}; X) &= L^q(S; L^2(0,\infty; \tfrac{du}{u})),\\ \gamma(\tfrac{du}{u}; X)^* &= L^{q'}(S; L^2(0,\infty; \tfrac{du}{u})),\\ \gamma(\tfrac{du}{u}; L^p(\mathbb{R}, v; X)) &= L^p(\mathbb{R}, v; L^q(S; L^2(0,\infty; \tfrac{du}{u}))).\end{aligned}$$

The γ-multiplier theorem (cf. Theorem 3.A.7) which is applied below in (3.2.5) can be replaced by [152, 4a] in this case. Finally, the estimates in (3.2.6) can be found in [107] in this special case.

Proof of Theorem 3.2.4. Step 1: By density it suffices to prove

$$\|I_K f\|_{L^p(\mathbb{R},v;X)} \le C\|f\|_{L^p(\mathbb{R},v;X)}$$

with C independent of $f \in L^p(\mathbb{R}, v; D(A_1) \cap R(A_1))$. Note that by Lemma 3.2.3, I_K is well defined on this subspace.

Step 2: Fix $0 < \alpha < \alpha' \le \min\{\sigma_0' - \sigma_0, \sigma_1' - \sigma_1\}$. First, since $z \to \phi_0(zA_0)T(t,s)\phi_1(zA_1)$ is analytic and bounded on $\Sigma_{\alpha'}$, by Lemma 3.2.6, for $x \in D(A_1) \cap R(A_1)$ and $z > 0$,

$$\phi_0(zA_0)T(t,s)\phi_1(zA_1)x = \sum_{j\in\{-1,1\}} \frac{j}{2}\int_0^\infty \Phi_{0,j}(u)k_\alpha(u,z)T(t,s)\Phi_{1,j}(u)x\,du,$$

with $k_\alpha(u,t)$ as in (3.2.4) and $\Phi_{k,j}(u) = \phi_k(ue^{ij\alpha}A_k)$ for $j \in \{-1,1\}$ and $k \in \{0,1\}$. Together with (H2) this yields the following representation of $K(t,s)x$ for $x \in D(A_1) \cap R(A_1)$:

$$K(t,s)x = \sum_{j\in\{-1,1\}} \frac{j}{2}\int_0^\infty \Phi_{0,j}(u)S_u(t,s)\Phi_{1,j}(u)x\,\frac{du}{u},$$

where $S_u(t,s) := \tilde{k}_\alpha(u,t-s)T(t,s)$ with $\tilde{k}_\alpha(u,t) := k_\alpha(u,|t|)\frac{u}{t}$ and k_α is defined as in (3.2.4). Moreover, the kernels $\tilde{k}_\alpha(u,\cdot)$ satisfy

$$|\tilde{k}_\alpha(u,t)| \le \alpha^{-1}h_\alpha(\tfrac{t}{u})u^{-1}, \quad u,t>0,$$

where $h_\alpha(x) = \frac{x^{\beta-1}}{x^{2\beta}+1}$ and $\beta := \frac{\pi}{2\alpha} > 0$. Extending $k_\alpha(u,t)$ as zero for $t < 0$, by Example 3.1.1 we find that $\tilde{k}_\alpha(u,\cdot) \in \mathcal{K}$. Indeed, substituting $y = x^\beta$, we obtain

$$\|h\|_{L^1(0,\infty)} = \int_0^\infty \frac{x^{\beta-1}}{x^{2\beta}+1}\,dx = \frac{1}{\beta}\int_0^\infty \frac{1}{y^2+1}\,dy = \alpha.$$

Therefore, the following representation holds for the singular integral

$$I_K f = \sum_{j\in\{-1,1\}} \frac{j}{2}\int_0^\infty \Phi_{0,j}(u) I_{S_u}[\Phi_{1,j}(u)f]\,\frac{du}{u},$$

where $f \in L^p(\mathbb{R}, v; D(A_1)\cap R(A_1))$.

Step 3: Let $Y_1 = L^p(\mathbb{R}, v; X)$ and $Y_2 = L^{p'}(\mathbb{R}, v'; X^\#)$, where $X^\# := \overline{D(A_0^*)} \cap \overline{R(A_0^*)}$ is the moondual[2] of X with respect to A_0 (see [104, Appendix A]) and $v' = v^{-\frac{1}{p-1}}$. For $g \in Y_2$ write $\langle f, g\rangle_{Y_1,Y_2} = \int_{\mathbb{R}} \langle f(t), g(t)\rangle\,dt$. In this way Y_2 can be identified with an isometric closed subspace of Y_1^*. Note that by [104, Proposition 15.4], $X^\#$ is norming for X, i.e. $\|x\|_\# := \sup\{|\langle x, x^\#\rangle| : x^\# \in X^\#, \|x^\#\| \le 1\}$ defines a norm on X which is equivalent to the original norm. Hence, Lemma 2.2.6 implies that Y_2 is norming for Y_1. For fixed $g \in Y_2$ it follows from Fubini's theorem and γ-duality (see [74, Sections 2.3 and 2.6] and [95, Section 5]),

$$\begin{aligned}
|\langle I_K f, g\rangle_{Y_1,Y_2}| &\le \sum_{j\in\{-1,1\}} \frac{1}{2}\Big|\int_{\mathbb{R}}\int_0^\infty \langle \Phi_{0,j}(u) I_{S_u}[\Phi_{1,j}(u)f](t), g(t)\rangle\,\frac{du}{u}\,dt\Big| \\
&= \sum_{j\in\{-1,1\}} \frac{1}{2}\Big|\int_0^\infty \langle I_{S_u}[\Phi_{1,j}(u)f], \Phi_{0,j}(u)^\# g\rangle\,\frac{du}{u}\Big| \\
&= \sum_{j\in\{-1,1\}} \frac{1}{2}\|I_{S_u}[\Phi_{1,j}(u)f]\|_{\gamma(\mathbb{R}_+,\frac{du}{u};Y_1)}\|\Phi_{0,j}(u)^\# g\|_{\gamma(\mathbb{R}_+,\frac{du}{u};Y_1)^*}.
\end{aligned}$$

Here $\Phi_{0,j}(u)^\# := \phi_0(ue^{ij\alpha}A_0^\#)$. By (H4) the family $\{I_{S_u} : u > 0\}$ is $\mathscr{R}$-bounded by some constant C_T. Therefore, by the Kalton-Weis γ-multiplier theorem (cf. Theorem 3.A.7)

$$\|I_{S_u}[\Phi_{1,j}(u)f]\|_{\gamma(\mathbb{R}_+,\frac{du}{u};Y_1)} \le C_T \|\Phi_{1,j}(u)f\|_{\gamma(\mathbb{R}_+,\frac{du}{u};Y_1)}. \tag{3.2.5}$$

Here we used that X does not contain an isomorphic copy of c_0 as it has finite cotype (see (H3)). The remaining two square function norms can be estimated by the square function estimates of Kalton and Weis. Indeed, by (H3) and [74, Theorem 4.11] or [95, Section 7] (here we again use the finite cotype of X) and the γ-Fubini property (see [121, Theorem 13.6]), we obtain

$$\begin{aligned}
\|\Phi_{1,j}(u)f\|_{\gamma(\mathbb{R}_+,\frac{du}{u};Y_1)} &\eqsim \|\Phi_{1,j}(u)f\|_{L^p(\mathbb{R},v;\gamma(\mathbb{R}_+,\frac{du}{u};X))} \le C_{A_1}\|f\|_{Y_1}, \\
\|\Phi_{0,j}(u)^\# g\|_{\gamma(\mathbb{R}_+,\frac{du}{u};Y_1)^*} &\eqsim \|\Phi_{0,j}(u)^\# g\|_{L^{p'}(\mathbb{R},v';\gamma(\mathbb{R}_+,\frac{du}{u};X)^*)} \le C_{A_0}\|g\|_{Y_2}.
\end{aligned} \tag{3.2.6}$$

Combining all the estimates yields

$$|\langle I_K f, g\rangle_{Y_1,Y_2}| \le C_T C_{A_0} C_{A_1}\|f\|_{Y_1}\|g\|_{Y_2}.$$

Taking the supremum over all $g \in L^{p'}(\mathbb{R}, v'; X^\#)$ with $\|g\|_{Y_2} \le 1$ we find $\|I_K f\|_{Y_1} \le C_T C_{A_0} C_{A_1}\|f\|_{Y_1}$. This proves the L^p-boundedness. □

[2] In the special case $X = L^q(S)$ with $q \in (1,\infty)$, one can use the usual adjoint $*$ instead of the moon adjoint $\#$. The advantage in using the moondual comes in the case in which X is a non-reflexive space. We refer to [104, Section 15] for details.

Remark 3.2.8. One can also apply standard extrapolation techniques to obtain weighted boundedness results for singular integrals from the unweighted case (see [26, 79]). However, for this one needs Hörmander conditions on the kernel. As our proof gives a result in the more general setting, we can avoid smoothness assumptions on the kernel.

3.3. MAXIMAL L^p-REGULARITY

In this section we will apply Theorem 3.2.4 to obtain maximal L^p-regularity for the following evolution equation on a Banach space X_0:

$$\begin{aligned} u'(t) + A(t)u(t) &= f(t), \ t \in (0,T) \\ u(0) &= x. \end{aligned} \tag{3.3.1}$$

As explained in the introduction no abstract L^p-theory is available for (3.3.1) outside the case where $t \mapsto A(t)$ is continuous.

The following assumption will be made throughout this whole section.

(A) Let X_0 be a Banach space and assume the Banach space X_1 embeds densely and continuously in X_0. Let $p \in [1,\infty)$ and $v \in A_p$ with the convention that $v \equiv 1$ if $p = 1$. Let $A : \mathbb{R} \to \mathscr{L}(X_1, X_0)$ be such that for all $x \in X_1$, $t \mapsto A(t)x$ is strongly measurable, and there is a constant $C > 0$ such that

$$C^{-1}\|x\|_{X_1} \le \|x\|_{X_0} + \|A(t)x\|_{X_0} \le C\|x\|_{X_1}.$$

The above implies that each $A(t)$ is a closed operator on X_0 with $D(A(t)) = X_1$. Note that whenever A is given on an interval $I \subseteq \mathbb{R}$, we may always extend it constantly or periodically to all of $\mathbb{R}$.

Before we state the main result we will present some preliminary results on evolution equations with time-dependent A.

3.3.1. Preliminaries on evolution equations

Preliminaries on evolution equations and evolution families were given in Section 2.4. We explain here some parts that are different in our set-up.

For a strongly measurable function $f : (a,b) \to X_0$ we consider:

$$\begin{cases} u'(t) + A(t)u(t) = f(t), \ t \in (a,b) \\ u(a) = x, \end{cases} \tag{3.3.2}$$

where $u(a) = x$ is omitted if $a = -\infty$.

1. Assume $-\infty < a < b < \infty$. The function u is said to be a *strong solution* of (3.3.2) if $u \in W^{1,1}(a,b;X_0) \cap L^1(a,b;X_1) \cap C([a,b];X_0)$, $u(a) = x$ and (3.3.2) holds for almost all $t \in (a,b)$.

2. Assume $a = -\infty$ and $b < \infty$. The function u is said to be a *strong solution* of (3.3.2) if $u \in W^{1,1}_{\mathrm{loc}}(a,b;X_0) \cap L^1_{\mathrm{loc}}(a,b;X_1) \cap C((a,b];X_0)$ and $\lim_{s\to a} u(s) = 0$ and (3.3.2) holds for almost all $t \in (a,b)$.

3. Assume $b = \infty$. The function u is said to be a *strong solution* of (3.3.2) if for every $T > a$ the restriction to $[a,T]$ or $(a,T]$ yield strong solutions in the sense of (1) and (2) respectively.

Note the following simple embedding result for general A_p-weights.

Lemma 3.3.1. *Let $p \in [1,\infty)$ and let $v \in A_p$, where $v \equiv 1$ if $p = 1$. For $-\infty < a < b < \infty$, $W^{1,p}((a,b),v;X_0) \hookrightarrow C([a,b];X_0)$ and*

$$\|u\|_{C([a,b];X_0)} \le C\|u\|_{W^{1,p}((a,b),v;X_0)}.$$

Proof. Since $L^p((a,b),v;X_0) \hookrightarrow L^1(a,b;X_0)$, and $u(t)-u(s) = \int_s^t u'(r)\,dr$, the continuity of u is immediate. Moreover,

$$\|u(t)\| \le \|u(s)\| + \int_s^t \|u'(r)\|\,dr \le \|u(s)\| + C\|u'\|_{L^p((a,b),v;X_0)}.$$

Taking $L^p((a,b),v)$-norms with respect to the s-variable yields the result. □

There is a correspondence between the evolution problem (3.3.2) and evolution families as defined below. Recall that a function is called strongly continuous if it is continuous in the strong operator topology.

Definition 3.3.2. Let $(A(t))_{t\in\mathbb{R}}$ be as in (A). A two parameter family of bounded linear operators $S(t,s)$, $s \le t$, on a Banach space X_0 is called an *evolution family for A* if the following conditions are satisfied:

(i) $S(s,s) = I$, $S(t,r)S(r,s) = S(t,s)$ for $s \le r \le t$;

(ii) $(t,s) \to S(t,s)$ is strongly continuous for $s \le t$.

(iii) For all $s \in \mathbb{R}$ and $T \in (s,\infty)$, for all $x \in X_1$, the function $u:[s,T] \to X_0$ defined by $u(t) = S(t,s)x$ is in $L^1(s,T;X_1)\cap W^{1,1}(s,T;X_0)$ and satisfies $u'(t)+A(t)S(t,s)x = 0$ for almost all $t \in (s,T)$.

(iv) For all $t \in \mathbb{R}$ and $T \in (-\infty,t]$ for all $x \in X_1$, the function $u:[T,t] \to X_0$ defined by $u(s) = S(t,s)x$ is in $L^1(T,t;X_1)\cap W^{1,1}(T,t;X_0)$ and satisfies $u'(s) = S(t,s)A(s)x$.

Note that (iii) says that u is a strong solution of (3.3.2) with $f = 0$.

The above definition differs from the usual one from the literature, because $t \mapsto A(t)$ is only assumed to be measurable in time. Therefore, one cannot expect $S(t,s)x$ to be differentiable in the classical sense.

Example 3.3.3. Assume $A:\mathbb{R}\to\mathscr{L}(X_1,X_0)$ is strongly measurable and satisfies (A). Define a family of operators $\mathscr{A}$ by

$$\mathscr{A} = \{A(t) : t\in\mathbb{R}\}\cup\Big\{\frac{1}{t-s}\int_s^t A(r)\,dr : s<t\Big\}.$$

Here we use the strong operator topology to define the integral. Assume there exist ϕ, M and N such that all $B\in\mathscr{A}$ are all sectorial of angle $\phi<\pi/2$ and for all $\lambda\in\Sigma_\phi$,

$$\|\lambda(\lambda+B)^{-1}\|\leq M \quad\text{and}\quad \|x\|_{X_1}\leq N(\|x\|_{X_0}+\|Bx\|_{X_0})$$

Assume for every $B_1,B_2\in\mathscr{A}$ and $\lambda,\mu\in\Sigma_\phi$, the operators $(\lambda+B_1)^{-1}$ and $(\mu+B_2)^{-1}$ commute. Define

$$S(t,s)=e^{-(t-s)A_{st}},\ \text{where } A_{st}=\frac{1}{t-s}\int_s^t A(r)\,dr.$$

Then S is an evolution family for A. Here the exponential operator is defined by the usual Cauchy integral (see [113, Chapter 2]). Usually, no simple formula for S is available if the operators in $\mathscr{A}$ do not commute.

Note that in this special case the kernel $K(t,s)=\mathbf{1}_{\{s<t\}}A(0)e^{-\lambda(t-s)}S(t,s)$ satisfies the Calderón-Zygmund estimates of [79]. Indeed, note that

$$\frac{\partial K}{\partial t}=-\mathbf{1}_{\{s<t\}}(\lambda+A(t))A(0)e^{-\lambda(t-s)}S(t,s)$$

and

$$\frac{\partial K}{\partial s}=\mathbf{1}_{\{s<t\}}(\lambda+A(s))A(0)e^{-\lambda(t-s)}S(t,s).$$

Now since for all $r\in\mathbb{R}$ and $B\in\mathscr{A}$, $\|A(r)x\|\leq NC(\|x\|_{X_0}+\|Bx\|_{X_0})$, we find that for all $r,\tau\in\mathbb{R}$ and $s<t$ letting $\sigma=(t+s)/2$,

$$\begin{aligned}\|A(r)A(\tau)S(t,s)\| &= \|A(r)S(t,\sigma)\|\,\|A(\tau)S(\sigma,s)\|\\ &\leq N^2C^2(1+\|A_{\sigma t}S(t,\sigma)\|)(1+\|A_{s\sigma}S(\sigma,s)\|)\\ &\leq C'(1+(t-s)^{-1})^2\leq \tfrac{3}{2}C'(1+(t-s)^{-2}).\end{aligned}$$

Therefore, the extrapolation results from the unweighted case to the weighted case of Remark 3.2.8 does hold in this situation.

Proposition 3.3.4. *Let S be an evolution family for A. Fix $x\in X_0$ and $f\in L^1(a,b;X_0)$. If* (3.3.2) *has a strong solution $u\in L^1(a,b;X_1)\cap W^{1,1}(a,b;X_0)\cap C([a,b];X_0)$, then it satisfies*

$$u(t)=S(t,s)u(s)+\int_s^t S(t,r)f(r)\,dr,\quad a<s\leq t<b, \tag{3.3.3}$$

where we allow $s=a$ and $t=b$ whenever these are finite numbers. In particular, strong solutions are unique if $a>-\infty$. In the case $a=-\infty$ this remains true if $\lim_{s\to-\infty}\|S(t,s)\|=0$.

A partial converse is used and proved in Theorem 3.3.8.

Proof. Fix $a < s < t < b$. We claim that for $u \in C^1([s,b];X_1)$, $r \mapsto S(t,r)u(r)$ is in $W^{1,1}(s,b;X_0)$ and

$$\frac{d}{dr}[S(t,r)u(r)] = -S(t,r)A(r)u(r) + S(t,r)u'(r), \quad r \in (s,T). \tag{3.3.4}$$

Indeed, let $\phi \in C_c^1(s,t)$ be a test function. For a function $u = \psi x$ with $\psi \in C^1([s,b])$ and $x \in X_1$, it follows from $\frac{d}{dr}S(t,r)x = -S(t,s)A(r)x$ that

$$\begin{aligned}\int_s^t S(t,r)\frac{d}{dr}[u(r)\phi(r)]\,dr &= \int_s^t S(t,r)x\frac{d}{dr}[\psi(r)\phi(r)]\,dr\\ &= \int_s^t S(t,r)A(r)x\psi(r)\phi(r)\,dr\\ &= \int_s^t S(t,r)A(r)u(r)\phi(r)\,dr.\end{aligned}$$

Now (3.3.4) follows for this special choice of u since $\frac{d}{dr}[u(r)\phi(r)] = u'(r)\phi(r) + u(r)\phi'(r)$. By linearity and density (3.3.4) extends to all $u \in C^1([s,b];X_1)$.

Again by density (3.3.4) extends to all $u \in W^{1,1}([s,b];X_0) \cap L^1(s,b;X_1)$. Indeed, given such a u we can extend it to a function $u \in W^{1,1}(\mathbb{R};X_0) \cap L^1(\mathbb{R};X_1)$. Now a simple mollifier argument shows that we can approximate u by a sequence of functions in $C^1([s,b];X_1)$ in the norm of $W^{1,1}(\mathbb{R};X_0) \cap L^1(\mathbb{R};X_1)$.

Applying (3.3.4) to the strong solution u of (3.3.2), yields

$$\frac{d}{dr}[S(t,r)u(r)] = S(t,r)f(r).$$

Integrating this identity over (s,t), we find (3.3.3).

If $a > -\infty$, then we may take $s = a$ in the above proof and hence we can replace $u(s) = u(a)$ by the initial value x. If $a = \infty$, the additional assumption on S allows us to let $s \to -\infty$ to obtain

$$u(t) = \int_{-\infty}^t S(t,r)f(r)\,dr, \quad t < b.$$

□

Corollary 3.3.5. *If S_1 and S_2 are both evolution families for A, then $S_1 = S_2$.*

Proof. Fix $x \in X_1$ and $a < s < T < b$. By definition $u(t) = S_1(t,s)x$ for $t \in [s,T]$, is a strong solution of $u'(t) = A(t)u(t)$ and $u(s) = x$. As S_2 is an evolution family for A, Proposition 3.3.4 yields that $u(t) = S_2(t,s)x$ for $t \in [s,T]$. Therefore, $S_1(t,s)x = S_2(t,s)x$ and the result follows from the fact that X_1 is dense in X_0. □

3.3.2. *Assumptions on* A

The following condition can be interpreted as an abstract ellipticity condition.

(E) Assume that X_0 has finite cotype and assume that there exists $A_0 \in \mathscr{L}(X_1, X_0)$ which has a bounded H^∞-calculus of angle $\sigma < \pi/2$ and that there is a constant $C > 0$ such that

$$C^{-1}\|x\|_{X_1} \le \|x\|_{X_0} + \|A_0 x\|_{X_0} \le C\|x\|_{X_1}.$$

Assume moreover that there exists a strongly continuous evolution family $(T(t,s))_{s\le t}$ for $(A(t) - A_0)_{t\in\mathbb{R}}$ such that e^{-rA_0} commutes with $T(t,s)$ for every $t \ge s$ and $r \in \mathbb{R}_+$ and assume there exists an $\omega \in \mathbb{R}$ such that

$$\|T(t,s)\|_{\mathscr{L}(X_0)} \le Me^{\omega(t-s)}, \quad s \le t.$$

Set $T(t,s) = 0$ for $t < s$. The following $\mathscr{R}$-boundedness condition will be used.

(Rbdd) Assume that the family $\mathscr{I} := \{I_{\omega,kT} : \ k \in \mathscr{K}\} \subseteq \mathscr{L}(L^p(\mathbb{R}, v, X_0))$ is $\mathscr{R}$-bounded, where for $k \in \mathscr{K}$ and $f \in L^p(\mathbb{R}, v; X_0)$,

$$I_{\omega,kT} f(t) := \int_{\mathbb{R}} k(t-s)e^{-\omega|t-s|}T(t,s)f(s)ds.$$

Remark 3.3.6.

1. By (A) and (E) there is a constant C such that

$$\begin{aligned} C^{-1}(\|A(t)x\|_{X_0} + \|x\|_{X_0}) &\le \|A_0x\|_{X_0} + \|x\|_{X_0} \\ &\le C(\|A(t)x\|_{X_0} + \|x\|_{X_0}), \quad t \in \mathbb{R} \end{aligned} \tag{3.3.5}$$

   and both norms are equivalent to $\|x\|_{X_1}$.

2. For m even, if the $A(t)$ are m-th order elliptic operators with x-independent coefficients one typically takes $A_0 = \delta(-\Delta)^m$ with $\delta > 0$ small enough.

3. For $p, q \in (1,\infty)$, $v \in A_p$ and $X = L^q$, the $\mathscr{R}$-boundedness assumption follows from the weighted boundedness of $T(t,s)$ for all $w \in A_q$ (see Theorem 3.1.4).

4. Although we allow $p = 1$ and $v = 1$ in the above assumptions, checking the assumption (Rbdd) seems more difficult in this limiting case.

Lemma 3.3.7. *Under the assumptions (A) and (E) the evolution family S for A uniquely exists and satisfies*

$$\begin{aligned} S(t,s) &= e^{-(t-s)A_0}T(t,s) \\ &= T(t,s)e^{-(t-s)A_0} = e^{-\frac{1}{2}(t-s)A_0}T(t,s)e^{-\frac{1}{2}(t-s)A_0}, \quad s \le t, \end{aligned} \tag{3.3.6}$$

and there is a constant C such that for all $s \le t$, $\|S(t,s)\|_{\mathscr{L}(X_0)} \le Ce^{\omega(t-s)}$. Moreover, there is a constant C such that,

$$\|S(t,s)\|_{\mathscr{L}(X_1)} \le C\|S(t,s)\|_{\mathscr{L}(X_0)}, \quad s \le t.$$

Proof. The second identity follows from (E). To prove the first identity, we check that $S(t,s)$ given by (3.3.6) is an evolution family for A. By Corollary 3.3.5 this would complete the proof. It is simple to check properties (i) and (ii) of Definition 3.3.2 and it remains to check (iii) and (iv). Let $x \in X_1$. By the product rule for weak derivatives and (E) we find

$$\begin{aligned}\frac{d}{dt}S(t,s)x &= -A_0 e^{-(t-s)A_0}T(t,s)x - (A(t)-A_0)T(t,s)e^{-(t-s)A_0}x \\ &= -A_0S(t,s)x - (A(t)-A_0)S(t,s)x = -A(t)S(t,s)x.\end{aligned}$$

Similarly, one checks that $\frac{d}{ds}S(t,s)x = S(t,s)A(s)x$. The fact that $S(t,s)$ satisfies the same exponential estimate as $T(t,s)$ follows from the estimate (2.3.1) applied to A_0.

By assumptions, for every $x \in X_1$, $e^{-rA_0}S(t,s)x = S(t,s)e^{-rA_0}x$. Thus, by differentiation we find $-A_0S(t,s)x = -S(t,s)A_0x$ and therefore

$$\begin{aligned}\|S(t,s)x\|_{X_1} &\le C(\|A_0S(t,s)x\|_{X_0} + \|S(t,s)x\|_{X_0}) \\ &\le C(\|S(t,s)A_0x\|_{X_0} + \|S(t,s)x\|_{X_0}) \\ &\le C\|S(t,s)\|_{\mathscr{L}(X_0)}(\|A_0x\|_{X_0} + \|x\|_{X_0}) \le C'\|S(t,s)\|_{\mathscr{L}(X_0)}\|x\|_{X_1}.\end{aligned}$$

□

3.3.3. *Main result on maximal L^p-regularity*

Next we will present our main abstract result on the regularity of the strong solution to the problem

$$u'(t) + (A(t)+\lambda)u(t) = f(t), \quad t \in \mathbb{R}. \tag{3.3.7}$$

Theorem 3.3.8. *Assume (A), (E), and (Rbdd). For any $\lambda > \omega$ and for every $f \in L^p(\mathbb{R}, v; X_0)$ there exists a unique strong solution $u \in W^{1,p}(\mathbb{R}, v; X_0) \cap L^p(\mathbb{R}, v; X_1)$ of (3.3.7). Moreover, there is a constant C independent of f and λ such that*

$$\begin{aligned}(\lambda-\omega)\|u\|_{L^p(\mathbb{R},v,X_0)} + \|A_0u\|_{L^p(\mathbb{R},v;X_0)} &\le C\|f\|_{L^p(\mathbb{R},v;X_0)} \\ \|u'\|_{L^p(\mathbb{R},v;X_0)} &\le \tfrac{C(\lambda-\omega+1)}{\lambda-\omega}\|f\|_{L^p(\mathbb{R},v;X_0)}.\end{aligned} \tag{3.3.8}$$

Remark 3.3.9. Parts of the theorem can be extended to $\lambda = \omega$, but we will not consider this in detail. The constant in the estimate (3.3.8) for u' can be improved if one knows $\|A(t)x\|_{X_0} \le C\|A_0x\|_{X_0}$ or when taking $\lambda \ge \omega + 1$ for instance.

Before, we turn to the proof of Theorem 3.3.8 we introduce some shorthand notation. Let $S_\lambda(t,s) = e^{-\lambda(t-s)}S(t,s)$ and $T_\lambda(t,s) = e^{-\lambda(t-s)}T(t,s)$. Since by Lemma

3.3.7, S is an evolution family for A, also S_λ is the evolution family for $A(t)+\lambda$. Similarly, $T_\lambda(t,s)$ is an evolution family for $A(t)-A_0+\lambda$. By (3.3.3) if the support of $f\in L^1(\mathbb{R};X_0)$ is finite, a strong solution of (3.3.7) satisfies

$$u(t)=\int_{-\infty}^{t} S_\lambda(t,r)f(r)\,dr,\quad t\in\mathbb{R}. \tag{3.3.9}$$

Proof. Replacing $A(t)$ and $T(t,s)$ by $A(t)+\omega$ and $e^{-(t-s)\omega}T(t,s)$ one sees that without loss of generality we may assume $\omega=0$ in (E) and (Rbdd). We first prove that u given by (3.3.9), is a strong solution and (3.3.8) holds. First let $f\in L^p(\mathbb{R},v;X_1)$ and such that f has support on the finite interval $[a,b]$. Later on we use a density argument for general $f\in L^p(\mathbb{R},v;X_0)$. Let u be defined as in (3.3.9). Note that $u=0$ on $(-\infty,a]$.

Step 1: By Lemma 3.3.7 the function u defined by (3.3.9) satisfies

$$\begin{aligned}\|u(t)\|_{X_1}&\le\int_{-\infty}^{t}\|S_\lambda(t,s)\|_{\mathscr{L}(X_1)}\|f(s)\|_{X_1}\,ds\\&\le C'\|f\|_{L^1(a,b;X_1)}\le C_v\|f\|_{L^p(\mathbb{R},v;X_1)}.\end{aligned}$$

We show that u is a strong solution of (3.3.7). Observe that from Fubini's Theorem and $\frac{d}{ds}S_\lambda(s,r)x=-(\lambda+A(s))S_\lambda(s,r)x$ for $x\in X_1$, we deduce

$$\begin{aligned}\int_{-\infty}^{t}(\lambda+A(s))u(s)\,ds&=\int_{-\infty}^{t}\int_{-\infty}^{s}(\lambda+A(s))S_\lambda(s,r)f(r)\,dr\,ds\\&=\int_{-\infty}^{t}\int_{r}^{t}(\lambda+A(s))S_\lambda(s,r)f(r)\,ds\,dr\\&=\int_{-\infty}^{t}(-S_\lambda(t,r)f(r)+f(r))dr=-u(t)+\int_{-\infty}^{t}f(r)dr.\end{aligned}$$

Therefore, u is a strong solution of (3.3.7).

Step 2: In this step we show there exists a $C\ge0$ independent of λ and f such that

$$\|A_0u\|_{L^p(\mathbb{R},v;X_0)}\le C\|f\|_{L^p(\mathbb{R},v;X_0)}. \tag{3.3.10}$$

By (3.3.6) and (3.3.9) we can write $A_0u=I_Kf$, where

$$K(t,s)=\frac{\phi((t-s)A_0)T_\lambda(t,s)\phi((t-s)A_0)}{t-s}.$$

Here $\phi\in H_0^\infty(\Sigma_{\sigma'})$ for $\sigma'<\pi/2$ is given by $\phi(z)=z^{1/2}e^{-z/2}$. In order to apply Theorem 3.2.4, we note that all assumptions (H1)-(H4) are satisfied. Only the $\mathscr{R}$-boundedness condition (H4) requires some comment. Note that $k\in\mathscr{K}$ implies that for all $\lambda\ge0$, $k_\lambda\in\mathscr{K}$ where $k_\lambda(t)=e^{-\lambda t}\mathbf{1}_{\{t>0\}}k(t)$. Therefore, it follows from (Rbdd) that for all $\lambda\ge0$,

$$\mathscr{R}(I_{kT_\lambda}:k\in\mathscr{K})=\mathscr{R}(I_{k_\lambda T}:k\in\mathscr{K})\le\mathscr{R}(I_{kT}:k\in\mathscr{K})<\infty$$

which gives (H4) with a uniform estimate in λ. Now (3.3.10) follows from Theorem 3.2.4.

Step 3: In this step we show there exists a $C \ge 0$ independent of λ and f such that

$$\lambda \|u\|_{L^p(\mathbb{R},v;X_0)} \le C\|f\|_{L^p(\mathbb{R},v;X_0)}. \tag{3.3.11}$$

Using (3.3.9) and $\|S(t,s)\| \le C$ we find

$$\lambda \|u\|_{X_0} \le \int_{-\infty}^{t} \lambda \|S_\lambda(t,s)f(s)\|_{X_0}\, ds \le C\int_{-\infty}^{t} \lambda e^{-\lambda(t-s)}\|f(s)\|_{X_0} ds \le C r_\lambda * g(t),$$

where $r_\lambda(t) = \lambda e^{-\lambda|t|}$ and $g(s) = \|f(s)\|_{X_0}$ As $r_1 \in L^1(\mathbb{R})$ is radially decreasing by [70, Theorem 2.1.10] and [71, Theorem 9.1.9],

$$\begin{aligned}\lambda \|u\|_{L^p(\mathbb{R}_+,v;X_0)} &\le C\|r_\lambda * g\|_{L^p(\mathbb{R},v)} \\ &\le C\|Mg\|_{L^p(\mathbb{R},v)} \le C'\|g\|_{L^p(\mathbb{R},v)} = C'\|f\|_{L^p(\mathbb{R},v;X_0)}\end{aligned}$$

in the case $p > 1$. The case $p = 1$ follows from Fubini's theorem and the convention $v \equiv 1$. This estimate yields (3.3.11).

Step 4: To prove the estimate for u' note that $u' = -\lambda u - Au + f$, and hence writing $Z = L^p(\mathbb{R}, v; X_0)$, by (3.3.5) and (3.3.8), we obtain

$$\begin{aligned}\|u'\|_Z &\le \lambda\|u\|_Z + \|Au\|_Z + \|f\|_Z \\ &\le (\lambda + C)\|u\|_Z + C\|A_0 u\|_Z + \|f\|_Z \le K\Big(\frac{\lambda + C}{\lambda - \omega} + 1\Big)\|f\|_Z.\end{aligned}$$

This finishes the proof of (3.3.8) for $f \in L^p(\mathbb{R}; X_1)$ with support in $[a,b]$

Step 5: Now let $f \in L^p(\mathbb{R}, v; X_0)$. Choose for $n \ge 1$, $f_n \in L^p(\mathbb{R}, v; X_1)$ with compact support and such that $f_n \to f$ in $L^p(\mathbb{R}, v; X_0)$. For each $n \ge 1$ let u_n be the corresponding strong solution of (3.3.7) with f replaced by f_n. From (3.3.8) applied to $u_n - u_m$ we can deduce that $(u_n)_{n\ge 1}$ is a Cauchy sequence and hence convergent to some $\overline{u}$ in $L^p(\mathbb{R}, v; X_1) \cap W^{1,p}(\mathbb{R}, v; X_0)$. On the other hand, for u defined as in (3.3.9) one can show in the same way as in Step 3 that for almost all $t \in \mathbb{R}$,

$$\begin{aligned}\|u(t) - u_n(t)\| &\le \int_{-\infty}^{t} \|S_\lambda(t,s)\|\,\|f(s) - f_n(s)\|\, ds \\ &\le C\int_{-\infty}^{t} e^{-\lambda(t-s)}\|f(s) - f_n(s)\|\, ds \le C\, M(\|f - f_n\|)(t),\end{aligned}$$

where M is the Hardy-Littlewood maximal operator. Taking $L^p(v)$-norms and using the boundedness of the maximal operator we find $u_n \to u$ in $L^p(\mathbb{R}, v; X_0)$ and hence $u = \overline{u}$ if $p \in (1,\infty)$. Taking limits (along a subsequence), (3.3.7) and (3.3.8) follow if $p \in (1,\infty)$. The case $p = 1$ is proved similarly using Young's inequality. $\square$

It will be convenient to restate our results in terms of maximal L^p_v-regularity. For $-\infty \le a < b \le \infty$, let

$$\mathrm{MR}^p((a,b),v) = W^{1,p}((a,b),v;X_0) \cap L^p((a,b),v;X_1).$$

Definition 3.3.10. Let $-\infty \le a < b \le \infty$. Assume (A) holds and let $p \in [1,\infty)$ and $v \in A_p$ with the convention that $v \equiv 1$ if $p = 1$. The operator-valued function A is said to have *maximal L^p_v-regularity on (a,b)* if for all $f \in L^p((a,b),v;X_0)$, the problem

$$\begin{cases} u'(t) + A(t)u(t) = f(t), & t \in (a,b) \\ u(a) = 0, \end{cases} \tag{3.3.12}$$

has a unique strong solution $u : (a,b) \to X_0$ and there is a constant C independent of f such that

$$\|u\|_{\mathrm{MR}^p((a,b),v)} \le C\|f\|_{L^p((a,b),v;X_0)}. \tag{3.3.13}$$

Here we omit the condition $u(a) = 0$ if $a = -\infty$.

Of course, the reverse estimate of (3.3.13) holds trivially. Note that maximal L^p_v-regularity on (a,b) implies maximal L^p_v-regularity on $(c,d) \subseteq (a,b)$. It is also easy to check that if $|b-a| < \infty$, the maximal L^p_v-regularity on (a,b) for A and $\lambda + A$ are equivalent (see first part of the proof of Proposition 3.3.11). If one additionally assumes that A generates an evolution family then one can obtain a uniform estimate in the additional parameter λ.

Proposition 3.3.11. *Assume $p \in (1,\infty)$. Assume $-\infty < a < b < \infty$. Assume (A) and assume A generates a strongly continuous evolution family S. If there is a $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 + A$ has maximal L^p_v-regularity on (a,b), then for every $\lambda \in \mathbb{R}$, $\lambda + A$ has maximal L^p_v-regularity on (a,b). Moreover, there is a constant C such that for every $\lambda \ge \lambda_0$ and $f \in L^p((a,b),v;X_0)$, the unique strong solution $u \in MR^p((a,b),v)$ satisfies*

$$|\lambda|\|u\|_{L^p((a,b),v;X_0)} + \|u\|_{MR^p((a,b),v)} \le C\|f\|_{L^p((a,b),v;X_0)}. \tag{3.3.14}$$

When it is unclear whether A generates an evolution family, then the above result still holds, but it becomes unclear whether (3.3.14) holds with a uniform estimate in λ.

Proof. In all evolution equations below we assume zero initial conditions.

The existence and uniqueness of a strong solution follows from the observation that the solutions of $u'(t) + (\lambda + A(t))u(t) = f(t)$ and $w'(t) + A(t)w(t) = e^{\lambda t}f(t)$ are connected by the identity $u(t) = e^{-\lambda t}w(t)$.

To prove the required estimate let $\lambda \in \mathbb{R}$ be arbitrary and note that the strong solution of $u'(t) + (\lambda + A(t))u(t) = f(t)$ satisfies

$$u'(t) + (\lambda_0 + A(t))u(t) = f(t) + (\lambda_0 - \lambda)u(t).$$

Therefore, the estimate (3.3.13) yields

$$\|u\|_{\mathrm{MR}^p((a,b),v)} \le C\Big[\|f\|_{L^p((a,b),v;X_0)} + |\lambda_0-\lambda|\|u\|_{L^p((a,b),v;X_0)}\Big].$$

It remains to estimate $|\lambda_0-\lambda|\|u\|_{L^p((a,b),v;X_0)}$. First consider $\lambda > 0$. Since $u(t) = \int_0^t e^{-\lambda(t-s)} S(t,s) f(s)\,ds$ it follows in the same way as in the proof of (3.3.11) that

$$\lambda|u(t)|_{X_0} \le C r_\lambda * g(t),$$

where $g = \mathbf{1}_{(a,b)} f$ and we used the uniform boundedness of S on finite time intervals. Now the proof can be finished in the same way as was done for (3.3.11). For $\lambda \in [\lambda_0, 0]$ the result is obvious as $|\lambda_0-\lambda| \le |\lambda_0|$. □

Remark 3.3.12. A version of this result holds on infinite time intervals if one takes $\lambda > \omega$ where ω is such that $\|S(t,s)\| \le Me^{\omega(t-s)}$ for $-a < s \le t < b$. This follows from the same argument as in the above proof. However, one needs to start with f with compact support, and use a density argument at the end. This is in order to make sure that $t \mapsto e^{\lambda t} f(t)$ is in $L^p((a,b),v;X_0)$ again.

The result of Theorem 3.3.8 immediately implies that

Corollary 3.3.13. *Assume (A), (E) and (Rbdd). For any $\lambda > \omega$, $\lambda + A$ has maximal L^p_v-regularity on $\mathbb{R}$.*

Actually the constant in the estimate can be taken uniformly in λ. Indeed, for fixed $\lambda_0 > \omega$ by (3.3.8) and Remark 3.3.9, there is a constant C such that for all $\lambda \ge \lambda_0$ and for all $f \in L^p(\mathbb{R}_+, v; X_0)$,

$$\|u\|_{\mathrm{MR}^p(\mathbb{R},v)} \le C\|f\|_{L^p(\mathbb{R},v;X_0)}. \tag{3.3.15}$$

This is a maximal regularity estimate with constant which is uniform in λ.

Remark 3.3.14. If A is time independent and has an H^∞-calculus of angle $< \pi/2$, then setting $A_0 = A$, and $T(t,s) = I$, Theorem 3.3.8 yields a maximal regularity result for autonomous equations. There are much more suitable ways to derive maximal L^p-regularity results in the autonomous case (see [94, 104, 152, 153]), using less properties of the operator A. Indeed, only $\mathscr{R}$-sectoriality of A is needed, but the Banach space X_0 is assumed to be a UMD space. We assume more on the operator but less on the space as we only require finite cotype of X_0 and the $\mathscr{R}$-boundedness of a certain integral operator. Another theory where no assumptions on the Banach space are made but even more on the operator, can be found in [92]. In the above mentioned works only maximal L^p-regularity on $\mathbb{R}_+$ is considered, but by a standard trick due to Kato one can always reduce to this case (see for instance the proof of [53, Theorem 7.1]). For the case of time-dependent operators this is no longer true.

For convenience of the reader we state the following consequence in the autonomous case.

Corollary 3.3.15. *Let $p \in [1,\infty)$ and $v \in A_p$. Assume $A \in H^\infty(\Sigma_\sigma)$ with $\sigma < \pi/2$. Assume the family $\mathscr{I} := \{I_k : k \in \mathscr{K}\} \subset \mathscr{L}(L^p(\mathbb{R}, v, X_0))$ is $\mathscr{R}$-bounded, where for $k \in \mathscr{K}$ and $f \in L^p(\mathbb{R}, v; X_0)$,*

$$I_k f(t) := \int_{\mathbb{R}} k(t-s) f(s) ds. \tag{3.3.16}$$

Then for every $f \in L^p(\mathbb{R}_+, v; X_0)$ there exists a unique strong solution $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}_+, v; X_0) \cap L^p_{\mathrm{loc}}(\mathbb{R}_+, v; X_1)$ of the problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in \mathbb{R}_+ \\ u(0) = 0, \end{cases} \tag{3.3.17}$$

Moreover, there is a constant C independent of f such that

$$\|u'\|_{L^p(\mathbb{R}_+, v, X_0)} + \|Au\|_{L^p(\mathbb{R}_+, v; X_0)} \leq C \|f\|_{L^p(\mathbb{R}_+, v; X_0)}. \tag{3.3.18}$$

We show with the following proposition that one can use standard extrapolation techniques to obtain maximal L^p_v-regularity from the unweighted case. This is a consequence of Theorem 3.3.8 combined with the extrapolation results of [79].

Proposition 3.3.16. *Assume (A), (E) and (Rbdd) with $v = 1$ and $p \in (1,\infty)$. Assume $\|A_0 S(t,s)\| \leq C(t-s)^{-1}$. Then for any $v \in A_p(\mathbb{R})$ and any $\lambda > \omega$, $\lambda + A$ has maximal L^p_v-regularity on $\mathbb{R}$.*

Proof. As in the proof of Theorem 3.3.8 it is enough to prove the estimate

$$\|A_0 u\|_{L^p(\mathbb{R}, v; X_0)} \leq C \|f\|_{L^p(\mathbb{R}, v; X_0)}, \quad \forall\ v \in A_p(\mathbb{R}).$$

The result then follows in the same way as Corollary 3.3.13 from Theorem 3.3.8.

Assume first $v = 1$. As in the proof of Theorem 3.3.8, we can write $A_0 u = I_K f$, with

$$K(t,s) = \mathbf{1}_{\{s<t\}} A_0 e^{-\lambda(t-s)} S(t,s).$$

Note that in the special case in which $\|A_0 S(t,s)\| \leq C(t-s)^{-1}$, the kernel $K(t,s)$ satisfies the Calderón-Zygmund estimates of [79, Definition 2.5]. Moreover, by Theorem 3.3.8 it holds that $\|A_0 u\|_{L^p(\mathbb{R}; X_0)} \leq C\|f\|_{L^p(\mathbb{R}; X_0)}$, which implies that I_K is a vector valued Calderón-Zygmund operator (see [79, Definition 2.6]). We can thus apply [79, Corollary 2.10] to get that for all $v \in A_p$, $\|A_0 u\|_{L^p(\mathbb{R}, v; X_0)} \leq C\|f\|_{L^p(\mathbb{R}, v; X_0)}$, with C independent of f and λ, but depending on the A_p constant $[v]_p$. $\square$

3.3.4. Traces and initial values

Recall from Lemma 3.3.1 that any $u \in W^{1,p}((a,b),v;X_0)$ has a continuous version. We introduce certain interpolation spaces in order to give a more precise description of traces. Let $X_{v,p}$ be the space of all $x \in X_0$ for which there is a $u \in \mathrm{MR}^p(\mathbb{R}_+,v)$ such that $u(0) = x$. Let

$$\|x\|_{X_{v,p}} = \inf\{\|u\|_{\mathrm{MR}^p(\mathbb{R}_+,v)} : u(0) = x\}. \tag{3.3.19}$$

Spaces of this type have been studied in the literature (see [18, 23, 91] and references therein). Obviously, one has $X_1 \hookrightarrow X_{v,p} \hookrightarrow X_0$.

For $t \in \mathbb{R}$ and a weight v, let $v_t = v(\cdot - t)$. The following trace estimate on $\mathbb{R}_+$ is a direct consequence of the definitions. A similar assertions holds for $u \in \mathrm{MR}^p(\mathbb{R},v)$ for all $t \in \mathbb{R}$.

Proposition 3.3.17 (Trace estimate)**.** *For $u \in MR^p(\mathbb{R}_+,v)$, one has*

$$\|u(t)\|_{X_{v_t,p}} \le \|u\|_{MR^p(\mathbb{R}_+,v)}, \quad t \in [0,\infty).$$

A simple application of maximal regularity is that one can automatically consider non-zero initial values. Note that without loss of generality we can let $a = 0$.

Proposition 3.3.18. *Assume (A) and let $T \in (0,\infty]$. Assume A has maximal L^p_v-regularity on $(0,T)$ with constant K_A. For $x \in X_0$ and $f : (0,T) \to X_0$ strongly measurable the following are equivalent:*

(1) *The data satisfies $x \in X_{v,p}$ and $f \in L^p((0,T),v;X_0)$*

(2) *There exists a unique strong solution $u \in MR^p((0,T),v)$ of*

$$\begin{cases} u'(t) + A(t)u(t) = f(t), & t \in (0,T) \\ u(0) = x. \end{cases} \tag{3.3.20}$$

In this case there is a constant $c_{v,p,T}$ such that the following estimate holds:

$$\begin{aligned} \max\{c_{v,p,T}\|x\|_{X_{v,p}}, \|f\|_{L^p((0,T),v;X_0)}\} &\le \|u\|_{MR^p((0,T),v)} \\ &\le K_A\|x\|_{X_{v,p}} + K_A\|f\|_{L^p((0,T),v;X_0)}. \end{aligned} \tag{3.3.21}$$

Proof. (1) $\Rightarrow$ (2): Let $w \in \mathrm{MR}^p(\mathbb{R}_+,v)$ be such that $w(0) = x$. Let $g(t) = -(w'(t) + A(t)w(t))$. Then $g \in L^p((0,T),v;X_0)$. Let $\tilde{u}$ be the solution to (3.3.20) with zero initial value and with f replaced by $f+g$. Now $u(t) = \tilde{u}(t) + w(t)$ is the required strong solution of (3.3.20). Indeed, clearly $u(0) = x$ and

$$\begin{aligned} u'(t) + A(t)u(t) &= \tilde{u}'(t) + A(t)\tilde{u}(t) + w'(t) + A(t)w(t) \\ &= f + g - g = f. \end{aligned}$$

Moreover,

$$\begin{aligned}\|u\|_{\mathrm{MR}^p((0,T),v)} &\le \|\tilde{u}\|_{\mathrm{MR}^p((0,T),v)} + \|w\|_{\mathrm{MR}^p((0,T),v)} \\ &\le K_A\|f\|_{L^p((0,T),v;X_0)} + K_A\|w\|_{\mathrm{MR}^p(\mathbb{R}_+,v)}.\end{aligned}$$

Taking the infimum over all $w \in \mathrm{MR}^p(\mathbb{R}_+, v)$ with $w(0) = x$ also yields the second part of (3.3.21).

(2) $\Rightarrow$ (1): As u' and Au are both in $L^p((0,T), v; X_0)$, the identity in (3.3.20) yields that $f \in L^p((0,T), v; X_0)$ with the estimate as stated. To obtain the required properties for x note that $u \in \mathrm{MR}^p((0,T), v)$ can be extended to a function $u \in \mathrm{MR}^p(\mathbb{R}_+, v)$ with $c_{v,p,T}\|u\|_{\mathrm{MR}^p(\mathbb{R}_+,v)} \le \|u\|_{\mathrm{MR}^p((0,T),v)}$. In the case $T = \infty$ we can take $c_{v,p} = 1$. □

It can be difficult to identify $X_{v,p}$. For power weights this is possible. Including a power weight has become an important standard technique to allow non-smooth initial data and to create compactness properties. At the same time, the regularity properties of the solution to (3.3.20) for $t > 0$ are unchanged. Moreover, maximal regularity results with power weights are important in the study of non-linear PDEs. For more details and applications to evolution equations we refer to [72, 97, 113, 119, 120, 129].

Example 3.3.19. Assume $v(t) = t^\alpha$ with $\alpha \in (-1, p-1)$. Then $v \in A_p$ and $X_{v,p} = (X_0, X_1)_{1-\frac{1+\alpha}{p},p}$ (see [148, Theorem 1.8.2]). Here $(X_0, X_1)_{\theta,p}$ stands for the real interpolation space between X_0 and X_1. In the limiting cases $\alpha \uparrow p-1$ and $\alpha \downarrow -1$, one sees that the endpoint X_1 and X_0 can almost be reached.

As in [129] we find that for $\alpha \in [0, p-1)$, any $u \in \mathrm{MR}^p(\mathbb{R}_+, v)$ has a continuous version with values in $(X_0, X_1)_{1-\frac{1+\alpha}{p},p}$ and

$$\sup_{t\in\mathbb{R}_+} \|u(t)\|_{(X_0,X_1)_{1-\frac{1+\alpha}{p},p}} \le C\|u\|_{\mathrm{MR}^p(\mathbb{R}_+,v)}. \tag{3.3.22}$$

Indeed, this follows from the boundedness and strong continuity of the left-translation in $L^p(\mathbb{R}_+, v; (X_0, X_1)_{1-\frac{1+\alpha}{p},p})$ and Proposition 3.3.17.

On the other hand, for every $-1 < \alpha < p-1$ one has $u \in C((0,\infty); (X_0, X_1)_{1-\frac{1}{p},p})$ and for every $\varepsilon > 0$,

$$\sup_{t\in[\varepsilon,\infty)} t^{\alpha/p}\|u(t)\|_{(X_0,X_1)_{1-\frac{1}{p},p}} \le C\|t \mapsto t^{\alpha/p}u(t)\|_{\mathrm{MR}^p(\varepsilon,\infty)} \le C_\varepsilon\|u\|_{\mathrm{MR}^p(\mathbb{R}_+,v)},$$

where we used $t^{-p} \le \max\{1, \varepsilon^{-p}\}$. If additionally $u(0) = 0$, then by Hardy's inequality (see [80, p. 245-246]) we can take $\varepsilon = 0$ in the last estimate.

In the special case X_0 is a Hilbert space, Theorem 3.3.8 implies the following result.

Theorem 3.3.20. *Let X_0 be a Hilbert space. Assume $A : (0,\tau) \to \mathscr{L}(X_1, X_0)$ is such that for all $x \in X_1$, $t \mapsto A(t)x$ is measurable and*

$$c_1\|x\|_{X_1} \le \|x\|_{X_0} + \|A(t)x\|_{X_0} \le c_2\|x\|_{X_1}, \quad t \in (0,\tau), \; x \in X_1.$$

Assume there is an operator A_0 on X_0 with $D(A_0) = X_1$ which generates a contractive analytic semigroup $(e^{-zA_0})_{z\in\Sigma_\theta}$ which is such that $(A(t)-A_0)_{t\in(0,\tau)}$ generates an evolution family $(T(t,s))_{0\le s\le t\le\tau}$ on X_0 which commutes with $(e^{-rA_0})_{r\ge 0}$.

$$e^{-rA_0}T(t,s) = T(t,s)e^{-rA_0}, \;\; 0 \le s \le t \le \tau, \;\; r \ge 0.$$

Then A has maximal L^p-regularity for every $p \in (1,\infty)$, i.e. for every $f \in L^p(0,\tau;X_0)$ and $x \in (X_0,X_1)_{1-\frac{1}{p},p}$ there exists a unique strong solution $u \in L^p(0,\tau;X_1) \cap W^{1,p}(0,\tau;X_0) \cap C([0,\tau];(X_0,X_1)_{1-\frac{1}{p},p})$ of (3.3.20) *and there is a constant C independent of f and x such that*

$$\begin{aligned}\|u\|_{L^p(0,\tau;X_1)} + \|u\|_{W^{1,p}(0,\tau;X_0)} &+ \|u\|_{C([0,\tau];(X_0,X_1)_{1-\frac{1}{p},p})} \\ &\le C\|f\|_{L^p(0,\tau;X_0)} + C\|x\|_{(X_0,X_1)_{1-\frac{1}{p},p}}.\end{aligned}$$

Proof. First of all we may use a constant extension of A to an operator family on $\mathbb{R}$. Clearly, we can do this in such a way that $T(t,s)$ is uniformly bounded in $-\infty < s \le t < \infty$ say by a constant M. For instance one can take $A(t) = A_0$ for $t \notin (0,\tau)$. Assumption (A) is clearly satisfied. Note that by the assumption and [104, Theorem 11.13], A_0 has a bounded H^∞-calculus of angle $< \pi/2$ and hence (E) is satisfied.

By Proposition 3.1.2 $\{I_{kT} : k \in \mathscr{K}\}$ is uniformly bounded. For $p = 2$, this implies $\mathscr{R}$-boundedness of $\{I_{kT} : k \in \mathscr{K}\} \subseteq \mathscr{L}(L^2(\mathbb{R},v;X_0))$, because $L^2(\mathbb{R},v;X_0)$ is a Hilbert space. By Proposition 3.1.3 this implies that $\{I_{kT} : k \in \mathscr{K}\} \subseteq \mathscr{L}(L^p(\mathbb{R},v;X_0))$ is $\mathscr{R}$-bounded as well and hence condition (Rbdd) holds. Therefore, all the conditions of Theorem 3.3.8 are satisfied, and we find that A has maximal L^p_v-regularity on $\mathbb{R}$. This implies that A has maximal L^p_v-regularity on $(0,\tau)$, and hence the required result follows from Proposition 3.3.18 and Example 3.3.19. □

3.3.5. Perturbation and approximation

In this section we will illustrate how the additional parameter λ from (3.3.15) can be used to solve the perturbed problem

$$\begin{cases} u'(t) + A(t)u(t) + B(t,u(t)) = f(t), \; t \in (0,T) \\ u(0) = x. \end{cases} \tag{3.3.23}$$

Here $B : [0,T] \times X_1 \to X_0$ is such that there exists a constant $\varepsilon > 0$ small enough and constants $C, L \ge 0$ such that for all $x, y \in X_1$ and $t \in (0,T)$,

$$\begin{aligned}\|B(t,x) - B(t,y)\|_{X_0} &\le \varepsilon\|x-y\|_{X_1} + L_B\|x-y\|_{X_0}, \\ \|B(t,x)\|_{X_0} &\le C_B(1 + \|x\|_{X_1}).\end{aligned} \tag{3.3.24}$$

Recall that $\mathrm{MR}^p((0,T),v) = W^{1,p}((0,T),v;X_0) \cap L^p((0,T),v;X_1)$.

Proposition 3.3.21. *Assume $T<\infty$. Assume (A) holds and assume there is a λ_0 such that for all $\lambda \geq \lambda_0$, $\lambda + A$ has maximal L^p_v-regularity on $(0,T)$ and there is a constant $C_A > 0$ such that for all $\lambda \geq \lambda_0$ and $f \in L^p((0,T),v;X_0)$, the strong solution u to* (3.3.12) *satisfies*

$$\lambda \|u\|_{L^p((0,T),v;X_0)} + \|u\|_{MR^p((0,T),v)} \leq C_A \|f\|_{L^p((0,T),v;X_0)}. \tag{3.3.25}$$

Assume the constant from (3.3.24) *satisfies $\varepsilon < \frac{1}{C_A}$. Then for every $f \in L^p((0,T),v;X_0)$ and $x \in X_{v,p}$, there exists a unique strong solution $u \in MR^p((0,T),v)$ of* (3.3.23) *and*

$$\|u\|_{MR^p((0,T),v)} \leq C(1 + \|x\|_{X_{v,p}} + \|f\|_{L^p((0,T),v;X_0)}), \tag{3.3.26}$$

where C is independent of f and x.

The proof of this proposition is a standard application of the regularity estimate (3.3.25) combined with the Banach fixed point theorem.

Proof. Let $\lambda > 0$ be so large that $\frac{C_A L_B}{\lambda} < C_A \varepsilon := 1 - \theta$ and define the following equivalent norm on $\mathrm{MR}^p((0,T),v)$:

$$\|u\|_\lambda = \lambda \|u\|_{L^p((0,T),v;X_0)} + \|u\|_{\mathrm{MR}^p((0,T),v)}.$$

We will prove that for all $g \in L^p((0,T),v;X_0)$ and $x \in X_{v,p}$ there exists a unique strong solution $w \in \mathrm{MR}^p((0,T),v)$ of

$$w'(t) + (A(t) + \lambda) w(t) + \tilde{B}(t, w(t)) = g(t), \quad w(0) = x. \tag{3.3.27}$$

and that w satisfies the estimate (3.3.26) with (u,f) replaced by (w,g). Here $\tilde{B}(t,x) = e^{-\lambda t} B(t, e^{\lambda t} x)$ and note that $\tilde{B}$ satisfies the same Lipschitz estimate (3.3.24) as B. To see that the required result for (3.3.23) follows from this, note that there is a one-to-one correspondence between both problems given by $u(t) = e^{\lambda t} w(t)$ and $f = e^{\lambda t} g$. Therefore, from now it suffices to consider (3.3.27).

In order to solve (3.3.27) we use the maximal regularity estimate (3.3.25) combined with Proposition 3.3.18 and the special choice of λ. For $\phi \in \mathrm{MR}^p((0,T),v)$ we write $w = L(\phi)$, where $w \in \mathrm{MR}^p((0,T),v)$ is the unique strong solution of

$$w'(t) + (A(t) + \lambda) w(t) = g(t) - \tilde{B}(t, \phi(t)), \quad w(0) = x. \tag{3.3.28}$$

Then for $\phi_1, \phi_2 \in \mathrm{MR}^p((0,T),v)$, by (3.3.25) one has

$$\begin{aligned}
\|L(\phi_1) - L(\phi_2)\|_\lambda &\leq C_A \|\tilde{B}(\cdot,\phi_1) - \tilde{B}(\cdot,\phi_2)\|_{L^p((0,T),v;X_0)} \\
&\leq C_A \varepsilon \|\phi_1 - \phi_2\|_{L^p((0,T),v;X_1)} + C_A L_B \|\phi_1 - \phi_2\|_{L^p((0,T),v;X_0)} \\
&\leq (1-\theta) \|\phi_1 - \phi_2\|_\lambda.
\end{aligned}$$

Hence L is a contraction on $\mathrm{MR}^p((0,T),v)$ with respect to the norm $\|\cdot\|_\lambda$. Therefore, by the Banach fixed point theorem there is a unique $w \in \mathrm{MR}^p((0,T),v)$ such that

$L(w) = w$. It is clear that w is the required strong solution of (3.3.27). To prove the required estimate note that by (3.3.25) and Proposition 3.3.18 one has

$$\begin{aligned}\|w\|_\lambda = \|L(w)\|_\lambda &\le \|L(w) - L(0)\|_\lambda + \|L(0)\|_\lambda \\ &\le (1-\theta)\|w\|_\lambda + C_A(\|g\|_{L^p((0,T),v;X_0)} + C_B) + C\|x\|_{X_{v,p}}.\end{aligned}$$

Subtracting $(1-\theta)\|w\|_\lambda$ on both sides, and rewriting the estimate in terms of f and u gives the required result. □

A similar result holds on infinite time intervals if we replace the assumption (3.3.24) by the following condition:

$$\begin{aligned}\|B(t,x) - B(t,y)\|_{X_0} &\le \varepsilon\|x-y\|_{X_1} + L_B\|x-y\|_{X_0}, \\ \|B(t,x)\|_{X_0} &\le C_B\|x\|_{X_1}.\end{aligned} \tag{3.3.29}$$

This is to make sure that $t \mapsto B(t,u(t)) \in L^p(\mathbb{R},v;X_0)$ if $u \in L^p(\mathbb{R},v;X_1)$. Consider for λ large enough:

$$u'(t) + (\lambda + A(t))u(t) + B(t,u(t)) = f(t), \ t \in \mathbb{R}. \tag{3.3.30}$$

Proposition 3.3.22. *Assume (A) holds and assume there is a λ_0 such that for all $\lambda \ge \lambda_0$, $\lambda + A$ has maximal L^p_v-regularity on $\mathbb{R}$ and there is a constant $C_A > 0$ such that for all $\lambda \ge \lambda_0$ and $f \in L^p(\mathbb{R},v;X_0)$, the strong solution u to* (3.3.12) *satisfies*

$$\lambda\|u\|_{L^p(\mathbb{R},v;X_0)} + \|u\|_{MR^p(\mathbb{R},v)} \le C_A\|f\|_{L^p(\mathbb{R},v;X_0)}. \tag{3.3.31}$$

Assume the constant from (3.3.29) *satisfies $\varepsilon < \frac{1}{C_A}$. Then there exists a λ_0' such that for every $\lambda \ge \lambda_0'$ for every $f \in L^p(\mathbb{R},v;X_0)$ there exists a unique strong solution $u \in L^p(\mathbb{R},v;X_1) \cap W^{1,p}(\mathbb{R},v;X_0)$ of* (3.3.30) *and*

$$\|u\|_{MR^p(\mathbb{R},v)} \le C\|f\|_{L^p(\mathbb{R},v;X_0)}, \tag{3.3.32}$$

where C is independent of λ and f.

The proof follows the line of that of Proposition 3.3.21, so we omit the details. With a similar method as in Proposition 3.3.21 one obtains the following perturbation result which will be used in Section 3.5.

Proposition 3.3.23. *Assume $T < \infty$. Assume (A) holds and $A(\cdot)$ has maximal L^p_v-regularity on $(0,T)$ and the estimate* (3.3.13) *holds with constant C_A. Let $\varepsilon < C_A$. If $B : [0,T] \to \mathscr{L}(X_1, X_0)$ satisfies $\|B(t)x\|_{X_0} \le \varepsilon\|x\|_{X_1}$ for all $x \in X_1$ and $t \in [0,T]$, then $A + B$ has maximal L^p_v-regularity with constant $\frac{C_A}{1-C_A\varepsilon}$.*

Proof. One can argue as in the proof of Proposition 3.3.21 with $\lambda = 0$, $g = f$, $\tilde{B} = B$ and $1-\theta = C_A\varepsilon$. Moreover, if $w = L(w)$, then

$$\begin{aligned}\|w\|_{\mathrm{MR}^p((0,T),v)} &= \|L(w) - L(0)\|_{\mathrm{MR}^p((0,T),v)} + \|L(0)\|_{\mathrm{MR}^p((0,T),v)} \\ &\le (1-\theta)\|w\|_{\mathrm{MR}^p((0,T),v)} + C_A\|f\|_{L^p((0,T),v;X_0)},\end{aligned}$$

and the required estimate result follows. □

Consider now the sequence of problems:

$$\begin{cases} u_n'(t) + A_n(t)u(t) = f_n(t), \ t \in (a,b) \\ u(a) = x_n. \end{cases} \tag{3.3.33}$$

Here we omit the initial condition if $a = -\infty$.

Recall that $v_a = v(\cdot - a)$. The following approximation result holds.

Proposition 3.3.24. *Assume (A) holds for A and A_n for $n \geq 1$ with uniform estimates in n. Assume A and A_n for $n \geq 1$ have maximal L^p_v-regularity on (a,b) with uniform estimates in n. Let $f_n, f \in L^p((a,b), v; X_0)$ and $x_n, x \in X_{v_a,p}$ for $n \geq 1$. Then if u and u_n are the solutions to* (3.3.2) *and* (3.3.33) *respectively, then there is a constant C only dependent on the maximal L^p_v regularity constants and the constants in (A) such that*

$$\begin{aligned} \|u_n - u\|_{MR^p((a,b),v)} \leq C\Big[&\|x_n - x\|_{X_{v_a,p}} + \|f_n - f\|_{L^p((a,b),v;X_0)} \\ &+ \|(A_n - A)u\|_{L^p((a,b),v;X_0)}\Big]. \end{aligned} \tag{3.3.34}$$

In particular if $x_n \to x$ in $X_{v_a,p}$, for all $z \in X_1$, $A_n(t)z \to A(t)z$ in X_0 a.e. and $f_n \to f$ in $L^p((a,b),v;X_0)$, then $u_n \to u$ in $MR^p((a,b),v)$.

Typically, one can take $A_n = \varphi_n * A$ where $(\varphi_n)_{n\geq 1}$ is an approximation of the identity. If φ_n are smooth functions, then A_n will also be smooth and therefore, A_n will generate an evolution family with many additional properties (see [113, 143]).

Proof. The last assertion follows from (3.3.34) and the dominated convergence theorem. To prove the estimate (3.3.34) note that $w_n = u_n - u$ satisfies the following equation

$$w_n' + A_n w_n = (f_n - f) + (A_n - A)u, \quad w_n(a) = x_n - x.$$

Therefore, the (3.3.34) follows immediately from the maximal L^p_v-regularity estimate. $\square$

3.4. AN EXAMPLE: m-TH ORDER ELLIPTIC OPERATORS

In this section let $p, q \in (1,\infty)$, $m \in \{1,2,\dots\}$ and consider the usual multi-index notation $D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_d^{\alpha_d}$, $\xi^\alpha = (\xi^1)^{\alpha_1} \cdot \ldots \cdot (\xi^d)^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_d^0$. Below we let $X_0 = L^q(\mathbb{R}^d, w)$ and $X_1 = W^{m,q}(\mathbb{R}^d, w)$.

Recall that $f \in W^{m,q}(\mathbb{R}^d, w)$ if $f \in L^q(\mathbb{R}^d, w)$ and for all $|\alpha| \leq m$, $\|D^\alpha f\|_{L^q(\mathbb{R}^d,w)} < \infty$. In this case we let

$$[f]_{W^{m,q}(\mathbb{R}^d,w)} = \sum_{|\alpha|=m} \|D^\alpha f\|_{L^q(\mathbb{R}^d,w)}, \quad \|f\|_{W^{m,q}(\mathbb{R}^d,w)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^q(\mathbb{R}^d,w)}.$$

The weights in space will be used in combination with Theorem 3.1.4 to obtain $\mathscr{R}$-boundedness of the integrals operators arising in (Rbdd).

Consider an m-th order elliptic differential operator A given by

$$(A(t)u)(t,x) := \sum_{|\alpha|\le m} a_\alpha(t,x) D^\alpha u(t,x), \quad t\in\mathbb{R}_+,\ x\in\mathbb{R}^d, \tag{3.4.1}$$

where $D_j := -i\frac{\partial}{\partial_j}$ and $a_\alpha : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$.

In this section we will give conditions under which there holds maximal L^p_v-regularity for A or equivalently we will prove optimal L^p_v-regularity results for the solution to the problem

$$\begin{cases} u'(t,x) + (\lambda + A(t))u(t,x) = f(t,x), \ t\in(a,b),\ x\in\mathbb{R}^d \\ u(a,x) = u_0(x), \quad x\in\mathbb{R}^d. \end{cases} \tag{3.4.2}$$

A function u will be called a *strong* $L^p_v(L^q_w)$*-solution* of (3.4.2) if $u \in \mathrm{MR}^p((a,b),v)$ and (3.4.2) holds almost everywhere.

With slight abuse of notation we write A for the realization of A on $X_0 = L^q(\mathbb{R}^d, w)$ with domain $D(A) = X_1$. In this way (3.4.2) can be modeled as a problem of the form (3.3.20). Also, we have seen in Section 3.3 (and in particular Proposition 3.3.18) that it is more general to study maximal L^p_v-regularity on $\mathbb{R}$. Therefore, we will focus on this case below.

3.4.1. Preliminaries on elliptic equations

In this section we present some results for elliptic equations which will be needed below. Recall the definition of $A \in Ell(\theta,\kappa,K)$ from Section 2.1:

Definition 3.4.1. Let

$$A := \sum_{|\alpha|\le m} a_\alpha D^\alpha,$$

with $a_\alpha \in \mathbb{C}$ constant. We say that A is *uniformly elliptic* of angle $\theta\in(0,\pi)$ if there exists a constant $\kappa\in(0,1)$ such that

$$A_\sharp(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha \subset \Sigma_\theta \text{ and } |A_\sharp(\xi)| \ge \kappa, \quad \xi\in\mathbb{R}^d,\ |\xi|=1$$

and there is a constant K such that $|a_\alpha| \le K$ for all $|\alpha|\le m$. In this case we write $A\in \mathrm{Ell}(\theta,\kappa,\mathrm{K})$.

The following result is on the sectoriality of the operator in the x-independent case. The proof is an application of the Mihlin multiplier theorem.

Theorem 3.4.2. *Let* $1<q<\infty$ *and* $w\in A_q$. *Assume* $A\in\mathrm{Ell}(\theta_0,\kappa,\mathrm{K})$ *with* $\theta_0\in(0,\pi)$. *Then for every* $\theta>\theta_0$ *there exists an* A_q*-consistent constant* C *depending on the parameters* $m,d,\theta_0-\theta,\kappa,K,q$ *such that*

$$\|\lambda^{1-\frac{|\beta|}{m}} D^\beta(\lambda+A)^{-1}\|_{\mathscr{L}(L^q(\mathbb{R}^d,w))} \le C, \quad |\beta|\le m,\ \lambda\in\Sigma_{\pi-\theta}. \tag{3.4.3}$$

In particular, there is a constant $\tilde{C}$ *depending only on* θ *and* C *such that* $\|e^{-tA}\|\le\tilde{C}$.

The case of x-dependent coefficients can be derived by standard localization arguments, but we will not need this case below (see [77, Theorem 3.1] and [102, Section 6]).

Proof. For (3.4.3) we need to check that for every $\lambda \in \Sigma_{\pi-\theta}$, and $|\beta| \le m$, the symbol $\mathcal{M}: \mathbb{R}^d \to \mathbb{C}$ given by

$$\mathcal{M}(\xi) = \lambda^{1-\frac{|\beta|}{m}} \xi^\beta (\lambda + A_\sharp(\xi))^{-1}$$

satisfies the following: for every multiindex $\alpha \in \mathbb{N}_0^d$, there is a constant C_α which only depends on $d, \alpha, \theta - \theta_0, K, \kappa$ such that

$$|\xi^\alpha D^\alpha \mathcal{M}(\xi)| \le C_\alpha, \quad \xi \in \mathbb{R}^d. \tag{3.4.4}$$

Indeed, as soon as this is checked, the result is a consequence of the weighted version of Mihlin's multiplier theorem (see [65, Theorem IV.3.9]).

In order to check the condition for $\ell \ge 0$ let F_ℓ be the span of functions of the form $\lambda^\eta g h^{-1}$, where $\eta \in [0,1]$, $g : \mathbb{R}^d \to \mathbb{C}$ is polynomial which is homogeneous of degree $\nu \in \mathbb{N}_0$ and $h = (\lambda + A_\sharp)^\mu$ with $\mu \in \mathbb{N}$ and $\ell = m(\mu - \eta) - \nu$. It is clear that $\mathcal{M} \in F_0$. Using induction one can check that for $f \in F_\ell$ one has $D^\alpha f \in F_{\ell+|\alpha|}$.

We claim that for $f \in F_\ell$ the mapping $\xi \mapsto |\xi|^\ell f(\xi)$ is uniformly bounded. In order to prove this it suffices to consider $f = \lambda^\eta g h^{-1}$ with g and h as before, and $\ell = m(\mu - \eta) - \nu$. As $\xi \mapsto |\xi|^{-\nu} g(\xi)$ is bounded it remains to estimate

$$\lambda^\eta h(\xi)^{-1} |\xi|^{\ell+\nu} = s^\eta (s + A_\sharp(\xi^*))^{-\mu},$$

where $\xi^* = \xi/|\xi|$ and $s = \lambda|\xi|^{-m}$.

Write $A_\sharp(\xi^*) = re^{i\varphi}$ with $r = |A_\sharp(\xi^*)|$ and $|\varphi| < \theta_0$ and $s = \rho e^{i\psi}$ with $\rho = |s|$ and $|\psi| < \pi - \theta$. Then

$$|s^\eta (s + A_\sharp(\xi^*))^{-\mu}| = \rho^\eta |\rho e^{i\psi} + re^{i\varphi}|^{-\mu} = \rho^\eta (\rho^2 + r^2 + 2\rho r \cos(\psi - \varphi))^{-\mu/2}.$$

Since $\cos(\psi - \varphi) \ge \cos(\pi - (\theta - \theta_0)) = -\cos(\theta - \theta_0) = -(1 - \varepsilon^2)$ with $\varepsilon \in (0,1)$ and $-2\rho r \ge -(\rho^2 + r^2)$ and we find

$$|s^\eta (s + A_\sharp(\xi^*))^{-\mu}| \le \rho^\eta (\rho^2 + r^2)^{-\mu/2} \varepsilon^{-\mu} \le \kappa^{\mu-\eta} \varepsilon^{-\mu},$$

where in the last step we used $r \ge \kappa$ and $\mu \ge \eta$. This proves the claim.

In order to check (5.2.3) note that $\mathcal{M} \in F_0$ and hence by the above $D^\alpha \mathcal{M}(\xi) \in F_{|\alpha|}$. Therefore, the bound follows from the claim about F_ℓ and the observation that the functions g arising in the linear combinations of the form $\lambda^\eta g h^{-1}$ satisfy $|g(\xi)| \le C_{K,d,\alpha}$.

The assertion for e^{-tA} follows from (2.3.1) and the estimate (3.4.3) with $\beta = 0$. ☐

As a consequence we obtain the following:

Corollary 3.4.3. *Let $\lambda_0 > 0$. Under the conditions of Theorem 3.4.2, the operator A is closed and for every $\lambda \geq \lambda_0$,*

$$c\|u\|_{W^{m,q}(\mathbb{R}^d,w)} \leq \|(\lambda + A)u\|_{L^q(\mathbb{R}^d,w)} \leq (K+\lambda)\|u\|_{W^{m,q}(\mathbb{R}^d,w)},$$

where c^{-1} is A_q-consistent and only depends on $m, d, \theta_0 - \theta, \kappa, K, q$ and λ_0.

Corollary 3.4.3 for x-dependent coefficients will be derived from Theorem 3.4.5 in Remark 3.4.7.

Corollary 3.4.4. *Let $m \geq 1$, $1 < q < \infty$ and $w \in A_q$. If $m \geq 2$, then there is an A_q-consistent constant C depending only on d, q and m such that for all $|\beta| \leq m-1$*

$$\begin{aligned}\|D^\beta f\|_{L^q(\mathbb{R}^d,w)} &\leq C\|f\|_{L^q(\mathbb{R}^d,w)}[f]_{W^{m,q}(\mathbb{R}^d,w)}\\ &\leq C'\lambda^{\frac{\beta}{m}}\|f\|_{L^q(\mathbb{R}^d,w)} + C'\lambda^{-\frac{m-|\beta|}{|m|}}[f]_{W^{m,q}(\mathbb{R}^d,w)}.\end{aligned}$$

Proof. Note that for $|\beta| = 1$,

$$\|D^\beta f\|_{L^q(\mathbb{R}^d,w)} \leq C\lambda^{\frac{1}{2}}\|f\|_{L^q(\mathbb{R}^d,w)} + \lambda^{-\frac{1}{2}}[f]_{W^{2,q}(\mathbb{R}^d,w)}$$

follows from Theorem 3.4.2 with $A = -\Delta$ and the required estimate follows by minimizing over all $\lambda > 0$. The case $m > 2$ can be obtained by induction (see [102, Exercise 1.5.6]). The final estimate follows from Young's inequality. ☐

3.4.2. Main result on $\mathbb{R}^d$

For A of the form (3.4.1) and $x_0 \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}$ let us introduce the notation:

$$A(t_0, x_0) := \sum_{|\alpha|\leq m} a_\alpha(t_0, x_0)D^\alpha.$$

for the operator with constant coefficients.

(C) Let A be given by (3.4.1) and assume each $a_\alpha : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ is measurable. We assume there exist $\theta_0 \in [0, \pi/2)$, κ and K such that for all $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$, $A(t_0, x_0) \in \mathrm{Ell}(\theta_0, \kappa, \mathrm{K})$. Assume there exists an increasing function $\omega : (0,\infty) \to (0,\infty)$ with the property $\omega(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and such that

$$|a_\alpha(t,x) - a_\alpha(t,y)| \leq \omega(|x-y|), \ \ |\alpha| = m, \ \ t \in \mathbb{R}, \ \ x, y \in \mathbb{R}^d.$$

As $\theta_0 < \pi/2$, the above ellipticity condition implies that m is even in all the results below.

The set of parameters on which all constant below will depend is given by

$$\mathscr{P} = \{\kappa, K, \omega, [v]_{A_p}, [w]_{A_q}, p, q, d, m, \theta_0\}. \tag{3.4.5}$$

Moreover, all the dependence on the weights will be in an A_p and A_q-consistent way.

Theorem 3.4.5. *Let $p,q \in (1,\infty)$. Let $v \in A_p(\mathbb{R})$ and $w \in A_q(\mathbb{R}^d)$. Assume condition (C) on A. Then there exists a $\lambda_0 \in \mathbb{R}$ depending on the parameters in $\mathscr{P}$ such that for all $\lambda \geq \lambda_0$ the operator $\lambda + A$ has maximal L^p_v-regularity on $\mathbb{R}$. Moreover, for every $\lambda \geq \lambda_0$ and for every $f \in L^p(\mathbb{R}, v; X_0)$ there exists a unique $u \in MR^p(\mathbb{R}, v)$ which is a strong $L^p(L^q)$ solution of*

$$u'(t,x) + (\lambda + A(t))u(t,x) = f(t,x),\ a.e.\ t \in \mathbb{R},\ x \in \mathbb{R}^d$$

and there is constant C depending on the parameters in $\mathscr{P}$ such that

$$\lambda \|u\|_{L^p(\mathbb{R},v;X_0))} + \|u\|_{MR^p(\mathbb{R},v)} \leq C\|f\|_{L^p(\mathbb{R},v;X_0)}. \tag{3.4.6}$$

Recall that $\mathrm{MR}^p(\mathbb{R}, v) = W^{1,p}(\mathbb{R}, v; X_0) \cap L^p(\mathbb{R}, v; X_1)$.
Also note that the estimate (3.4.6) also holds if one replaces $\mathbb{R}$ by $(-\infty, T)$ for some $T \in \mathbb{R}$. The above result also implies that $\lambda + A$ has maximal L^p_v-regularity on $(0,T)$ for every $T < \infty$ and every $\lambda \in \mathbb{R}$.

The proof of the above result is a based on Theorem 3.3.8, standard PDE techniques and extrapolation arguments. The proof of Theorem 3.4.5 is divided in several steps of which some are standard, but we prefer to give a complete proof for convenience of the reader. In Steps 1 and 2 we assume $a_\alpha = 0$ for $|\alpha| < m$ and show how to include these lowers order terms later on.

Step 1: Consider the case where the coefficients $a_\alpha : \mathbb{R} \to \mathbb{C}$ are x-independent. Choose $\delta > 0$ small enough and set $A_0 = \delta(-\Delta)^{m/2}$. Note that by Corollary 3.4.3 $D(A_0) = X_1$. We write

$$A(t) = \sum_{|\alpha|=m} a_\alpha(t) D^\alpha,\quad \tilde{A}(t) = A(t) - A_0.$$

It is a simple exercise to see that there exist $\delta_0 > 0$, $\theta' \in (\theta, \frac{\pi}{2})$ and $\kappa' > 0$ depending on κ and θ that for all $\delta \in (0,\delta_0]$, $\tilde{A}(t) \in \mathrm{Ell}(\theta', \kappa', \mathrm{K})$. Therefore, each $\tilde{A}(t)$ satisfies the conditions of Theorem 3.4.2 with constants only depending on $\delta_0, \kappa, \theta, K$. The same holds for operators of the form $\tilde{A}_{ab} := \frac{1}{b-a}\int_a^b \tilde{A}(t)\,dt$, where $0 \leq a < b < \infty$. Note that $\tilde{A}_{ab}$ and $\tilde{A}(t)$ are resolvent commuting and have domain X_1. Therefore, by Example 3.3.3 the evolution family for $\tilde{A}$ exists and is given by

$$T(t,s) = \exp\Big(-(t-s)\tilde{A}_{st}\Big),\quad 0 \leq s \leq t < \infty.$$

Moreover, for all $\lambda > 0$,

$$\|T(t,s)\|_{\mathscr{L}(L^q(\mathbb{R}^d,w))} \leq C, \quad 0 \leq s \leq t, \tag{3.4.7}$$

where C only depends on $\delta_0, \kappa, \theta, \theta_0, K, q, [w]_{A_q}$. Since A_0 is also resolvent commuting with $\tilde{A}_{ab}$ and $\tilde{A}(t)$, it follows from Lemma 3.3.7 that the evolution family generated by A factorizes as

$$S(t,s) = e^{-\frac{1}{2}(t-s)A_0} T(t,s) e^{-\frac{1}{2}(t-s)A_0},\quad 0 \leq s \leq t < \infty.$$

We check the hypothesis (A), (E) and (Rbdd) of Theorem 3.3.8. Condition (A) follows Corollary 3.4.3 with $\lambda = 1$. For condition (E), recall from Section 2.3 that A_0 has a bounded H^∞-calculus of angle $< \pi/2$. Moreover, $X_0 = L^q(\mathbb{R}^d)$ has finite cotype (see [45, Chapter 11]). Finally, (Rbdd) follows from Theorem 3.1.4 and (3.4.7). Therefore, by Theorem 3.3.8 we find there is a constant C such that (3.4.6) holds for all $\lambda \geq 1$.

Step 2: Next we consider the case where the coefficients of A are also x-dependent, but still with $a_\alpha = 0$ for $\alpha < m$. We start with a standard freezing lemma.

Lemma 3.4.6 (Freezing lemma). *Let $\varepsilon > 0$ be such that $\omega(\varepsilon) \leq \frac{1}{2C}$, where C is the constant for* (3.4.6) *obtained in Step 1. If $u \in MR^p(\mathbb{R}, v)$ and for some $x_0 \in \mathbb{R}^d$ for each $t \in \mathbb{R}$, $u(t,\cdot)$ has support in a ball $B(x_0,\varepsilon) = \{x : |x - x_0| < \varepsilon\}$, then for all $\lambda \geq 1$, the following estimate holds:*

$$\lambda \|u\|_{L^p(\mathbb{R},v;X_0)} + \|u\|_{MR^p(\mathbb{R},v)} \leq 2C\|(\lambda + A)u + u'\|_{L^p(\mathbb{R},v;X_0)}. \tag{3.4.8}$$

Proof. Let $f := (\lambda + A)u + u'$ and observe that $u' + (A(\cdot, x_0) + \lambda)u = f + (A(\cdot, x_0) - A)u$. By (3.4.6), we find

$$\lambda \|u\|_{L^p(\mathbb{R},v;X_0)} + \|u\|_{\mathrm{MR}^p(\mathbb{R},v)} \leq C\|f\|_{L^p(\mathbb{R},v;X_0)} + C\|(A(\cdot, x_0) - A)u\|_{L^p(\mathbb{R},v;X_0)}.$$

Note that by the support condition on u and the continuity of $x \mapsto a_\alpha(\cdot, x)$,

$$\|(A(t, x_0) - A(t))u(t)\|_{X_0} \leq \omega(\varepsilon)\|u(t)\|_{X_1}.$$

Therefore, $C\|(A(\cdot, x_0) - A)u\|_{L^p(\mathbb{R},v;X_0)} \leq \frac{1}{2}\|u\|_{\mathrm{MR}^p(\mathbb{R},v)}$ and hence

$$\lambda \|u\|_{L^p(\mathbb{R},v;X_0)} + \|u\|_{\mathrm{MR}^p(\mathbb{R},v)} \leq C\|f\|_{L^p(\mathbb{R},v;X_0)} + \frac{1}{2}\|u\|_{\mathrm{MR}^p(\mathbb{R},v)}.$$

and the result follows from this. □

Step 3: In this step we use a localization argument in the case $p = q$ to show that there is a constant C such that for all $u \in \mathrm{MR}^p(\mathbb{R}, v)$,

$$\lambda \|u\|_{L^q(\mathbb{R},v;X_0)} + \|u\|_{L^q(\mathbb{R},v;X_1)} \leq C\|(\lambda + A)u + u'\|_{L^q(\mathbb{R},v;X_0)}. \tag{3.4.9}$$

(a) Take a $\phi \in C^\infty(\mathbb{R}^d)$ with $\phi \geq 0$, $\|\phi\|_{L^q(\mathbb{R}^d)} = 1$ and support in the ball $B_\varepsilon = \{x : |x| < \varepsilon\}$ where $\varepsilon > 0$ is as in Lemma 3.4.6. Note that

$$|\nabla^m u(t,x)|^q = \int_{\mathbb{R}^d} |\nabla^m u(t,x)\phi(x - \xi)|^q \, d\xi. \tag{3.4.10}$$

By the product rule, we can write

$$\nabla^m[u(t,x)\phi(x-\xi)] = \nabla^m u(t,x)\cdot\phi(x-\xi) + \sum_{|\alpha|\leq m-1} c_\alpha D^\alpha u(t,x) D^{g(\alpha)}\phi(x-\xi),$$

with $|g(\alpha)| \le m$ and $c_\alpha \ge 0$. Therefore,

$$|\nabla^m u(t,x)\cdot \phi(x-\xi)| \le |\nabla^m[u(t,x)\phi(x-\xi)] + \tilde{C}\sum_{|\alpha|\le m-1}|D^\alpha u(t,x)|,$$

where used $\sum_{|\alpha|\le m-1} c_\alpha |D^{g(\alpha)}\phi(x)| \le \tilde{C}$. Taking $L^q(\mathbb{R},v)$-norms on both sides gives

$$\begin{aligned}\|\nabla^m u\|_{L^q(\mathbb{R},v;X_0)} &= \Big(\int_{\mathbb{R}^d}\|\nabla^m u\phi(\cdot-\xi)\|^q_{L^q(\mathbb{R};X_0)}d\xi\Big)^{1/q}\\ &\le \Big(\int_{\mathbb{R}^d}\|\nabla^m(u\phi(\cdot-\xi))\|^q_{L^q(\mathbb{R},v;X_0)}d\xi\Big)^{1/q} + L,\end{aligned} \tag{3.4.11}$$

where $L = \tilde{C}\sum_{|\alpha|\le m-1}\|D^\alpha u\|_{L^q(\mathbb{R},v;X_0)}$. For each fixed ξ in the case $p=q$, Lemma 3.4.6 applied to $x\mapsto u(t,x)\phi(x-\xi)$ yields

$$\begin{aligned}&\|\nabla^m(u(t)\phi)\|_{L^q(\mathbb{R},v;X_0)} \le \Big(\int_{\mathbb{R}^d}\|\nabla^m(u\phi(\cdot-\xi))\|^q_{L^q(\mathbb{R},v;X_0)}d\xi\Big)^{1/q} + L\\ &\le C\Big(\int_{\mathbb{R}^d}\|(\lambda+A)(u\phi(\cdot-\xi)) + u'\phi(\cdot-\xi)\|^q_{L^q(\mathbb{R},v;X_0)}\,d\xi\Big)^{\frac{1}{q}} + L,\end{aligned} \tag{3.4.12}$$

Note that for each $\xi\in\mathbb{R}^d$,

$$\begin{aligned}(\lambda+A)(u\phi(\cdot-\xi)) &= \sum_{|\alpha|=m} a_\alpha D^\alpha[u\phi(\cdot-\xi)] + \lambda u\phi(\cdot-\xi)\\ &= (\lambda+A)u\cdot\phi(\cdot-\xi) + \sum_{|\alpha|\le m-1} c_\alpha a_\alpha D^\alpha u D^{g(\alpha)}\phi(\cdot-\xi).\end{aligned}$$

Thus we also have

$$\begin{aligned}&\Big(\int_{\mathbb{R}^d}\|(\lambda+A)(u\phi(\cdot-\xi)) + u'\phi(\cdot-\xi)\|^q_{L^q(\mathbb{R},v;X_0)}\,d\xi\Big)^{\frac{1}{q}}\\ &\qquad\le \Big(\int_{\mathbb{R}^d}\|[(\lambda+A)u + u']\phi(\cdot-\xi)\|^q_{L^q(\mathbb{R},v;X_0)}\,d\xi\Big)^{\frac{1}{q}} + KL.\\ &\qquad= \|(\lambda+A)u + u'\|_{L^q(\mathbb{R},v;X_0)} + KL.\end{aligned}$$

Combining the latter with (3.4.11) and (3.4.12) gives

$$\|\nabla^m u\|_{L^q(\mathbb{R},v;X_0)} \le C\|(\lambda+A)u + u'\|_{L^q(\mathbb{R},v;X_0)} + (K+1)L,$$

where K is as in condition (C). We may conclude that

$$\|u\|_{L^q(\mathbb{R},v;X_1)} \le C\|(\lambda+A)u + u'\|_{L^q(\mathbb{R},v;X_0)} + C\|u\|_{L^q(\mathbb{R},v;W^{m-1,q}(\mathbb{R}^d,w))}. \tag{3.4.13}$$

To include the lower order terms, let

$$B(t)u(t,x) = \sum_{|\alpha|\le m-1} a_\alpha(t,x)D^\alpha u.$$

By (3.4.13) with $f = (A+B+\lambda)u + u'$ and the triangle inequality, we find

$$\|u\|_{L^q(\mathbb{R},v;X_1)} \leq C\|f\|_{L^q(\mathbb{R},v;X_0)} + C(K+1)\|u\|_{L^q(\mathbb{R},v;W^{m-1,q}(\mathbb{R}^d,w))}. \tag{3.4.14}$$

In a similar way, one sees that for all $\lambda \geq 1$

$$\lambda\|u\|_{L^q(\mathbb{R},v;X_0)} \leq C\|f\|_{L^q(\mathbb{R},v;X_0)} + C(K+1)\|u\|_{L^q(\mathbb{R},v;W^{m-1,q}(\mathbb{R}^d,w))}. \tag{3.4.15}$$

In order to obtain (3.4.9) from (3.4.14) and (3.4.15) note that it follows from the interpolation inequality from Corollary 3.4.4 that for all $\nu > 0$

$$\|u\|_{W^{m-1,q}(\mathbb{R}^d,w)} \leq C\nu^{m-1}\|u\|_{L^q(\mathbb{R}^d,w)} + C\nu^{-1}\|u\|_{W^{m,q}(\mathbb{R}^d,w)}. \tag{3.4.16}$$

Therefore, choosing ν small enough we can combine the latter with (3.4.15) to obtain

$$\begin{aligned}\lambda\|u\|_{L^q(\mathbb{R},v;X_0)} + \|u\|_{L^q(\mathbb{R},v;X_1)} &\leq C\|f\|_{L^q(\mathbb{R},v;X_0)} \\ &\quad + \frac{1}{2}\|u\|_{L^q(\mathbb{R},v;X_1)} + C_\nu\|u\|_{L^q(\mathbb{R},v;X_0)}.\end{aligned}$$

Setting $\lambda_0 = \max\{2C_\nu, 1\}$, it follows that for all $\lambda \geq \lambda_0$,

$$\frac{1}{2}\lambda\|u\|_{L^q(\mathbb{R},v;X_0)} + \frac{1}{2}\|u\|_{L^q(\mathbb{R},v;X_1)} \leq C\|f\|_{L^q(\mathbb{R},v;X_0)}.$$

This clearly implies (3.4.9).

Step 4: To extrapolate the estimate from the previous step to $p \neq q$, let $u : \mathbb{R} \to X_1$ be a Schwartz function. Then by (3.4.9) we have for all $v \in A_q$ there exists A_q-consistent constants $\lambda_0, C > 0$ such that for all $\lambda \geq \lambda_0$

$$\|F_\lambda\|_{L^q(\mathbb{R},v)} \leq C\|G_\lambda\|_{L^q(\mathbb{R},v)},$$

where $F_\lambda = \|u\|_{X_1}$, $G_\lambda = \|(\lambda + A)u + u'\|_{X_0}$. Therefore, by the extrapolation result Theorem 2.2.5 it follows that for all $v \in A_p$ there exist a A_p-consistent constants λ_0' and C' such that for all $\lambda \geq \lambda_0'$,

$$\|F_\lambda\|_{L^p(\mathbb{R},v)} \leq C'\|G_\lambda\|_{L^p(\mathbb{R},v)},$$

This yields

$$\|u\|_{L^p(\mathbb{R},v;X_1)} \leq C'\|(\lambda + A)u + u'\|_{L^p(\mathbb{R},v;X_0)}.$$

Similarly, one proves the estimate for $\lambda\|u\|_{L^p(\mathbb{R},v;X_0)}$. As $u' = (\lambda + A)u + u' - (\lambda + A)u$, (3.4.6) with righthand side $f = (\lambda + A)u + u'$ follows.

Step 5: Let A be as in the theorem. For $s \in [0,1]$ let $A_s = sA + (1-s)(-\Delta)^{m/2}$, where we recall that m is even. Then A_s satisfies condition (C) with constants κ and K replaced by $\min\{\kappa, 1\}$ and $\max\{K, 1\}$, respectively. Therefore, for all $\lambda \geq \lambda_0$, (3.4.6) holds with right-hand side $f = (\lambda + A_s)u + u'$ with a constant C which does not dependent on s. For $s = 0$ for all $\lambda \geq \lambda_0$, for every $f \in L^p(\mathbb{R}, v; X_0)$, one has existence and uniqueness of a strong solution $u \in \mathrm{MR}^p(\mathbb{R}, v)$ to $u' + (\lambda + A_s)u = f$ by step 1. Therefore, the method of continuity (see [68, Theorem 5.2]) yields existence and uniqueness of a strong solution for every $s \in [0,1]$. Taking $s = 1$, the required result follows and this completes the proof of Theorem 3.4.5.

Remark 3.4.7. In the proof of Theorem 3.4.5 we applied Corollary 3.4.3 only for the case of x-independent coefficients. It is rather simple to derive Corollary 3.4.3 with x-dependent coefficients from Theorem 3.4.5 (cf. [102, Exercise 4.3.13]). Indeed, let A be t-independent but such that (C) holds and let $u \in W^{m,q}(\mathbb{R}^d, w)$. For $\tilde{u} : \mathbb{R} \to X_1$ given by $\tilde{u}(t) = e^{-\mu|t|}u$ with $\mu > 0$, let

$$f(t) = \mathbf{1}_{(-\infty,0)}(t)(\tilde{u}'(t) + (\lambda + A)\tilde{u}(t)) = \mathbf{1}_{(-\infty,0)}(t)e^{\mu t}[-\mu + (\lambda + A)u].$$

Then, applying (3.4.6) to $\tilde{u}$ with $v = 1$, we get that for every $\lambda \geq \lambda_0$,

$$\begin{aligned}(\mu p)^{-1/p}[\lambda\|u\|_{X_0} + \|u\|_{X_1}] &= \lambda\|u\|_{L^p((-\infty,0),X_0)} + \|u\|_{\mathrm{MR}^p((-\infty,0))}\\ &\leq C\|f\|_{L^p(\mathbb{R};X_0)} = C(\mu p)^{-1/p}\| -\mu + (\lambda + A)u\|_{X_0}.\end{aligned}$$

Comparing the left and right-hand side and letting $\mu \downarrow 0$, we obtain

$$\lambda\|u\|_{L^q(\mathbb{R}^d,w)} + \|u\|_{W^{m,q}(\mathbb{R}^d,w)} \leq C\|(\lambda + A)u\|_{L^q(\mathbb{R}^d,w)}.$$

Finally, as a consequence of Theorem 3.4.5, we show the following maximal regularity result with non-zero initial value.

Theorem 3.4.8. *Let $T \in (0,\infty)$. Assume condition (C) on the family of operators $(A(t))_{t\in(0,T)}$ given by* (3.4.1). *Let $p, q \in (1,\infty)$. Then the operator A has maximal L^p-regularity on $(0,T)$, i.e. for every $f \in L^p(0,T;L^q(\mathbb{R}^d))$ and $u_0 \in \mathscr{B}^s_{q,p}(\mathbb{R}^d)$ with $s = m(1-\frac{1}{p})$, there exists a unique*

$$u \in W^{1,p}(0,T;L^q(\mathbb{R}^d)) \cap L^p(0,T;W^{m,q}(\mathbb{R}^d)) \cap C([0,T];\mathscr{B}^s_{q,p}(\mathbb{R}^d))$$

such that

$$\begin{aligned}u'(t,x) + A(t)u(t,x) &= f(t,x), \ t \in (0,T), \ x \in \mathbb{R}^d,\\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^d.\end{aligned} \tag{3.4.17}$$

holds a.e. and there is a $C > 0$ independent of u_0 and f such that

$$\begin{aligned}\|u\|_{L^p(0,T;W^{m,q}(\mathbb{R}^d))} + \|u\|_{W^{1,p}(0,T;L^q(\mathbb{R}^d))} + \|u\|_{C([0,T];\mathscr{B}^s_{q,p}(\mathbb{R}^d))}&\\ \leq C\big(\|f\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} + \|u_0\|_{\mathscr{B}^s_{q,p}(\mathbb{R}^d)}\big).&\end{aligned} \tag{3.4.18}$$

Proof. By Theorem 3.4.5 there is a $\lambda \in \mathbb{R}$ such that $\lambda + A$ has maximal L^p-regularity on $\mathbb{R}$ and hence on $(0,T)$ as well. By the observation after Definition 3.3.10 this implies that A has maximal L^p-regularity on $(0,T)$ and hence we can find a unique solution

$$u \in W^{1,p}(0,T;L^q(\mathbb{R}^d)) \cap L^p(0,T;W^{m,q}(\mathbb{R}^d))$$

of (1.2.1) with $u_0 = 0$. By Proposition 3.3.18 with $v \equiv 1$, we can allow non-zero initial values $u_0 \in X_{v,p} = (L^q(\mathbb{R}^d), W^{m,q}(\mathbb{R}^d))_{1-\frac{1}{p},p}$ (see Example 3.3.19). By [19, Theorem 6.2.4] or [148, Remark 2.4.2.4] this real interpolation space can be identified with $\mathscr{B}^s_{q,p}(\mathbb{R}^d)$ with $s = m(1-1/p)$. Finally, the fact that $u \in C([0,T];\mathscr{B}^s_{q,p}(\mathbb{R}^d))$ follows from Example 3.3.19 as well. □

Remark 3.4.9. In the case A is time-independent and $v \equiv 1$, Theorem 3.4.5 reduces to [77, Theorem 3.1] in case of scalar equations.

3.5. QUASILINEAR EVOLUTION EQUATIONS

In this section we illustrate how the results of Section 3.3 can be used to study nonlinear PDEs. We extend the result of [28] and [127] (see [97] for the weighted setting) to the case of time-dependent operators A without continuity assumptions. Our proof slightly differs from the previous ones since we can immediately deal with the non-autonomous setting. For notational simplicity we consider the unweighted setting only.

3.5.1. Abstract setting

Let X_0 be a Banach space and $X_1 \hookrightarrow X_0$ densely, $0 < T \le T_0 < \infty$, $J = [0,T]$, $J_0 = [0,T_0]$ and $p \in (1,\infty)$. Let $X_p = (X_0, X_1)_{1-\frac{1}{p},p}$ equipped with the norm from (3.3.19). Consider the quasi-linear problem

$$\begin{cases} u'(t) + A(t,u(t))u(t) = F(t,u(t)), & t \in J \\ u(0) = x. \end{cases} \tag{3.5.1}$$

where $x \in X_p$ and

- $A : J_0 \times X_p \to \mathscr{L}(X_1, X_0)$ is such that for each $y \in X_1$ and $x \in X_p$, $t \to A(t,x)y$ is strongly measurable and satisfies the following continuity condition: for each $R > 0$ there is a constant $C(R) > 0$ such that

$$\|A(t,x_1)y - A(t,x_2)y\|_{X_0} \le C(R)\|x_1 - x_2\|_{X_p}\|y\|_{X_1}, \tag{3.5.2}$$

  with $t \in J_0$, $x_1, x_2 \in X_p$, $\|x_1\|_{X_p}, \|x_2\|_{X_p} \le R$, $y \in X_1$.

- $F : J_0 \times X_p \to X_0$ is such that $F(\cdot,x)$ is measurable for each $x \in X_p$, $F(t,\cdot)$ is continuous for a.a. $t \in J_0$ and $F(\cdot,0) \in L^p(J_0;X_0)$ and F satisfies the following condition on Lipschitz continuity: for each $R > 0$ there is a function $\phi_R \in L^p(J_0)$ such that

$$\|F(t,x_1) - F(t,x_2)\|_{X_0} \le \phi_R(t)\|x_1 - x_2\|_{X_p},$$

for a.a. $t \in J_0,\ x_1, x_2 \in X_p,\ \|x_1\|_{X_p}, \|x_2\|_{X_p} \le R$.

Theorem 3.5.1. *Assume the above conditions on A and F. Let $x_0 \in X_p$ and assume that $A(\cdot, x_0)$ has maximal L^p-regularity. Then there is a $T \in (0, T_0]$ and radius $\varepsilon > 0$ both depending on x_0 such that for all $x \in B_\varepsilon = \{y \in X_p : \|y - x_0\|_{X_p} \le \varepsilon\}$, (3.5.1) admits a unique solution $u \in \mathrm{MR}(J) := W^{1,p}(J; X_0) \cap L^p(J; X_1)$. Moreover, there is a constant C such that for all $x, y \in B_\varepsilon$ the corresponding solutions u^x and u^y satisfy*

$$\|u^x - u^y\|_{\mathrm{MR}(J)} \le C\|x - y\|_{X_p}.$$

The proof will be given in Section 3.5.3.

Remark 3.5.2. This result can be extended to the weighted setting, with power weight $v(t) = t^\alpha$, $\alpha \in (-1, p-1)$ (see Example 3.3.19). In this case, the usual reflection argument does not work, since the extension becomes dependent on the weight and the constant in (3.5.7) becomes T-dependent. Instead, one can use a linear and bounded extension operator from $\mathrm{MR}((0,T), v)$ to $\mathrm{MR}(\mathbb{R}_+, v)$ whose norm is independent on T (see [119, Lemma 2.5]).

3.5.2. Example of a quasilinear second order equation

Let $T_0 > 0$ and $J_0 = [0, T_0]$. In this section we will give conditions under which there exists a local solution of the problem:

$$u'(t,x) + \sum_{|\alpha|=2} a_\alpha(t,x,u(t,x),\nabla u(t,x)) D^\alpha u(t,x) = f(t,x,u(t,x),\nabla u(t,x)), \tag{3.5.3}$$

with initial value $u(0,x) = u_0(x)$, $t \in J_0$, $x \in \mathbb{R}^d$ and where $D_j := -i\frac{\partial}{\partial_j}$. The main new feature here is that the above functions a_α are only measurable in time. Note that possible lower order terms a_α can be included in f. We will provide an $L^p(L^q)$-theory for (3.5.3) under the following conditions on p and q:

(i) Let $X_0 = L^q(\mathbb{R}^d),\ X_1 = W^{2,q}(\mathbb{R}^d),\ X_p = \mathscr{B}^{2(1-\frac{1}{p})}_{q,p}(\mathbb{R}^d)$ where $p, q \in (1,\infty)$ satisfy

$$2\Big(1 - \frac{1}{p}\Big) - \frac{d}{q} > 1. \tag{3.5.4}$$

This condition is to ensure the following continuous embedding holds (see [148, Theorem 2.8.1])

$$\mathscr{B}^{2(1-\frac{1}{p})}_{q,p}(\mathbb{R}^d) \hookrightarrow C^{1+\delta}(\mathbb{R}^d), \quad \text{for all } 0 < \delta < 2\Big(1 - \frac{1}{p}\Big) - \frac{d}{q} - 1. \tag{3.5.5}$$

Also note that $\mathscr{B}_{q,p}^{2(1-\frac{1}{p})}(\mathbb{R}^d) = (X_0, X_1)_{1-\frac{1}{p},p}$ by [148, 2.4.2(16)].

On a and f we assume the following conditions:

(ii) Assume each $a_\alpha : J_0 \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is a measurable function such that $\sup_{t,x,y,z} |a_\alpha(t,x,y,z)| < \infty$ and there is an $\theta \in (0,\pi)$ and $\kappa \in (0,1)$ such that for all $t \in J_0,\ x, z \in \mathbb{R}^d,\ y \in \mathbb{R}$,

$$\sum_{|\alpha|=2} a_\alpha(t,x,y,z)\xi^\alpha \subset \Sigma_\theta \text{ and } \Big|\sum_{|\alpha|=2} a_\alpha(t,x,y,z)\xi^\alpha\Big| \geq \kappa, \quad \xi \in \mathbb{R}^d,\ |\xi| = 1.$$

(iii) Assume that for every $R > 0$ there exists a function $\omega_R : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon \downarrow 0} \omega_R(\varepsilon) = 0$ such that for all $t \in J_0,\ x_1, x_2 \in \mathbb{R}^d,\ |y|, |z| \leq R$,

$$|a_\alpha(t,x_1,y,z) - a_\alpha(t,x_2,y,z)| \leq \omega_R(|x_1 - x_2|).$$

(iv) Assume that for each $|\alpha| = 2$ for every $R > 0$ there exists a constant $C_\alpha(R)$ such that for all $t \in J_0,\ x \in \mathbb{R}^d,\ |y_1|, |y_2| \leq R$, and $|z_1|, |z_2| \leq R$,

$$|a_\alpha(t,x,y_1,z_1) - a_\alpha(t,x,y_2,z_2)| \leq C_\alpha(R)(|y_1 - y_2| + |z_1 - z_2|), \tag{3.5.6}$$

(v) Assume $f : J_0 \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is a measurable function such that

$$\int_{J_0} \Big(\int_{\mathbb{R}^d} |f(t,x,0,0)|^q \, dx\Big)^{\frac{p}{q}} \, dt < \infty.$$

For every $R > 0$ there exists a function $\phi(R) \in L^p(J_0)$ such that for all $t \in J_0$, $x \in \mathbb{R}^d$, $|y_1|, |y_2| \leq R$ and $|z_1|, |z_2| \leq R$,

$$|f(t,x,y_1,z_1) - f(t,x,y_2,z_2)|_{X_0} \leq \phi(R)(t)(|y_1 - y_2| + |z_1 - z_2|).$$

Let $\mathrm{MR}^p(J) = W^{1,p}(J; L^q(\mathbb{R}^d)) \cap L^p(J; W^{2,q}(\mathbb{R}^d))$ and note that by (3.5.5)

$$\mathrm{MR}^p(J) \hookrightarrow C(J; X_p) \hookrightarrow C(J; C^{1+\delta}(\mathbb{R}^d)).$$

In order to apply Theorem 3.5.1 to obtain local well-posedness define $A : J_0 \times X_p \to \mathscr{L}(X_1, X_0)$ and $F : J_0 \times X_p \to X_0$ by

$$(A(t,v)u)(x) = \sum_{|\alpha| \leq 2} a_\alpha(t,x,v(x),\nabla v(x)) D^\alpha u(x),$$
$$F(t,u)(x) = f(t,x,u(x),\nabla u(x)).$$

Then A and F satisfy the conditions of Theorem 3.5.1. Indeed, applying (3.5.6) we find that for $R > 0$ and $\|v_1\|_{X_p}, \|v_2\|_{X_p} \leq R$ and $u \in X_1$,

$$\|A(t,v_1)u - A(t,v_2)u\|_{X_0} \leq K(R)\Big(\|v_1 - v_2\|_{X_0} + \|\nabla v_1 - \nabla v_2\|_{X_0}\Big)\|u\|_{X_1}$$

$$\leq K(R)C\|v_1 - v_2\|_{X_p}\|u\|_{X_1}.$$

Here we have used that for $k \in \{0,1\}$ and $v \in X_p$, $\|D^k v\|_\infty \leq C\|v\|_{X_p}$ by (3.5.5).

Next we check that for every $g \in X_p$, $A(\cdot, g)$ has maximal L^p-regularity. In order to do so we check that $A(t,g)$ satisfies the conditions of Theorem 3.4.5. Indeed, let $R = \|g\|_{C^1(\mathbb{R}^d)} < \infty$. By (3.5.5), $g \in C^{1+\delta}(\mathbb{R}^d)$ and therefore,

$$\begin{aligned}
&|a_\alpha(t,x_1,g(x_1),\nabla g(x_1)) - a_\alpha(t,x_2,g(x_2),\nabla g(x_2))| \\
&\quad\leq \quad |a_\alpha(t,x_1,g(x_1),\nabla g(x_1)) - a_\alpha(t,x_2,g(x_1),\nabla g(x_1))| \\
&\quad\quad + \ |a_\alpha(t,x_2,g(x_1),\nabla g(x_1)) - a_\alpha(t,x_2,g(x_2),\nabla g(x_2))| \\
&\quad\leq \omega_R(|x_1 - x_2|) + |g(x_1) - g(x_2)| + |\nabla g(x_1) - \nabla g(x_2)| \\
&\quad\leq \omega_R(|x_1 - x_2|) + \|g\|_{C^{1+\delta}(\mathbb{R}^d)}(|x_1 - x_2| + |x_1 - x_2|^\delta).
\end{aligned}$$

Thus $A(t,g)$ satisfies the required continuity condition in the space variable. Hence Theorem 3.4.5 yields that $A(\cdot, u_0)$ has maximal L^p-regularity. The conditions on F can be checked in a similar way and we obtain the following result as a consequence of Theorem 3.5.1.

Theorem 3.5.3. *Assume the above conditions on $p, q \in (1,\infty)$ and a_α and f. Let $g \in X_p := \mathscr{B}_{q,p}^{2(1-\frac{1}{p})}(\mathbb{R}^d)$ be arbitrary. Then there is a $T \in (0, T_0]$ and radius $\varepsilon > 0$ both depending on g such that for all $u_0 \in B_\varepsilon = \{v \in X_p : \|v - g\|_{X_p} \leq \varepsilon\}$,* (3.5.3) *admits a unique solution*

$$u \in W^{1,p}(J; L^q(\mathbb{R}^d)) \cap L^p(J; W^{2,q}(\mathbb{R}^d)) \cap C(J; X_p).$$

Moreover, there is a constant C such that for all $u_0, v_0 \in B_\varepsilon$ the corresponding solutions u and v satisfy

$$\|u - v\|_{W^{1,p}(J;L^q(\mathbb{R}^d))} + \|u - v\|_{L^p(J;W^{2,q}(\mathbb{R}^d))} + \|u - v\|_{C(J;X_p)} \leq C\|x - y\|_{X_p}.$$

Remark 3.5.4.

1. If a_α only depends on u and not on its derivatives, then one can replace (3.5.4) by the condition $2\left(1 - \frac{1}{p}\right) - \frac{d}{q} > 0$.
2. Theorem 3.5.3 can be extended to higher-order equations. Then a_α is allowed to depend on the $(m-1)$-th derivatives of u. Moreover, by the results that will be introduced in Chapter 5 one can also consider higher-order systems.

3.5.3. Proof of Theorem 3.5.1

From the trace estimate (3.3.22) and a simple reflection argument one sees see that there exists a constant C independent of T such that for all $u \in \mathrm{MR}^p(J)$ with $u(0) = 0$ one has

$$\|u\|_{C([0,T];X_p)} \leq C_{\mathrm{Tr}}\|u\|_{\mathrm{MR}^p(J)}. \tag{3.5.7}$$

Without the assumption $u(0) = 0$, one still has the above estimate but with a constant which blows up as $T \downarrow 0$ (see (3.3.21) and use a translation argument).

For $u, v \in \mathrm{MR}(J)$ with $u(0) = v(0) \in X_p$, the following consequence of (3.5.7) will be used frequently:

$$\begin{aligned} \|u\|_{C(J;X_p)} &\le \|u - v\|_{C(J;X_p)} + \|v\|_{C(J;X_p)} \\ &\le C_{\mathrm{Tr}}\|u - v\|_{\mathrm{MR}^p((0,T))} + \|v\|_{C(J;X_p)}. \end{aligned} \tag{3.5.8}$$

Proof of Theorem 3.5.1. We modify the presentation in [97] to our setting. By the assumption and Proposition 3.3.18 we know that for each $x \in X_p$, there exists a unique solution $w^x \in \mathrm{MR}^p(J)$ of the problem

$$\begin{cases} w'(t) + A(t, x_0)w(t) = F(t, x_0), & t \in J_0 \\ w(0) = x. \end{cases}$$

Moreover, by linearity

$$\|w^x - w^y\|_{\mathrm{MR}^p(J_0)} \le C_0\|x - y\|_{X_p}.$$

By (3.3.21) and a translation argument we see that

$$\|w^x - w^y\|_{C(J_0;X_p)} \le C_1\|w^x - w^y\|_{\mathrm{MR}^p(J_0)} \le C_1C_2\|x - y\|_{X_p}. \tag{3.5.9}$$

Step 1. Let C_A be the maximal L^p-regularity constant of $A(\cdot, u_0)$. We show that for a certain set of function $u \in \mathrm{MR}^p(J)$ maximal L^p-regularity holds with constant $2C_A$. Fix $R > 0$. Since $w^{x_0} : [0, T] \to X_p$ is continuous we can find $T \in (0, T_0]$ such that

$$\|w^{x_0}(t) - x_0\|_{X_p} \le \frac{1}{4C(R)C_A}, \quad t \in [0, T]. \tag{3.5.10}$$

Let

$$r_0 := \frac{1}{4C(R)C_A(C_{\mathrm{Tr}} + C_{\mathrm{Tr}}C_2 + C_1C_2)}$$

and write

$$\mathbb{B}_{r_0} = \{v \in \mathrm{MR}^p(J) : \|v(0) - x_0\|_{X_p} \le r_0 \text{ and } \|v - w^{x_0}\|_{\mathrm{MR}^p(J)} \le r_0\}.$$

From the assumptions we see that for all $v \in \mathbb{B}_{r_0}$ and $t \in [0, T]$, writing $x = v(0)$,

$$\begin{aligned} &\|v(t) - x_0\|_{X_p} \\ &\le \|v(t) - w^x(t)\|_{X_p} + \|w^x(t) - w^{x_0}(t)\|_{X_p} + \|w^{x_0}(t) - x_0\|_{X_p} \\ &\le C_{\mathrm{Tr}}\|v - w^x\|_{\mathrm{MR}^p(J)} + C_1C_2\|x - x_0\|_{X_p} + \frac{1}{4C(R)C_A} \\ &\le C_{\mathrm{Tr}}r_0 + C_{\mathrm{Tr}}\|w^{x_0} - w^x\|_{\mathrm{MR}^p(J)} + C_1C_2\|x - x_0\|_{X_p} + \frac{1}{4C(R)C_A} \\ &\le C_{\mathrm{Tr}}r_0 + C_{\mathrm{Tr}}C_2r_0 + C_1C_2r_0 + \frac{1}{4C(R)C_A} \le \frac{1}{2C(R)C_A}, \end{aligned} \tag{3.5.11}$$

where we used (3.5.7) and (3.5.9). Therefore by (3.5.11) and the assumption

$$\|A(t,v(t))-A(t,x_0)\|_{\mathscr{L}(X_1,X_0)} \le C(R)\|v(t)-x_0\|_{X_p} \le \frac{1}{2C_A}.$$

Now Proposition 3.3.23 yields that $A(\cdot,v(\cdot))$ has maximal L^p-regularity with constant $2C_A$ for each $v\in\mathbb{B}_{r_0}$.

Step 2. Let $R = 1+C_{\mathrm{Tr}}+C_{\mathrm{Tr}}C_2+C_1C_2+\|w^{x_0}\|_{C(J_0;X_p)}$. Fix $0<r\le\min\{1,r_0\}$ and T as in Step 1. Note that by (3.5.8) and (3.5.9) for $v\in\mathbb{B}_r$ and $x=v(0)$,

$$\begin{aligned}
\|v\|_{C(J;X_p)} &\le C_{\mathrm{Tr}}\|v-w^x\|_{\mathrm{MR}^p(J)}+\|w^x-w^{x_0}\|_{C(J;X_p)}+\|w^{x_0}\|_{C(J;X_p)}\\
&\le C_{\mathrm{Tr}}\|v-w^{x_0}\|_{\mathrm{MR}^p(J)}+C_{\mathrm{Tr}}\|w^{x_0}-w^x\|_{\mathrm{MR}^p(J)}+C_1C_2\|x-x_0\|_{X_p}+\|w^{x_0}\|_{C(J;X_p)}\\
&\le C_{\mathrm{Tr}}r+(C_{\mathrm{Tr}}C_2+C_1C_2)\|x-x_0\|_{X_p}+\|w^{x_0}\|_{C(J;X_p)}\\
&\le C_{\mathrm{Tr}}r+(C_{\mathrm{Tr}}C_2+C_1C_2)r+\|w^{x_0}\|_{C(J;X_p)}\le R,
\end{aligned}$$

where we used $r\le 1$. Similarly, for $x\in B_r$, $\|x\|_{X_p}\le r+\|x_0\|_{X_p}\le R$.

For $x\in B_r$, let $\mathbb{B}_{r,x}\subseteq\mathbb{B}_r$ be defined by

$$\mathbb{B}_{r,x}=\{u\in\mathrm{MR}^p(J): u(0)=x \text{ and } \|u-w^{x_0}\|_{\mathrm{MR}^p(J)}\le r\}.$$

Before we introduce a fixed point operator argument on $\mathbb{B}_{r,x}$, let

$$\begin{aligned}
f(v_1,v_2)&=F(t,v_1(t))-F(t,v_2(t)),\\
a(v_1,v_2,v_3)(t)&=(A(t,v_2(t))-A(t,v_1(t)))v_3(t).
\end{aligned}$$

for $v_j\in\mathbb{B}_{r,x_j}$ with $x_j\in B_r$ for $j\in\{1,2\}$ and $v_3\in\mathrm{MR}^p(J)$. Observe that by (3.5.8) and (3.5.9)

$$\begin{aligned}
\|v_1-v_2\|_{C(J;X_p)} &\le C_{\mathrm{Tr}}\|v_1-v_2-(w^{x_1}-w^{x_2})\|_{\mathrm{MR}^p(J)}+\|w^{x_1}-w^{x_2}\|_{C(J;X_p)}\\
&\le C_{\mathrm{Tr}}\|v_1-v_2\|_{\mathrm{MR}^p(J)}+(C_{\mathrm{Tr}}C_2+C_1C_2)\|x_1-x_2\|_{X_p}.
\end{aligned}$$

Let $C_J=\|\phi_R\|_{L^p(J)}$. For f we find

$$\begin{aligned}
\|f(v_1,v_2)\|_{L^p(J;X_0)} &\le \|\phi_R(v_1-v_2)\|_{L^p(J;X_p)}\le C_J\|v_1-v_2\|_{C(J;X_p)}\\
&\le C_JC_{\mathrm{Tr}}\|v_1-v_2\|_{\mathrm{MR}^p(J)}+C_JC_3\|x_1-x_2\|_{X_p},
\end{aligned}$$

where $C_3=(C_{\mathrm{Tr}}C_2+C_1C_2)$. Similarly, applying the estimate for v_1-v_2 again,

$$\begin{aligned}
\|a(v_1,v_2,v_3)\|_{L^p(J;X_0)} &\le C(R)\big\|\|v_2-v_1\|_{X_p}\|v_3\|_{X_1}\big\|_{L^p(J;X_0)}\\
&\le C(R)\|v_1-v_2\|_{C(J;X_p)}\|v_3\|_{\mathrm{MR}^p(J)}\\
&\le C(R)\|v_3\|_{\mathrm{MR}^p(J)}\big[C_{\mathrm{Tr}}\|v_1-v_2\|_{\mathrm{MR}^p(J)}+C_3\|x_1-x_2\|_{X_p}\big].
\end{aligned}$$

For $v\in\mathbb{B}_{r,x}$ and $x\in B_r$ let $L_x(v)=u\in\mathrm{MR}^p(J)$ denote the solution of

$$\begin{cases}u'(t)+A(t,v(t))u(t)=F(t,v(t)), & t\in J_0\\ u(0)=x.\end{cases}$$

For v_1, v_2 as before let $u_j := L_{x_j}(v_j)$ for $j \in \{1,2\}$. We find that $u := u_1 - u_2$ in $\mathrm{MR}^p(J)$ satisfies $u(0) = x_1 - x_2$ and

$$u'(t) + A(t, v_1(t))u(t) = f(v_1, v_2)(t) + a(v_1, v_2, u_2)(t), \quad t \in J.$$

Therefore, by Step 1, Proposition 3.3.18 and the previous estimates, we find

$$\begin{aligned} &\|L_{x_1}(v_1) - L_{x_2}(v_2)\|_{\mathrm{MR}^p(J)} \\ &\le 2C_A\Big(\|x_1 - x_2\|_{X_p} + \|f(v_1, v_2)\|_{L^p(J;X_0)} + \|a(v_1, v_2, u_2)\|_{L^p(J;X_0)}\Big) \\ &\le K_1(\|u_2\|_{\mathrm{MR}^p(J)})\|x_1 - x_2\|_{X_p} + K_2(\|u_2\|_{\mathrm{MR}^p(J)})\|v_1 - v_2\|_{\mathrm{MR}^p(J)}, \end{aligned} \tag{3.5.12}$$

where for $s \ge 0$,

$$\begin{aligned} K_1(s) &= 2C_A(1 + C_J C_3 + C(R)C_3 s), \\ K_2(s) &= 2C_A\big[C_J C_{\mathrm{Tr}} + C(R)C_{\mathrm{Tr}} s\big]. \end{aligned}$$

Extending the definitions of L, f and a in the obvious way we can write $w^{x_0} = L_{x_0}(x_0)$. Estimating as before, one sees that for $x \in B_r$ and $v \in \mathbb{B}_{r,x}$,

$$\begin{aligned} &\|L_x(v) - L_{x_0}(x_0)\|_{\mathrm{MR}^p(J)} \\ &\le 2C_A\Big(\|x - x_0\|_{X_p} + \|f(v, x_0\|_{L^p(J;X_0)} + \|a(v, x_0, w^{x_0})\|_{L^p(J;X_0)}\Big) \\ &\le 2C_A\Big(\|x - x_0\|_{X_p} + \big[C_J + C(R)\|w^{x_0}\|_{\mathrm{MR}^p(J)}\big]\|v - x_0\|_{C(J;X_p)}\Big), \\ &\le 2C_A\Big(\|x - x_0\|_{X_p} + \big[C_J + C(R)\|w^{x_0}\|_{\mathrm{MR}^p(J)}\big]\frac{1}{2C(R)C_A}\Big), \end{aligned} \tag{3.5.13}$$

where in the last step we used (3.5.11).
Choose $0 < r \le \min\{1, r_0\}$ such that

$$4C_A r C(R) C_{\mathrm{Tr}} \le \frac{1}{4}.$$

Choose T such that (3.5.10) holds,

$$2C_A C_J C_{\mathrm{Tr}} \le \frac{1}{4}, \quad \frac{C_J}{C(R)} \le \frac{r}{4}, \quad \text{and} \quad \|w^{x_0}\|_{\mathrm{MR}^p(J)} \le \frac{r}{4}.$$

Let $\varepsilon = \min\{\frac{r}{4C_A}, r\}$. Then from (3.5.13) we obtain that for all $x \in B_\varepsilon$, L_x maps $\mathbb{B}_{r,x}$ into itself. In particular, for all $x \in B_\varepsilon$ and $v \in \mathbb{B}_{r,x}$,

$$\|L_x(v)\|_{\mathrm{MR}^p(J)} \le \|L_x(v) - w^{x_0}\|_{\mathrm{MR}^p(J)} + \|w^{x_0}\|_{\mathrm{MR}^p(J)} \le r + \frac{r}{4} \le 2r. \tag{3.5.14}$$

Moreover, for all $x_j \in B_\varepsilon$ and $v_j \in \mathbb{B}_{r,x_j}$ for $j \in \{1,2\}$,

$$\|L_{x_1}(v_1) - L_{x_2}(v_2)\|_{\mathrm{MR}^p(J)} \le K_1(2)\|x_1 - x_2\|_{X_p} + \frac{1}{2}\|v_1 - v_2\|_{\mathrm{MR}^p(J)}, \tag{3.5.15}$$

where we used (3.5.12) and (3.5.14). In particular, L_x defines a contraction on $\mathbb{B}_{r,x}$ and by the Banach contraction principle we find that there exists a unique $u \in \mathbb{B}_{r,x}$ such that $L_x(u) = u$. This yields the required result.

The final estimate of the theorem follows from (3.5.15). □

3.A. APPENDIX: γ-BOUNDEDNESS

We give here a brief description of γ-boundedness and related notions. The interested reader can find more details in [74, 85, 104, 121].

3.A.1. Type and cotype

We first define type and cotype of a Banach space X. Most of this material is taken from [85].

Definition 3.A.1. A Banach space X has *type* $p \in [1,2]$ if there exists a constant $\tau \ge 0$ such that for all sequences $(x_n)_{n=1}^N \in X$, $N \in \mathbb{N}$, we have

$$(\mathbb{E}\|\sum_{n=1}^N \varepsilon_n x_n\|^p)^{1/p} \le \tau(\|\sum_{n=1}^N x_n\|^p)^{1/p},$$

and X has *cotype* $q \in [2,\infty]$ if there exists a constant $c \ge 0$ such that for all sequences $(x_n)_{n=1}^N \in X$, $N \in \mathbb{N}$, we have

$$(\|\sum_{n=1}^N x_n\|^q)^{1/q} \le c(\mathbb{E}\|\sum_{n=1}^N \varepsilon_n x_n\|^q)^{1/q},$$

with the obvious modifications if $q = \infty$.

The least admissible constants in the estimates above are called respectively the *type p constant* $\tau_{p,X}$ and the *cotype q constant* $c_{q,X}$ of X. By the Kahane-Khintchine inequalities (cf. Theorem 2.7.2), the exponents p and q (with exception of $q = \infty$) could be replaced by any $r \in [1,\infty)$, up to different constants.

We say that X has non-trivial type if it has type $p \in (1,2]$. finite cotype if it has cotype $q \in [2,\infty)$.

Example 3.A.2. We list some examples of type and cotype. We refer to [85] for details.

1. Every Banach space X has type 1 and cotype ∞, with type and cotype constants equal to 1.
2. The space c_0 does not have any non-trivial type.
3. Every Hilbert space H has type 2 and cotype 2, with type and cotype constants equal to 1. This follows by the properties of the inner product.

The next proposition states a duality relation between type and cotype (see [85, Proposition 7.7])

Proposition 3.A.3. *If X has type $p \in [1,2]$, then X^* has cotype $q \in [2,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $c_{q,X^*} \le \tau_{p,X}$.*

The proof follows by a simple duality argument. For more properties of type and cotype, we refer the reader to [45, 85].

3.A.2. γ-boundedness

Let now $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. Let $(\gamma_n)_{n=1}^N$ be a finite *Gaussian sequence*, i.e. a sequence of independent Gaussian random variables on Ω. The following definitions can be found in [121] and [85].

Definition 3.A.4. A family of operators $\mathscr{T} \in \mathscr{L}(X)$ is said to be *γ-bounded* if there exists a constant $C \geq 0$ such that for all $N \in \mathbb{N}$, $(T_n)_{n=1}^N \in \mathscr{T}$, $(x_n)_{n=1}^N \in X$

$$\Big\| \sum_{n=1}^N \gamma_n T_n x_n \Big\|_{L^2(\Omega;X)} \leq C \Big\| \sum_{n=1}^N \gamma_n x_n \Big\|_{L^2(\Omega;X)}.$$

The least admissible constant C is called the *γ-bound* of $\mathscr{T}$, and it is denoted by $\gamma(\mathscr{T})$.

Every γ-bounded family $\mathscr{T}$ is uniformly bounded with $\sup_{T\in\mathscr{T}} \|T\| \leq \gamma(\mathscr{T})$, and the reverse holds if X is a Hilbert space.

Replacing the Gaussian random variables by Rademachers functions in the above definitions, we arrive at the related notion of $\mathscr{R}$-boundedness. In particular, every $\mathscr{R}$-bounded family is γ-bounded, and the reverse holds if X has finite cotype.

An important result concerning γ-boundedness is the so-called γ-multiplier theorem, due to Kalton and Weis in [95, Proposition 4.11]. The version stated below in Theorem 3.A.7 is taken from [121, Theorem 5.2], and it is the one of interest for us.

Definition 3.A.5. Let H be a Hilbert space. An operator $T \in \mathscr{L}(H;X)$ is called *γ-summing* if

$$\|T\|^2_{\gamma_\infty(H;X)} := \sup_h \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n T\, h_n \Big\|^2 < \infty,$$

where the supremum is taken over all finite orthonormal systems $h = \{h_1, \dots, h_n\}$ in H. The space of all γ-summing operators from H to X is denoted by $\gamma_\infty(H;X)$.

With respect to the norm $\|T\|_{\gamma_\infty(H;X)}$, the space $\gamma_\infty(H;X)$ is a Banach space. We denote as $\gamma(H;X)$ the closure of the finite rank operators in $\gamma_\infty(H;X)$. The operators $T \in \gamma(H;X)$ are called *γ-radonifying*. The following properties hold.

Proposition 3.A.6. *Let $T \in \mathscr{L}(H;X)$. Then the following hold.*

(i) Every γ-radonifying operator T is γ-summing and $\|T\|_{\gamma_\infty(H;X)} = \|T\|_{\gamma(H;X)}$.

(ii) If X does not contain a closed subspace isomorphic to c_0, then $\gamma_\infty(H;X) = \gamma(H;X)$;

(iii) On $\gamma_\infty(H;X)$ the equivalent norm

$$\|T\|^p_{\gamma_{\infty,p}(H;X)} := \sup_h \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n T h_n \Big\|^p$$

holds.

We remark that (ii) fails for $H=\ell^2$ and $X=c_0$. For the proofs, we refer to [121, Proposition 3.15] for *(i)* and [121, Theorem 4.3] for *(ii)*, while statement *(iii)* follows directly from Kahane's inequality.

Let now $(\mathscr{A},\Sigma,\mu)$ be a σ-finite measure space. For a bounded and strongly measurable function $\phi:\mathscr{A}\to\mathscr{L}(H,X)$ we define the operator $T_\phi\in\mathscr{L}(L^2(\mathscr{A};H),X)$ by

$$T_\phi f := \int_\Omega \phi f \, d\mu.$$

If ϕ is a finite rank simple function, i.e. a simple function with values in $\gamma(H;X)$, then $T_\phi\in\gamma(L^2(\mathscr{A};H);X)$.

Theorem 3.A.7 (Kalton-Weis γ-multiplier theorem)**.** *Let X,Y be Banach spaces and $(\mathscr{A},\Sigma,\mu)$ be a σ-finite measure space. Suppose that $M:\mathscr{A}\to\mathscr{L}(X,Y)$ is strongly measurable and that $\mathscr{T}:=\{M(t):t\in\mathscr{A}\}$ is γ-bounded. Then for every finite rank simple functions $\phi:\mathscr{A}\to\gamma(H;X)$, the operator $T_{M\phi}$ belongs to $\gamma_\infty(L^2(\mathscr{A};H);Y)$ and*

$$\|T_{M\phi}\|_{\gamma_\infty(L^2(\mathscr{A};H);Y)}\le\gamma(\mathscr{T})\|T_\phi\|_{\gamma_\infty(L^2(\mathscr{A};H);X)}$$

As a result, the map $\tilde{M}:T_\phi\to T_{M\phi}$ has a unique extension to a bounded operator $\tilde{M}:\gamma_\infty(L^2(\mathscr{A};H);X)\to\gamma_\infty(L^2(\mathscr{A};H);Y)$ of norm $\|\tilde{M}\|\le\gamma(\mathscr{T})$.

Moreover, the following γ-Fubini property holds. This is taken from [121, Theorem 13.6].

Theorem 3.A.8. *The isomorphism*

$$\gamma(H;L^p(\mathscr{A};X))\simeq L^p(\mathscr{A};\gamma(H;X))$$

holds for every $p\in[1,\infty)$.

4

On the ℓ^s_H-boundedness of a family of integral operators

In this chapter we consider operators with values in a Hilbert space H and we prove an ℓ^s_H-boundedness result for integral operators with operator-valued kernels. The proofs are based on extrapolation techniques with weights due to Rubio de Francia (Section 4.2). As a consequence of the main result Theorem 4.3.5, in the special case $H = \mathbb{C}$ we will prove Theorem 3.1.4, which gives a sufficient condition for the $\mathscr{R}$-boundedness of a family of integral operators and has been already applied in Chapter 3. The generalization to ℓ^s_H-boundedness will be needed in Chapter 5, in order to prove a maximal regularity result for systems of parabolic equations. The results here presented are base on [62].

4.1. Preliminaries on ℓ^s-boundedness

In this section we introduce the notions of ℓ^s-boundedness and ℓ^s_H-boundedness, where H is an Hilbert space, and we present some simple example.

4.1.1. ℓ^s-boundedness

In this section we introduce ℓ^s-boundedness. For this we will use the notion of a Banach lattice (see [109]). An example of a Banach lattice is L^p or any Banach function space (see [159, Section 63]). In our main results only iterated L^p-spaces will be needed. They will be introduced later in (4.2.1).

Although ℓ^s-boundedness is used implicitly in the literature for operators on L^p-spaces, on Banach functions spaces it was introduced in [152] under the name $\mathscr{R}_s$-boundedness. An extensive study can be found in [103, 149].

Definition 4.1.1. Let X and Y be Banach lattices and let $s \in [1,\infty]$. We call a family of operators $\mathscr{T} \subseteq \mathscr{L}(X,Y)$ *ℓ^s-bounded* if there exists a constant C such that for all integers N, for all sequences $(T_n)_{n=1}^N$ in $\mathscr{T}$ and $(x_n)_{n=1}^N$ in X,

$$\left\| \left(\sum_{n=1}^{N} |T_n x_n|^s \right)^{\frac{1}{s}} \right\|_Y \leq C \left\| \left(\sum_{n=1}^{N} |x_n|^s \right)^{\frac{1}{s}} \right\|_X$$

with the obvious modification for $s = \infty$. The least possible constant C is called the *ℓ^s-bound* of $\mathscr{T}$ and is denoted by $\mathscr{R}^{\ell^s}(\mathscr{T})$ and often abbreviated as $\mathscr{R}^s(\mathscr{T})$.

Example 4.1.2. Take $p \in (1,\infty)$ and let $\mathscr{T} \subseteq \mathscr{L}(L^p(\mathbb{R}^d))$ be uniformly bounded by a constant C. Then $\mathscr{T}$ is ℓ^p-bounded with $\mathscr{R}^p(\mathscr{T}) \leq C$.

The following basic properties will be needed later on.

Proposition 4.1.3. *Let $\mathscr{T} \subseteq \mathscr{L}(X,Y)$, where X and Y are Banach function spaces.*

1. *Let $1 \leq s_0 < s_1 \leq \infty$ and assume that X and Y have an order continuous norm. If $\mathscr{T} \subseteq \mathscr{L}(X,Y)$ is ℓ^{s_j}-bounded for $j = 0,1$, then $\mathscr{T}$ is ℓ^s-bounded for all $s \in [s_0, s_1]$ and with $\theta = \frac{s-s_0}{s_1-s_0}$, the following estimate holds:*

$$\mathscr{R}^s(\mathscr{T}) \leq \mathscr{R}^{s_0}(\mathscr{T})^{1-\theta}\mathscr{R}^{s_1}(\mathscr{T})^{\theta} \leq \max\{\mathscr{R}^{s_0}(\mathscr{T}), \mathscr{R}^{s_1}(\mathscr{T})\}$$

2. *If $\mathscr{T}$ is ℓ^s-bounded, then the adjoint family $\mathscr{T}^* = \{T^* \in \mathscr{L}(Y^*, X^*) : T \in \mathscr{T}\}$ is $\ell^{s'}$-bounded and $\mathscr{R}^{s'}(\mathscr{T}^*) = \mathscr{R}^s(\mathscr{T})$.*

Proof. (1) follows from Calderón's theory of complex interpolation of vector-valued function spaces (see [25] and [103, Proposition 2.14]). For (2) we refer to [103, Proposition 2.17] and [123, Proposition 3.4]. □

Remark 4.1.4. Below we will only need Proposition 4.1.3 in the case $X = Y = L^{\overline{q}}(\Omega)$. To give the details of the proof of Proposition 4.1.3 in this situation one first needs to know that $X^* = L^{\overline{q}'}(\Omega)$ which can be obtained by elementary arguments (see Proposition 4.A.1 below). As a second step one needs to show that $X(\ell^s_N)^* = X^*(\ell^{s'}_N)$ and this is done in Lemma 4.A.2.

Example 4.1.5. Let $1 \leq s_0 \leq q \leq s_1 \leq \infty$. Let $X = L^q(\Omega)$ and let $\mathscr{T} \subset \mathscr{L}(X)$ be ℓ^{s_j}-bounded for $j \in \{0,1\}$. Then for $s \in [s_0, q]$, $\mathscr{R}^s(\mathscr{T}) \leq \mathscr{R}^{s_0}(\mathscr{T})$ and for $s \in [q, s_1]$, $\mathscr{R}^s(\mathscr{T}) \leq \mathscr{R}^{s_1}(\mathscr{T})$. Indeed, note that by Example 4.1.2,

$$\mathscr{R}^q(\mathscr{T}) = \sup_{T \in \mathscr{T}} \|T\| \leq \mathscr{R}^{s_j}(\mathscr{T}), \quad j \in \{0,1\}.$$

Now the estimates follow from Proposition 4.1.3 by interpolating with exponents (s_0, q) and (q, s_1). In particular, it follows that the function $s \mapsto \mathscr{R}^s(\mathscr{T})$, is decreasing on $[s_0, q]$ and increasing on $[q, s_1]$.

4.1.2. ℓ^s_H-boundedness

In order to deal with systems of operators in Chapter 5, we need to generalize the definition of ℓ^s-boundedness to the setting of operators with values in a Hilbert space H, i.e. $X(H) = L^q(\mathbb{R}^d; H)$. In the case H has finite dimension N, one could apply Definiton 4.1.1 coordinate-wise, but this only yields estimates with N-dependent constants. To avoid this, in this chapter we directly consider H-valued operators and we introduce the notion of ℓ^s_H-boundedness, which is an extension of ℓ^s-boundedness to this setting. For $H = \mathbb{C}^N$, this will give a sufficient condition to obtain maximal L^p-regularity for systems of operators (see Theorems 4.3.12 and 5.1.3), while Theorem 3.1.4 will follow directly taking $H = \mathbb{C}$.

In the following we introduce the definition of ℓ^s_H-boundedness, for an Hilbert space H.

Definition 4.1.6. Let H be a Hilbert space and let $s \in [1,\infty]$. We call a family of operators $\mathscr{T} \subseteq \mathscr{L}(X(H), Y(H))$ ℓ^s_H*-bounded* if there exists a constant C such that for all integers N, for all sequences $(T_n)_{n=1}^N$ in $\mathscr{T}$ and $(x_n)_{n=1}^N$ in $X(H)$,

$$\Big\| \Big(\sum_{n=1}^N \|T_n x_n\|_H^s \Big)^{\frac{1}{s}} \Big\|_Y \leq C \Big\| \Big(\sum_{n=1}^N \|x_n\|_H^s \Big)^{\frac{1}{s}} \Big\|_X$$

with the obvious modification for $s = \infty$. The least possible constant C is called the ℓ^s_H*-bound* of $\mathscr{T}$ and is denoted by $\mathscr{R}^{\ell^s}_H(\mathscr{T})$ and often abbreviated as $\mathscr{R}^s_H(\mathscr{T})$.

Proposition 4.1.3 and Example 4.1.5 can be directly generalized to this setting. Below we will only need the case $X(H) = Y(H) = L^q(\mathbb{R}^d; H)$.

4.2. EXTRAPOLATION IN $L^p(L^q)$-SPACES

Take $n \in \mathbb{N}$ and let for $i = 1, \cdots, n$ the triple $(\Omega_i, \Sigma_i, \mu_i)$ be a σ-finite measure space. Define the product measure space

$$(\Omega, \Sigma, \mu) = (\Omega_1 \times \cdots \times \Omega_n, \Sigma_1 \times \cdots \times \Sigma_n, \mu_1 \times \cdots \times \mu_n)$$

Then of course (Ω, Σ, μ) is also σ-finite. For $\overline{q} \in (1,\infty)^n$ we write

$$L^{\overline{q}}(\Omega) = L^{q_1}(\Omega_1, \cdots L^{q_n}(\Omega_n)). \tag{4.2.1}$$

In this section we extend Theorem 2.2.3 to values in the above mixed $L^{\overline{q}}(\Omega)$ spaces. For the case $\Omega = \mathbb{N}$ this was already done in [32, Corollary 3.12]. This will be needed later on in the proofs.

Theorem 4.2.1. *Let $f, g : \mathbb{R}^d \times \Omega \to \mathbb{R}_+$ be a pair of nonnegative, measurable functions and suppose that for some $p_0 \in (1,\infty)$ there exists an increasing function α on $\mathbb{R}_+$ such that for all $w_0 \in A_{p_0}$*

$$\|f(\bullet, s)\|_{L^{p_0}(\mathbb{R}^d, w_0)} \leq \alpha([w_0]_{A_{p_0}}) \|g(\bullet, s)\|_{L^{p_0}(\mathbb{R}^d, w_0)} \tag{4.2.2}$$

for all $s \in \Omega$. Then for all $p \in (1,\infty)$ and $\overline{q} \in (1,\infty)^n$ there exist $c_{p,\overline{q},d} > 0$ and $\beta_{p_0,p,\overline{q}} > 0$ such that for all $w \in A_p$,

$$\|f\|_{L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))} \leq 4^n \alpha\Big(c_{p,\overline{q},d} [w]_{A_p}^{\beta_{p_0,p,\overline{q}}} \Big) \|g\|_{L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))}. \tag{4.2.3}$$

Proof. We will prove this theorem by induction. The base case $n = 0$ is just weighted extrapolation, as covered in Theorem 2.2.3.

Now take $n \in \mathbb{N} \cup \{0\}$ arbitrary and assume that the assertion holds for all pairs $f, g : \mathbb{R}^d \times \Omega \to \mathbb{R}_+$ of nonnegative, measurable functions. Let $(\Omega_0, \Sigma_0, \mu_0)$ be a σ-finite measure space and take nonnegative, measurable functions $f, g : \mathbb{R}^d \times \Omega_0 \times \Omega \to \mathbb{R}_+$. Assume that (4.2.2) holds for p_0, all $w \in A_{p_0}$ and all $s \in \Omega_0 \times \Omega$.

Now take $(s_0, s_1, \cdots, s_n) \in \Omega_0 \times \Omega$ arbitrary. Let $\overline{q} \in (1,\infty)^n$ be given and take $r \in (1,\infty)$ arbitrary. Define $\overline{r} = (r, q_1, \cdots, q_n)$ and the pair of functions $F, G : \mathbb{R}^d \to [0,\infty]$ as

$$F(x) = \left\| f(x,\bullet) \right\|_{L^{\overline{r}}(\Omega\times\Omega_0)} \qquad G(x) = \left\| g(x,\bullet) \right\|_{L^{\overline{r}}(\Omega\times\Omega_0)}$$

By our induction hypothesis we know for all $p \in (1,\infty)$ there exist $c_{p,\overline{q},d}$ and $\beta_{p_0,p,\overline{q}}$ such that for all $w \in A_p$

$$\|f(\bullet, s_0, \bullet)\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))} \leq 4^n \alpha(c_{p,\overline{q},d}[w]_{A_p}^{\beta_{p_0,p,\overline{q}}})\|g(\bullet, s_0, \bullet)\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))}$$

Now taking $p = r$ we obtain

$$\begin{aligned}
\|F\|_{L^r(\mathbb{R}^d,w)} &= \Big(\int_{\Omega_0}\int_{\mathbb{R}^d} \|f(x,s_0,\bullet)\|^r_{L^{\overline{q}}(\Omega)}\, w(x)\,\mathrm{d}x\,\mathrm{d}\mu_0\Big)^{\frac{1}{r}} \\
&\leq 4^n \alpha(c_{r,\overline{q},d}[w]_{A_r}^{\beta_{p_0,r,\overline{q}}})\Big(\int_{\Omega_0}\int_{\mathbb{R}^d} \|g(x,s_0,\bullet)\|^r_{L^{\overline{q}}(\Omega)}\, w(x)\,\mathrm{d}x\,\mathrm{d}\mu_0\Big)^{\frac{1}{r}} \\
&= 4^n \alpha(c_{r,\overline{q},d}[w]_{A_r}^{\beta_{p_0,r,\overline{q}}})\|G\|_{L^r(\mathbb{R}^d,w)}
\end{aligned}$$

using Fubini's theorem in the first and third step. So with Theorem 2.2.3 using $p_0 = r$ we obtain for all $p \in (1,\infty)$ that there exist $c_{r,p,\overline{q},d} > 0$ and $\beta_{p_0,p,\overline{r}} > 0$ such that for all $w \in A_p$,

$$\begin{aligned}
\|f\|_{L^p(\mathbb{R}^d,w;L^{\overline{r}}(\Omega_0\times\Omega))} &= \|F\|_{L^p(\mathbb{R}^d,w)} \\
&\leq 4^{n+1}\alpha\Big(c_{r,p,\overline{q},d}[w]_{A_p}^{\beta_{p_0,p,\overline{r}}}\Big)\|G\|_{L^p(\mathbb{R}^d,w)} \\
&= 4^{n+1}\alpha\Big(c_{r,p,\overline{q},d}[w]_{A_p}^{\beta_{p_0,p,\overline{r}}}\Big)\|g\|_{L^p(\mathbb{R}^d,w;L^{\overline{r}}(\Omega_0\times\Omega))}.
\end{aligned}$$

This proves (4.2.3) for $n+1$. □

Remark 4.2.2. Note that in the application of Theorem 4.2.1 it will often be necessary to use an approximation by simple functions to check the requirements, since point evaluations in (4.2.2) are not possible in general. Furthermore note that in the case that $f = Tg$ with T a bounded linear operator on $L^p(\mathbb{R}^d, w)$ for all $w \in A_p$ this theorem holds for all UMD Banach function spaces, which is one of the deep results of Rubio de Francia and can be found in [135, Theorem 5].

As an application of Theorem 4.2.1 we will present a short proof of the boundedness of the Hardy-Littlewood maximal operator on mixed $L^{\overline{q}}$-spaces.

Definition 4.2.3. Let $p \in (1,\infty)$ and $w \in A_p$. For $f \in L^p(\mathbb{R}^d, w; X)$ with $X = L^{\overline{q}}(\Omega)$ we define the maximal function $\widetilde{M}$ as

$$\widetilde{M} f(x,s) = \sup_{Q \ni x} \fint_Q |f(y,s)| \, \mathrm{d}y$$

with Q all cubes in $\mathbb{R}^d$ as before.

We can see that $\widetilde{M}$ is measurable, as the value of the supremum in the definition stays the same if we only consider rational cubes. We will show that the maximal function is bounded on the space $X = L^{\overline{q}}(\Omega)$. Note that if $\Omega = \mathbb{N}$, the result below reduces to the weighted version of the Fefferman-Stein theorem [12].

Theorem 4.2.4. *$\widetilde{M}$ is bounded on $L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))$ for all $p \in (1,\infty)$ and $w \in A_p$.*

Proof. Let M be the Hardy-Littlewood maximal operator and assume that $f \in L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))$ is simple. By Proposition 2.2.1 and the definition of the Hardy-Littlewood maximal operator we know that

$$\|\widetilde{M} f(\bullet, s)\|_{L^p(\mathbb{R}^d,w)} = \|M f(\bullet, s)\|_{L^p(\mathbb{R}^d,w)} \leq C_{p,d} \cdot [w]_{A_p}^{\frac{1}{p-1}} \|f(\bullet, s)\|_{L^p(\mathbb{R}^d,w)}$$

Then by Theorem 4.2.1 we get that

$$\|\widetilde{M} f\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))} \leq \alpha_{p,\overline{q},d}([w]_{A_p}) \|f\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))}$$

with $\alpha_{p,\overline{q},d}$ an increasing function on $\mathbb{R}_+$. With a density argument we then get that $\widetilde{M}$ is bounded on $L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))$. $\square$

Remark 4.2.5. Using deep connections between harmonic analysis with weights and martingale theory, Theorem 4.2.4 was obtained in [21] and [135, Theorem 3] for UMD Banach function spaces in the case $w = 1$. It has been extended to the weighted setting in [146]. As our main result Theorem 4.3.5 is formulated for iterated $L^{\overline{q}}(\Omega)$-spaces we prefer the above more elementary treatment.

4.3. MAIN RESULT

In this section we present the main results of this chapter, Theorem 4.3.5 and Corollary 4.3.9, and we prove Theorem 3.1.4 as a consequence. We will first obtain some preliminary results on convolution operators and integral operators which will be needed in the proofs.

4.3.1. Convolution operators

Recall the class of kernels $\mathcal{K}$ from Definition 3.1.1:

$$\mathcal{K} = \{k \in L^1(\mathbb{R}^d) : \text{for all simple } f : \mathbb{R}^d \to \mathbb{R}_+ \text{ one has } |k * f| \leq M f \text{ a.e.}\}.$$

To keep the presentation as simple as possible we only consider the iterated space $X = L^{\overline{q}}(\Omega, H)$ with $\overline{q} \in (1,\infty)^n$ below (see (4.2.1)). For a kernel $k \in L^1(\mathbb{R}^d)$, $p \in (1,\infty)$ and $w \in A_p$ define the convolution operator T_k on $L^p(\mathbb{R}^d, w; X)$ as $T_k f = k * f$. Of course by the definition of $\widetilde{M}$ we also have $|k * f| \le \widetilde{M} f$ almost everywhere for all simple $f : \mathbb{R}^d \to X$.

Proposition 4.3.1. *Let $\overline{q} \in (1,\infty)^n$ and $X(H) = L^{\overline{q}}(\Omega, H)$. For all $s \in [1,\infty]$ and $p \in (1,\infty)$ and $w \in A_p$, the family of convolution operators $\mathscr{T} = \{T_k : k \in \mathscr{K}\}$ on $L^p(\mathbb{R}^d, w; X(H))$ is ℓ^s_H-bounded and there is an increasing function $\alpha_{p,\overline{q},s,d}$ such that*

$$\mathscr{R}^s(\mathscr{T}) \le \alpha_{p,\overline{q},s,d}([w]_{A_p}).$$

Proof. Let $1 < s < \infty$. Assume that $f_1, \cdots, f_N$ are simple. Take $t \in \Omega$ and $i \in \{1, \cdots, N\}$ arbitrary. Note that we have $f_i(\bullet, t) \in L^p(\mathbb{R}^d, w)$. Then since $|T_{k_i} f_i(x,t)| \le \widetilde{M} f_i(x,t)$ for almost all $x \in \mathbb{R}^d$, the result follows from Theorem 4.2.4 using the vector $(q_1, \cdots, q_n, s)$ and the measure space

$$\big(\Omega \times \{1, \cdots, N\}, \Sigma \times P(\{1, \cdots, N\}), \mu \times \lambda\big)$$

with λ the counting measure. Now the result follows by the density of the simple functions in $L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega, H))$.

The proof of the cases $s = 1$ and $s = \infty$ follow the lines of [123, Theorem 4.7], where the unweighted setting is considered. In the case $s = \infty$ also assume that $f_1, \cdots, f_N$ are simple. With the boundedness of $\widetilde{M}$ from Theorem 4.2.4 we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \Big\| \sup_{1 \le n \le N} |T_{k_n} f_n(x)| \Big\|^p_{L^{\overline{q}}(\Omega,H)} w(x)\, \mathrm{d}x &\le \int_{\mathbb{R}^d} \Big\| \sup_{1 \le n \le N} \widetilde{M} f_n(x) \Big\|^p_{L^{\overline{q}}(\Omega,H)} w(x)\, \mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \Big\| \widetilde{M}\Big(\sup_{1 \le n \le N} |f_n| \Big)(x) \Big\|^p_{L^{\overline{q}}(\Omega,H)} w(x)\, \mathrm{d}x \\
&\le \alpha_{p,\overline{q},d}([w]_{A_p})^p \int_{\mathbb{R}^d} \Big\| \Big(\sup_{1 \le n \le N} |f_n| \Big)(x) \Big\|^p_{L^{\overline{q}}(\Omega,H)} w(x)\, \mathrm{d}x
\end{aligned}$$

with $\alpha_{p,\overline{q},d}$ an increasing function on $\mathbb{R}_+$. The claim now follows by the density of the simple functions in $L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega, H))$.

For $s = 1$ we use duality. For $f \in L^p(\mathbb{R}^d, w; X(H))$ and $g \in L^{p'}(\mathbb{R}^d, w'; X^*)$, let

$$\langle f, g \rangle = \int_{\mathbb{R}^d} \langle f(x), g(x) \rangle_{X, X(H)^*}\, dx.$$

It follows from Proposition 4.A.1 that in this way $L^p(\mathbb{R}^d, w; X(H))^* = L^{p'}(\mathbb{R}^d, w'; X(H)^*)$. Moreover, one has $T_k^* = T_{\tilde{k}}$ with $\tilde{k}(x) = k(-x)$. Now since $k \in \mathscr{K}$ if and only if $\tilde{k} \in \mathscr{K}$ we know by the second case that the adjoint family $\mathscr{T}^* = \{T^* : T \in \mathscr{T}\}$ is ℓ^∞-bounded on $L^{p'}(\mathbb{R}^d, w'; X(H)^*)$. Now the result follows from Proposition 4.1.3. $\square$

Remark 4.3.2. Proposition 4.3.1 is an extension of [123, Theorem 4.7] to the weighted setting. The result remains true for UMD Banach function spaces X and can be proved using the same techniques of [123] where one needs to apply the weighted extension of [135, Theorem 3] which is obtained in [146].

The endpoint case $s = 1$ of Proposition 4.3.1 plays a crucial rôle in the proof of Theorems 4.3.5 and 4.3.12. Quite surprisingly the case $s = 1$ plays a central rôle in the proof of [123, Theorem 7.2] as well, where it is used to prove $\mathscr{R}$-boundedness of a family of stochastic convolution operators.

4.3.2. Integral operators with operator valued kernel

In this section H is a Hilbert space and (Ω, Σ, μ) is a σ-finite measure space such that $L^q(\Omega; H)$ is separable for some (for all) $q \in (1, \infty)$.

Definition 4.3.3. Let $\mathscr{J}$ be an index set. For each $j \in \mathscr{J}$, let $T_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathscr{L}(L^q(\Omega; H))$ be such that for all $\phi \in L^q(\Omega; H)$, $(x, y) \mapsto T_j(x, y)\phi$ is measurable and $\|T_j(x, y)\| \le 1$. For $k \in \mathscr{K}$ define the operator I_{k,T_j} on $L^p(\mathbb{R}^d, v; L^q(\Omega; H))$ as

$$I_{k,T_j} f(x) = \int_{\mathbb{R}^d} k(x - y) T_j(x, y) f(y)\, \mathrm{d}y \tag{4.3.1}$$

and denote the family of all such operators by $\mathscr{I}_T$.

In the above definition we consider a slight generalization of the setting of Theorem 3.1.4: We allow different operators T_j for $j \in \mathscr{J}$ in the ℓ^s_H-boundedness result of Theorem 4.3.5.

We first prove that the family of operators $\mathscr{I}_T$ is uniformly bounded.

Lemma 4.3.4. *Let $1 < p, q < \infty$ and write $X(H) = L^q(\Omega; H)$. Assume that for all $\phi \in X(H)$ and $j \in \mathscr{J}$, $(x, y) \mapsto T_j(x, y)\phi$ is measurable and $\|T_j(x, y)\| \le 1$. Then there exists an increasing function $\alpha_{p,d}$ on $\mathbb{R}_+$ such that for all $I_{k,T_j} \in \mathscr{I}_T$,*

$$\left\| I_{k,T_j} \right\|_{\mathscr{L}(L^p(\mathbb{R}^d, v; X(H)))} \le \alpha_{p,d}([v]_{A_p}), \quad v \in A_p.$$

Proof. Let $f \in L^p(\mathbb{R}^d, v; X(H))$ arbitrary. Then by Minkowski's inequality for integrals in (i), the properties of $k \in \mathscr{K}$ in (ii) and boundedness of M on $L^p(\mathbb{R}^d, v)$ in (iii), we get

$$\begin{aligned}
\|I_{k,T_j} f\|_{L^p(\mathbb{R}^d, v; X(H))} &= \left(\int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}^d} k(x - y) T_j(x, y) f(y)\, \mathrm{d}y \right\|^p_{X(H)} v(x)\, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\overset{(i)}{\le} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |k(x - y)| \|T_j(x, y) f(y)\|_{X(H)}\, \mathrm{d}y \right)^p v(x)\, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |k(x - y)| \|f(y)\|_{X(H)}\, \mathrm{d}y \right)^p v(x)\, \mathrm{d}x \right)^{\frac{1}{p}}
\end{aligned}$$

$$\overset{(ii)}{\leq} \left(\int_{\mathbb{R}^d} \big(M(\|f\|_{X(H)})(x)\big)^p v(x)\,\mathrm{d}x \right)^{\frac{1}{p}} \overset{(iii)}{\leq} \alpha_{p,d}([v]_{A_p}) \, \|f\|_{L^p(\mathbb{R}^d,v;X(H))}$$

with $\alpha_{p,d}$ an increasing function on $\mathbb{R}_+$. This proves the lemma. □

We can now state the main result of this chapter.

Theorem 4.3.5. *Let $1 < p, q < \infty$ and write $X(H) = L^q(\Omega; H)$. Assume the following conditions*

(1) For all $\phi \in X(H)$ and $j \in \mathscr{J}$, $(x, y) \mapsto T_j(x, y)\phi$ is measurable.

(2) For all $s \in (1,\infty)$, $\mathscr{T} = \{T_j(x, y) : x, y \in \mathbb{R}^d, j \in \mathscr{J}\}$ is ℓ^s_H-bounded,

Then for all $v \in A_p$ and all $s \in (1,\infty)$, the family of operators $\mathscr{I}_T \subseteq L^p(\mathbb{R}^d, v; X(H))$ as defined in (4.3.1)*, is ℓ^s_H-bounded with $\mathscr{R}^s_H(\mathscr{I}_T) \leq C$ where C depends on $p, q, d, s, [v]_{A_p}$ and on $\mathscr{R}^\sigma_H(\mathscr{T})$ for $\sigma \in (1,\infty)$ and is A_p-consistent.*

Example 4.3.6. When $\Omega = \mathbb{R}^e$ with μ the Lebesgue measure and $q_0 \in (1,\infty)$, then the weighted boundedness of each of the operators $T_j(x, y)$ on $L^{q_0}(\mathbb{R}^e, w; H)$ for all A_{q_0}-weights w in an A_{q_0}-consistent way, is a sufficient condition for the ℓ^s_H-boundedness which is assumed in Theorem 4.3.5. Indeed, this follows from the extension of [32, Corollary 3.12] (also see Theorem 4.2.1) to the H-valued setting.

Usually, the weighted boundedness is simple to check with [65, Theorem IV.3.9] or [71, Theorem 9.4.6], because often for each $x, y \in \mathbb{R}^d$ and $j \in \mathscr{J}$, $T_j(x, y)$ is given by a Fourier multiplier operator in $\mathbb{R}^e$.

Example 4.3.7.

(i) Let $H = \mathbb{C}$ and let $q \in (1,\infty)$. Let $T(t) = e^{t\Delta}$ for $t \geq 0$ be the heat semigroup, where Δ is the Laplace operator on $\mathbb{R}^e$. Then it follows from the weighted Mihlin multiplier theorem ([65, Theorem IV.3.9]) that for all $w \in A_q$, $\|T(t)\|_{\mathscr{L}(L^q(\mathbb{R}^e,w))} \leq C$, where C is A_q-consistent. Therefore, as in Example 4.3.6, $\{T(t) : t \in \mathbb{R}_+\}$ is ℓ^s-bounded on $L^q(\mathbb{R}^d, w)$ by an A_q-consistent $\mathscr{R}^s$-bound.

(ii) In order to give an example of an operator $I_{k,T}$ as in (4.3.1), we could let $T(x, y) = T(\phi(x, y))$, where $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is measurable. Other examples can be given if one replaces the heat semigroup by a two parameter evolution family $T(t, s)$, which was the setting of Chapter 3.

To prove Theorem 4.3.5 we will first show a result assuming ℓ^s_H-boundedness for a fixed $s \in (1,\infty)$. Here we can also include $s = 1$.

Proposition 4.3.8. *Let $1 \leq s < q < \infty$ and write $X(H) = L^q(\Omega; H)$. Assume the following conditions*

(1) For all $\phi \in X(H)$ and all $j \in \mathscr{J}$, $(x, y) \mapsto T_j(x, y)\phi$ is measurable.

(2) $\mathscr{T} = \{T_j(x,y) : x, y \in \mathbb{R}^d, j \in \mathscr{J}\}$ *is* ℓ^s_H*-bounded.*

Then for all $p \in (s,\infty)$ *and all* $v \in A_{\frac{p}{s}}$ *the family of operators* $\mathscr{I}_T \subseteq L^p(\mathbb{R}^d, v; X(H))$ *defined as in* (4.3.1)*, is* ℓ^s_H*-bounded and there exist an increasing function* $\alpha_{s,p,q,d}$ *such that*

$$\mathscr{R}^s_H(\mathscr{I}_T) \le \mathscr{R}^s_H(\mathscr{T})\alpha_{s,p,q,d}([v]_{A_{\frac{p}{s}}}).$$

Proof. Without loss of generality we can assume $\mathscr{R}^s_H(\mathscr{T}) = 1$. We start with a preliminary observation. By the extensions of [65, Theorem VI.5.2] or [71, Theorem 9.5.8] to the H-valued setting, the ℓ^s_H-boundedness is equivalent to the following: for every $u \ge 0$ in $L^{\frac{q}{q-s}}(\Omega)$ there exists a $U \in L^{\frac{q}{q-s}}(\Omega)$ such that

$$\begin{aligned} &\|U\|_{L^{\frac{q}{q-s}}(\Omega)} \le \|u\|_{L^{\frac{q}{q-s}}(\Omega)}, \\ &\int_\Omega \|T_j(x,y)\phi\|^s_H u\, d\mu \le \int_\Omega \|\phi\|^s_H U\, d\mu, \quad x, y \in \Omega,\ j \in \mathscr{J},\ \phi \in L^q(\Omega; H). \end{aligned} \tag{4.3.2}$$

For $n = 1, \cdots, N$ take $I_{k_n, T_{j_n}} \in \mathscr{I}_T$ and let $I_n = I_{k_n, T_{j_n}}$ where $j_1, \ldots, j_N \in \mathscr{J}$. Take $f_1, \cdots, f_N \in L^p(\mathbb{R}^d, v; X)$ and note that

$$\Big\| \Big(\sum_{n=1}^N \|I_n f_n\|^s_H \Big)^{\frac{1}{s}} \Big\|_{L^p(\mathbb{R}^d, v; L^q(\Omega))} = \Big\| \sum_{n=1}^N \|I_n f_n\|^s_H \Big\|^{\frac{1}{s}}_{L^{\frac{p}{s}}\left(\mathbb{R}^d, v; L^{\frac{q}{s}}(\Omega)\right)}.$$

Let $r \in (1,\infty)$ be such that $\frac{1}{r} + \frac{s}{q} = 1$ and fix $x \in \mathbb{R}^d$. As $L^r(\Omega) = L^{\frac{q}{s}}(\Omega)^*$, we can find a function $u \in L^r(\Omega)$, which will depend on x, with $u \ge 0$ and $\|u\|_{L^r(\Omega)} = 1$ such that

$$\Big\| \sum_{n=1}^N \|I_n f_n(x)\|^s_H \Big\|_{L^{\frac{q}{s}}(\Omega)} = \sum_{n=1}^N \int_\Omega \|I_n f_n(x)\|^s_H u \,\mathrm{d}\mu. \tag{4.3.3}$$

By the observation in the beginning of the proof, there is a function $U \ge 0$ in $L^r(\Omega)$ (which depends on x again) such that (4.3.2) holds. Since $\|k_n\|_{L^1(\mathbb{R}^d)} \le 1$, Hölder's inequality yields

$$\|I_n f_n(x)\|^s_H \le \int_{\mathbb{R}^d} |k_n(x-y)| \|T_{j_n}(x,y) f_n(y)\|^s_H \,\mathrm{d}y. \tag{4.3.4}$$

Applying (4.3.4) in (i), estimate (4.3.2) in (ii), and Hölder's inequality in (iii), we get:

$$\begin{aligned} \sum_{n=1}^N \int_\Omega \|I_n f_n(x)\|^s_H u \,\mathrm{d}\mu &\overset{(i)}{\le} \sum_{n=1}^N \int_\Omega \int_{\mathbb{R}^d} |k_n(x-y)| \|T_{j_n}(x,y) f_n(y)\|^s_H \,\mathrm{d}y\, u \,\mathrm{d}\mu \\ &= \sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(x-y)| \int_\Omega \|T_{j_n}(x,y) f_n(y)\|^s_H u \,\mathrm{d}\mu \,\mathrm{d}y \\ &\overset{(ii)}{\le} \sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(x-y)| \int_\Omega \|f_n(y)\|^s_H U \,\mathrm{d}\mu \,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(x-y)| \, \|f_n(y)\|_H^s \,\mathrm{d}y \, U \,\mathrm{d}\mu \\
&\overset{(iii)}{\le} \Big\| \sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(x-y)| \, \|f_n(y)\|_H^s \,\mathrm{d}y \Big\|_{L^{\frac{q}{s}}(\Omega)}.
\end{aligned}
$$

Combining (4.3.3) with the above estimate and applying the ℓ^1-boundedness result of Proposition 4.3.1 to $\|f_n\|_H^s \in L^{\frac{p}{s}}\Big(\mathbb{R}^d, v; L^{\frac{q}{s}}(\Omega)\Big)$ (here we use $v \in A_{\frac{p}{s}}$), we get

$$
\begin{aligned}
\Big\| \Big(\sum_{n=1}^N \|I_n f_n\|_H^s \Big)^{\frac{1}{s}} \Big\|_{L^p(\mathbb{R}^d, v; L^q(\Omega))} &\le \Big\| \sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(\bullet - y)| \|f_n(y)\|_H^s \,\mathrm{d}y \Big\|^{\frac{1}{s}}_{L^{\frac{p}{s}}\left(\mathbb{R}^d, v; L^{\frac{q}{s}}(\Omega)\right)} \\
&= \Big\| \sum_{n=1}^N \Big| k_n * \|f_n\|_H^s \Big| \, \Big\|^{\frac{1}{s}}_{L^{\frac{p}{s}}\left(\mathbb{R}^d, v; L^{\frac{q}{s}}(\Omega)\right)} \\
&\le \alpha_{p,q,s,d}([v]_{A_{\frac{p}{s}}}) \Big\| \sum_{n=1}^N \|f_n\|_H^s \Big\|^{\frac{1}{s}}_{L^{\frac{p}{s}}\left(\mathbb{R}^d, v; L^{\frac{q}{s}}(\Omega)\right)} \\
&= \alpha_{p,q,s,d}([v]_{A_{\frac{p}{s}}}) \Big\| \Big(\sum_{n=1}^N \|f_n\|_H^s \Big)^{\frac{1}{s}} \Big\|_{L^p(\mathbb{R}^d, v; L^q(\Omega))}
\end{aligned}
$$

with $\alpha_{p,q,s,d}$ an increasing function on $\mathbb{R}_+$. This proves the ℓ^s_H-boundedness. □

Next we prove Theorem 4.3.5. For a constant ϕ depending on a parameter $t \in I \subset \mathbb{R}$, we write $\phi \propto t$ if $\phi_t \le \phi_s$ whenever $t \le s$ and $s, t \in I$.

Proof of Theorem 4.3.5. Fix $q \in (1, \infty)$, $p = q$, $v \in A_q$ and $\kappa = 2[v]_{A_q} \ge 2$. The case $p \ne q$ will be considered at the end of the proof.

Step 1. First we prove the theorem for very small $s \in (1, q)$. Proposition 2.2.1 gives $\sigma_1 = \sigma_{q,\kappa,d} \in (1, q)$ and $C_{q,\kappa,d}$ such that for all $s \in (1, \sigma_1]$ and all weights $u \in A_q$ with $[u]_{A_q} \le \kappa$,

$$
[u]_{A_{\frac{q}{s}}} \le [u]_{A_{\frac{q}{\sigma}}} \le C_{q,\kappa,d}.
$$

Moreover, $\sigma_1 \propto \kappa^{-1}$ and $C \propto \kappa$.

By Proposition 4.3.8, $\mathscr{I}_T \subseteq \mathscr{L}(L^q(\mathbb{R}^d, v; X(H)))$ is ℓ^s_H-bounded for all $s \in (1, \sigma_1)$ and

$$
\mathscr{R}^s_H(\mathscr{I}_T) \le \mathscr{R}^s(\mathscr{T}) \alpha_{s,q,d}([v]_{A_{\frac{q}{s}}}) \le \mathscr{R}^s_H(\mathscr{T}) \beta_{q,s,d,\kappa}, \tag{4.3.5}
$$

with $\beta_{q,s,d,\kappa} = \alpha_{q,s,d}(C_{q,\kappa,d})$. Note that $\beta \propto \kappa$ and $\beta \propto s'$.

Step 2. Now we use a duality argument to prove the theorem for large $s \in (q, \infty)$. By Proposition 2.2.1, $v' \in A_{q'}$ and $\tilde{\kappa} = 2[v']_{A_{q'}} = 2[v]_{A_q}^{\frac{1}{q-1}} = 2(\kappa^{\frac{1}{q-1}})$. Note that we can identify $X(H)^* = L^{q'}(\Omega; H)$ and $L^q(\mathbb{R}^d, v; X(H))^* = L^{q'}(\mathbb{R}^d, v'; X(H)^*)$ by Proposition 4.A.1. Define $\mathscr{I}_T^* = \{I^* : I \in \mathscr{I}_T\}$.

It is standard to check that for $I_{k,T_j} \in \mathscr{I}_T$ the adjoint I^*_{k,T_j} satisfies

$$
I^*_{k,T_j} g(x) = \int_{\mathbb{R}^d} \tilde{k}(y-x) \tilde{T}_j(x,y) g(y) \,\mathrm{d}y = I_{\tilde{k},\tilde{T}_j} g(x)
$$

with $\tilde{k}(x) = k(-x)$ and $\tilde{T}_j(x,y) = T_j^*(y,x)$. Indeed, for all $f \in L^q(\mathbb{R}^d, v; X(H))$ and $g \in L^{q'}(\mathbb{R}^d, v'; X(H)^*)$ we have

$$\begin{aligned}
&\left\langle f, I_{k,T_j}^* g \right\rangle_{L^q(\mathbb{R}^d,v;X(H)),L^{q'}(\mathbb{R}^d,v';X(H)^*)} = \left\langle I_{k,T_j} f, g \right\rangle_{L^q(\mathbb{R}^d,v;X(H)),L^{q'}(\mathbb{R}^d,v';X(H)^*)}\\
&= \int_{\mathbb{R}^d} \left\langle \int_{\mathbb{R}^d} k(x-y) T_j(x,y) f(y)\,\mathrm{d}y, g(x) \right\rangle_{X(H),X(H)^*} \mathrm{d}x\\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x-y) \langle T_j(x,y) f(y), g(x) \rangle_{X(H),X(H)^*} \,\mathrm{d}y\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\langle f(y), k(x-y) T_j^*(x,y) g(x) \right\rangle_{X(H),X(H)^*} \mathrm{d}y\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d} \left\langle f(y), \int_{\mathbb{R}^d} k(x-y) T_j^*(x,y) g(x)\,\mathrm{d}x \right\rangle_{X(H),X(H)^*} \mathrm{d}y\\
&= \left\langle f, y \mapsto \int_{\mathbb{R}^d} k(x-y) T_j^*(x,y) g(x)\,\mathrm{d}x \right\rangle_{L^q(\mathbb{R}^d,v;X(H)),L^{q'}(\mathbb{R}^d,v';X(H)^*)}
\end{aligned}$$

As already noted before we have $\tilde{k} \in \mathcal{K}$. Furthermore, by Proposition 4.1.3 the adjoint family $\mathcal{T}^*$ is $\mathcal{R}_H^{s'}$-bounded with $\mathcal{R}_H^{s'}(\mathcal{T}^*) = \mathcal{R}_H^{s}(\mathcal{T})$. Therefore, it follows from Step 1 that there is a $\sigma_2 = \sigma_{q',\tilde{\kappa},d} \in (1, q')$ such that for all $s' \in (1, \sigma_2]$, $\mathcal{I}_T^*$ is $\ell_H^{s'}$-bounded on $L^{q'}(\mathbb{R}^d, v'; X(H)^*)$ and using Proposition 4.1.3 again, we obtain $\mathcal{I}_T$ is ℓ_H^s-bounded and

$$\mathcal{R}_H^s(\mathcal{I}_T) = \mathcal{R}_H^{s'}(\mathcal{I}_T^*) \le \mathcal{R}_H^{s'}(\mathcal{T}^*)\beta_{q',s',d,\tilde{\kappa}} = \mathcal{R}_H^s(\mathcal{T})\beta_{q',s',d,\tilde{\kappa}}. \tag{4.3.6}$$

Therefore, Proposition 4.1.3 yields that $\mathcal{I}_T$ is ℓ_H^s-bounded on $L^q(\mathbb{R}^d, v; X(H))$ for all $s \in [\sigma_2', \infty)$.

Step 3. We can now finish the proof in the case $p = q$ by an interpolation argument. In the previous steps 1 and 2 we have found $1 < \sigma_1 < q < \sigma_2' < \infty$ such that $\mathcal{I}_\alpha$ is ℓ_H^s-bounded for all $s \in (1, \sigma_1] \cup [\sigma_2', \infty)$ with

$$\mathcal{R}_H^s(\mathcal{I}_T) \le \mathcal{R}_H^s(\mathcal{T})\gamma_{q,s,d,\kappa}. \tag{4.3.7}$$

where $\gamma_{q,s,d,\kappa} = \beta_{q,s,d,\kappa}$ if $s \le \sigma_1$ and $\gamma_{q,s,d,\kappa} = \beta_{q',s',d,\tilde{\kappa}}$ if $s \ge \sigma_2'$. Clearly, $\gamma := \gamma_{q,s,d,\kappa}$ satisfies $\gamma \propto \kappa$, $\gamma \propto s'$ for $s \in (1, \sigma_1]$ and $\gamma \propto s$ for $s \in [\sigma_2', \infty)$. Moreover, $\sigma_1 \propto \frac{1}{\kappa}$ and $\sigma_2' \propto \kappa$.

Now Proposition 4.1.3 yields the ℓ_H^s-boundedness and the required estimates for the remaining $s \in [\sigma_1, \sigma_2']$ and by (4.3.7) we find

$$\begin{aligned}
\mathcal{R}_H^s(\mathcal{I}_T) &\le \max\{\mathcal{R}_H^{\sigma_1}(\mathcal{I}_T), \mathcal{R}_H^{\sigma_2'}(\mathcal{I}_T)\}\\
&\le \max\{\mathcal{R}_H^{\sigma_1}(\mathcal{T}), \mathcal{R}_H^{\sigma_2'}(\mathcal{T})\}\gamma.
\end{aligned}$$

where $\gamma = \max\{\gamma_{q,\sigma_1,d,\kappa}, \gamma_{q',\sigma_2,d,\tilde{\kappa}}\}$. By Example 4.1.5, $\mathcal{R}_H^{\sigma_1}(\mathcal{T}) \propto \kappa$ and $\mathcal{R}_H^{\sigma_2'}(\mathcal{T}) \propto \kappa$. Also $\gamma \propto \kappa$ in the above. Therefore, the obtained $\mathcal{R}_H^s$-bound is A_q-consistent.

Step 4. Next let $p, q \in (1,\infty)$. Fix $s \in (1,\infty)$. For $n = 1, \cdots, N$ take $I_{k_n, T_{j_n}} \in \mathscr{I}_T$ and let $I_n = I_{k_n, T_{j_n}}$. Take $f_1, \cdots, f_N \in L^p(\mathbb{R}^d, v; X(H)) \cap L^q(\mathbb{R}^d, v; X(H))$ and let

$$F = \Big\| \Big(\sum_{n=1}^N \|I_n f_n\|_H^s \Big)^{\frac{1}{s}} \Big\|_{L^q(\Omega)} \text{ and } G = \Big\| \Big(\sum_{n=1}^N \|f_n\|_H^s \Big)^{\frac{1}{s}} \Big\|_{L^q(\Omega)}.$$

By the previous step we know that for all $v \in A_q$,

$$\|F\|_{L^q(\mathbb{R}^d, v)} \le C \|G\|_{L^q(\mathbb{R}^d, v)},$$

where C depends on d, s, q, and $[v]_{A_p}$ and is A_p-consistent. Therefore, by Theorem 2.2.3 we can extrapolate to obtain for all $p \in (1,\infty)$ and $v \in A_p$,

$$\|F\|_{L^p(\mathbb{R}^d, v)} \le \tilde{C} \|G\|_{L^p(\mathbb{R}^d, v)},$$

where $\tilde{C}$ depends on C, p and $[v]_{A_p}$ and is again A_p-consistent. This implies the required $\mathscr{R}^s_H$-boundedness for all $p, q \in (1,\infty)$ with constant $\tilde{C}$. □

Corollary 4.3.9. *Let $\Omega \subseteq \mathbb{R}^e$ be an open set and H be an Hilbert space. Let $1 < p, q, q_0 < \infty$. Assume the following conditions*

(1) For all $\phi \in L^q(\Omega; H)$ and $j \in \mathscr{J}$, $(x, y) \mapsto T_j(x, y)\phi$ is measurable.

(2) For all $w \in A_{q_0}$, $\sup_{j \in \mathscr{J}, x, y \in \Omega} \|T_j(x, y)\|_{\mathscr{L}(L^{q_0}(\Omega, w; H))} \le C$, where C is A_{q_0}-consistent.

Then for every $v \in A_p$, $w \in A_q$ and $s \in (1,\infty)$, the family of operators

$$\mathscr{I}_T \subseteq L^p(\mathbb{R}^d, v; L^q(\Omega, w; H)) \text{ as defined in } (4.3.1),$$

is ℓ^s_H-bounded with $\mathscr{R}^s_H(\mathscr{I}_T) \le \tilde{C}$ where $\tilde{C}$ depends on $p, q, d, s, [v]_{A_p}, [w]_{A_q}$ and on $\mathscr{R}^\sigma_H(\mathscr{T})$ for $\sigma \in (1,\infty)$ and is A_p- and A_q-consistent.

Proof. In the case $\Omega = \mathbb{R}^e$, note that Example 4.3.6 yields that for each $q \in (1,\infty)$ and each $w \in A_q$ and $s \in (1,\infty)$, $\mathscr{T}$ considered on $L^q(\Omega, w; H)$ is ℓ^s_H-bounded. Moreover, $\mathscr{R}^s_H(\mathscr{T}) \le K$, where K depends on q, s, e and $[w]_{A_q}$ in an A_q-consistent way. Therefore, the result follows from Theorem 4.3.5.

In the case $\Omega \subseteq \mathbb{R}^e$, we reduce to the case $\mathbb{R}^e$ by a restriction-extension argument. For convenience we sketch the details. Let $E : L^q(\Omega, w; H) \to L^q(\mathbb{R}^e, w; H)$ be the extension by zero and let $R : L^q(\mathbb{R}^e, w; H) \to L^q(\Omega, w; H)$ be the restriction to Ω. For every $x, y \in \mathbb{R}^d$ and $j \in \mathscr{J}$, let $\tilde{T}_j(x, y) = E T_j(x, y) R \in \mathscr{L}(L^q(\mathbb{R}^e, w; H))$ and let $\tilde{\mathscr{T}} = \{\tilde{T}_j(x, y) : x, y \in \mathbb{R}^d\}$.

Since $\|\tilde{T}_j(x, y)\|_{\mathscr{L}(L^q(\mathbb{R}^e, w; H))} \le \|T_j(x, y)\|_{\mathscr{L}(L^q(\Omega, w; H))} \le C$, it follows from the case $\Omega = \mathbb{R}^e$ that $\mathscr{I}_{\tilde{T}} \subseteq L^p(\mathbb{R}^d, v; L^q(\mathbb{R}^e, w; H))$ is ℓ^s-bounded with $\mathscr{R}^s_H(\mathscr{I}_{\tilde{T}}) \le \tilde{C}$. Now it remains to observe that the restriction of $I_{k, \tilde{T}_j}$ to $L^p(\mathbb{R}^d, v; L^q(\Omega, w; H))$ is equal to I_{k, T_j} and hence $\mathscr{R}^s_H(\mathscr{I}_T) \le \mathscr{R}^s_H(\mathscr{I}_{\tilde{T}}) \le \tilde{C}$. □

Next we will prove Theorem 3.1.4. We first introduce the notion of $\mathscr{R}_H$-boundedness for operators with values in a Hilbert space H.

Definition 4.3.10. Let H be a Hilbert space. We call a family of operators $\mathscr{T} \subseteq \mathscr{L}(X(H), Y(H))$ *$\mathscr{R}_H$-bounded* if there exists a constant C such that for all $N \in \mathbb{N}$, all sequences $(T_n)_{n=1}^N$ in $\mathscr{T}$ and $(x_n)_{n=1}^N$ in X,

$$\Big\| \sum_{n=1}^N r_n T_n x_n \Big\|_{L^2(\Omega;Y(H))} \leq C \Big\| \sum_{n=1}^N r_n x_n \Big\|_{L^2(\Omega;X(H))}. \tag{4.3.8}$$

The least possible constant C is called the *$\mathscr{R}_H$-bound* of $\mathscr{T}$ and is denoted by $\mathscr{R}_H(\mathscr{T})$.

Remark 4.3.11. For $X(H) = Y(H) = L^{\overline{q}}(\Omega; H)$ with $q \in (1,\infty)^n$, the notions ℓ^2_H-boundedness and $\mathscr{R}_H$-boundedness of any family $\mathscr{S} \subseteq \mathscr{L}(X, Y)$ coincide and $C^{-1}\mathscr{R}^2_H(\mathscr{S}) \leq \mathscr{R}_H(\mathscr{S}) \leq C\mathscr{R}^2_H(\mathscr{S})$, where C is a constant which only depends on $\overline{q}$. This assertion follows from the Kahane-Khintchine inequalities (see [45, 1.10 and 11.1]), and in the case $X = Y = L^q$ it was already shown in Section 2.7.

Theorem 4.3.12. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set, H be a Hilbert space. Let $p, q \in (1,\infty)$. Assume that for all A_q-weights w,*

$$\| T(t,s)\|_{\mathscr{L}(L^q(\Omega,w;H))} \leq C, \quad s, t \in \mathbb{R}, \tag{4.3.9}$$

where C depends on the A_q-constant of w in a consistent way. Then the family of integral operators $\{I_k : k \in \mathscr{K}\} \subseteq \mathscr{L}(L^p(\mathbb{R}; L^q(\Omega; H)))$ as defined in (3.1.2) *is $\mathscr{R}_H$-bounded.*

Proof. The result follows directly from Corollary 4.3.9 and Remark 4.3.11 with $X(H) = L^p(\mathbb{R}; L^q(\Omega; H))$. ☐

Theorem 3.1.4 follows directly from Theorem 4.3.12 with $H = \mathbb{C}$.

4.A. APPENDIX: DUALITY OF ITERATED $L^{\overline{q}}$-SPACES

In this section we consider $H = \mathbb{C}$ for simplicity. Let $(\Omega_i, \Sigma_i, \mu_i)$ for $i = 1, \dots n$ be σ-finite measure spaces. The dual of the iterated space $L^{\overline{q}}(\Omega)$ as defined in (4.2.1), is exactly what one would expect. In a general setting one can prove that $L^p(\Omega; X)^* = L^{p'}(\Omega, X^*)$ for reflexive Banach function spaces X from which the duality for $L^{\overline{q}}(\Omega)$ follows, as is done in [46, Chapter IV] using the so-called Radon-Nikodym property of Banach spaces. Here we present an elementary proof just for $L^{\overline{q}}(\Omega)$.

Proposition 4.A.1. *Let $\overline{q} \in (1,\infty)^n$. For every bounded linear functional Φ on $L^{\overline{q}}(\Omega)$ there exists a unique $g \in L^{\overline{q}'}(\Omega)$ such that:*

$$\Phi(f) = \int_\Omega f g \, \mathrm{d}\mu \tag{4.A.1}$$

for all $f \in L^{\overline{q}}$ and $\|\Phi\| = \|g\|_{L^{\overline{q}'}(\Omega)}$, i.e. $L^{\overline{q}}(\Omega)^* = L^{\overline{q}'}(\Omega)$.

Proof. We follow the strategy of proof from [136, Theorem 6.16]. The uniqueness proof is as in [136, Theorem 6.16]. Also by repeatedly applying Hölder's inequality we have for any g satisfying (4.A.1) that

$$\|\Phi\| \le \|g\|_{L^{\overline{q}'}(\Omega)}. \tag{4.A.2}$$

So it remains to prove that g exists and that equality holds in (4.A.2). As in [136, Theorem 6.16] one can reduce to the case $\mu(\Omega) < \infty$. Define $\lambda(E) = \Phi(\chi_E)$ for $E \in \Sigma$. Then one can check that λ is a complex measure which is absolutely continuous with respect to μ. So by the Radon-Nikodym Theorem [136, Theorem 6.10] we can find a $g \in L^1(\Omega)$ such that for all measurable $E \subseteq \Omega$

$$\Phi(\chi_E) = \int_E g \,\mathrm{d}\mu = \int_\Omega \chi_E g \,\mathrm{d}\mu$$

and from this we get by linearity $\Phi(f) = \int_\Omega f g \,\mathrm{d}\mu$ for all simple functions f. Now take a $f \in L^\infty(\Omega)$ arbitrary and let f_i be simple functions such that $\|f_i - f\|_{L^\infty(\Omega)} \to 0$ for $i \to \infty$. Then since $\mu(\Omega) < \infty$ we have $\|f_i - f\|_{L^{\overline{q}}(\Omega)} \to 0$ for $i \to \infty$. Hence

$$\Phi(f) = \lim_{i\to\infty} \Phi(f_i) = \lim_{i\to\infty} \int_\Omega f_i g \,\mathrm{d}\mu = \int_\Omega f g \,\mathrm{d}\mu. \tag{4.A.3}$$

We will now prove that $g \in L^{\overline{q}'}(\Omega)$ and that equality holds in (4.A.2). Take $k \in \mathbb{N}$ arbitrary. Let $E_k^1 = \{s \in \Omega : \frac{1}{k} \le |g(s)| \le k\}$ and define for $i = 2, \cdots, n$

$$E_k^i = \left\{ s \in \Omega : \left\| g_k(s_1, \cdots, s_{i-1}, \bullet) \right\|_{L^{q_i'}(\Omega_i, \cdots L^{q_n'}(\Omega_n))} \ge \frac{1}{k} \right\}$$

Now take $g_k = g \prod_{i=1}^n \chi_{E_k^i}$ and let α be its complex sign function, i.e. $|\alpha| = 1$ and $\alpha |g_k| = g_k$. Take

$$f(s) = \overline{\alpha} |g_k(s)|^{q_n' - 1} \prod_{i=2}^n \left\| g_k(s_1, \cdots, s_{i-1}, \bullet) \right\|_{L^{q_i'}(\Omega_i, \cdots L^{q_n'}(\Omega_n))}^{q_{i-1}' - q_i'}$$

where we define $0 \cdot \infty = 0$. Then $f \in L^\infty(\Omega)$ and one readily checks that

$$\int_\Omega f g_k \,\mathrm{d}\mu = \|g_k\|_{L^{\overline{q}'}(\Omega)}^{q_1'} \quad \text{and} \quad \left\| f \right\|_{L^{\overline{q}}(\Omega)} = \|g_k\|_{L^{\overline{q}'}(\Omega)}^{\frac{q_1'}{q_1}}. \tag{4.A.4}$$

So from (4.A.4) we obtain

$$\|g_k\|_{L^{\overline{q}'}(\Omega)}^{q_1'} = \int_\Omega f g_k \,\mathrm{d}\mu = \Phi(f) \le \left\| f \right\|_{L^{\overline{q}}(\Omega)} \|\Phi\| = \|g_k\|_{L^{\overline{q}'}(\Omega)}^{\frac{q_1'}{q_1}} \|\Phi\|$$

which means $\|g_k\|_{L^{\overline{q}'}(\Omega)} \le \|\Phi\|$. Since this holds for all $k \in \mathbb{N}$ we obtain by Fatou's lemma that $\|g\|_{L^{\overline{q}'}(\Omega)} \le \|\Phi\|$, which proves that $g \in L^{\overline{q}'}(\Omega)$ and $\|g\|_{L^{\overline{q}'}(\Omega)} = \|\Phi\|$. From this we also get (4.A.3) for all $f \in L^{\overline{q}'}(\Omega)$ by Hölders inequality and the dominated convergence theorem. This proves the required result. $\square$

To obtain the duality result in Proposition 4.1.3 for $s=1$ and $s=\infty$, one also needs the following end-point duality result. Let $X(\ell^s_N)$ be the space of all N-tuples $(f_n)_{n=1}^N \in X^N$ with

$$\|(f_n)_{n=1}^N\|_{X(\ell^s_N)} = \Big\|\Big(\sum_{n=1}^N |f_n|^s\Big)^{1/s}\Big\|_X$$

with the usual modification if $s=\infty$.

Lemma 4.A.2. *Define $X = L^{\overline{q}}(\Omega)$. Take $s \in [1,\infty]$ and $N \in \mathbb{N}$. Then for every bounded linear functional Φ on $X(\ell^s_N)$ there exists a unique $g \in X^*(\ell^{s'}_N)$ such that*

$$\Phi(f) = \sum_{i=1}^N \langle f_i, g_i\rangle_{X,X^*}$$

for all $f \in X(\ell^s_N)$ and $\|\Phi\| = \|g\|_{X^*(\ell^{s'}_N)}$, i.e. $X(\ell^s_N)^* = X^*(\ell^{s'}_N)$.

Also this result can be proved with elementary arguments. Indeed, for $r_1, r_2 \in [1,\infty]$ we have $X(\ell^{r_1}_N) = X(\ell^{r_2}_N)$ as sets and the following inequalities hold for all $f \in X(\ell^r_N)$ and $r \in [1,\infty]$

$$\begin{aligned} \|f\|_{X(\ell^r_N)} &\le \|f\|_{X(\ell^1_N)} \le N^{1-\frac{1}{r}}\|f\|_{X(\ell^r_N)} \\ \|f\|_{X(\ell^\infty_N)} &\le \|f\|_{X(\ell^r_N)} \le N^{\frac{1}{r}}\|f\|_{X(\ell^\infty_N)}. \end{aligned} \tag{4.A.5}$$

Now the lemma readily follows from $X(\ell^r_N)^* = X^*(\ell^{r'}_N)$ for $r \in (1,\infty)$ and letting $r \downarrow 1$ and $r \uparrow \infty$.

III

Application to elliptic differential operators

5

Evolution families and maximal regularity for systems of parabolic equations

In this chapter we prove maximal L^p-regularity for systems of parabolic PDEs, where the elliptic operator A has coefficients which depend on time in a measurable way and are continuous in the space variables. The proof is an application of the characterization of maximal L^p-regularity introduced in Chapter 3 and it is based on operator-theoretic methods. One of the main ingredients in the proof is the construction of an evolution family on weighted L^q-spaces, in Section 5.2. This application generalizes the one in Section 3.4. In this chapter we will consider operators in divergence forms as well. The results here presented are based on [63].

5.1. Assumptions and main results

5.1.1. Ellipticity

Consider an operator A of the form

$$A = \sum_{|\alpha|\le m, |\beta|\le m} a_{\alpha\beta} D^\alpha D^\beta$$

where $a_{\alpha\beta} \in \mathbb{C}^{N\times N}$ are constant matrices and $D = -i(\partial_1, \dots, \partial_d)$. The *principal symbol* of A is defined as

$$A_\#(\xi) := \sum_{|\alpha|=|\beta|=m} \langle \xi^\alpha,\ a_{\alpha\beta}\, \xi^\beta \rangle, \quad \xi \in \mathbb{R}^d. \tag{5.1.1}$$

Definition 5.1.1. We say that A is *uniformly elliptic* of angle $\theta \in (0,\pi)$ if there exists a constant $\kappa \in (0,1)$ such that

$$\sigma(A_\#(\xi)) \subseteq \Sigma_\theta \cap \{\xi : |\xi| \ge \kappa\}, \xi \in \mathbb{R}^d, |\xi| = 1, \tag{5.1.2}$$

and there is a constant $K \ge 1$ such that $\|a_{\alpha\beta}\| \le K$ for all $|\alpha|, |\beta| \le m$. In this case we write $A \in \mathrm{Ell}(\theta, \kappa, \mathrm{K})$.

Observe that in the case $N = 1$, the above definition is equivalent to Definition 2.5.2.

Definition 5.1.2. We say that A is *elliptic in the sense of Legendre–Hadamard* (see [51, 60, 67]) if there exists a constant $\kappa > 0$ such that

$$\mathrm{Re}(\langle x, A_{\#}(\xi)x\rangle) \geq \kappa\|x\|^2, \quad \xi \in \mathbb{R}^d,\ |\xi| = 1, x \in \mathbb{C}^N \tag{5.1.3}$$

and there is a constant K such that $\|a_{\alpha\beta}\| \leq K$ for all $|\alpha|,|\beta| \leq m$. In this case we write $A \in \mathrm{Ell}^{\mathrm{LH}}(\kappa, \mathrm{K})$.

Obviously, (5.1.3) implies (6.2.3) with $\theta = \arccos(\kappa/\tilde{K}) \in (0, \pi/2)$, where $\tilde{K}$ depends only on m and K.

5.1.2. $L^p(L^q)$-theory for systems of PDEs with time-dependent coefficients.

In order to state the main result consider the following system of PDEs

$$u'(t,x) + (\lambda + A(t))u(t,x) = f(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^d \tag{5.1.4}$$

where $u, f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^N$ and A is the following differential operator of order $2m$:

$$(A(t)u)(x) = \sum_{|\alpha| \leq m, |\beta| \leq m} a_{\alpha\beta}(t,x) D^\alpha D^\beta u(x), \tag{5.1.5}$$

where $a_{\alpha\beta} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^{N\times N}$. A function $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^N$ is called a *strong solution* to (5.1.4) when all the above derivatives (in distributional sense) exist in $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d; \mathbb{C}^N)$ and (5.1.4) holds almost everywhere.

For A of the form (5.1.5), $x_0 \in \mathbb{R}^d$, and $t_0 \in \mathbb{R}$ let us introduce the notation:

$$A(t_0, x_0) := \sum_{|\alpha| \leq m, |\beta| \leq m} a_{\alpha\beta}(t_0, x_0) D^\alpha D^\beta.$$

for the operator with constant coefficients.

The coefficients of A are only assumed to be measurable in time. More precisely, the following conditions on the coefficients are supposed to hold:

(C) Let A be given by (5.1.5) and assume each $a_{\alpha\beta} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^{N\times N}$ is measurable. We assume there exist κ, K such that for all $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$, $A(t_0, x_0) \in \mathrm{Ell}^{\mathrm{LH}}(\kappa, \mathrm{K})$. Assume there exists an increasing function $\omega : (0,\infty) \to (0,\infty)$ with the property $\omega(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and such that

$$\|a_{\alpha\beta}(t,x) - a_{\alpha\beta}(t,y)\| \leq \omega(|x-y|), \quad |\alpha| = |\beta| = m,\ t \in \mathbb{R},\ x, y \in \mathbb{R}^d.$$

The first main result is on the maximal regularity for (5.1.4).

Theorem 5.1.3 (Non-divergence form)**.** *Let $p, q \in (1,\infty)$, $v \in A_p(\mathbb{R})$, $w \in A_q(\mathbb{R}^d)$, $X_0 = L^q(\mathbb{R}^d, w; \mathbb{C}^N)$ and $X_1 = W^{2m,q}(\mathbb{R}^d, w; \mathbb{C}^N)$. Assume condition (C) holds. Then there exists an A_p-A_q-consistent constant λ_0 such that for all $\lambda \geq \lambda_0$ and every $f \in L^p(\mathbb{R}, v; X_0)$ there* *exists a unique strong solution* $u \in W^{1,p}(\mathbb{R}, v; X_0) \cap L^p(\mathbb{R}, v; X_1)$ *of* (5.1.4)*. Moreover, there is an* A_p*-*A_q*-consistent constant* C *depending on* v, w, p, q, d, m, κ, K *and* ω *such that for all* $\lambda \geq \lambda_0$,

$$\lambda \|u\|_{L^p(\mathbb{R},v;X_0)} + \|u\|_{L^p(\mathbb{R},v;X_1)} + \|u\|_{W^{1,p}(\mathbb{R},v;X_0)} \leq C\|f\|_{L^p(\mathbb{R},v;X_0)}. \tag{5.1.6}$$

By setting $\mathrm{MR}^p(\mathbb{R}, v) := W^{1,p}(\mathbb{R}, v; X_0) \cap L^p(\mathbb{R}, v; X_1)$ and $\|u\|_{\mathrm{MR}^p(\mathbb{R},v)} := \|u\|_{L^p(\mathbb{R},v;X_1)} + \|u\|_{W^{1,p}(\mathbb{R},v;X_0)}$, the above result states that A has maximal L^p_v-regularity on $\mathbb{R}$ for all $f \in L^p(\mathbb{R}, v; X_0)$, according to Definition 3.3.10.

Note that the constant C does not depend on the dimension N. Actually, our proof allows a generalization to infinite dimensional systems, but we will not consider this here.

A similar result holds in the case $A(t)$ is in divergence form:

$$u'(t,x) + (\lambda + A_{\mathrm{div}}(t))u(t,x) = \sum_{|\alpha|\leq m} D^\alpha f_\alpha(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^d \tag{5.1.7}$$

where $u, f_\alpha, g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^N$. Here A_{div} is the following differential operator of order $2m$:

$$(A_{\mathrm{div}}(t)u)(x) = \sum_{|\alpha|\leq m, |\beta|\leq m} D^\alpha\big(a_{\alpha\beta}(t,x)D^\beta u(x)\big), \tag{5.1.8}$$

where $a_{\alpha\beta} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^{N\times N}$. Again we assume the same condition (C). We say that $u \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$ is a *weak solution* of (5.1.7) if $\nabla^m u \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$ exists in the weak sense and for all $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} -\langle u, \varphi'\rangle + \lambda\langle u, \varphi\rangle + (-1)^{|\alpha|}\langle a_{\alpha\beta} D^\beta u, D^\alpha \varphi\rangle \, d(t,x) \\ = \int_{\mathbb{R}^{d+1}} (-1)^{|\alpha|}\langle f, D^\alpha \varphi\rangle \, d(t,x), \end{aligned}$$

where we used the summation convention.

Theorem 5.1.4 (Divergence form)**.** *Let* $p, q \in (1,\infty)$, $v \in A_p(\mathbb{R})$, $w \in A_q(\mathbb{R}^d)$, $X_0 = L^q(\mathbb{R}^d, w; \mathbb{C}^N)$ *and* $X_{1/2} = W^{m,q}(\mathbb{R}^d, w; \mathbb{C}^N)$*. Assume condition (C) holds for* A_{div} *as in* (5.1.8)*. Then there exists an* A_p*-*A_q*-consistent constant* λ_0 *such that for all* $\lambda \geq \lambda_0$ *and every* $(f_\alpha)_{|\alpha|\leq m}$ *in* $L^p(\mathbb{R}, v; X_0)$ *there exists a unique weak solution* $u \in L^p(\mathbb{R}, v; X_{\frac{1}{2}})$ *of* (5.1.7)*. Moreover, there is an* A_p*-*A_q*-consistent constant* C *depending on* v, w, p, q, d, m, κ,K *and* ω *such that*

$$\sum_{|\alpha|\leq m} \lambda^{1-\frac{|\alpha|}{m}} \|D^\alpha u\|_{L^p(\mathbb{R},v;X_0)} \leq C \sum_{j=1}^{d} \lambda^{\frac{|\alpha|}{m}} \|f_\alpha\|_{L^p(\mathbb{R},v;X_0)}. \tag{5.1.9}$$

5.2. GENERATION OF EVOLUTION FAMILIES

In this section we will show that in the case $A(t)$ has x-independent coefficient it generates a strongly continuous evolution family $S(t,s)$ (see Definition 3.3.2).

5.2.1. On the sectoriality of the operator

First consider the case $a_{\alpha\beta}$ is time and space independent:

$$(Au)(x) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha} D^{\beta} u(x), \tag{5.2.1}$$

The next result can be found in [77, Theorem 3.1], where the case of x-dependent coefficients is considered as well.

Theorem 5.2.1. *Let A be of the form* (5.2.1) *and assume there exist θ, $\kappa > 0$ and $K > 0$ such that $A \in \mathrm{Ell}(\theta, \kappa, \mathrm{K})$. Let $1 < q < \infty$ and $w \in A_q$ and let $X_0 = L^q(\mathbb{R}^d, w; \mathbb{C}^N)$. Then there exists an A_q-consistent constant C depending on the parameters m, d, κ, K, q such that*

$$\|\lambda^{1-\frac{|\beta|}{m}} D^{\beta}(\lambda + A)^{-1}\|_{\mathscr{L}(X_0)} \le C, \quad |\beta| \le m, \ \lambda \in \Sigma_{\pi-\theta}. \tag{5.2.2}$$

Later on the above result will be applied to the operator $A(t)$ for fixed $t \in \mathbb{R}$. To prove (5.2.2) it suffices to check that for every $\lambda \in \Sigma_{\pi-\theta}$, and $|\beta| \le m$, the symbol $\mathscr{M} : \mathbb{R}^d \to \mathbb{C}$ given by

$$\mathscr{M}(\xi) = \lambda^{1-\frac{|\beta|}{m}} \xi^{\beta}(\lambda + A_{\#}(\xi))^{-1}$$

satisfies the following type of Mihlin's condition: for every multiindex $\alpha \in \mathbb{N}^d$, there is a constant C_α which only depends on $d, \alpha, \theta, \theta_0, K, \kappa$ such that

$$|\xi|^{\alpha} |D^{\alpha} \mathscr{M}(\xi)| \le C_\alpha, \quad \xi \in \mathbb{R}^d. \tag{5.2.3}$$

Indeed, then the result is a consequence of the weighted version of Mihlin's multiplier theorem as in [65, Theorem IV.3.9]. Note that this extends to the $\mathscr{L}(\mathbb{C}^N)$-valued case (see [19, Theorem 6.1.6] for the unweighted case). The proof of (5.2.3) follows from elementary calculus and the following lemma taken from [56, Proposition 3.1]. For convenience and in order to track the constants in the estimates, we present the details.

Lemma 5.2.2. *Let $A \in \mathrm{Ell}(\theta_0, \kappa, \mathrm{K})$ be of the form* (5.2.1) *with $\kappa \in (0,1)$, $K > 0$ and $\theta_0 \in (0, \pi)$. Let $\theta \in (\theta_0, \pi)$ be fixed. Then there is a positive constant $C = C(\kappa, \theta_0, \theta)$ such that*

$$\|(A_{\#}(\xi) + \lambda)^{-1}\| \le C(|\xi|^m + |\lambda|)^{-1}, \quad (\lambda, \xi) \in \Sigma_{\pi-\theta} \setminus \{0\} \times \mathbb{R}^d, \tag{5.2.4}$$

where $A_{\#}$ is the principal symbol of A.

Proof. To start, we recall a general observation from [11, Lemma 4.1]. If $B \in \mathscr{L}(\mathbb{C}^N)$ with $\sigma(B) \subseteq \{z : |z| \ge r\}$ for some $r > 0$, then one has

$$\|B^{-1}\| \le \|B\|^n r^{-n-1}, \quad n \ge 0. \tag{5.2.5}$$

Indeed, to show this it suffices to consider the case $r = 1$. Since $\|B^*B\| = \|B\|^2$, it is sufficient to consider self-adjoint B. Let $\lambda_{\min}, \lambda_{\max} \geq 1$ be the smallest and largest eigenvalue of B respectively. The observation follows from

$$\|B^{-1}\| = \frac{1}{\lambda_{\min}} \leq 1 \leq (\lambda_{\max})^n = \|B\|^n.$$

We then claim that with $\varepsilon = \sqrt{\frac{1-b}{2}}$ and $b = |\cos(\theta - \theta_0)|$,

$$|\lambda + \mu| \geq \varepsilon(|\lambda| + |\mu|), \ \forall\ \xi \in \mathbb{R}^d,\ \lambda \in \Sigma_{\pi-\theta}\setminus\{0\},\ \mu \in \sigma(A_\#(\xi)). \tag{5.2.6}$$

To prove the claim, write $\mu = |\mu|e^{i\varphi}$ with $|\varphi| \leq \theta_0$ and $\lambda = |\lambda|e^{i\psi}$ with $|\psi| \leq \pi - \theta$. Clearly, $|\psi - \varphi| \leq \pi - (\theta - \theta_0)$, from which we see $\cos(\psi - \varphi) \geq -b$. Therefore, the claim follows from the elementary estimates

$$\begin{aligned}|\lambda + \mu|^2 &= |\lambda|^2 + |\mu|^2 + 2\mathrm{Re}(\lambda\overline{\mu}) = |\lambda|^2 + |\mu|^2 + 2|\lambda|\,|\mu|\cos(\psi - \varphi)\\ &\geq |\lambda|^2 + |\mu|^2 - 2b|\lambda|\,|\mu| \geq (1 - b)(|\lambda|^2 + |\mu|^2) \geq \varepsilon(|\lambda| + |\mu|)^2.\end{aligned}$$

The assumptions on $A_\#$ and homogeneity yield

$$\sigma(A_\#(\xi)) \subseteq \Sigma_{\theta_0} \cap \{z : |z| \geq \kappa|\xi|^{2m}\}, \quad \xi \in \mathbb{R}^d. \tag{5.2.7}$$

This implies that for all $(\lambda,\xi) \in \Sigma_{\pi-\theta}\setminus\{0\} \times \mathbb{R}^d$ with $|\lambda| + |\xi|^{2m} = 1$,

$$\sigma(\lambda + A_\#(\xi)) \subseteq \{z : |z| \geq \varepsilon\kappa\}. \tag{5.2.8}$$

Indeed, if $\mu \in A_\#(\xi)$, then from (5.2.7) and (5.2.6) we see that

$$|\lambda + \mu| \geq \varepsilon(|\lambda| + |\mu|) \geq \varepsilon(|\lambda| + \kappa|\xi|^{2m}) \geq \kappa\varepsilon\Big(\frac{|\lambda|}{\kappa} + |\xi|^{2m}\Big) \geq \varepsilon\kappa$$

From (5.2.5) and (5.2.8) we can conclude $\|(\lambda + A_\#(\xi))^{-1}\| \leq (\varepsilon\kappa)^{-1}$, with $(\lambda,\xi) \in \Sigma_{\pi-\theta}\setminus\{0\} \times \mathbb{R}^d$ and $|\lambda| + |\xi|^{2m} = 1$. By homogeneity we obtain (5.2.4) with $C = (\varepsilon\kappa)^{-1}$. □

As a consequence we obtain the following:

Corollary 5.2.3. *Let $\lambda_0 > 0$. Under the conditions of Theorem 5.2.1, the operator A on X_0 with domain $X_1 = W^{2m,q}(\mathbb{R}^d, w; \mathbb{C}^N)$ is closed and for every $\lambda \geq \lambda_0$,*

$$c\|u\|_{X_1} \leq \|(\lambda + A)u\|_{X_0} \leq (2K + \lambda)\|u\|_{X_1},$$

where c^{-1} is A_q-consistent and only depends on $m, d, \theta_0, \theta, \kappa, K, q$.

5.2.2. *Generation theorem*

Consider A with time-dependent coefficients:

$$(A(t)u)(x) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(t) D^\alpha D^\beta u(x), \tag{5.2.9}$$

with $A(t) \in \mathrm{Ell}^{\mathrm{LH}}(\kappa, \mathrm{K})$ for some $\kappa, K > 0$ independent of $t \in \mathbb{R}$. It follows from Theorem 5.2.1 that $A(t)$ is a sectorial operator and by Corollary 5.2.3 the graph norm of $\|u\|_{D(A(t))}$ is equivalent to the norm $\|u\|_{W^{2m,q}(\mathbb{R}^d,w;\mathbb{C}^N)}$ with uniform estimates and constants which only depend on w, q, d, κ, K, m.

The main result of this section is that $(A(t))_{t\in\mathbb{R}}$ generates a strongly continuous evolution family $(S(t,s))_{-\infty<s\le t<\infty}$ on $L^q(\mathbb{R}^d, w; \mathbb{C}^N)$ for all $q \in (1,\infty)$ and $w \in A_q$. The precise definition of evolution family was given in Definition 3.3.2. Recall that $u(t) = S(t,s)g$ if and only if

$$\begin{aligned} u'(t) + A(t)u(t) &= 0, \quad \text{for almost all } t \in (s,\infty), \\ u(s) &= g. \end{aligned} \tag{5.2.10}$$

Theorem 5.2.4 (Generation of the evolution family)**.** *Let $q \in (1,\infty)$, $w \in A_q$ and set $X_0 = L^q(\mathbb{R}^d, w; \mathbb{C}^N)$ and $X_1 = W^{2m,q}(\mathbb{R}^d, w; \mathbb{C}^N)$. Assume that there exists $\kappa, K > 0$ such that for each $t \in \mathbb{R}$, $A(t) \in \mathrm{Ell}^{\mathrm{LH}}(\kappa, \mathrm{K})$. Then, the operator family $(A(t))_{t\in\mathbb{R}}$ with $D(A(t)) = X_1$ generates a unique strongly continuous evolution family $(S(t,s))_{s\le t}$ on X_0. Moreover, the evolution family satisfies the following properties.*

1. *$(t,s) \mapsto S(t,s) \in \mathscr{L}(X_0)$ is continuous on $\{(t,s) : s < t\}$.*
2. *for all $\alpha \in \mathbb{N}^d$ there is a constant C such that*

   $$\|D^\alpha S(t,s)\|_{\mathscr{L}(X_0)} \le C|t-s|^{-|\alpha|/(2m)}, \quad s < t,$$

   where C only depends on q, d, κ, K, m and on w in an A_q-consistent way.
3. *for all $k \in \mathbb{N}$, and multiindices α with $|\alpha| \le k$,*

   $$D^\alpha S(t,s)u = S(t,s)D^\alpha u, \quad \text{for all } u \in W^{k,q}(\mathbb{R}^d, w; \mathbb{C}^N), s < t.$$

4. *The following weak derivatives exists for almost every $s < t$,*

   $$D_t S(t,s) = -A(t)S(t,s) \text{ on } \mathscr{L}(X_0) \tag{5.2.11}$$

   $$D_s S(t,s) = S(t,s)A(s) \text{ on } \mathscr{L}(X_1, X_0). \tag{5.2.12}$$

As far as we know the existence and uniqueness of the evolution family was unknown even in the case $w = 1$ and $q = 2$. The main difficulty in obtaining the evolution family is that the operators $A(t)$ and $A(s)$ do not commute in general. If they were commuting, then a more explicit formula for the evolution family exists (see Example 3.3.3).

Example 5.2.5. An example where the operators are not commuting can already be given in the case $m = d = 1$, $N = 2$ by taking $A(t) = a(t)D_1^2$, where

$$a(t) = \begin{pmatrix} 1 & \mathbf{1}_{(0,\infty)}(t) \\ \mathbf{1}_{(-\infty,0)}(t) & 1 \end{pmatrix}$$

One can check that $a(1)$ and $a(-1)$ are not commuting. Furthermore, one can check that the ellipticity condition (5.1.3) holds.

In the proof below we use Fourier multiplier theory. It turns out that the symbol is only given implicitly as the solution to a system of differential equation. In order to check the conditions of Mihlin's theorem we apply the implicit function theorem (see [47, Theorem 10.2.1]). The following simple lemmas will be needed.

Lemma 5.2.6 (Gronwall for weak derivatives)**.** *Let $-\infty < s < T < \infty$, $f \in L^1(s,T)$, $a \in L^\infty(s,T)$ and $x \in \mathbb{R}$. Assume $u \in W^{1,1}(s,T) \cap C([s,T])$ satisfies*

$$u'(t) \le a(t)u(t) + f(t), \quad \textit{for almost all } t \in (s,T),$$

and $u(s) = x$. Let $\sigma(t,r) = e^{a_{tr}}$ and $a_{tr} = \int_r^t a(\tau)\,d\tau$ for $s \le r < t \le T$. Then

$$u(t) \le \sigma(t,s)x + \int_s^t \sigma(t,r)f(r)\,dr, \quad t \in (s,T).$$

This follows if one integrates the estimate $\frac{d}{dr}\big[u(r)e^{-a_{rs}}\big] \le e^{-a_{rs}}f(r)$ over (s,t).

Proof. By the product rule for weak derivatives we find

$$\frac{d}{dr}\big[u(r)e^{-a_{rs}}\big] = e^{-a_{rs}}[u'(r) - a(r)u(r)] \le e^{-a_{rs}}f(r).$$

for almost all $r \in (s,T)$. Integrating over $r \in [s,t]$ we obtain

$$u(t)e^{-a_{ts}} \le x + \int_s^t e^{-a_{rs}}f(r)\,dr.$$

from which the result follows. □

The following existence and uniqueness result will be needed.

Lemma 5.2.7. *Let X be a Banach space and $p \in [1,\infty)$. Let $Q : \mathbb{R} \times X \to X$ be measurable and assume there are constants K_1 and K_2 such that for all $t \in \mathbb{R}$ and $x, y \in X$, $\|Q(t,x) - Q(t,y)\| \le K_1\|x - y\|$ and $\|Q(t,x)\| \le K_2(1 + \|x\|)$. Let $u_0 \in X$ and $f \in L^p(\mathbb{R};X)$. Fix $s \in \mathbb{R}$. Then there is a unique function $u \in C([s,\infty);X)$ such that*

$$u(t) - u_0 = \int_s^t Q(s,u(s)) + f(s)\,ds, \quad t \ge s.$$

Moreover, with $\lambda = K_1 + 1$, there is a $C \ge 0$ independent of f and u_0 such that

$$\sup_{t \ge s} e^{-\lambda(t-s)}\|u(t)\| \le C\big(1 + \|u_0\| + \|f\|_{L^p(s,\infty;X)}\big).$$

Proof. The uniqueness follows from Gronwall's inequality. Indeed, let u and v be solutions. Let $w = u - v$. Then

$$\|w'(r)\| \le \|Q(r,u(r)) - Q(t,v(r))\| \le K_1 \|w(r)\|.$$

Therefore, integration over $[s,t]$ yields that

$$\|w(t)\| \le K_1 \int_s^t \|w(r)\|\,dr,$$

since $\int_s^t |w'(r)|dr = |w(t) - w(s)|$ and $w(s) = u(s) - v(s) = u_0 - u_0 = 0$. Now the classical integral form of Gronwall's lemma gives $w = 0$.

The existence is a simple consequence of the Banach fixed point theorem. For convenience we give the details. Fix $\lambda > K_1$. Let E_λ be the space of continuous functions $u : [s,\infty) \to X$ for which $\|u\|_{E_\lambda} = \sup_{t\ge s} e^{-\lambda(t-s)}\|u(t)\| < \infty$. Let $L : E_\lambda \to E_\lambda$ be given by

$$L(u)(t) = u_0 + \int_s^t Q(r,u(r))\,dr + \int_s^t f(r)\,dr.$$

We will show that the mapping L is a contraction on E_λ. Note that for $u \in E_\lambda$,

$$\begin{aligned}
\|e^{-\lambda(t-s)}L(u)(t)\| &\le e^{-\lambda(t-s)}\|u_0\| + e^{-\lambda(t-s)}\int_s^t K_2(1+\|u(r)\|)\,dr \\
&\quad + (t-s)^{1/p'} e^{-\lambda(t-s)}\|f\|_{L^p(\mathbb{R};X)} \\
&\le e^{-\lambda(t-s)}(\|u_0\| + (t-s)^{1/p'}\|f\|_{L^p(\mathbb{R};X)}) + e^{-\lambda(t-s)}K_2(t-s) \\
&\quad + e^{-\lambda(t-s)}K_2\|u\|_{E_\lambda}\int_s^t e^{\lambda(r-s)}\,dr \\
&\le e^{-\lambda(t-s)}(\|u_0\| + (t-s)^{1/p'}\|f\|_{L^p(\mathbb{R};X)}) + e^{-\lambda(t-s)}K_2(t-s) \\
&\quad + K_2\|u\|_{E_\lambda}\frac{1}{\lambda},
\end{aligned}$$

and hence $L(u) \in E_\lambda$. Similarly,

$$\begin{aligned}
e^{-\lambda(t-s)}\|L(u)(t) - L(v)(t)\| &\le e^{-\lambda(t-s)}\int_s^t K_1\|u(r)-v(r)\|\,dr \\
&\le e^{-\lambda(t-s)}K_1\|u-v\|_{E_\lambda}\int_s^t e^{\lambda(r-s)}\,dr \\
&\le \frac{K_1}{\lambda}\|u-v\|_{E_\lambda}.
\end{aligned}$$

Hence, $\|L(u) - L(v)\|_{E_\lambda} \le \frac{K_1}{\lambda}\|u-v\|_{E_\lambda}$. Since $\lambda > K_1$ it follows that L is a contraction. Therefore, there is a unique $u \in E_\lambda$ such that $L(u) = u$, from which we get that

$$u(t) = u_0 + \int_s^t Q(r,u(r))\,dr, \quad t \ge s.$$

Finally observe that

$$\begin{aligned}\|u\|_{E_\lambda} &= \|L(u)\|_{E_\lambda} \le \|L(0)\|_{E_\lambda} + \|L(u)-L(0)\|_{E_\lambda}\\ &\le e^{-\lambda(t-s)}\|u_0\| + (t-s)^{1/p'} e^{-\lambda(t-s)}\|f\|_{L^p(\mathbb{R};X)} + e^{-\lambda(t-s)}K_2(t-s) + \frac{K_1}{\lambda}\|u\|_{E_\lambda}\\ &\le \|u_0\| + (\lambda p')^{-\frac{1}{p'}}\|f\|_{L^p(\mathbb{R};X)} + \frac{K_2}{\lambda} + \frac{K_1}{\lambda}\|u\|_{E_\lambda}.\end{aligned}$$

Hence

$$\|u\|_{E_\lambda} \le \left[\|u_0\| + (\lambda p')^{-\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}_+;X)} + \frac{K_2}{\lambda}\right]\frac{\lambda}{\lambda - K_1}.$$

Taking $\lambda = K_1 + 1$, yields the required result. □

Remark 5.2.8. One can also allow $f \in L^1(\mathbb{R};X)$. In this case, as before we get

$$\|u\|_{E_\lambda} \le |u_0| + \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R};X)} + \frac{K_2}{\lambda} + \frac{K_1}{\lambda}\|u\|_{E_\lambda}.$$

Now we can prove the generation result.

Proof of Theorem 5.2.4. The proof is divided in several steps. Let $B = \mathbb{C}^{N\times N}$ with the operator norm and let $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$.

Step 1: Fix $s \in \mathbb{R}$. Let I denote the $N \times N$ identity matrix. We will first construct the operators $S(t,s)$ and check that (2) holds for $|\alpha| = 0$. For this we show that the function v given by

$$\begin{aligned} v_t(t,\xi) + A_\#(t,\xi)v(t,\xi) &= 0,\\ v(s,\xi) &= I, \end{aligned} \tag{5.2.13}$$

is an $L^q(\mathbb{R}^d, w; \mathbb{C}^N)$-Fourier multiplier by applying a Mihlin multiplier theorem for weighted L^q-spaces (see [65, Theorem IV.3.9] for the case $N = 1$). The solution u to (5.2.10) is then given by

$$u(t) = S(t,s)g = \mathscr{F}^{-1}(v(t,\cdot)\hat{g}),$$

where $\hat{g}$ denotes the Fourier transform of g. Note that by Lemma 5.2.7 for each $\xi \in \mathbb{R}^d_*$ there exists a unique solution $v(\cdot,\xi) \in C([s,\infty);B)$ of (5.2.13). Conversely, if $S(t,s)$ is an evolution family for $A(t)$, then by applying the Fourier transform, one sees that $\mathscr{F}(S(t,s))$ has to satisfy (5.2.13) for almost all $t > s$. This yields the uniqueness of the evolution family.

To check the conditions of the multiplier theorem it suffices to prove the following claim: It holds that $v(t,\cdot) \in C^\infty(\mathbb{R}^d_*;B)$ and for all multiindices $\gamma \in \mathbb{N}^d$, and $j \ge 0$,

$$\|D^\gamma v(t,\xi)\|_B \le C|\xi|^{-|\gamma|}, \quad \xi \in \mathbb{R}^d_*, \tag{5.2.14}$$

where C only depends γ, d, m, κ and K. The estimate (5.2.14) will be proved by induction on the length of γ by using the implicit function theorem.

Step 2: As a preliminary result we first prove an estimate for the problem

$$\begin{aligned} v_t(t,\xi) + A_\#(t,\xi)v(t,\xi) &= f(t,\xi), \\ v(s,\xi) &= M, \end{aligned} \tag{5.2.15}$$

where $f : (s,\infty)\times\mathbb{R}^d_* \to B$ is measurable and for each $\xi \in \mathbb{R}^d_*$, $f(\cdot,\xi) \in L^2(s,\infty;B)$ and $M \in B$. Note that the existence and uniqueness of a solution $v(\cdot,\xi) \in W^{1,2}(s,T;B) \cap C([s,T];B)$ for fixed $\xi \neq 0$ and $T > s$ follows from Lemma 5.2.7. Moreover, since $v(\cdot,\xi)$ is obtained from a sequential limiting procedure in the Banach fixed point theorem, the function v is measurable on $[s,T]\times\mathbb{R}^d_*$. Choosing T arbitrary large, it follows that there is a unique measurable $v : [s,\infty)\times\mathbb{R}^d_* \to B$ for which the restriction to $[s,T]$ satisfies $v(\cdot,\xi) \in W^{1,2}(s,T;B) \cap C([s,T];B)$ and is a solution to (5.2.15).

Fix $\xi \in \mathbb{R}^d_*$, $\varepsilon \in (0,\kappa)$ and $x \in \mathbb{R}^N$. From the ellipticity condition (5.1.3) and (5.2.15) we infer that

$$\begin{aligned} \frac{1}{2}D_t|v(t,\xi)x|^2 &= -\mathrm{Re}\big(\langle v(t,\xi)x, A_\#(t,\xi)v(t,\xi)x\rangle\big) + \mathrm{Re}\big(\langle v(t,\xi)x, f(t,\xi)x\rangle\big) \\ &\le -\kappa|\xi|^{2m}|v(t,\xi)x|^2 + |v(t,\xi)x|\,|f(t,\xi)x| \\ &\le (\varepsilon-\kappa)|\xi|^{2m}|v(t,\xi)x|^2 + \frac{1}{4\varepsilon}|\xi|^{-2m}|f(t,\xi)x|^2, \end{aligned}$$

where we used $2ab \le a^2 + b^2$ on the last line. Thus Lemma 5.2.6 yields:

$$|v(t,\xi)x|^2 \le e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-s)}|Mx|^2 + \frac{1}{2\varepsilon}\int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)}|\xi|^{-2m}|f(r,\xi)x|^2\,dr.$$

Taking the supremum over all $|x| \le 1$, we find that

$$\|v(t,\xi)\|^2 \le e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-s)}\|M\|^2 + \frac{1}{2\varepsilon}\int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)}|\xi|^{-2m}\|f(r,\xi)\|^2\,dr. \tag{5.2.16}$$

Note that if $f = 0$, then the second term vanishes and we can take $\varepsilon = 0$ in (5.2.16). In this case $\|v(t,\xi)\| \le e^{(\varepsilon-\kappa)|\xi|^{2m}(t-s)} \le 1$ and hence (5.2.14) holds for $|\gamma| = 0$. Also note that if v_j is the solution to (5.2.15) with (M,f) replaced by (M_j,f_j) for $j = 1,2$, then by the previous estimates also

$$\|v_1(t,\xi) - v_2(t,\xi)\|^2 \le \|M_1 - M_2\|^2 + \frac{1}{2\varepsilon}|\xi|^{-2m}\|f_1(\cdot,\xi) - f_2(\cdot,\xi)\|^2_{L^2((s,\infty);B)}.$$

Consequently, since $D_t v_1 - D_t v_2 = -A_\#(t,\xi)(v_1 - v_2) + (f_1 - f_2)$ we deduce that

$$\begin{aligned} &\|v_1(\cdot,\xi) - v_2(\cdot,\xi)\|_{W^{1,2}(s,T;B)} \\ &\le C(1+|\xi|^{2m})\|M_1 - M_2\| + C\sum_{j=-1}^{1}|\xi|^{jm}\|f_1(\cdot,\xi) - f_2(\cdot,\xi)\|_{L^2((s,\infty);B)}, \end{aligned}$$

where C does not depend on $\xi \in \mathbb{R}^d_*$. Thus the solution depends in a Lipschitz continuous way on the data.

Step 3: Fix $T > 0$. Define $\Psi : \mathbb{R}^d_* \times W^{1,2}(s,T;B) \to L^2(s,T;B) \times B$ by

$$(\Psi(\xi)v)(t) := (v'(t) + A_\#(t,\xi)v(t), v(s)).$$

Clearly, v is a solution to (5.2.15) if and only if $\Psi(\xi)v(t) = (f,M)$. Therefore, by the previous step for each $\xi \neq 0$, $\Psi(\xi)$ is an homeomorphism and $\Psi(\xi)^{-1}(f,M) = v(t,\xi)$.

For fixed $M \in B$ and $f \in C^\infty(\mathbb{R}^d_*; L^2(\mathbb{R};B))$, let

$$\Phi^{f,M} : \mathbb{R}^d_* \times W^{1,2}(s,T;B) \to L^2(s,T;B) \times B$$

be given by

$$\Phi^{f,M}(\xi, v) := (\Psi(\xi)v) - (f(\xi), M).$$

Now for fixed $\bar{\xi} \in \mathbb{R}^d_*$, $\Phi^{f,M}(\bar{\xi}, v) = 0$ holds if and only if v is a solution to (5.2.15). Therefore, by the previous step there exists a unique $(\bar{\xi}, \bar{v}) \in \mathbb{R}^d_* \times W^{1,2}(s,t;B)$ such that $\Phi^{f,M}(\bar{\xi}, \bar{v}) = 0$. The Fréchet derivative with respect to the second coordinate satisfies

$$D_2\Phi^{f,M}(\bar{\xi}, \bar{v})v = \big(v'(t) + A_\#(t,\xi)v(t), v(s)\big) = (\Psi(\xi)v)(t,\xi). \tag{5.2.17}$$

Thus, also $D_2\Phi^{f,M}(\bar{\xi}, \bar{v})$ is an homeomorphism. Moreover, since $A_\#(t,\cdot)$ and f are C^∞ on $\mathbb{R}^d_*$, it follows that for every $v \in W^{1,2}(s,T;B)$, $\xi \mapsto \Phi^{f,M}(\xi, v)$ is C^∞ on $\mathbb{R}^d_*$. Now by the implicit function theorem (see [47, Theorem 10.2.1]) there exists a unique continuous mapping $\zeta : \mathbb{R}^d_* \to W^{1,2}(s,T;B)$ such that $\zeta(\xi) = v(\cdot,\xi)$, $(\xi,\zeta(\xi)) \in \mathbb{R}^d_* \times W^{1,2}(s,T;B)$ and $\Phi^{f,M}(\xi,\zeta(\xi)) = 0$ for every $\xi \in \mathbb{R}^d_*$. From this we obtain that the unique solution of (5.2.15) can be expressed by

$$v(\cdot,\xi) := \zeta(\xi) = \Psi(\xi)^{-1}(f,M).$$

Moreover, by the implicit function theorem ζ is C^∞ on $\mathbb{R}^d_*$ as an $W^{1,2}(s,T;B)$-valued function and

$$D_{\xi_j}\zeta(\xi) = -(\Psi(\xi))^{-1} \circ \Psi^{f,M}_{\xi_j}(\xi,\zeta(\xi))) = \Psi(\xi)^{-1}\Big(\tilde{f}, 0\Big),$$

where $\tilde{f}(t,\xi) = -D_{\xi_j}A_\#(t,\xi)y(\xi) + D_{\xi_j}f(t,\xi)$ and where we applied (5.2.17). This means that $D_{\xi_j}\zeta(\xi)$ is a solution to (5.2.15) with $M = 0$ and f replaced by $\tilde{f}(t,\xi)$ and that by (5.2.16) the following estimate holds

$$\|D_{\xi_j}\zeta(\xi)(t)\|^2 \leq \frac{1}{2\varepsilon}\int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)}|\xi|^{-2m}\|\tilde{f}(r,\xi)\|^2\,dr. \tag{5.2.18}$$

Step 4. We are now in position to do the induction step. Assume that $\forall\ |\gamma| \leq n$ the problem

$$\begin{aligned} v'_\gamma(t,\xi) + A_\#(t,\xi)v_\gamma(t,\xi) &= f(t,\xi) \\ v_\gamma(s,\xi) &= M, \end{aligned} \tag{5.2.19}$$

has a unique solution given by $v_\gamma(t,\xi) = D^\gamma v(t,\xi)$, where $M = 0$ if $|\gamma| \geq 1$, $M = I$ if $|\gamma| = 0$ and f is given by

$$f(t,\xi) = -\sum_{\substack{\eta_1+\eta_2=\gamma \\ \eta_2\neq\gamma}} c_{\eta_1,\eta_2} D^{\eta_1} A_\#(t,\xi) D^{\eta_2} v(t,\xi)$$

and assume that $\forall\ |\gamma| \leq n$,

$$\|D^\gamma v(t,\xi)\| \leq C_\gamma |\xi|^{-|\gamma|}. \tag{5.2.20}$$

Of course these assertions hold in the case $|\gamma| = 0$, by Step 2.

Fix $|\gamma| = n+1$ and write $\gamma = \tilde{\gamma} + \beta$, with $|\tilde{\gamma}| = n$, $|\beta| = 1$. By Step 3, the function $w = D^\beta v_\gamma$ satisfies

$$\begin{aligned} w'(t,\xi) + A_\#(t,\xi) w(t,\xi) &= \tilde{f}(t,\xi) \\ w(s,\xi) &= 0, \end{aligned} \tag{5.2.21}$$

and for suitable $\tilde{c}_{\eta_1,\eta_2}$,

$$\tilde{f}(t,\xi) = -\sum_{\substack{\eta_1+\eta_2=\gamma \\ \eta_2\neq\gamma}} \tilde{c}_{\eta_1,\eta_2} D^{\eta_1} A_\#(t,\xi) D^{\eta_2} v(t,\xi)$$

Moreover, by (5.2.18), the fact that $\xi \mapsto D^{\eta_1} A_\#(t,\xi)$ is a $(2m-|\eta_1|)$-homogenous polynomial, $|\eta_1| + |\eta_2| = n+1$, and (5.2.20) we find

$$\begin{aligned} \|D^\gamma v(t,\xi)\|^2 &\leq \frac{1}{2\varepsilon} \int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)} |\xi|^{-2m} \|\tilde{f}(r,\xi)\|^2\, dr \\ &\leq \frac{1}{2\varepsilon} \sum_{\substack{\eta_1+\eta_2=\gamma \\ \eta_2\neq\gamma}} \tilde{c}_{\eta_1,\eta_2} \int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)} |\xi|^{2m-2|\eta_1|} \|D^{\eta_2} v(r,\xi)\|^2\, dr \\ &\leq \tilde{C}_\gamma |\xi|^{-2|\gamma|} \int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)} |\xi|^{2m}\, dr \leq C_\gamma |\xi|^{-2|\gamma|}. \end{aligned}$$

This completes the induction step and hence (5.2.14) follows.

Step 5: To prove (2) for general α, fix $k \geq 0$. Since $\|v(t)\| \leq e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-s)}$, there is a constant C such that

$$\|v(t)\| \leq C|\xi|^{-2mk} |t-s|^{-k},\ t \geq s.$$

Now if we replace the induction hypothesis (5.2.20) by

$$\|D^\gamma v(t,\xi)\| \leq C_\gamma |\xi|^{-|\gamma|-2mk} |t-s|^{-k},\ \ s < t, \xi \neq 0 \tag{5.2.22}$$

for all $|\gamma| \leq n$, then for $\gamma = \tilde{\gamma} + \beta$ with $|\tilde{\gamma}| = n$ and $|\beta| = 1$, we find

$$\|D^\gamma v(t,\xi)\|^2 \leq \frac{1}{2\varepsilon} \int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)} |\xi|^{-2m} \|\tilde{f}(r,\xi)\|^2\, dr$$

$$\begin{aligned}
&\leq \frac{1}{2\varepsilon} \sum_{\substack{\eta_1+\eta_2=\gamma \\ \eta_2\neq\gamma}} \tilde{c}_{\eta_1,\eta_2} \int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)} |\xi|^{2m-2|\eta_1|} \|D^{\eta_2} v(r,\xi)\|^2 \, dr \\
&\leq \tilde{C}_\gamma |\xi|^{-2|\gamma|} \int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)} |\xi|^{2m} |\xi|^{-4mk} (r-s)^{-2k} \, dr \\
&\leq \tilde{C}_\gamma |\xi|^{-2|\gamma|-4mk} (t-s)^{-2k} \int_s^t e^{2(\varepsilon-\kappa)|\xi|^{2m}(t-r)} |\xi|^{2m} \, dr \\
&\leq C_{\tilde{\gamma}} |\xi|^{-2|\gamma|-4mk} (t-s)^{-2k}.
\end{aligned}$$

Hence by induction, (5.2.22) holds for all integers $n \geq 0$.

By (5.2.22), $w(t,\xi) = (-i\xi)^\alpha v(t,\xi)$ satisfies the conditions of the Mihlin multiplier theorem, with constant $\lesssim (t-s)^{-|\alpha|/(2m)}$ and therefore we find that

$$\|D^\alpha S(t,s)\|_{X_0} \leq C(t-s)^{-|\alpha|/(2m)},$$

which proves (2). The identity in (3) is a direct consequence of the fact that $v(t,\xi)(-i\xi)^\alpha = (-i\xi)^\alpha v(t,\xi)$.

Step 6: Next we prove that $S(t,s)$ is a strongly continuous evolution family for $A(t)$, i.e. that it satisfies Definition 3.3.2. The identities $S(t,t) = I$ and $S(t,s)S(s,r) = S(t,r)$ are clear from the definition of v and Lemma 5.2.7. To prove strong continuity of the evolution family, consider $(t,s) \mapsto S(t,s)g = \mathscr{F}^{-1}(v_s(t)\hat{g})$ for $g \in X_1$, where v_s is the solution to (5.2.13). Setting $f(r) = -A(r)S(r,s)g$ it follows from (3) that for all $r \geq s$, $\|f(r)\|_{X_0} \leq C\|g\|_{X_1}$. Moreover,

$$S(t,s)g - g = \mathscr{F}^{-1}(v_s(t,\cdot)\hat{g} - \hat{g}) = \mathscr{F}^{-1}\Big(\int_s^t \hat{f}(r)\, dr\Big) = \int_s^t f(r)\, dr$$

in $\mathscr{S}'(\mathbb{R}^d;\mathbb{C}^N)$ and hence in X_0. This proves Definition 3.3.2 (iii). Moreover, we find

$$\|S(t,s)g - g\|_{X_0} \leq (t-s) \sup_{r\in[s,t]} \|f(r)\|_{X_0} \leq C(t-s)\|g\|_{X_1}$$

which implies the continuity of $(t,s) \mapsto S(t,s)g$ for $g \in X_1$. The general case follows by approximation and the uniform boundedness of $S(t,s)$. It remains to prove Definition 3.3.2 (iv) and this will be done in the next step.

Step 7: To prove (5.2.11) fix $r \in (s,t)$. Note that by (2), $f = S(r,s)g \in W^{\ell,q}(\mathbb{R}^d, w;\mathbb{C}^N)$ for any $\ell \in \mathbb{N}$. Therefore, it follows from the previous step and (3) that

$$\begin{aligned}
S(t,s)g - S(r,s)g = S(t,r)f - f &= -\int_r^t A(\tau)S(\tau,r)f \, d\tau \\
&= -\sum_{|\alpha|=|\beta|=m} \int_r^t a_{\alpha,\beta} D^\alpha S(\tau,r) D^\beta f \, d\tau
\end{aligned} \tag{5.2.23}$$

and since by (2), $\|D^\alpha S(\tau,r)\| \leq C(\tau-r)^{-1/2}$ for $|\alpha| = m$ we find that

$$\|S(t,s)g - S(r,s)g\|_{X_0} \leq C\int_r^t (\tau-r)^{-1/2}(r-s)^{-1/2}\, d\tau \|g\|_{X_0}$$

$$\leq C(t-r)^{1/2}(r-s)^{-1/2}\|g\|_{X_0}.$$

This implies that $t \mapsto S(t,s) \in \mathscr{L}(X_0)$ is Hölder continuous on $[s+\varepsilon,\infty)$ for any $\varepsilon > 0$. Moreover, since A is strongly measurable also $\tau \mapsto A(\tau)S(\tau,s)$ is a strongly measurable function. By (2), $\|A(\tau)S(\tau,s)\| \leq C(\tau - s)^{-1}$ and hence it is locally integrable on $[s,\infty)$ as an $\mathscr{L}(X_0)$-valued function. Therefore, (5.2.23) implies that for $s < r < t$,

$$S(t,s) - S(r,s) = -\int_r^t A(\tau)S(\tau,s)\, d\tau$$

and thus $D_t S(t,s) = -A(t)S(t,s)$ in $\mathscr{L}(X_0)$ for almost all $s < t$.

To prove (5.2.12) we use a similar duality argument as in [4, Section 6] and [2, Proposition 2.9]. Fix $t_0 \in \mathbb{R}$. Clearly, $A(t_0-\tau)^*$ has symbol $A_\#(t_0-\tau,\xi)^*$ and hence generates a strongly continuous evolution family, $(W(t_0;\tau,s))_{s\leq\tau}$. Now as in [2, Proposition 2.9] one can deduce $S(t,s)^* = W(t;t-s,0)$. Therefore, applying (5.2.11) to $W(t;t-s,0)$, we see that for almost all $s < t$

$$D_s S(t,s)^* = D_s W(t;t-s,0) = A(t-(t-s))^* W(t;t-s,0) = A(s)^* S(t,s)^*,$$

and hence for almost all $s < t$,

$$D_s S(t,s) = (A(s)^* S(t,s)^*)^* \text{ on } \mathscr{L}(X_0). \tag{5.2.24}$$

Now the result follows since the identity $(A(s)^* S(t,s)^*)^* = S(t,s)A(s)$ holds on X_1. In particular, we find that for $g \in X_1$,

$$S(t,s)g - S(t,s-\varepsilon)g = \int_s^t S(t,r)A(r)S(s,s-\varepsilon)g\, dr$$

and letting $\varepsilon \downarrow 0$, yields

$$S(t,s)g - g = \int_s^t S(t,r)A(r)g\, dr$$

from which we obtain Definition 3.3.2 (iv).

From the above construction and the properties of W one sees that $D_s S(t,s)$ is locally integrable on $(-\infty,t)$, and that $s \mapsto S(t,s) \in \mathscr{L}(X_0)$ is Hölder continuous on $(-\infty,-\varepsilon+t)$ for any $\varepsilon > 0$. Combining this with the Hölder continuity of $t \mapsto S(t,s)$, we see that $(t,s) \mapsto S(t,s) \in \mathscr{L}(X_0)$ is continuous on $\{(t,s) : s < t\}$. □

5.3. PROOFS THEOREMS 5.1.3 AND 5.1.4

To prove Theorems 5.1.3 and 5.1.4 we check the conditions of Theorem 3.3.8. We remark that in these conditions we now need to replace the scalar field by $H = \mathbb{C}^N$. In particular, we will need to use the notion of ℓ^s_H-boundedness, as defined in Section 4.1.2.

We first consider the case in which the coefficients of A are x-independent.

Proposition 5.3.1. *Let $q \in (1,\infty)$, $w \in A_q$ and set $X_0 = L^q(\mathbb{R}^d, w; \mathbb{C}^N)$ and $X_1 = W^{2m,q}(\mathbb{R}^d, w; \mathbb{C}^N)$. Assume A is of the form (5.2.9). Let $\kappa, K > 0$ be such that for all $t \in \mathbb{R}$, $A(t) \in \mathrm{Ell}(\kappa, \mathrm{K})$. Let $A_0 := \delta(-\Delta)^m I_N$ for $\delta \in (0,\kappa)$ fixed. Then the following properties hold:*

1. *A_0 has a bounded H^∞-calculus of any angle $\sigma \in (0,\pi/2)$.*
2. *$A(t) - A_0 \in \mathrm{Ell}(\kappa-\delta, \mathrm{K}+\delta)$ and generates a unique evolution family $T(t,s)$ with the property that*
$$\|T(t,s)\|_{\mathscr{L}(L^q(\mathbb{R}^d, w; \mathbb{C}^N))} \le C, \quad s \le t,$$
where C is A_q-consistent.
3. *$T(t,s)$ commutes with e^{-rA_0} for all $s \le t$ and $r \ge 0$.*

Proof. (1): The symbol of A_0 is $\delta|\xi|^{2m} I$, where I is the $N \times N$ identity matrix and the fact that the operator A_0 has a bounded H^∞-calculus follows from the weighted version of the Mihlin multiplier theorem (see [104, Example 10.2b] for the unweighted case).

(2): For $|\xi| = 1$ and $x \in \mathbb{C}^N$,
$$\mathrm{Re}(\langle x, (A_\#(\xi) - \delta|\xi|^{2m})x\rangle) \ge (\kappa-\delta)\|x\|^2.$$
Also the coefficients of the symbol of A_0 are δ or 0, so indeed $\mathrm{Ell}(\kappa-\delta, \mathrm{K}+\delta)$ and the required result follows from Theorem 5.2.4.

(3) From the proof of Theorem 5.2.4 we see that $T(t,s)$ is given by a Fourier multiplier operator. Also e^{-rA_0} is given by a Fourier multiplier with symbol $e^{-r|\xi|^{2m}} I_N$. This symbol clearly commutes with any matrix in $\mathbb{C}^{N\times N}$, and hence with the symbol of $T(t,s)$. Therefore, the operators $T(t,s)$ and e^{-rA_0} commute. □

Proof of Theorem 5.1.3. Step 1: First assume A is of the form (5.2.9), i.e. it has x-independent coefficients. Then by Theorem 4.3.12 and Proposition 5.3.1, the conditions of Theorem 3.3.8 are satisfied. Therefore, the existence and uniqueness result and (5.1.6) follow for any fixed $\lambda_0 > 0$ and the constant in (5.1.6) is A_p-A_q-consistent.

Step 2: In order to complete the proof, one can repeat the argument of Theorem 3.4.5 by replacing the scalar field by $\mathbb{C}^N$. Note that to apply the localization argument and to include the lower order terms, one has to use the interpolation estimate from Theorem 5.2.1. □

Proof of Theorem 5.1.4. Step 1: First assume A is of the form (5.2.9) again. Now we use the result from Theorem 5.1.3 in the x-independent case in a similar way as in [102, Theorem 4.4.2]. Let $\lambda \ge \lambda_0$, where $\lambda_0 > 0$ is fixed. For each $|\alpha| \le m$, let $v_\alpha \in W^{1,p}(\mathbb{R}, v; X_0) \cap L^p(\mathbb{R}, v; X_1)$ be the unique solution to
$$v'(t,x) + (\lambda + A(t))v(t,x) = f_\alpha(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^d.$$

Then by Theorems 5.2.1 and 5.1.3

$$\sum_{|\beta|\le m} \lambda^{1-\frac{|\beta|}{2m}} \|D^{\beta+\alpha} v\|_{L^p(\mathbb{R},v;X_0)} \le C\lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L^p(\mathbb{R},v;X_0)}.$$

Therefore, setting $u = \sum_{|\alpha|\le m} D^\alpha v_\alpha$ and using the fact D^α and A commute in distributional sense, we find that u is a weak solution to (5.1.7) and that (5.1.9) holds. Uniqueness follows from (5.1.9) as well.

Step 2: To obtain the result for general A, one can use a localization argument with weights and extrapolation as in Theorem 3.4.5 in the non-divergence form case. This argument works in the divergence form case as well (see [102, Section 13.6] for the elliptic setting). □

5.3.1. *Consequences for the initial value problem*

In this section we consider the initial value problem

$$\begin{cases} u'(t,x) + A(t)u(t,x) = f(t,x), \ t \in (0,T), \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}^d, \end{cases} \tag{5.3.1}$$

where A is in non-divergence form and satisfies the same condition (C) as in Theorem 5.1.3. A function $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^N$ is a strong solution of (5.3.1) when all the above derivatives (in the sense distributions) exist, (5.3.1) holds almost everywhere and for all bounded sets $Q \subseteq \mathbb{R}^d$, $u(t,\cdot) \to u_0$ in $L^1(Q;\mathbb{C}^N)$.

In order to make the next result more transparent we only consider power weights in the time variable, as in Section 3.3.4 (see remark before Example 3.3.19).

Theorem 5.3.2. *Let $T \in (0,\infty)$. Let $p,q \in (1,\infty)$, $\gamma \in [0,p-1)$, $v_\gamma(t) = t^\gamma$, $w \in A_q(\mathbb{R}^d)$, $X_0 = L^q(\mathbb{R}^d,w;\mathbb{C}^N)$ and $X_1 = W^{2m,q}(\mathbb{R}^d,w;\mathbb{C}^N)$. Assume condition (C) holds and let $s = 2m\Big(1-\frac{1+\gamma}{p}\Big)$. Then for every $f \in L^p(0,T,v_\gamma;X_0)$ and every $u_0 \in B^s_{q,p}(\mathbb{R}^d,w)$ there exists a unique strong solution $u \in W^{1,p}(0,T,v_\gamma;X_0) \cap L^p(0,T,v_\gamma;X_1) \cap C([0,T];B^s_{q,p}(\mathbb{R}^d,w))$ of* (5.3.1). *Moreover, there is a constant C depending on γ, w, p, q, d, m, κ, K, ω and T such that*

$$\begin{aligned} \|u\|_{L^p(0,T,v_\gamma;X_1)} + \|u\|_{W^{1,p}(0,T,v_\gamma;X_0)} + \|u\|_{C([0,T];B^s_{q,p}(\mathbb{R}^d,w))} \\ \le C\|f\|_{L^p(0,T,v_\gamma;X_0)} + C\|u_0\|_{B^s_{q,p}(\mathbb{R}^d,w)}. \end{aligned}$$

Proof. Substituting $v(t,\cdot) = e^{-\lambda t}u(t,\cdot)$ it follows that we may replace A by $\lambda + A$ for an arbitrary λ. Therefore, extending f as zero outside $(0,T)$, by Theorem 5.1.3 we may assume that A has maximal L^p_v-regularity as defined in Definition 3.3.10 for any $v \in A_p$. Recall from [71, Example 9.1.7] that $v_\gamma \in A_p$.

By [129] and the maximal L^p-regularity estimate from Theorem 5.1.3 (also see Section 3.3.4), we need that $u_0 \in (X_0,X_1)_{1-\frac{1+\gamma}{p},p}$ to obtain the well-posedness result and the estimate. The latter real interpolation space can be identified with

$B^s_{q,p}(\mathbb{R}^d, w)$. Indeed, in the case $w = 1$, this follows from [19, Theorem 6.2.4] or [148, Remark 2.4.2.4]. In the weighted setting this follows from the inhomogeneous case of [24, Theorem 3.5]. $\square$

6

Maximal L^p-regularity for parabolic equations with general boundary conditions

This short chapter is devoted to the study of maximal L^p-regularity for systems of parabolic PDEs on the upper half-space with homogeneous general boundary conditions. The proof is an application of the characterization of maximal L^p-regularity introduced in Chapter 3. The main result Theorem 6.2.4 is based on the assumption that the system of operators under consideration generates an evolution family which is bounded on weighted L^q-spaces. In Section 6.4 we show as an example that this condition is satisfied by a second-order elliptic differential operator. The results in this chapter are based on an ongoing research project with Mark Veraar.

6.1. Preliminaries: weights on d-dimensional intervals

In this chapter we will consider A_p-weights in which the supremum in the A_p-constant is taken over d-dimensional intervals, instead of cubes. The reason is that in this case the A_p constant is invariant under dilations $w(x) \mapsto w(\delta x)$, $\delta = (\delta_1, \dots, \delta_d) \in (\mathbb{R}_+)^d$. This property will be needed in the proof of our main result Theorem 6.2.4. We introduce these weights below, and we refer the reader to [65, IV.6] for details.

Let $\mathscr{R}$ be the family of all bounded d-dimensional intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d].$$

Define the *strong maximal function* by

$$M_S f := \sup_{x \in R \in \mathscr{R}} \frac{1}{|R|} \int_R |f(y) dy.$$

The following proposition is a combination of the results in [65, IV.6]

Proposition 6.1.1. *The following properties hold.*

(i) For $p \in (1, \infty)$, M_S is a bounded operator from $L^p(\mathbb{R}^d)$ to itself.

(ii) $M_S(f^\delta) = (M_S f)^\delta$, *with* $\delta = (\delta_1, \dots, \delta_d) \in (\mathbb{R}_+)^d$ *and* $f^\delta(x) = f(\delta(x)) = f(\delta_1 x_1, \dots, \delta_d x_d)$

(iii) $M_S f(x) \le M^1 \circ \dots \circ M^d f(x)$, *where for every* $j = 1, \dots, d$, M^j *denotes the one dimensional Hardy-Littlewood maximal operator*

$$M^j f(x) = \sup_{a < x_j < b} \frac{1}{b-a} \int_a^b |f(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_d)| dy$$

Moreover, the convolution and approximation result stated in [86, Proposition 2.3.9] holds with the strong maximal function. This will be needed in Section 6.4.

In the following, we define a class of A_p-weights in which we consider d-dimensional rectangles instead of cubes.

Definition 6.1.2. Let $w(x)$ be a weight in $\mathbb{R}^d$. For $p \in [1, \infty]$, we say that $w \in A_p^*(\mathbb{R}^d)$ if $w^\delta \in A_p(\mathbb{R}^d)$ uniformly in $\delta = (\delta_1, \dots, \delta_d) \in (\mathbb{R}_+)^d$, i.e. $\sup_\delta [w^\delta]_{A_p(\mathbb{R}^d)} < \infty$.

Since dilations by $\delta = (\delta_1, \dots, \delta_d) \in (\mathbb{R}_+)^d$ trasform cubes in arbitrary d-dimensional intervals, a change of variables proves that, for $p \in (1, \infty)$, $w \in A_p^*(\mathbb{R}^d)$ if and only if

$$[w]_{A_p^*} := [w]_{p^*} := \sup_{R \in \mathscr{R}} \Big(\frac{1}{|R|} \int_R w(x) dx \Big) \Big(\frac{1}{|R|} \int_R w(x)^{-\frac{1}{p-1}} \, dx \Big)^{p-1} < \infty.$$

It follows directly from the definition that the A_p^* constant is invariant under dilations $w(x) \mapsto w(\delta x)$. Observe that in the case of the half-space $\mathbb{R}^d_+$, the above definition hold if one replace the d-dimensional rectangle R with $R \cap \mathbb{R}^d_+$.

The following properties hold. We refer to [65, Theorem IV.6.2] for the proof.

Theorem 6.1.3. *Given a weight* $w(x)$ *in* $\mathbb{R}^d$ *and* $p \in (1, \infty)$, *the following are equivalent:*

(i) $w \in A_p^*(\mathbb{R}^d)$;

(ii) there exists $C > 0$ *such that for every* $j = 1, 2, \dots, n$ *we have*

$$[w(x_1, x_2, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_d)]_{A_p(\mathbb{R})} \le C$$

for a.e. $(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in \mathbb{R}^{d-1}$;

(iii) M_S *is a bounded operator from* $L^p(\mathbb{R}^d, w)$ *to itself.*

Therefore, there is an equivalence between A_p^* and the one-dimensional A_p-condition on each variable (uniformly in the remaining variables). Moreover, as stated in [65, Theorem IV.6.9], the well-known extrapolation theorem of Rubio de Francia (see e.g. [65, Theorem IV.5.19] and [32, Theorem 3.9]) and the weighted version of the Mihlin multiplier theorem [65, Theorem IV.3.9] hold with $w \in A_p^*(\mathbb{R}^d)$.

6.2. ASSUMPTIONS AND MAIN RESULT

Let $p,q \in (1,\infty)$, $v \in A_p(\mathbb{R})$ and $w \in A_q^*(\mathbb{R}_+^d)$. Let A be a 2m-th order differential operator given by

$$A(t) = \sum_{|\alpha|\leq 2m} a_\alpha(t,x)D^\alpha, \tag{6.2.1}$$

where $D = -i(\partial_1,\dots,\partial_d)$, $a_\alpha : \mathbb{R}\times\mathbb{R}_+^d \to \mathbb{C}^{N\times N}$.

For $j = 1,\dots,m$ and $m_j \in \{0,\dots,2m-1\}$, we consider the boundary differential operators B_j of order m_j given by

$$B_j u = \sum_{|\beta|\leq m_j} b_{j\beta}D^\beta u, \quad b_{j\beta} \in \mathbb{C}.$$

Define

$$W_B^{2m,q}(\mathbb{R}_+^d, w;\mathbb{C}^N) := \{u \in W^{2m,q}(\mathbb{R}_+^d, w;\mathbb{C}^N) : B_j u = 0,\ j = 0\dots,m\}.$$

Set $X_0 = L^q(\mathbb{R}_+^d, w;\mathbb{C}^N)$ and $X_1 = W_B^{2m,q}(\mathbb{R}_+^d, w;\mathbb{C}^N)$. Consider the system of PDEs with boundary condition

$$\begin{cases} u'(t,x) + (\lambda + A(t))u(t,x) = f(t,x) & \text{in } \mathbb{R}\times\mathbb{R}_+^d \\ B_j u(t,x)\big|_{x_1=0} = 0 & \text{on } \mathbb{R}\times\mathbb{R}^{d-1} \quad j = 1,\dots,m, \end{cases} \tag{6.2.2}$$

where $u,f : \mathbb{R}\times\mathbb{R}_+^d \to \mathbb{C}^N$. A function $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}^N$ is called a *strong solution* to (6.2.2) when all the above derivatives (in distributional sense) exist in $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+^d)$ and (6.2.2) holds almost everywhere.

Recall the following definition from Section 5.1.1.

Definition 6.2.1. We say that A is *uniformly elliptic* of angle $\theta \in (0,\pi)$ if there exists a constant $\kappa \in (0,1)$ such that

$$\sigma(A_\#(\xi)) \subseteq \Sigma_\theta \cap \{\xi : |\xi| \geq \kappa\}, \xi \in \mathbb{R}^d, |\xi| = 1, \tag{6.2.3}$$

where $A_\#(\xi)$ denotes the pryncipal symbol of A, and there is a constant $K \geq 1$ such that $\|a_\alpha\| \leq K$ for all $|\alpha| \leq 2m$. In this case we write $A \in \mathrm{Ell}(\theta,\kappa,\mathrm{K})$.

For $t \geq s$ we introduce the operator $S(t,s)$ so that for $g \in C_0^\infty(\mathbb{R}_+^d)$, $u(t) = S(t,s)g$ is the solution to the initial boundary valued problem

$$\begin{cases} u'(t,x) + A(t)u(t,x) = 0, & \text{in } (s,\infty)\times\mathbb{R}_+^d \\ B_j u(t,x)\big|_{x_1=0} = 0 & \text{on } \mathbb{R}\times\mathbb{R}^{d-1} \quad j = 1,\dots,m, \\ u(s,x) = g(x). \end{cases} \tag{6.2.4}$$

In this case we say that $S(t,s)$ is the evolution family generated by $A(t)$. The precise definition of an evolution family for $(A(t))_{t\in\mathbb{R}}$ was given in Definition 3.3.2.

For A of the form (6.2.1) and $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d_+$ let us introduce the notation

$$A(t_0, x_0) := \sum_{|\alpha| \le 2m} a_\alpha(t_0, x_0) D^\alpha.$$

for the operator with constant coefficients. The principal part of $A(t_0, x_0)$ is denoted by $A_\#(t_0, x_0)$.

In what follows we denote as I_N the $N \times N$ identity matrix and we set $D_1 = -i\partial_{x_1}$. We assume the following conditions on the coefficients of A:

(C) Let A be given by (6.2.1) and assume each $a_\alpha : \mathbb{R} \times \mathbb{R}^d_+ \to \mathbb{C}^{N\times N}$ is measurable. We assume there exist $\theta_0 \in [0, \pi/2)$, κ and K such that for all $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d_+$, $A(t_0, x_0) \in \mathrm{Ell}(\theta_0, \kappa, \mathrm{K})$. Assume there exists an increasing function $\omega : (0, \infty) \to (0, \infty)$ with the property $\omega(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and such that

$$|a_\alpha(t, x) - a_\alpha(t, y)| \le \omega(|x - y|), \quad t \in \mathbb{R}, \ x, y \in \mathbb{R}^d_+.$$

We assume the following condition on the operator family $(A(t))_{t\in\mathbb{R}}$:

(T) Let $A_0 := \delta(\sum_{j=2}^d D_j^2)^m I_N$, $\delta \in (0, \kappa)$. For every $x = x_0 \in \mathbb{R}^d_+$ fixed assume that $A_\#(t, x_0) - A_0$ generates a strongly continuous evolution family $T(t, s)$ which is ℓ^s-bounded on $L^q(\Omega, w; \mathbb{C}^N)$ for every $q \in (1, \infty)$ and $w \in A_q^*(\mathbb{R}^d_+)$.

We moreover impose the following maximal regularity condition for the time independent situation. The definition of maximal L^p_v-regularity was given in Definition 3.3.10.

(MR) Let $p, q \in (1, \infty)$. Fix $w \in A_q^*(\mathbb{R}_+)$. We assume that the problem

$$\begin{cases} u'(t,x) + (\lambda + (-\Delta)^m I_N) u(t,x) = f(t,x) & \text{in } \mathbb{R} \times \mathbb{R}^d_+ \\ B_j u(t,x)|_{x_1=0} = 0 & \text{on } \mathbb{R} \times \mathbb{R}^{d-1}, \quad j = 1, \dots, m, \end{cases} \tag{6.2.5}$$

has the property of maximal L^p_v-regularity on $\mathbb{R}$ for every $v \in A_p(\mathbb{R})$ and every $f \in L^q(\mathbb{R}^d_+, w; \mathbb{C}^N)$. Moreover, we assume that the operator $(-\Delta)^m I_N$ has a bounded H^∞-functional calculus for any angle $< \pi/2$.

Example 6.2.2. Let $B_j = D_{x_1}^{j-1}$, $j = 1, \dots, m$. Then, the (MR)-condition is satisfied by the parabolic problem (6.2.5) with the Dirichlet boundary condition $B_j u = 0$. This result is well known in the literature. An example can be found in [104, Section 7.18]. The (MR)-condition holds also if one assumes more general boundary conditions, for instance of Lopatinskii–Shapiro type. See for example [40, Part II].

Before stating the main result, we need the following lemma.

Lemma 6.2.3. *Let $p \in (1, \infty)$, $\tilde{\alpha} = (m, 0, \dots, 0)$, $v \in A_p(\mathbb{R})$, $w \in A_q^*(\mathbb{R}^d_+)$. Let $X_0 = L^q(\mathbb{R}^d_+, w; \mathbb{C}^N)$ and $X_1 = W_B^{2m,q}(\mathbb{R}^d_+, w; \mathbb{C}^N)$. Assume (MR) to hold. Let $\tilde{A}(t) = a_{\tilde{\alpha}\tilde{\alpha}}(t) D_1^{2m}$ and assume* *that $a_{\tilde\alpha\tilde\alpha}:\mathbb{R}\to\mathbb{C}^{N\times N}$ is measurable. Assume moreover that for every $t=t_0\in\mathbb{R}$ fixed, $\sigma(\tilde A_\#(t_0,\xi))\subseteq\Sigma_\theta\cap\{\xi:|\xi|\geq\kappa\},\xi\in\mathbb{R}^d,|\xi|=1$, and there is a constant $K\geq 1$ such that $\|a_{\tilde\alpha\tilde\alpha}\|\leq K$. Then there exists $\lambda_0=\lambda_0(m,d,q,p,K,\theta_0,[v]_p,[w]_{q^*})>0$ such that for every $\lambda\geq\lambda_0$ and every $u\in W^{1,p}(\mathbb{R},v;X_0)\cap L^p(\mathbb{R},v;X_1)$ satisfying*

$$\begin{cases} u'(t,x)+(\lambda+\tilde A(t))u(t,x)=f(t,x) & \text{in } \mathbb{R}\times\mathbb{R}^d_+\\ B_j u(t,x)\big|_{x_1=0}=0 & \text{on } \mathbb{R}\times\mathbb{R}^{d-1}\quad j=1,\dots,m,\end{cases}$$

where $f\in L^p(\mathbb{R},v;X_0)$, we have

$$\lambda\|u\|_{L^p(\mathbb{R},v;X_0)}+\|u\|_{W^{1,p}(\mathbb{R},v;X_0)}+\|u\|_{L^p(\mathbb{R},v;X_1)}\leq C\|f\|_{L^p(\mathbb{R},v;X_0)}+C\sum_{j=2}^d\|D_j^{2m}u\|_{L^p(\mathbb{R},v;X_0)},$$

with a constant $C=(m,d,q,p,K,\theta_0,[v]_p,[w]_{q^*})>0$.

As an alternative proof of Lemma 6.2.3 one could apply Theorem 3.3.8, with a proper choice of A_0. This will not be treated here. We consider instead the following clearer argument.

Proof. By the extrapolation theorem of Rubio de Francia [32, Theorem 3.9] it suffices to take $p=q$. We then proceed by steps.

Step 1. We apply a substitution argument to reduce to the case $a_{\tilde\alpha\tilde\alpha}(t)\equiv 1$. Let

$$\phi(t)=\begin{cases}\int_0^t a_{\tilde\alpha\tilde\alpha}(\tau)\,d\tau, & t\geq 0\\ -\int_t^0 a_{\tilde\alpha\tilde\alpha}(\tau)\,d\tau, & t<0\end{cases}$$

Observe that ϕ is invertible and $\phi'(t)=a_{\tilde\alpha\tilde\alpha}(t)$. Consider

$$\tilde u(s,x)=u(\phi^{-1}(s),x).$$

Then a simple calculation shows that $\tilde u$ satisfies the problem

$$\begin{cases}\tilde u'(s,x)+(\lambda+(D_1^{2m})I_N)\tilde u(s,x)=\tilde f(s,x) & \text{in } \mathbb{R}\times\mathbb{R}^d_+\\ B_j\tilde u(s,x)\big|_{x_1=0}=0 & \text{on } \mathbb{R}\times\mathbb{R}^{d-1}\quad j=1,\dots,m,\end{cases}\tag{6.2.6}$$

where $\tilde f(s,x)=f(\phi^{-1}(s),x)/(a_{\tilde\alpha\tilde\alpha}(\phi^{-1}(s),x))$. Moreover, we get the weighted transformation $\tilde v=v(\phi^{-1}(s))\in A_p(\mathbb{R})$ with $[\tilde v]_p\simeq[v]_p$.

Step 2. By adding the term $(\sum_{j=2}^d D_j^{2m})I_N\tilde u$ on both sides of (6.2.6) we get

$$\begin{cases}\tilde u'(s,x)+(\lambda+(-\Delta)^m I_N)\tilde u(s,x)=\tilde g(s,x) & \text{in } \mathbb{R}\times\mathbb{R}^d_+\\ B_j\tilde u(s,x)\big|_{x_1=0}=0 & \text{on } \mathbb{R}\times\mathbb{R}^{d-1}\quad j=1,\dots,m,\end{cases}\tag{6.2.7}$$

where $\tilde g(s,x)=\tilde f(s,x)+(\sum_{j=2}^d D_j^{2m})I_N\tilde u$. By the (MR)-assumption it holds that

$$\begin{aligned}\lambda\|\tilde u\|_{L^p(\mathbb{R},v;X_0)}+\|\tilde u\|_{W^{1,p}(\mathbb{R},v;X_0)}+\|\tilde u\|_{L^p(\mathbb{R},v;X_1)}&\leq C\|\tilde g\|_{L^p(\mathbb{R},v;X_0)}\\ &\leq C\|\tilde f\|_{L^p(\mathbb{R},v;X_1)}+C\sum_{j=2}^d\|D_j^{2m}\tilde u\|_{L^p(\mathbb{R},v;X_1)}.\end{aligned}$$

Step 3. Substituting back to u and observing that the norms are equivalent, we obtain the required estimate. □

The first main result is on the maximal L^p_v-regularity for (6.2.2).

Theorem 6.2.4. *Let $p,q \in (1,\infty)$, $v \in A_p(\mathbb{R})$ and $w \in A_q^*(\mathbb{R}^d_+)$. Let $X_0 = L^q(\mathbb{R}^d_+, w; \mathbb{C}^N)$ and $X_1 = W_B^{2m,q}(\mathbb{R}^d_+, w; \mathbb{C}^N)$. Assume conditions (C) and (T) on A. Assume condition (MR) to holds. Then there exists a constant $\lambda_0 = \lambda_0(d,p,q,m,\kappa,K,\theta_0,\omega,[v]_p,[w]_{q^*}) > 0$ such that for all $\lambda \geq \lambda_0$ and for every $f \in L^p(\mathbb{R}, v; X_0)$ there exists a unique strong solution $u \in W^{1,p}(\mathbb{R}, v; X_0) \cap L^p(\mathbb{R}, v; X_1)$ of (6.2.2). Moreover, there is a constant $C = C(d,p,q,m,\kappa,K,\theta_0,\omega,[v]_p,[w]_{q^*})$ such that*

$$\lambda \|u\|_{L^p(\mathbb{R},v;X_0))} + \|u\|_{W^{1,p}(\mathbb{R},v;X_0)} + \|u\|_{L^p(\mathbb{R},v;X_1)} \leq C\|f\|_{L^p(\mathbb{R},v;X_0)}. \tag{6.2.8}$$

In the same way as Corollary 4.3.9 follows from Theorem 4.3.5, the following corollary can be derived from Theorem 6.2.4.

We assume the following condition on the operator family $(A(t))_{t\in\mathbb{R}}$:

(T′) Let $A_0 := \delta(\sum_{j=2}^d D_j^2)^m I_N$, $\delta \in (0,\kappa)$. For every $x = x_0 \in \mathbb{R}^d_+$ fixed assume that $A_\#(t,x_0) - A_0$ generates a strongly continuous evolution family $T(t,s)$ such that for every $q \in (1,\infty)$ and $w \in A_q^*(\mathbb{R}^d_+)$,

$$\|T(t,s)\|_{\mathscr{L}(L^q(\mathbb{R}^d_+,w))} \leq C,$$

where C is A_q^*-consistent.

Corollary 6.2.5. *Let $p,q \in (1,\infty)$, $v \in A_p(\mathbb{R})$ and $w \in A_q^*(\mathbb{R}^d_+)$. Let $X_0 = L^q(\mathbb{R}^d_+, w; \mathbb{C}^N)$ and $X_1 = W_B^{2m,q}(\mathbb{R}^d_+, w; \mathbb{C}^N)$. Assume condition (C) and (T′) on A. Assume condition (MR) to hold. Then there exists a constant $\lambda_0 = \lambda_0(d,p,q,m,\kappa,K,\theta_0,\omega,[v]_p,[w]_{q^*})$ such that for all $\lambda \geq \lambda_0$ and for every $f \in L^p(\mathbb{R}, v; X_0)$ there exists a unique strong solution $u \in W^{1,p}(\mathbb{R}, v; X_0) \cap L^p(\mathbb{R}, v; X_1)$ of (6.2.2). Moreover, there is a constant $C = C(d,p,q,m,\kappa,K,\theta_0,\omega,[v]_p,[w]_{q^*})$ such that*

$$\lambda \|u\|_{L^p(\mathbb{R},v;X_0))} + \|u\|_{W^{1,p}(\mathbb{R},v;X_0)} + \|u\|_{L^p(\mathbb{R},v;X_1)} \leq C\|f\|_{L^p(\mathbb{R},v;X_0)}.$$

6.3. PROOF OF THEOREM 6.2.4

The proof of Theorem 6.2.4 will be done as an application of Theorem 3.3.8 and standard PDE arguments such as localization procedure and method of continuity.

Proof of Theorem 6.2.4. Let $f \in L^p(\mathbb{R}, v; X_1)$ and such that f has support on the finite interval $[a,b]$. The general case $f \in L^p(\mathbb{R}, v; X_0)$ follows from a density argument as in the proof of Theorem 3.3.8.

Without loss of generality, we may assume the lower order terms of A to be all zero. In fact, they can be added later on via interpolation arguments, as in the proof of Theorem 3.4.5

We assume first the coefficients of A to be x-independent. To show the maximal regularity estimate (6.2.8) we proceed by steps.

Step 1: We first want to apply Theorem 3.3.8 to obtain that

$$\|D_j^{2m} u\|_{L^p(\mathbb{R},v;X_0)} \le C\|f\|_{L^p(\mathbb{R},v;X_0)}, \quad \forall\, 2 \le j \le d. \tag{6.3.1}$$

For this, we check that the assumptions of Theorem 3.3.8 are satisfied. Observe that the norm equivalence

$$C^{-1}\|x\|_{X_1} \le \|x\|_{X_0} + \|A(t)x\|_{X_0} \le C\|x\|_{X_1}$$

for $C > 0$ follows by taking a suitable extension operator and applying Corollary 3.4.3. Let now

$$A_0 := \delta(\sum_{j=2}^{d} D_j^2)^m I_N, \quad \delta \in (0,\kappa).$$

The principal symbol of A_0 is given by $\delta|\xi'|^{2m} I_N$, where $\xi' \in \mathbb{R}^{d-1}$. By the (MR)-assumption, A_0 has an H^∞-functional calculus for any angle $< \pi/2$. Observe now that for $|\xi| = 1$ and $x \in \mathbb{C}^N$,

$$\mathrm{Re}(\langle x, (A_\sharp(\xi) - \delta|\xi'|^{2m} x\rangle) \ge (\kappa - \delta)\|x\|^2$$

and the coefficients of the symbol of A_0 are δ or 0. So, $A(t) - A_0 \in \mathrm{Ell}(\theta, \kappa - \delta, K + \delta)$. Since by assumption $A(t) - A_0$ generates an evolution family $T(t,s)$ which is ℓ^s-bounded on $L^q(\mathbb{R}_+^d, w)$, then by Theorem 4.3.5 we get the $\mathscr{R}$-boundedness condition of Theorem 3.3.8. Moreover, e^{-rA_0} commutes with $T(t,s)$ for all $s \le t$ and $r \ge 0$. All the conditions of Theorem 3.3.8 are thus satisfied. Therefore, the existence and uniqueness result and (6.3.1) follow for any fixed $\lambda_0 > 0$ and the constant C is A_p-A_q^*-consistent.

Step 2: We now state a mixed-norm estimate, which constitute a key result for the proof. For simplicity of notation we consider $N = 1$, but the general case follows in the same way. Here we use the same method as in [51, Proposition 1].

Lemma 6.3.1. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}_+^d$ and $w \in A_q^*(\Omega)$ for $q \in (1,\infty)$. For all $u \in W^{2m,q}(\Omega, w)$ and all $\varepsilon > 0$, there exists a constant $C = C([w]_{q^*})$ such that for $k = 0,\dots,2m-1$ it holds*

$$\|D_1^k D_{x'}^{2m-k} u\|_{L^q(\Omega,w)} \le C\varepsilon^{2m}\|D_1^{2m} u\|_{L^q(\Omega,w)} + C\varepsilon^{-2m}\sum_{j=2}^{d}\|D_j^{2m} u\|_{L^q(\Omega,w)}.$$

Proof. Without loss of generality we can focus on $\Omega = \mathbb{R}^d$, since the proof for $\Omega = \mathbb{R}_+^d$ directly follows by using a suitable extension operator. In fact, let $\overline{u} \in W^{2m,q}(\mathbb{R}^d, \overline{w})$ be a suitable extension for u (see e.g. [5, Theorem 5.5.19]), where $\overline{w} \in A_q^*(\mathbb{R}^d)$ is such that $\overline{w} = w$ if $x \in R^d_+$ and $\overline{w} = 1$ elsewhere. Now if the case $\Omega = \mathbb{R}^d$ holds, then

$$
\begin{aligned}
\|D_1^k D_{x'}^{2m-k} u\|_{L^q(\mathbb{R}^d_+,w)} &\le \|D_1^k D_{x'}^{2m-k} \overline{u}\|_{L^q(\mathbb{R}^d,\overline{w})} \\
&\le C\varepsilon^{2m} \|D_1^{2m}\overline{u}\|_{L^p(\mathbb{R}^d,\overline{w})} + C\varepsilon^{-2m} \sum_{j=2}^d \|D_j^{2m}\overline{u}\|_{L^q(\mathbb{R}^d,\overline{w})} \\
&= 2C\varepsilon^{2m} \|D_1^{2m} u\|_{L^p(\mathbb{R}^d_+,w)} + 2C\varepsilon^{-2m} \sum_{j=2}^d \|D_j^{2m} u\|_{L^q(\mathbb{R}^d_+,w)}.
\end{aligned}
$$

To prove the case $\Omega = \mathbb{R}^d$, let $m(\xi) = \dfrac{\xi_1^k (\xi')^{2m-k}}{|\xi|^{2m}}$, $\xi = (\xi_1, \xi') \in \mathbb{R}^d$, with $\xi' := (\xi_2, \dots, \xi_d) \in \mathbb{R}^{d-1}$. This defines an $L^q(\mathbb{R}^d, w)$-Fourier multiplier. In fact,

$$
T_m f := D_1^k D_{x'}^{2m-k} f = \mathscr{F}^{-1}\Big(\frac{\xi_1^k (\xi')^{2m-k}}{|\xi|^{2m}} \widehat{f}(\xi)\Big)
$$

and since $m(\xi)$ is in $C^\infty(\mathbb{R}^d \setminus \{0\})$ and it is homogeneous of degree zero, i.e. $m(\rho\xi) = \rho m(\xi)$ for every $\xi \in \mathbb{R}^d$ and $\rho > 0$, it satisfies the Mihlin's condition $|\xi|^\gamma |D_\xi^\gamma m(\xi)| \le C_{\gamma,d}$ for all multi-indices $\gamma \in \mathbb{N}^d$ (see e.g. [104, Lemma 6.3] and [55, Example 8.8.12]). Then, also the dilation

$$
(\xi_1, \xi') \mapsto m_\varepsilon(\xi_1, \xi') := m(\varepsilon\xi_1, \varepsilon^{-1}\xi') = \frac{\xi_1^k (\xi')^{2m-k}}{\varepsilon^{2m}\xi_1^{2m} + \varepsilon^{-2m}|\xi'|^{2m}}
$$

is an $L^q(\mathbb{R}^d, \overline{w}_\varepsilon)$-multiplier, where $\overline{w}_\varepsilon(x) = \overline{w}(\varepsilon x_1, \varepsilon^{-1}x') \in A_q^*(\mathbb{R}^d)$ is such that $[\overline{w}]_{q^*} = [\overline{w}_\varepsilon]_{q^*}$, since weights in the class A_q^* are dilation invariant. By the weighted version of Mihlin's multiplier theorem [65, Theorem IV.3.9] it holds that

$$
\|T_{m_\varepsilon} f\|_{L^q(\mathbb{R}^d,\overline{w})} = \|T_m f_\varepsilon\|_{L^q(\mathbb{R}^d,\overline{w}_\varepsilon)} \le C_{[\overline{w}_\varepsilon]_{q^*}} \|\mathscr{F}^{-1}\widehat{f}_\varepsilon\|_{L^q(\mathbb{R}^d,\overline{w}_\varepsilon)}
$$

Choosing now h such that $h(x) = f(\varepsilon x_1, \varepsilon^{-1}x')$ and $(\varepsilon^{2m}\xi_1^{2m} + \varepsilon^{-2m}|\xi'|^{2m})\widehat{h} = \widehat{f}_\varepsilon$, we get

$$
\begin{aligned}
\|\mathscr{F}^{-1}\widehat{f}_\varepsilon\|_{L^q(\mathbb{R}^d,\overline{w}_\varepsilon)} &= \|\mathscr{F}^{-1}((\varepsilon^{2m}\xi_1^{2m} + \varepsilon^{-2m}|\xi'|^{2m})\widehat{h})\|_{L^q(\mathbb{R}^d,\overline{w}_\varepsilon)} \\
&\le C_{[\overline{w}_\varepsilon]_{q^*}} \|(\varepsilon^{2m} D_1^{2m} + \varepsilon^{-2m} \sum_{j=2}^d D_j^{2m}) h\|_{L^q(\mathbb{R}^d,\overline{w}_\varepsilon)} \\
&= C_{[w]_{q^*}} \|(\varepsilon^{2m} D_1^{2m} + \varepsilon^{-2m} \sum_{j=2}^d D_j^{2m}) f\|_{L^q(\mathbb{R}^d,\overline{w})},
\end{aligned}
$$

which yields

$$
\|D_1^k D_{x'}^{2m-k}\|_{L^q(\mathbb{R}^d,\overline{w})} \le C_{[w]_{q^*}} \|(\varepsilon^{2m} D_1^{2m} + \varepsilon^{-2m} \sum_{j=2}^d D_j^{2m}) f\|_{L^q(\mathbb{R}^d,\overline{w})}
$$

$\square$

Step 3: We now follow a procedure as in [51, Theorem 4]. We move all the spatial derivatives except $a_{\tilde{\alpha}\tilde{\alpha}}(t)D_1^{2m}u$, $\tilde{\alpha} = (m, 0, \ldots, 0)$, to the right-hand side of (6.2.2) and we consider

$$\begin{cases} u' + (\lambda + a_{\tilde{\alpha}\tilde{\alpha}}(t)D_1^{2m})u = F(t,x) & \text{in } \mathbb{R} \times \mathbb{R}^d_+ \\ B_j u(t,x)\big|_{x_1=0} = 0 & \text{on } \mathbb{R} \times \mathbb{R}^{d-1}, \quad j = 1, \ldots, m, \end{cases}$$

where $F = f - \sum_{|\alpha|=2m, \alpha\neq 2\tilde{\alpha}} a_\alpha(t)D^\alpha u$. By Lemma 6.2.3 we get the estimate

$$\lambda\|u\|_{L^p(\mathbb{R},v;X_0)} + \|u\|_{W^{1,p}(\mathbb{R},v;X_0)} + \|u\|_{L^p(\mathbb{R},v;X_1)} \leq C\|F\|_{L^p(\mathbb{R},v;X_0)} + C\sum_{j=2}^d \|D_j^{2m}u\|_{L^p(\mathbb{R},v;X_0)}, \tag{6.3.2}$$

for any $\lambda \geq \lambda_0 > 0$. Observe that by Lemma 6.3.1 the norm of F can be estimated as

$$\begin{aligned} \|F\|_{L^p(\mathbb{R},v;X_0)} &\leq \|f\|_{L^p(\mathbb{R},v;X_0)} + \sum_{\substack{|\alpha|=2m \\ \alpha\neq 2\tilde{\alpha}}} \|D^\alpha u\|_{L^p(\mathbb{R},v;X_0)} \\ &\leq \|f\|_{L^p(\mathbb{R},v;X_0)} + C\varepsilon^{2m}\|D_1^{2m}u\|_{L^p(\mathbb{R},v;X_0)} + C\varepsilon^{-2m}\sum_{j=2}^d \|D_j^{2m}u\|_{L^p(\mathbb{R},v;X_0)}. \end{aligned}$$

Now by *Step 1* we find

$$\lambda\|u\|_{L^p(\mathbb{R},v;X_0)} + \|u\|_{W^{1,p}(\mathbb{R},v;X_0)} + \|u\|_{L^p(\mathbb{R},v;X_1)} \leq C_\varepsilon\|f\|_{L^p(\mathbb{R},v;X_0)} + C\varepsilon^{2m}\|D_1^{2m}u\|_{L^p(\mathbb{R},v;X_0)}.$$

By choosing now $\varepsilon > 0$ such that $C\varepsilon^{2m} = \frac{1}{2}$, we can incorporate $D_1^{2m}u$ on the left-hand side of (6.3.2) and we obtain the estimate

$$\lambda\|u\|_{L^p(\mathbb{R},v;X_0)} + \|u\|_{W^{1,p}(\mathbb{R},v;X_0)} + \|u\|_{L^p(\mathbb{R},v;X_1)} \leq C\|f\|_{L^p(\mathbb{R},v;X_0)}.$$

Step 4: In order to complete the proof, one can repeat word by word the argument of Theorem 5.1.3. □

6.4. AN EXAMPLE

In this section we show that if A is a second order elliptic differential operator with x-independent coefficients and zero lower order terms, then it generates a strongly continuous evolution family on $L^q(\mathbb{R}^d_+, w)$, for all $q \in (1, \infty)$, $w \in A_q^*(\mathbb{R}^d_+)$, for the parabolic problem with the Dirichlet boundary condition, satisfying a weighted norm estimate.

Let A be a second order differential operator of the form

$$A(t)u = \sum_{i,j=1}^d a_{ij}(t)D_iD_ju. \tag{6.4.1}$$

with $a_{ij} : \mathbb{R} \to \mathbb{C}$ measurable. Assume that for every fixed $t = t_0 \in \mathbb{R}$, $A(t_0)$ is uniformly elliptic, i.e. there exists costants $K > 1, \kappa \in (0,1)$ such that for all $\xi \in \mathbb{R}^d$ we have the ellipticity condition

$$\kappa|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(t_0)\xi^i\xi^j \leq K|\xi|^2,$$

It follows from Theorem 5.2.1, with $N = 1$ there, that $A(t)$ is a sectorial operator. The following proposition is on the evolution family $S_D(t,s)$ generated by $A(t)$. Recall $u(t) = S_D(t,s)g$ if and only if

$$\begin{cases} u'(t,x) + A(t)u(t,x) = 0 & \text{in } (s,\infty) \times \mathbb{R}^d_+ \\ u(t,0,x') = 0 & \text{on } (s,\infty) \times \mathbb{R}^{d-1} \\ u(s,x) = g(x) \end{cases} \tag{6.4.2}$$

Let $W^{2,q}_D(\mathbb{R}^d_+, w) := \{u \in W^{2m,q}(\mathbb{R}^d_+, w) : \ u(t,0,x') = 0, \ t \in \mathbb{R}, x' \in \mathbb{R}^{d-1}\}$.

Proposition 6.4.1. *Let $q \in (1,\infty)$, $w \in A^*_q(\mathbb{R}^d_+)$, $X_0 = L^q(\mathbb{R}^d_+, w)$ and $X_1 = W^{2,q}_D(\mathbb{R}^d_+, w)$. Assume that there exists $\theta \in [0,\pi/2)$ and $\kappa, K > 0$ such that $A(t) \in \mathrm{Ell}(\theta,\kappa,\mathrm{K})$. Assume that the operator family $(A(t))_{t\in\mathbb{R}}$ with $D(A(t)) = X_1$ generates a strongly continuous evolution family $(S_D(t,s))_{s\leq t}$ on X_0 for the problem* (6.4.2) *such that*

$$|S_D(t,s)g(x)| \leq CM_S g^0(x), \tag{6.4.3}$$

where $C = C(q,d,\kappa,K)$ and

$$g^0(x) = \begin{cases} g(x) & x_1 \geq 0 \\ 0 & x_1 < 0. \end{cases}$$

Then, for $q \in (1,\infty)$ and any $w \in A^*_q(\mathbb{R}^d_+)$ there is a constant C depending only on q, d, κ, K and on w in an A^*_q-consistent way, such that

$$\|S_D(t,s)\|_{\mathcal{L}(L^q(\mathbb{R}^d_+,w))} \leq C, \ \ s < t. \tag{6.4.4}$$

Proof. By assumption (6.4.3), taking the $L^q(\mathbb{R}^d_+, w)$-norms on both sides for $w \in A^*_q(\mathbb{R}^d_+)$ and applying the weighted boundedness of the strong maximal function stated in Theorem 6.1.3 we get

$$\|S_D(t,s)g\|_{L^q(\mathbb{R}^d_+,w)} \leq C\|M_S g^0\|_{L^q(\mathbb{R}^d_+,w)} \leq C\|g^0\|_{L^q(\mathbb{R}^d_+,w)} = C\|f\|_{L^q(\mathbb{R}^d_+,w)},$$

with $C = C(q,d,\kappa,K,[w]_{A^*_q})$. □

Lemma 6.4.2. *Let $q \in (1,\infty)$, $w \in A^*_q(\mathbb{R}^d_+)$, $X_0 = L^q(\mathbb{R}^d_+, w)$ and $X_1 = W^{2,q}_D(\mathbb{R}^d_+, w)$. Assume that there exists $\theta \in [0,\pi/2)$ and $\kappa, K > 0$ such that $A(t) \in \mathrm{Ell}(\theta,\kappa,\mathrm{K})$. Then the operator family $(A(t))_{t\in\mathbb{R}}$ with $D(A(t)) = X_1$ generates a strongly continuous evolution family $(S_D(t,s))_{s\leq t}$ on X_0 for the problem* (6.4.2) *that satisfies* (6.4.3).

Proof. Following the lines of the proof of [142, Proposition 2.3] one can directly show that the operator $A(t)$ generates a strongly continuous evolution family $S_D(t,s)$ with the integral representation

$$S_D(t,s)g(x) = \int_{\mathbb{R}^d_+} \Gamma_D(t,s,x,y)g(y)dy,$$

where Γ_D denotes the Green's Kernel in the half-space with the Dirichlet boundary condition. Moreover, by [142, Theorem 2.3] it holds that Γ_D satisfies a Gaussian estimate, i.e.,

$$|\Gamma_D(t,s,x,y)| \leq C(t-s)^{-d/2} e^{-b\frac{|x-y|^2}{t-s}},$$

with constants $C, b > 0$. Then, the estimate (6.4.3) follows directly by [86, Proposition 2.3.9] (see also [70, Theorem 2.1.10]). □

Remark 6.4.3. (i) The results in [142] are stated in the setting of the whole space $\mathbb{R}^d$. However, as it is stated in [142, Remark 1.4 (iv)] this general set-up can be slightly modified in order to treat also initial boundary value problems with the Dirichlet boundary condition. As an example in the time-independent setting one can consider [36, Theorem 3.2.7 and Corollary 3.2.8]. Details are still under investigation.

(ii) As it can be evinced from the above proofs, the main ingredient to show the weighted boundedness of the evolution family generated by $A(t)$ is that its Green's kernel satisfies certain Gaussian estimates. We believe that the same method as [142] could give new informations on Gaussian estimates for higher-order differential operators with coefficients measurable in time and more general boundary conditions. In the time-independent setting, positive results for second-order elliptic operators were given by Arendt-ter Elst in [15] where they showed that the assertions of [36, Theorem 3.2.7] remains true also for more general type of boundary conditions. In the non-autonomous case, if the dependence on time is assumed to be Hölder-continuous, then it is a result of [90] and [140] that second order operators satisfying boundary conditions of Lopatinskii–Shapiro type generate an evolution family whose kernel satisfies a Gaussian bound. This will be subject of further investigation.

(iii) In the case of systems of differential operators, different methods need to be discovered to prove both the existence of the evolution family and the weighted estimates.

IV

Higher order parabolic equations with VMO assumptions and general boundary conditions

7

Higher order elliptic and parabolic equations with VMO assumptions and general boundary conditions

In this chapter we prove mixed $L^p(L^q)$-estimates, with $p, q \in (1,\infty)$, for higher order elliptic and parabolic equations on the half space $\mathbb{R}^{d+1}_+$ with general boundary conditions of Lopatinskii-Shapiro type. We assume that the elliptic operators A have coefficients in the class of vanishing mean oscillation both in the time variable and in the space variable. In the proof, we apply and extend the technique developed by Krylov in [100] as well as Dong and Kim in [51] to produce mean oscillation estimates for equations on the half space with general boundary conditions. The results here presented are based on [48].

7.1. Preliminaries

7.1.1. *Function spaces and notation*

In this section we introduce some function spaces and notation to be use throughout the chapter.

We denote

$$\mathbb{R}^{d+1}_+ = \mathbb{R} \times \mathbb{R}^d_+.$$

The parabolic distance between $X = (t,x)$ and $Y = (s,y)$ in $\mathbb{R}^{d+1}_+$ is defined by $\rho(X,Y) = |x-y| + |t-s|^{\frac{1}{2m}}$. For a function f on $\mathcal{D} \subset \mathbb{R}^{d+1}_+$, we set

$$(f)_{\mathcal{D}} = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} f(t,x)\,dx\,dt = \fint_{\mathcal{D}} f(t,x)\,dx\,dt.$$

For $m = 1,2,\ldots$ fixed depending on the order of the equations under consideration, we denote by

$$Q^+_r(t,x) = ((t-r^{2m},t) \times B_r(x)) \cap \mathbb{R}^{d+1}_+ \tag{7.1.1}$$

the parabolic cylinders, where

$$B_r(x) = \left\{y \in \mathbb{R}^d : |x-y| < r\right\} \subset \mathbb{R}^d$$

denotes the ball of radius r and center x. We use Q_r^+ to indicate $Q_r^+(0,0)$. We also define

$$B_r^+(x) = B_r(x) \cap \mathbb{R}^d_+.$$

We define mean oscillations of f on a parabolic cylinder as

$$\operatorname{osc}(f, Q_r^+(t,x)) := \fint_{Q_r^+(t,x)} \left| f(s,y) - (f)_{Q_r^+(t,x)} \right| ds\, dy$$

and we denote for $R \in (0,\infty)$,

$$(f)^\sharp_R := \sup_{(t,x)\in\mathbb{R}^{d+1}} \sup_{r\le R} \operatorname{osc}(f, Q_r^+(t,x)).$$

Next, we introduce the function spaces which will be used in the chapter. Due to the multitude of parameters needed, we will use a different notation than the one use previously in this thesis, i.e. we will use subscript L_p instead of L^p. We give the precise definitions below.

For $p \in (1,\infty)$ and $k \in \mathbb{N}_0$, we define the standard Sobolev space as

$$W_p^k(\mathbb{R}^d_+) = \left\{ u \in L_p(\mathbb{R}^d_+) :\ D^\alpha u \in L_p(\mathbb{R}^d_+) \quad \forall |\alpha| \le k \right\}.$$

where $L_p(\mathbb{R}^d_+) = \{ f : \mathbb{R}^d \to \mathbb{C} :\ \|f\|_{L_p(\mathbb{R}^d_+)} = (\int_{\mathbb{R}^d_+} |f(x)|^p \, dx)^{1/p} < \infty \}$.

For $p, q \in (1,\infty)$, we denote

$$L_p(\mathbb{R}^{d+1}_+) = L_p(\mathbb{R}; L_p(\mathbb{R}^d_+))$$

and mixed-norm spaces

$$L_{p,q}(\mathbb{R}^{d+1}_+) = L_p(\mathbb{R}; L_q(\mathbb{R}^d_+)).$$

For parabolic equations we denote for $k = 1, 2, \ldots$,

$$W_p^{1,k}(\mathbb{R}^{d+1}_+) = W_p^1(\mathbb{R}; L_p(\mathbb{R}^d_+)) \cap L_p(\mathbb{R}; W_p^k(\mathbb{R}^d_+))$$

and mixed-norm spaces

$$W_{p,q}^{1,k}(\mathbb{R}^{d+1}_+) = W_p^1(\mathbb{R}; L_q(\mathbb{R}^d_+)) \cap L_p(\mathbb{R}; W_q^k(\mathbb{R}^d_+)).$$

We will use the following weighted Sobolev spaces. For $v \in A_p(\mathbb{R})$ and $w \in A_q(\mathbb{R}^d_+)$, we denote

$$L_{p,q,v,w}(\mathbb{R}^{d+1}_+) = L_p(\mathbb{R}, v; L_q(\mathbb{R}^d_+, w))$$

and

$$W_{p,q,v,w}^{1,k}(\mathbb{R}^{d+1}_+) = W_p^1(\mathbb{R}, v; L_q(\mathbb{R}^d_+, w)) \cap L_p(\mathbb{R}, v; W_q^k(\mathbb{R}^d_+, w)),$$

where by $f \in L_{p,q,v,w}(\mathbb{R}^{d+1}_+)$ we mean

$$\|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d_+} |f(t,x)|^q \, w(x)\, dx \right)^{p/q} v(t)\, dt \right)^{1/p} < \infty.$$

7.1.2. Interpolation and trace

The following function spaces from the interpolation theory will be needed. For more information and proofs we refer the reader to [117, 147, 148].

For $p \in (1,\infty)$ and $s = [s] + s_* \in \mathbb{R}_+ \backslash \mathbb{N}_0$, where $[s] \in \mathbb{N}_0$, $s_* \in (0,1)$, we define the Slobodetskii space W_p^s by real interpolation as

$$W_p^s = (W_p^{[s]}, W_p^{[s]+1})_{s_*,p}.$$

For $m \in \mathbb{N}$ and $s \in (0,1]$ we consider anisotropic spaces of the form

$$W_p^{s,2ms}(\mathbb{R}_+^{d+1}) = W_p^s(\mathbb{R}; L_p(\mathbb{R}_+^d)) \cap L_p(\mathbb{R}; W_p^{2ms}(\mathbb{R}_+^d)).$$

For $p \in (1,\infty)$, $q \in [1,\infty]$, $r \in \mathbb{R}$, and X a Banach space, we introduce the X-valued Triebel–Lizorkin space $F_{p,q}^r(\mathbb{R}^d, X)$ as defined below.

Definition 7.1.1. Let $\Phi(\mathbb{R}^d)$ be as in Section 2.1. Given $(\varphi_k)_{k\geq 0} \in \Phi(\mathbb{R}^d)$, we define the *X-valued Triebel–Lizorkin space* as

$$\begin{aligned} &F_{p,q}^r(\mathbb{R}^d, X) \\ &= \big\{ f \in \mathscr{S}'(\mathbb{R}^d, X) : \ \|f\|_{F_{p,q}^r(\mathbb{R}^d,X)} := \|(2^{kr}\mathscr{F}^{-1}(\widehat{\varphi}_k \hat{f}))_{k\geq 0}\|_{L_p(\mathbb{R}^d,\ell_q(X))} < \infty \big\}. \end{aligned}$$

Observe that by Fubini's theorem $\mathscr{B}_{p,p}^r(\mathbb{R}^d) = F_{p,p}^r(\mathbb{R}^d)$, where $\mathscr{B}_{p,p}^r$ denotes the Besov space as in Definition 2.1.1 Moreover, we have the following equivalent definition of Slobodetskii space

$$W_p^s(\mathbb{R}^d) = \begin{cases} W_p^k(\mathbb{R}^d), & s = k \in \mathbb{N} \\ \mathscr{B}_{p,p}^s(\mathbb{R}^d), & s \in \mathbb{R}_+ \backslash \mathbb{N}_0. \end{cases}$$

Later on we will consider X-valued Triebel-Lizorkin spaces on an interval $(-\infty, T) \subset \mathbb{R}$. We define these spaces by restriction.

Definition 7.1.2. Let $T \in (-\infty, \infty]$ and let X be a Banach space. For $p \in (1,\infty)$, $q \in [1,\infty)$ and $r \in \mathbb{R}$ we denote by $F_{p,q}^r((-\infty,T);X)$ the collection of all restrictions of elements of $F_{p,q}^r(\mathbb{R};X)$ on $(-\infty,T)$. If $f \in F_{p,q}^r((-\infty,T);X)$ then

$$\|f\|_{F_{p,q}^r((-\infty,T);X)} = \inf \|g\|_{F_{p,q}^r(\mathbb{R};X)}$$

where the infimum is taken over all $g \in F_{p,q}^r(\mathbb{R};X)$ whose restriction on $(-\infty,T)$ coincides with f.

The following spatial traces and interpolation inequalities will be needed in our proofs. For full details, we refer the reader respectively to [41, Lemma 3.5 and Lemma 3.10] (see also [117, Lemma 1.3.11 and Lemma 1.3.13]).

Theorem 7.1.3. *Let $p \in (1,\infty)$, $m \in \mathbb{N}$, and $s \in (0,1]$ so that $2ms \in \mathbb{N}$. Then the map*

$$\mathrm{tr}_{x_1=0} : W_p^{s,2ms}(\mathbb{R}_+^{d+1}) \hookrightarrow W_p^{s-\frac{1}{2mp},2ms-\frac{1}{p}}(\mathbb{R} \times \mathbb{R}^{d-1})$$

is continuous.

Lemma 7.1.4. *Let $p \in (1,\infty)$ and let $m \in \mathbb{N}$ and $s \in [0,1)$ be given. Then for every $\varepsilon > 0$, for $\beta \in \mathbb{N}_0^n$ with $s + \frac{|\beta|}{2m} + \frac{1}{2mp} < 1$, it holds that for $u \in W_p^{1,2m}(\mathbb{R} \times \mathbb{R}_+^d)$,*

$$\|\mathrm{tr}_\Omega \nabla^\beta u\|_{W_p^{s,2ms}(\mathbb{R}\times\mathbb{R}^{d-1})} \le \varepsilon \|D^{2m} u\|_{L_p(\mathbb{R}\times\mathbb{R}_+^d)} + \varepsilon \|u_t\|_{L_p(\mathbb{R}\times\mathbb{R}_+^d)} + C_\varepsilon \|u\|_{L_p(\mathbb{R}\times\mathbb{R}_+^d)}.$$

The following results for $p, q \in (1,\infty)$ will be important tools in the proof of Theorem 7.2.4.

Theorem 7.1.5. *Let $p, q \in (1,\infty)$. Let for $j = 1,\dots,m$ and $m_j \in \{0,\dots,2m-1\}$, $k_j = 1 - m_j/(2m) - 1/(2mq)$. Then the map*

$$\begin{aligned}\mathrm{tr}_{x_1=0} : W_p^{1-\frac{m_j}{2m}}(\mathbb{R}; L_q(\mathbb{R}_+^d)) \cap L_p(\mathbb{R}; W_q^{2m-m_j}(\mathbb{R}_+^d)) \\ \hookrightarrow F_{p,q}^{k_j}(\mathbb{R}; L_q(\mathbb{R}^{d-1})) \cap L_p(\mathbb{R}; \mathscr{B}_{q,q}^{2mk_j}(\mathbb{R}^{d-1}))\end{aligned}$$

is continuous.

Proof. The proof is essentially contained in the proof of [41, Proposition 6.4], so we only give a sketched proof for the sake of completeness. Let

$$u \in L_p(\mathbb{R}; W_q^{2m-m_j}(\mathbb{R}_+^d)).$$

Taking traces in x_1 and applying [148, Theorem 2.9.3] (applied pointwise almost everywhere in time), we get

$$u|_{x_1=0} \in L_p(\mathbb{R}; \mathscr{B}_{q,q}^{2m-m_j-\frac{1}{q}}(\mathbb{R}^{d-1})).$$

For the time regularity, let $u \in W_{p,q}^{1,2m}(\mathbb{R} \times \mathbb{R}_+^d)$ and define B as in [41, Proposition 6.4] by

$$B = (\partial_t)^{\frac{1}{2m}} \quad \text{with} \quad D(B) = W_p^{\frac{1}{2m}}(\mathbb{R}; L_q(\mathbb{R}_+^d)).$$

Set $u_j = B^{2m-m_j-1}u$. Then, $u_j \in W_p^{\frac{1}{2m}}(\mathbb{R}; L_q(\mathbb{R}_+^d)) \cap L_p(\mathbb{R}; W_q^1(\mathbb{R}_+; L_q(\mathbb{R}^{d-1})))$. Following the line of the proof of [41, Proposition 6.4], one can show that

$$u_j|_{x_1=0} \in F_{p,q}^{\frac{1}{2m}-\frac{1}{2mq}}(\mathbb{R}; L_q(\mathbb{R}^{d-1})).$$

This yields

$$D^{m_j} u|_{x_1=0} \in F_{p,q}^{k_j}(\mathbb{R}; L_q(\mathbb{R}^{d-1})),$$

which completes the proof. $\square$

Lemma 7.1.6. *Let $p,q\in(1,\infty)$ and let $m\in\mathbb{N}$ and $s\in[0,1)$ be given. Then for every $\varepsilon>0$, for $\beta\in\mathbb{N}_0^n$ with $s+\frac{|\beta|}{2m}+\frac{1}{2mq}<1$, it holds that for $u\in W^{1,2m}_{p,q}(\mathbb{R}^{d+1}_+)$,*

$$\begin{aligned}\|\mathrm{tr}_{\mathbb{R}^d_+}\nabla^\beta u\|_{F^s_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R},v;\mathscr{B}^{2ms}_{q,q}(\mathbb{R}^{d-1}))}&\\ \le \varepsilon\|D^{2m}u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))}+\varepsilon\|u_t\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))}&+C_\varepsilon\|u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))}.\end{aligned}$$

The proof follows the line of [41, Lemma 3.10], by considering $p\neq q$ there and applying Theorem 7.1.5.

7.1.3. Anisotropic Sobolev embedding theorem

We will use the following parabolic Sobolev embedding theorem. Details about the proof can be found in [20, Section 18.12].

We denote

$$\begin{aligned}W^{k,2m,h}_{t,x_1,x';p}(\mathbb{R}^{d+1}_+)=W^k_p(\mathbb{R};L_p(\mathbb{R}^d_+))\cap L_p(\mathbb{R};W^{2m}_p(\mathbb{R}_+;L_p(\mathbb{R}^{d-1})))&\\ \cap L_p(\mathbb{R};L_p(\mathbb{R}_+;W^h_p(\mathbb{R}^{d-1}))).&\end{aligned}$$

Theorem 7.1.7. *Let $p\in(1,\infty)$ and $m\in\mathbb{N}$. Then it holds for k,h sufficiently large that*

$$W^{k,2m,h}_{t,x_1,x';p}(Q_1^+)\hookrightarrow C^{\frac{2m-1/p}{2m},2m-1/p}(Q_1^+).$$

Moreover,

$$\|u\|_{C^{\frac{2m-1/p}{2m},2m-1/p}(Q_1^+)}\le C\|u\|_{W^{k,2m,h}_{t,x_1,x';p}(Q_1^+)},$$

with $C>0$ independent of u.

7.1.4. Maximal function theorems on mixed $L_p(L_q)$-spaces

The classical Hardy–Littlewood maximal function theorem and the Fefferman–Stein theorem (see [71, Theorem 9.1.9 and Corollary 7.4.6]) have been recently generalized to mixed $L_p(\mathbb{R},v;L_q(\mathbb{R}^d_+,w))$ spaces by Dong and Kim in Corollaries 2.6 and 2.7 of [52]. Their proofs are based on the extrapolation theorem of Rubio de Francia (see [132–134], or [65, Chapter IV]), that allows one to extrapolate from weighted L_p-estimates for a single $p\in(1,\infty)$ to weighted L_q-estimates for all $q\in(1,\infty)$. These results will play an important role in the proof of Theorem 7.2.4, and thus we state them below for completeness.

Let $Q_r^+(t,x)$ be a parabolic cylinder as in (7.1.1) and let $\mathcal{Q}=\{Q_r^+(t,x):(t,x)\in\mathbb{R}^{d+1}_+,\ r\in(0,\infty)\}$. Define for $p,q\in(1,\infty)$ the parabolic maximal function and sharp function of a function $f\in L_p(\mathbb{R};L_q(\mathbb{R}^d_+))$ by

$$\mathscr{M}f(t,x)=\sup_{\substack{Q\in\mathcal{Q}\\(t,x)\in Q}}\fint_Q|f(s,y)|\,dy\,ds$$

and

$$f^\sharp(t,x) = \sup_{\substack{Q\in\mathcal{Q}\\(t,x)\in Q}} \fint_Q |f(s,y)-(f)_Q|\,dy\,ds.$$

Theorem 7.1.8 (Corollary 2.6 of [52]). *Let $p,q\in(1,\infty)$, $v\in A_p(\mathbb{R})$ and $w\in A_q(\mathbb{R}^d_+)$. Then for any $f\in L_p(\mathbb{R},v;L_q(\mathbb{R}^d_+,w))$, we have*

$$\|\mathcal{M}f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \le C\|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)},$$

where $C = C(d,p,q,[v]_p,[w]_q) > 0$.

Theorem 7.1.9 (Corollary 2.7 of [52]). *Let $p,q\in(1,\infty)$, $v\in A_p(\mathbb{R})$ and $w\in A_q(\mathbb{R}^d_+)$. Then for any $f\in L_p(\mathbb{R},v;L_q(\mathbb{R}^d_+,w))$, we have*

$$\|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \le C\|f^\sharp\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)},$$

where $C = C(d,p,q,[v]_p,[w]_q) > 0$.

7.2. ASSUMPTIONS AND MAIN RESULTS

In this section let $p,q\in(1,\infty)$, $m=1,2,\dots$ and consider a $2m$-th order elliptic differential operator A given by

$$Au = \sum_{|\alpha|\le 2m} a_\alpha(t,x)D^\alpha u,$$

where $a_\alpha:\mathbb{R}\times\mathbb{R}^d_+\to\mathbb{C}$. For $j=1,\dots,m$ and $m_j\in\{0,\dots,2m-1\}$, consider the boundary differential operators B_j of order m_j given by

$$B_j u = \sum_{|\beta|=m_j} b_{j\beta}D^\beta u + \sum_{|\beta|<m_j} b_{j\beta}(t,x)D^\beta u,$$

where $b_{j\beta}\in\mathbb{C}$ if $|\beta|=m_j$, and $b_{j\beta}:\mathbb{R}\times\mathbb{R}^d_+\to\mathbb{C}$ if $\beta|<m_j$.

In this section we will give conditions on the operators A and B_j under which there holds $L_p(L_q)$-estimates for the solution to the parabolic problem

$$\begin{cases} u_t(t,x) + (A+\lambda)u(t,x) = f(t,x) & \text{in } \mathbb{R}\times\mathbb{R}^d_+ \\ B_j u(t,x)\Big|_{x_1=0} = g_j(t,x) & \text{on } \mathbb{R}\times\mathbb{R}^{d-1} \quad j=1,\dots,m, \end{cases} \tag{7.2.1}$$

and to the elliptic problem

$$\begin{cases} (A+\lambda)u = f & \text{in} \quad \mathbb{R}^d_+ \\ B_j u\Big|_{x_1=0} = 0 & \text{on} \quad \mathbb{R}^{d-1}, \quad j=1,\dots,m, \end{cases} \tag{7.2.2}$$

where, for the elliptic case, the coefficients of the operators involved are functions independent on $t\in\mathbb{R}$, i.e., defined on $\mathbb{R}^d_+$.

7.2.1. *Assumptions on* A *and* B_j.

We first introduce a parameter–ellipticity condition in the sense of [40, Definition 5.1]. Here $A_\sharp(t,x,\xi) = \sum_{|\alpha|=2m} a_\alpha(t,x)\xi^\alpha$ denotes the *principal symbol* of the operator A.

(E)$_\theta$ Let $\theta \in (0,\pi)$. For all $t \in \mathbb{R}$, $x \in \mathbb{R}^d_+$ it holds that

$$\sigma(A_\sharp(t,x,\xi)) \subset \Sigma_\theta, \quad \forall\, \xi \in \mathbb{R}^n,\ |\xi| = 1,$$

for the spectrum of the operator $A_\sharp(t,x,\xi)$, where $\Sigma_\theta = \{z \in \mathbb{C}\backslash\{0\} : \ |\arg(z)| < \theta\}$ and $\arg : \mathbb{C}\backslash\{0\} \to (-\pi,\pi]$.

The following (LS)$_\theta$-condition is a condition of Lopatinskii–Shapiro type. Before stating it, we need to introduce some notation.

Denote by

$$A_\sharp(t,x,D) := \sum_{|\alpha|=2m} a_\alpha(t,x)D^\alpha \quad \text{and} \quad B_{j,\sharp}(D) := \sum_{|\beta|=m_j} b_{j\beta}D^\beta$$

the *principal part* of $A(t,x)$ and B_j respectively. Let $t_0 \in \mathbb{R}$ and x_0 be in a neighborhood of $\partial\mathbb{R}^{d+1}_+$ of width $2R_0$, i.e., $x_0 \in B_{2R_0}(x') \cap \mathbb{R}^d_+$ for some $x' \in \partial\mathbb{R}^d_+$, and consider the operator $A_\sharp(t_0,x_0,D)$. Taking the Fourier transform $\mathscr{F}_{x'}$ with respect to $x' \in \mathbb{R}^{d-1}$ and letting $v(x_1,\xi) := \mathscr{F}_{x'}(u(x_1,\cdot))(\xi)$, we obtain

$$\begin{aligned} A_\sharp(t_0,x_0,\xi,D_{x_1})v &:= \mathscr{F}_{x'}(A_\sharp(t_0,x_0,D)u(x_1,\cdot))(\xi) \\ &= \sum_{k=0}^{2m}\sum_{|\beta|=k} a_{(\beta,k)}(t_0,x_0)\xi^\beta D^{2m-k}_{x_1} v \end{aligned}$$

and

$$B_{j,\sharp}(\xi,D_{x_1})v := \mathscr{F}_{x'}(B_{j,\sharp}(D)u(x_1,\cdot))(\xi) = \sum_{k=0}^{m_j}\sum_{|\gamma|=k} b_{(\gamma,k)j}\xi^\gamma D^{m_j-k}_{x_1} v.$$

where we denote $D_{x_1} = -i\frac{\partial}{\partial x_1}$.

(LS)$_\theta$ Let $\theta \in (0,\pi)$. For each $(h_1,\dots,h_m)^T \in \mathbb{R}^m$ and each $\xi \in \mathbb{R}^{d-1}$ and $\lambda \in \overline{\Sigma}_{\pi-\theta}$, such that $|\xi| + |\lambda| \neq 0$, the ODE problem in $\mathbb{R}_+$

$$\begin{cases} \lambda v + A_\sharp(t_0,x_0,\xi,D_{x_1})v = 0, & x_1 > 0, \\ B_{j,\sharp}(\xi,D_{x_1})v\Big|_{x_1=0} = h_j, & j = 1,\dots,m \end{cases} \tag{7.2.3}$$

admits a unique solution $v \in C^\infty(\mathbb{R}_+)$ such that $\lim_{x\to\infty} v(x) = 0$.

We now introduce a regularity condition on the leading coefficients, where ρ is a parameter to be specified.

Assumption 7.2.1 (ρ). *There exist a constant* $R_0 \in (0,1]$ *such that* $(a_\alpha)^\sharp_{R_0} \le \rho$.

Throughout the chapter, we impose the following assumptions on the coefficients of A and B_j.

(A) The coefficients a_α are functions $\mathbb{R} \times \mathbb{R}^d_+ \to \mathbb{C}$ and satisfy Assumption 7.2.1 (ρ) with a parameter $\rho \in (0,1)$ to be determined later. Moreover there exists a constant $K > 0$ such that $\|a_\alpha\| \le K$, $|\alpha| \le 2m$, and there exists $\theta \in (0, \frac{\pi}{2})$ such that A satisfies condition $(\mathrm{E})_\theta$.

(B) For each $j = 1, \ldots, m$, the coefficients $b_{j\beta}$ are such that

$$\begin{cases} b_{j\beta} \in \mathbb{C} & \text{if } |\beta| = m_j, \\ b_{j\beta} : \mathbb{R} \times \mathbb{R}^d_+ \to \mathbb{C} & \text{if } |\beta| < m_j, \end{cases}$$

and for $|\beta| < m_j$, $b_{j\beta} \in C^{1-\frac{m_j}{2m}, 2m-m_j}(\mathbb{R}^{d+1}_+)$ and there exists $K > 0$ such that

$$\|b_{j\beta}\|_{C^{1-\frac{m_j}{2m}, 2m-m_j}} \le K.$$

Remark 7.2.2. The $(\mathrm{LS})_\theta$-condition is essentially of algebraic nature, as it can be reformulated as a condition on the roots of a homogeneous polynomial. For further details, we refer the reader to [155] and [131]. It is not difficult to verify this condition in applications. For instance, see [42, Section 3] or [117, Section 5.2].

Example 7.2.3. (i) Assume A has order $2m$ and $B_j = D^{j-1}_{x_1}$, $j = 1, \ldots, m$. Then, the Dirichlet boundary condition $B_j u|_{x_1=0} = g_j$ on $\partial\mathbb{R}^d_+$ satisfies the $(\mathrm{LS})_\theta$-condition. We refer the reader to [6, Section I.2] for the proof. We remark that the complementing condition in [6] is equivalent to the $(\mathrm{LS})_\theta$-condition.

(ii) Let $A = \sum_{|\alpha|=2} a_\alpha D^\alpha$, with $a_\alpha \in \mathbb{C}$ and let $B = \sum_{|\beta|=1} b_\beta D^\beta$ with $0 \ne b_{(1,0,\ldots,0)} \in \mathbb{C}$. Then the $(\mathrm{LS})_\theta$-condition is equivalent to the algebraic condition that for each $\xi \in \mathbb{R}^{d-1}$ and $\lambda \in \overline{\Sigma}_{\pi-\theta}$ such that $|\xi| + |\lambda| \ne 0$, the characteristic polynomial

$$a_0(\xi)\mu^2 + a_1(\xi)\mu + a_0(\xi) + \lambda = 0$$

of (7.2.3), has two distinct roots $\mu_\pm$ with $\mathrm{Im}\mu_+ > 0 > \mathrm{Im}\mu_-$, where $a_k(\xi) = \sum_{|\alpha|=k} a_{(k,\alpha)} \xi^\alpha$. The proof follows the line of [104, Section 7.4].

We can now state our main result.

Theorem 7.2.4. *Let* $T \in (-\infty, \infty]$, $p, q \in (1, \infty)$. *Let* $v \in A_p((-\infty, T))$ *and* $w \in A_q(\mathbb{R}^d_+)$. *There exists*

$$\rho = \rho(\theta, m, d, K, p, q, [v]_p, [w]_q, b_{j\beta}) \in (0,1)$$

such that under the assumptions (A), (B), and $(\mathrm{LS})_\theta$ *for some* $\theta \in (0, \pi/2)$, *the following hold.*

(i) Assume the lower-order terms of B_j to be all zero and $g_j \equiv 0$, with $j = 1, \ldots, m$. Then there exists $\lambda_0 = \lambda_0(\theta, m, d, K, p, q, R_0, [v]_p, [w]_q, b_{j\beta}) \geq 0$ such that for every $\lambda \geq \lambda_0$, for

$$u \in W_p^1((-\infty, T), v; L_q(\mathbb{R}_+^d, w)) \cap L_p((-\infty, T), v; W_q^{2m}(\mathbb{R}_+^d, w))$$

satisfying (7.2.1) on $(-\infty, T) \times \mathbb{R}_+^d$, where $f \in L_p((-\infty, T), v; L_q(\mathbb{R}_+^d, w))$, it holds that

$$\begin{aligned} \|u_t\|_{L_p((-\infty,T),v;L_q(\mathbb{R}_+^d,w))} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p((-\infty,T),v;L_q(\mathbb{R}_+^d,w))} \\ \leq C\|f\|_{L_p((-\infty,T),v;L_q(\mathbb{R}_+^d,w))}, \end{aligned} \tag{7.2.4}$$

with a constant $C = C(\theta, m, d, K, p, q, [v]_p, [w]_q, b_{j\beta}) > 0$.

(ii) Let $v = w = 1$. Then there exists $\lambda_0 = \lambda_0(\theta, m, d, K, p, q, R_0, b_{j\beta}) \geq 0$ such that for every $\lambda \geq \lambda_0$, for

$$u \in W_p^1((-\infty, T); L_q(\mathbb{R}_+^d)) \cap L_p((-\infty, T); W_q^{2m}(\mathbb{R}_+^d))$$

satisfying (7.2.1) on $(-\infty, T)$, where $f \in L_p((-\infty, T); L_q(\mathbb{R}_+^d))$ and

$$g_j \in F_{p,q}^{k_j}((-\infty, T); L_q(\mathbb{R}^{d-1})) \cap L_p((-\infty, T); \mathscr{B}_{q,q}^{2mk_j}(\mathbb{R}^{d-1}))$$

with $k_j = 1 - m_j/(2mq) - 1/(2mq)$, it holds that

$$\begin{aligned} \|u_t\|_{L_p((-\infty,T);L_q(\mathbb{R}_+^d))} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p((-\infty,T);L_q(\mathbb{R}_+^d))} \\ \leq C\|f\|_{L_p((-\infty,T);L_q(\mathbb{R}_+^d))} + C\|g_j\|_{F_{p,q}^{k_j}((-\infty,T);L_q(\mathbb{R}^{d-1}))\cap L_p((-\infty,T);\mathscr{B}_{q,q}^{2mk_j}(\mathbb{R}^{d-1}))}, \end{aligned} \tag{7.2.5}$$

with a constant $C = C(\theta, m, d, K, p, q, b_{j\beta}) > 0$.

From the a priori estimates for the parabolic equation in Theorem 7.2.4, we obtain the a priori estimates for the higher-order elliptic equation as well, by using the arguments in [52, Theorem 5.5] and [100, Theorem 2.6]. The key idea is that the solutions to elliptic equations can be viewed as steady state solutions to the corresponding parabolic cases.

We state below the elliptic version of Theorem 7.2.4. Here, the coefficients of A and B_j are now independent of t.

Theorem 7.2.5. *Let $q \in (1, \infty)$ and $w \in A_q(\mathbb{R}_+^d)$. There exists*

$$\rho = \rho(\theta, m, d, K, q, [w]_q) \in (0, 1)$$

such that under assumptions (A), (B), and $(LS)_\theta$ for some $\theta \in (0, \pi/2)$, the following hold.

(i) Assume the lower-order terms of B_j to be all zero and consider homogeneous boundary conditions. Then, there exists $\lambda_0 = \lambda_0(\theta, m, d, K, q, R_0, [v]_q, b_{j\beta}) \geq 0$ such that for $u \in W_q^{2m}(\mathbb{R}_+^d; w)$ satisfying (7.2.2) where $f \in L_q(\mathbb{R}_+^d, w)$, it holds that

$$\sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_q(\mathbb{R}_+^d,w)} \leq C\|f\|_{L_q(\mathbb{R}_+^d,w)}, \tag{7.2.6}$$

with a constant $C = C(\theta, m, d, K, q, [w]_q, b_{j\beta}) > 0$.

(ii) Let $w = 1$. *Then there exists* $\lambda_0 = \lambda_0(\theta, m, d, K, q, R_0, b_{j\beta}) \geq 0$ *such that for every* $\lambda \geq \lambda_0$, *for* $u \in W_q^{2m}(\mathbb{R}_+^d)$ *satisfying*

$$\begin{cases} (A+\lambda)u = f & \text{in } \mathbb{R}_+^d \\ B_j u\Big|_{x_1=0} = g_j & \text{on } \mathbb{R}^{d-1}, \end{cases}$$

where $f \in L_q(\mathbb{R}_+^d)$ *and* $g_j \in \mathscr{B}_{q,q}^{2mk_j}(\mathbb{R}^{d-1})$, *with* $k_j = 1 - m_j/(2m) - 1/(2mq)$, *it holds that*

$$\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_q(\mathbb{R}_+^d)} \leq C\|f\|_{L_q(\mathbb{R}_+^d)} + C\|g_j\|_{\mathscr{B}_{q,q}^{2mk_j}(\mathbb{R}^{d-1})}, \tag{7.2.7}$$

with a constant $C = C(\theta, m, d, K, q, b_{j\beta}) > 0$.

Remark 7.2.6. (i) In Theorems 7.2.4 and 7.2.5 we focus only on the a priori estimates. The solvability of the corresponding equations will be derived in Section 7.5.

(ii) For notational simplicity, in this chapter we focus only on the scalar case. However, similar to [40], with the same proofs both Theorems 7.2.4 and 7.2.5 hold if one considers systems of operators, i.e., the coefficients a_α and $b_{j\beta}$ are $N \times N$ complex matrix-valued functions.

7.3. MEAN OSCILLATION ESTIMATES FOR u_t AND $D^\alpha u$, $0 \leq |\alpha| \leq 2m$, EXCEPT $D_1^{2m} u$

The main result of this section is stated in Lemma 7.3.5, and it shows mean oscillation estimates for u_t and $D^\alpha u$, for all $0 \leq |\alpha| \leq 2m$ except $D_{x_1}^{2m} u$. The proof of this lemma is the main novelty of the chapter, and it generalizes some results in [52] to general boundary conditions.

For a function f defined on $\mathscr{D} \subset \mathbb{R}_+^{d+1}$, we set

$$[f]_{C^{\frac{\nu}{2m},\nu}(\mathscr{D})} = \sup_{\substack{(t,x),(s,y)\in\mathscr{D} \\ (t,x)\neq(s,y)}} \frac{|f(t,x) - f(s,y)|}{|t-s|^{\frac{\nu}{2m}} + |x-y|^\nu}.$$

Throughout the section, we assume that A and B_j consist only of their principal part. Let

$$A_0 = \sum_{|\alpha|=2m} \bar{a}_\alpha D^\alpha$$

be an operator with constant coefficients satisfying $|\overline{a}_\alpha| \leq K$ for a constant $K > 0$ and satisfying condition $(\mathrm{E})_\theta$ with $\theta \in (0, \pi/2)$.

We first prove an auxiliary estimate, which is derived from a result in [41].

Lemma 7.3.1. *Let* $T \in (-\infty, +\infty]$ *and* $p, q \in (1, \infty)$. *Let* A_0 *and* B_j *be as above. Assume that for some* $\theta \in (0, \pi/2)$ *the* (LS)$_\theta$*-condition is satisfied. Then for every* $f \in L_{p,q}((-\infty, T) \times \mathbb{R}_+^d)$ *and*

$$g_j \in F_{p,q}^{k_j}((-\infty, T); L_q(\mathbb{R}^{d-1})) \cap L_p((-\infty, T); \mathscr{B}_{q,q}^{2mk_j}(\mathbb{R}^{d-1}))$$

with $j \in \{1,\dots,m\}$, $m_j \in \{0,\dots,2m-1\}$, $k_j = 1-m_j/(2m)-1/(2mq)$ *and* $u \in W^{1,2m}_{p,q}((-\infty,T)\times\mathbb{R}^d_+)$ *satisfying*

$$\begin{cases} u_t(t,x)+(\lambda+A_0)u(t,x)=f(t,x) & \text{in } (-\infty,T)\times\mathbb{R}^d_+ \\ B_j u(t,x)\Big|_{x_1=0}=g_j(t,x) & \text{on } (-\infty,T)\times\mathbb{R}^{d-1}, \end{cases} \tag{7.3.1}$$

with $\lambda \ge 0$, *we have*

$$\begin{aligned} &\|u_t\|_{L_{p,q}((-\infty,T)\times\mathbb{R}^d_+)}+\sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u\|_{L_{p,q}((-\infty,T)\times\mathbb{R}^d_+)} \\ &\quad\le C\|f\|_{L_{p,q}((-\infty,T)\times\mathbb{R}^d_+)}+\sum_{j=1}^m\|g_j\|_{F^{k_j}_{p,q}((-\infty,T);L_q(\mathbb{R}^{d-1}))\cap L_p((-\infty,T);\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))}, \end{aligned} \tag{7.3.2}$$

with $C=C(\theta,m,d,K,p,q,b_{j\beta})>0$. *Moreover, for any* $\lambda>0$, $f\in L_{p,q}((-\infty,T)\times\mathbb{R}^d_+)$ *and*

$$g_j \in F^{k_j}_{p,q}((-\infty,T);L_q(\mathbb{R}^{d-1}))\cap L_p((-\infty,T);\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))$$

with $j\in\{1,\dots,m\}$, $m_j\in\{0,\dots,2m-1\}$, $k_j=1-m_j/(2m)-1/(2mq)$, *there exists a unique solution* $u\in W^{1,2m}_{p,q}((-\infty,T)\times\mathbb{R}^d_+)$ *to* (7.3.1).

Proof. We divide the proof into several steps. First we assume that $T=\infty$.

Step 1. Let $u\in W^{1,2m}_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)$ be a solution to

$$\begin{cases} u_t(t,x)+(\lambda+A_0)u(t,x)=f(t,x) & \text{in } \mathbb{R}_+\times\mathbb{R}^d_+ \\ B_j u(t,x)\Big|_{x_1=0}=g_j(t,x) & \text{on } \mathbb{R}_+\times\mathbb{R}^{d-1},\quad j=1,\dots,m \\ u(0,x)=0 & \text{on } \mathbb{R}^d_+, \end{cases} \tag{7.3.3}$$

with $\lambda>0$. By applying [41, Proposition 6.4] to (7.3.3), it holds that

$$\begin{aligned} &\|u_t\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)}+\|D^{2m}u\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)} \\ &\le C\|f\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)}+C\sum_{j=1}^m\|g_j\|_{F^{k_j}_{p,q}(\mathbb{R}_+;L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R}_+;\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))}, \end{aligned} \tag{7.3.4}$$

with $C=C(\lambda,\theta,m,d,K,p,q,b_{j\beta})$. We remark that although the estimate is not explicitly stated in this reference, it can be extracted from the proofs there. We want to show that the estimate (7.3.4) also holds when $\lambda=0$.

For this, observe that in [41, Proposition 6.4], the coefficients of the operators under consideration are time and space dependent. In our case, since A_0 has constant coefficients, using a scaling $t\to\lambda^{-1}t$, $x\to\lambda^{-1/2m}x$, we obtain that the estimate (7.3.4) holds for any $\lambda\in(0,1)$ and with constant C uniform in λ. In fact, for a general $\lambda\in(0,1)$, let $v(t,x):=u(\lambda^{-1}t,\lambda^{-1/2m}x)$. Then v satisfies

$$\begin{cases} v_t(t,x)+A_0v(t,x)+v(t,x)=\tilde f(t,x) & \text{in } \mathbb{R}_+\times\mathbb{R}^d_+ \\ B_j v(t,x)\Big|_{x_1=0}=\tilde g_j(t,x) & \text{on } \mathbb{R}_+\times\mathbb{R}^{d-1} \\ v(0,x)=0 & \text{on } \mathbb{R}^d_+. \end{cases} \tag{7.3.5}$$

where

$$\tilde{f}(t,x) = \lambda^{-1} f(\lambda^{-1} t, \lambda^{-1/2m} x)$$

and

$$\tilde{g}_j(t,x) = \lambda^{-m_j/2m} g_j(\lambda^{-1} t, \lambda^{-1/2m} x).$$

Applying (7.3.4) with $\lambda = 1$ to (7.3.5) we get that

$$\begin{aligned}&\|v_t\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)} + \|D^{2m} v\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)}\\ &\le C\|\tilde{f}\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)} + C\sum_{j=1}^m \|\tilde{g}_j\|_{F^{k_j}_{p,q}(\mathbb{R}_+;L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R}_+;\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))},\end{aligned} \tag{7.3.6}$$

with $C = C(\theta, m, d, K, p, q, b_{j\beta})$. Now, scaling back and using the definition of the Besov space and Triebel–Lizorkin space, it is easily seen

$$\begin{aligned}&\|u_t\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)} + \|D^{2m} u\|_{L_{p,q}((0,\infty)\times\mathbb{R}^d_+)}\\ &\le C\|f\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)} + C\sum_{j=1}^m \|g_j\|_{F^{k_j}_{p,q}(\mathbb{R}_+;L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R}_+;\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))},\end{aligned} \tag{7.3.7}$$

where C is independent of $\lambda \in (0,1)$. Sending $\lambda \to 0$, we obtain that the estimate (7.3.4) holds when $\lambda = 0$. Finally, by applying a procedure of S. Agmon as in [100, Theorem 4.1], from (7.3.4) with $\lambda = 0$ it follows that when $\lambda > 0$,

$$\begin{aligned}&\|u_t\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)}\\ &\le C\|f\|_{L_{p,q}(\mathbb{R}_+\times\mathbb{R}^d_+)} + C\sum_{j=1}^m \|g_j\|_{F^{k_j}_{p,q}(\mathbb{R}_+;L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R}_+;\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))},\end{aligned} \tag{7.3.8}$$

with constant $C = C(\theta, m, d, K, p, q, b_{j\beta})$.

Step 2. Take $\eta \in C^\infty(\mathbb{R})$ such that $\eta = 1$ for $t > 1$ and $\eta = 0$ for $t < 0$. Define $u_n = \eta(t+n)u$. From (7.3.1), we see that u_n satisfies

$$\begin{cases} (u_n)_t(t,x) + (\lambda + A_0)u_n(t,x) = f_n(t,x) & \text{in } (-n,\infty)\times\mathbb{R}^d_+ \\ B_j u_n(t,x)\Big|_{x_1=0} = g_{n,j}(t,x) & \text{on } (-n,\infty)\times\mathbb{R}^{d-1} \\ u_n(-n,x) = 0 & \text{on } \mathbb{R}^d_+, \end{cases} \tag{7.3.9}$$

for $j = 1,\dots,m$, where $\lambda > 0$ and

$$f_n = \eta(t+n) f + u\eta_t(t+n) \quad \text{and} \quad g_{n,j}(t,x) = \eta(t+n) g_j(t,x).$$

By applying (7.3.8) to (7.3.9), we get that

$$\|(u_n)_t\|_{L_{p,q}((-n,\infty)\times\mathbb{R}^d_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_n\|_{L_{p,q}((-n,\infty)\times\mathbb{R}^d_+)}$$

$$\begin{aligned}&\le C\|f_n\|_{L_{p,q}((-n,\infty)\times\mathbb{R}^d_+)}\\&\quad+C\sum_{j=1}^m\|g_{n,j}\|_{F^{k_j}_{p,q}((-n,\infty);L_q(\mathbb{R}^{d-1}))\cap L_p((-n,\infty);\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))},\end{aligned}\tag{7.3.10}$$

with $C=C(\theta,m,d,K,p,q,b_{j\beta})$. Now, taking the limit as $n\to\infty$ yields (7.3.2), i.e., for any $\lambda>0$,

$$\begin{aligned}&\|u_t\|_{L_{p,q}(\mathbb{R}\times\mathbb{R}^d_+)}+\sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u\|_{L_{p,q}(\mathbb{R}\times\mathbb{R}^d_+)}\\&\le C\|f\|_{L_{p,q}(\mathbb{R}\times\mathbb{R}^d_+)}+C\sum_{j=1}^m\|g_j\|_{F^{k_j}_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R};\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))},\end{aligned}$$

with $C=C(\theta,m,d,K,p,q,b_{j\beta})$.

Step 3. For the solvability, let $f\in L_{p,q}(\mathbb{R}^{d+1}_+)$ and

$$g_j\in F^{k_j}_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R};\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1})),\quad j=1\dots,m.$$

For integer $n>0$, define

$$f_n=\eta(t+n)f\quad\text{and}\quad g_{n,j}=\eta(t+n)g_j$$

so that $f_n\to f$ in $L_{p,q}(\mathbb{R}^{d+1}_+)$ and

$$g_{n,j}\to g_j\quad\text{in } F^{k_j}_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R};\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1})).$$

Now let $u_n\in W^{1,2m}_{p,q}((-n,\infty)\times\mathbb{R}^d_+)$ be the solution to the initial-boundary value problem with f_n and $g_{n,j}$ and zero initial value at $t=-n$, the existence of which is guaranteed by [41, Proposition 6.4]. We extend u_n to be zero for $t<-n$. It is easily seen that u_n satisfies (7.3.1) with f_n and $g_{n,j}$ in place of f and g_j, respectively. Applying the a priori estimate obtained in the argument above to u_m-u_n, we get that $\{u_n\}$ is a Cauchy sequence. Then the limit $u\in W^{1,2m}_{p,q}(\mathbb{R}^{d+1}_+)$ is a solution to (7.3.1).

Step 4. For general $T<\infty$, we may assume $T=0$ by shifting the t-coordinate. We first take the even extensions of u with respect to $t=0$. Then $u\in W^{1,2m}_{p,q}(\mathbb{R}\times\mathbb{R}^d_+)$. Next we take the even extension of f and g_j with respect to $t=0$. Let $v\in W^{1,2m}_{p,q}(\mathbb{R}\times\mathbb{R}^d_+)$ be the solution to

$$\begin{cases}v_t(t,x)+(\lambda+A_0)v(t,x)=f(t,x) & \text{in } \mathbb{R}^{d+1}_+\\ B_jv(t,x)\Big|_{x_1=0}=g_j(t,x) & \text{on } \mathbb{R}\times\mathbb{R}^{d-1},\ j=1,\dots,m,\end{cases}$$

the existence of which is guaranteed by the argument above. Observe that $w:=u-v\in W^{1,2m}_{p,q}(\mathbb{R}\times\mathbb{R}^d_+)$ satisfies

$$\begin{cases}w_t(t,x)+(\lambda+A_0)w(t,x)=0 & \text{in } (-\infty,0)\times\mathbb{R}^d_+\\ B_jw(t,x)\Big|_{x_1=0}=0 & \text{on } (-\infty,0)\times\mathbb{R}^{d-1},\ j=1,\dots,m.\end{cases}$$

We claim that $w=0$ on $t<0$. Indeed, for any $T_1<0$, we solve the equation of w in $(T_1,\infty)\times\mathbb{R}^d_+$ with the zero initial data to get w_1, and extend $w_1=0$ for $t<T_1$. It is easily seen that the extended function w_1 satisfies the same equation of w in $\mathbb{R}\times\mathbb{R}^d_+$. By the uniqueness of the solution, $w=w_1$. Therefore, $w=0$ when $t<T_1$ for any $T_1<0$. Then,

$$
\begin{aligned}
&\|u_t\|_{L_{p,q}((-\infty,0)\times\mathbb{R}^d_+)}+\sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u\|_{L_{p,q}((-\infty,0)\times\mathbb{R}^d_+)}\\
&=\|v_t\|_{L_{p,q}((-\infty,0)\times\mathbb{R}^d_+)}+\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha v\|_{L_{p,q}((-\infty,0)\times\mathbb{R}^d_+)}\\
&\le C\|f\|_{L_{p,q}(\mathbb{R}\times\mathbb{R}^d_+)}+\sum_{j=1}^m\|g_j\|_{F^{k_j}_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R};\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))}\\
&=C\|f\|_{L_{p,q}((-\infty,0)\times\mathbb{R}^d_+)}+\sum_{j=1}^m\|g_j\|_{F^{k_j}_{p,q}((-\infty,0);L_q(\mathbb{R}^{d-1}))\cap L_p((-\infty,T);\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))}.
\end{aligned}
$$

The solvability is obtained by taking the even extension of g_j and f, and then solve the equation in $\mathbb{R}\times\mathbb{R}^d_+$. The uniqueness follows from the a priori estimate. □

Remark 7.3.2. In Lemma 7.3.1 as well as Theorem 7.2.4, we assumed $\theta\in(0,\pi/2)$. However, in [41, 117], it is shown that in the case of operators with constant leading coefficients, or operators with uniformly continuous leading coefficients in a bounded domain, it is enough that the conditions $(\mathrm{E})_\theta$ and $(\mathrm{LS})_\theta$ are satisfied for $\theta=\pi/2$, which are slightly weaker. The condition $(\mathrm{E})_{\pi/2}$ is also referred to as normal ellipticity condition.

From Lemma 7.3.1, we obtain the following Hölder estimate.

Lemma 7.3.3. *Let $0<r_1<r_2<\infty$. Let $v\in W^{1,2m}_p(Q^+_{r_2})$ be a solution to the homogeneous problem*

$$
\begin{cases}
v_t+A_0v=0 & \text{in } Q^+_{r_2}\\
B_jv\Big|_{x_1=0}=0 & \text{on } Q_{r_2}\cap\{x_1=0\},\ j=1,\dots,m.
\end{cases}
\tag{7.3.11}
$$

Assume that for some $\theta\in(0,\pi/2)$ the $(LS)_\theta$-condition is satisfied. Then there exists a constant $C=C(\theta,K,p,d,m,r_1,r_2,b_{j\beta})>0$ such that

$$
\|v_t\|_{L_p(Q^+_{r_1})}+\|D^{2m}v\|_{L_p(Q^+_{r_1})}\le C\|v\|_{L_p(Q^+_{r_2})}.
\tag{7.3.12}
$$

Furthermore, for $\nu=1-\frac{1}{p}$,

$$
[v_t]_{C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})}+[D^{2m-1}D_{x'}v]_{C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})}\le C\|v_t\|_{L_p(Q^+_{r_2})}+C\|D^{2m}v\|_{L_p(Q^+_{r_2})},
\tag{7.3.13}
$$

with $C=C(\theta,K,p,d,m,r_1,r_2,b_{j\beta})>0$.

Proof. Set $R_0 = r_1$ and $R_i = r_1 + (r_2 - r_1)(1 - 2^{-i})$, for $i = 1, 2, \ldots$. For each $i = 0, 1, 2, \ldots$, take $\eta_i \in C_0^\infty(\overline{\mathbb{R}^{d+1}_+})$ satisfying

$$\begin{cases} \eta_i = 1 & \text{in } Q^+_{R_i} \\ \eta_i = 0 & \text{outside } (-R_i^{2m}, R_i^{2m}) \times B_{R_{i+1}} \end{cases}$$

and

$$|D^k \eta_i| \le C 2^{ki} (r_2 - r_1)^{-k}, \quad |(\eta_i)_t| \le C 2^{2mi} (r_2 - r_1)^{-2m} \tag{7.3.14}$$

where $k = 0, 1, \ldots, 2m$. It is easily seen that $v\eta_i \in W_p^{1,2m}(\mathbb{R}^{d+1}_+)$ satisfies

$$\begin{cases} (v\eta_i)_t + A_0(v\eta_i) = f & \text{in } \mathbb{R}^{d+1}_+ \\ B_j(v\eta_i)\Big|_{x_1=0} = \operatorname{tr}_{x_1=0} G_j & \text{on } \partial\mathbb{R}^{d+1}_+, \ j = 1, \ldots, m \\ (v\eta_i)(-r_2^{2m}, \cdot) = 0, \end{cases} \tag{7.3.15}$$

where

$$f = v(\eta_i)_t + \sum_{|\alpha|=2m} \sum_{|\gamma| \le 2m-1} \binom{\alpha}{\gamma} a_\alpha(t_0, x_0) D^\gamma v D^{\alpha-\gamma} \eta_i$$

and

$$G_j = \sum_{|\beta|=m_j} \sum_{|\tau| \le m_j - 1} \binom{\beta}{\tau} b_{j\beta} D^\tau v D^{\beta-\tau} \eta_i, \quad j = 1, \ldots, m.$$

Thus we extended (7.3.11) to a system on $\mathbb{R} \times \mathbb{R}^d_+$ without changing the value of v on $Q^+_{r_1}$. Now let

$$g_j = \operatorname{tr}_{x_1=0} G_j \in W_p^{1-\frac{m_j}{2m}-\frac{1}{2mp}, 2m-m_j-\frac{1}{p}}(\mathbb{R} \times \mathbb{R}^{d-1}).$$

By applying Lemma 7.3.1 with $p = q$, we get

$$\begin{aligned} &\|(v\eta_i)_t\|_{L_p(\mathbb{R}^{d+1}_+)} + \|D^{2m}(v\eta_i)\|_{L_p(\mathbb{R}^{d+1}_+)} \\ &\qquad \le C\|f\|_{L_p(\mathbb{R}^{d+1}_+)} + C\sum_{j=1}^m \|g_j\|_{W_p^{1-\frac{m_j}{2m}-\frac{1}{2mp}, 2m-m_j-\frac{1}{p}}(\mathbb{R}\times\mathbb{R}^{d-1})}, \end{aligned}$$

where $C = C(\theta, K, d, m, p, b_{j\beta})$.

By Theorem 7.1.3 with $s = 1 - \frac{m_j}{2m} \in (0, 1]$, $m_j \in \{0, \ldots, 2m-1\}$, we have

$$\|g_j\|_{W_p^{1-\frac{m_j}{2m}-\frac{1}{2mp}, 2m-m_j-\frac{1}{p}}(\mathbb{R}\times\mathbb{R}^{d-1})} \le C\|G_j\|_{W_p^{1-\frac{m_j}{2m}, 2m-m_j}(\mathbb{R}^{d+1}_+)}.$$

Observe that

$$\|f\|_{L_p(\mathbb{R}^{d+1}_+)} \le C\|(\eta_i)_t v\|_{L_p(\mathbb{R}^{d+1}_+)} + C \sum_{|\alpha|=2m} \sum_{|\gamma|\le 2m-1} \|D^\gamma v D^{\alpha-\gamma}\eta_i\|_{L_p(\mathbb{R}^{d+1}_+)}$$

and

$$\|G_j\|_{W_p^{1-\frac{m_j}{2m},2m-m_j}(\mathbb{R}^{d+1}_+)} \le C \sum_{|\beta|=m_j} \sum_{|\tau|\le m_j-1} \|D^\tau v D^{\beta-\tau}\eta_i\|_{W_p^{1-\frac{m_j}{2m},2m-m_j}(\mathbb{R}^{d+1}_+)},$$

where the constant $C = C(\theta, K, p, d, m)$ may vary from line to line. By (7.3.14), it holds that

$$\|(\eta_i)_t v\|_{L_p(\mathbb{R}^{d+1}_+)} \le C2^{2mi}(r_2-r_1)^{-2m}\|v\|_{L_p(Q^+_{r_2})}.$$

By (7.3.14) and interpolation inequalities (see e.g. [102] and the proof of [50, Lemma 3.2]), for $\varepsilon > 0$ small enough and $|\gamma| \le 2m-1$ we get

$$\begin{aligned}
&\|D^\gamma v D^{\alpha-\gamma}\eta_i\|_{L_p(\mathbb{R}^{d+1}_+)} \le \|D^\gamma(v\eta_{i+1})D^{\alpha-\gamma}\eta_i\|_{L_p(\mathbb{R}^{d+1}_+)}\\
&\le C2^{(2m-|\gamma|)i}(r_2-r_1)^{-(2m-|\gamma|)}\|D^\gamma(v\eta_{i+1})\|_{L_p(\mathbb{R}^{d+1}_+)}\\
&\le \varepsilon\|D^{2m}(v\eta_{i+1})\|_{L_p(\mathbb{R}^{d+1}_+)} + C_\varepsilon 2^{2mi}(r_2-r_1)^{-2m}\|v\|_{L_p(Q^+_{r_2})},
\end{aligned}$$

where $C_\varepsilon = C\varepsilon^{\frac{|\gamma|}{|\gamma|-2m}}$. Moreover, by Lemma 7.1.4 and (7.3.14), for $\varepsilon > 0$ small enough and $|\tau| \le m_j - 1$ we get

$$\begin{aligned}
&\|D^\tau v D^{\beta-\tau}\eta_i\|_{W_p^{1-\frac{m_j}{2m},2m-m_j}(\mathbb{R}^{d+1}_+)}\\
&\le C2^{(m_j-|\tau|)i}(r_2-r_1)^{-(m_j-|\tau|)}\|D^\tau(v\eta_{i+1})\|_{W_p^{1-\frac{m_j}{2m},2m-m_j}(\mathbb{R}^{d+1}_+)}\\
&\le \varepsilon\|D^{2m}(v\eta_{i+1})\|_{L_p(\mathbb{R}^{d+1}_+)} + \varepsilon\|(v\eta_{i+1})_t\|_{L_p(\mathbb{R}^{d+1}_+)} + C_\varepsilon 2^{2mi}(r_2-r_1)^{-2m}\|v\|_{L_p(Q^+_{r_2})}
\end{aligned}$$

where $C_\varepsilon = C\varepsilon^{\frac{2m+|\tau|-m_j}{|\tau|-m_j}}$.

Combining the above inequalities yields

$$\begin{aligned}
&\|(v\eta_i)_t\|_{L_p(\mathbb{R}^{d+1}_+)} + \|D^{2m}(v\eta_i)\|_{L_p(\mathbb{R}^{d+1}_+)} \le (C+C_\varepsilon)2^{2mi}(r_2-r_1)^{-2m}\|v\|_{L_p(Q^+_{r_2})}\\
&+ C\varepsilon\|D^{2m}(v\eta_{i+1})\|_{L_p(\mathbb{R}^{d+1}_+)} + C\varepsilon\|(v\eta_{i+1})_t\|_{L_p(\mathbb{R}^{d+1}_+)}.
\end{aligned}$$

We multiply both sides by ε^i and we sum with respect to i to get

$$\begin{aligned}
&\sum_{i=0}^\infty \varepsilon^i\Big(\|(v\eta_i)_t\|_{L_p(\mathbb{R}^{d+1}_+)} + \|D^{2m}(v\eta_i)\|_{L_p(\mathbb{R}^{d+1}_+)}\Big)\\
&\le (C+C_\varepsilon)(r_2-r_1)^{-2m}\sum_{i=0}^\infty (2^{2m}\varepsilon)^i\|v\|_{L_p(Q^+_{r_2})}\\
&\quad + C\sum_{i=1}^\infty \varepsilon^i\Big(\|D^{2m}(v\eta_i)\|_{L_p(\mathbb{R}^{d+1}_+)} + \|(v\eta_i)_t\|_{L_p(\mathbb{R}^{d+1}_+)}\Big).
\end{aligned}$$

We choose $\varepsilon = 2^{-2m-1}$ and observe that the above summations are finite. Then, the above estimate gives

$$\|(v\eta_0)_t\|_{L_p(\mathbb{R}^{d+1}_+)} + \|D^{2m}(v\eta_0)\|_{L_p(\mathbb{R}^{d+1}_+)} \le C(r_2-r_1)^{-2m}\|v\|_{L_p(Q^+_{r_2})}. \tag{7.3.16}$$

Since the left-hand side of (7.3.16) is greater than that of (7.3.12), we can conclude

$$\|v_t\|_{L_p(Q^+_{r_1})} + \|D^{2m} v\|_{L_p(Q^+_{r_1})} \le C(r_2 - r_2)^{-2m} \|v\|_{L_p(Q^+_{r_2})},$$

with $C = C(\theta, K, p, d, m, b_{j\beta})$.

To show the Hölder estimate for v, we proceed as follows. First, observe that from (7.3.12) and interpolation inequalities, it holds that

$$\|v\|_{W^{1,2m}_p(Q^+_{r_1})} \le C\|v\|_{L_p(Q^+_{r_2})}. \tag{7.3.17}$$

Observe now that for $k, h > 0$, the derivatives $D^k_t D^h_{x'} v$ satisfy the same equation as v. Hence, from (7.3.17) and a standard bootstrap argument, it holds that $v \in W^{k,2m,h+2m}_{t,x_1,x';p}(Q^+_{r_1})$ with

$$\|v\|_{W^{k,2m,h+2m}_{t,x_1,x';p}(Q^+_{r_1})} \le C\|v\|_{L_p(Q^+_{r_2})}.$$

Observe that Theorem 7.1.7 implies for $\nu = 1 - \frac{1}{p}$,

$$v,\, D^{2m-1} v \in C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})$$

and

$$[v]_{C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})} + [D^{2m-1} v]_{C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})} \le C\|v\|_{W^{k,2m,h+2m}_{t,x_1,x';p}(Q^+_{r_1})} \le C\|v\|_{L_p(Q^+_{r_2})}. \tag{7.3.18}$$

Since v_t satisfies the same equation as v, we have

$$[v_t]_{C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})} \le C\|v_t\|_{L_p(Q^+_{r_2})}. \tag{7.3.19}$$

In order to show (7.3.13), we need to apply the following Poincaré type inequality for solutions to equations satisfying the Lopatinskii–Shapiro condition. Its proof is postponed to the end of this section.

Lemma 7.3.4. *There exists a polynomial P of order $2m-2$ such that $v-P$ satisfies* (7.3.11) *and there exists a constant $C = C(d, m, p, K, b_{j\beta}, r_2) > 0$ such that*

$$\|D^\alpha(v - P)\|_{L_p(Q^+_{r_2})} \le C\|D^{2m-1} v\|_{L_p(Q^+_{r_2})} \tag{7.3.20}$$

for $|\alpha| \in \{0, \dots, 2m-2\}$.

By (7.3.18) and Lemma 7.3.4 there exists a polynomial P of order $2m-2$ such that

$$\begin{aligned}[D^{2m-1} v]_{C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})} &= [D^{2m-1}(v - P)]_{C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})} \\ &\le \|v - P\|_{L_p(Q^+_{r_2})} \le C\|D^{2m-1} v\|_{L_p(Q^+_{r_2})},\end{aligned}$$

from which, since $D_{x'} v$ satisfies the same equation as v, we get that

$$[D^{2m-1} D_{x'} v]_{C^{\frac{\nu}{2m},\nu}(Q^+_{r_1})} \le C\|D^{2m} v\|_{L_p(Q^+_{r_2})}.$$

Together with (7.3.19), the above inequality yields (7.3.13). □

Similar to [51, Corollary 5], from Lemma 7.3.3 we obtain the following mean oscillation estimates for u_t and $D^\alpha u$, for all $0 \le |\alpha| \le 2m$ except $D_{x_1}^{2m} u$.

Lemma 7.3.5. *Let $\kappa \ge 16$ and $p \in (1,\infty)$. Let $f \in L_{p,loc}(\overline{\mathbb{R}^{d+1}_+})$, $X_0 = (t_0, x_0) \in \overline{\mathbb{R}^{d+1}_+}$, and $\lambda \ge 0$. Assume that for $r \in (0,\infty)$, $u \in W^{1,2m}_{p,loc}(\overline{\mathbb{R}^{d+1}_+})$ satisfies $u_t + (A_0 + \lambda)u = f$ in $Q^+_{\kappa r}(X_0)$ and $B_j u|_{x_1=0} = 0$ on $Q_{\kappa r}(X_0) \cap \{x_1 = 0\}$, $j = 1, \ldots, m$. Assume that for some $\theta \in (0, \pi/2)$ the (LS)$_\theta$-condition is satisfied. Then*

$$
\begin{aligned}
&(|u_t - (u_t)_{Q^+_r(X_0)}|)_{Q^+_r(X_0)} + \sum_{\substack{|\alpha| \le 2m \\ \alpha_1 < 2m}} \lambda^{1-\frac{|\alpha|}{2m}} (|D^\alpha u - (D^\alpha u)_{Q^+_r(X_0)}|)_{Q^+_r(X_0)} \\
&\qquad \le C\kappa^{-(1-\frac{1}{p})} \sum_{|\alpha| \le 2m} \lambda^{1-\frac{|\alpha|}{2m}} (|D^\alpha u|^p)^{\frac{1}{p}}_{Q^+_{\kappa r}(X_0)} + C\kappa^{\frac{d+2m}{p}} (|f|^p)^{\frac{1}{p}}_{Q^+_{\kappa r}(X_0)}, \qquad (7.3.21)
\end{aligned}
$$

where the constant $C = C(\theta, d, m, K, p, b_{j\beta}) > 0$.

Proof. Using a scaling argument, it suffices to prove (7.3.21) only for $r = 8/\kappa$. Indeed, assume that the inequality (7.3.21) holds true for $r = 8/\kappa$. For a given $r \in (0,\infty)$, let $r_0 = 8/\kappa$, $R = r/r_0$ and $v(t,x) = u(R^{2m}t, Rx)$. Then v satisfies $B_j v = 0$ on $Q^+_{\kappa r_0}(Z_0) \cap \{x_1 = 0\}$ and

$$
v_t(t,x) + \sum_{|\alpha|=2m} a_\alpha(R^{2m}t_0, Rx_0) D^\alpha v(t,x) + \lambda R^{2m} v(t,x) = R^{2m} f(R^{2m}t, Rx) \qquad (7.3.22)
$$

on $Q^+_{\kappa r_0}(Z_0)$, where $Z_0 = (R^{-2m}t_0, R^{-1}x_0) \in \overline{\mathbb{R}^{d+1}_+}$. Then, by (7.3.21) applied to (7.3.22), we have

$$
\begin{aligned}
&(|v_t - (v_t)_{Q^+_{r_0}(Z_0)}|)_{Q^+_{r_0}(Z_0)} + \sum_{\substack{|\alpha| \le 2m \\ \alpha_1 < 2m}} \lambda^{1-\frac{|\alpha|}{2m}} R^{2m-|\alpha|} (|D^\alpha v - (D^\alpha v)_{Q^+_{r_0}(Z_0)}|)_{Q^+_{r_0}(Z_0)} \\
&\le C\kappa^{-(1-\frac{1}{p})} \sum_{|\alpha| \le 2m} \lambda^{1-\frac{|\alpha|}{2m}} R^{2m-|\alpha|} (|D^\alpha v|^p)^{\frac{1}{p}}_{Q^+_{\kappa r_0}(Z_0)} + C\kappa^{\frac{d+2m}{p}} R^{2m} (|f|^p)^{\frac{1}{p}}_{Q^+_{\kappa r_0}(Z_0)}.
\end{aligned}
$$

Note that

$$
(D^\alpha v)_{Q^+_{r_0}(Z_0)} = R^{|\alpha|} (D^\alpha u)_{Q^+_r(X_0)} \quad \text{and} \quad (v_t)_{Q^+_{r_0}(Z_0)} = R^{2m} (u_t)_{Q^+_r(X_0)},
$$

so the above inequality implies (7.3.21) for arbitrary $r \in (0,\infty)$.

We now assume $r = 8/\kappa$ and consider two cases, where we denote by x_0^1 the first coordinate of x_0.

Case 1: $x_0^1 \ge 1$. In this case, $Q^+_{\kappa r/8}(X_0) = Q_{\kappa r/8}(X_0)$. The proof of (7.3.21) then follows from [52, Lemma 5.7], with $\kappa \ge 2$ instead of $\kappa \ge 8$ there. Note that in this case, the (LS)$_\theta$-condition is not needed.

Case 2: $x_0^1 \in [0,1]$. We denote $Y_0 := (t_0, 0, x_0')$ and we set $Q'_{\kappa r}(Y_0) := (t_0 - (\kappa r)^{2m}, t_0) \times B_{\kappa r}(x_0')$. Observe that

$$
Q^+_r(X_0) \subset Q^+_2(Y_0) \subset Q^+_4(Y_0) \subset Q^+_6(Y_0) \subset Q^+_{\kappa r}(X_0).
$$

To prove (7.3.21), we proceed by three steps.

Step 1. We assume for simplicity $Y_0 = (0,0)$, since a translation in t and x' then gives the result for general Y_0. Decompose $u = v + w$ where:

- $w \in W_p^{1,2m}(\mathbb{R}_+^{d+1})$ is the solution to the inhomogeneous problem

$$\begin{cases} w_t + (A_0 + \lambda)w = f\zeta & \text{in } \mathbb{R} \times \mathbb{R}_+^d \\ B_j w\Big|_{x_1=0} = 0 & \text{on } \partial\mathbb{R}_+^{d+1},\ j = 1,\dots,m \\ w(-6^{2m},\cdot) = 0. \end{cases} \tag{7.3.23}$$

  where $\zeta \in C_0^\infty(\mathbb{R}_+^{d+1})$ satisfies $\zeta = 1$ in $(-4^{2m},0)\times B_4$ and $\zeta = 0$ outside $(-6^{2m},6^{2m})\times B_6$.

- $v \in W_{p,loc}^{1,2m}(\mathbb{R}_+^{d+1})$ is the solution to the homogeneous problem

$$\begin{cases} v_t + (A_0 + \lambda)v = 0 & \text{in } Q_4^+ \\ B_j v\Big|_{x_1=0} = 0 & \text{on } Q_4 \cap \{x_1 = 0\},\ j = 1,\dots,m. \end{cases} \tag{7.3.24}$$

Step 2. It follows directly from Lemma 7.3.1 with $g_j \equiv 0$ that there exists a unique solution $w \in W_p^{1,2m}(\mathbb{R}_+^{d+1})$ of (7.3.23) that satisfies

$$\begin{aligned} \|w_t\|_{L_p(\mathbb{R}_+^{d+1})} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha w\|_{L_p(\mathbb{R}_+^{d+1})} &\le C\|f\zeta\|_{L_p(\mathbb{R}_+^{d+1})} \\ &\le C\|f\|_{L_p(Q_6^+)} \le C\|f\|_{L_p(Q_{\kappa r}^+(X_0))}, \end{aligned} \tag{7.3.25}$$

where $C = C(\theta, K, d, m, p, b_{j\beta})$. In particular,

$$(|w_t|^p)_{Q_r^+}^{1/p} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} (|D^\alpha w|^p)_{Q_r^+}^{1/p} \le C\kappa^{\frac{d+2m}{p}} (|f|^p)_{Q_{\kappa r}^+}^{1/p}. \tag{7.3.26}$$

Step 3. We claim that there exists a constant $C = C(\theta, p, K, d, m, b_{j\beta})$ such that

$$\begin{aligned} (|v_t - (v_t)_{Q_r^+}|)_{Q_r^+(X_0)} + \sum_{\substack{|\alpha|\le 2m \\ \alpha_1 < 2m}} \lambda^{1-\frac{|\alpha|}{2m}} (|D^\alpha v - (D^\alpha v)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} \\ \le C\kappa^{-\nu} \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} (|D^\alpha v|^p)_{Q_{\kappa r}^+(X_0)}^{1/p}. \end{aligned} \tag{7.3.27}$$

To show the claim, we first assume $\lambda = 0$. We apply Lemma 7.3.3 with the choice $r_1 = 2$ and $r_2 = 4$, and we get

$$[v_t]_{C^{\frac{\nu}{2m},\nu}(Q_2^+)} + [D^{2m-1}D_{x'}v]_{C^{\frac{\nu}{2m},\nu}(Q_2^+)} \le C\|v_t\|_{L_p(Q_4^+)} + C\|D^{2m}v\|_{L_p(Q_4^+)}, \tag{7.3.28}$$

where $\nu = 1 - \frac{1}{p}$ and $C = C(\theta, p, K, d, m, b_{j\beta})$.

For $\lambda > 0$ we follow the proof of [51, Lemma 3], based on an idea by S. Agmon. Consider for $y \in \mathbb{R}$,

$$\zeta(y) = \cos(\lambda^{\frac{1}{2m}} y) + \sin(\lambda^{\frac{1}{2m}} y).$$

Note that

$$D_y^{2m}\zeta(y) = \lambda\zeta(y), \quad \zeta(0) = 1, \quad |D^{2m-|\alpha|}\zeta(0)| = \lambda^{1-\frac{|\alpha|}{2m}}.$$

Denote by $(t,z) = (t,x,y) \in \mathbb{R}_+^{d+2}$, where $z = (x,y) \in \mathbb{R}_+^{d+1}$ with $x \in \mathbb{R}_+^d$, and set

$$\tilde{v}(t,z) = v(t,x)\zeta(y), \quad \tilde{Q}_r^+ = (-r^{2m},0) \times \left\{|z| < r, z \in \mathbb{R}_+^{d+1}\right\}.$$

Since v satisfies (7.3.24) on Q_4^+, $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_t + A_0\tilde{v} + D_y^{2m}\tilde{v} = 0 & \text{in } \tilde{Q}_4^+ \\ B_j\tilde{v}|_{x_1=0} = 0 & \text{on } \tilde{Q}_4^+ \cap \{x_1 = 0\}. \end{cases}$$

Thus, we can proceed as in (7.3.28) and get for $r = 8/\kappa$, $\kappa \geq 16$, and $|\alpha| \leq 2m$ with $\alpha_1 < 2m$,

$$[\tilde{v}_t]_{C^{\frac{\nu}{2m},\nu}(\tilde{Q}_2^+)} + [D_y^{2m-|\alpha|}D^\alpha\tilde{v}]_{C^{\frac{\nu}{2m},\nu}(\tilde{Q}_2^+)} \leq C\|\tilde{v}_t\|_{L_p(\tilde{Q}_4^+)} + C\|D^{2m}\tilde{v}\|_{L_p(\tilde{Q}_4^+)}. \tag{7.3.29}$$

Since $|D^{2m-|\alpha|}\zeta(0)| = \lambda^{1-\frac{|\alpha|}{2m}}$,

$$\lambda^{1-\frac{|\alpha|}{2m}}[D^\alpha v]_{C^{\frac{\nu}{2m},\nu}(Q_2^+)} \leq [D_y^{2m-|\alpha|}D^\alpha\tilde{v}]_{C^{\frac{\nu}{2m},\nu}(\tilde{Q}_2^+)}.$$

Observe now that

$$\begin{aligned}(|D^\alpha v - (D^\alpha v)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} &\leq C\kappa^{-\nu}[D^\alpha v]_{C^{\frac{\nu}{2m},\nu}(Q_r^+(X_0))} \\ &\leq C\kappa^{-\nu}[D^\alpha v]_{C^{\frac{\nu}{2m},\nu}(Q_2^+)}\end{aligned}$$

and the same holds for v_t. This implies that

$$\begin{aligned}&(|v_t - (v_t)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} + \lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha v - (D^\alpha v)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} \\ &\leq C\kappa^{-\nu}[v_t]_{C^{\frac{\nu}{2m},\nu}(Q_2^+)} + C\kappa^{-\nu}\lambda^{1-\frac{|\alpha|}{2m}}[D^\alpha v]_{C^{\frac{\nu}{2m},\nu}(Q_2^+)} \\ &\leq C\kappa^{-\nu}[\tilde{v}_t]_{C^{\frac{\nu}{2m},\nu}(\tilde{Q}_2^+)} + C\kappa^{-\nu}[D_y^{2m-|\alpha|}D^\alpha\tilde{v}]_{C^{\frac{\nu}{2m},\nu}(\tilde{Q}_2^+)}.\end{aligned}$$

Therefore, the left-hand side of (7.3.27) is bounded by that of (7.3.29).

Since $D^{2m}\tilde{v}$ is a linear combination of terms such as

$$\lambda^{\frac{1}{2}-\frac{k}{2m}}\cos(\lambda^{\frac{1}{2m}}y)D_x^k u(t,x), \quad \lambda^{\frac{1}{2}-\frac{k}{2m}}\sin(\lambda^{\frac{1}{2m}}y)D_x^k u(t,x), \quad k = 0,\ldots,2m,$$

we have

$$\|D^{2m}\tilde{v}\|_{L_p(\tilde{Q}_4^+)} \leq \sum_{|\alpha|\leq 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha v\|_{L_p(Q_{\kappa r}^+(X_0))}.$$

This together with $v_t = -A_0 v$ yields

$$C\kappa^{-\nu}\|\tilde{v}_t\|_{L_p(\tilde{Q}_4^+)} + C\kappa^{-\nu}\|D^{2m}\tilde{v}\|_{L_p(\tilde{Q}_4^+)} \le C\kappa^{-\nu}\sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha v\|_{L_p(Q^+_{\kappa r}(X_0))},$$

which shows that the right-hand side of (7.3.29) is bounded by that of (7.3.27).

Step 4. Since $u = w + v$, by (7.3.26) and (7.3.27) we get

$$\begin{aligned}
&(|u_t - (u_t)_{Q_r}|)_{Q_r^+(X_0)} + \sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u - (D^\alpha u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)}\\
&\overset{(i)}{\le} C(|u_t - (v_t)_{Q_r}|)_{Q_r^+(X_0)} + C\sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u - (D^\alpha v)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)}\\
&\le C(|v_t - (v_t)_{Q_r}|)_{Q_r^+(X_0)} + C\sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha v - (D^\alpha v)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)}\\
&\quad + C(|w_t|^p)^{1/p}_{Q_r^+(X_0)} + C\sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha w|^p)^{1/p}_{Q_r^+(X_0)}\\
&\le C\kappa^{-\nu}\sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha v|^p)^{1/p}_{Q^+_{\kappa r}(X_0)} + C\kappa^{\frac{d+2m}{p}}(|f|^p)^{1/p}_{Q^+_{\kappa r}(X_0)}\\
&\overset{(ii)}{\le} C\kappa^{-\nu}\sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u|^p)^{1/p}_{Q^+_{\kappa r}(X_0)} + C\kappa^{\frac{d+2m}{p}}(|f|^p)^{1/p}_{Q^+_{\kappa r}(X_0)},
\end{aligned}$$

where in (i) we used the fact that for any constant c_1, c_2 it holds

$$(|u_t - (u_t)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} \le 2(|u_t - c_1|)_{Q_r^+(X_0)},$$
$$(|D^\alpha u - (D^\alpha u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} \le 2(|D^\alpha u - c_2|)_{Q_r^+(X_0)},$$

and we took $c_1 = (v_t)_{Q_r^+(X_0)}$, $c_2 = (D^\alpha v)_{Q_r^+(X_0)}$, while in (ii) we used $v = u - w$ and (7.3.25). □

We now use the idea of freezing the coefficients as in [52, Lemma 5.9], to obtain the following mean oscillation estimate on $Q_r^+(X_0)$ for operators with variable coefficients when r is small.

Lemma 7.3.6. *Let $\lambda \ge 0$ and $\kappa \ge 16$. Assume that A and B_j, $j = 1,\ldots,m$, satisfy conditions (A), (B), and (LS)$_\theta$ for some $\theta \in (0,\pi/2)$, and assume the lower-order coefficients of A and B_j to be all zero. Let $\mu,\varsigma \in (1,\infty)$, $\frac{1}{\varsigma}+\frac{1}{\mu} = 1$. Then, for $r \in (0, R_0/\kappa]$, $X_0 \in \overline{\mathbb{R}^{d+1}_+}$ and $u \in W^{1,2m}_{p\mu,loc}(\overline{\mathbb{R}^{d+1}_+})$ satisfying $u_t + (A(t)+\lambda)u = f$ in $Q^+_{\kappa r}(X_0)$ and $B_j u|_{x_1=0} = 0$ on $Q_{\kappa r}(X_0)\cap\{x_1 = 0\}$, $j = 1,\ldots,m$, where $f \in L_{p,loc}(\overline{\mathbb{R}^{d+1}_+})$, we have*

$$\begin{aligned}
&(|u_t - (u_t)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} + \sum_{\substack{|\alpha|\le 2m\\ \alpha_1<2m}}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u - (D^\alpha u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)}\\
&\qquad \le C\kappa^{-(1-\frac{1}{p})}\sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u|^p)^{\frac{1}{p}}_{Q^+_{\kappa r}(X_0)} + C\kappa^{\frac{d+2m}{p}}(|f|^p)^{\frac{1}{p}}_{Q^+_{\kappa r}(X_0)}
\end{aligned}$$

$$+ C\kappa^{\frac{d+2m}{p}} \rho^{\frac{1}{p\varsigma}} (|D^{2m}u|^{p\mu})^{\frac{1}{p\mu}}_{Q^+_{\kappa r}(X_0)},$$

where $C = C(\theta, d, m, \mu, K, p, b_{j\beta}) > 0$.

Proof. Fix $(t_0, x_0) \in \overline{\mathbb{R}^{d+1}_+}$. For any $(s, y) \in Q^+_{\kappa r}(t_0, x_0)$, set

$$A_{s,y}u = \sum_{|\alpha|=2m} a_\alpha(s, y) D^\alpha u.$$

Then u satisfies

$$\begin{cases} u_t + (A_{s,y} + \lambda)u = g & \text{in } Q^+_{\kappa r} \\ B_j u|_{x_1=0} = 0 & \text{on } Q^+_{\kappa r} \cap \{x_1 = 0\}, \end{cases}$$

where

$$g := f + \sum_{|\alpha|=2m} (a_\alpha(s, y) - a_\alpha(t, x)) D^\alpha u.$$

Note that when $x_0^1 \le R_0$, we have $y_1 \le 2R_0$ so that the $(\mathrm{LS})_\theta$-condition is satisfied for $A_{s,y}$ and B_j. It follows from Lemma 7.3.5 that

$$\begin{aligned} &(|u_t - (u_t)_{Q^+_r(X_0)}|)_{Q^+_r(X_0)} + \sum_{|\alpha|\le 2m, \alpha_1<2m} \lambda^{1-\frac{|\alpha|}{2m}} (|D^\alpha u - (D^\alpha u)_{Q^+_r(X_0)}|)_{Q^+_r(X_0)} \\ &\qquad \le C\kappa^{-(1-\frac{1}{p})} \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} (|D^\alpha u|^p)^{\frac{1}{p}}_{Q^+_{\kappa r}(X_0)} + C\kappa^{\frac{d+2m}{p}} (|g|^p)^{\frac{1}{p}}_{Q^+_{\kappa r}(X_0)}, \end{aligned} \tag{7.3.30}$$

where $C = C(\theta, d, m, K, p, b_{j\beta})$. Note that

$$(|g|^p)^{\frac{1}{p}}_{Q^+_{\kappa r}(X_0)} \le (|f|^p)^{\frac{1}{p}}_{Q^+_{\kappa r}(X_0)} + I^{\frac{1}{p}}, \tag{7.3.31}$$

where

$$I = (|(a_\alpha(s, y) - a_\alpha(t, x)) D^\alpha u|^p)_{Q^+_{\kappa r}(X_0)}.$$

Take now the average of I with respect to (s, y) in $Q^+_{\kappa r}(X_0)$. By Hölder's inequality it holds that

$$\begin{aligned} &\left(\fint_{Q^+_{\kappa r}(X_0)} I \, ds\, dy\right)^{\frac{1}{p}} \le \left(\fint_{Q^+_{\kappa r}(X_0)} (|(a_\alpha(s, y) - a_\alpha(t, x)) D^\alpha u|^p)_{Q^+_{\kappa r}(X_0)} \, ds\, dy\right)^{\frac{1}{p}} \\ &\le \left(\fint_{Q^+_{\kappa r}(X_0)} (|(a_\alpha(s, y) - a_\alpha(t, x))|^{p\varsigma})^{\frac{1}{\varsigma}}_{Q^+_{\kappa r}(X_0)} \, ds\, dy\right)^{\frac{1}{p}} (|D^{2m}u|^{p\mu})^{\frac{1}{p\mu}}_{Q^+_{\kappa r}(X_0)}. \end{aligned}$$

Moreover, by the boundedness of the coefficients a_α, the assumption $r \le R_0/\kappa$ and Assumption 7.2.1 (ρ), we get

$$\begin{aligned} &\left(\fint_{Q^+_{\kappa r}(X_0)} (|(a_\alpha(s, y) - a_\alpha(t, x)|^{p\varsigma})_{Q^+_{\kappa r}(X_0)})^{\frac{1}{\varsigma}}\right)^{\frac{1}{p}} \\ &\le \left(\fint_{Q^+_{\kappa r}(X_0)} (|a_\alpha(s, y) - a_\alpha(t, x)|)_{Q^+_{\kappa r}(X_0)} \, ds\, dy\right)^{\frac{1}{p\varsigma}} \\ &\le C(\mathrm{osc}(a_\alpha, Q^+_{\kappa r}))^{\frac{1}{p\varsigma}} \le C((a_\alpha)^\sharp_{R_0})^{\frac{1}{p\varsigma}} \le C\rho^{\frac{1}{p\varsigma}}. \end{aligned}$$

This together with (7.3.30) and (7.3.31) gives the desired estimate. When $x_0^1 > R_0$, the results follows directly by [51, Lemma 5], since in this case there are no boundary conditions involved. $\square$

We conclude this section with the proof of Lemma 7.3.4.

Proof of Lemma 7.3.4. Without loss of generality we can take $r_2 = 1$. We take for simplicity the center X_0 of Q_1^+ to be $(0,0)$. A translation of the coordinates then gives the result for general $X_0 \in \partial\mathbb{R}_+^{d+1}$.

Assume that the polynomial P has the form

$$P = \sum_{|\alpha|\le 2m-2} \frac{c_\alpha}{\alpha!} x^\alpha, \quad x = (x_1, x') \in \mathbb{R}_+^n, \quad \alpha! = \alpha_1! \cdots \alpha_d!$$

and satisfies the boundary conditions

$$B_j P\Big|_{x_1=0} = \sum_{|\beta|=m_j} b_{j\beta} D^\beta P\Big|_{x_1=0} = 0, \tag{7.3.32}$$

where $j = 1, \ldots, m$ and $0 \le m_j \le 2m-1$. Since P is of order $2m-2$, we only need to consider the boundary conditions whose order is $m_j \le 2m-2$.

Assume that the $(\mathrm{LS})_\theta$-condition is satisfied. Then, the boundary operators $B_1, \ldots, B_m$ are linearly independent, and so are their tangential derivatives $D_{x'}^\gamma B_j$.

To determine the coefficients c_α of the polynomial, we proceed by induction on the value of $|\alpha|$. For this, we introduce two subgroups of multi-indices:

$$I_{|\alpha|} := \{\alpha \in \mathbb{N}_0^d : \ c_\alpha \text{ are determined using the boundary conditions}\}$$
$$J_{|\alpha|} := \{\alpha \in \mathbb{N}_0^d : \ c_\alpha \text{ are determined using the condition } (D^\alpha P)_{Q_1^+} = (D^\alpha v)_{Q_1^+}\}.$$

Step 1. Let $|\alpha| = 2m-2$ and $m_j \le 2m-2$. We will first determine the coefficients c_α and then prove the Poincaré type inequality

$$\|D^\alpha(v-P)\|_{L_p(Q_1^+)} \le C\|D^{2m-1}v\|_{L_p(Q_1^+)}. \tag{7.3.33}$$

For this, we take the $2m-2-m_j$-th tangential derivatives of each boundary condition in (7.3.32) and setting $x' = 0$ we get a system of equations of the form

$$\sum_{|\beta|=m_j} b_{j\beta} c_{\beta+\gamma} = 0, \tag{7.3.34}$$

each γ satisfying $|\gamma| = 2m-2-m_j$, so that $|\beta+\gamma| = 2m-2$, and $\gamma_1 = 0$.

We rewrite the above system as the product of the $r \times n$ matrix $\mathbb{B} = [b_{j\beta}^{i,\ell}]_{i=1,\ell=1}^{r,n}$ of the coefficients $b_{j\beta}$ by the vector $\mathbf{C} = (c_\alpha^\ell : \ |\alpha| = 2m-2)_{\ell=1}^n$ of the coefficients c_α, where n denotes the number of the unknown c_α's and r the number of the equations in (7.3.34).

By the (LS)$_\theta$-condition, the r rows of $\mathbb{B}$ are linearly independent. This implies that there exists an $r \times r$ submatrix $\mathbb{B}_1$ of $\mathbb{B}$ such that $rank(\mathbb{B}_1) = r$. Define $\mathbb{B}_2 := \mathbb{B} - \mathbb{B}_1$. Consider the vectors $\mathbf{C}_1 := (c^i_\alpha : \alpha \in I_{2m-2})_{i=1}^r$ and $\mathbf{C}_2 := (c^k_\alpha : \alpha \in J_{2m-2})_{k=1}^{n-r}$.

We then rewrite the equation $\mathbb{B}\mathbf{C} = 0$ as $\mathbb{B}_1\mathbf{C}_1 = -\mathbb{B}_2\mathbf{C}_2$, and we get

$$\mathbf{C}_1 = -\mathbb{B}_1^{-1}\mathbb{B}_2\mathbf{C}_2.$$

From this we obtain that the coefficients c_α with $\alpha \in I_{2m-2}$ depends on the coefficients c_α with $\alpha \in J_{2m-2}$.

We determine the last ones by requiring

$$(D^\alpha P)_{Q_1^+} = (D^\alpha v)_{Q_1^+}, \quad \alpha \in JmathcalJ_{2m-2}.$$

We then apply the interior Poincaré inequality as in [50, Lemma 3.3] and we get

$$\begin{aligned} \|D^\alpha(v-P)\|_{L_p(Q_1^+)} &\le C_0\|D^{2m-1}(v-P)\|_{L_p(Q_1^+)} \\ &= C_0\|D^{2m-1}v\|_{L_p(Q_1^+)}, \end{aligned} \tag{7.3.35}$$

with $\alpha \in J_{2m-2}$ and $C_0 = C_0(d, m, p)$.

Now let $\mathbf{D}^\alpha(v-P)$ be the vector of the derivatives $D^\alpha(v-P)$ for any multi-index α, $\mathbf{B}(v-P)$ be the vector with components $B_j(v-P)$, and $\mathbf{D}^\gamma_{x'}\mathbf{B}(v-P)$ be the vector with components $D^\gamma_{x'}B_j(v-P)$. Observe that

$$\mathbb{B}\mathbf{D}^\alpha(v-P) = \mathbf{D}^\gamma_{x'}\mathbf{B}(v-P), \tag{7.3.36}$$

where $|\gamma| + m_j = |\alpha| = 2m-2$.

Furthermore, let $\mathbf{D}^\alpha_\mathbf{I}(v-P)$ and $\mathbf{D}^\alpha_\mathbf{J}(v-P)$ denote the vectors with components $D^\alpha(v-P)$ with respectively $\alpha \in I_{2m-2}$ and $\alpha \in J_{2m-2}$. Observe that the order of their components depends respectively on the order of the components in the vectors $\mathbf{C}_1$ and $\mathbf{C}_2$ defined above. Thus, for $\mathbb{B}_1$ and $\mathbb{B}_2$ introduced above, it holds that

$$\mathbb{B}\mathbf{D}^\alpha(v-P) = \mathbb{B}_1\mathbf{D}^\alpha_\mathbf{I}(v-P) + \mathbb{B}_2\mathbf{D}^\alpha_\mathbf{J}(v-P).$$

This, combined with (7.3.36), implies that

$$\mathbb{B}_1\mathbf{D}^\alpha_I(v-P) = \mathbf{D}^\gamma_{x'}\mathbf{B}(v-P) - \mathbb{B}_2\mathbf{D}^\alpha_\mathbf{J}(v-P). \tag{7.3.37}$$

Since $D^\gamma_{x'}B_j(v-P) = 0$ on the boundary, we can apply the boundary Poincaré inequality and we get

$$\|D^\gamma_{x'}B_j(v-P)\|_{L_p(Q_1^+)} \le C_1\|D^{2m-1}v\|_{L_p(Q_1^+)}, \quad C_1 = C_1(d, m, p, K). \tag{7.3.38}$$

By (7.3.37) and combining (7.3.35) and (7.3.38), we get

$$\|D^\alpha(v-P)\|_{L_p(Q_1^+)} \le (\det(\mathbb{B}_1))^{-1}C_2\|D^{2m-1}v\|_{L_p(Q_1^+)}, \quad \alpha \in I_{2m-2},$$

where $C_2 = C_2(d,m,p,K)$. Since $\mathbb{B}_1$ has dimension $r \times r$ and $rank(\mathbb{B}_1) = r$, $\det(\mathbb{B}_1) \neq 0$. Thus, there exists $\delta > 0$ small enough and depending on $b_{j\beta}$, such that $\det(\mathbb{B}_1) > \delta$. Therefore, we obtain (7.3.33), i.e.,

$$\|D^\alpha(v-P)\|_{L_p(Q_1^+)} \le C\|D^{2m-1}v\|_{L_p(Q_1^+)}, \quad |\alpha| = 2m-2,$$

with C depending only on d,m,p,K and $b_{j\beta}$.

Step 2. Let $|\alpha| = 2m-3$ and $m_j \le 2m-3$. By taking the $(2m-3-m_j)$-th tangential derivatives of each boundary condition in (7.3.32) and setting $x' = 0$ we get a system of equation of the form

$$\sum_{|\beta|=m_j} b_{j\beta}c_{\beta+\gamma} = 0,$$

each γ satisfying $|\gamma| = 2m-3-m_j$, so that $|\beta+\gamma| = 2m-3$, and $\gamma_1 = 0$. As before, we determine the coefficients c_α with $\alpha \in I_{2m-3}$ in terms of the coefficients c_α with $\alpha \in J_{2m-3}$. The last one are determined as in the previous step by requiring

$$(D^\alpha P)_{Q_1^+} = (D^\alpha v)_{Q_1^+}, \quad \alpha \in J_{2m-3}.$$

Observe that in the average condition there are coefficients c_α with $|\alpha| = 2m-2$, but they have been already determined in *Step 1*. From this, proceeding as in *Step 1* and applying the PoincarPoincaré type inequality (7.3.33) we get

$$\|D^\alpha(v-P)\|_{L_p(Q_1^+)} \le C\|D^{2m-2}(v-P)\|_{L_p(Q_1^+)} \le C\|D^{2m-1}v\|_{L_p(Q_1^+)},$$

with $|\alpha| = 2m-3$ and C depending only on d,m,p,K and $b_{j\beta}$.

Step k. Let $|\alpha| = 2m-1-k$ and $m_j \le 2m-1-k$. We proceed by induction.

By taking the $(2m-1-k-m_j)$-th tangential derivatives of each boundary condition in (7.3.32) and setting $x' = 0$ we get a system of equation of the form

$$\sum_{|\beta|=m_j} b_{j\beta}c_{\beta+\gamma} = 0,$$

each γ satisfying $|\gamma| = 2m-1-k-m_j$, so that $|\beta+\gamma| = 2m-1-k$, and $\gamma_1 = 0$. Proceeding as before, we determine the coefficients c_α, $\alpha \in I_{2m-1-k}$ in terms of the coefficients c_α, $\alpha \in J_{2m-1-k}$. The last ones are determined by requiring

$$(D^\alpha P)_{Q_1^+} = (D^\alpha v)_{Q_1^+}, \quad \alpha \in J_{2m-1-k}.$$

Observe that by induction we have determined the coefficients c_α, $|\alpha| \in \{2m-2,\dots,2m-k\}$. Therefore, proceeding as in *Step 1*, using induction for $|\alpha| \in \{2m-2,\dots,2m-k\}$ and applying the Poincaré type inequalities obtained at any induction step, we get

$$\|D^\alpha(v-P)\|_{L_p(Q_1^+)} \le C\|D^{2m-k}(v-P)\|_{L_p(Q_1^+)} \le \cdots \le C\|D^{2m-1}v\|_{L_p(Q_1^+)},$$

with $|\alpha| = 2m-1-k$ and C depending only on d, m, p, K and $b_{j\beta}$.

Step 2m-1. Let $|\alpha| = 0$. If $P(x)|_{x_1=0} = 0$ is a boundary condition, then $c_0 = 0$. Otherwise, we determine c_0 by using the average condition $(P)_{Q_1^+} = (v)_{Q_1^+}$.

This concludes the construction of the required polynomial P. Moreover, by induction we get (7.3.20).

To conclude the proof, observe that the polynomial P satisfies the boundary conditions. In fact, by the construction above, at each step one can show by induction that the tangential derivatives of the boundary conditions are equal to zero. Since the boundary conditions are satisfied at the origin $x' = 0$, they must then be satisfied for any $x' \in \mathbb{R}^{d-1}$. The assertion follows. □

7.4. $L_p(L_q)$-ESTIMATES FOR SYSTEMS WITH GENERAL BOUNDARY CONDITION

We are now ready to prove Theorem 7.2.4. For this, we will follow the procedure of [52, Theorem 5.4] and we will need two intermediate results. The first one follows from Lemma 7.3.6.

Lemma 7.4.1. *Let $p, q \in (1,\infty)$, $v \in A_p(\mathbb{R})$, $w \in A_q(\mathbb{R}^d_+)$, $\lambda \geq 0$ and $t_1 \in \mathbb{R}$. Assume that A and B_j, $j = 1,\ldots,m$, satisfy conditions (A), (B), and (LS)$_\theta$ for some $\theta \in (0,\pi/2)$, and assume the lower-order coefficients of A and B_j to be all zero. Then, there exists constants $R_1, \rho \in (0,1)$, depending only on θ, m, d, K, p, q, $[v]_p$, $[w]_q$, and $b_{j\beta}$, such that for $u \in W^{1,2m}_{p,q,v,w}(\mathbb{R}^{d+1}_+)$ vanishing outside $(t_1-(R_0R_1)^{2m}, t_1)\times\mathbb{R}^d_+$ and satisfying (7.2.1) in $\mathbb{R}^{d+1}_+$, where $f \in L_{p,q,v,w}(\mathbb{R}^{d+1}_+)$, it holds that*

$$\sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \leq C\|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}, \tag{7.4.1}$$

where $C = C(\theta, d, m, K, p, q, [v]_p, [w]_q, b_{j\beta}) > 0$.

Proof. For the given $v \in A_p(\mathbb{R})$ and $w \in A_q(\mathbb{R}^d_+)$, using reverse Hölder's inequality (see [71, Corollary 9.2.4 and Remark 9.2.3]) we find $\sigma_1 = \sigma_1(p,[v]_p)$, $\sigma_2 = \sigma_2(q,[w]_q)$ such that $p-\sigma_1 > 1$, $q-\sigma_2 > 1$ and

$$v \in A_{p-\sigma_1}(\mathbb{R}), \quad w \in A_{q-\sigma_2}(\mathbb{R}^d_+).$$

Take $p_0, \mu \in (1,\infty)$ satisfying $p_0\mu = \min\Big\{\dfrac{p}{p-\sigma_1}, \dfrac{q}{q-\sigma_2}\Big\} > 1$. Note that

$$v \in A_{p-\sigma_1} \subset A_{p/(p_0\mu)} \subset A_{p/p_0}(\mathbb{R}),$$

$$w \in A_{q-\sigma_2} \subset A_{q/(p_0\mu)} \subset A_{q/p_0}(\mathbb{R}^d_+).$$

Then it holds that

$$u \in W^{1,2m}_{p_0\mu,\mathrm{loc}}(\mathbb{R}^{d+1}_+), \quad f \in L_{p_0\mu,\mathrm{loc}}(\mathbb{R}^{d+1}_+).$$

Indeed, by [52, Lemma 3.1], for any $g \in L_{p_0\mu,loc}$ and for any half-ball $B_1^+ \subset \mathbb{R}^d_+$ and interval $B_2 \subset \mathbb{R}$,

$$\begin{aligned}
&\frac{1}{|B_1^+||B_2|}\int_{B_1^+\times B_2} |g|^{p_0\mu}\,dx\,dt = \frac{1}{|B_2|}\int_{B_2}\frac{1}{|B_1^+|}\int_{B_1^+}|g|^{p_0\mu}\,dx\,dt\\
&\le \frac{1}{|B_2|}\int_{B_2}\Big(\frac{[w]_{q/(p_0\mu)}}{w(B_1^+)}\int_{B_1^+}|g|^q w(x)\,dx\Big)^{\frac{p_0\mu}{q}}\,dt\\
&\le \Big(\frac{[v]_{p/(p_0\mu)}}{v(B_2)}\int_{B_2}\Big(\frac{[w]_{q/(p_0\mu)}}{w(B_1^+)}\int_{B_1^+}|g|^q w(x)\,dx\Big)^{\frac{p}{q}} v(t)\,dt\Big)^{\frac{p_0\mu}{p}}.
\end{aligned}$$

Let $\kappa \ge 16$ be a large constant to be specified. For $u \in W^{1,2m}_{p_0\mu,\mathrm{loc}}(\mathbb{R}^{d+1}_+)$, if $r > \frac{R_0}{\kappa}$, since u vanishes outside $(t_1-(R_0R_1)^{2m}, t_1)\times\mathbb{R}^d_+$, for $0\le|\alpha|\le 2m$, we have

$$\begin{aligned}
&(|D^\alpha u - (D^\alpha u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} \le 2(D^\alpha u)_{Q_r^+(X_0)}\\
&\le 2(I_{(t_1-(R_0R_1)^{2m},t_1)}(s))^{1-\frac{1}{p_0}}_{Q_r^+(X_0)}(|D^\alpha u|^{p_0})^{\frac{1}{p_0}}_{Q_r^+(X_0)}\\
&\le C_{d,m,p_0}\kappa^{2m(1-\frac{1}{p_0})}R_1^{2m(1-\frac{1}{p_0})}(|D^\alpha u|^{p_0})^{\frac{1}{p_0}}_{Q_r^+(X_0)}.
\end{aligned} \tag{7.4.2}$$

If $r \in (0, R_0/\kappa]$, then by Lemma 7.3.6 with $p = p_0$, there exists a constant $C = C(\theta, d, m, \mu, K, p_0, b_{j\beta})$ such that, for $\frac{1}{\mu}+\frac{1}{\varsigma}=1$,

$$\begin{aligned}
&(|u_t-(u_t)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} + \sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u-(D^\alpha u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)}\\
&\le C\kappa^{-(1-\frac{1}{p_0})}\sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u|^{p_0})^{\frac{1}{p_0}}_{Q_{\kappa r}^+(X_0)} + C\kappa^{\frac{d+2m}{p_0}}(|f|^{p_0})^{\frac{1}{p_0}}_{Q_{\kappa r}^+(X_0)}\\
&\quad + C\kappa^{\frac{d+2m}{p_0}}\rho^{\frac{1}{p_0\varsigma}}(|D^{2m}u|^{p_0\mu})^{\frac{1}{p_0\mu}}_{Q_{\kappa r}^+(X_0)}.
\end{aligned} \tag{7.4.3}$$

Combining (7.4.2) and (7.4.3) we get

$$\begin{aligned}
&(|u_t-(u_t)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} + \sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u-(D^\alpha u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)}\\
&\le C(\kappa^{2m(1-\frac{1}{p_0})}R_1^{2m(1-\frac{1}{p_0})}+\kappa^{-(1-\frac{1}{p_0})})\sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u|^{p_0})^{\frac{1}{p_0}}_{Q_{\kappa r}^+(X_0)}\\
&\quad + C\kappa^{\frac{d+2m}{p_0}}(|f|^{p_0})^{\frac{1}{p_0}} + C\kappa^{\frac{d+2m}{p_0}}\rho^{\frac{1}{p_0\varsigma}}(|D^{2m}u|^{p_0\mu})^{\frac{1}{p_0\mu}}_{Q_{\kappa r}^+(X_0)}.
\end{aligned}$$

Observe that

$$\begin{aligned}
(u_t)^\sharp(t,x) + \sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(D^\alpha u)^\sharp(t,x) &\le \sup(|u_t-(u_t)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)}\\
&\quad + \sup\sum_{|\alpha|\le 2m,\alpha_1<2m}\lambda^{1-\frac{|\alpha|}{2m}}(|D^\alpha u-(D^\alpha u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)},
\end{aligned}$$

where the supremum is taken over all the $Q_r^+(X_0)$ with $(t,x) \in Q_r^+(X_0)$. This implies

$$
\begin{aligned}
&(u_t)^\sharp(t,x) + \sum_{|\alpha|\le 2m,\alpha_1<2m} \lambda^{1-\frac{|\alpha|}{2m}} (D^\alpha u)^\sharp(t,x) \\
&\le C(\kappa^{2m(1-\frac{1}{p_0})} R_1^{2m(1-\frac{1}{p_0})} + \kappa^{-(1-\frac{1}{p_0})}) \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} [\mathcal{M}(|D^\alpha u|^{p_0})(t,x)]^{\frac{1}{p_0}} \\
&\quad + C\kappa^{\frac{d+2m}{p_0}} [\mathcal{M}(|f|^{p_0})(t,x)]^{\frac{1}{p_0}} + C\kappa^{\frac{d+2m}{p_0}} \rho^{\frac{1}{p_0\varsigma}} [\mathcal{M}(|D^{2m} u|^{p_0\mu})(t,x)]^{\frac{1}{p_0\mu}}.
\end{aligned}
\tag{7.4.4}
$$

By taking the $L_{p,q,v,w}(\mathbb{R}^{d+1}_+)$-norms on both sides of (7.4.4) and applying Theorems 7.1.8 and 7.1.9, we get for $C = C(\theta, d, m, K, p, q, [v]_p, [w]_q, b_{j\beta})$,

$$
\begin{aligned}
&\|u_t\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m,\alpha_1<2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\
&\le C\kappa^{\frac{d+2m}{p_0}} \|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + C\kappa^{\frac{d+2m}{p_0}} \rho^{\frac{1}{p_0\varsigma}} \|D^{2m} u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\
&\quad + C(\kappa^{2m(1-\frac{1}{p_0})} R_1^{2m(1-\frac{1}{p_0})} + \kappa^{-(1-\frac{1}{p_0})}) \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)},
\end{aligned}
\tag{7.4.5}
$$

where we used

$$
\begin{aligned}
&\|[\mathcal{M}(D^{2m} u)^{p_0\mu}]^{\frac{1}{p_0\mu}}\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} = \|\mathcal{M}(D^{2m} u)^{p_0\mu}\|^{\frac{1}{p_0\mu}}_{L_{p/(p_0\mu),q/(p_0\mu),v,w}(\mathbb{R}^{d+1}_+)} \\
&\le C\|(D^{2m} u)^{p_0\mu}\|^{\frac{1}{p_0\mu}}_{L_{p/(p_0\mu),q/(p_0\mu),v,w}(\mathbb{R}^{d+1}_+)} = C\|D^{2m} u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)},
\end{aligned}
$$

with $C = C(d, p/(p_0\mu), q/(p_0\mu), [v]_p, [w]_q)$.

It follows from the equation that

$$
a_{\tilde\alpha\tilde\alpha}(t,x) D_{x_1}^{2m} u = f - u_t - \sum_{|\alpha|=2m,\alpha_1<2m} a_\alpha(t,x) D^\alpha u - \lambda u,
$$

where $\tilde\alpha = (m, 0, \ldots, 0)$. Thus, by taking the $L_{p,q,v,w}$-norms and by the assumptions on the coefficients, it holds that for $C = C(\theta, d, m, K, p, q, [v]_p, [w]_q)$,

$$
\begin{aligned}
\|D_{x_1}^{2m} u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} &\le C\|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + C\|u_t\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\
&\quad + C \sum_{|\alpha|\le 2m,\alpha_1<2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}.
\end{aligned}
\tag{7.4.6}
$$

Combining (7.4.5) and (7.4.6), we get

$$
\begin{aligned}
&\sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\
&\le C\kappa^{\frac{d+2m}{p_0}} \|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + C\kappa^{\frac{d+2m}{p_0}} \rho^{\frac{1}{p_0\varsigma}} \|D^{2m} u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\
&\quad + C(\kappa^{2m(1-\frac{1}{p_0})} R_1^{2m(1-\frac{1}{p_0})} + \kappa^{-(1-\frac{1}{p_0})}) \sum_{|\alpha|\le 2m,\ \alpha_1\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}.
\end{aligned}
$$

Finally by first taking $\kappa \geq 16$ sufficiently large and then ρ and R_1 sufficiently small such that

$$C\kappa^{-(1-\frac{1}{p_0})} \leq \frac{1}{6}, \quad C\kappa^{2m(1-\frac{1}{p_0})} R_1^{2m(1-\frac{1}{p_0})} \leq \frac{1}{6}, \quad \text{and} \quad C\kappa^{\frac{d+2m}{p_0}} \rho^{\frac{1}{p_0\varsigma}} \leq \frac{1}{6},$$

we get (7.4.1). $\square$

From Lemma 7.4.1 and using a partition of unity argument with respect to only the time variable, we can prove the second intermediate result.

Proposition 7.4.2. *Assume that A and B_j, $j = 1, \ldots, m$, satisfy conditions (A), (B), and (LS)$_\theta$ for some $\theta \in (0, \pi/2)$, and assume the lower-order terms of B_j to be all zero. Then there exists $\rho = \rho(\theta, m, d, K, p, q, [v]_p, [w]_q, b_{j\beta}) \in (0,1)$ such that for $\lambda \geq 0$, $f \in L_{p,q,v,w}(\overline{\mathbb{R}^{d+1}_+})$ and $u \in W^{1,2m}_{p,q,v,w}(\overline{\mathbb{R}^{d+1}_+})$ satisfying* (7.2.1), *we have*

$$\begin{aligned} &\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\ &\leq C_1 \|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + C_2 \sum_{|\alpha| \leq 2m-1} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}, \end{aligned} \tag{7.4.7}$$

where

$$\begin{aligned} C_1 &= C_1(\theta, d, m, K, p, q, [v]_p, [w]_q, b_{j\beta}), \\ C_2 &= C_2(\theta, d, m, K, p, q, [v]_p, [w]_q, R_0, b_{j\beta}). \end{aligned}$$

Proof. Without loss of generality, we can assume the lower-order coefficients of A to be zero. To see this, just move the terms $\sum_{|\alpha|<2m} a_\alpha(t,x) D^\alpha$ to the right-hand side of (7.2.1), i.e., consider

$$u_t + \sum_{|\alpha|=2m} a_\alpha(t,x) D^\alpha u = f - \sum_{|\alpha| \leq 2m-1} a_\alpha(t,x) D^\alpha u$$

and recall that the lower-order coefficients of A are bounded by K, so that

$$\sum_{|\alpha| \leq 2m-1} \|a_\alpha D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \leq C_K \sum_{|\alpha| \leq 2m-1} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}.$$

If (7.4.7) holds for $A = \sum_{|\alpha|=2m} a_\alpha(t,x) D^\alpha$, we thus get

$$\begin{aligned} &\sum_{|\alpha| \leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\ &\leq C_1 \|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + C_1 C_K \sum_{|\alpha| \leq 2m-1} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\ &\quad + C_2 \sum_{|\alpha| \leq 2m-1} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\ &\leq C_1 \|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + C_2 \sum_{|\alpha| \leq 2m-1} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}. \end{aligned}$$

Take now $R_1 \in (0,1)$ from Lemma 7.4.1 and fix a non-negative infinitely differentiable function $\zeta(t)$ defined on $\mathbb{R}$ such that $\zeta(t)$ vanishes outside $(-(R_0R_1)^{2m}, 0)$ and

$$\int_{\mathbb{R}} \zeta(t)^p \, dt = 1.$$

Then, $u(t,x)\zeta(t-s)$ satisfies

$$\begin{cases} (u(t,x)\zeta(t-s))_t + (\lambda + A)(u(t,x)\zeta(t-s)) & \\ \quad = \zeta(t-s) f(t,x) + \zeta_t(t-s) u(t,x) & \text{on } \mathbb{R}^{d+1}_+ \\ B_j(u(t,x)\zeta(t-s))\Big|_{x_1=0} = 0 & \text{on } \mathbb{R} \times \mathbb{R}^{d-1}. \end{cases} \tag{7.4.8}$$

For each $s \in \mathbb{R}$, since $u(t,x)\zeta(t-s)$ vanishes outside $(s-(R_0R_1)^{2m}, s) \times \mathbb{R}^d_+$, by Lemma 7.4.1 applied to (7.4.8) we get

$$\begin{aligned} &\sum_{|\alpha|<2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha(u\zeta(\cdot - s))\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \\ &\le C\|f\zeta(\cdot - s)\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + C\|u\zeta_t(\cdot - s)\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}, \end{aligned} \tag{7.4.9}$$

where $C = C(d,m,K,p,q,[v]_p,[w]_q,b_{j\beta})$. Note that

$$\begin{aligned} \|D^\alpha u(t,\cdot)\|^p_{L_{q,w}(\mathbb{R}^d_+)} &= \int_{\mathbb{R}} \|D^\alpha u(t,\cdot)\|^p_{L_{q,w}(\mathbb{R}^d_+)} \zeta(t-s)^p \, ds \\ &= \int_{\mathbb{R}} \|D^\alpha u(t,\cdot)\zeta(t-s)\|^p_{L_{q,w}(\mathbb{R}^d_+)} \, ds. \end{aligned}$$

Thus, by integrating with respect to t,

$$\|D^\alpha u\|^p_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} = \int_{\mathbb{R}} \|D^\alpha(u\zeta(\cdot - s))\|^p_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \, ds.$$

From this and (7.4.9) it follows that

$$\sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \le C_1\|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} + C_2\|u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)},$$

where $C_1 = C_1(\theta, d, m, K, p, q, [v]_p, [w]_q, b_{j\beta}) > 0$ and C_2 depends on R_0R_1 and the same parameters as C_1 does. □

Now Theorem 7.2.4 follows from Proposition 7.4.2.

Proof of Theorem 7.2.4. It suffices to consider $T = \infty$. For the general case when $T \in (-\infty, \infty]$, we can follow the proof of Lemma 7.3.1 with the obvious changes in the weighted setting, so we omit the details.

(i) In Proposition 7.4.2 we take $\lambda_0 \ge 0$ depending only on C_2 such that

$$\frac{1}{2} \sum_{|\alpha|\le 2m-1} \lambda^{1-\frac{|\alpha|}{2m}} \le \sum_{|\alpha|\le 2m-1} \left(\lambda^{1-\frac{|\alpha|}{2m}} - C_2\right)$$

for any $\lambda \ge \lambda_0$. By (7.4.7) we get

$$\frac{1}{2}\sum_{|\alpha|\le 2m-1}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}+\|D^{2m}u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \le C\|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}$$

and thus

$$\sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)} \le C\|f\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}. \tag{7.4.10}$$

Finally, the estimate of $\|u_t\|_{L_{p,q,v,w}(\mathbb{R}^{d+1}_+)}$ follows by noting that $u_t = f-(A+\lambda)u$ and (7.4.10). This proves (7.2.4).

(ii) As in the proof of Proposition 7.4.2, we can assume the lower-order coefficients of A to be zero. Let

$$A(0,0)u := \sum_{|\alpha|=2m} a_\alpha(0,0)D^\alpha.$$

By Lemma 7.3.1, we first solve

$$\begin{cases}\partial_t u_1 + (\lambda + A(0,0))u_1 = 0 & \text{in } \mathbb{R}^{d+1}_+\\ \sum_{|\beta|=m_j} b_{j\beta}D^\beta u_1\Big|_{x_1=0} = -\sum_{|\beta|<m_j} b_{j\beta}(t,x)D^\beta u\Big|_{x_1=0} + g_j & \text{on } \mathbb{R}\times\mathbb{R}^{d-1},\end{cases}$$

and by Theorem 7.1.5 we get

$$\begin{aligned}\|\partial_t u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} &+ \sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))}\\ &\le C\Big\|\sum_{|\beta|<m_j} b_{j\beta}D^\beta u\Big\|_{W_p^{(2m-m_j)\frac{1}{2m}}(\mathbb{R};L_q(\mathbb{R}^d_+))\cap L_p(\mathbb{R};W_q^{2m-m_j}(\mathbb{R}^d_+))}\\ &\qquad + \|g_j\|_{F^{k_j}_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R};\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))}.\end{aligned} \tag{7.4.11}$$

Next $u_2 = u - u_1$ satisfies the equation

$$\begin{cases}\partial_t u_2 + (\lambda + A)u_2 = f - (A - A(0,0))u_1 & \text{in } \mathbb{R}^{d+1}_+\\ \sum_{|\beta|=m_j} b_{j\beta}D^\beta u_2\Big|_{x_1=0} = 0 & \text{on } \mathbb{R}\times\mathbb{R}^{d-1},\end{cases}$$

to which we can apply statement (i) with $v = w = 1$ to get

$$\begin{aligned}&\|\partial_t u_2\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + \sum_{|\alpha|\le 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_2\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))}\\ &\le C\|f\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + C\|(A-A(0,0))u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))}\\ &\le C\|f\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + C_K\|D^{2m}u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))},\end{aligned} \tag{7.4.12}$$

with $\lambda \geq \lambda_0$, where λ_0 depends only on the constant C_2 from Proposition 7.4.2. Now, since $u = u_1 + u_2$, by (7.4.12) and (7.4.11),

$$\begin{aligned}
&\|u_t\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} \\
&\leq \|\partial_t u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} \\
&\quad + \|\partial_t u_2\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_2\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} \\
&\leq \|\partial_t u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + \sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} \\
&\quad + C_K \|D^{2m} u_1\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + C\|f\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} \\
&\leq C\|f\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + C_K \|g_j\|_{F^{k_j}_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R};\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))} \\
&\quad + C_K \Big\| \sum_{|\beta|<m_j} b_{j\beta}(t,x) D^\beta u \Big\|_{W_p^{(2m-m_j)\frac{1}{2m}}(\mathbb{R};L_q(\mathbb{R}^d_+))\cap L_p(\mathbb{R};W_q^{2m-m_j}(\mathbb{R}^d_+))} \\
&\overset{\text{(i)}}{\leq} C\|f\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + C_K \|g_j\|_{F^{k_j}_{p,q}(\mathbb{R};L_q(\mathbb{R}^{d-1}))\cap L_p(\mathbb{R};\mathscr{B}^{2mk_j}_{q,q}(\mathbb{R}^{d-1}))} \\
&\quad + C_K\big(C\varepsilon \|D^{2m} u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + C\varepsilon \|u_t\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))} + C_\varepsilon \|u\|_{L_p(\mathbb{R};L_q(\mathbb{R}^d_+))}\big),
\end{aligned}$$

where (i) follows from the smoothness the coefficients $b_{j\beta}(t,x)$ for $|\beta| < m_j$ and by using interpolation estimates as in Lemma 7.1.6. Now, taking ε small enough so that $C_K C\varepsilon \leq 1/2$ and λ such that $\lambda \geq \max\{\lambda_0, 2C_K C_\varepsilon\}$, we get (7.2.5). □

From Theorem 7.2.4, we now prove Theorem 7.2.5.

Proof of Theorem 7.2.5. (i) Take $\zeta \in C_0^\infty(\mathbb{R})$ and set $v(t,x) = \zeta(t/n)u(x)$, $n \in \mathbb{Z}$, which satisfies, in $\mathbb{R}^{d+1}_+$

$$\begin{cases} v_t(t,x) + (A+\lambda)v(t,x) = h & \text{in } \mathbb{R}\times\mathbb{R}^d_+ \\ B_j v(t,x)\Big|_{x_1=0} = 0 & \text{on } \mathbb{R}\times\mathbb{R}^{d-1}, \end{cases} \tag{7.4.13}$$

with $h := \frac{1}{n}\zeta_t(\frac{t}{n})u(x) + \zeta(\frac{t}{n})f$. If we now apply Theorem 7.2.4 to (7.4.13) with $v = 1$ we get

$$\sum_{|\alpha|\leq 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha v\|_{L_p(\mathbb{R};L_{q,w}(\mathbb{R}^d_+))} \leq C\|h\|_{L_p(\mathbb{R};L_{q,w}(\mathbb{R}^d_+))}, \tag{7.4.14}$$

with $C = C(\theta, m, d, K, p, q, R_0, [w]_q, b_{j\beta})$. Observe now that

$$\begin{aligned}
\|h\|_{L_p(\mathbb{R};L_{q,w}(\mathbb{R}^d_+))} &\leq \frac{1}{n}\Big(\int_R \big|\zeta_t(t/n)\big|^p\,dt\Big)^{1/p} \|u\|_{L_{q,w}(\mathbb{R}^d_+)} \\
&\quad + \Big(\int_R \big|\zeta(t/n)\big|^p\,dt\Big)^{1/p} \|f\|_{L_{q,w}(\mathbb{R}^d_+)},
\end{aligned}$$

and
$$\|D^\alpha v\|_{L_p(\mathbb{R};L_{q,w}(\mathbb{R}^d_+))} = \left(\int_R |\zeta(t/n)|^p\,dt\right)^{1/p} \|D^\alpha u\|_{L_{q,w}(\mathbb{R}^d_+)}.$$
Thus, combining the above estimates with (7.4.14) and letting $n \to +\infty$, we get (7.2.6). (ii) The estimate (7.2.7) follows in the same way from (7.2.5). □

7.5. EXISTENCE OF SOLUTIONS

The a priori estimates of Theorems 7.2.4 and 7.2.5 can be used to derive the existence of solutions to the corresponding equations. In this section we focus on the solvability of the parabolic problem (7.2.1). The elliptic case follows in the same way from the a priori estimates in Theorem 7.2.5.

As in the proof of Lemma 7.3.1, via a standard argument it suffices to consider $T = \infty$ (see e.g. [100, Theorem 2.1]). Under the conditions in Theorem 7.2.4(ii), from the a priori estimate (7.2.5), the standard method of continuity (see [68, Theorem 5.2]) combined with Lemma 7.3.1, yields existence and uniqueness of a strong solution to (7.2.1).

We now assume that the conditions in Theorem 7.2.4(i) are satisfied and we show the solvability of (7.2.1) via a density argument as in [52, Section 8]. By reverse Hölder's inequality and the doubling property of A_p-weights, one can find a sufficiently large constant p_1 and small constants $\varepsilon_1, \varepsilon_2 \in (0,1)$ depending on d, p, q, $[v]_p$, $[w]_q$ such that
$$1 - \frac{p}{p_1} = \frac{1}{1+\varepsilon_1}, \quad 1 - \frac{q}{p_1} = \frac{1}{1+\varepsilon_2},$$
and both $v^{1+\varepsilon_1}$ and $w^{1+\varepsilon_2}$ are locally integrable and satisfy the doubling property, i.e. for every $r > 0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d_+$,
$$\int_{I_{2r}(t_0)} v^{1+\varepsilon_1}\,dt \le C_0 \int_{I_r(t_0)} v^{1+\varepsilon_1}\,dt, \tag{7.5.1}$$
$$\int_{B^+_{2r}(x_0)} w^{1+\varepsilon_1}\,dt \le C_0 \int_{B^+_r(x_0)} w^{1+\varepsilon_1}\,dt, \tag{7.5.2}$$
where C_0 is independent of r, t_0, x_0 and $I_r(t_0) = (t_0 - r^{2m}, t_0 + r^{2m})$ denotes an interval in $\mathbb{R}$. By Hölder's inequality, any function $f \in L_{p_1}(\mathbb{R}^{d+1}_+)$ is locally in $L_{p,q,v,w}(\mathbb{R}^{d+1}_+)$ and for any $r > 0$,
$$\|f\|_{L_{p,q,v,w}(\mathcal{Q}^+_r)} \le C\|f\|_{L_{p_1}(\mathcal{Q}^+_r)}, \tag{7.5.3}$$
where $\mathcal{Q}^+_r = ((-r^{2m}, r^{2m}) \times B_r) \cap \mathbb{R}^{d+1}_+$, with B_r being a ball of radius r in $\mathbb{R}^d$, and C depends also on r.

Now if $f \in L_{p,q,v,w}(\mathbb{R}^{d+1}_+)$, by the denseness of $C_0^\infty(\overline{\mathbb{R}^{d+1}_+})$ in $L_{p,q,v,w}(\mathbb{R}^{d+1}_+)$, we can find a sequence of smooth functions $\{f_k\}_{k=0,1,\ldots}$ with bounded supports such that
$$f_k \to f \text{ in } L_{p,q,v,w}(\mathbb{R}^{d+1}_+) \text{ as } k \to \infty. \tag{7.5.4}$$

Since for each k, $f_k \in L_{p_1}(\mathbb{R}^{d+1}_+)$, by the solvability in the unweighted setting of Theorem 7.2.4(ii) with p_1 instead of $p = q$, zero lower-order coefficients for B_j and $g_j \equiv 0$, there exists a unique solution $u_k \in W^{1,2m}_{p_1}(\mathbb{R}^{d+1}_+)$ to

$$\begin{cases} (u_k)_t(t,x) + (A+\lambda)u_k(t,x) = f_k(t,x) & \text{in } \mathbb{R}\times\mathbb{R}^d_+ \\ B_j u_k(t,x)\Big|_{x_1=0} = 0 & \text{on } \mathbb{R}\times\mathbb{R}^{d-1}, \quad j = 1,\dots,m, \end{cases}$$

provided that $\lambda \geq \lambda_1(\theta, m, d, p_1, K, R_0, b_{j\beta})$ and $\rho \leq \rho_1(\theta, m, d, p_1, K, b_{j\beta})$.

We claim that if $\lambda \geq \max\{\lambda_0, \lambda_1\}$, then $u_k \in W^{1,2m}_{p,q,v,w}(\mathbb{R}^{d+1}_+)$. If the claim is proved, it follows from the a priori estimate (7.2.4) and from (7.5.4) that $\{u_k\}$ is a Cauchy sequence in $W^{1,2m}_{p,q,v,w}(\mathbb{R}^{d+1}_+)$. Let u be its limit. Then, by taking the limit of the equation for u_k, it follows that u is the solution to (7.2.1).

In order to prove the claim, we fix a $k \in \mathbb{N}$ and we assume that f_k is supported in $\mathcal{Q}^+_R$ for some $R \geq 1$. By (7.5.3) we have

$$\|D^\alpha u_k\|_{L_{p,q,v,w}(\mathcal{Q}^+_{2R})} < \infty, \; 0 \leq |\alpha| \leq 2m \tag{7.5.5}$$

and

$$\|(u_k)_t\|_{L_{p,q,v,w}(\mathcal{Q}^+_{2R})} < \infty. \tag{7.5.6}$$

For $j \geq 0$, we take a sequence of smooth functions η_j such that $\eta_j \equiv 0$ in $\mathcal{Q}^+_{2^j R}$, $\eta_j \equiv 1$ outside $\mathcal{Q}^+_{2^{j+1}R}$ and

$$|D^\alpha \eta_j| \leq C2^{-j|\alpha|}, \quad |\alpha| \leq 2m, \quad |(\eta_j)_t| \leq C2^{-2mj}.$$

Observe that $u_k\eta_j \in W^{1,2m}_{p_1}(\mathbb{R}^{d+1}_+)$ satisfies

$$\begin{cases} \partial_t(u_k\eta_j) + (A+\lambda)(u_k\eta_j) = f_j & \text{in } \mathbb{R}^{d+1}_+ \\ B_j(u_k\eta_j)\Big|_{x_1=0} = \operatorname{tr}_{x_1=0} G_j & \text{on } \partial\mathbb{R}^{d+1}_+, \; j = 1,\dots,m, \end{cases}$$

where by Leibnitz's rule

$$f_j = u_k(\eta_j)_t + \sum_{1\leq|\alpha|\leq 2m} \sum_{|\gamma|\leq|\alpha|-1} \binom{\alpha}{\gamma} a_\alpha D^\gamma u_k D^{\alpha-\gamma}\eta_j$$

and

$$G_j = \sum_{|\beta|=m_j} \sum_{|\tau|\leq m_j-1} \binom{\beta}{\tau} b_{j\beta} D^\tau u_k D^{\beta-\tau}\eta_j.$$

Now let

$$g_j = \operatorname{tr}_{x_1=0} G_j \in W_{p_1}^{1-\frac{m_j}{2m}-\frac{1}{2mp_1},\, 2m-m_j-\frac{1}{p_1}}(\mathbb{R}\times\mathbb{R}^{d-1}).$$

By applying the a priori estimate (7.2.5), with p_1 instead of $p = q$ there, to $u_k\eta_j$, we get

$$\|\partial_t(u_k\eta_j)\|_{L_{p_1}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha(u_k\eta_j)\|_{L_{p_1}(\mathbb{R}^{d+1}_+)}$$
$$\le C\|f_j\|_{L_{p_1}(\mathbb{R}^{d+1}_+)} + C\sum_{j=1}^m \|g_j\|_{W_{p_1}^{1-\frac{m_j}{2m}-\frac{1}{2mp_1},2m-m_j-\frac{1}{p_1}}(\mathbb{R}\times\mathbb{R}^{d-1})},$$

with a constant $C = C(\theta, m, d, K, p_1, b_{j\beta}) > 0$. By Theorem 7.1.3 with $s = 1 - m_j/(2m) \in (0,1]$, $m_j \in \{0,\dots,2m-1\}$, we have

$$\|g_j\|_{W_{p_1}^{1-\frac{m_j}{2m}-\frac{1}{2mp_1},2m-m_j-\frac{1}{p_1}}(\mathbb{R}\times\mathbb{R}^{d-1})} \le C\|G_j\|_{W_{p_1}^{1-\frac{m_j}{2m},2m-m_j}(\mathbb{R}^{d+1}_+)}.$$

Observe that

$$\|f_j\|_{L_{p_1}(\mathbb{R}^{d+1}_+)} \le C\|(\eta_j)_t u_k\|_{L_{p_1}(\mathbb{R}^{d+1}_+)} + C\sum_{1\le|\alpha|\le 2m}\sum_{|\gamma|\le|\alpha|-1} \|D^\gamma u_k D^{\alpha-\gamma}\eta_j\|_{L_{p_1}(\mathbb{R}^{d+1}_+)}$$

and

$$\|G_j\|_{W_{p_1}^{1-\frac{m_j}{2m},2m-m_j}(\mathbb{R}^{d+1}_+)} \le C\sum_{|\beta|=m_j}\sum_{|\tau|\le m_j-1} \|D^\tau u_k D^{\beta-\tau}\eta_j\|_{W_{p_1}^{1-\frac{m_j}{2m},2m-m_j}(\mathbb{R}^{d+1}_+)}.$$

This implies that

$$\|\partial_t(u_k\eta_j)\|_{L_{p_1}(\mathbb{R}^{d+1}_+)} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha(u_k\eta_j)\|_{L_{p_1}(\mathbb{R}^{d+1}_+)}$$
$$\le C\|(\eta_j)_t u_k\|_{L_{p_1}(\mathbb{R}^{d+1}_+)} + C\sum_{1\le|\alpha|\le 2m}\sum_{|\gamma|\le|\alpha|-1} \|D^\gamma u_k D^{\alpha-\gamma}\eta_j\|_{L_{p_1}(\mathbb{R}^{d+1}_+)}$$
$$+ C\sum_{|\beta|=m_j}\sum_{|\tau|\le m_j-1} \|D^\tau u_k D^{\beta-\tau}\eta_j\|_{W_{p_1}^{1-\frac{m_j}{2m},2m-m_j}(\mathbb{R}^{d+1}_+)},$$

from which it follows that

$$\|(u_k)_t\|_{L_{p_1}(\mathbb{R}^{d+1}_+\setminus\mathcal{Q}^+_{2^{j+1}R})} + \sum_{|\alpha|\le 2m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u_k\|_{L_{p_1}(\mathbb{R}^{d+1}_+\setminus\mathcal{Q}^+_{2^{j+1}R})}$$
$$\le C2^{-j}\|u_k\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})} + C2^{-j}\sum_{1\le|\alpha|\le 2m}\sum_{|\gamma|\le|\alpha|-1} \|D^\gamma u_k\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}$$
$$+ C2^{-j}\sum_{|\beta|=m_j}\sum_{|\tau|\le m_j-1} \|D^\tau u_k\|_{W_{p_1}^{1-\frac{m_j}{2m},2m-m_j}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}.$$

By standard interpolation inequalities (see e.g. [102]),

$$\|D^\gamma u_k\|_{L_p(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})} \le C\|D^{2m}u_k\|_{L_p(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})} + C\|u_k\|_{L_p(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})},$$

and by the interpolation estimates as in Lemma 7.1.4,

$$\|D^\tau u_k\|_{W_p^{1-\frac{m_j}{2m},2m-m_j}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}$$

$$\leq C\|D^{2m}u_k\|_{L_p(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}+C\|(u_k)_t\|_{L_p(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}+C\|u_k\|_{L_p(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}.$$

Thus, we get

$$\begin{aligned}\|(u_k)_t\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+2}R}\setminus\mathcal{Q}^+_{2^{j+1}R})}&+\sum_{|\alpha|\leq 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_k\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+2}R}\setminus\mathcal{Q}^+_{2^{j+1}R})}\\ &\leq C2^{-j}(\|(u_k)_t\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}+\|D^{2m}u_k\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}\\ &\qquad+\|u_k\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}).\end{aligned}$$

By induction, we obtain for each $j\geq 1$,

$$\begin{aligned}\|(u_k)_t\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}&+\sum_{|\alpha|\leq 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_k\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}\\ &\leq C^j2^{-\frac{j(j-1)}{2}}(\|(u_k)_t\|_{L_{p_1}(\mathcal{Q}^+_{2R})}+\|D^{2m}u_k\|_{L_{p_1}(\mathcal{Q}^+_{2R})}+\|u_k\|_{L_{p_1}(\mathcal{Q}^+_{2R})}).\end{aligned}\tag{7.5.7}$$

Finally, by Holder's inequality, (7.5.1), (7.5.2) and (7.5.7), we get for each $j\geq 1$,

$$\begin{aligned}&\|(u_k)_t\|_{L_{p,q,v,w}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}+\sum_{|\alpha|\leq 2m}\lambda^{1-\frac{|\alpha|}{2m}}\|D^\alpha u_k\|_{L_{p,q,v,w}(\mathcal{Q}^+_{2^{j+2}R}\setminus\mathcal{Q}^+_{2^{j+1}R})}\\ &\leq\|v\|^{\frac{1}{p}}_{L_{1+\varepsilon_1}(I_{2^{j+1}R})}\|w\|^{\frac{1}{q}}_{L_{1+\varepsilon_2}(B^+_{2^{j+1}R})}\Big(\|(u_k)_t\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+2}R}\setminus\mathcal{Q}^+_{2^{j+1}R})}\\ &\quad+\|D^{2m}u_k\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}+\|u_k\|_{L_{p_1}(\mathcal{Q}^+_{2^{j+1}R}\setminus\mathcal{Q}^+_{2^jR})}\Big)\\ &\leq CC^{j(1+\frac{1}{p}+\frac{1}{q})}2^{-\frac{j(j-1)}{2}}\Big(\|(u_k)_t\|_{L_{p_1}(\mathcal{Q}^+_{2R})}+\|D^{2m}u_k\|_{L_{p_1}(\mathcal{Q}^+_{2R})}+\|u_k\|_{L_{p_1}(\mathcal{Q}^+_{2R})}\Big).\end{aligned}$$

The above inequality together with (7.5.5) and (7.5.6) implies that $u_k\in W^{1,2m}_{p,q,v,w}(\mathbb{R}^{d+1}_+)$, which proves the claim.

Remark 7.5.1. Under certain compatibility condition, the solvability of the corresponding initial-boundary value problem can also be obtained. See, for instance, [102, Sect. 2.5] and [41] for details.

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SUMMARY

The subject of this thesis is the study of maximal L^p-regularity of the Cauchy problem

$$\begin{aligned} u'(t) + A(t)u(t) &= f(t), \quad t \in (0,T), \\ u(0) &= x. \end{aligned} \tag{1}$$

We assume $(A(t))_{t\in(0,T)}$ to be a family of closed operators on a Banach space X_0, with constant domain $D(A(t)) = X_1$ for every $t \in (0,T)$. Maximal L^p-regularity means that for all $f \in L^p(0,T;X_0)$, the solution of the evolution problem (1) is such that u', Au are both in $L^p(0,T;X_0)$. In the autonomous setting, using a new vector-valued Mihlin multiplier theorem, Lutz Weis in [152] characterize the maximal L^p-regularity property of an operator in terms of $\mathcal{R}$-sectoriality. In the non-autonomous setting, maximal L^p-regularity holds if $t \mapsto A(t)$ is assumed to be continuous and each operator $A(t_0)$ has maximal regularity for $t_0 \in (0,T)$ fixed, as showed by Prüss and Schnaubelt in [128]. The disadvantage is that continuity in time is not a natural assumption in the L^p-setting.

In the first part of the thesis, we introduce a new operator-theoretic approach to maximal L^p-regularity in the case the dependence $t \mapsto A(t)$ is just measurable. This approach is based on the L^p-boundedness of a new class of vector-valued singular integrals of non-convolution type and the $\mathcal{R}$-boundedness of a family of integral operators. In our main result we consider the more general case of maximal L^p-regularity on $\mathbb{R}$ and weighted L^p-spaces, and we also give results for the initial value problems. We then extend the result of Clément and Li [28] and Köhne, Prüss and Wilke [97] on quasilinear equations to the time-dependent setting.

The abstract method is then applied to concrete parabolic PDEs. As a first application, we show that an elliptic differential operator of even order, with coefficients measurable in the time variable and continuous in the space variables, enjoys maximal L^p-regularity on $L^q(\mathbb{R}^d)$, for every $p, q \in (1,\infty)$. The proof is an application of our abstract result, combined with standard PDE techniques as the localization procedure and the method of continuity. This gives an alternative approach to several PDE results obtained by Krylov, Dong and Kim in [51, 96, 101, 102], where only the cases $p = q$ or $q \le p$ were considered.

In order to apply the abstract result in the space-independent setting, we need a sufficient condition for the $\mathcal{R}$-boundedness of a certain family of integral operators. Such a condition is obtained, in the generalized setting of ℓ^s-boundedness, in the first part of this work.

In the second part, we apply the abstract method to systems of elliptic operators. For this, we construct the evolution family $S(t,s)$ on $L^q(\mathbb{R}^d, \mathbb{C}^N)$ generated by

$A(t)$ in the case the coefficients are space-independent.

The last part of this thesis is devoted to the study of maximal L^p-regularity on $L^q(\mathbb{R}^d_+)$ of an elliptic operator A with coefficients in the class of VMO (Vanishing Mean Oscillation) in the time and the space variables, and Lopatinskii-Shapiro assumptions on the boundary. The proof is based on a different approach, in particular it is an application of the results obtained by Denk, Hieber and Prüss in [40, 41], and an extension of the techniques of Dong and Kim [51, 52].

Samenvatting

Het onderwerp van dit proefschrift is de studie van maximale L^p-regulariteit van het Cauchy probleem

$$\begin{aligned} u'(t) + A(t)u(t) &= f(t), \quad t \in (0,T), \\ u(0) &= x. \end{aligned} \tag{1}$$

We nemen aan dat $(A(t))_{t\in(0,T)}$ een familie is van gesloten operatoren op een Banach ruimte X_0, met constant domein $D(A(t)) = X_1$ voor elke $t \in (0,T)$. Maximale L^p-regulariteit betekent dat voor alle $f \in L^p(0,T;X_0)$, de oplossing van het evolutieprobleem (1) zodanig is dat u, Au beide in $L^p(0,T;X_0)$ zijn. In de niet-autonome setting, geldt maximale L^p-regulariteit als $t \mapsto A(t)$ continu is en elke operator $A(t_0)$ maximale regulariteit heeft voor $t_0 \in (0,T)$ vast, zoals aangetoond door Prüss and Schnaubelt in [128]. Het nadeel is dat continuïteit in tijd niet een natuurlijke aanname is in de L^p-setting.

In het eerste deel van dit proefschrift introduceren we een nieuwe operatortheoretische benadering voor maximale L^p-regulariteit in het geval de afhankelijkhaid $t \mapsto A(t)$ slechts meetbaar is. Deze benadering is gebaseerd op de L^p-begrensdheid van een nieuwe klasse van vectorwaardige singuliere integraloperatoren van niet-convolutietype en de $\mathscr{R}$-begrensdheid van een familie van integraaloperatoren. In ons hoofdresultaat beschouwen we het meer algemene geval van maximale L^p-regulariteit op $\mathbb{R}$ en gewogen L^p-ruimten, and we geven ook resultaten voor beginwaardeproblemen. We breiden dan de resultaten van Clément en Li [28] en Köhne, Prüss en Wilke [97] over quasilineaire vergelijkingen uit naar de tijdsafhankelijke setting.

De abstracte methode wordt vervolgens toegepast op concrete parabolische PDVen. Als een eerste toepassing laten we zien dat een elliptische operator van even grod, met coefficienten meetbaar in de tijdsvariabele en continu in de ruimtevariabelen, maximale L^p-regulariteit op $L^q(\mathbb{R}^d)$ heeft, voor elke $p, q \in (1,\infty)$. Het bewijs is een toepassing van ons abstracte resultaat, gecombineerd met standaard PDV-technieken als de localisatieprocedure en de methode van continuiteit. Dit geeft een alternatieve benadering voor verscheidene resultaten voor PDVen verkregen door Krylov, Dong en Kim in [51, 96, 101, 102], waar enkel de gevallen $p = q$ of $q \le p$ worden beschouwd.

Om het abstracte resultaat toe te passen in de ruimteonafhankelijke setting, hebben we een voldoende voorwarde nodig voor de $\mathscr{R}$-begrensdheid van een zekere familie van integraaloperatoren. Een dergelijke voorwaarde wordt ook verkregen, in de veralgemeniseerde setting van ℓ^s-begrensdheid, in het eerste deel van dit werk.

In het tweede gedeelte, passen we de abstracte methode toe op systemen van elliptische operatoren. Hiervoor construeren we de evaluatiefamilie $S(t,s)$ op $L^q(\mathbb{R}^d, \mathbb{C}^N)$ voortgebracht door $A(t)$ in het geval dat de coefficiënten ruimteonafhankelijk zijn.

Het laaste gedaalte van dit proefschrift wordt dat gewijd aan de studie van maximale L^p-regulariteit op $L^q(\mathbb{R}^d_+)$ van een elliptische operator A met coefficiënten in de klasse van VMO (Vanishing Mean Oscillation) in de tijds- en ruimtevariabelen, en Lopatinskii-Shapiro aannamen op de rand. Het bewijs is gebaseerd op een andere benadering, in het bijzonder is het een toepassing van de resultaten verkregen door Denk, Hieber en Prüss in [40, 41] en een uitbreiding van de technieken van Dong en Kim [51, 52].

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In Italy we say "Quando si chiude una porta, si apre un portone" (When you close a door, you open a gate). Let's see where this new door will lead to.

Delft, October 2016

Chiara Gallarati

Curriculum Vitæ

Chiara Gallarati

Chiara Gallarati was born on August 19, 1988, in Genova, Italy. She completed her scientific high school in the small town of Erba, in 2007. In the same year she began her studies in Mathematics at the University of Milano-Bicocca. In 2011, during her master's degree, she obtained an Erasmus Scholarship and spent a semester at the Universidad Autonoma of Madrid, Spain. She obtained her MSc. degree in Mathematics *cum laude* in 2012. In January 2013 she started her PhD research under the supervision of Mark Veraar and Jan van Neerven at the Delft University of Technology. Part of this research was carried out during a five month stay at Brown University in Providence, United States of America.

List of Publications

(1) C. Gallarati, E. Lorist and M.C. Veraar. On the ℓ^s-boundedness of a family of integral operators. *Revista Matemàtica Iberoamericana*, 32(4):1277–1294, 2016.

(2) C. Gallarati and M.C. Veraar. Maximal regularity for non-autonomous equations with measurable dependence on time. Online first in *Potential Analysis*, 2016.

(3) C. Gallarati and M.C. Veraar. Evolution families and maximal regularity for systems of parabolic equations. To appear in *Advances in Differential Equations*, 2016.

(4) H. Dong and C. Gallarati. Higher order elliptic and parabolic equations with VMO assumptions and general boundary conditions. Submitted. See arxiv preprint server, https://arxiv.org/abs/1702.03254.

(5) C. Gallarati and M.C. Veraar. Maximal L^p-regularity for parabolic equations with measurable dependence on time and general boundary conditions. In preparation, 2017.

(6) H. Dong and C. Gallarati. Higher order parabolic equations with VMO assumptions and general boundary conditions with variable leading coefficients. In preparation, 2017.