CHAPTER 1

ORTHOGONAL COORDINATES FOR THE ANALYSIS OF LONG GRAVITY WAVES NEAR ISLANDS[1]

R. O. Reid
Professor of Oceanography
Department of Oceanography and Department of Civil Engineering
Texas A&M University, College Station, Texas

A. C. Vastano
Doctoral Candidate in Oceanography, Department of Oceanography
Texas A&M University, College Station, Texas

ABSTRACT

A method is presented for the evaluation of an orthogonal coordinate system of particular use in the study of the diffraction of long gravity waves near islands of irregular shape. The problem involves a conformal mapping of the island onto a coordinate plane in which the island contour is a unit circle. The mapping relation is evaluated by an iterative procedure which is reminiscent of a method introduced by Theodorsen, but has the advantage that it is applicable to an island contour in which the island contour relation $r(\theta)$, in polar coordinates, is multivalued for a given range of θ. This generality is achieved by employing a parametric representation of the island contour in which the arc length on the island contour enters as the parameter. Application is made for two specific cases, Wake Island and Kauai.

INTRODUCTION

In the numerical solution of diffraction of long waves near an island of irregular shape, it is convenient to employ a two-dimensional orthogonal coordinate system (ρ, β) in which the island contour at MSL is represented by a constant value of one of the coordinates (e.g., $\rho = \rho_o$), while preserving a simple polar system in the far field. Such a system facilitates the application of the boundary condition on the island and of the radiation condition in the far field. If the transformation is conformal, then the cononical form of the wave equation is preserved except for the introduction of a variable scale factor.

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A discrete mesh based on uniform increments of ρ, β can have the advantage of providing the greatest spatial resolution in the immediate vicinity of the island. This is pertinent to the finite difference approximation of the wave equation for the near field representation of waves scattered by the island.

In the simple case of a circular island, polar coordinates are an obvious choice. If x and y are cartesian coordinates with origin at the center of the circular island, then the relations between x, y and the coordinates ρ, β are simply

$$x = f(\rho) \sin \beta,$$

$$y = f(\rho) \cos \beta,$$

and

$$r \equiv \sqrt{x^2 + y^2} = f(\rho).$$

Clearly a constant value of ρ corresponds to the island boundary for this case.

For a less trivial case of an island of elliptical shape, with major axis oriented along the y-axis, the following coordinate transformation relations are convenient

$$x = c \sinh \alpha \cos \beta,$$

$$y = c \cosh \alpha \sin \beta,$$

where $\alpha = f(\rho)$ and c is a constant with the dimension of length. For constant value of $\alpha = \alpha_o$, the above relations are simply the parametric equations for an ellipse with minor and major semi-axes (a, b) given by

$$a = c \sinh \alpha_o,$$

$$b = c \cosh \alpha_o,$$

respectively. Note that $b^2 - a^2 = c^2$ which is constant. For very large α_o the eccentricity of the ellipse approaches zero. The isolines of β are hyperbolas which are everywhere normal to the elliptical isolines of α. Moreover for large α, the isolines of β are asymptotic to the radial lines $\theta = \beta$, θ being the angle measured clockwise from the x-axis.

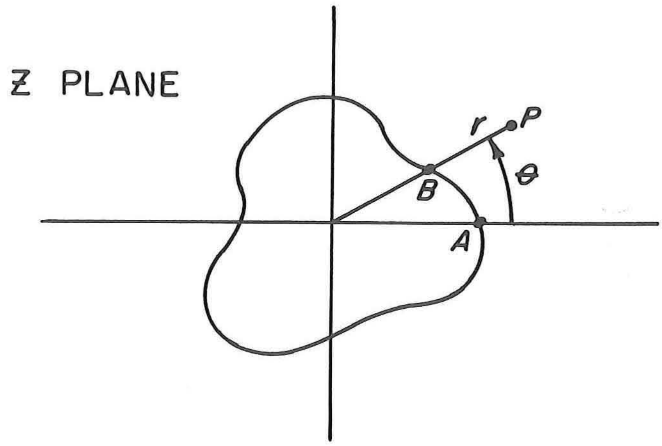

Figure 1 Conformal mapping planes

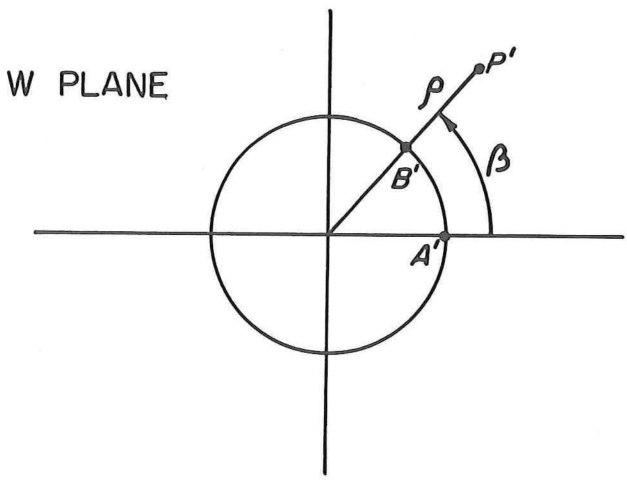

Figure 2 Conformal mapping planes

For the more general case of an island contour for which r is any periodic, single-valued function of θ, methods exist for the evaluation of suitable conformal transformations by which the island can be mapped into a circle in the ρ, β-plane. An excellent summary of such methods is given by Birkhoff et al., (1953). A particularly useful method for regions bounded by a nearly circular contour is that of Theodorsen (as described by Warschawski, 1945).

Theodorsen's method does not apply to regions for which the contour relation $r = f(\theta)$ is multivalued, since the method presumes that $f(\theta)$ can be represented in terms of a Fourier series. This difficulty can be circumvented by employing a parametric representation of the island contour, using a (non-dimensional) arc length (s) as a parameter. As expected, this generalization is achieved only at the expense of considerable complication in the computational process. Such a generalized scheme is described in this paper. It is shown to be convergent for several examples. Like most other methods, the conditions for which the method fails to converge cannot be set forth a priori. Qualitatively, the more nearly the island contour conforms to a circle the more rapid is the convergence process for the evaluation of the appropriate transformation relations.

THE CONFORMAL MAPPING PROBLEM

The construction of an orthogonal coordinate system for an island is essentially a conformal mapping problem in which the region of the z-plane, exterior to the island shoreline, is mapped one to one into the region exterior to a circle in the w-plane (Figures 1 and 2). Let z denote the complex position vector with its origin within the island (Figure 1) and let w represent the associated complex vector in that conformal image plane which maps the island contour as a circle of radius ρ_0 (Figure 2). We can regard a point P' in the w-plane as a unique image of point P in the z-plane or vice versa.

Now let $z = f(w)$ be the analytic mapping function representing the transformation between the planes. We choose this form rather than the inverse $w = g(z)$ since a knowledge of the explicit form of $f(w)$ expedites the plotting of the contours of ρ and β in the z-plane. In particular the curve $\rho = \rho_0$ maps as the island contour in the z-plane.

The polar coordinate representations in the two planes are

$$w = \rho e^{i\beta}$$

and

$$z = r e^{i\theta} . \qquad (1)$$

Alternatively we can write $z = x + iy$ where $x = r\cos\theta$ and $y = r\sin\theta$.

We require that the transformation $z = f(w)$ has the following properties:

(A) $\rho = \rho_o$ represents the island contour, a circle in the w-plane;

(B) $\theta \to (\beta + C_o)$, as ρ approaches infinity, where C_o is a constant;

(C) $r \to C_1 \rho$, as ρ approaches infinity, where C_1 is a constant.

As will be seen presently, one of the two constants, ρ_o or C_1 can be chosen arbitrarily; C_o is also arbitrary.

It will be convenient to deal with a non-dimensional or normalized radial coordinate R. An appropriate scale factor would be some mean radius of the island; specifically we take as the scale factor, the radius of a circular island with the same total perimeter. Thus if L is the total perimeter we define R, the non-dimensional coordinate, by

$$R = \frac{2\pi r}{L}, \tag{2}$$

where $L/2\pi$ is the effective mean radius. As a consequence,

$$\zeta \equiv \frac{2\pi z}{L} = R e^{i\theta}. \tag{3}$$

We further take ℓ as the arc length from some fixed reference point A on the island to any point B such that $\oint d\ell = L$. Then the non-dimensional arc length is

$$s = \frac{2\pi \ell}{L} \tag{4}$$

such that $\oint ds = 2\pi$.

A transform which possesses the desired behavior in the far field and is capable of satisfying the desired condition on the island is discussed by Lamb (p. 75, 1945). In complex notation

$$\ln \zeta = \ln w + C_o + \sum_{n=1}^{\infty} C_n w^{-n}, \tag{5}$$

where C_o and the C_n are complex constants, $A_n + i B_n$. Inserting the polar forms of ζ and w, the complex expression of the C_n and splitting into real and imaginary parts yields

$$\ln R = \ln \rho + A_o + \sum_{n=1}^{\infty} (A_n \cos n\beta + B_n \sin n\beta)\, \rho^{-n}, \tag{6a}$$

and

$$\theta = \beta + B_o + \sum_{n=1}^{\infty} (B_n \cos n\beta - A_n \sin n\beta)\, \rho^{-n}. \tag{6b}$$

Note that for large ρ, conditions (B) and (C) are satisfied. Now if the A_n, B_n are known, the plotting of contours of ρ and β in the ζ plane can be readily carried out, i.e., the values of R, θ, for selected ρ, β can be found explicitly.

The above relations are for the exterior mapping problem; for the case of a basin one can map the interior of the basin onto the interior of a circle in the w-plane using a relation similar to (5) but with positive exponents (w^n).

Consider for a moment an island for which R vs. θ is multivalued. Clearly it would be impossible to represent R as a function of θ by a Fourier series in the vicinity of any shoreline fold-back. However, the parametric relations $\theta = f_1(s)$ and $R = f_2(s)$ would be single valued for their entire range and therefore can be so represented. To incorporate this feature we will let

$$\begin{aligned} \ln R &= F(s) \\ \theta &= G(s) \end{aligned} \tag{7}$$

be the specified parametric relations for the island contour. In view of the definition of s it follows that

$$(ds)^2 = (R d\theta)^2 + (dR)^2 \tag{8}$$

on the island contour. Hence

$$e^{F(s)} \left[[F'(s)]^2 + [G'(s)]^2 \right]^{1/2} = 1, \tag{9}$$

which represents a basic constraint on the function F(s) and G(s) for all s regardless of the shape of the island contour.

On the island contour the first of relations (6) involves the constant $A_o + \ln \rho_o$. Note that one of the terms, A_o or ρ_o, is arbitrary and it will be convenient to take $\rho_o = 1$. Thus, the relations (6) on the island contour reduce to

$$\ln R = F(s) = A_o + \sum_{n=1}^{\infty} (A_n \cos n\beta + B_n \sin n\beta), \tag{10a}$$

$$\theta = G(s) = \beta + B_o + \sum_{n=1}^{\infty} (B_n \cos n\beta - A_n \sin n\beta), \tag{10b}$$

where it is implied that s is a single valued function of β. If one selects $B_o = 0$, then $\theta \to \beta$ as $\rho \to \infty$; an alternative is to select B_o such that $\theta = 0$ for $\beta = 0$ on the island contour. The latter choice is made in the examples presented later.

The problem is to determine the function $s(\beta)$ such that the equations (10) are compatible; this requires that

$$A_m = \frac{1}{\pi} \int_o^{2\pi} F(s) \cos m\beta \, d\beta = \frac{-1}{\pi} \int_o^{2\pi} (G(s) - \beta) \sin m\beta \, d\beta \tag{11a}$$

$$B_m = \frac{1}{\pi} \int_o^{2\pi} [G(s) - \beta] \cos m\beta \, d\beta = \frac{1}{\pi} \int_o^{2\pi} F(s) \sin m\beta \, d\beta \tag{11b}$$

for all $m \geq 1$. Moreover $s(\beta)$ must be compatible with (8). Using (10a, b) we find

$$\frac{ds}{d\beta} = e^{A_o} \left[\left[1 - \sum_{n=1}^{\infty} n (A_n \cos n\beta + B_n \sin s\beta) \right]^2 + \left[\sum_{n=1}^{\infty} n (B_n \cos n\beta - A_n \sin n\beta) \right]^2 \right]^{1/2} \times \exp \left[\sum_{n=1}^{\infty} (A_n \cos n\beta + B_n \sin n\beta) \right]. \tag{12}$$

If we take $s(0) = 0$, then $s(2\pi) = 2\pi$ which imposes a condition on A_o independent of the relation

$$A_o = \frac{1}{2\pi} \int_o^{2\pi} F(s)\, d\beta \qquad (13)$$

implied by (10a). We have assumed that the termwise differentiation of the series (10a, b) yields convergent series in relation (12).

Finally if one employs the condition $\theta = 0$ for $\beta = 0$, then we obtain the further constraint on $s(\beta)$:

$$B_o = - \sum_{n=1}^{\infty} B_n \cos n\beta - A_n \sin n\beta = \frac{1}{2\pi} \int_o^{2\pi} (G(s) - \beta)\, d\beta \,. \qquad (14)$$

ITERITIVE EVALUATION OF A_n, B_n and $s(\beta)$

Clearly one must employ truncated versions of the series (5), (6), (10), (12) in practical computation. Let N designate the maximum harmonic in the series. If $F(s)$ and $G(s)$ are represented in discrete steps then it is logical to demand at least 2N values of F and G at nominally uniform intervals of s as specified information.

Let A_n^k, B_n^k designate the k^{th} approximation for the coefficients A_n, B_n and let $s^k(\beta)$ be the associated k^{th} approximation of $s(\beta)$ determined from (12) with A_n, B_n replaced by A_n^k, B_n^k for $n = 1, 2, \ldots N$. In the evaluation of $s^k(\beta)$ we choose $s^k(0) = 0$ and take e^{Ao} such that the mean value of $ds^k / d\beta$ over the range 2π is exactly unity. Thus

$$s^k(\beta) = 2\pi \, \frac{\Gamma^k(\beta)}{\Gamma^k(2\pi)} \qquad (15)$$

where

$$\Gamma^k(\beta) = \int_0^\beta \left[\left[1 - \sum_{n=1}^{N} n\,(A_n^k \cos n\beta + B_n^k \sin n\beta)\right]^2 + \left[\sum_{n=1}^{N} n\,(B_n^k \cos n\beta - A_n^k \sin n\beta)\right]^2\right]^{1/2} \times \exp\left[\sum_{n=1}^{N}(A_n^k \cos n\beta + B_n^k \sin n\beta\right] d\beta. \tag{16}$$

The value of A_o required for normalization of $s(\beta)$ at the k^{th} iterative is

$$A_o^k = \ln\left[\,2\pi/\Gamma^k(\beta)\,\right]. \tag{17}$$

We will choose the A_n^k, B_n^k for a given $s^{k-1}(\beta)$ such that the following error function is a minimum:

$$E_N^k = \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{n=1}^{N}(A_n^k \cos n\beta + B_n^k \sin n\beta) - F(s^{k-1})\right]^2 d\beta + \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{n=1}^{N}(B_n^k \cos n\beta - A_n^k \sin n\beta) - \left[G(s^{k-1}) - \beta\right]\right]^2 d\beta. \tag{18}$$

This will be a minimum provided that

$$A_n^k = \frac{1}{2\pi}\int_0^{2\pi}\left[F(s^{k-1}) \cos n\beta - \left[G(s^{k-1}) - \beta\right] \sin n\beta\right] d\beta \tag{19a}$$

and

$$B_n^k = \frac{1}{2\pi}\int_0^{2\pi}\left[F(s^{k-1}) \sin n\beta + \left[G(s^{k-1}) - \beta\right] \cos n\beta\right] d\beta \tag{19b}$$

for $n = 0, 1, \ldots, N$. Moreover, the value of E_N^k will be

$$E_N^k = \frac{1}{2\pi}\int_0^{2\pi}\left[\left[F(s^{k-1})\right]^2 + \left[G(s^{k-1}) - \beta\right]^2\right] d\beta - \sum_{n=1}^{N}\left[\left[A_n^k\right]^2 + \left[B_n^k\right]^2\right]. \tag{20}$$

The procedure will be convergent provided that E_N^k approaches zero as k and N grow large. At each iterative level a decision must be made as to the upper limit N. This is based on the degree to which the variance of F and (G - β) is represented by the Fourier coefficients. Once the A_n^k and B_n^k are determined up to the desired N, the function $s^k(\beta)$ is evaluated by (15) and (16). The coefficients for the k + 1 level are then calculated from (19a, b) and the process repeats or ends with a satisfactory solution as determined by a suitably small value of E_N^k.

The initial estimate of s(β) is taken as β, i.e.,

$$s^o(\beta) = \beta \tag{21}$$

from which A_n^1 and B_n^1 are obtained. Note that (21) is consistent with the conditions $s(0) = 0$, $s(2\pi) = 2\pi$. In the evaluation of the integrals over β, numerical quadrature using a simple trapazoidal rule is employed and values of F and G at the appropriate discrete steps of β are determined by quadratic interpolation using the (k - 1) estimate of s(β).

TEST OF THE ITERATIVE PROCEDURE

Two tests of the procedure have been carried out for cases in which the transformation is known independently. The first case treated was a "delta" shaped island for which

$$\begin{aligned} \ln r &= 0.30 \cos 3\beta - 0.03 \cos 6\beta \\ \theta &= \beta - 0.30 \sin 3\beta + 0.03 \sin 6\beta. \end{aligned} \tag{22}$$

In this case $R(s)$ and $\theta(s)$ could be evaluated at discrete s as input using (22). Because of the symmetry, computation could be carried out for the range $0 \leqslant \beta \leqslant \pi/3$. Computations for this case were carried out on a desk computer from r and θ specified at 12 discrete β in the above range. The successive calculations of A_n^k up to four iterations are presented in table 1. The B_n values are identically zero and were not calculated. In this case the value of E_N^k decreased by a factor of about four per iteration. The number of harmonics chosen in the successive representation of $\ln R$ and θ was 12 (only A_o, A_3, A_6 . . . have non-zero value for this case).

Table 1

Calculations for Delta Island

	k = 1	2	3	4	Exact
A_o*	-.082	-.086	-.0925	-.0928	-.0924
A_3	.218	.279	.2927	.2971	.3000
A_6	.010	-.012	-.0232	-.0280	-.0300
A_9	.005	.0011	.0004	.0001	0
A_{12}	.0006	-.0010	-.0002	.0001	0

* A_o values given above and in Table 2 are an average of the values calculated by (13) and (17).

As a second test for which the transformation is known exactly, an elliptical island with a ratio of major axis to minor axis of two was selected. The exact mapping relation for this test is

$$Z = c \sinh (\ln w \sqrt{3}) \qquad (23)$$

where c is an arbitrary scale factor. In this case the minor and major, semi axes of the ellipse $\rho = 1$ are

$$a = \frac{c}{2} \; (3^{1/2} - 3^{-1/2}) ,$$

$$b = \frac{c}{2} \; (3^{1/2} + 3^{-1/2}) ,$$

which corresponds to $b/a = 2$.

If (23) is expanded in a power series we obtain for $\rho = 1$:

$$\ln r = \ln \frac{c\sqrt{3}}{2} - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{3}\right)^m \cos 2\, m\beta \tag{24a}$$

$$\theta = \beta - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{3}\right)^m \sin 2\, m\beta . \tag{24b}$$

Thus for $n > 0$

$$A_n = \begin{cases} 0 \text{ for n odd} \\ \frac{1}{n} \left(\frac{1}{3}\right)^n \text{ for n even} \end{cases}$$

and (25)

$$B_n = 0.$$

The value of L in this case can be evaluated in terms of an elliptic integral of the second kind

$$L = 4\, b\, E\, (c/b), \tag{26}$$

where $c/b = \sqrt{3/4}$ for this case. It follows from (24a) that

$$A_o = \ln \frac{\pi c}{L} \sqrt{3} = \ln [\; (c/b)^2 \pi/2\, E\, (c/b) \;] . \tag{27}$$

The numerical calculations were carried out on a digital computer from R and θ specified at 60 discrete points at uniform increments of β. The resulting values for A_n after five iterations are given in table 2 in comparison with the exact values as calculated from (25) and (27) up to eight harmonics.

Table 2

Calculations for the elliptical island b/a = 2 after 5 iterations

	Numerical procedure	Exact Values
A_o	-.02798	-.02795
A_2	-.33320	-.33333
A_4	-.05491	-.05555
A_6	-.01221	-.01235
A_8	-.00286	-.00308

All odd A_n and all B_n are zero for this case.

CALCULATIONS FOR REAL ISLANDS

The computer program employing the foregoing method has been applied to two actual islands in the Pacific: Wake Island and Kauai.

In practice the island shoreline is placed on cards in (x,y) form for data input. Both island cases utilized slightly more than three hundred such points (308 for Kauai and 330 for Wake). The values are at nearly uniform spacing of arc length. Associated values of s needed for the interpolational procedure are calculated by a finite difference version of (8) with appropriate normalization so as to yield a total range of 2π once around the island.

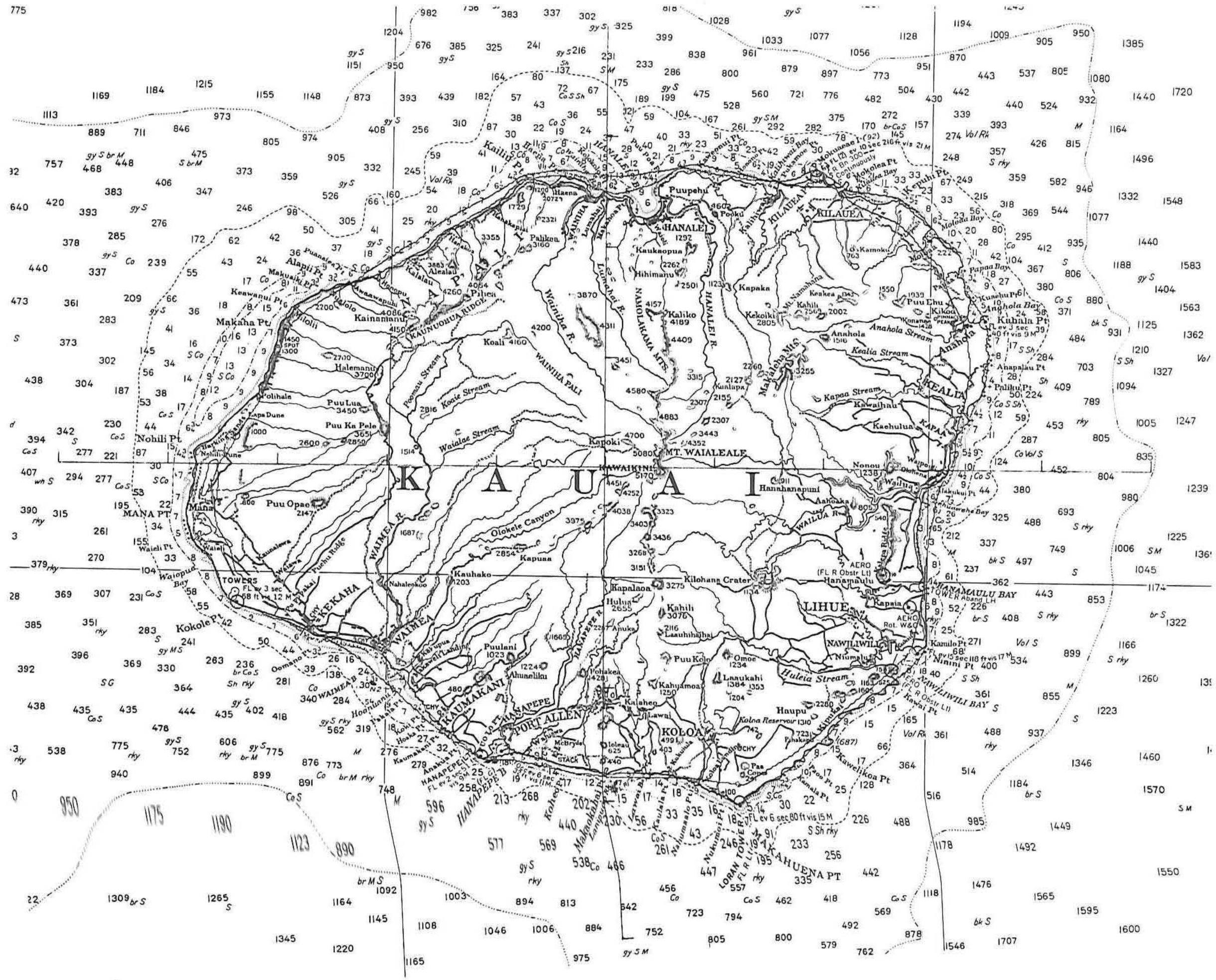

Figure 3 Kauai, Hawaii

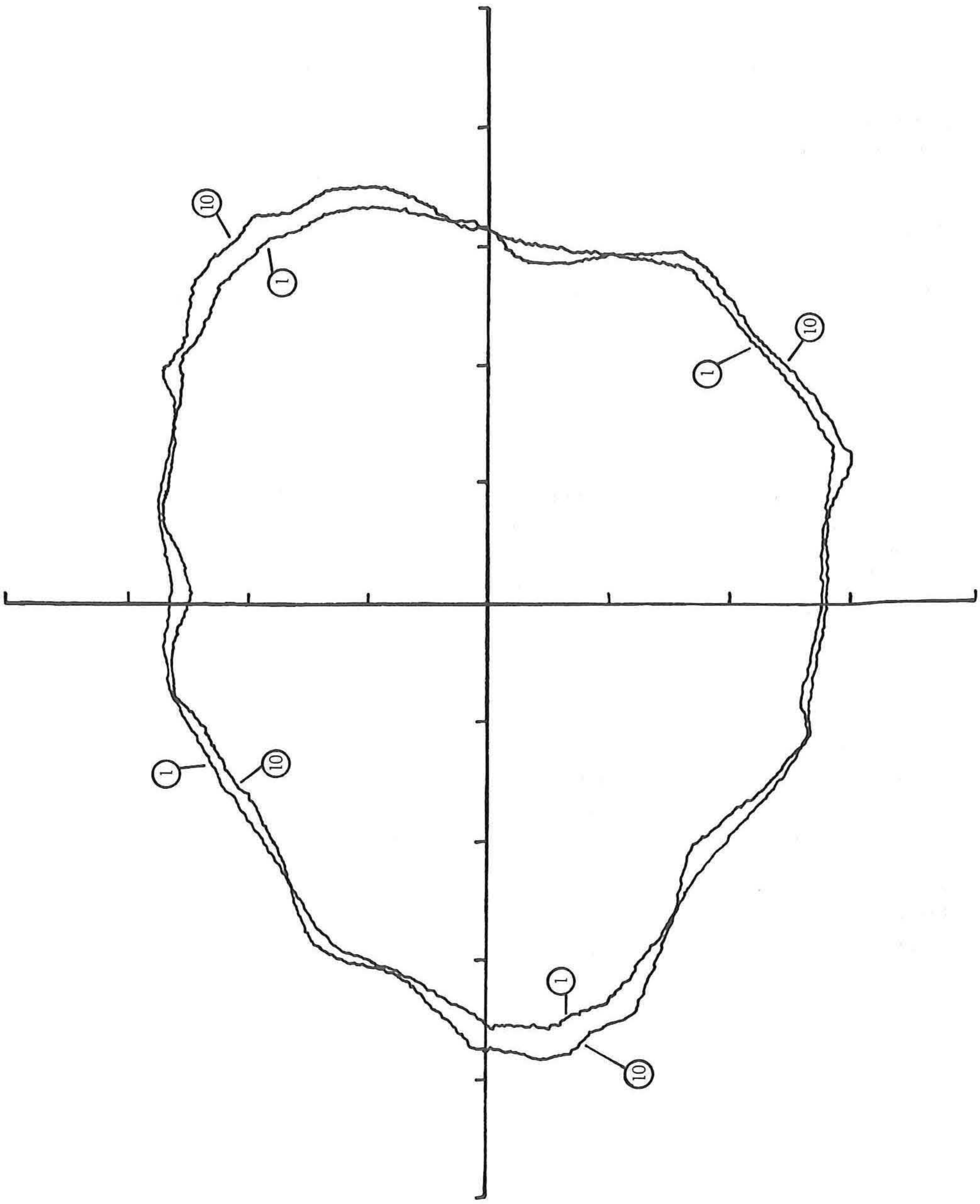

Figure 4 First, (1), and tenth, (10), computer approximations to the perimeter of Kauai.

Figure 3 shows Kauai Island as taken from the Coast and Geodetic Survey Chart 4117. A total of ten iterations were carried out to assure convergence. The results of the first and the tenth iterations are shown in figure 4. The tenth iteration essentially reproduces the island contour given in figure 3. In order to do this a total of 129 harmonics were selected by the program in the fitting process. This number of harmonics is less than half the number of points,which would be a logical upper limit.

Figure 5 shows the Wake Island area as taken from U. S. Navy Oceanographic Chart H. O. 6034. In this case the outer reef line was employed as the effective island contour. The results for the first and the tenth iterations are shown in figure 6. The total number of harmonics chosen by the program for the tenth and final iteration was 56. It is of interest to note that this is much less than the number of harmonics employed in the first iteration.

For each island, the machine time using an IBM 7094 was less than three minutes for the ten iterations.

The grid in figure 7 reproduces the orthogonal coordinate system (ρ,β) in which the reef line contour of Wake Island is represented by the value $\rho = 1$. Contours of ρ at intervals of 0.5 and β at intervals of 20° are shown. Note the desirable characteristic of the grid approaching a far field polar coordinate system within a few island diameters.

CONCLUDING REMARKS

The examples given above show that the computational procedure for the evaluation of mapping relations is successful for fairly irregular shaped islands. None of the above examples contain shoreline foldbacks, were r is multivalued for given θ. Experimentation with such systems indicates that moderate indentations will be tolerated by this method. Accurate reproduction of extreme features (a U shaped island for example) would require considerable increase in the total number of points around the island in the evaluation of the Fourier coefficients by numerical quadrature. Moreover a commensurate increase in the number of harmonics is also required. The difficulty in such cases is that a very small range of β is represented by the indentation region.

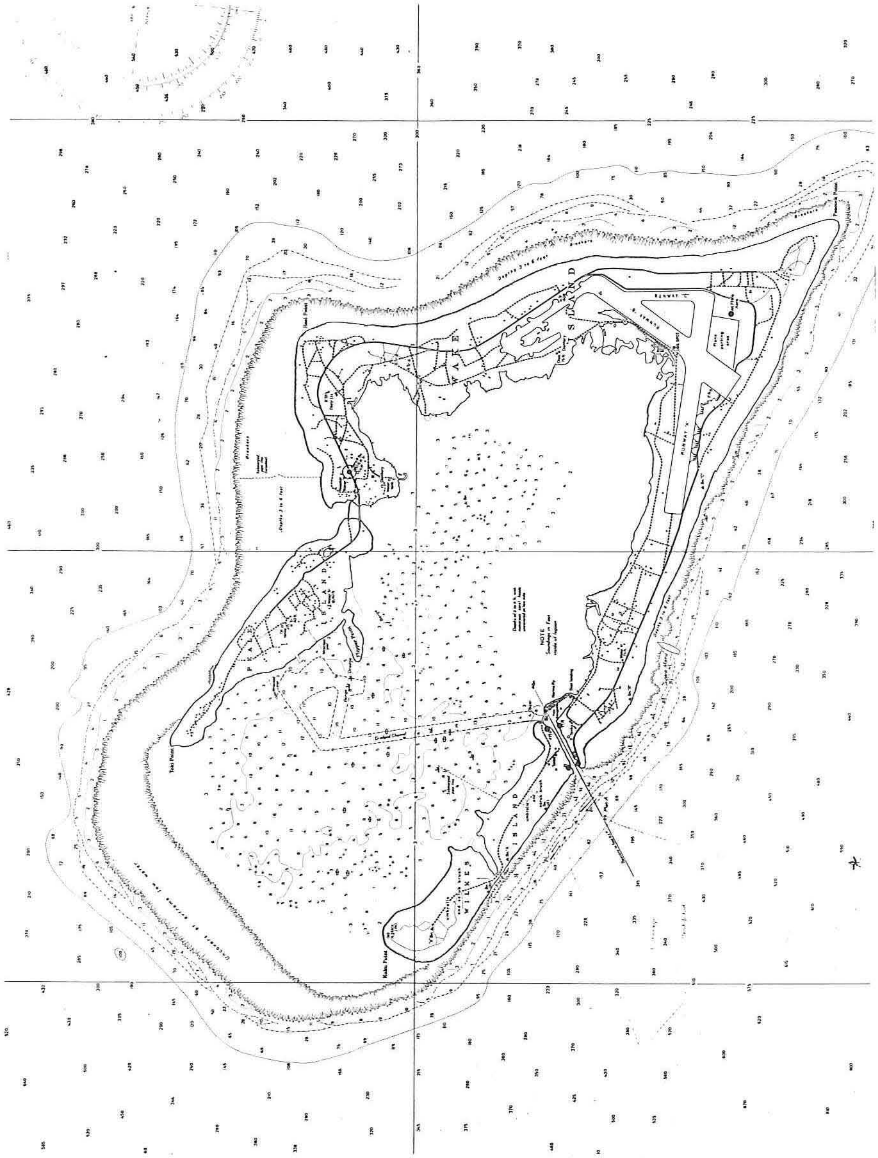

Figure 5 Wake Island

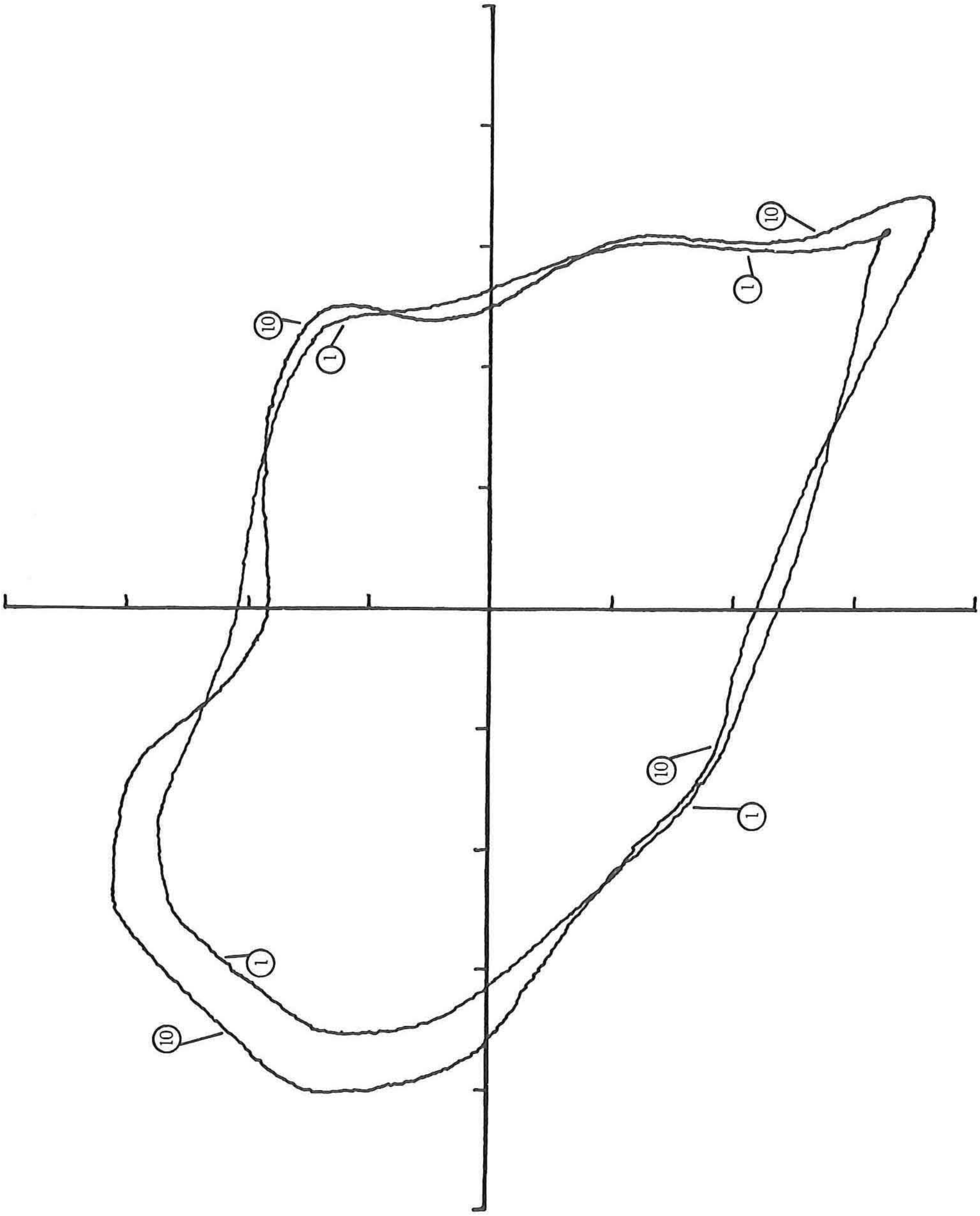

Figure 6 First, (1), and tenth, (10), computer approximations to the reef line contour of Wake Island.

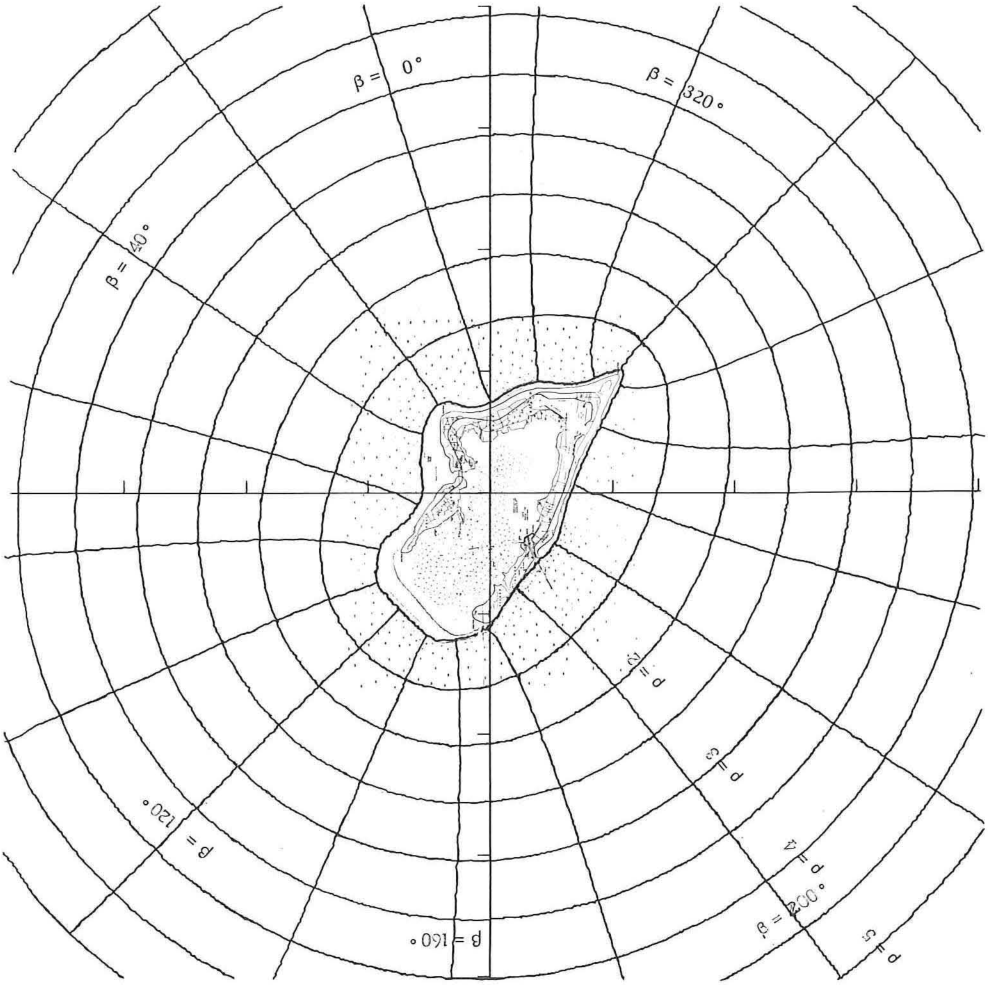

Figure 7 Special orthogonal coordinate system for Wake Island.

REFERENCES

Birkhoff, G., Young, D. M., and Zarantonello, E. H. (1953). Numerical method in conformal mapping. Proc. of Symposia in Applied Mathematics, vol. IV, Fluid Dynamics, McGraw-Hill Book Co., Inc.

Warschawski, S. E. (1945). On Theodorsen's method of conformal mapping of nearly circular regions. Quart. Appl. Math. vol. 3, pp. 12-28.

Lamb, Horace (1945). Hydrodynamics, Dover. Pub.