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Reflections of waves induced by a nonlinear spring at the boundary

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“Reflections of waves induced by a nonlinear spring at the boundary”

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Preface

This is a BSc thesis for Applied Mathematics at the TU Delft. I would like to thank my supervisor Dr.ir. W.T. van Horssen for the guidance throughout this research.

Contents

Frequently used Notation

u	displacement of string
x	place indicator
t	time
T	tension in string
k	linear spring stiffness
k_2	nonlinear spring stiffness
$f(x)$	initial displacement of string
$g(x)$	intital velocity of string
ϕ	$f(-t) + f(t) + \int_{-t}^{t} g(s)ds$
$A(t)$	$2(f'(t) + g(t))$
$\hat{u}$, $\hat{f}$, $\hat{g}$, $\hat{\phi}$, $\hat{A}$	ϵu, ϵf, ϵg, $\epsilon \phi$, ϵA, for $\epsilon > 0$ small
MSP	Multiple Scales Perturbations
RK4	Fourth Order Runge Kutta
$\hat{\phi}_a$	approximated $\hat{\phi}$ using MSP method
$\hat{\phi}_n$	approximated $\hat{\phi}$ using RK4 method

Summary

In this thesis the nonlinear spring system is considered. This system contains a semi-infinite string that is modelled by the wave equation with a pair of initial conditions and a nonlinear boundary condition. The goal of this thesis is to find a good approximation to the solution of this system. Furthermore we study the behaviour of the string by plotting the reflected waves.
In chapter 2 the nonlinear spring system is described more thoroughly. We also introduce D'Alembert's formula. This is a general solution to the wave equation with the corresponding initial conditions, which is used throughout this thesis for intermediate calculations.
First the nonlinear term is neglected. This results in the linear spring system, which is studied in chapter 3. Various levels of spring stiffness are considered for which the solution is derived using D'Alembert's formula. For a chosen set of initial conditions, the reflected waves are plotted and studied.
In chapter 4 the nonlinear system is considered. D'Alembert's formula is applied, which results in a simpler differential equation.
To solve this equation, an analytical approach is taken using the Multiple Scales Perturbations method (MSP) in chapter 5. For this method it is necessary to consider only small displacements. This assumption again leads to a new differential equation. This is the differential equation which is continued to use in the rest of this thesis.
The MSP method is applied to three different pairs of initial conditions. For these three examples the approximations are plotted to study the reflected waves at multiple moments in time. A proof for the accuracy of an approximation using the MSP method is given as well. This proof holds for a small parameter value, but in practice we do not know what value is small enough. The accuracy is to be estimated by comparing it to approximations of another method, which is done in chapter 6
Here, the Fourth Order Runge Kutta method (RK4) is applied to the same three examples of initial conditions as in the previous chapter. Since this method is not unconditionally stable, the stability has to be checked for all plots. A few examples on how this is done are also discussed in this chapter.
The findings of the two methods are compared to each other. By comparing this, we draw conclusions on the accuracy of the approximations. The final conclusions are drawn in chapter 7.

1 Introduction

1.1 Problem description

In this thesis we consider a vibrating string described by the well-known wave equation. This is a second-order partial differential equation. We complement this equation with initial conditions and boundary conditions. Sometimes this system of equations can be solved exactly, but there are many other cases where this is impossible. Both cases will be considered in this thesis.
The boundary conditions to this equation will be nonlinear. It describes a spring that is attached at the beginning of the string and has a nonlinear behaviour.
The goal of this thesis is to find accurate approximations of the solution of this system. We use an analytical method and a numerical method. We check the accuracy of these methods by comparing the approximations to each other.
Furthermore we will study the behaviour of the string for several initial conditions. We analyse this by plotting the reflected waves.

1.2 Thesis structure

In chapter 2 the above mentioned system of equations will be explained more thoroughly.
Before we dive into the nonlinear system, we will start with a spring that behaves in a linear way. In chapter 3 we find the exact solution of this linear system and analyse the reflected waves.
In chapter 4 we move on to the nonlinear behaviour of the spring. Here we derive a new differential equation for the nonlinear system, which we will use in the rest of the thesis.
From this ordinary differential equation we want to approximate the solution. We will do this with two different methods.
The first method is the Multiple Scales Perturbations method (MSP) in chapter 5. With this method we will analytically find an approximation of the solution. We apply it to three different pairs of initial conditions and study the reflections. At the end of the chapter a proof for the accuracy of this method is given.
We will also use the Runge Kutta Fourth Order method (RK4) in chapter 6 to numerically approximate the solution of the nonlinear system. We apply this method to the same pairs of initial conditions and again study the reflections. We will compare the results from the RK4 method to the MSP method.
Finally, we draw conclusions in chapter 7 based on the obtained results.

2 Equations of Motion

We consider a second-order partial differential equation known as the wave equation. This equation describes the motion of a semi-infinite string, placed from $x = 0$ to infinity. $u(x,t)$ is the vertical displacement of the string, t is the time. To the wave equation we add a pair of initial conditions and a boundary condition. This results in the following system:

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x, \\ T u_x(0,t) = k u(0,t) + k_2 u^2(0,t), \quad t \geq 0. \end{aligned}$$

where $f(x)$ is a function that describes the initial displacement of the string, and $g(x)$ describes the initial velocity. Without loss of generality it is assumed throughout this thesis that the wave speed is given by $c^2 = 1$.
In this thesis we consider a nonlinear boundary condition. We attach a nonlinear spring to the string at $x = 0$. We can remove the nonlinear behaviour by taking $k_2 = 0$, which results in a linear spring system with the spring constant k. T is a value for the tension in the spring. This system is portrayed in figure 1.
Throughout this research we will make use of D'Alembert's formula. This is a well-known formula that describes the general solution of the wave equation with the corresponding initial velocity and displacement. The expression for the displacement is

$$u(x,t) = \frac{1}{2} f(x+t) + \frac{1}{2} f(x-t) + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds,$$

given the arguments of f and g are greater than zero.
Before we examine the wave reflections due to the nonlinear boundary conditions, we remove the nonlinear term by taking $k_2 = 0$ and look into the linear system.

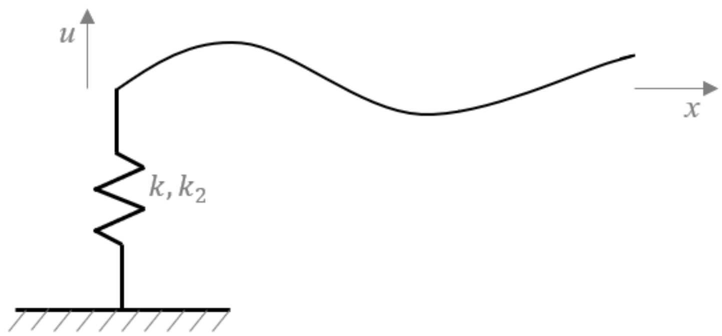


Figure 1: The nonlinear spring system

3 The Linear Spring System

3.1 Introduction

First we want to know how the string behaves when the attached spring acts linearly. This is when $k_2 = 0$. This linear system looks as follows:

$$\begin{aligned} u_{tt} - u_{xx} = 0, \qquad 0 < x, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x, \\ Tu_x(0,t) = ku(0,t). \end{aligned}$$

Now we distinguish 2 special cases before we solve the linear spring system.
The first is when we let k go to infinity. When we rewrite the boundary condition like this:

$$\frac{T}{k}u_x(0,t) = u(0,t),$$

we see that for $k \to \infty$ it follows that $u(0,t) \to 0$. We can interpret this as the string being fixed at $x = 0$.
For the second case we take $k = 0$, which basically means that there is no spring attached. Now we rewrite the boundary condition like this:

$$\frac{k}{T}u(0,t) = u_x(0,t).$$

When $k = 0$ it follows that $u_x(0,t) = 0$. This means that at $x = 0$ the string moves freely in the u-direction.

3.2 String fixed at $x = 0$

We consider the linear spring system with the string being fixed at $x = 0$:

$$\begin{aligned} u_{tt} - u_{xx} = 0, \qquad 0 < x, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x, \\ u(0,t) = 0. \end{aligned}$$

Now we are going to apply D'Alembert's formula to the boundary condition $u(0,t) = 0$:

$$u(0,t) = \frac{1}{2}f(t) + \frac{1}{2}f(-t) + \frac{1}{2}\int_{-t}^{t} g(s)ds = 0.$$

Remember that f and g are only defined for positive arguments. So we will look for an expression of f and g for which they are defined for negative arguments. Since f is the initial displacement and g the initial velocity, f and g are independent functions. We determine g by taking $f = 0\ \forall t$ and vice versa.

$$\begin{aligned} f = 0: \qquad & \int_{-t}^{t} g(s)ds = 0, \\ g = 0: \qquad & f(t) = -f(-t). \end{aligned}$$

It follows that f and g are both odd functions. The solution to the problem is:

$$u(x,t) = \begin{cases} \frac{1}{2}f(x+t) + \frac{1}{2}f(x-t) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds & x - t > 0, \\ \frac{1}{2}f(x+t) - \frac{1}{2}f(t-x) + \frac{1}{2}\int_{t-x}^{x+t} g(s)ds & x - t < 0. \end{cases}$$

3.3 String freely moving at $x = 0$

Now we consider the linear spring system which has a free movement in the u-direction:

$$\begin{aligned} u_{tt} - u_{xx} = 0, \qquad 0 < x, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x, \\ u_x(0,t) = 0. \end{aligned}$$

Just like we did in the previous case, we apply D'Alembert's formula to the boundary condition:

$$u_x(0,t) = \frac{1}{2}f'(t) + \frac{1}{2}f'(-t) + \frac{1}{2}g(t) - \frac{1}{2}g(-t) = 0.$$

Again f and g are only defined for positive arguments. Considering f and g as independent functions, we get:

$$\begin{aligned} f = 0: \qquad & g(t) = g(-t), \\ g = 0: \qquad & -f'(t) = f'(-t). \end{aligned}$$

We integrate:

$$\begin{aligned} \int_0^t -f'(s)ds = \int_0^t f'(-s)ds \iff & -f(t) + f(0) = -f(-t) + f(0), \\ \iff & f(t) = f(-t). \end{aligned}$$

We see that now f and g are both even functions. Hereby the solution to the problem is:

$$u(x,t) = \begin{cases} \frac{1}{2}f(x+t) + \frac{1}{2}f(x-t) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds & x - t > 0, \\ \frac{1}{2}f(x+t) + \frac{1}{2}f(t-x) + \frac{1}{2}\int_0^{x+t} g(s)ds + \frac{1}{2}\int_0^{t-x} g(s)ds & x - t < 0. \end{cases}$$

3.4 Solving the linear spring system

We have seen that for $k \to \infty$ the linear system is described by odd functions, whereas for $k = 0$ it is described by even functions. Now we want to know what the solution looks like if we have $0 < k < \infty$.

$$\begin{aligned} u_{tt} - u_{xx} = 0, \qquad 0 < x, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x, \\ Tu_x(0,t) = ku(0,t). \end{aligned}$$

By substituting D'Alembert's formula into the boundary condition, we get

$$T(\frac{1}{2}f'(t) + \frac{1}{2}f'(-t) + \frac{1}{2}g(t) - \frac{1}{2}g(-t)) = k(\frac{1}{2}f(t) + \frac{1}{2}f(-t) + \frac{1}{2}\int_{-t}^{t} g(s)ds).$$

Again we make use of the independence of f and g for finding the general solution for negative arguments.
First we take $g = 0$:

$$\begin{aligned} T(\frac{1}{2}f'(t) + \frac{1}{2}f'(-t)) = k(\frac{1}{2}f(t) + \frac{1}{2}f(-t)), \\ \iff f'(-t) - \frac{k}{T}f(-t) = -f'(t) + \frac{k}{T}f(t). \end{aligned}$$

Define $y(t) = f(-t)$.

$$y'(t) + \frac{k}{T}y(t) = f'(t) - \frac{k}{T}f(t).$$

This is a first order differential equation, where the right-hand side is a known function. We solve this using the Integrating Factor method. The differential equation is of the form:

$$y'(t) + p(t)y(t) = q(t), \qquad \text{with} \quad p(t) = \frac{k}{T} \quad \text{and} \quad q(t) = f'(t) - \frac{k}{T}f(t).$$

Then the following equation holds

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)q(t), \tag{1}$$

where the integrating factor is given by:

$$\mu(t) = e^{\int_0^t p(s)ds} = e^{\frac{k}{T}t}.$$

We integrate equation (1) on both sides:

$$\begin{aligned}
\mu(t)y(t) - \mu(0)y(0) &= \int_0^t \mu(s)q(s)ds, \\
\iff y(t) &= y(0)e^{-\frac{k}{T}t} + e^{-\frac{k}{T}t}\int_0^t e^{\frac{k}{T}s}q(s)ds, \\
\iff f(-t) &= f(0)e^{-\frac{k}{T}t} + e^{-\frac{k}{T}t}\int_0^t e^{\frac{k}{T}s}(f'(s) - \frac{k}{T}f(s))ds.
\end{aligned}$$

We can split this into two integrals and apply integration by parts to obtain the expression for f for negative arguments:

$$f(-t) = f(t) - 2\frac{k}{T}e^{-\frac{k}{T}t}\int_0^t e^{\frac{k}{T}s}f(s)ds.$$

We do the same for $f = 0$ to obtain the expression for g.

$$\begin{aligned}
T(\frac{1}{2}g(t) - \frac{1}{2}g(-t)) &= k(\frac{1}{2}\int_{-t}^t g(s)ds), \\
\iff g'(-t) - \frac{k}{T}g(-t) &= -g'(t) + \frac{k}{T}g(t).
\end{aligned}$$

Again we substitute $y(t) = g(-t)$ to get a first order differential equation:

$$y'(t) + \frac{k}{T}y(t) = g'(t) - \frac{k}{T}g(t).$$

The right-hand side contains only known functions. We again solve this equation with the Integrating Factor method and obtain the solution:

$$g(-t) = g(t) - 2\frac{k}{T}e^{-\frac{k}{T}t}\int_0^t e^{\frac{k}{T}s}g(s)ds.$$

Using the expressions we found for $f(-t)$ and $g(-t)$, we get the general solution to the problem:

$$u(x,t) = \begin{cases} \frac{1}{2}f(x+t) + \frac{1}{2}f(x-t) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds & x-t > 0, \\ \frac{1}{2}f(x+t) + \frac{1}{2}f(t-x) - \frac{k}{T}e^{\frac{k}{T}(x-t)}\int_0^{t-x} e^{\frac{k}{T}s}f(s)ds + & \\ \frac{1}{2}\int_0^{x+t} g(s)ds - \frac{1}{2}\int_0^{t-x} g(s)ds + e^{\frac{k}{T}(x-t)}\int_0^{t-x} e^{\frac{k}{T}s}g(s)ds & x-t < 0. \end{cases}$$

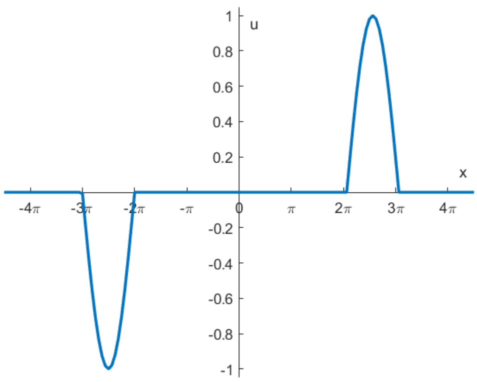
3.5 Visualizing the reflected waves

Until now we worked with arbitrary initial conditions. To get an idea of how a wave is reflected, we choose functions for $f(x)$ and $g(x)$. When $g(x) = f'(x)$, the entire wave moves to the left, so to $x = 0$. We take the following functions:

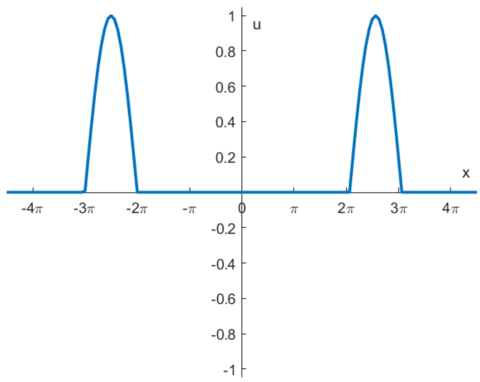
$$f(x) = \begin{cases} 0 & 0 < x < 2\pi, \\ sin(x) & 2\pi \leq x \leq 3\pi, \\ 0 & 3\pi < x, \end{cases}$$

$$g(x) = \begin{cases} 0 & 0 < x < 2\pi, \\ cos(x) & 2\pi \leq x \leq 3\pi, \\ 0 & 3\pi < x. \end{cases}$$

Figure 2a shows the initial wave on the positive x-axis and the reflected wave for $k \to \infty$ on the negative x-axis. Earlier we found that for $k \to \infty$, f and g are extended as odd functions. This is in line with the odd reflection of the wave in the plot.
In figure 2b the wave is reflected for the case $k = 0$. For this case f and g were extended as even functions. We see this again in the even reflection of the wave. Figure 3 shows the reflected wave



(a) $k \to \infty$



(b) $k = 0$

Figure 2: Reflected waves for cases $k \to \infty$ and $k = 0$

for the linear system for multiple values of $\frac{k}{T}$. For the tension in the string we take $T = 8000$, and we take different levels of stiffness for the linear spring. $\frac{k}{T} = 0$ shows an even reflection. This corresponds with the findings in the case $k = 0$. $\frac{k}{T} = 5$ describes a really stiff spring. This reflection is close to matching the case $k \to \infty$, where the wave has an odd reflection. In figure 4a it is shown more clearly that indeed for a larger k the wave is reflected oddly.
In figure 3 we see that for $\frac{k}{T} = \frac{1}{4}$, the amplitude is smaller on both sides of the x-axis compared to the initial wave. But when $\frac{k}{T} = 1$ the amplitude underneath the x-axis gets even bigger then the amplitude of the initial wave. We look at figure 4b. Here we see that when $\frac{k}{T} > \frac{1}{2}$, the amplitude remains big below the x-axis. But for values $0 < \frac{k}{T} < \frac{1}{2}$, the absolute displacement is smaller than the initial displacement.

3.6 Conclusions

In this chapter we studied the linear spring system. We expressed the displacement $u(x, t)$ as a combination of functions f and g. For the initial conditions we took a singular sine wave

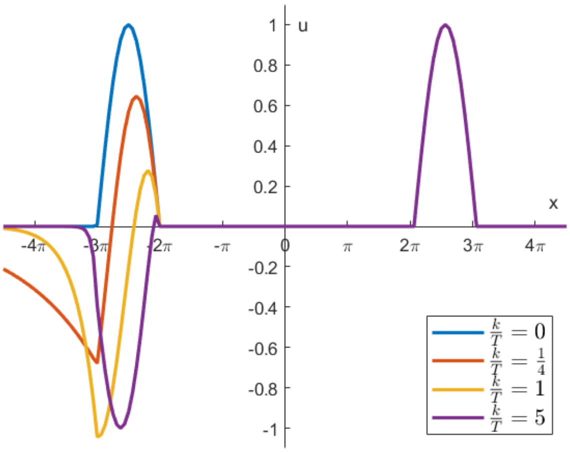
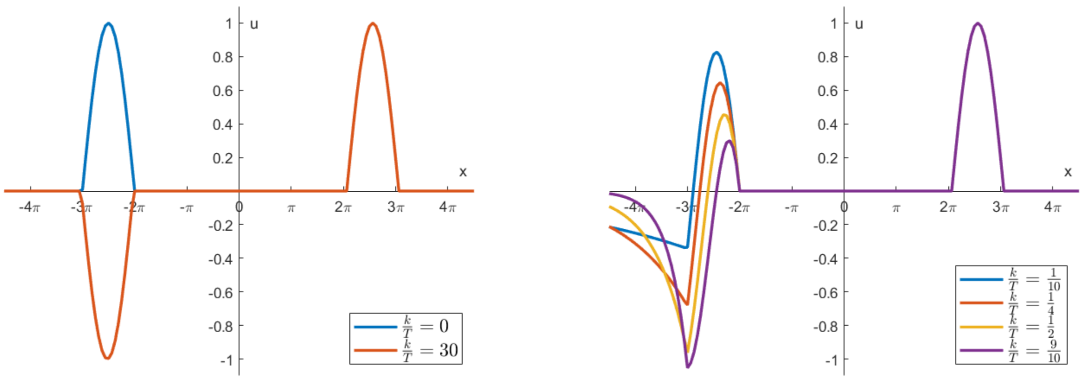


Figure 3: Reflected waves for the linear system



(a) (b)

Figure 4: Reflected waves for the linear system

and studied the reflected waves. We found that for a large $\frac{k}{T}$, the wave had an odd reflection. Whereas for $\frac{k}{T} = 0$ the wave is reflection evenly.

4 The Nonlinear Spring System

4.1 Introduction

After analysing the linear spring system, we now allow the nonlinear term to play a part in the system. The nonlinear system is described in the following manner:

$$\begin{aligned} u_{tt} - u_{xx} = 0, \qquad 0 < x, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x, \\ Tu_x(0,t) = ku(0,t) + k_2u^2(0,t). \end{aligned}$$

We won't be able to find an exact solution to this system, but we will be looking for an accurate approximation of the solution. We will do this analytically by using the Multiple Scales Perturbations method (MSP), and numerically by using the Fourth Order Runge Kutta method (RK4). We will compare the approximations of both methods, so we can draw conclusions about the accuracy.

4.2 Deriving the displacement function

We apply D'Alembert's formula to the boundary conditions.

$$\begin{aligned} T(f'(t) + f'(-t) + g(t) - g(-t)) = k(f(t) + f(-t) + \int_{-t}^{t} g(s)ds) + \\ k_2(\frac{1}{2}(f(t))^2 + f(t)f(-t) + \frac{1}{2}(f(t))^2 + f(t)\int_{-t}^{t} g(s)ds + f(-t)\int_{-t}^{t} g(s)ds + \frac{1}{2}(\int_{-t}^{t} g(s)ds)^2). \end{aligned} \tag{2}$$

For simplification we define:

$$\phi(t) = f(-t) + f(t) + \int_{-t}^{t} g(s)ds.$$

Then equation (2) can be rewritten to:

$$\phi'(t) + \frac{k}{T}\phi(t) + \frac{k_2}{2T}\phi^2(t) = A(t), \qquad \text{with } A(t) = 2(f'(t) + g(t)).$$

The left-hand side contains the unknown function ϕ, but the right-hand side is a known function. Notice that this differential equation is still nonlinear. The goal is to find an approximation for ϕ, which we call $\hat{\phi}_a$ (analytical) or $\hat{\phi}_n$ (numerical). We write this in the definition of ϕ:

$$\hat{\phi}_{a,n}(t) \approx f(t) + \int_{0}^{t} g(s)ds + f(-t) + \int_{-t}^{0} g(s)ds.$$

We can rewrite this to let the unknown functions be expressed in known functions:

$$f(-t) + \int_{-t}^{0} g(s)ds \approx \hat{\phi}_{a,n}(t) - f(t) - \int_{0}^{t} g(s)ds.$$

The approximation for the displacement $u(x,t)$ then is

$$u(x,t) \approx \begin{cases} \frac{1}{2}f(x+t) + \frac{1}{2}f(x-t) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds & x - t > 0, \\ \frac{1}{2}f(x+t) + \frac{1}{2}\int_{0}^{x+t} g(s)ds + \frac{1}{2}(\hat{\phi}_{a,n}(t-x) - f(t-x) - \int_{0}^{t-x} g(s)ds) & x - t < 0. \end{cases} \tag{3}$$

Now that we have found an expression for the approximated displacement, it is time to find $\hat{\phi}_{a,n}$. In chapter 5 we will use the MSP method for the analytical approach and in chapter 6 the RK4 method for the numerical approach.

4.3 Conclusions

In this section we considered the nonlinear spring system. We defined the function $\phi(t)$ to simplify the problem to a first order differential equation. From this we found an expression for $u(x,t)$ in functions of f, g and $\hat{\phi}_{a,n}$.

5 Analytical Approach using the MSP Method

5.1 Introduction to perturbation theory

From now on we consider only small displacements. We set $u = \epsilon \hat{u}$ for a small $\epsilon > 0$ and as a consequence $\phi = \epsilon\hat{\phi}$, $f = \epsilon \hat{f}$, $g = \epsilon \hat{g}$ and $A = \epsilon \hat{A}$. We obtain the new differential equation:

$$\hat{\phi}'(t) + \frac{k}{T}\hat{\phi}(t) + \epsilon\frac{k_2}{2T}\hat{\phi}^2(t) = \hat{A}(t). \tag{4}$$

Now the nonlinear term is just a small disturbance of order ϵ, so equation (4) is close to a solvable equation. This type of a problem, we can address with perturbation theory.
With perturbation theory you will find an approximation that is a composition of the exact solution of the solvable part and higher-order terms caused by the perturbed part. We do this by writing $\hat{\phi}(t)$ as the power series

$$\hat{\phi} = \hat{\phi}(t) = \hat{\phi}_0(t) + \epsilon\hat{\phi}_1(t) + \epsilon^2\hat{\phi}_2(t) + \cdots .$$

The difference between $\hat{\phi}_0(t)$ and the exact solution to equation (4) should be of the order of the next term, $\epsilon\hat{\phi}_1(t)$, so order ϵ. The difference between $\hat{\phi}_0(t) + \epsilon\hat{\phi}_1(t)$ and the exact solution is of order ϵ^2, and so on. A problem occurs when for an arbitrary k, $\phi_k(t)$ is unbounded on a time scale of $t = \mathcal{O}(\frac{1}{\epsilon})$. These so-called secular terms cause resonance in the system. The Multiple Scales Perturbations method, (Murdock, 1999, pp. 227–238), is a method that can remove these unwanted secular terms.

5.2 Introduction to the Multiple Scales Perturbations method

To prevent resonance on a time scale of $t = \mathcal{O}(\frac{1}{\epsilon})$, we use a two-scale expansion. We introduce the variable $\tau = \epsilon t$, which we treat independently from t. When $t = \mathcal{O}(\frac{1}{\epsilon})$, the τ is still of order 1. The approximation $\hat{\phi}$ will now be of the form:

$$\hat{\phi} = \hat{\phi}(t,\tau) = \hat{\phi}_0(t,\tau) + \epsilon\hat{\phi}_1(t,\tau) + \epsilon^2\hat{\phi}(t,\tau) + \cdots .$$

We will substitute this in the differential equation (4), where we will get a new equality for each time scale. From this we could derive $\hat{\phi}_k(t,\tau)$ for each $\mathcal{O}(\epsilon^k)$. Here it becomes clear which are the unwanted secular terms. The conditions to get rid of those terms can be solved and will provide an approximation to the solution.
We will only be deriving $\hat{\phi}_0(t,\tau)$, as this has an error of order ϵ when $t = \mathcal{O}(\frac{1}{\epsilon})$. This will be shown in chapter 5.6.1. Note that this only holds for an ϵ that is small enough, since the perturbation theory holds for $\epsilon \to 0$. In practice we don't know which value of $\epsilon > 0$ is small enough for the wanted accuracy. We will find out by comparing the approximations to the results from the RK4 method in chapter 6.3. If it turns out that $\hat{\phi}_0$ is not a very good approximation, one could add the term $\epsilon\hat{\phi}_1$. The downside to this is that it brings along more calculations.

5.3 Solving the nonlinear spring system using the MSP method

We substitute $\hat{\phi}(t,\tau)$ in equation (4):

$$\hat{\phi}_{0t} + \frac{k}{T}\hat{\phi}_0 + (\hat{\phi}_{1_t} + \hat{\phi}_{0_\tau} + \frac{k}{T}\hat{\phi}_1 + \frac{k_2}{2T}\hat{\phi}^2)\epsilon + (\hat{\phi}_{2_t} + \hat{\phi}_{1_\tau} + \frac{k}{T}\hat{\phi}_2 + \frac{k_2}{T}\hat{\phi}_0\hat{\phi}_1)\epsilon^2 + \cdots = \hat{A}(t).$$

From this we derive the following equations for each time scale:

$$\begin{aligned}
\mathcal{O}(1) \qquad & \hat{\phi}_{0_t} + \frac{k}{T}\hat{\phi}_0 = \hat{A}(t),\\
\mathcal{O}(\epsilon) \qquad & \hat{\phi}_{1_t} + \frac{k}{T}\hat{\phi}_1 = -\hat{\phi}_{0_\tau} - \frac{k_2}{2T}\hat{\phi}_0^2,\\
\mathcal{O}(\epsilon^2) \qquad & \hat{\phi}_{2_t} + \frac{k}{T}\hat{\phi}_2 = -\hat{\phi}_{1_\tau} - \frac{k_2}{T}\hat{\phi}_0\hat{\phi}_1.\\
& \cdots
\end{aligned} \tag{5}$$

We are only interested in the first two problems of equation (5). These are needed to construct a $\hat{\phi}_0$ that is bounded on a time scale of $t = \mathcal{O}(\frac{1}{\epsilon})$. Adding the next $\hat{\phi}_k$'s to the approximation will add a perturbation that is negligible small, assuming we choose an ϵ sufficiently small.
First we solve equation $\mathcal{O}(1)$:

$$\hat{\phi}_{0_t} + \frac{k}{T}\hat{\phi}_0 = \hat{A}(t). \tag{6}$$

This is a partial differential equation without a derivative to τ. This means we can treat it like an ordinary differential equation. First we find the homogeneous solution:

$$\hat{\phi}_{0,hom}(t,\tau) = B_0(\tau)e^{-\frac{k}{T}t},$$

where $B_0(\tau)$ is a function of τ.
To find the particular solution, we use the Variation of Constants method.
Assume $B_0(\tau)$ depends not only on τ, but also on t. By filling $\hat{\phi}_0(t,\tau) = B_0(t,\tau)e^{-\frac{k}{T}t}$ in equation (6), we get the solution:

$$\begin{aligned}\hat{\phi}_0(t,\tau) =& B_0(t,\tau)e^{-\frac{k}{T}t},\\ =&(B_0(0,\tau) + \int_0^t \hat{A}(s)e^{\frac{k}{T}s}ds)e^{-\frac{k}{T}t},\end{aligned}$$

where $B_0(0,\tau)$ is yet to be determined. For that we must use the equation $\mathcal{O}(\epsilon)$:

$$\hat{\phi}_{1_t} + \frac{k}{T}\hat{\phi}_1 = -\hat{\phi}_{0_\tau} - \frac{k_2}{2T}\hat{\phi}_0^2. \tag{7}$$

When we fill in $\hat{\phi}_0(t,\tau)$ in this expression, it becomes clear which terms in the right-hand side of equation (7) lead to unbounded terms in $\hat{\phi}_1$ on a time scale of $t = \mathcal{O}(\frac{1}{\epsilon})$. We then choose $B_0(0,\tau)$ in such a way that $\hat{\phi}_1$ will not go past its order for large t.

5.4 Removing secular terms

Here are three examples of initial conditions. In all examples it is shown how to remove the secular terms and subsequently how the approximation is derived.

5.4.1 Example 1: string with constant speed

The string starts with zero displacement. It has an initial velocity of 1 at all $x > 0$. Therefore the initial conditions are as follows:

$$\begin{aligned}u(x,0) &= \hat{f}(x) = 0,\\ u_t(x,0) &= \hat{g}(x) = 1.\end{aligned}$$

This means

$$\hat{A}(t) = 2(\hat{f}'(t) + \hat{g}(t)) = 2.$$

We derive $\hat{\phi}_0$:

$$\begin{aligned}\hat{\phi}_0(t,\tau) &= (B_0(0,\tau) + \int_0^t \hat{A}(s)e^{\frac{k}{T}s}ds)e^{-\frac{k}{T}t},\\ &= (B_0(0,\tau) - 2\frac{T}{k})e^{-\frac{k}{T}t} + 2\frac{T}{k},\\ &= C(\tau)e^{-\frac{k}{T}t} + 2\frac{T}{k} \quad \text{with } C(\tau) = B_0(0,\tau) - 2\frac{T}{k}.\end{aligned}$$

We fill this in equation (7) to find out which terms cause resonance.

$$\begin{aligned}\hat{\phi}_{1_t} + \frac{k}{T}\hat{\phi}_1 &= -C'(\tau)e^{-\frac{k}{T}t} - \frac{k_2}{2T}(C(\tau)e^{-\frac{k}{T}t} + 2\frac{T}{k})^2, \\ &= (-C'(\tau) - \frac{2k_2}{k}C(\tau))e^{-\frac{k}{T}t} - \frac{2Tk_2}{k^2} - \frac{k_2}{2T}C^2(\tau)e^{-\frac{2k}{T}t}.\end{aligned}$$

We want to choose $C(\tau)$ such that

$$-C'(\tau) - \frac{2k_2}{k}C(\tau) = 0,$$

or else $\hat{\phi}_1$ will gain unbounded terms. The following $C(\tau)$ suffices:

$$C(\tau) = C(0)e^{-\frac{2k_2}{k}\tau}.$$

Then $\hat{\phi}_0$ becomes

$$\begin{aligned}\hat{\phi}_0(t,\tau) &= C(0)e^{-\frac{2k_2}{k}\tau}e^{-\frac{k}{T}t} + 2\frac{T}{k}, \\ &= -2\frac{T}{k}e^{-\frac{2k_2}{k}\tau}e^{-\frac{k}{T}t} + 2\frac{T}{k}.\end{aligned}$$

Note that we took $C(0) = -2\frac{T}{k}$ so $\hat{\phi}_0(t=0) = 0$ holds. Then $\hat{\phi}_0$ is continuous in zero. The obtained approximation is:

$$\hat{\phi}_a(t) = -2\frac{T}{k}e^{-\frac{2k_2}{k}\epsilon t}e^{-\frac{k}{T}t} + 2\frac{T}{k}.$$

5.4.2 Example 2: singular sine wave

We take the initial conditions as the ones we also used for the linear spring system:

$$\hat{u}(x,0) = \hat{f}(x) = \begin{cases} 0 & 0 < x < 2\pi, \\ sin(x) & 2\pi \le x \le 3\pi, \\ 0 & 3\pi < x, \end{cases}$$

$$\hat{u}_t(x,0) = \hat{g}(x) = \begin{cases} 0 & 0 < x < 2\pi, \\ cos(x) & 2\pi \le x \le 3\pi, \\ 0 & 3\pi < x. \end{cases}$$

Then

$$\hat{A}(t) = 2(\hat{f}'(t) + \hat{g}(t)) = \begin{cases} 0 & 0 < t < 2\pi, \\ 4cos(t) & 2\pi \le t \le 3\pi, \\ 0 & 3\pi < t. \end{cases}$$

We derive $\hat{\phi}_0$:

$$\begin{aligned}\hat{\phi}_0(t,\tau) &= (B_0(0,\tau) + \int_0^t \hat{A}(s)e^{\frac{k}{T}s}ds)e^{-\frac{k}{T}t}, \\ &= \begin{cases} B_0(0,\tau)e^{-\frac{k}{T}t} & 0 < t < 2\pi, \\ (B_0(0,\tau) - \frac{4kT}{T^2+k^2}e^{2\frac{k}{T}\pi})e^{-\frac{k}{T}t} + \frac{4T}{T^2+k^2}(T\sin(t) + k\cos(t)) & 2\pi \le t \le 3\pi, \\ (B_0(0,\tau) - \frac{4kT}{T^2+k^2}(e^{3\frac{k}{T}\pi} + e^{2\frac{k}{T}\pi}))e^{-\frac{k}{T}t} & 3\pi < t. \end{cases}\end{aligned}$$

For equation (7) we need $\hat{\phi}_{0_\tau}(t,\tau)$ and $\hat{\phi}_0^2(t,\tau)$:

$$\hat{\phi}_{0_\tau}(t,\tau) = B_0'(0,\tau)e^{-\frac{k}{T}t},$$

$$\hat{\phi}_0^2(t,\tau) = \begin{cases} B_0^2(0,\tau)e^{-\frac{2k}{T}t} & 0 < t < 2\pi, \\ (B_0(0,\tau) - \frac{4kT}{T^2+k^2}e^{2\frac{k}{T}\pi})^2 e^{-\frac{2k}{T}t} + (\frac{4T}{T^2+k^2}(T\sin(t) + k\cos(t)))^2 & 2\pi \leq t \leq 3\pi, \\ (B_0(0,\tau) - \frac{4kT}{T^2+k^2}(e^{3\frac{k}{T}\pi} + e^{2\frac{k}{T}\pi}))^2 e^{-\frac{2k}{T}t} & 3\pi < t. \end{cases}$$

We see that $\hat{\phi}_{0_\tau}$ is the only problematic part, which gives rise to unbounded terms in $\hat{\phi}_1$.
If we take a constant for $B_0(0,\tau)$, then $B_0'(0,\tau)e^{-\frac{k}{T}t} = 0$ and $\hat{\phi}_1$ will be bounded for $t = \mathcal{O}(\frac{1}{\epsilon})$.
We pick $B_0(0,\tau) = 0$ for the continuity of $\hat{\phi}_0$ in zero. Now that we have removed the secular terms, we obtain the approximation

$$\hat{\phi}_a(t) = \begin{cases} 0 & 0 < t < 2\pi, \\ -\frac{4kT}{T^2+k^2}e^{2\frac{k}{T}\pi}e^{-\frac{k}{T}t} + \frac{4T}{T^2+k^2}(T\sin(t) + k\cos(t)) & 2\pi \leq t \leq 3\pi, \\ -\frac{4kT}{T^2+k^2}(e^{3\frac{k}{T}\pi} + e^{2\frac{k}{T}\pi})e^{-\frac{k}{T}t} & 3\pi < t. \end{cases}$$

5.4.3 Example 3: continuous sine wave

In this example we consider a string that is a continuously moving sine wave. This matches the following initial conditions:

$$\hat{u}(x,0) = \hat{f}(x) = \sin(x),$$
$$\hat{u}_t(x,0) = \hat{g}(x) = \cos(x).$$

This means

$$A(t) = 2(\hat{f}'(t) + \hat{g}(t)) = 4\cos(t).$$

We derive $\hat{\phi}_0$

$$\begin{aligned}\hat{\phi}_0(t,\tau) &= (B_0(0,\tau) + \int_0^t A(s)e^{\frac{k}{T}s}ds)e^{-\frac{k}{T}t}, \\ &= (B_0(0,\tau) - 4\frac{4Tk}{T^2+k^2})e^{-\frac{k}{T}t} + 4\frac{T(T\sin(t) + k\cos(t))}{T^2+k^2}.\end{aligned}$$

Substitute this in equation (7):

$$\begin{aligned}\hat{\phi}_{1_t} + \frac{k}{T}\hat{\phi}_1 = &- B_0'(0,\tau)e^{-\frac{k}{T}t} - \frac{k_2}{2T}[(B_0(0,\tau) - 4\frac{4Tk}{T^2+k^2})^2 e^{-2\frac{k}{T}t} \\ &+ 8\frac{T(T\sin(t) + k\cos(t))}{T^2+k^2}(B_0(0,\tau) - \frac{4Tk}{T^2+k^2})e^{-\frac{k}{T}t} + 16(\frac{T(T\sin(t) + k\cos(t))}{T^2+k^2})^2].\end{aligned}$$

Just as in the previous example, $B_0'(0,\tau)$ has to be zero, or else $\hat{\phi}_1$ has a term that goes past its order for $t = \mathcal{O}(\frac{1}{\epsilon})$. And again for continuity of $\hat{\phi}_0$ we choose $B_0(0,\tau) = 0$.
We obtain the approximation

$$\hat{\phi}_a(t) = -\frac{4Tk}{T^2+k^2}e^{-\frac{k}{T}t} + 4\frac{T(T\sin(t) + k\cos(t))}{T^2+k^2}.$$

5.5 Visualising the reflected waves

We have found approximations of the solution using the Multiple Scales Perturbations method for three different examples. The next step is to visualize them and see how the wave is reflected. From equation (3) we deduce the approximated displacement:

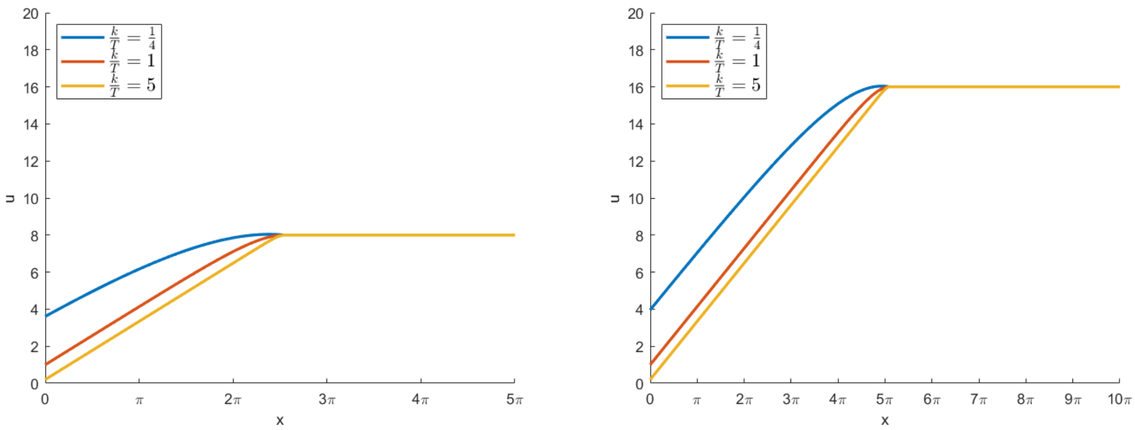
$$\hat{u}(x,t) = \begin{cases} \frac{1}{2}\hat{f}(x+t) + \frac{1}{2}\hat{f}(x-t) + \frac{1}{2}\int_{x-t}^{x+t}\hat{g}(s)ds & x - t > 0, \\ \frac{1}{2}\hat{f}(x+t) + \frac{1}{2}\int_0^{x+t}\hat{g}(s)ds + \frac{1}{2}(\hat{\phi}_a(t-x) - \hat{f}(t-x) - \int_0^{t-x}\hat{g}(s)ds) & x - t < 0. \end{cases}$$

5.5.1 Example 1: string with constant speed

For equation (3) we need:

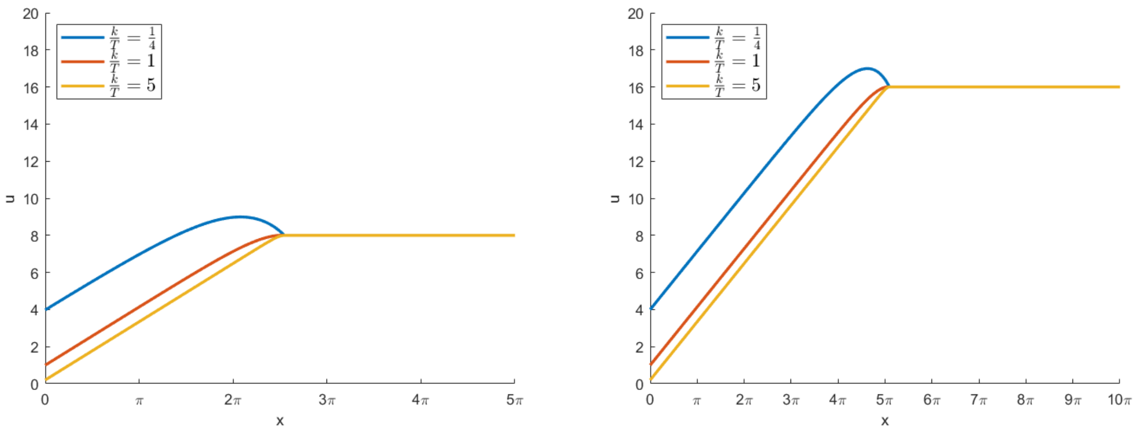
$$\hat{\phi}_a(t-x) = -2\frac{T}{k}e^{-\frac{2k_2}{k}\epsilon(t-x)}e^{-\frac{k}{T}(t-x)} + 2\frac{T}{k}.$$

We will consider two different values of ϵ : 0.01 and 0.1. For each ϵ, three values of $\frac{k}{T}$ are plotted. $k_2 = 4000$ and $T = 8000$ are fixed.
First we consider $\epsilon = 0.01$ in figure 5. The string starts at $x = 0\ \forall x > 0$ and moves up as a whole with a constant speed. Even though the string will continue to move upwards, the spring will try to hold it down. In figure 5a this is shown for $t = 8$. At $t = 16$ in figure 5b the string has moved even further up, but the spring still holds it in place at $x = 0$. A really stiff spring, $\frac{k}{T} = 5$, will keep the string down at almost a zero displacement. Whereas a softer spring, $\frac{k}{T} = \frac{1}{4}$, keeps the string a little bit higher, at $u = 4$.
For $\epsilon = 0.1$ in figure 6 the same is happening, except for one difference. The string bounces up before it is pulled down.



(a) $t = 8$ (b) $t = 16$

Figure 5: Example 1: reflected waves for $\epsilon = 0.01$



(a) $t = 8$ (b) $t = 16$

Figure 6: Example 1: reflected waves for $\epsilon = 0.1$

5.5.2 Example 2: singular sine wave

In this example we have:

$$\hat{\phi}_a(t-x) = \begin{cases} 0 & 0 < t-x < 2\pi, \\ -\frac{4kT}{T^2+k^2} e^{2\frac{k}{T}\pi} e^{\frac{k}{T}(x-t)} + \frac{4T}{T^2+k^2}(T\sin(t-x) + k\cos(t-x)) & 2\pi \leq t-x \leq 3\pi, \\ -\frac{4kT}{T^2+k^2}(e^{3\frac{k}{T}\pi} + e^{2\frac{k}{T}\pi}) e^{\frac{k}{T}(x-t)} & 3\pi < t-x. \end{cases}$$

The obtained solution does not depend on k_2 neither does it on ϵ. Because $\forall \epsilon > 0$ the solutions will be equivalent, it follows that the obtained approximation will be the solution of the linear spring system. Notice that figure 7 shows the same results as in figure 3 when we plot the reflected waves using the same values.
To take the perturbed part into consideration, one should derive $\hat{\phi}_1$ as well to obtain the approximation $\hat{\phi}_b = \hat{\phi}_0 + \epsilon\hat{\phi}_1$. However, in this thesis we limit the approximation to $\hat{\phi}_a = \hat{\phi}_0$.
Figure 8 shows the transition from the initial wave to the reflected wave. In figure 8a we see that the spring first expands to double the amplitude of the initial wave. After that it transitions through figure 8b to the reflected wave.

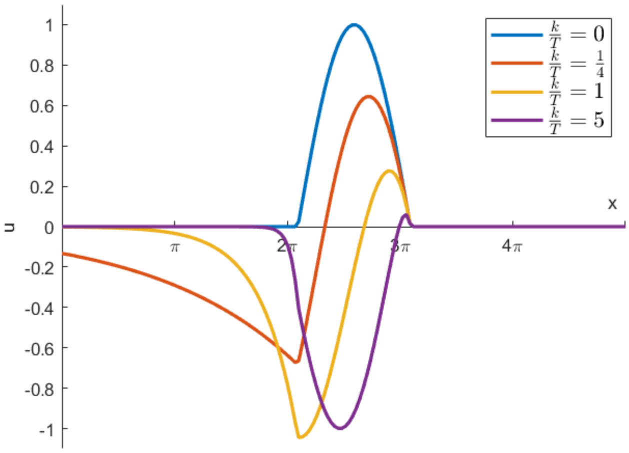


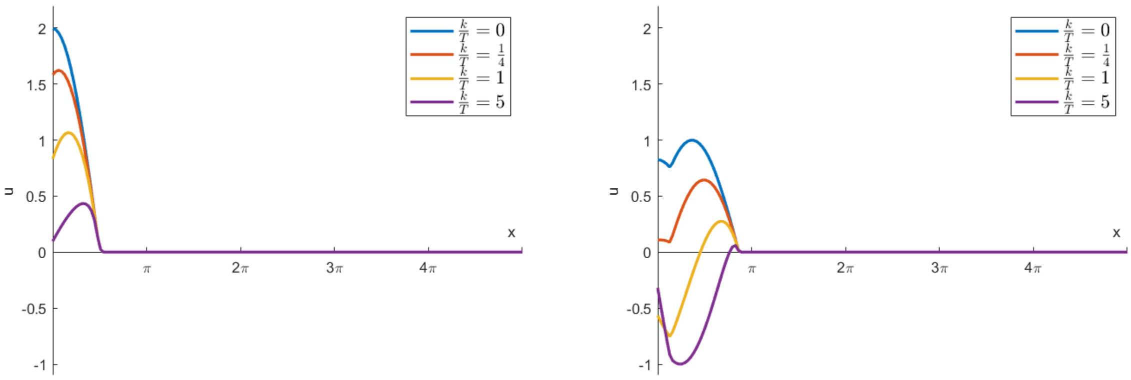
Figure 7: Example 2: reflected waves at $t = 16$

5.5.3 Example 3: continuous sine wave

For the continuous sine wave we have:

$$\hat{\phi}_a(t-x) = -\frac{4Tk}{T^2+k^2} e^{-\frac{k}{T}(t-x)} + 4\frac{T(T\sin(t-x) + k\cos(t-x))}{T^2+k^2}.$$

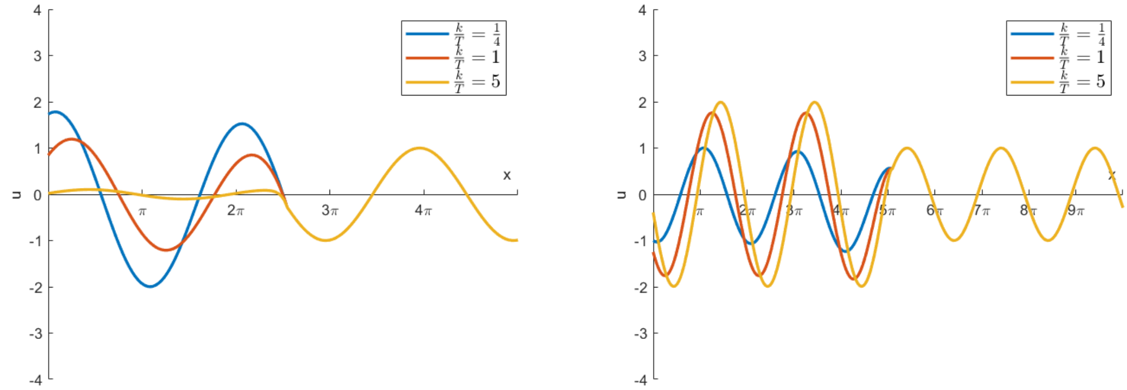
Again the solution does not depend on k_2 or on ϵ. Therefore the nonlinear term has no effect on the system.
For multiple values of $\frac{k}{T}$ the approximations are shown in figure 9. We again took $T = 8000$. The wave is reflected with sinusoidal functions which amplitude changes back and forth. The amplitude goes from about $u = 2$ to $u = 0$ and then mirrors in the x-axis. For $\frac{k}{T} = 5$ in figure 9a we see the moment when the amplitude has come down to almost zero.
In figure 9b we see that the the amplitude of the wave caused by stiffer springs are not necessarily lower then those caused by the softer springs.



(a) $t = 7.8$

(b) $t = 9$

Figure 8: Example 2: transition to reflected wave



(a) $t = 8$

(b) $t = 16$

Figure 9: Example 3: reflected waves

5.6 Proof accuracy $\hat{\phi}_a$

5.6.1 Introduction

For this proof we assume that $\epsilon > 0$ is small enough for the perturbation theory to hold. Until now we assumed that $\hat{\phi}_a = \hat{\phi}_0$ was an accurate approximation of the solution. We suspected $\hat{\phi}_b = \hat{\phi}_0 + \epsilon\hat{\phi}_1$ would make a change that is significantly small. In this section, we are going to prove this conjecture. We do this by showing $|\hat{\phi} - \hat{\phi}_a| = \mathcal{O}(\epsilon)$, where $\hat{\phi}$ is the exact solution of the differential equation (4).
We find an upper estimate using the triangle inequality.

$$|\hat{\phi} - \hat{\phi}_a| = |\hat{\phi} - \hat{\phi}_0| = |\hat{\phi} - (\hat{\phi}_0 + \epsilon\hat{\phi}_1) + \epsilon\hat{\phi}_1| \leq |\hat{\phi} - (\hat{\phi}_0 + \epsilon\hat{\phi}_1)| + |\epsilon\hat{\phi}_1| = |\hat{\phi} - \hat{\phi}_b| + |\epsilon\hat{\phi}_1|.$$

In section 5.4, we chose $B_0(0, \tau)$ such that $|\epsilon\hat{\phi}_1| = \mathcal{O}(\epsilon)$. So it suffices to show that $|\hat{\phi} - \hat{\phi}_b| = \mathcal{O}(\epsilon)$. First we fill in $\hat{\phi}_b$ in equation (4) to find the error which we will call the remainder. After that we find an expression for the exact solution $\hat{\phi}$ and the approximation $\hat{\phi}_b$. We do this by taking the differential equations and transforming them into integral equations. Lastly, we subtract these expressions and look for an upper estimate using Gronwall's Theorem.

5.6.2 Determine the remainder

We can rewrite equation (4) to

$$\begin{aligned} L\hat{\phi}(t) - \hat{A}(t) &= \hat{\phi}'(t) + \frac{k}{T}\hat{\phi}(t) + \epsilon\frac{k_2}{2T}\hat{\phi}^2(t) - \hat{A}(t), \\ &= 0. \end{aligned} \tag{8}$$

Using that $\hat{\phi}_b$ depends on t and τ both, we get

$$\begin{aligned} L\hat{\phi}_b(t) - \hat{A}(t) &= \hat{\phi}_{0_t} + \frac{k}{T}\hat{\phi}_0 - \hat{A}(t) + \epsilon(\hat{\phi}_{1_t} + \hat{\phi}_{0_\tau} + \frac{k}{T}\hat{\phi}_1 + \frac{k_2}{2T}\hat{\phi}_0^2), \\ &= -\epsilon^2(\hat{\phi}_{1_t} + \frac{k_2}{T}\hat{\phi}_0\hat{\phi}_1) - \epsilon^3\frac{k_2}{2T}\hat{\phi}_1^2. \end{aligned} \tag{9}$$

To obtain the remainder R, we subtract equation (9) from (8):

$$\begin{aligned} L\hat{\phi}(t) - L\hat{\phi}_b(t) &= \epsilon^2(\hat{\phi}_{1_t} + \frac{k_2}{T}\hat{\phi}_0\hat{\phi}_1) + \epsilon^3\frac{k_2}{2T}\hat{\phi}_1^2, \\ &= \epsilon^2 R(t, \tau, \epsilon). \end{aligned}$$

We can see that R is bounded for $t = \mathcal{O}(\frac{1}{\epsilon})$.

5.6.3 Finding expressions for $\hat{\phi}$ and $\hat{\phi}_b$

When we convert the differential equations for $\hat{\phi}$ and $\hat{\phi}_b$ to integral equations, we get the following expressions:

$$\begin{aligned} \hat{\phi}'(t) &= -\frac{k}{T}\hat{\phi}(t) - \epsilon\frac{k_2}{2T}\hat{\phi}^2(t) + \hat{A}(t), \\ \Rightarrow \hat{\phi}(t) &= e^{-\frac{k}{T}t}\hat{\phi}(0) - \epsilon e^{-\frac{k}{T}t}\frac{k_2}{T}\int_0^t \hat{\phi}^2(s)e^{\frac{k}{T}s}ds + e^{-\frac{k}{T}t}\int_0^t \hat{A}(s)e^{\frac{k}{T}s}ds, \\ \hat{\phi}_b'(t) &= -\frac{k}{T}\hat{\phi}_b(t) - \epsilon\frac{k_2}{2T}\hat{\phi}_b^2(t) + \hat{A}(t) + \epsilon^2 R(t, \tau, \epsilon), \\ \Rightarrow \hat{\phi}_b(t) &= e^{-\frac{k}{T}t}\hat{\phi}_b(0) - \epsilon e^{-\frac{k}{T}t}\frac{k_2}{T}\int_0^t \hat{\phi}_b^2(s)e^{\frac{k}{T}s}ds + e^{-\frac{k}{T}t}\int_0^t \hat{A}(s)e^{\frac{k}{T}s}ds + \epsilon^2 e^{-\frac{k}{T}t}\int_0^t R(s, \tau, \epsilon)e^{\frac{k}{T}s}ds. \end{aligned}$$

By subtracting these two, we get an estimate of the form:

$$|\hat{\phi} - \hat{\phi}_b| \leq e^{-\frac{k}{T}t}|\hat{\phi}(0) - \hat{\phi}_b(0)| + \epsilon e^{-\frac{k}{T}t}\frac{k_2}{2T}\int_0^t |\hat{\phi}^2(s) - \hat{\phi}_b^2(s)|e^{\frac{k}{T}s}ds + \epsilon^2 e^{-\frac{k}{T}t}\int_0^t |R(s, \tau, \epsilon)|e^{\frac{k}{T}s}ds.$$

We will find an upper estimate for each term. By adding these up, we will obtain an upper estimate for $|\hat{\phi} - \hat{\phi}_0|$.

5.6.4 Finding an upper estimate for $|\hat{\phi} - \hat{\phi}_a|$

The first thing we know is that $|\hat{\phi}(0) - \hat{\phi}_b(0)| = \mathcal{O}(\epsilon^2)$.
Second, we know that $\forall t : \hat{\phi}(t)$ exists and is unique. Therefore, $\hat{\phi}^2(t)$ is a Lipschitz continuous function. Then $\exists c_1 > 0 : |\hat{\phi}^2(s) - \hat{\phi}_b^2(s)| \leq c_1|\hat{\phi}(s) - \hat{\phi}_b(s)|$.
Earlier we found that $R(t, \tau, \epsilon)$ is bounded for $t = \mathcal{O}(\frac{1}{\epsilon})$, so the third thing we know is that $\exists c_2 > 0 : t \leq \frac{c_2}{\epsilon}$. We use this for finding an upper estimate of the remainder term:

$$\epsilon^2 e^{-\frac{k}{T}t}\int_0^t |R(s, \tau, \epsilon)|e^{\frac{k}{T}s}ds \leq \epsilon^2 t \max|R(t, \tau, \epsilon)| \leq c_2\epsilon \max|R(t, \tau, \epsilon)| = c_2\mathcal{O}(\epsilon).$$

Combining these three estimates, we obtain the following:

$$|\hat{\phi} - \hat{\phi}_b| \leq c_1 \epsilon \frac{k_1}{2T} \int_0^t |\hat{\phi}(s) - \hat{\phi}_b(s)| ds + \mathcal{O}(\epsilon^2) + c_2 \mathcal{O}(\epsilon).$$

Now we can apply Gronwall's Theorem. We use the theorem as it was stated by Verhulst (1996, p. 7):
Gronwall's Theorem
Assume that for $t_0 \leq t \leq t_0 + a$, with a a positive constant, we have the estimate

$$\Phi(t) \leq \delta_1 \int_{t_0}^t \psi(s)\Phi(s) ds + \delta_3,$$

in which, for $t_0 \leq t \leq t_0 + a, \Phi(t)$ and $\psi(t)$ are continuous functions, $\Phi(t) \geq 0$ and $\psi(t) \geq 0; \delta_1$ and δ_3 are positive constants. Then we have for $t_0 \leq t \leq t_0 + a$

$$\Phi(t) \leq \delta_3 e^{\delta_1 \int_{t_0}^t \psi(s) ds}.$$

We apply this theorem, setting $\psi(t) = 1$. We get

$$|\hat{\phi} - \hat{\phi}_b| \leq (\mathcal{O}(\epsilon^2) + c_2 \mathcal{O}(\epsilon)) e^{\epsilon c_1 \frac{k_2}{2T} t}.$$

Then for $t = \mathcal{O}(\frac{1}{\epsilon}) : |\hat{\phi} - \hat{\phi}_b| = \mathcal{O}(\epsilon)$, and therefore $|\hat{\phi} - \hat{\phi}_a| = \mathcal{O}(\epsilon)$.
This proves the conjecture.

5.7 Conclusions

In this chapter we considered only small displacements. By doing that we could apply the Multiple Scales Perturbations method to find an approximated solution to the nonlinear spring system.
We considered three examples of initial conditions. For all three we applied the method and found the approximated solution $\hat{\phi}_a$ and therefore also $\hat{u}(x, t)$. In the first example, the solution was dependent on the nonlinear term, so the perturbed part. Whereas in the second and third example, the nonlinear term did not play any role in the solution.
At last we proved that for an $\epsilon > 0$ small enough, the error to the solution of this method is of order ϵ.

6 Numerical Approach using the RK4 Method

6.1 Introduction

In the previous section we found the approximated solution $\hat{\phi}_a$ for three examples of initial conditions. We want to find out how good these approximations were. A way to do this is to apply another method to the same problems and thereafter compare the results to each other. In this section, we look for a solution with a numerical approach. We will use the Fourth Order Runge-Kutta method. It is not necessary here to consider only small displacements. But by sticking to the differential equation (4), we can study the solution for the same values of ϵ. In this way we can compare the solutions to those we obtained from the Multiple Scales Perturbations method.

6.2 Introduction to the Fourth Order Runge Kutta method

The Fourth Order Runge-Kutta method, (C.Vuik, Vermolen, Gijzen, & M.J. Vuik, 2016, pp. 77–78), is an iterative method for solving systems of the form:

$$\begin{cases} y'(t) &= F(t, y) \qquad t > t_0, \\ y(t_0) &= y_0. \end{cases}$$

Let w denote the numerical approximation of y. First the predictors must be computed:

$$\begin{aligned} k_1 &= hF(t_n, w_n), \\ k_2 &= hF(t_n + \frac{1}{2}h, w_n + \frac{1}{2}k_1), \\ k_3 &= hF(t_n + \frac{1}{2}h, w_n + \frac{1}{2}k_2), \\ k_4 &= hF(t_n + h, w_n + k_3), \end{aligned}$$

where h is the step size. At t_{n+1} the following solution is approximated:

$$w_{n+1} = w_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

The truncation errors made in this method rely on the chosen time step. The local truncation error, the error made in a certain iteration, is $\mathcal{O}(h^5)$. The global truncation error, $|y_n - w_n|$, is $\mathcal{O}(h^4)$. These errors only hold when the method is stable. In chapter 6.4 we will check the stability of this method on our differential equation.

6.3 Comparing MSP to RK4

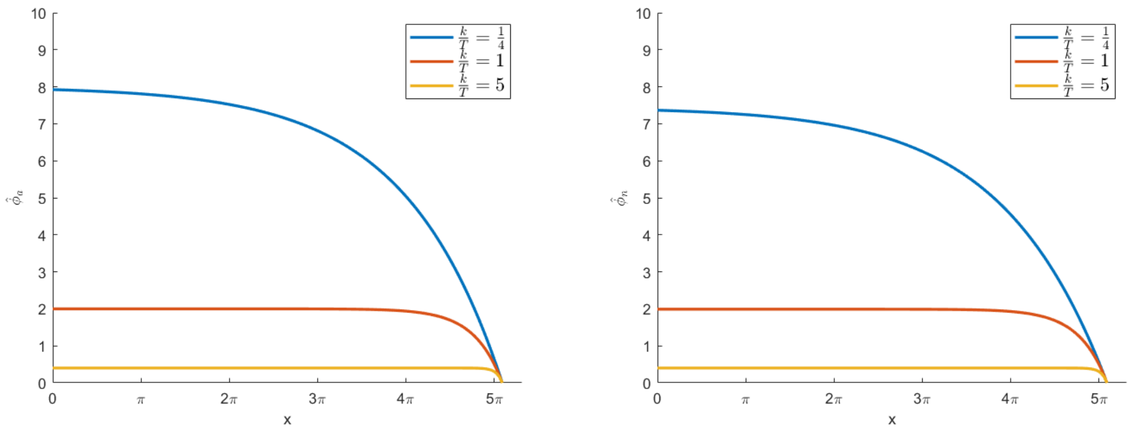
The initial-value problem is as follows:

$$\begin{cases} \hat{\phi}'(t) &= F(t, \hat{\phi}) = -\frac{k}{T}\hat{\phi}(t) - \epsilon\frac{k_2}{2T}\hat{\phi}^2(t) + \hat{A}(t) \qquad t > 0, \\ \hat{\phi}(0) &= 0. \end{cases}$$

We take step size $h = 1/10$. Then when stability holds, the global truncation error is $\mathcal{O}(10^{-4})$. We will take the same three examples from the previous chapter and solve these numerically. By plotting the solution $\hat{\phi}_n$ from the RK4 method and $\hat{\phi}_a$ from the MSP method, we can study the difference in the two approximations. In these plots we use the same values as in the previous chapter.

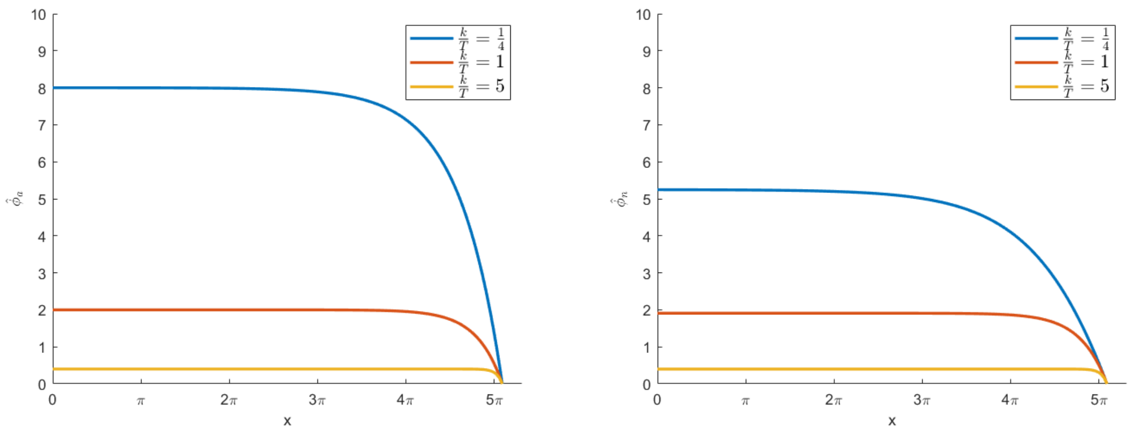
6.3.1 Example 1: string with constant speed

In figure 10 the $\hat{\phi}_a$ from the MSP method and the $\hat{\phi}_n$ from the RK4 method are shown for $\epsilon = 0.01$. For $\frac{k}{T} = 1$ and $\frac{k}{T} = 5$ the approximations are about the same. For $\frac{k}{T} = \frac{1}{4}$ the approximation is a bit higher for the MSP method then for the RK4 method.
The two approximations for $\epsilon = 0.1$ are plotted in figure 11. Again they look the same for $\frac{k}{T} = 1$ and $\frac{k}{T} = 5$. For $\frac{k}{T} = \frac{1}{4}$ the difference has become a lot bigger.
For both values of ϵ there is a difference in the plots. But we definitely see that the approximations of the solution with $\epsilon = 0.01$ are a lot better then the approximations with $\epsilon = 0.1$.
We also see that for a stiffer spring, the value of ϵ barely affects the accuracy of the approximation. Remember that for the MSP method we considered only small displacements. This could be a reason why the approximations differ so much for the softer spring.



(a) $\hat{\phi}_a$ (b) $\hat{\phi}_n$

Figure 10: Example 1: approximations $\hat{\phi}_a$ and $\hat{\phi}_n$ for $\epsilon = 0.01$



(a) $\hat{\phi}_a$ (b) $\hat{\phi}_n$

Figure 11: Example 1: approximations $\hat{\phi}_a$ and $\hat{\phi}_n$ for $\epsilon = 0.1$

6.3.2 Example 2: singular sine wave

In this example the approximation $\hat{\phi}_a$ shown in figure 12 did not depend on ϵ. This means that in this approximation the nonlinear term is fully neglected.

In figure 13b we see that $\hat{\phi}_n$ for $\epsilon = 0.1$ takes lower values than $\hat{\phi}_a$ for $\frac{k}{T} < 1$. Whereas for $\epsilon = 0.01$ in figure 13a the approximation $\hat{\phi}_n$ is about the same as $\hat{\phi}_a$ for all values of $\frac{k}{T}$. Therefore we say that these are accurate approximations to the solution.
We see that the more ϵ approaches zero, the more $\hat{\phi}_n$ approaches $\hat{\phi}_a$.
Notice that we did not plot $\frac{k}{T} = 0$ for $\hat{\phi}_n$. This is because the method was unstable for these values. Instead we took the smallest values for which the method was stable. The fact that we used different values is another reason why the solutions might not look the same.

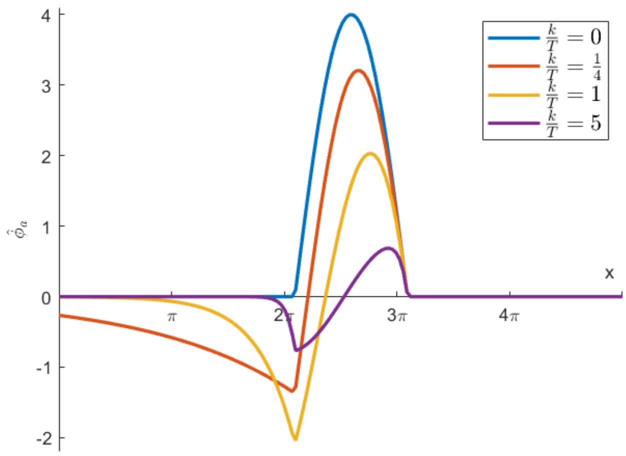
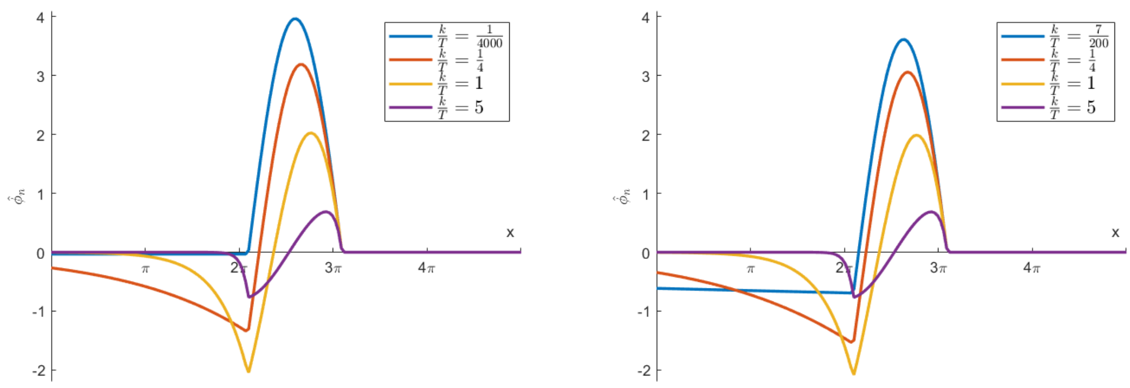


Figure 12: Example 2: approximations $\hat{\phi}_a$



(a) $\epsilon = 0.01$

(b) $\epsilon = 0.1$

Figure 13: Example 2: approximations $\hat{\phi}_n$

6.3.3 Example 3: continuous sine wave

Just like in example 2, the $\hat{\phi}_a$ in figure 14 does not depend on ϵ. Therefore we should again have that for $\epsilon \to 0 \Rightarrow \hat{\phi}_n \to \hat{\phi}_a$.
In figure 15b $\hat{\phi}_n$ is shown for $\epsilon = 0.1$. We see that the approximation is about the same as $\hat{\phi}_a$ for the stiff springs. But for $\frac{k}{T} = \frac{1}{4}$, $\hat{\phi}_n$ takes lower values than $\hat{\phi}_a$.
In figure 15a the approximation $\hat{\phi}_n$ is very accurate to the approximation $\hat{\phi}_a$. These findings are in line with our suspicion.

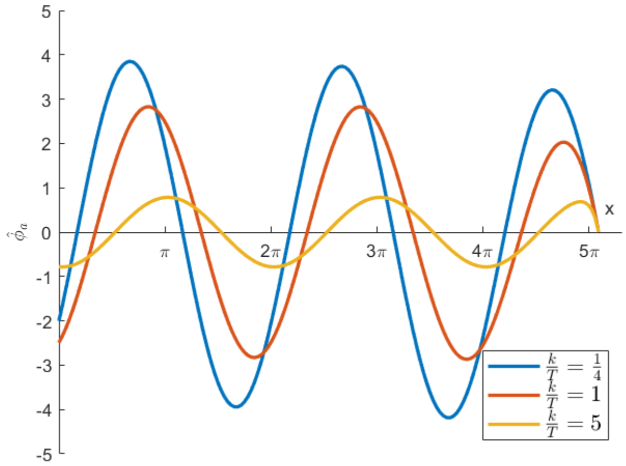
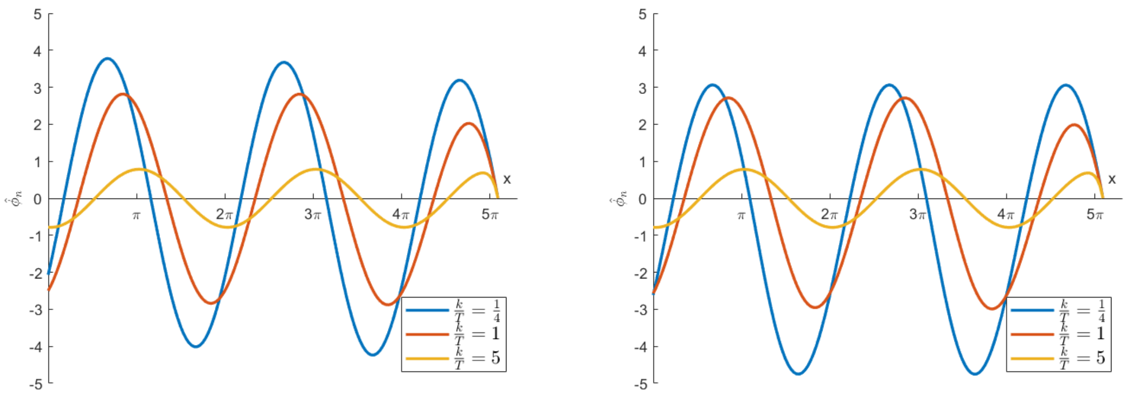


Figure 14: Example 3: approximations $\hat{\phi}_a$



(a) $\epsilon = 0.01$

(b) $\epsilon = 0.1$

Figure 15: Example 3: approximations $\hat{\phi}_n$

6.4 Checking the stability conditions

To determine the stability of the numerical method, we check the condition $|Q(\lambda h)| \leq 1$ for all $\lambda = \frac{\partial f}{\partial \hat{\phi}}(t_i, \hat{\phi}_{n_i})$, where $\hat{\phi}_{n_i}$ is the numerical approximation at t_i. $Q(\lambda h)$ is called the amplification factor. For the RK4 method, the amplification factor is given by

$$|Q(\lambda h)| = |1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24}|.$$

We consider two cases for checking if the method is stable or not. We consider the system with the singular sine wave used for example 2. We take the values $\epsilon = 0.01$, $k_2 = 4000$, and $T = 8000$. λ is given by

$$\begin{aligned}\lambda &= \frac{\partial f}{\partial \hat{\phi}}(t_i, \hat{\phi}_{n_i}),\\ &= -\frac{k}{T} - \epsilon \frac{k_2}{T} \hat{\phi}_{n_i}.\end{aligned}$$

In the first case we take $\frac{k}{T} = 0$. We find that $\forall i : -0.0297 \leq \hat{\phi}_{n_i} \leq 3.9646$. But for the lowest value of $\hat{\phi}_{n_i}$ it holds that

$$\lambda = -0 - 0.01 * \frac{1}{2} * -0.0297 = 0.0001485.$$

Substituting this value for λ it follows that

$$|Q(\lambda h)| \approx 1.000015 > 1.$$

Therefore the method is not stable for these values.
In the second case we consider $\frac{k}{T} = \frac{1}{4}$. Now $\forall i : -1.3367 \leq \hat{\phi}_{n_i} \leq 3.1879$. Then

$$\lambda = -\frac{1}{4} - 0.01 * \frac{1}{2} * \hat{\phi}_{n_i},$$

and so

$$|Q(\lambda h)| < 1 \quad \forall \hat{\phi}_{n_i}.$$

This means the method is stable and therefore the global truncation error is $\mathcal{O}(h^4)$.

6.5 Conclusions

In this chapter we used the Fourth Order Runge Kutta method to find the numerical approximation $\hat{\phi}_n$. We made sure stability of this method held for our system. Therefore we concluded that the global truncation error of $\hat{\phi}_n$ to the exact solution is $\mathcal{O}(h^4)$.
We compared the plots of $\hat{\phi}_a$ from the analytical approach to $\hat{\phi}_n$ from the numerical approach to determine the accuracy of the two approximations.
In example 1, where the approximation is influenced by the nonlinear part, we saw that for a stiff spring the approximations were the same and therefore accurate. This was not true for the softer spring, which was approximated poorly for $\epsilon = 0.1$. For $\epsilon = 0.01$ the approximation was better, but still not too accurate.
In example 2 and 3, where the approximation is not influenced by the perturbed part, we found that when ϵ approaches zero, the approximation $\hat{\phi}_n$ approaches $\hat{\phi}_a$.
In the cases where $\hat{\phi}_a$ lacked accuracy, one could derive $\hat{\phi}_1$ or even $\hat{\phi}_2$ to add to the approximation. This provides an extra order term to the approximated solution.

7 Conclusions

In this thesis we considered a semi-infinite string described by the wave equation which was complemented with initial conditions and boundary conditions. The boundary condition described a spring that behaves in a nonlinear way.
The goal of this thesis was to find accurate approximations of the solution of this nonlinear spring system.
The solution was approximated using two different approaches.
First we tried an analytical approach, using the Multiple Scales Perturbations method. Before we applied this method, we had to consider only small displacements. Second we took a numerical approach, using the Fourth Order Runge Kutta method. For this method we had to check the stability of the solution. We compared the findings of the two methods to determine the accuracy of the approximations and concluded the following.

- We found that for bigger displacements, the two approximations could differ. Remember that we considered only small displacements for the MSP method. This is a logical explanation why the approximations could be off.
- For some pairs of initial conditions, the nonlinear part was fully neglected in $\hat{\phi}_a$. For those cases the MSP method did not deliver any insight on the nonlinear behaviour of the system. To avoid this problem, one could add the next order term to the approximation.
- For some values, the RK4 method was not stable. To avoid studying the unstable solution, we plotted the nearest values for which the solution was stable. This was another reason why the approximations differed from each other.
- We found that the more ϵ approached zero, the more the numerical approximation $\hat{\phi}_n$ and analytical approximation $\hat{\phi}_a$ were similar. From this we conclude that for ϵ approaching zero, the approximations are very accurate. But of course the smaller ϵ is, the less the nonlinear behaviour is included.
  Taking a bigger ϵ to include the nonlinear behaviour more would decrease the accuracy of $\hat{\phi}_a$. One should consider the approximation $\hat{\phi}_b = \hat{\phi}_0 + \epsilon\hat{\phi}_1$ instead, or even add more terms of extra orders if the accuracy is to be maintained.

If the reflections of waves induced by a nonlinear spring at the boundary are to be studied, the solution to the nonlinear spring system is to be approximated. By the research done in this thesis we attained knowledge on the accuracy of the approximations found by the considered methods.

8 Discussion

Since in practice the approximation $\hat{\phi}_a = \hat{\phi}_0$ from the MSP method did not always acquire the wanted accuracy, one could consider the approximation $\hat{\phi}_b = \hat{\phi}_0 + \epsilon\hat{\phi}_1$. This requires more calculations, since $\hat{\phi}_2$ has to be derived in such a way that it is bounded on a time scale of $\mathcal{O}(\frac{1}{\epsilon})$. The upside is that it will probably produce an approximation closer to the solution. Again, this can be checked by comparing $\hat{\phi}_b$ to another method.

In this thesis we approximated $\hat{\phi}_n$ using the RK4 method. A problem occurred when the method was not stable. Therefore the approximation could not always be studied for the wanted values. One could use another numerical method that has unconditional stability. In this way the approximations can be compared using the wanted values.

Finally, further research could be done by changing the boundary condition. For example, one could change the behaviour of the nonlinear spring by adding an extra nonlinear term:

$$Tu_x(0,t) = ku(0,t) + k_2u^2(0,t) + k_3u^3(0,t).$$

Or one could add a damper to the system:

$$Tu_x(0,t) = ku(0,t) + k_2u^2(0,t) + \alpha u_t(0,t).$$

9 Bibliography

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