QUASI-GROUP EXTENSIONS OF ABELIAN GROUPS

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PROEFSCHRIFT

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CONTENTS

TABLE OF NOTATIONS

$A \times B$	The Cartesian product of a set A and a set B.
$A \otimes B$	The direct product of a group A and a group B.
$A \oplus B$	The direct sum of an abelian group A and an abelian group B.
$A \underset{\sim}{\times} B$	A subdirect product of the groups A and B.
$A \underset{\sim}{+} B$	A subdirect sum of the abelian groups A and B.
$A < B$	A is a subgroup of a group B.
$A \triangleleft B$	A is a normal subgroup of a group B.
$\mathfrak{A}(A)$	The *automorphism group* of a group A.
$\mathfrak{A}^*(A)$	The factor group of $\mathfrak{A}(A)$ modulo the normal subgroup of inner automorphisms of A.
$Z(G)$	The centre of a group G.
$A[n]$	Subgroup of A consisting of elements annihilated by n.
$e, e', \varepsilon, \varepsilon'$	The identity element of a multiplicative group.
$0, 0'$	The identity element of an additive group.
θ_0	The mapping of all the elements of a group A onto the identity element of a group B.
$K(\varphi)$	The kernel of a homomorphism φ.
R_0	The additive group of the rational numbers.
$C(p^\infty)$	The quasi cyclic group belonging to the prime p.
V_4	Klein's four-group.

CHAPTER I

THE EXTENSION PROBLEM

§ 1.1 Introduction

Given two groups A and B, then a group G containing a normal subgroup $A' \cong A$ such that $G/A' \cong B$, is called an *extension of* A *by* B (A' and A may be identified for a fixed isomorphism between them). The direct product $A \otimes B$ constitutes an extension of A by B for arbitrary groups A and B.

The problem to find for given groups A and B all extensions G in this sense was first posed by O. Hölder [11]. In a later article [12], he developed a method for the construction of all possible extensions G for finite groups A and B and he applied it to find all groups of a certain given order. Hölder also derived conditions under which G possesses a subgroup isomorphic to B ([12], p. 329). Extensions that have this property are called *splitting* (§ 1.4).

O. Schreier made a large contribution towards the development of the extension theory in his articles [15] and [16]. He considered extensions of arbitrary groups A and B and expressed the problem of finding all extensions of A by B in terms of so-called *factor systems* (§ 1.2) and of systems of automorphisms of A satisfying specified conditions. In time the problem became known as the *Hölder-Schreier extension problem.*

R. Baer has vastly contributed to the literature of the extension theory. In [1] he pointed out that with every extension of A by B corresponds in a particular way a homomorphism (the so-called *associated homomorphism*) of B into the factor group $\mathfrak{A}^*(A)$ of the group of all automorphisms of A modulo the normal subgroup of all inner automorphisms of A. Conversely, he proved by means of a counter example that if A is non-commutative, then an arbitrary given homomorphism of B into $\mathfrak{A}^*(A)$ need not be associated with some extension of A by B. If A is abelian, however, then each homomorphism of B into the automorphism group $\mathfrak{A}(A)$ of A is associated with some extension of A by B. The description of all extensions of A by B by means of factor systems and homomorphisms of B into $\mathfrak{A}^*(A)$ is therefore reduced to the case where A is abelian. In the sequel this restriction on A will always be assumed.

Eilenberg and MacLane [6] established the connection between the Ext group (§ 3.1) and certain groups of homomorphisms.

The problem may be approached along two distinct lines: by means of factor systems (§ 1.2) or by means of *cohomology groups* (§ 1.3).

The concept of a group extension can be generalized by means of *quasi-groups*. G is called a quasi-group if it is closed under a binary operation, if it has an identity element e and if each element $g \in G$ has a unique inverse g^{-1} in G. (e.g. Hausmann and Ore [10]). Such a generalization is made in § 2.1 where also the notion of a factor system is generalized to that of a *quasi factor system*. In chapter II some properties of quasi factor systems are being studied and a few special classes of quasi factor systems are introduced. In chapter III a generalization of the classical Ext group is made and the structure of this generalized group is studied using *inter alia* some results obtained in chapter II.

§ 1.2 Factor systems

We shall always take the operation in an abelian group to be addition. In view of this convention, we consider any extension G of an abelian group A to be additive even if G is non-commutative.

Every element g of a given extension G of the abelian group A by B induces an automorphism in A through the mapping $a \rightarrow -g + a + g$ for all $a \in A$. Elements of the same coset of A in G induce the same automorphism in A. The elements of A itself induce the identity automorphism in A. To each coset of A in G an element of $\mathfrak{A}(A)$ is associated in this way. The mapping is a homomorphism and by $G/A \cong B$ we now have a homomorphism $\theta: B \rightarrow \mathfrak{A}(A)$, called the *associated homomorphism of the extension* G. The homomorphism mapping every element of B onto the identity element of $\mathfrak{A}(A)$, will always be denoted by θ_0. We shall refer to θ_0 as the *trivial homomorphism*.

If we choose for every $\alpha \in B$ a representative $g(\alpha)$ in the corresponding coset of A in G ($g(\varepsilon) = 0$), then $g(\alpha) + g(\beta)$ lies in the same coset of A in G as $g(\alpha\beta)$, i.e. $g(\alpha) + g(\beta) = g(\alpha\beta) + f(\alpha, \beta)$ where $f(\alpha, \beta) \in A$. The set of all elements $f(\alpha, \beta)$ with α and β running through B and the representatives $g(\alpha)$ fixed, satisfies the condition

$$f(\alpha, \varepsilon) = f(\varepsilon, \alpha) = 0 \text{ for all } \alpha \in B \tag{1.2.1}$$

and by the associative law in G we have

$$f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) = f(\alpha, \beta\gamma) + f(\beta, \gamma) \tag{1.2.2}$$

for all $\alpha, \beta, \gamma \in B$. A function $f: B \times B \rightarrow A$ satisfying conditions (1.2.1) and (1.2.2) is called a (B, A, θ)-*factor system.*

Conversely, given an abelian group A $\{0, a, b, c, \ldots\}$ and an arbitrary group B $\{\varepsilon, \alpha, \beta, \gamma, \ldots\}$, a homomorphism $\theta: B \rightarrow \mathfrak{A}(A)$ and a (B, A, θ)-factor system f, then the set $G = B \times A$ forms a group under the operation

$$(\alpha, a) + (\beta, b) = (\alpha\beta, a(\beta\theta) + b + f(\alpha, \beta)). \tag{1.2.3}$$

The set of all elements of the form (ε, a) is a subgroup $A' \subseteq G$ and $A' \cong A$ under the mapping $a \longleftrightarrow (\varepsilon, a)$. G is an extension of A by B with θ as associated homomorphism (See [15]). We say G is defined by the *extension system* (θ, f).

Let $F(B, A, \theta)$ denote the set of all possible (B, A, θ)-factor systems. Define an operation in $F(B, A, \theta)$ as follows:

$$(f + g)(\alpha, \beta) = f(\alpha, \beta) + g(\alpha, \beta). \tag{1.2.4}$$

The commutative law in A implies the closure of $F(B, A, \theta)$ under this operation and the associative law holds in $F(B, A, \theta)$ by force of its validity in A. The factor system f_0 with $f_0(\alpha, \beta) = 0$ for all $\alpha, \beta \in B$ is the identity element in $F(B, A, \theta)$. If $f \in F(B, A, \theta)$, then also $-f$ defined by $(-f)(\alpha, \beta) = -f(\alpha, \beta)$ for all $\alpha, \beta \in B$, therefore $F(B, A, \theta)$ is an abelian group.

Two extensions G_1 and G_2 of A by B are called *equivalent,* $G_1 \sim G_2$, if there exists an isomorphism between them leaving both A and B invariant. Equivalent extensions have the same associated homomorphism. If G_1 and G_2 are equivalent extensions of A by B determined by the extension systems (θ, f) and (θ, g) resp., then there exists a function ω: $B \to A$ such that

$$\omega(\varepsilon) = 0 \tag{1.2.5}$$

and

$$h(\alpha, \beta) = f(\alpha, \beta) - g(\alpha, \beta) = \{\omega(\alpha)\}(\beta\theta) + \omega(\beta) - \omega(\alpha\beta) \tag{1.2.6}$$

for all $\alpha, \beta \in B$. A factor system h for which there exists a function ω: $B \to A$ satisfying (1.2.5) and (1.2.6) is called a *transformation system*. The set $T(B, A, \theta)$ of all (B, A, θ)-transformation systems is a subgroup of $F(B, A, \theta)$. From (1.2.6) it follows that the factor systems of equivalent extensions of A by B with associated homomorphism θ belong to the same coset of $T(B, A, \theta)$ in $F(B, A, \theta)$. Therefore, the non-equivalent extensions of an abelian group A by an arbitrary group B with associated homomorphism θ correspond in a one-one way with the elements of the group

$$\mathrm{Ext}(B, A, \theta) = F(B, A, \theta)/T(B, A, \theta). \tag{1.2.7}$$

We call $\mathrm{Ext}(B, A, \theta)$ the *group of extensions of* A *by* B *with associated homomorphism* θ. Note that $\mathrm{Ext}(B, A, \theta)$ is abelian.

§ 1.3 **Cohomology groups**

The notion of the group $\mathrm{Ext}(B, A, \theta)$ defined in § 1.2 may also be developed as follows:

If B^n denotes the cartesian product of n copies of an arbitrary group B and A is an abelian group, then every function f: $B^n \to A$ is called an *n-dimensional cochain*. In particular, the zero-dimensional cochains are the elements of A. The set of all n-dimensional cochains is an abelian group $C^n(B, A)$ after introduction of the following operation:

$$(f_1 + f_2)(\alpha_1, \alpha_2, \ldots, \alpha_n) = f_1(\alpha_1, \alpha_2, \ldots, \alpha_n) + f_2(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

In particular $C^0(B, A) = A$.

Let $\theta : B \to \mathfrak{A}(A)$ be a fixed homomorphism. Then with every n-dimensional cochain f we associate an $(n + 1)$-dimensional cochain δf, called the *coboundary* of f and defined by:

$$(\delta f)(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) = f(\alpha_2, \ldots, \alpha_{n+1}) +$$
$$+ \sum_{k=1}^{n} (-1)^k f(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k \alpha_{k+1}, \ldots, \alpha_{n+1}) +$$
$$+ (-1)^{n+1} \{ f(\alpha_1, \ldots, \alpha_n) \} (\alpha_{n+1}\theta). \qquad (1.3.1)$$

From this definition it follows directly that

$$\delta(f_1 + f_2) = \delta f_1 + \delta f_2, \qquad (1.3.2)$$

so that the mapping $f \rightarrow \delta f$ is a homomorphism of $C^n(B,A)$ into $C^{(n+1)}(B,A)$. It can also be proved that

$$\delta(\delta f) = 0. \qquad (1.3.3)$$

An n-dimensional cochain with zero coboundary is called a *cocycle*. The n-dimensional cocycles form a subgroup $Z^n(B, A, \theta)$ of $C^n(B, A)$.

Consider on the other hand, for $n > 0$, the n-dimensional cochains that are coboundaries of some $(n-1)$-dimensional cochains. They form by virtue of (1.3.2) a subgroup $D^n(B, A, \theta)$ of $C^n(B, A)$. $D^0(B, A, \theta) = 0$ by definition.

By (1.3.3) we have $D^n(B, A, \theta) \subseteq Z^n(B, A, \theta)$.

The group $$H^n(B, A, \theta) = Z^n(B, A, \theta) / D^n(B, A, \theta) \qquad (1.3.4)$$

is called the *n-th cohomology group of* B *over* A *relative to* θ.

The coboundary of the two-dimensional cochain $f(\alpha_1, \alpha_2)$ is given by $(\delta f)(\alpha_1, \alpha_2, \alpha_3) = f(\alpha_2, \alpha_3) - f(\alpha_1\alpha_2, \alpha_3) + f(\alpha_1, \alpha_2\alpha_3) - \{ f(\alpha_1, \alpha_2) \} (\alpha_3\theta)$.
i.e. f is a cocycle if and only if

$f(\alpha_1\alpha_2, \alpha_3) + \{ f(\alpha_1, \alpha_2) \} (\alpha_3\theta) = f(\alpha_1, \alpha_2\alpha_3) + f(\alpha_2, \alpha_3)$, i.e. if $f \in F(B, A, \theta)$.

Therefore $Z^2(B, A, \theta) = F(B, A, \theta)$.

On the other hand, $f(\alpha_1, \alpha_2) \in D^2(B, A, \theta)$ if and only if there exists a one-dimensional cochain $\omega(\alpha_1)$ such that $f(\alpha_1, \alpha_2) = \{ \omega(\alpha_1) \} (\alpha_2\theta) + \omega(\alpha_2) - \omega(\alpha_1\alpha_2)$, i.e. if $f \in T(B, A, \theta)$.

Thus $D^2(B, A, \theta) = T(B, A, \theta)$.
Hence $H^2(B, A, \theta) = \mathrm{Ext}(B, A, \theta)$.

§ 1.4 Splitting extensions

If A and B are subgroups of an arbitrary group G such that each $g \in G$ may be written in the form $g = ab$ with $a \in A$, $b \in B$, then we say G *is generated by* A *and* B and write $G = \{A, B\}$. If

(a) $A \lhd G$,
(b) $B \lhd G$,
(c) $A \cap B = e$,
(d) $G = \{A, B\}$,

then we call G the *direct product* of A and B and write $G = A \otimes B$. If both A and B are abelian, we call G the *direct sum* of A and B and write $G = A \oplus B$. When conditions (c) and (d) hold for the subgroups A and B but (a) and (b) are not satisfied, we call G a *semi-direct product* of A and B.

If G is a group containing an abelian subgroup A and a subgroup B such that conditions (a), (c) and (d) are satisfied, then we call G a *splitting extension of* A *by* B. Thus, a splitting extension is a direct or certain semi-direct product.

An extension G *of* A $\{0, a, b, c, \ldots\}$ *by* B $\{\varepsilon, \alpha, \beta, \gamma, \ldots\}$ *with extension system* (θ, f_0) *is splitting for every* θ, (1.4.1)
because the subgroup A′ of all elements of the form (ε, a) and the subgroup B′ of all elements of the form $(\alpha, 0)$ satisfy conditions (a), (c) and (d).

If G *is an extension of* A *by* B *with extension system* (θ, f), *then* G *is splitting if and only if* $f \in T(B, A, \theta)$. (1.4.2)

Suppose G is splitting.

Then G contains a subgroup $B' \cong B$; let $\alpha' \to \alpha$ under this isomorphism. Suppose $\omega : B \to A$ is an arbitrary function with $\omega(\varepsilon) = 0$. Choose as representative of the coset of A in G corresponding to $\alpha \in B$ the element $\alpha' + \omega(\alpha)$ for all $\alpha \in B$. Then, by the definition of f we have:

$$\alpha' + \omega(\alpha) + \beta' + \omega(\beta) = \alpha' + \beta' + \omega(\alpha\beta) + f(\alpha, \beta) \text{ for all } \alpha, \beta \in B.$$

$$\text{i.e. } f(\alpha, \beta) = -\omega(\alpha\beta) - \beta' + \omega(\alpha) + \beta' + \omega(\beta),$$
$$= \{\omega(\alpha)\}(\beta\theta) + \omega(\beta) - \omega(\alpha\beta), \text{ i.e. } f \in T(B, A, \theta).$$

Conversely, suppose $f \in T(B, A, \theta)$.

Thus if G and G_0 are determined by the extension systems (θ, f) and (θ, f_0) resp., then G and G_0 are equivalent.

Therefore G is splitting by (1.4.1).

Clearly, the non-equivalent splitting extensions of A by B stand in a one-one correspondence with the distinct homomorphisms of B into $\mathfrak{A}(A)$.

§ 1.5 **Central extensions**

G is called a *central extension* of A by B if A lies in the centre of G. If G is a central extension of B by an arbitrary group B, then all automorphisms $a \to -g + a + g$, $g \in G$ of A are equal to the identity automorphism, i.e. G has θ_0 as associated homomorphism. Suppose, conversely, that G is an extension of A $\{0, a, b, c, \ldots\}$ by an arbitrary group B $\{\varepsilon, \alpha, \beta, \gamma, \ldots\}$ with extension system (θ_0, f), $f \in F(B, A, \theta_0)$. With A′ as in § 1.2, we have

$$A' \subseteq Z(G),$$

because

$$(\varepsilon, a) + (\beta, b) = (\beta, a + b + f(\varepsilon, \beta)) = (\beta, b) + (\varepsilon, a) \text{ for}$$

all $(\varepsilon, a) \in A'$, $(\beta, b) \in G$, i.e.

G *is a central extension of* A *by* B *if and only if it has* θ_0 *as associated homomorphism.* (1.5.1)

Conrad [4] illustrated the relationship between central extensions and bilinear mappings, defined as follows:

If A is an abelian group and B an arbitrary group, then a mapping $g: B \times B \to A$ is called *bilinear* if for all $\alpha, \beta, \gamma \in B$

$$g(\alpha\beta, \gamma) = g(\alpha, \gamma) + g(\beta, \gamma)$$

and

$$g(\alpha, \beta\gamma) = g(\alpha, \beta) + g(\alpha, \gamma).$$

For a bilinear mapping $g: B \times B \to A$ and arbitrary $\alpha \in B$ we have

$$g(\alpha, \alpha) = g(\alpha\varepsilon, \alpha) = g(\alpha, \alpha) + g(\varepsilon, \alpha)$$

or $g(\varepsilon, \alpha) = 0$. Similarly $g(\alpha, \varepsilon) = 0$ for all $\alpha \in B$.

Moreover, $g(\alpha\beta, \gamma) + g(\alpha, \beta)$

$= g(\alpha, \gamma) + g(\beta, \gamma) + g(\alpha, \beta)$

$= g(\alpha, \beta\gamma) + g(\beta, \gamma)$, since A is abelian.

i.e. $g \in F(B, A, \theta_0)$.

Every bilinear mapping of $B \times B$ *into* A *is a* (B, A, θ_0)*-factor system.* (1.5.2)

Conversely, Conrad [4] proved that if a (B, A, θ_0)-factor system f is linear in one variable, then it is bilinear.

Under certain conditions, the group $F(B, A, \theta_0)$ is splitting. In order to prove this, we define:

If for a given $f \in F(B, A, \theta)$ we have that $f(\alpha, \beta) = f(\beta, \alpha)$ (resp. $f(\alpha, \beta) = -f(\beta, \alpha)$) for all $\alpha, \beta \in B$, then f is called *symmetric* (resp. *skew-symmetric*).

The set $P(B, A, \theta)$ of all symmetric (B, A, θ)-factor systems as well as the set $Q(B, A, \theta)$ of all skew-symmetric (B, A, θ)-factor systems are subgroups of $F(B, A, \theta)$.

Conrad [3] proved that if

(a) $A[2] = 0$,

(b) B is abelian,

then every $f \in Q(B, A, \theta)$ is bilinear. By (1.5.2) he thus proved that in this case, $Q(B, A, \theta_0)$ is the set of all bilinear skew-symmetric mappings of $B \times B$ into A.

Assume that we have in addition to conditions (a) and (b), also

(c) $A = 2A$,

and let $f \in F(B, A, \theta_0)$. Define:

$$p : B \times B \to A; \; p(\alpha, \beta) = \tfrac{1}{2}[f(\alpha, \beta) + f(\beta, \alpha)]$$

and $$q : B \times B \to A; \; q(\alpha, \beta) = \tfrac{1}{2}[f(\alpha, \beta) - f(\beta, \alpha)].$$

It is readily shown that $p, q \in F(B, A, \theta_0)$. Clearly p is symmetric, q skew-symmetric and

$$f = p + q. \tag{1.5.3}$$

Moreover, this representation of f as the sum of a symmetric and a skew-symmetric function is unique. Therefore

$$F(B, A, \theta_0) = P(B, A, \theta_0) \oplus Q(B, A, \theta_0). \tag{1.5.4}$$

Furthermore, if $f \in T(B, A, \theta_0)$, then there exists a function $\omega : B \to A$ such that $f(\alpha, \beta) = \omega(\alpha) + \omega(\beta) - \omega(\alpha\beta) =$
$= \omega(\beta) + \omega(\alpha) - \omega(\beta\alpha) = f(\beta, \alpha)$ for all $\alpha, \beta \in B$.

$$\text{i.e.} \quad T(B, A, \theta_0) \subseteq P(B, A, \theta_0). \tag{1.5.5}$$

All abelian extensions are of course central extensions. Suppose B is abelian and let G be an extension of A by B with extension system (θ_0, f), $f \in F(B,A,\theta_0)$. Then for arbitrary (α, a), $(\beta, b) \in G$, we have

$$(\alpha, a) + (\beta, b) = (\alpha + \beta, \; a + b + f(\alpha, \beta))$$

and $$(\beta, b) + (\alpha, a) = (\beta + \alpha, \; b + a + f(\beta, \alpha)),$$

i.e. G is abelian if and only if $f(\alpha, \beta) = f(\beta, \alpha)$ for all $\alpha, \beta \in B$, that is if $f \in P(B, A, \theta_0)$.

An extension G *of* A *by an abelian group* B *with extension system* (θ_0, f) *is abelian if and only if* f *is symmetric.* (1.5.6)

A group is called *locally cyclic* if all of its finitely generated subgroups are cyclic (e.g. the groups R_0 and $C(p^\infty)$).

If B *is locally cyclic, then an extension* G *of* A *by* B *is abelian if and only if* G *has the trivial homomorphism as associated homomorphism.* [1]) (1.5.7)

If G is abelian, then it has θ_0 as associated homomorphism by (1.5.1).

Conversely, Loonstra [14] proved that if B is cyclic and G has extension system (θ_0, f), then f is symmetric, thus G is abelian.

If B is locally cyclic, then there exists a cyclic subgroup $C \subseteq B$ containing any two given elements $\alpha_0, \beta_0 \in B$. The method applied by Loonstra [14] is now applicable in proving that $f(\alpha, \beta) = f(\beta, \alpha)$ for all $\alpha, \beta \in C$, in particular $f(\alpha_0, \beta_0) = f(\beta_0, \alpha_0)$. Since $\alpha_0, \beta_0 \in B$ are arbitrary, $f(\alpha, \beta) = f(\beta, \alpha)$ for all $\alpha, \beta \in B$.

i.e. G is abelian.

To conclude this survey of the relationship between central extensions and bilinear mappings, we mention that Conrad [4] proved the following:

Assume groups A and B satisfy conditions (a) — (c) of this paragraph and that $f \in F(B, A, \theta_0)$ is bilinear. Let $f = p + q$ be the representation (1.5.3) and suppose G and H are the extensions of A by B with extension systems (θ_0, f) and (θ_0, q) resp. Then G and H are equivalent. If, moreover, f is symmetric, then $q = f_0$ and

$$G \sim H = A \oplus B.$$

1) This result with B cyclic was proved by Baer [1].

CHAPTER II

QUASI FACTOR SYSTEMS

§ 2.1 Introduction

In this paragraph, we generalize the notion of a factor system. To this end, we consider an abelian group A $\{0, a, b, c, \dots\}$, an arbitrary group B $\{\varepsilon, \alpha, \beta, \gamma, \dots\}$ and a fixed homomorphism $\theta : B \to \mathfrak{A}(A)$. Let $f : B \times B \to A$ be an *arbitrary function* only satisfying condition (1.2.1). Consider the set $G = B \times A$ with operation (1.2.3) using f as defined above.

It follows that for any $(\alpha, a), (\beta, b), (\gamma, c) \in G$,

$$\{(\alpha, a) + (\beta, b)\} + (\gamma, c) = [(\alpha, a) + \{(\beta, b) + (\gamma, c)\}] + \\ + (\varepsilon, f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma)),$$

i.e. the operation (1.2.3) in G is associative if and only if

$$f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma) = 0, \quad (2.1.1)$$

thus if and only if f is a (B, A, θ)-factor system.

It is our aim to point out how a generalization of the concept of an extension of A by B as defined in chapter I results if we sacrifice the associativity of the operation (1.2.3) by replacing (2.1.1) by a pair of weaker conditions. Consider a fixed subgroup $C \subseteq A$.

DEFINITION 1:

A function $f : B \times B \to A$ *satisfying the conditions* (1.2.1) and also

$$f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma) \in C \quad (2.1.2)$$

and $\{f(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) = f(\alpha, \alpha^{-1})$ (2.1.3) *for all* $\alpha, \beta, \gamma \in B$, *is called a* $(B, A, \theta; C)$*-quasi factor system.*

We note that every (B, A, θ)-factor system is a $(B, A, \theta; C)$-quasi factor system for arbitrary $C \subseteq A$. The set $F(B, A, \theta; C)$ of all $(B, A, \theta; C)$-quasi factor systems is an abelian group under the operation

$$(f_1 + f_2)(\alpha, \beta) = f_1(\alpha, \beta) + f_2(\alpha, \beta) \text{ for all } \alpha, \beta \in B$$

and $F(B, A, \theta) \subseteq F(B, A, \theta; C)$.

If $f \in F(B, A, \theta; C)$, then the set $G = B \times A$ is closed under the operation (1.2.3) and $(\varepsilon, 0)$ is the identity element of G while

$$\begin{aligned} -(\alpha, a) &= (\alpha^{-1}, -a(\alpha\theta)^{-1} - \{f(\alpha^{-1}, \alpha)\}(\alpha\theta)^{-1}) \\ &= (\alpha^{-1}, -a(\alpha\theta)^{-1} - f(\alpha, \alpha^{-1})). \end{aligned}$$

We show that G may be considered as an "extension" (henceforth called a *quasi-group extension*) of A by B. If we denote the subset of G consisting of all elements of the form (ε, a) by A', then the mapping $a \rightarrow (\varepsilon, a)$ establishes an isomorphism $A \cong A'$. Furthermore, we divide G into mutually disjunct classes by the rule:

(α, a) *and* (β, b) *belong to the same class if and only if* $\alpha = \beta$. (2.1.4)

Since $(\alpha, a) + A' = (\alpha, b) + A'$ for all $a, b \in A$, we denote the class to which (α, a) belongs by $(\alpha, a) + A'$ $(= A' + (\alpha, a))$. The set of all such classes is denoted by G/A'.

Making use of the fact that (2.1.4) is a regular partition, we define an operation in G/A' as follows:

$$\{(\alpha, a) + A'\}\{(\beta, b) + A'\} = (\alpha, a) + (\beta, b) + A'.$$

The mapping ψ: $G/A' \rightarrow B$; $\{(\alpha, a) + A'\}\psi = \alpha$ is readily seen to be an isomorphism. Since we now have $A \cong A'$ and $G/A' \cong B$, we call G a *quasi-group extension of* A *by* B.

A convenient concept was introduced by Garrison [9] who called an element c of a quasi-group H *centre associative in* H if $x(cy) = (xc)y$ for all $x, y \in H$. *Left associative* and *right associative* elements are defined similarly.

The elements of A' *are centre, left and right associative in* G. (2.1.5)

The notion of an associated homomorphism for quasi-group extensions is introduced in the same way as for group extensions

(§ 1.2). In the present case, G has θ as associated homomorphism, since for arbitrary $(\varepsilon, b) \in A'$ and all $(\alpha, a) \in G$ we have

$$- (\alpha, a) + (\varepsilon, b) + (\alpha, a) \qquad \text{(Applying (2.1.5.))}$$
$$= (\alpha^{-1}, - a(\alpha\theta)^{-1} - \{ f(\alpha^{-1}, \alpha) \} (\alpha\theta)^{-1}) + (\alpha, b(\alpha\theta) + a)$$
$$= (\varepsilon, b(\alpha\theta)).$$

Suppose, conversely, that G is a quasi-group with the following properties:

(a) G contains an abelian subgroup A.

(b) The elements of A are left, centre and right associative in G.

(c) $\{(-g + a) + (g + b)\} + (-g + c) = (-g + a) + \{(g + b) + (-g + c)\}$ for all $g \in G$; $a, b, c \in A$.

(d) $(-h - g) + a + (g + h) = -h + (-g + a + g) + h$ for all $g, h \in G$; $a \in A$.

(e) There exists an epimorphism $\varphi: G \rightarrow B$ with $K(\varphi) = A$.

Then it is readily shown that with this quasi-group extension G of A by B, a homomorphism $\theta: B \rightarrow \mathfrak{A}(A)$ is associated as well as a $(B, A, \theta; C)$-quasi factor system f, where $C \subseteq A$ denotes the associator group of G.

In the present chapter, we shall be concerned with the construction of factor systems; we shall deduce a few properties of quasi factor systems in general and we shall introduce some special types of quasi factor systems. In the next chapter, the results obtained will be applied to investigate the structure of the group of all quasi-group extensions.

It will be convenient to make the following convention:

Let A $\{0, a, b, c, \ldots\}$ be an abelian group with a subgroup C, B $\{\varepsilon, \alpha, \beta, \gamma, \ldots\}$ an arbitrary group and $\theta: B \rightarrow \mathfrak{A}(A)$ a fixed homomorphism. This system will henceforth be refered to as *the general extension system* $\{B, A, \theta; C\}$.

If in the general extension system $\{B, A, \theta; C\}$, we define
$$A(B\theta) = \{a \mid a \in A \text{ and } a\psi = a \text{ for all } \psi \in B\theta\}, \qquad (2.1.6)$$
then $A(B\theta)$ is a subgroup of A. This subgroup plays an important role in the sequel.

§ 2.2 The construction of factor systems

In the general extension system $\{B, A, \theta; C\}$, the problem of constructing all possible $(B, A, \theta; C)$-quasi factor systems arises. Viljoen [17] gave methods of construction for certain special cases. In what follows, methods of construction will be developed for the case

(a) $B = C(n)$, A an abelian group, $C = 0$ and θ arbitrary,
(b) $B = V_4$, A an abelian group with $A[2] = 0$, $C = 0$ and θ to be specified.

(a) Suppose in this case θ is generated by $1 \to \alpha$.

DEFINITION 2:

If for a given function $f: B \times B \to A$ *we have for the fixed elements* $r, s, t \in B$ *that*
$f(r + s, t) + \{f(r, s)\}\alpha^t = f(r, s + t) + f(s, t)$, *then we call* (r, s, t) *a* (C, f, θ)*-system and write* $(r, s, t) \in (C, f, \theta)$.

Note that if f satisfies the condition (1.2.1), then $(r, s, t) \in (C, f, \theta)$ for all $r, s, t \in B$ if and only if $f \in F(B, A, \theta)$.

Any function $f: B \times B \to A$ can be represented by a $n \times n$ matrix $M = (m_{ij})$ over A with $m_{r,s} = f(r, s)$. By (1.2.1), a necessary condition that $f \in F(B, A, \theta)$ is that the first row and first column of M contain only zero's. After the remainder of the second row has been chosen in some way or other, the matrix can be completed inductively by means of (1.2.2), i.e. in such a way that the condition

$$(r, 1, t) \in (C, f, \theta) \text{ for all } r, t \in B \qquad (2.2.1)$$

holds. In order that $f \in F(B,A, \theta)$, however, it is necessary that

$$(r, s, t) \in (C, f, \theta) \text{ for all } r, s, t \in B. \qquad (2.2.2)$$

The following lemma states the conditions under which (2.2.1) implies (2.2.2).

LEMMA 1:

Let in the general extension system $\{B, A, \theta; C\}$, $B = C(n)$ *and* θ *be induced by* $1 \to \alpha$. *If* $C = 0$ *and* $f: B \times B \to A$ *is any given* *function, then* $(r,s,t) \in (C,f,\theta)$ *for all* $r,s,t \in B$ *if* $(r,1,t) \in (C,f,\theta)$ *for all* $r,t \in B$.

PROOF:

Suppose $(r,k,t) \in (C,f,\theta)$ for all $r,t \in B$ and some fixed $k \in B$. Then we prove $(r,k+1,t) \in (C,f,\theta)$ for all $r,t \in B$. We have

(i) $(r+k,1,t) \in (C,f,\theta) \Longrightarrow$
$f(r+k+1,t) + \{f(r+k,1)\}\alpha^t = f(r+k,1+t) + f(1,t),$

(ii) $(r,k,1) \in (C,f,\theta) \Longrightarrow$
$f(r+k,1) + \{f(r,k)\}\alpha = f(r,k+1) + f(k,1),$

(iii) $(r,k,t+1) \in (C,f,\theta) \Longrightarrow$
$f(r+k,t+1) + \{f(r,k)\}\alpha^{t+1} = f(r,k+t+1) + f(k,t+1).$

(iv) $(k,1,t) \in (C,f,\theta) \Longrightarrow$
$f(k+1,t) + \{f(k,1)\}\alpha^t = f(k,1+t) + f(1,t).$

From (i) and (ii) follows

$$\begin{aligned} & f(r+k+1,t) + \{f(r,k+1)\}\alpha^t \\ = {} & f(r+k,1+t) + f(1,t) + \{f(r,k)\}\alpha^{t+1} - \{f(k,1)\}\alpha^t \\ = {} & f(r,k+1+t) + f(k,t+1) + f(1,t) - \{f(k,1\}\alpha^t \text{ (by (iii))} \\ = {} & (r,k+1+t) + f(k+1,t) \text{ (by (iv))}, \end{aligned}$$

i.e. $(r,k+1,t) \in (C,f,\theta)$ for all $r,t \in B$.

i.e. $(r,s,t) \in (C,f,\theta)$ for all $r,s,t \in B$.

THEOREM 1:

Let in the general extension system $\{B, A, \theta; C\}$, $B = C(n)$ *and* θ *be induced by* $1 \rightarrow \alpha$. *If* $C = 0$ *and* $f: B \times B \rightarrow A$ *satisfies the conditions:*

(i) $f(r,0) = f(0,r) = 0$ *for all* $r \in B$,

(ii) $(\sum_{i=1}^{n-1} f(1,i))\alpha = \sum_{i=1}^{n-1} f(1,i),$

(iii) $f(r+1,s) = f(r,1+s) + f(1,s) - \{f(r,1)\}\alpha^s$

for $r = 1, \ldots, n-2$; $s = 1, \ldots, n-1$,

then $f \in F(B, A, \theta)$.

PROOF:

By definition $(r, 1, s) \in (C, f, \theta)$ for $r = 0, 1, \ldots, n-2$ and all $s \in B$.

Furthermore $(n-1, 1, s) \in (C, f, \theta)$ for all $s \in B$ if and only if $\{f(n-1, 1)\}\alpha^s = f(n-1, 1+s) + f(1, s)$ for all $s \in B$. (a)

But $(n-2, 1, s+1) \in (C, f, \theta) \Longrightarrow$

$$f(n-1, s+1) + \{f(n-2, 1)\}\alpha^{s+1} = f(n-2, s+2) + f(1, s+1)$$

and $(n-2, 1, 1) \in (C, f, \theta) \Longrightarrow$

$$f(n-1, 1) + \{f(n-2, 1)\}\alpha = f(n-2, 2) + f(1, 1).$$

Thus (a) becomes:

$$\{f(n-2, 2)\}\alpha^s + \{f(1, 1)\}\alpha^s = f(n-2, s+2) + f(1, s+1) + f(1, s). \qquad \text{(b)}$$

But $(n-3, 1, s+2) \in (C, f, \theta) \Longrightarrow$

$$f(n-2, s+2) + \{f(n-3, 1)\}\alpha^{s+2} = f(n-3, s+3) + f(1, s+2)$$

and $(n-3, 1, 2) \in (C, f, \theta) \Longrightarrow$

$$f(n-2, 2) + \{f(n-3, 1)\}\alpha^2 = f(n-3, 3) + f(1, 2).$$

Now (b) becomes

$$\{f(n-3, 3)\}\alpha^s + \{f(1, 2)\}\alpha^s + \{f(1, 1)\}\alpha^s = f(n-3, s+3) + f(1, s+2) + f(1, s+1) + f(1, s). \qquad \text{(c)}$$

Proceeding in this way, we finally have:

$$\left\{\sum_{i=1}^{n-1} f(1, i)\right\} \alpha^s = f(1, s+n-1) + \ldots + f(1, s+1) + f(1, s) = \sum_{i=1}^{n-1} f(1, i) \text{ for all } s \in B. \qquad \text{(d)}$$

But (d) is satisfied for all $s \in B$ by the definition of f.

i.e. $(r, 1, s) \in (C, f, \theta)$ for all $r, s \in B$.

Applying Lemma 1 we have

$(r, s, t) \in (C, f, \theta)$ for all $r, s, t \in B$.
i.e. $f \in F(B, A, \theta)$.

REMARK:

Under the stated conditions, theorem 1 yields all possible (B, A, θ)-factor systems. To prove this, we consider an arbitrary $f \in F(B, A, \theta)$. Then f satisfies conditions (i) and (iii) by the definition of a factor system.

Furthermore, we have

$$f(2, 1) + \{f(1, 1)\}\alpha = f(1, 2) + f(1, 1).$$
$$f(3, 1) + \{f(1, 2)\}\alpha = f(1, 3) + f(2, 1).$$

....

$$f(n-1, 1) + \{f(1, n-2)\}\alpha = f(1, n-1) + f(n-2, 1).$$
$$\{f(1, n-1)\}\alpha = f(n-1, 1).$$

i.e. $\{\sum_{i=1}^{n-1} f(1, i)\}\alpha + \sum_{i=1}^{n-1} f(1, i).$

Thus f satisfies condition (ii) also and can therefore be constructed in the way described in theorem 1.

(b) Suppose A is an abelian group with $A[2] = 0$.
Let $R = \{r\} \cong C(2)$ and $S = \{s\} \cong C(2)$, thus $R \oplus S \cong V_4$. (2.2.3)

Denote by α the automorphism $a \rightarrow -a$ for all $a \in A$, and consider the homomorphisms:

$$\varphi : R \rightarrow \mathfrak{A}(A);\ r\varphi = \alpha$$
and $$\psi : S \rightarrow \mathfrak{A}(A);\ s\psi = \alpha.$$

Denote the homomorphisms of V_4 into $\mathfrak{A}(A)$ generated by the pairs (θ_0, θ_0), (θ_0, ψ) and (φ, ψ) by θ_i $(i = 0, 1, 2)$ resp..

The following three theorems give a method of construction of all possible (V_4, A, θ_i)-factor systems $(i = 0, 1, 2)$. The proofs are very similar and we only prove theorem 2(c). If for the function $f: V_4 \times V_4 \rightarrow A$ we have $f(x, 0) = f(0, x) = 0$ for all $x \in V_4$, then

THEOREM 2(a):

$$f \in F(V_4, A, \theta_0) \Longleftrightarrow \begin{array}{l}(1)\ f(r, s) = f(s, r)\\ (2)\ f(r, r+s) = f(r+s, r)\\ (3)\ f(s, r+s) = f(r+s, s)\end{array}$$

(4) $f(r, r) = f(r, s) + f(r, r + s)$
(5) $f(s, s) = f(r, s) + f(r + s, s)$
(6) $f(r + s, r + s) = f(r, r + s) + f(s, r + s)$

THEOREM 2(b):

$f \in F(V_4, A, \theta_1) \Longleftrightarrow$ (1) $f(r, r) + f(r, s) + f(r, r + s) = 0$
(2) $f(r, r) = f(r + s, r) + f(s, r)$
(3) $f(s, r) + f(s, r + s) = 0$
(4) $f + s, s) = f(r, s)$
(5) $f(r + s, r + s) = f(r + s, r) + f(r + s, s)$
(6) $f(r, s) + f(r + s, r) = 0$
(7) $f(s, s) = 0$

THEOREM 2(c):

$f \in F(V_4, A, \theta_2) \Longleftrightarrow$ (1) $f(r, r) = 0$
(2) $f(s, s) = 0$
(3) $f(r, s) + f(r, r + s) = 0$
(4) $f(r + s, r) = f(s, r)$
(5) $f(s, r + s) + f(s, r) = 0$
(6) $f(r + s, s) = f(r, s)$
(7) $f(r + s, r) + f(r + s, s) +$
$+ f(r + s, r + s) = 0$

PROOF:

Suppose $f \in F(V_4, A, \theta_2)$.

(1) and (2): Since $(r, r, r), (s, s, s) \in (0, f, \theta_2)$ in the notation of Definition 2, we have

$-f(r, r) = f(r, r)$ and $-f(s, s) = f(s, s)$.
i.e. $f(r, r) = f(s, s) = 0$ since $A[2] = 0$.

(3): $f(r, r + s) + f(r, s) = 0$ for $(r, r, s) \in (0, f, \theta_2)$.

(4): $(s, r, r) \in (0, f, \theta_2) \Longrightarrow f(r + s, r) - f(s, r) = 0$.

(5): By $(s, s, r) \in (0, f, \theta_2)$, we have

$$f(s, r + s) + f(s, r) = 0$$

(6): $(r, s, s) \in (0, f, \theta_2)$, i.e. $f(r + s, s) - f(r, s) = 0$

(7): $-f(r + s, r + s) = f(r + s, s) + f(r + s, r)$,
because $(r + s, r + s, r) \in (0, f, \theta_2)$.

Conversely, suppose the function f satisfies the conditions (1) — (7). We now show that $(x, y, z) \in (0, f, \theta_2)$ for all $x, y, z \in V_4$ and thus that $f \in F(V_4, A, \theta_2)$. We note that $(x, y, z) \in (0, f, \theta_2)$ if either $x = 0$, $y = 0$ or $z = 0$.

$(r, r, r) \in (0, f, \theta_2)$ by (1).

$(r, r, s) \in (0, f, \theta_2)$ by (1) and (3).

$(r, r, r + s) \in (0, f, \theta_2)$ by (1) and (3).

$(r, s, r) \in (0, f, \theta_2)$:

$f(r, s) + f(r, r + s) = f(r + s, r) - f(s, r)$ by (3) and (4),

i.e. $f(r + s, r) - f(r, s) = f(r, r + s) + f(s, r)$.

$(r, s, s) \in (0, f, \theta_2)$ by (2) and (6).

$(r, s, r + s) \in (0, f, \theta_2)$:

$f(r + s, r + s) + f(r, s) =$

$= - f(r + s, s) - f(r + s, r) + f(r + s, s)$ by (6) and (7)

$= - f(s, r)$ by (4)

$= f(s, r + s)$ by (5).

$(r, r + s, r) \in (0, f, \theta_2)$:

$f(r, s) + f(r, r + s) = - f(r + s, r) + f(s, r)$ by (3) and (4),

i.e. $f(s, r) - f(r, r + s) = f(r, s) + f(r + s, r)$.

$(r, r + s, s) \in (0, f, \theta_2)$:

$- f(r, r + s) = f(r, s)$ by (3)

$= f(r + s, s)$ by (6).

$(r, r + s, r + s) \in (0, f, \theta_2)$:

$f(r + s, r + s) = - f(r + s, r) - f(r + s, s)$ by (7)

$= - f(s, r) - f(r, s)$ by (4) and (6)

$= f(s, r + s) + f(r, r + s)$ by (3) and (5).

$(s, r, r) \in (0, f, \theta_2)$ by (4).

$(s, r, s) \in (0, f, \theta_2)$:

$f(r + s, s) - f(r, s) = f(s, r + s) + f(s, r)$ by (5) and (6),

i.e. $f(r + s, s) - f(s, r) = f(s, r + s) + f(r, s)$.

$(s, r, r + s) \in (0, f, \theta_2)$:

$f(r + s, r + s) + f(r + s, r) + f(r + s, s) + f(s, r) - f(r + s, r)$

$= f(r, s) + f(r, r + s)$ by (3), (4) and (7),

i.e. $f(r + s, r + s) + f(s, r) = f(r, r + s)$ by (6).

$(s, s, r) \in (0, f, \theta_2)$: by (5).

$(s, s, s) \in (0, f, \theta_2)$ by (2).

$(s, s, r + s) \in (0, f, \theta_2)$ by (5).

$(s, r + s, r) \in (0, f, \theta_2)$:

$- f(s, r + s) = f(s, r)$ by (5)

$= f(r + s, r)$ by (4).

$(s, r + s, s) \in (0, f, \theta_2)$:
$f(r, s) - f(r + s, s) = f(s, r + s) + f(s, r)$ by (5) and (6),
i.e. $f(r, s) - f(s, r + s) = f(s, r) + f(r + s, s)$.
$(s, r + s, r + s) \in (0, f, \theta_2)$:

$$\begin{aligned} f(r, r + s) + f(s, r + s) &= -f(r, s) - f(s, r) \text{ by (3) and (5)} \\ &= -f(r + s, s) - f(r + s, r) \\ &\qquad\qquad \text{by (4) and (6)} \\ &= f(r + s, r + s) \text{ by (7).} \end{aligned}$$

$(r + s, r, r) \in (0, f, \theta_2)$ by (4).
$(r + s, r, s) \in (0, f, \theta_2)$ by (2), (6) and (7).
$(r + s, r, r + s) \in (0, f, \theta_2)$:

$$\begin{aligned} f(s, r + s) + f(r + s, r) &= 0 \text{ by (4) and (5)} \\ &= f(r + s, s) + f(r, r + s) \\ &\qquad\qquad \text{by (3) and (6).} \end{aligned}$$

$(r + s, s, r) \in (0, f, \theta_2)$:

$$\begin{aligned} f(r + s, r + s) + f(s, r) &= f(r + s, r + s) + f(r + s, r) \text{ by (4)} \\ &= -f(r + s, s) \text{ by (7).} \end{aligned}$$

$(r + s, s, s) \in (0, f, \theta_2)$: by (6).
$(r + s, s, r + s) \in (0, f, \theta_2)$:
$f(r, r + s) + f(r + s, s) = f(r + s, r) + f(s, r + s)$,
(See the case $(r + s, r, r + s) \in (0, f, \theta_2)$).
$(r + s, r + s, r) \in (0, f, \theta_2)$ by (7).
$(r + s, r + s, s) \in (0, f, \theta_2)$ by (7).
$(r + s, r + s, r + s) \in (0, f, \theta_2)$ trivially.

This proves the theorem.

§ 2.3 A relation between quasi factor systems and factor systems

At the beginning of this chapter, we introduced the concept of a quasi factor system and then gave a method for the construction of such systems in some special cases. In this paragraph, we point out some relations existing between quasi factor systems in general and factor systems.

We start off with the general extension system $\{B, A, \theta; C\}$ and suppose

$$C \subseteq A \ (B\,\theta). \tag{2.3.1}$$

If $\psi \in B\theta$ and we define $\bar{\psi}$ as follows

$$\bar{\psi} : A/C \to A/C;\ (a + C)\bar{\psi} = a\psi + C,$$

then $\bar{\psi} \in \mathfrak{A}(A/C)$ because of (2.3.1). The mapping

$$\chi : B\theta \rightarrow \mathfrak{A}(A/C);\ \psi\chi = \bar{\psi}$$

is a homomorphism and induces the homomorphism

$$\bar{\theta} : B \rightarrow \mathfrak{A}(A/C);\ \alpha\bar{\theta} = \alpha\theta\chi.$$

If we define the group

$$F^*(B, A, \theta; C) = \{f \mid f \in F(B, A, \theta; C) \text{ and } f(\alpha, \beta) \in C \text{ for all } \alpha, \beta \in B\},$$

then $F^*(B, A, \theta; C) \subseteq F(B, A, \theta; C)$. We can now prove

THEOREM 3:

$$F(B, A, \theta; C)/F^*(B, A, \theta; C) \cong F(B, A/C, \bar{\theta}).$$

PROOF:

Let $f \in F(B, A, \theta; C)$.

Define $\bar{f} : B \times B \rightarrow A/C;\ \bar{f}(\alpha, \beta) = f(\alpha, \beta) + C.$ (2.3.2)

Then $\bar{f}(\alpha, \varepsilon) = \bar{f}(\varepsilon, \alpha) = C$ for all $\alpha \in B$ and

$$\begin{aligned} & \bar{f}(\alpha\beta, \gamma) + \{\bar{f}(\alpha, \beta)\}(\gamma\bar{\theta}) \\ = {} & f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) + C \\ = {} & f(\alpha, \beta\gamma) + f(\beta, \gamma) + C \\ = {} & \bar{f}(\alpha, \beta\gamma) + \bar{f}(\beta, \gamma) \text{ for all } \alpha, \beta, \gamma \in B \end{aligned}$$

i.e. $\bar{f} \in F(B, A/C, \bar{\theta})$.

The mapping

$\Phi : F(B, A, \theta; C) \rightarrow F(B, A/C, \bar{\theta});\ f\Phi = \bar{f}$ ($\bar{f}$ defined by (2.3.2)) is clearly a homomorphism. That Φ is even an epimorphism is seen as follows:

Let $\bar{f} \in F(B, A/C, \bar{\theta})$ and choose fixed representatives $\bar{a}$ in the cosets of C in A, $\bar{a}(\alpha, \beta) \in \bar{f}(\alpha, \beta)$, taking 0 as the representative of C. Since $\{\bar{f}(\alpha^{-1}, \alpha)\}(\alpha^{-1}\bar{\theta}) = \bar{f}(\alpha, \alpha^{-1})$ for all $\alpha \in B$, we have

$$\{\bar{a}(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) - \bar{a}(\alpha, \alpha^{-1}) \in C \text{ for all } \alpha \in B$$

and we are therefore able to select the representatives $\bar{a}$ so as to satisfy

$$\{\bar{a}(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) = \bar{a}(\alpha, \alpha^{-1}) \text{ for all } \alpha \in B \tag{2.3.3}$$

We now define a function

$f\colon B \times B \to A;\ f(\alpha, \beta) = \bar{a}(\alpha, \beta)$ for all $\alpha, \beta \in B$.

Then $f(\alpha, \varepsilon) = f(\varepsilon, \alpha) = 0$ for all $\alpha \in B$ and

$$\begin{aligned} & f(\alpha\beta, \gamma) + \{f(\alpha, \beta\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma) + C \\ = {} & \bar{a}(\alpha\beta, \gamma) + \{\bar{a}(\alpha, \beta\}(\gamma\theta) - \bar{a}(\alpha, \beta\gamma) - \bar{a}(\beta, \gamma) + C \\ = {} & C \end{aligned}$$

since $\bar{f} \in F(B, A/C, \bar{\theta})$. Furthermore,

$$\{f(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) = f(\alpha, \alpha^{-1}) \text{ for all } \alpha \in B \text{ by (2.3.3).}$$

$$\text{i.e. } f \in F(B, A, \theta; C) \text{ and } f\Phi = \bar{f}.$$

Denote the identity element of $F(B, A/C, \bar{\theta})$ by $\bar{f}_0$, i.e. $\bar{f}_0(\alpha, \beta) = C$ for all $\alpha, \beta \in B$. To prove that $K(\Phi) = F^*(B, A, \theta; C)$, we first note that $f\Phi = \bar{f}_0$ for all $f \in F^*(B, A, \theta; C)$. If conversely $f\Phi = \bar{f}_0$, then

$$f(\alpha, \beta) + C = \bar{f}_0(\alpha, \beta) = C \text{ for all } \alpha, \beta \in B.$$

i.e. $f(\alpha, \beta) \in C$ for all $\alpha, \beta \in B$.

i.e. $f \in F^*(B, A, \theta; C)$.

This proves the theorem.

§ 2.4 Splitting criteria

In this paragraph, we impose the restriction $C = 0$ on the general extension system $\{B, A, \theta; C\}$ and deduce properties of factor systems which are applied in chapter III to investigate the structure of certain groups of extensions.

If $f \in F(B, A, \theta)$ and G is an extension of A by B with extension system (θ, f), then G is splitting if and only if f is a transformation system (See 1.4.2)). Therefore, it is convenient to have necessary and sufficient conditions that a given $f \in F(B, A, \theta)$ is a transformation system.

We treat the cases (a) and (b) of § 2.2.

(a) In this section, we need a concept given by

DEFINITION 3:

If $\alpha \in \mathfrak{A}(A)$, *then an element of the form* $a + a\alpha + a\alpha^2 + \ldots + a\alpha^{n-1}$; $a \in A$, n *a positive integer, is called an* n-α-*norm of* A.

We denote the set of all n-α-norms of A by $N(n, \alpha)$. Note that $N(n, \alpha)$ is a subgroup of A.

Returning to the special case of the general extension system $\{B, A, \theta; C\}$ under consideration, we suppose θ is induced by $1 \to \beta$, i.e. $\beta^n = \varepsilon$. We then prove

THEOREM 4:

$$f \in T(B, A, \theta) \text{ if and only if } \sum_{i=1}^{n-1} f(1, i) \in N(n, \beta).$$

PROOF:

Let $f \in T(B, A, \theta)$.

Then there exists a function $\omega : B \to A$ with $\omega(0) = 0$, and

$$f(r, s) = \{\omega(r)\}\beta^s + \omega(s) - \omega(r + s) \text{ for all } r, s \in B.$$

In particular:

$$f(1, 1) = \{\omega(1)\}\beta + \omega(1) - \omega(2)$$
$$f(1, 2) = \{\omega(1)\}\beta^2 + \omega(2) - \omega(3)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f(1, n-1) = \{\omega(1)\}\beta^{n-1} + \omega(n-1)$$

$$\text{i.e. } \sum_{i=1}^{n-1} f(1, i) = \omega(1) + \{\omega(1)\}\beta + \{\omega(1)\}\beta^2 + \ldots + \{\omega(1)\}\beta^{n-1} \in N(n, \beta).$$

Conversely, suppose $f \in F(B, A, \theta)$ such that

$$\sum_{i=n}^{n-1} f(1, i) = a + a\beta + \ldots + a\beta^{n-1}, \ a \in A \qquad (2.4.1)$$

Define a function $\omega : B \to A$ as follows:

$$\omega(0) = 0$$
$$\omega(1) = a$$
$$\omega(2) = a + a\beta - f(1, 1)$$

....

$$\omega(k) = \sum_{i=0}^{k-1} \{a\beta^i - f(1, i)\}$$

....

$$\omega(n-1) = a + a\beta + \ldots + a\beta^{n-2} - \sum_{i=1}^{n-1} f(1,i) + f(1,n-1)$$
$$= f(1, n-1) - a\beta^{n-1} \text{ by } (2.4.1).$$

We now show that indeed $f \in T(B, A, \theta)$:

In the first place:

$$f(1, 1) = a\beta + a - \omega(2)$$
$$= \{\omega(1)\}\beta + \omega(1) - \omega(2).$$

In general, if

$f(1, j) = \{\omega(1)\}\beta^j + \omega(j) - \omega(1 + j)$ for all j such that $1 \leqq j < s \leqq n - 2$, then since

$$\omega(s + 1) = \sum_{i=0}^{s} \{a\beta^i - f(1, i)\}$$

we have

$$f(1, s) = \sum_{i=0}^{s} a\beta^i - \sum_{i=1}^{s-1} f(1, i) - \omega(s + 1)$$

$$= \sum_{i=0}^{s} a\beta^i - \sum_{i=1}^{s-1} [\{\omega(1)\}\beta^i + \omega(i) - \omega(i + 1)] - \omega(s + 1)$$

$$= \{\omega(1)\}\beta^s + \omega(s) - \omega(s + 1).$$

Also

$$f(1, n-1) = a\beta^{n-1} + \omega(n-1)$$
$$= \{\omega(1)\}\beta^{n-1} + \omega(n-1).$$

Thus

$$f(1, i) = \{\omega(1)\}\beta^i + \omega(i) - \omega(1 + i),\ i = 0, 1, \ldots, n-1.$$

If now

$$f(s, i) = \{\omega(s)\}\beta^{i} + \omega(i) - \omega(s+i) \text{ for } i = 0, 1, \dots n-1$$

and fixed s such that $1 \leqq s \leqq n-2$, then applying (1.2.2) we have:

$$f(s+1, i) = \left\{ \begin{array}{l} \{\omega(s)\}\,\beta^{i+1} + \omega(i+1) - \omega(i+s+1) \\ \{\omega(1)\}\,\beta^{i} \;\; + \omega(i) - \omega(i+1) \\ -\{\omega(s)\}\beta^{i+1} - \{\omega(1)\}\beta^{i} + \{\omega(s+1)\}\beta^{i} \end{array} \right.$$

$$= \{\omega(s+1)\}\beta^{i} + \omega(i) - \omega(i+s+1).$$

Thus $f(i, j) = \{\omega(i)\}\beta^{j} + \omega(j) - \omega(i+j)$ for all $i, j \in B$.

i.e. $f \in T(B, A, \theta)$.

(b) Let V_4 be defined by (2.2.3). We then prove

LEMMA 5 :

$f \in F(V_4, A, \theta_0) \Longrightarrow f$ *is symmetric.*

PROOF :

By (1.2.2) we have:

$$f(r, r) = f(r, r+s) + f(r, s) \quad (2.4.2)$$

$$f(r+s, r) + f(s, r) = f(r, r) \quad (2.4.3)$$

$$f(r+s, r) + f(r, s) = f(r, r+s) + f(s, r) \quad (2.4.4)$$

$$f(r+s, s) + f(s, r) = f(s, r+s) + f(r, s). \quad (2.4.5)$$

From (2.4.2) and (2.4.3) follows:

$$f(r, r+s) + f(r, s) - f(r+s, r) - f(s, r) = 0 \quad (2.4.6)$$

and by (2.4.4) and (2.4.6) we have

$$2\{f(r, s) - f(s, r)\} = 0,$$

thus

$$f(r, s) = f(s, r)$$

since $A[2] = 0$. Substituting in (2.4.6) we find

$$f(r, r+s) = f(r+s, r)$$

and finally, by (2.4.5) we have

$$f(r+s, s) = f(s, r+s).$$

Thus f is symmetric.

With the homomorphisms θ_i (i = 0, 1, 2) defined as in § 2.2(b), we prove

THEOREM 5:

(1) $f \in T(V_4, A, \theta_0) \Longleftrightarrow f(r, r), f(s, s) \in 2A.$
(2) $f \in T(V_4, A, \theta_1) \Longleftrightarrow f(r, r) \in 2A.$
(3) $F(V_4, A, \theta_2) = T(V_4, A, \theta_2).$

PROOF:

(1) Suppose $f \in T(V_4, A, \theta_0)$.

Then there exists a function $\omega : V_4 \to A$ with $\omega(0) = 0$ and such that

$$f(r, r) = 2\omega(r) \in 2A$$

and $$f(s, s) = 2\omega(s) \in 2A.$$

Conversely, if $f \in F(V_4, A, \theta_0)$ and $f(r, r)$, $f(s, s) \in 2A$, we prove that $f \in T(V_4, A, \theta_0)$. In addition to the relations (2.4.2), (2.4.3) and (2.4.4), we shall also need

$$f(r + s, s) + f(r, s) = f(s, s) \qquad (2.4.7)$$

and $$f(r + s, r + s) + f(r, s) = f(r, r) + f(s, r + s). \qquad (2.4.8)$$

Suppose $$\left.\begin{aligned} f(r, r) &= 2a \\ f(s, s) &= 2a' \end{aligned}\right\} \quad a, a' \in A.$$

We then define a function $\omega : V_4 \to A$ as follows:

$$\begin{aligned} \omega(0) &= 0 \\ \omega(r) &= a \\ \omega(s) &= a' \\ \omega(r + s) &= \omega(r) + \omega(s) - f(r, s). \end{aligned}$$

From this definition it follows at once that

$$\begin{aligned} f(r, r) &= 2a = \omega(r) + \omega(r) \\ f(s, s) &= 2a' = \omega(s) + \omega(s) \\ f(r, s) &= \omega(r) + \omega(s) - \omega(r + s), \end{aligned}$$

and by Lemma 5

$$f(s, r) = \omega(s) + \omega(r) - \omega(r + s).$$

From (2.4.2) we deduce that

$$\begin{aligned} f(r, r + s) &= f(r, r) - f(r, s) \\ &= 2\omega(r) - \omega(r) - \omega(s) + \omega(r + s) \\ &= \omega(r) + \omega(r + s) - \omega(s) \end{aligned}$$

and by Lemma 5

$$f(r + s, r) = \omega(r + s) + \omega(r) - \omega(s).$$

Applying (2.4.7) we have

$$\begin{aligned} f(r + s, s) &= f(s, s) - f(r, s) \\ &= 2\omega(s) - \omega(r) - \omega(s) + \omega(r + s) \\ &= \omega(r + s) + \omega(s) - \omega(r), \end{aligned}$$

and again by Lemma 5

$$f(s, r + s) = \omega(s) + \omega(r + s) - \omega(r).$$

Finally, from (2.4.8) it follows that

$$f(r + s, r + s) = f(r, r) + f(s, r + s) - f(r, s)$$

$$= \left\{ \begin{array}{l} 2\omega(r) \\ \omega(s) + \omega(r + s) - \omega(r) \\ - \omega(r) - \omega(s) + \omega(r + s) \end{array} \right.$$

$$= 2\omega(r + s)$$

which proves that $f \in T(V_4, A, \theta_0)$.

(2) The proof of this part of the theorem is similar to that of the first part. We note in this case that by (1.2.2) we have

$$\begin{aligned} & - f(s, s) = f(s, s) \\ \text{i.e. } & 2f(s, s) = 0 \\ \text{i.e. } & f(s, s) = 0 \text{ since } A[2] = 0. \end{aligned}$$

If furthermore we have $f(r, r) = 2a$, then we define $\omega : V_4 \rightarrow A$ for this case as follows:

$$\begin{aligned} \omega(0) &= 0 \\ \omega(r) &= a \\ \omega(s) &= a' \text{ (}a'\text{ an arbitrary fixed element)} \\ \omega(r + s) &= - \omega(r) + \omega(s) - f(r, s). \end{aligned}$$

We now proceed exactly as in the previous case to show that $f \in T(V_4, A, \theta_1)$, using the relation (1.2.2).

(3) If $f \in F(V_4, A, \theta_2)$, then we have by (1.2.2) that

$$f(r, r) = f(s, s) = 0.$$

To show that $f \in T(V_4, A, \theta_2)$, select arbitrary fixed $a, a' \in A$ and define a function $\omega : V_4 \to A$ as follows:

$$\begin{aligned}
\omega(0) &= 0 \\
\omega(r) &= a \\
\omega(s) &= a' \\
\omega(r+s) &= -\omega(r) + \omega(s) - f(r, s).
\end{aligned}$$

From (1.2.2) we gather that:

$$-f(r, r) = f(r, r+s) + f(r, s). \tag{2.4.9}$$
$$f(r+s, r) - f(s, r) = f(r, r). \tag{2.4.10}$$
$$f(r+s, s) - f(r, s) = f(s, s). \tag{2.4.11}$$
$$-f(s, s) = f(s, r+s) + f(s, r). \tag{2.4.12}$$
$$f(r+s, r+s) + f(r, s) = f(r, r) + f(s, r+s). \tag{2.4.13}$$
$$f(s, r) - f(r+s, r) = f(r+s, r+s) + f(r, s). \tag{2.4.14}$$

By means of the definition of ω and the relations (2.4.9)—(2.4.14) we readily prove that $f \in T(V_4, A, \theta_2)$.

§ 2.5 **Isolated quasi factor systems:**

Our study of the structural properties of the group $\mathrm{Ext}(B, A, \theta; C)$ in the general extension system $\{B, A, \theta; C\}$ (see chapter III) will be facilitated by any knowledge concerning its structure in some special cases of the general extension system. In the previous paragraphs of this chapter, the ground has been prepared for this approach.

A second line of approach is to keep the variables in the general extension system $\{B, A, \theta; C\}$ as general as possible and to impose restrictions on the $(B, A, \theta; C)$-quasi factor systems so as to get information about the structure of some *subgroup* of $\mathrm{Ext}(B, A, \theta; C)$. With a view to this, we define some special types of quasi factor systems in the present paragraph.

We recall that in § 2.3 the restriction

$$f(\alpha, \beta) \in C \text{ for all } \alpha, \beta \in B \tag{2.5.1}$$

was imposed on $f \in F(B, A, \theta; C)$. This condition is sharpened if we define functions $f : B \times B \to A$ satisfying for *fixed* $c_f \in C$ depending on f the conditions

$$\begin{aligned} f(\alpha, \beta) &= 0 \text{ if } \alpha = \varepsilon \text{ or } \beta = \varepsilon \\ &= c_f \text{ if } \alpha \neq \varepsilon \text{ and } \beta \neq \varepsilon. \end{aligned} \tag{2.5.2}$$

A function $f : B \times B \to A$ satisfying (2.5.2) for some fixed $c_f \in C \subseteq A(B\theta)$ is clearly a $(B, A, \theta; C)$-quasi factor system. Furthermore, the set $F^{**}(B, A, \theta; C)$ of all such quasi factor systems is a subgroup of $F(B, A, \theta; C)$. If $f \in F(B, A, \theta; C)$, we denote the expression

$$f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma)$$

by $a(f; \alpha, \beta, \gamma)$.

Every $f \in F^{**}(B, A, \theta; C)$ has *the property* (*):

If $\alpha \neq \varepsilon$, $\beta \neq \varepsilon$, $\gamma \neq \varepsilon$, then

$$\begin{aligned} a(f; \alpha, \beta, \gamma) &= 0 \quad \text{if } \begin{cases} \alpha\beta = \varepsilon \text{ and } \beta\gamma = \varepsilon \text{ or} \\ \alpha\beta \neq \varepsilon \text{ and } \beta\gamma \neq \varepsilon \end{cases} \\ &= c_f \quad \text{if } \beta\gamma = \varepsilon \text{ and } \alpha\beta \neq \varepsilon \\ &= -c_f \text{ if } \beta\gamma \neq \varepsilon \text{ and } \alpha\beta = \varepsilon. \end{aligned}$$

We call a $(B, A, \theta; C)$-quasi factor system f with the property(*) for some fixed $c_f \in C$ an *isolated* $(B, A, \theta; C)$-*quasi factor system* with *isolation* c_f. The set $I(B, A, \theta; C)$ of all isolated $(B, A, \theta; C)$-quasi factor systems is a subgroup of $F(B, A, \theta; C)$.

Every (B, A, θ)-factor system is isolated with isolation 0 and this fact justifies the generalization from $F^{**}(B, A, \theta; C)$ to $I(B, A, \theta; C)$.

If G is a quasi-group extension of A by B with extension system (θ, f) and $f \in I(B, A, \theta; C)$ has isolation c, then we call G an *isolated quasi-group extension of* A *by* B *with isolation* c. A group extension G of A by B may therefore be considered to have isolation 0.

In § 3.3 we discuss a structural property of the group of all isolated quasi-group extensions in the general extension system $\{B, A, \theta; C\}$.

CHAPTER III

THE GROUP OF EXTENSIONS

§ 3.1 Introduction

If $g, h \in F(B, A, \theta; C)$ in the general extension system $\{B, A, \theta; C\}$, then the extension systems (θ, g) and (θ, h) determine quasi-group extensions G and H of A by B, both with associated homomorphism θ (see § 2.1). If in addition $g - h \in T(B, A, \theta)$, then there exists a function $\omega : B \to A$ with $\omega(\varepsilon) = 0$ and such that

$$(g - h)(\alpha, \beta) = \{\omega(\alpha)\}(\beta\theta) + \omega(\beta) - \omega(\alpha\beta)$$

for all $\alpha, \beta \in B$. Consider the mapping

$$\varphi : G \to H;\ (\alpha, a)\varphi = (\alpha, a + \omega(\alpha)).$$

The mapping is "onto" because for arbitrary $(\alpha, a) \in H$ we have $(\alpha, a - \omega(\alpha))\varphi = (\alpha, a)$. Since

$$\begin{aligned}\{(\alpha, a) + (\beta, b)\}\varphi &= (\alpha\beta, a(\beta\theta) + b + g(\alpha, \beta))\varphi \\ &= (\alpha\beta, a(\beta\theta) + b + g(\alpha, \beta) + \\ &\quad + \omega(\alpha\beta)) \\ &= (\alpha\beta, a(\beta\theta) + b + \{\omega(\alpha)\}(\beta\theta) + \\ &\quad + \omega(\beta) + h(\alpha, \beta)). \\ &= (\alpha, a + \omega(\alpha)) + (\beta, b + \omega(\beta)). \\ &= (\alpha, a)\varphi + (\beta, b)\varphi\end{aligned}$$

and $(\alpha, a)\varphi = (\varepsilon, 0)$ if and only if $\alpha = \varepsilon$, $a = 0$, we see φ is an isomorphism. Obviously φ leaves both A and B element-wise invariant and so G and H are equivalent in the sense of § 1.2. Suppose conversely that two quasi-group extensions G and H of A by B with extension systems (θ, g) and (θ, h) resp., $g, h \in F(B, A, \theta; C)$, are equivalent. Then there exists an isomorphism $\varphi : G \to H$ leaving both A and B invariant. Thus for arbitrary $(\alpha, a) \in G$ we have

$$(\alpha, a)\varphi = \{(\alpha, 0) + (\varepsilon, a)\}\varphi = (\alpha, 0)\varphi + (\varepsilon, a)$$
$$= (\alpha, \omega(\alpha)) + (\varepsilon, a) \text{ say, } \omega(\alpha) \in A$$
$$= (\alpha, a + \omega(\alpha)).$$

Collecting all $\omega(\alpha)$ with (α, a) running through G, we have a function $\omega : B \to A$ with the property $\omega(\varepsilon) = 0$. We have, moreover, for arbitrary (α, a), $(\beta, b) \in G$ that

$$\{(\alpha, a) + (\beta, b)\}\varphi = (\alpha\beta, a(\beta\theta) + b + g(\alpha, \beta))\varphi$$
$$= (\alpha\beta, a(\beta\theta) + b + g(\alpha, \beta) + \omega(\alpha\beta)). \quad (3.1.1)$$

But since φ is an isomorphism,

$$\{(\alpha, a) + (\beta, b)\}\varphi = (\alpha, a)\varphi + (\beta, b)\varphi$$
$$= (\alpha, a + \omega(\alpha)) + (\beta, b + \omega(\beta))$$
$$= (\alpha\beta, a(\beta\theta) + \{\omega(\alpha)\}(\beta\theta) + b + \omega(\beta) +$$
$$+ h(\alpha, \beta)). \quad (3.1.2)$$

(3.1.1) and (3.1.2) represent the same element of H and thus

$g(\alpha, \beta) + \omega(\alpha\beta) = \{\omega(\alpha)\}(\beta\theta) + \omega(\beta) + h(\alpha, \beta)$ for all a, $\beta \in B$.
i.e. $g - h \in T(B, A, \theta)$.

Two quasi-group extensions G *and* H *of* A *by* B *with extension systems* (θ, g) *and* (θ, h) *resp. are equivalent if and only if* $g - h \in T(B, A, \theta)$.

We thus denote by

$$\mathrm{Ext}(B, A, \theta; C) = F(B, A, \theta; C)/T(B, A, \theta) \quad (3.1.3)$$

the group of all non-equivalent C-*quasi-group extensions* of A by B with respect to the associated homomorphism θ.

We note the following trivial structural properties of $\mathrm{Ext}(B, A, \theta; C)$:

(i) *If in the general extension system* $\{B, A, \theta; C\}$ we have $C_1 < C_2 < A$,
then

$$\mathrm{Ext}(B, A, \theta; C_1) \subseteq \mathrm{Ext}(B, A, \theta; C_2). \quad (3.1.4)$$

(ii) *If* $C = C_1 \cap C_2$ for $C_1 < A$, $C_2 < A$, *then since* $F(B, A, \theta; C) = F(B, A, \theta; C_1) \cap F(B, A, \theta; C_2)$, *we have*

$$\mathrm{Ext}(B, A, \theta; C) =$$
$$= \mathrm{Ext}(B, A, \theta; C_1) \cap \mathrm{Ext}(B, A, \theta; C_2). \quad (3.1.5)$$

(iii) C *can always be embedded in* Ext(B, A, θ; C) *if* C $\subseteq$ A(Bθ).

PROOF:

Define a mapping

ψ : $F^{**}(B, A, \theta; C) \to C$; $f\psi = c_f$ with $F^{**}(B, A, \theta; C)$ as

defined in § 2.5 and c_f the isolation of f.

ψ is clearly an isomorphism.

We also have $F^{**}(B, A, \theta; C) \cap T(B, A, \theta) = f_0$, for suppose $f \in F^{**}(B, A, \theta; C)$ and $f(\alpha, \beta) = c \neq 0$ for $\alpha \neq \varepsilon$, $\beta \neq \varepsilon$. If $\alpha \neq \varepsilon$, $\beta \neq \varepsilon$, $\alpha\beta \neq \varepsilon$ we then have

$a(f; \alpha, \beta, \beta^{-1}) = c(\beta\theta)^{-1} \neq 0.$

i.e. f is not a (B, A, θ)-factor system.

Thus if $f, g \in F^{**}(B, A, \theta; C)$, then $f - g \in T(B, A, \theta)$ if and only if $f = g$.

i.e. $F^{**}(B, A, \theta; C)$ is isomorphic with a subgroup of Ext(B, A, θ; C).

In the special case C = 0, we have

$$\text{Ext}(B, A, \theta) = F(B, A, \theta)/T(B, A, \theta) \qquad (3.1.6)$$

which denotes the group of all *group extensions* of A by B with associated homomorphism θ. Thus

$$\text{Ext}(B, A, \theta) \subseteq \text{Ext}(B, A, \theta; C) \text{ for arbitrary } C < A.$$

The group

$$\text{Ext}(B, A, \theta_0) = F(B, A, \theta_0)/T(B, A, \theta_0) \qquad (3.1.7)$$

is by (1.5.1) the group of all non-equivalent *central group extensions* of A by B. Eilenberg and MacLane [7] denoted this group by Extcent(B, A).

If B is abelian, then $T(B, A, \theta_0) \subseteq P(B, A, \theta_0)$ by (1.5.5) and

$$\text{Ext}(B, A) = P(B, A, \theta_0)/T(B, A, \theta_0) \qquad (3.1.8)$$

denotes by (1.5.6) the group of all non-equivalent *abelian group extensions* of A by B.

Thus $\mathrm{Ext}(B, A) \subseteq \mathrm{Ext}(B, A, \theta_0)$.

If B is locally cyclic, then we have by (1.5.7) that

$$\mathrm{Ext}(B, A) = \mathrm{Ext}(B, A, \theta_0). \tag{3.1.9}$$

Structural properties of the group Ext(B, A) have been studied amongst others by Baer [1], [2], Eilenberg and MacLane [6].

Under the conditions (a), (b) and (c) of § 1.5 we have

$$F(B, A, \theta_0)/T(B, A, \theta_0) \cong P(B, A, \theta_0)/T(B, A, \theta_0) \oplus \oplus Q(B, A, \theta_0),$$

or

$$\mathrm{Ext}(B, A, \theta_0) \cong \mathrm{Ext}(B, A) \oplus Q(B, A, \theta_0). \tag{3.1.10}$$

In all cases where in addition $\mathrm{Ext}(B, A) = 0$ we thus have

$$\mathrm{Ext}(B, A, \theta_0) \cong Q(B, A, \theta_0).$$

§ 3.2 Extensions by cyclic groups

Every extension G of an arbitrary group A by a free group B is splitting. (Kurosh [13]). If therefore $B = C(\infty)$ in the general extension system $\{B, A, \theta; C\}$, then

$$\mathrm{Ext}(B, A, \theta) = 0.$$

The only element of Ext(B, A) in this case is a class of groups equivalent to the direct sum $A \oplus B$. If $\theta \neq \theta_0$, then all the extensions of A by B with θ as associated homomorphism are semi-direct products dependent on θ (See § 1.4).

We now turn to the case where B is a finite cyclic group C(n). Suppose θ is induced by $1 \rightarrow \alpha$, and let $A(B\theta)$ and $N(n, \alpha)$ be defined as in (2.1.6) and definition 3 resp.. We prove

THEOREM 6[1]):

$$\mathrm{Ext}(B, A, \theta) \cong A(B\theta)/N(n, \alpha).$$

---

[1]) This result was obtained in different ways by Baer [1] and Eilenberg [5].

PROOF:

Let $f \in F(B, A, \theta)$.

Then $\left\{ \sum_{i=1}^{n-1} f(1, i) \right\} \alpha = \sum_{i=1}^{n-1} f(1, i)$ by the remark of § 2.2.

i.e. $\sum_{i=1}^{n-1} f(1, i) \in A(B\theta)$.

Consider the mapping

$$\Phi\colon F(B, A, \theta) \rightarrow A(B\theta);\ f\Phi = \sum_{i=1}^{n-1}(f(1, i).$$

Then Φ is not only a homomorphism but even an epimorphism, because for a given $a \in A(B\theta)$ we can construct an $f \in F(B, A, \theta)$ with $f\Phi = a$ as follows:

(i) $f(r, 0) = f(0, r) = 0$ for $r = 0, 1, \ldots, n-1$.

(ii) $f(1, 1) = a$, $f(1, i) = 0$ for $i = 2, \ldots, n-1$.

(iii) $f(r+1, s) = f(r, 1+s) + f(1, s) - \{f(r, 1)\}\alpha^s$

for $r = 1, \ldots, n-2$; $s = 1, \ldots, n-1$.

That f is indeed a (B, A, θ)-factor system, follows from theorem 1. We also have by theorem 4 that

$$\{T(B, A, \theta)\}\Phi = N(n, \alpha).$$

This proves the theorem.

COROLLARY:

$$\mathrm{Ext}(B, A) = A/nA.$$

PROOF:

By (3.1.9), $\mathrm{Ext}(B, A) = \mathrm{Ext}(B, A, \theta_0)$ while

$$A(B\theta_0) = A(\varepsilon) = A \text{ and } N(n, \varepsilon) = nA.$$

§ 3.3 The group of isolated quasi-group extensions

In the general extension system $\{B, A, \theta; C\}$

$$\mathrm{Ext}(B, A, \theta; C(i)) = I(B, A, \theta; C)/T(B, A, \theta) \qquad (3.3.1)$$

is the group of all non-equivalent *isolated C-quasi-group extensions* of A by B with associated homomorphism θ (see § 2.5). Clearly

$$\mathrm{Ext}(B, A, \theta) \subseteq \mathrm{Ext}(B, A, \theta; C(i)) \subseteq \mathrm{Ext}(B, A, \theta; C)$$

for arbitrary C. If $C \subseteq A(B\theta)$ in the general extension system $\{B, A, \theta; C\}$, then we prove

THEOREM 7:

$$\mathrm{Ext}(B, A, \theta; C(i))/\mathrm{Ext}(B, A, \theta) \cong C.$$

PROOF:

Let $f \in I(B, A, \theta; C)$ has isolation c. The mapping

$$\Phi : I(B, A, \theta; C) \to C; \quad f\Phi = c$$

is clearly a homomorphism. On the other hand, if $c \in C$ is fixed, we consider the function $f : B \times B \to A$ defined by

(i) $f(\varepsilon, \alpha) = f(\alpha, \varepsilon) = 0$ for all $\alpha \in B$

(ii) $f(\alpha, \beta) = c$ for all $\alpha \neq \varepsilon$, $\beta \neq \varepsilon$.

Thus $f \in F^{**}(B, A, \theta; C)$ (see § 2.5) and therefore $f \in I(B, A, \theta; C)$ with isolation c so that $f\Phi = c$. Consequently Φ is an epimorphism.

Obviously $K(\Phi) = F(B, A, \theta)$.

i.e. $I(B, A, \theta; C)/F(B, A, \theta) \cong C$

and thus

$$I(B, A, \theta; C)/T(B, A, \theta) / F(B, A, \theta)/T(B, A, \theta) \cong C,$$

which proves the theorem.

COROLLARY:

If $\mathrm{Ext}(B, A, \theta) = 0$, then $\mathrm{Ext}(B, A, \theta; C(i)) \cong C$.

§ 3.4 On the divisibility of the group of extensions

If A is an additively written group and n is a natural number such that the equation $nx = a$ is solvable in A for arbitrary $a \in A$, then we say A is *divisible by* n and write $nA = A$. If $C < A$ and the equation $nx = c \in C$ is solvable in C whenever it is solvable in A, then we call C a *pure subgroup* of A.

In the next two theorems the conditions in the general extension system $\{B, A, \theta; C\}$ will be outlined under which a divisibility property of A (resp. B) is inherited by a certain group of extensions of A by B.

Suppose now that for a fixed natural number n

$$nA = A \tag{3.4.1}$$

in the general extension system $\{B, A, \theta; C\}$ and

$$C \text{ is a pure subgroup of } A \tag{3.4.2}$$

while $$A[n] = 0. \tag{3.4.3}$$

We then prove

THEOREM 8:

$$\mathrm{Ext}(B, A, \theta; C) = n\mathrm{Ext}(B, A, \theta; C).$$

PROOF:

It is trivially true that

$$n\mathrm{Ext}(B, A, \theta; C) \subseteq \mathrm{Ext}(B, A, \theta; C).$$

It only remains to prove that

$$F(B, A, \theta; C) \subseteq nF(B, A, \theta; C). \tag{3.4.4}$$

Suppose therefore $f \in F(B, A, \theta; C)$.

Then by (3.4.1), the equation

$$f(\alpha, \beta) = nx \tag{3.4.5}$$

is solvable for all $\alpha, \beta \in B$. For given $\alpha_0, \beta_0 \in B$, the solution $g(\alpha_0, \beta_0)$ of (3.4.5) is even unique, for suppose $g'(\alpha_0, \beta_0)$ also satisfies (3.4.5). Then

$$ng(\alpha_0, \beta_0) = ng'(\alpha_0, \beta_0).$$

$$\text{i.e. } n\{g(\alpha_0, \beta_0) - g'(\alpha_0, \beta_0)\} = 0$$

and thus $g(\alpha_0, \beta_0) = g'(\alpha_0, \beta_0)$ by (3.4.3).

Collecting the solutions $g(\alpha, \beta)$ of (3.4.5) with α and β running through B, we have a function $g : B \times B \to A$ which is a $(B, A, \theta; C)$-quasi factor system, for

1° $ng(\varepsilon, \alpha) = f(\varepsilon, \alpha) = 0$ and thus $g(\varepsilon, \alpha) = 0$ for all $\alpha \in B$ by the uniqueness of the solution of $nx = 0$.

Similarly $g(\alpha, \varepsilon) = 0$ for all $\alpha \in B$.

2° $$f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma)$$
$$= ng(\alpha\beta, \gamma) + \{ng(\alpha, \beta)\}(\gamma\theta) - ng(\alpha, \beta\gamma) - ng(\beta, \gamma)$$
$$= n[g(\alpha\beta, \gamma) + \{g(\alpha, \beta)\}(\gamma\theta) - g(\alpha, \beta\gamma) - g(\beta, \gamma)].$$

But $f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma) \in C$ and therefore

$g(\alpha\beta, \gamma) + \{g(\alpha, \beta)\}(\gamma\theta) - g(\alpha, \beta\gamma) - g(\beta, \gamma) \in C$ by (3.4.2).

3° $n[\{g(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta)] = ng(\alpha, \alpha^{-1})$ since $\{f(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) =$ $= f(\alpha, \alpha^{-1})$ for all $\alpha \in B$

and by (3.4.3) we have

$\{g(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) = g(\alpha, \alpha^{-1})$ for all $\alpha \in B$.

i.e. $f = ng \in nF(B, A, \theta; C)$,

which proves (3.4.4) and thus completes the proof of the theorem.

We next consider an abelian group $B = \{0, a, b, c, \ldots\}$ and prove

LEMMA 9:

If $f \in P(B, A, \theta_0)$ then
$2f(a, b) - f(2a, 2b) = f(a, a) + f(b, b) - f(a + b, a + b)$ for all $a, b \in B$.

PROOF:

From (1.2.2) it follows that for all $a, b \in B$

1° $f(a + b, a + b) + f(a, b) = f(a, a + 2b) + f(b, a + b)$.

2° $f(2a, 2b) + f(a, a) = f(a, a + 2b) + f(a, 2b)$.

3° $f(a + b, b) + f(a, b) = f(a, 2b) + f(b, b)$.

Thus

$$
\begin{aligned}
& 2f(a, b) - f(2a, 2b) \\
= \; & 2f(a, b) + f(a, a) - f(a, a + 2b) - f(a, 2b) \text{ by } 2^\circ \\
= \; & f(a, b) + f(b, a + b) - f(a + b, a + b) + f(a, a) - f(a, 2b) \\
& \text{by } 1^\circ \\
= \; & f(a, a) + f(b, b) - f(a, + b, a + b) \text{ by } 3^\circ \text{ and the symmetry} \\
& \text{of } f.
\end{aligned}
$$

THEOREM 9:

$$\text{If } 2B = B \tag{3.4.6}$$

$$\text{and } B[2] = 0, \tag{3.4.7}$$

then

$$2\mathrm{Ext}(B, A) = \mathrm{Ext}(B, A).$$

PROOF:

Consider an arbitrary $f + T(B, A, \theta_0) \in \mathrm{Ext}(B, A)$.
We then prove that the equation

$$2\{x + T(B, A, \theta_0)\} = f + T(B, A, \theta_0) \tag{3.4.8}$$

is solvable in Ext(B, A).

For given $a, b \in B$, suppose that by (3.4.6)

$$a = 2r \text{ and } b = 2s \tag{3.4.9}$$

and define a function

$$g : B \times B \rightarrow A; \; g(a, b) = f(r, s) \text{ for all } a, b \in B$$

(r and s as defined by (3.4.9)).

By (3.4.7), r and s and consequently $g(a, b)$ are uniquely defined. Moreover, $g \in P(B, A, \theta_0)$ for

1° $g(a, 0) = f(r, 0) = 0$ and also $g(0, a) = 0$ for all $a \in B$.

2°
$$
\begin{aligned}
& g(a + b, c) + g(a, b) \\
= \; & f(r + s, t) + f(r, s) \text{ where } a = 2r, \; b = 2s, \; c = 2t. \\
= \; & f(r, s + t) + f(s, t) \\
= \; & g(a, b + c) + g(b, c) \text{ for all } a, b, c \in B.
\end{aligned}
$$

We now prove that $x = g$ satisfies (3.4.8).

Suppose that for arbitrary $a, b \in B$ we have r and s as defined in (3.4.9). Then

$$\begin{aligned} 2g(a, b) - f(a, b) &= 2f(r, s) - f(2r, 2s) \\ &= f(r, r) + f(s, s) - f(r+s, r+s) \\ &\quad \text{by Lemma 9} \\ &= \omega(a) + \omega(b) - \omega(a+b), \end{aligned}$$

where $\omega : B \to A$ is defined by

$\omega(a) = f(r, r)$ for all $a \in B$.
i.e. $2g - f \in T(B, A, \theta_0)$

which proves that $x = g$ satisfies (3.4.8).

§ 3.5 The influence of homomorphisms in the general extension system $\{B, A, \theta; C\}$

Suppose $\varphi : B \to B'$ is an epimorphism with B in the general extension system $\{B, A, \theta; C\}$. Then the following question arises: What can be said of the group $Ext(B', A, \theta'; C)$ where θ' is related to θ in some way? A similar question may be asked if φ is an epimorphism of A to A'.

Firstly, we discuss the case

(a) $\varphi : B \to B'$:

We assume $K(\varphi) \subseteq K(\theta)$. (3.5.1)

Then $\theta' : B' \to \mathfrak{A}(A)$; $\alpha'\theta' = \alpha\theta$ with $\alpha \in \alpha'\varphi^{-1}$ is a homomorphism. The uniqueness of $\alpha'\theta'$ for a given α' is guarenteed by (3.5.1). Let

$$\begin{aligned} F'(B, A, \theta; C) = \{f \mid f \in F(B, A, \theta; C) \text{ with } \alpha_1 - \alpha_2, B_1 - B_2 \in K(\varphi) \\ \Longrightarrow f(\alpha_1, \beta_1) = f(\alpha_2, \beta_2)\}. \end{aligned}$$

Then $F'(B, A, \theta; C) < F(B, A, \theta; C)$.

Furthermore, for every $f \in T(B, A, \theta)$ there exists a function $\omega_f : B \to A$ with the properties (1.2.5) and (1.2.6).

With this notation, we define the group

$$\begin{aligned} T'(B, A, \theta) = \{f \mid f \in T(B, A, \theta) \\ \text{and } \alpha_1 - \alpha_2 \in K(\varphi) \Longrightarrow \omega_f(\alpha_1) = \omega_f(\alpha_2)\}. \end{aligned}$$

We now prove

THEOREM 10:

$$\mathrm{Ext}(B', A, \theta'; C) \cong F'(B, A, \theta; C)/T'(B, A, \theta).$$

PROOF:

Consider any $f \in F(B', A, \theta'; C)$ and define a function

$$f': B \times B \to A;\ f'(\alpha, \beta) = f(\alpha\varphi, \beta\varphi). \tag{3.5.2}$$

Then $f' \in F'(B, A, \theta; C)$, for

1° $f'(\varepsilon, \alpha) = f(\varepsilon', \alpha\varphi) = 0$ and similarly $f'(\alpha, \varepsilon) = 0$ for all $\alpha \in B$,

2° $f'(\alpha\beta, \gamma) + \{f'(\alpha, \beta)\}(\gamma\theta) - f'(\alpha, \beta\gamma) - f'(\beta, \gamma)$
$= f(\alpha\varphi\beta\varphi, \gamma\varphi) + \{f(\alpha\varphi, \beta\varphi)\}(\gamma\varphi\theta') - f(\alpha\varphi, \beta\varphi\gamma\varphi) - f(\beta\varphi, \gamma\varphi)$
$\in C$,

3° $\{f'(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) = \{f(\alpha^{-1}\varphi, \alpha\varphi)\}(\alpha^{-1}\varphi\theta') = f(\alpha\varphi, \alpha^{-1}\varphi)$
$= f'(\alpha, \alpha^{-1})$ for all $\alpha \in \beta$.

Also if $\alpha_1 - \alpha_2, \beta_1 - \beta_2 \in K(\varphi)$, then
$f'(\alpha_1, \beta_1) = f(\alpha_1\varphi, \beta_1\varphi) = f(\alpha_2\varphi, \beta_2\varphi) = f'(\alpha_2, \beta_2)$.

If we define

$$\psi : F(B', A, \theta'; C) \to F'(B, A, \theta; C);\ f\psi = f'$$

(f' defined as in (3.5.2)), then ψ is clearly a homomorphism, even an epimorphism, for let $g' \in F'(B, A, \theta; C)$ and put
$g : B' \times B' \to A;\ g(\alpha', \beta') = g'(\alpha, \beta),\ \alpha \in \alpha'\varphi^{-1},\ \beta \in \beta'\varphi^{-1}$.

Then $g(\alpha', \beta') \in A$ is unique for given $a', \beta' \in B'$ by the definition of $F'(B, A, \theta; C)$.

Furthermore $g \in F(B', A, \theta'; C)$, for

(i) $g(\varepsilon', \alpha') = g'(\varepsilon, \alpha) = 0 = g'(\alpha, \varepsilon) = g(\alpha', \varepsilon'),\ \alpha \in \alpha'\varphi^{-1}$ for all $\alpha' \in B'$.

(ii) $g(\alpha'\beta'\gamma') + \{g(\alpha', \beta')\}(\gamma'\theta') - g(\alpha', \beta'\gamma') - g(\beta', \gamma')$
$= g'(\alpha\beta, \gamma) + \{g'(\alpha, \beta)\}(\gamma\theta) - g'(\alpha, \beta\gamma) - g'(\beta, \gamma)$ where
$\alpha \in \alpha'\varphi^{-1}$, $\beta \in \beta'\varphi^{-1}$ and $\gamma \in \gamma'\varphi^{-1}$.
$\in C$ for all $\alpha', \beta', \gamma' \in B'$.

(iii) $\{g(\alpha'^{-1}, \alpha')\}(\alpha'^{-1}\theta') = \{g'(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta)$;
$\alpha \in \alpha'\varphi^{-1}$, $\alpha^{-1} \in \alpha'^{-1}\varphi^{-1}$
$= g'(\alpha, \alpha^{-1}) = g(\alpha', \alpha'^{-1})$
for all $\alpha' \in B'$.

Clearly $g\psi = g'$.

We now prove that $\{T(B', A, \theta')\}\psi = T'(B, A, \theta)$.

Let $f \in T(B', A, \theta')$.

If $f\psi = f'$, then $f'(\alpha, \beta) = f(\alpha\varphi, \beta\varphi) =$

$$= \{\omega_f(\alpha\varphi)\}(\beta\varphi\theta') + \omega_f(\beta\varphi) - \omega_f(\alpha\varphi\beta\varphi)$$
$$= \{\omega(\alpha)\}(\beta\theta) + \omega(\beta) - \omega(\alpha\beta),$$

where $\omega : B \to A$ is defined by $\omega(\alpha) = \omega_f(\alpha\varphi)$.

i.e. $f' \in T(B, A, \theta)$.

Furthermore, if $\alpha_1 - \alpha_2 \in K(\varphi)$, then

$\omega(\alpha_1) = \omega_f(\alpha_1\varphi) = \omega_f(\alpha_2\varphi) = \omega(\alpha_2)$, and so
$f' \in T'(B, A, \theta)$.

On the other hand, if $g' \in T'(B, A, \theta)$ and g is any original of g' under ψ, then we have

$$g(\alpha', \beta') = g'(\alpha, \beta) \text{ with } \alpha \in \alpha'\varphi^{-1}, \beta \in \beta'\varphi^{-1}$$
$$= \{\omega_{g'}(\alpha)\}(\beta\theta) + \omega_{g'}(\beta) - \omega_{g'}(\alpha\beta)$$
$$= \{\omega(\alpha')\}(\beta'\theta') + \omega(\beta') - \omega(\alpha'\beta'),$$

where $\omega : B' \to A$ is defined by $\omega(\alpha') = \omega_{g'}(\alpha)$, $\alpha \in \alpha'\varphi^{-1}$.

The uniqueness of $\omega(\alpha')$ for a given α' is guarenteed by the definition of $T'(B, A, \theta)$.

i.e. $g \in T(B' A, \theta')$.

This proves the theorem.

We now turn to the case

(b) $\varphi : A \rightarrow A'$:

Our first object is to construct in a natural way a homomorphism $\theta' : B \rightarrow \mathfrak{A}(A')$ related to θ in some sense. To this end we define $A(B\theta)$ as in (2.1.5) and we also assume

$$K(\varphi) \subseteq A(B\theta). \qquad (3.5.3)$$

For a given $\psi \in B\theta$ we define a mapping

$$\psi' : A' \rightarrow A';\ a'\psi' = a\psi\varphi,\ a \in a'\varphi^{-1}. \qquad (3.5.4)$$

(3.5.3) guarentees that the image of a′ under ψ' is unique and it is readily seen that

$$(a' + b')\psi' = a'\psi' + b'\psi' \text{ for arbitrary } a', b' \in A'.$$

If $a' \neq b'$ for $a', b' \in A'$ and $a \in a'\varphi^{-1}$, $b \in b'\varphi^{-1}$, then we have $a - b$ is not an element of $K(\varphi)$.

By (3.5.3) it follows that $(a - b)\psi$ is not an element of $K(\varphi)$ for any $\psi \in B\theta$.

i.e. $a\psi\varphi \neq b\psi\varphi$ or $a'\psi' \neq b'\psi'$

Therefore $\psi' \in \mathfrak{A}(A')$ if A′ is finite. In the case where A′ is infinite, we still have to show that every element of A′ is the image of some element of A′ under ψ'.

Consider, therefore, an arbitrary $a' \in A'$ and suppose $a \in a'\varphi^{-1}$. Then

$$(a\psi^{-1}\varphi)\psi' = a'.$$

Thus $\psi' \in \mathfrak{A}(A')$ in all cases.

Define a mapping

$$\theta^* : B\theta \rightarrow \mathfrak{A}(A');\ \psi\theta^* = \psi' \ (\psi' \text{ defined by } (3.5.4)).$$

Then θ^* is a homomorphism and we attain our object, mentioned at the beginning of this section, by defining the homomorphism:

$$\theta' : B \rightarrow \mathfrak{A}(A');\ \alpha\theta' = \alpha\theta\theta^*.$$

At this stage, we need

DEFINITION 4:

If for $f \in F(B, A, \theta; C)$ *in the general extension system* $\{B, A, \theta; C\}$ *there exists a function* $\omega : B \rightarrow A$ *with* $\omega(\varepsilon) = 0$ *such that*

$$f(\alpha, \beta) - \{\omega(\alpha)\}(\beta\theta) - \omega(\beta) + \omega(\alpha\beta) \in C,$$

then f *is called a* $(B, A, \theta; C)$*-quasi transformation system.*

If we denote the group of all $(B, A, \theta; C)$-quasi transformation systems by $T(B, A, \theta; C)$, then it is clear that

$$T(B, A, \theta) \subseteq T(B, A, \theta; C) \subseteq F(B, A, \theta; C).$$

We now prove

THEOREM 11:

$$\mathrm{Ext}(B, A', \theta') \cong F(B, A, \theta; K(\varphi))/T(B, A, \theta; K(\varphi)).$$

PROOF:

Consider an arbitrary $f \in F(B, A, \theta; K(\varphi))$ and define

$$f' : B \times B \rightarrow A';\ f'(\alpha, \beta) = \{f(\alpha, \beta)\}\varphi. \tag{3.5.5}$$

Then $f' \in F(B, A', \theta')$ because

1° $f'(\varepsilon, \alpha) = \{f(\varepsilon, \alpha)\}\varphi = 0$ and similarly $f'(\alpha, \varepsilon) = 0$ for all $\alpha \in B$,

2° $$\begin{aligned} & f'(\alpha\beta, \gamma) + \{f'(\alpha, \beta)\}(\gamma\theta') - f'(\alpha, \beta\gamma) - f'(\beta, \gamma) \\ &= [f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma)]\varphi \\ &= 0 \text{ since } f \in F(B, A, \theta; K(\varphi)). \end{aligned}$$

Define a mapping

$$\chi: F(B, A, \theta; K(\varphi)) \rightarrow F(B, A', \theta');\ f\chi = f' \ (f' \text{ defined by } (3.5.5)).$$

Then χ is clearly a homomorphism. That χ is even an epimorphism may be shown as follows:

Select fixed representatives in the cosets of $K(\varphi)$ in A with $r(a')$ *in the coset corresponding to a′ under the isomorphism* $A/K(\varphi) \cong A'$, and in particular $r(0') = 0$. Furthermore, since

$$[\{r(g'(\alpha^{-1}, \alpha))\}(\alpha^{-1}\theta)]\varphi = \{g'(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta'),$$

we can choose the representatives $r(a')$ so as to satisfy

$$[\{r(g'(\alpha^{-1}, \alpha))\}(\alpha^{-1}\theta)]\varphi = \{g'(\alpha^{-1}, \alpha)\}](\alpha^{-1}\theta), \quad \alpha \in B. \tag{3.5.6}$$

Then for any given $g' \in F(B, A', \theta')$ we define:

$$g : B \times B \to A; \; g(\alpha, \beta) = r(g'(\alpha, \beta)).$$

Thus $g \in F(B, A, \theta; K(\varphi))$, for

1° $g(\alpha, \varepsilon) = r(g'(\alpha, \varepsilon)) = 0$ and similarly $g(\varepsilon, \alpha) = 0$ for all $\alpha \in B$.

2°
$$\begin{aligned} & [g(\alpha\beta, \gamma) + \{g(\alpha, \beta)\}(\gamma\theta) - g(\alpha, \beta\gamma) - g(\beta, \gamma)]\varphi \\ = & [r(g'(\alpha\beta, \gamma)) + r\{(g'(\alpha, \beta))(\gamma\theta)\} + \\ & + k - r(g'(\alpha, \beta\gamma)) - r(g'(\beta, \gamma))]\varphi, \; k \in K(\varphi). \\ = & \; g'(\alpha\beta, \gamma) + \{g'(\alpha, \beta)\}(\gamma\theta') - g'(\alpha, \beta\gamma) - g'(\beta, \gamma) \\ = & \; 0, \end{aligned}$$

thus $g(\alpha\beta, \gamma) + \{g(\alpha, \beta)\}(\gamma\theta) - g(\alpha, \beta\gamma) - g(\beta, \gamma) \in K(\varphi)$.

3°
$$\begin{aligned} & \{g(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) \\ = & [r\{g'(\alpha^{-1}, \alpha)\}](\alpha^{-1}\theta) \\ = & \; r[\{g'(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta')] \\ = & \; r\{g'(\alpha, \alpha^{-1})\} \\ = & \; g(\alpha, \alpha^{-1}) \text{ for all } \alpha \in B. \end{aligned}$$

Also $g\chi = g'$.

We now prove that $\{T(B, A, \theta; K(\varphi))\}\chi = T(B, A', \theta')$. (3.5.7)

Suppose to this end that $f \in T(B, A, \theta; K(\varphi))$.
i.e. There exists a function $\omega : B \to A$ with $\omega(\varepsilon) = 0$ and such that

$$f(\alpha, \beta) - \{\omega(\alpha)\}(\beta\theta) - \omega(\beta) + \omega(\alpha\beta) \in K(\varphi).$$

If now $f' = f\chi$ then

$$f'(\alpha, \beta) = \{f(\alpha, \beta)\}\varphi$$
$$= [\{\omega(\alpha)\}(\beta\theta)]\varphi + \{\omega(\beta)\}\varphi - \{\omega(\alpha\beta)\}\varphi$$
$$= [\{\omega(\alpha)\}\varphi](\beta\theta') + \{\omega(\beta)\}\varphi - \{\omega(\alpha\beta)\}\varphi$$
$$= \{\omega'(\alpha)\}(\beta\theta') + \omega'(\beta) - \omega'(\alpha\beta) \text{ for all } \alpha, \beta \in B,$$

where $\omega' : B \to A'$ is defined by $\omega'(\alpha) = \{\omega(\alpha)\}\varphi$ for all $\alpha \in B$.
i.e. $f' \in T(B, A', \theta')$.

If, conversely, $g' \in T(B, A', \theta')$, then there exists a function $\omega : B \to A'$ with $\omega'(\varepsilon) = 0'$ and such that
$g'(\alpha, \beta) = \{\omega'(\alpha)\}(\beta\theta') + \omega'(\beta) - \omega'(\alpha\beta)$ for all $\alpha, \beta \in B$.

Suppose $g\chi = g'$.
i.e. For all $\alpha, \beta \in B$ we have $\{g(\alpha, \beta)\}\varphi = g'(\alpha, \beta)$.

Thus,

$$\{g(\alpha, \beta)\}\varphi - \{\omega'(\alpha)\}(\beta\theta') - \omega'(\beta) + \omega'(\alpha\beta) = 0. \qquad (3.5.8)$$

Define $\omega : B \to A;\ \omega(\alpha) = r\{\omega'(\alpha)\}$.

Then (3.5.8) becomes

$$[g(\alpha, \beta) - \{\omega(\alpha)\}(\beta\theta) - \omega(\beta) + \omega(\alpha\beta)]\varphi = 0$$
i.e. $g(\alpha, \beta) - \{\omega(\alpha)\}(\beta\theta) - \omega(\beta) + \omega(\alpha\beta) \in K(\varphi)$

and thus
$g \in T(B, A, \theta; K(\varphi))$.

This proves (3.5.7) and the theorem follows.

§ 3.6 **Direct products**

We have in the case where all groups considered are abelian that

$$\mathrm{Ext}(B, \sum_{\lambda}^{*} A) \cong \sum_{\lambda}^{*} \mathrm{Ext}(B, A), \qquad (3.6.1)$$

(Fuchs [8]). We are now interested in the possible existence of an analogy of (3.6.1) in the general extension system $\{B, A, \theta; C\}$.

Suppose, therefore, that $A = P \oplus Q$ where P and Q are $B\theta$-admissable subgroups of A. Let furthermore $P' < P$, $Q' < Q$ and put $C = P' \oplus Q'$. We then prove

THEOREM 12:

$$\text{Ext}(B, P \oplus Q, \theta; C) \cong \text{Ext}(B, P, \theta; P') \oplus \text{Ext}(B, Q, \theta; Q').$$

PROOF:

Consider arbitrary $(f, g) \in F(B, P, \theta; P') \oplus F(B, Q, \theta; Q')$ and define

$$h : B \times B \to A;\ h(\alpha, \beta) = f(\alpha, \beta) + g(\alpha, \beta). \qquad (3.6.2)$$

Then $h \in F(B, A, \theta; C)$, for

1° $h(\varepsilon, \alpha) = f(\varepsilon, \alpha) + g(\varepsilon, \alpha) = 0$ and also $h(\alpha, \varepsilon) = 0$ for all $\alpha \in B$.

2° $h(\alpha\beta, \gamma) + \{h(\alpha, \beta)\}(\gamma\theta) - h(\alpha, \beta\gamma) - h(\beta, \gamma)$
$= f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\theta) - f(\alpha, \beta\gamma) - f(\beta, \gamma)$
$+ g(\alpha\beta, \gamma) + \{g(\alpha, \beta)\}(\gamma\theta) - g(\alpha, \beta\gamma) - g(\beta, \gamma)$
$\in C$,

since $f \in F(B, P, \theta; P')$ and $g \in F(B, Q, \theta; Q')$.

3° $\{h(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta)$
$= \{f(\alpha^{-1}, \alpha) + g(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta)$
$= \{f(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) + \{g(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta)$
$= f(\alpha, \alpha^{-1}) + g(\alpha, \alpha^{-1})$
$= h(\alpha, \alpha^{-1})$ for all $\alpha \in B$.

If we define a mapping
$\Phi : F(B, P, \theta; P') \oplus F(B, Q, \theta; Q') \to F(B, A, \theta; C)$; $(f, g)\Phi = h$, (h defined by (3.6.2)), then Φ is clearly a homomorphism.

To prove that Φ is an epimorphism, we consider an arbitrary $h \in F(B, A, \theta; C)$. Then since $A = P \oplus Q$, we have that for all $\alpha, \beta \in B$, $h(\alpha, \beta)$ can be uniquely written in the form

$$h(\alpha, \beta) = h_p(\alpha, \beta) + h_q(\alpha, \beta) \qquad (3.6.3)$$

with $h_p(\alpha, \beta) \in P$ and $h_q(\alpha, \beta) \in Q$.

Collecting all $h_p(\alpha, \beta)$ (resp. $h_q(\alpha, \beta)$) with α and β running through B, we have a function $h_p : B \times B \to P$ (resp. $h^q : B \times B \to Q$) and we prove that

$$h_p \in F(B, P, \theta; P') \text{ and } h_q \in F(B, Q, \theta; Q'). \qquad (3.6.4)$$

(i) $h(\varepsilon, \alpha) = h_p(\varepsilon, \alpha) + h_q(\varepsilon, \alpha) = 0$ and by the uniqueness of the form (3.6.3), $h_p(\varepsilon, \alpha) = h_q(\varepsilon, \alpha) = 0$ for all $\alpha \in B$. Similarly $h_p(\alpha, \varepsilon) = h_q(\alpha, \varepsilon) = 0$ for all $\alpha \in B$.

(ii) $h(\alpha\beta, \gamma) + \{h(\alpha, \beta)\}(\gamma\theta) - h(\alpha, \beta\gamma) - h(\beta, \gamma)$

$= h_p(\alpha\beta, \gamma) + \{h_p(\alpha, \beta)\}(\gamma\theta) - h_p(\alpha, \beta\gamma) - h_p(\beta, \gamma)$

$+ h_q(\alpha\beta, \gamma) + \{h_q(\alpha, \beta)\}(\gamma\theta) - h(\alpha, \beta\gamma) - h_q(\beta, \gamma)$

$\in C$.

Again by the uniqueness of (3.6.3), we have

$h_p(\alpha\beta, \gamma) + \{h_p(\alpha, \beta)\}(\gamma\theta) - h_p(\alpha, \beta\gamma) - h_p(\beta, \gamma) \in P'$ and

$h_q(\alpha\beta, \gamma) + \{h_q(\alpha, \beta)\}(\gamma\theta) - h_q(\alpha, \beta\gamma) - h_q(\beta, \gamma) \in Q'$.

(iii) $\{h(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta)$

$= \{h_p(\alpha^{-1}, \alpha) + h_q(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta)$

$= \{h_p(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) + \{h_q(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta)$.

Furthermore, $h(\alpha, \alpha^{-1}) = h_p(\alpha, \alpha^{-1}) + h_q(\alpha, \alpha^{-1})$.

Since $\{h(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) = h(\alpha, \alpha^{-1})$ and by the uniqueness of the form (3.6.3), we have

$\{h_p(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) = h_p(\alpha, \alpha^{-1})$ and

$\{h_q(\alpha^{-1}, \alpha)\}(\alpha^{-1}\theta) = h_q(\alpha, \alpha^{-1})$

for all $\alpha \in B$.

This proves (3.6.4) and since $(h_p, h_q)\Phi = h$, we find that Φ is indeed an epimorphism. We still have to show that

$$\{T(B, P, \theta) \oplus T(B, Q, \theta)\}\Phi = T(B, A, \theta). \qquad (3.6.5)$$

If therefore $(f, g) \in T(B, P, \theta) \oplus T(B, Q, \theta)$, then there exist functions $\omega_f : B \to P$ and $\omega_g : B \to Q$ with $\omega_f(\varepsilon) = \omega_g(\varepsilon) = 0$ and such that if $(f, g)\Phi = h$, then

$h(\alpha, \beta) = f(\alpha, \beta) + g(\alpha, \beta)$

$= \{\omega_f(\alpha) + \omega_g(\alpha)\}(\beta\theta) + \omega_f(\beta) + \omega_g(\beta) - \omega_f(\alpha\beta) - \omega_g(\alpha\beta)$

$= \{\omega(\alpha)\}(\beta\theta) + \omega(\beta) - \omega(\alpha\beta)$ for all $\alpha, \beta \in B$,

where $\omega : B \to A$ is defined by $\omega(\alpha) = \omega_f(\alpha) + \omega_g(\alpha)$ for all $\alpha \in B$.

i.e. $h \in T(B, A, \theta)$.

Conversely, if $h \in T(B, A, \theta)$ and (f, g) is an arbitrary original of h under Φ, then by definition

$$f(\alpha, \beta) + g(\alpha, \beta) = h(\alpha, \beta)$$
$$= \{\omega(\alpha)\}(\beta\theta) + \omega(\beta) - \omega(\alpha\beta)$$

for all $\alpha, \beta \in B$ where $\omega : B \to A$ has the property $\omega(\varepsilon) = 0$.

Since $A = P \oplus Q$, we may express $\omega(\alpha)$ uniquely in the form

$$\omega(\alpha) = \omega_p(\alpha) + \omega_q(\alpha),\ \omega_p(\alpha) \in P,\ \omega_q(\alpha) \in Q \qquad (3.6.6)$$

for all $\alpha \in B$.

Then $\omega_p(\varepsilon) = \omega_q(\varepsilon) = 0$ and moreover

$$f(\alpha, \beta) + g(\alpha, \beta) = \{\omega_p(\alpha)\}(\beta\theta) + \omega_p(\beta) - \omega_p(\alpha\beta)$$
$$+ \{\omega_q(\alpha)\}(\beta\theta) + \omega_q(\beta) - \omega_q(\alpha\beta)$$

for all $\alpha, \beta \in B$.

By the uniqueness of the form (3.6.3) we now have

$$f(\alpha, \beta) = \{\omega_p(\alpha)\}(\beta\theta) + \omega_p(\beta) - \omega_p(\alpha\beta) \text{ and}$$
$$g(\alpha, \beta) = \{\omega_q(\alpha)\}(\beta\theta) + \omega_q(\beta) - \omega_q(\alpha\beta) \text{ for all } \alpha, \beta \in B$$

which proves (3.6.5) and thereby the theorem.

REMARK:

Theorem 12 is directly extendable to the case where A is the direct sum of a finite number of components, since the proof is entirely general.

Furthermore, if A and B are abelian groups and $B = \sum_{\lambda} B_\lambda$, then

$$\mathrm{Ext}(\sum_{\lambda} B_\lambda, A) \cong \sum_{\lambda} \mathrm{Ext}(B_\lambda, A) \qquad (3.6.7)$$

(Fuchs [8]). If we consider non-abelian extensions of A by B however, then (3.6.7) has no direct analogy as can be illustrated by the following counter example:

Let $A = C(\infty)$, $B_1 = C(3)$ and $B_2 = C(2)$, i.e. $B_1 \oplus B_2 \cong C(6)$. $\mathfrak{A}(A) \cong C(2)$ and denote by α the automorphism $n \to -n$ for all $n \in A$. Consider the homomorphisms

$$\theta : C(6) \to \mathfrak{A}(A);\ 1\theta = \alpha \text{ and } \varphi : C(2) \to \mathfrak{A}(A);\ 1\varphi = \alpha.$$

Then by theorem 6 we have

$$\mathrm{Ext}(B_1 \oplus B_2, A, \theta) = 0. \qquad (3.6.8)$$

On the other hand

$$\mathrm{Ext}(B_1, A, \theta_0) \oplus \mathrm{Ext}(B_2, A, \varphi) \cong A/3A \cong C(3). \quad (3.6.9)$$

Comparing (3.6.8) and (3.6.9) we conclude that in general, with θ induced by θ' and θ'', $\mathrm{Ext}(B' \otimes B'', A, \theta)$ and
$\mathrm{Ext}(B', A, \theta') \oplus \mathrm{Ext}(B'', A, \theta'')$
are not isomorphic.

The question arises under what conditions imposed on the general extension system $\{B, A, \theta; C\}$, a generalization of (3.6.7) is possible. We now illustrate that such generalizations are indeed possible. Consider an abelian group A with $A[2] = 0$ and let V_4 and the homomorphisms θ_i ($i = 0, 1, 2$) be as defined in § 2.2. We then prove

THEOREM 13:

(1) $\mathrm{Ext}(V_4, A, \theta_0) \cong \mathrm{Ext}(R, A, \theta_0) \oplus \mathrm{Ext}(S, A, \theta_0)$.

(2) $\mathrm{Ext}(V_4, A, \theta_1) \cong \mathrm{Ext}(R, A, \theta_0) \oplus \mathrm{Ext}(S, A, \psi)$.

(3) $\mathrm{Ext}(V_4, A, \theta_2) \cong \mathrm{Ext}(R, A, \varphi) \oplus \mathrm{Ext}(S, A, \psi)$.

PROOF:

(1) By Lemma 5 we have that

$$f \in F(V_4, A, \theta_0) \Longrightarrow f \text{ is symmetric.}$$

Thus every extension of A by V_4 having an extension system (θ_0, f) with $f \in F(V_4, A, \theta_0)$, is abelian.

i.e. $\mathrm{Ext}(V_4, A, \theta_0) = \mathrm{Ext}(V_4, A)$
$\cong \mathrm{Ext}(R, A) \oplus \mathrm{Ext}(S, A)$ by (3.6.7)
$= \mathrm{Ext}(R, A, \theta_0) \oplus \mathrm{Ext}(S, A, \theta_0)$ since R and S are cyclic.

(2) By theorem 6 and its corollary, we have
$\mathrm{Ext}(S, A, \psi) \cong A(S\psi)/N(2, \alpha) = 0$ since $A[2] = 0$ and
$\mathrm{Ext}(R, A, \theta_0) \cong A/2A$.

On the other hand, if we define a mapping

$$\Phi : F(V_4, A, \theta_1) \to A; \; f\Phi = f(r, r),$$

then Φ is not only a homomorphism but even an epimorphism.

For arbitrary $a \in A$ we namely define a function $f: V_4 \times V_4 \rightarrow A$ as follows:

$$\begin{aligned} f(x, 0) &= f(0, x) = 0 \text{ for all } x \in V_4, \\ f(s, s) &= f(r, s) = f(r+s, s) = f(r+s, r) = \\ &= f(r+s, r+s) = 0, \\ f(s, r) &= f(r, r) = a, \\ f(s, r+s) &= f(r, r+s) = -a. \end{aligned}$$

Then $f \in F(V_4, A, \theta_1)$ by theorem 2b and, moreover, $f\Phi = a$. Furthermore,

$$\{T(V_4, A, \theta_1)\}\Phi = 2A \text{ by theorem 5.}$$

$$\text{i.e. } F(V_4, A, \theta_1)/T(V_4, A, \theta_1) \cong A/2A.$$

(3) By theorem 6 and since $A[2] = 0$, we have

$$\text{Ext}(R, A, \varphi) \oplus \text{Ext}(S, A, \psi) = 0.$$

Also, by theorem 5 we have

$$F(V_4, A, \theta_2) = T(V_4, A, \theta_2).$$

$$\text{i.e. } \text{Ext}(V_4, A, \theta_2) = 0.$$

This proves the theorem.

§ 3.7. **Subdirect products**

In this paragraph we show that an analogy of property (3.6.7) holds for a special class of subdirect products $B = \underset{i=1}{\overset{n}{\tilde{\times}}} B_i$ in the general extension system $\{B, A, \theta; C\}$.

We treat the case $n = 2$ though our method is completely general. Suppose $B' = \{\varepsilon, \alpha, \beta, \gamma, \ldots\}$, $B'' = \{e, a, b, c, \ldots\}$ and $G = \{e, x, y, z, \ldots\}$ while $\lambda: B' \rightarrow G$ and $\mu: B'' \rightarrow G$ are epimorphisms. We also suppose that

B' *(resp.* B''*) is a splitting extension of* $K(\lambda)$ *(resp.* $K(\mu)$*) by* G. (3,7.1.)

The set $B = \{(\alpha, a) \mid \alpha \in B', a \in B'' \text{ with } \alpha\lambda = a\mu\}$ with operation $(\alpha, a)(\beta, b) = (\alpha\beta, ab)$ is a subdirect product of B′ and B″. Let $\theta^{\star} : G \to \mathfrak{A}(A)$ be an arbitrary fixed homomorphism and define the homomorphisms

$$\theta : B \to \mathfrak{A}(A);\ (\alpha, a)\theta = \alpha\lambda\theta^{\star} = a\mu\theta^{\star}.$$
$$\varphi : B' \to \mathfrak{A}(A);\ \alpha\varphi = \alpha\lambda.\theta^{\star}.$$
$$\psi : B'' \to \mathfrak{A}(A);\ a\psi = a\mu\theta^{\star}.$$

By (3.7.1), it is possible to choose representatives in the cosets of $K(\lambda)$ in B′ (resp. $K(\mu)$ in B″) in the following special way: if we denote the representative of the coset to which α (resp. a) belongs by α (resp. $\bar{a}$), then

$$\overline{\alpha\beta} = \bar{\alpha}.\bar{\beta} \text{ (resp. } \overline{ab} = \bar{a}.\bar{b}).$$

In particular, ε (resp. e) is the representative of $K(\lambda)$ (resp. $K(\mu)$). We now construct a subdirect sum $F(B', A, \varphi; C) \tilde{+} F(B'', A, \psi; C)$: for a given $f \in F(B', A, \varphi; C)$ we define a function $f' : G \times G \to A$ with

$$f'(x, y) = f(\bar{\alpha}, \bar{\beta}) \text{ where } \alpha \in x\lambda^{-1}, \beta \in y\lambda^{-1}. \qquad (3.7.2)$$

Thus

1° $f'(e, y) = f(\varepsilon, \bar{\alpha}) = 0$ and similarly $f'(y, e) = 0$ where $\alpha \in y\lambda^{-1}$ for all $y \in G$.

2° $f'(xy, z) + \{f'(x, y)\}(z\theta^{\star})$
$= f(\bar{\alpha}.\bar{\beta}, \bar{\gamma}) + \{f(\bar{\alpha}, \bar{\beta})\}(\bar{\gamma}\varphi)$ where $\alpha \in x\lambda^{-1}$, $\beta \in y\lambda^{-1}$, $\gamma \in z\lambda^{-1}$
$\equiv f(\bar{\alpha}, \bar{\beta}.\bar{\gamma}) + f(\bar{\beta}, \bar{\gamma})$ (modulo C)
$= f'(x, yz) + f'(y, z)$ for all $x, y, z \in G$.

3° $\{f'(x^{-1}, x)\}(x^{-1}\theta^{\star})$
$= \{f(\bar{\alpha}^{-1}, \bar{\alpha})\}(\bar{\alpha}^{-1}\varphi)$ where $\alpha \in x\lambda^{-1}$
$= f(\bar{\alpha}, \bar{\alpha}^{-1})$
$= f'(x, x^{-1})$ for all $x \in G$.

i.e. $f' \in F(G, A, \theta; C)$.

The mapping

$\bar{\lambda}: F(B', A, \varphi; C) \to F(G, A, \theta^{\star}; C);\ f\bar{\lambda} = f'$ (f′ defined by (3.7.2)) is an epimorphism. To prove this, we define for an arbitrary $f' \in F(G, A, \theta^*; C)$ a function

$$f : B' \times B' \to A; \; f(\alpha, \beta) = f'(\alpha\lambda, \beta\lambda).$$

Then

(1) $f(\varepsilon, \alpha) = f'(e, \alpha\lambda) = 0$ and similarly $f(\alpha, \varepsilon) = 0$ for all $\alpha \in B'$.

(2) $f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\varphi)$

$$= f'(\alpha\lambda.\beta\lambda, \gamma\lambda) + \{f'(\alpha\lambda, \beta\lambda)\}(\gamma\lambda.\theta^*)$$
$$\equiv f'(\alpha\lambda, \beta\lambda.\gamma\lambda) + f'(\beta\lambda, \gamma\lambda) \text{ (modulo C)}$$
$$= f(\alpha, \beta\gamma) + f(\beta, \gamma).$$

(3) $\{f(\alpha^{-1}, \alpha)\}(\alpha^{-1}\varphi) = \{f'(\alpha^{-1}\lambda, \alpha\lambda)\}(\alpha^{-1}\lambda.\theta^*)$

$$= f'(\alpha\lambda, \alpha^{-1}\lambda)$$
$$= f(\alpha, \alpha^{-1}) \text{ for all } \alpha \in B'.$$

i.e. $f \in F(B', A, \varphi; C)$ and $f\bar{\lambda} = f'$.

In an exactly similar manner an epimorphism

$$\bar{\mu} : F(B'', A, \psi; C) \to F(G, A, \theta^*; C)$$

is defined. Let

$S = \{(f, g) \mid f \in F(B', A, \varphi; C),\, g \in F(B'', A, \psi; C) \text{ and } f\bar{\lambda} = g\bar{\mu}\}$. Then we have a subdirect sum $S = F(B', A, \varphi; C) \mathbin{\tilde{+}} F(B'', A, \psi; C)$.

To establish an epimorphism of $F(B, A, \theta; C)$ onto S, we proceed as follows:

For arbitrary $h \in F(B, A, \theta; C)$ we define

$$f : B' \times B' \to A; \; f(\alpha, \beta) = h[(\alpha, \bar{a}), (\beta, \bar{b})]; \; a \in \alpha\lambda\mu^{-1}, \; b \in \beta\lambda\mu^{-1}. \tag{3.7.3}$$

Since

(i) $f(\varepsilon, \beta) = h[(\varepsilon, e), (\beta, \bar{b})] = 0$; $b \in \beta\lambda\mu^{-1}$ and similarly $f(\beta, \varepsilon) = 0$ for all $\beta \in B'$,

(ii) $f(\alpha\beta, \gamma) + \{f(\alpha, \beta)\}(\gamma\varphi)$

$$= h[(\alpha\beta, \bar{a}.\bar{b}), (\gamma, \bar{c})] + \{h[\alpha, \bar{a}), (\beta, \bar{b})]\}(\gamma, \bar{c})\theta$$
$$\equiv h[(\alpha, \bar{a}), (\beta\gamma, \bar{b}.\bar{c})] + h[(\beta, \bar{b}), (\bar{\gamma}, c)] \text{ (modulo C)}$$
$$= f(\alpha, \beta\gamma) + f(\beta, \gamma) \text{ for all } \alpha, \beta, \gamma \in B',$$

(iii) $\{f(\alpha^{-1}, \alpha)\}(\alpha^{-1}\varphi)$

$= [h\{(\alpha^{-1}, \overline{a^{-1}}), (\alpha, \bar{a})\}] (\alpha^{-1}, \overline{a^{-1}})\theta$ where $a \in \alpha\lambda\mu^{-1}$,

i.e. $\overline{a^{-1}} \in \alpha^{-1}\lambda\mu^{-1}$.

$= h\{(\alpha, \bar{a}), (\alpha^{-1}, \overline{a^{-1}})\}$

$= f(\alpha, \alpha^{-1})$ for all $\alpha \in B'$,

we have $f \in F(B', A, \varphi; C)$.

Similarly, for

$$g : B'' \times B'' \to A;\ g(a, b) = h[(\bar{\alpha}, a), (\bar{\beta}, b)];\ \alpha \in a\mu\lambda^{-1},\ \beta \in b\mu\lambda^{-1} \tag{3.7.4}$$

we have $g \in F(B'', A, \psi; C)$.

Moreover, $f\bar{\lambda} = g\bar{\mu}$, thus $(f, g) \in S$.

With f, g and h as above, we show that $\Phi : F(B, A, \theta; C) \to S$; $h\Phi = (f, g)$ is an epimorphism. For arbitrary $(f, g) \in S$ we define

$$h : B \times B \to A;\ h[(\alpha, a), (\beta, b)] = f(\alpha, \beta) + g(a, b) - f(\bar{\alpha}, \bar{\beta}). \tag{3.7.5}$$

Then we have

(a) $h[(\varepsilon, e), (\beta, b)] = f(\varepsilon, \beta) + g(e, b) - f(\varepsilon, \bar{\beta}) = 0$.
Similarly, $h[(\beta, b), (\varepsilon, e)] = 0$ for all $(\beta, b) \in B$.

(b) $h[(\alpha\beta, ab), (\gamma, c)] + \{h[(\alpha, a), (\beta, b)]\}(\gamma, c)\theta$

$= f(\alpha\beta, \gamma) + g(ab, c) - f(\bar{\alpha}.\bar{\beta}, \bar{\gamma}) + \{f(\alpha, \beta)\}(\gamma\varphi) + \{g(a, b)\}(c\psi) - \{f(\bar{\alpha}, \bar{\beta})\}(\bar{\gamma}\varphi)$.

$\equiv f(\alpha, \beta\gamma) + g(a, bc) - f(\bar{\alpha}, \bar{\beta}.\bar{\gamma}) + h[(\beta, b), (\gamma, c]. - f(\bar{\beta}, \bar{\gamma})$ (modulo C)

$= h[(\alpha, a), (\beta\gamma, bc)] + h[(\beta, b), (\gamma, c)]$.

(c) $[h\{(\alpha^{-1}, a^{-1}), (\alpha, a)\}] (\alpha^{-1}, a^{-1})\theta$

$= \{f(\alpha^{-1}, \alpha)\}(\alpha^{-1}\varphi) + \{g(a^{-1}, a)\}(a^{-1}\psi) - \{f(\overline{\alpha^{-1}}, \bar{\alpha})\}(\overline{\alpha^{-1}}\varphi)$

$= f(\alpha, \alpha^{-1}) + g(a, a^{-1}) - f(\bar{\alpha}, \overline{\alpha^{-1}})$

$= h\{(\alpha, a), (\alpha^{-1}, a^{-1})\}$ for all $(\alpha, a) \in B$.

i.e. $h \in F(B, A, \theta; C)$, and $h\Phi = (f, g)$.

Finally we show that

$$\{T(B, A, \theta)\}\Phi = T(B', A, \phi) \underset{\sim}{+} T(B'', A, \psi). \qquad (3.7.6)$$

If $h \in T(B, A, \theta)$, then there exists a function

$\omega : B \to A;\ \omega(\varepsilon, e) = 0$ and such that

$h[(\alpha, a), (\beta, b)] = \{\omega(\alpha, a)\}(\beta, b)\theta + \omega(\beta, b) - \omega(\alpha\beta, ab)$ for all $(\alpha, a), (\beta, b) \in B$. If then $f = h\Phi$ is defined by (3.7.3), we have

$$\begin{aligned} f(\alpha, \beta) &= h[(\alpha, \bar{a}), (\beta, \bar{b})] \\ &= [\omega(\alpha, \bar{a})](\beta, \bar{b})\theta + \omega(\beta, \bar{b}) - \omega(\alpha\beta, \bar{a}.\bar{b}) \\ &= \{\omega'(\alpha)\}(\beta\varphi) + \omega'(\beta) - \omega'(\alpha\beta) \text{ for all } \alpha, \beta \in B', \end{aligned}$$

where $\omega'(\alpha) = \omega(\alpha, \bar{a})$.

i.e. $f \in T(B', A, \varphi)$.

Similarly $g \in T(B'', A, \psi)$ for g defined by (3.7.4).

i.e. $h\Phi = (f, g) \in T(B', A, \varphi) \underset{\sim}{+} T(B'', A, \psi)$.

If conversely $(f, g) \in T(B', A, \varphi) \underset{\sim}{+} T(B'', A, \psi)$, then there exist functions $\omega_f : B' \to A$ and $\omega_g : B'' \to A$ such that $f(\alpha, \beta) = \{\omega_f(\alpha)\}(\beta\varphi) + \omega_f(\beta) - \omega_f(\alpha\beta)$ for all $\alpha, \beta \in B'$ and $g(a, b) = \{\omega_g(a)\}(b\psi) + \omega_g(b) - \omega_g(ab)$ for all $a, b \in B''$. Thus for $h : B \times B \to A$ defined by (3.7.5), we have:

$$\begin{aligned} h[(\alpha, a), (\beta, b)] = &\{\omega_f(\alpha) + \omega_g(a) - \omega_f(\bar{\alpha})\}(\beta, b)\theta \\ &+ \{\omega_f(\beta) + \omega_g(b) - \omega_f(\bar{\beta})\} \\ &- \{\omega_f(\alpha\beta) + \omega_g(ab) - \omega_f(\bar{\alpha}.\bar{\beta})\} \\ = &\{\omega(\alpha, a)\}(\beta, b)\theta + \omega(\beta, b) - \omega(\alpha\beta, ab) \end{aligned}$$

for all $(\alpha, a), (\beta, b) \in B$ where

$\omega(\alpha, a) = \omega_f(\alpha) + \omega_g(a) - \omega_f(\bar{\alpha})$.

i.e. $h \in T(B, A, \theta)$.

This proves (3.7.6) and we have thus proved

THEOREM 14:

$$\text{Ext}(B' \underset{\sim}{\times} B'', A, \theta; C) \cong \text{Ext}(B', A, \varphi; C) \underset{\sim}{+} \text{Ext}(B'', A, \psi; C).$$

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SAMENVATTING

Het begrip groepsuitbreiding G van een abelse groep A door een groep B wordt in dit proefschrift gegeneraliseerd door te veronderstellen dat G een quasi-groep is, die aan bepaalde eisen voldoet. Het begrip factorstelsel wordt eveneens gegeneraliseerd en wel tot dat van quasi-factorstelsel (§ 2.1). Het blijkt dat het probleem, om bij een gegeven abelse groep A en een willekeurige groep B alle quasi-groepuitbreidingen G van A door B te vinden, kan worden herleid tot de vraag om alle homomorfismen van B in de automorfismengroep van A en om alle quasi-factorstelsels van B $\times$ B in A te bepalen. Het Hölder-Schreier uitbreidingsprobleem is dus een bijzonder geval van het quasi-groepuitbreidingsprobleem.

Er worden constructies van factorstelsels gegeven in twee speciale gevallen (§ 2.2), waarbij tevens nodige en voldoende voorwaarden worden afgeleid opdat een gegeven factorstelsel een transformatiestelsel is (§ 2.4).

Als C een ondergroep is van een abelse groep A en B een willekeurige groep, terwijl θ een homomorfisme van B in de automorfismengroep van A is, dan is Ext(B, A, θ; C) de groep van alle niet-equivalente C-quasi-groepuitbreidingen van A door B met geassociëerde homomorfisme θ (§ 3.1). De groep Ext(B, A, θ) van alle groepsuitbreidingen van A door B met geassociëerde homomorfisme θ blijkt een ondergroep te zijn van Ext(B, A, θ; C) voor willekeurige C.

In Hoofdstuk III wordt een onderzoek ingesteld naar de invloed die bepaalde eigenschappen van A, B, C en θ op de structuur van Ext(B, A, θ; C) uitoefenen. Vervolgens worden structurele eigenschappen van bepaalde ondergroepen van Ext(B, A, θ; C) bestudeerd.

BIOGRAPHY

The author was born in South Africa in 1933. He completed his school career at the Vereeniging High School in 1950 and attended the Potchefstroom University and the Potchefstroom College of Education. After a few years as teacher in mathematics at the high schools of Nylstroom and Kempton Park, he was appointed lecturer in mathematics at the University of Pretoria where he also obtained the M.Sc. degree in mathematics in 1961. In 1963 he became assistant to Professor Loonstra in Delft.

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STELLINGEN

I

Gebruik makende van de notatie in dit proefschrift veronderstellen we, dat $f \in F(C(n), C(\infty), \theta)$. Indien de uitbreiding G van $C(\infty)$ door $C(n)$ wordt bepaald door middel van het uitbreidingssysteem (θ_0, f), dan is G cyclisch dan en slechts dan, als $\sum_{i=1}^{n-1} f(1, i) \equiv 1 \pmod{n}$.

II

De in dit proefschrift ingevoerde klasse van quasigroepen is het meest algemene algebraïsche systeem waarvoor een uitbreidingstheorie van een abelse groep A door een groep B langs de lijnen van de Hölder-Schreier theorie nog kan worden doorgevoerd.

III

Het is voor de quasigroepuitbreidingstheorie van abelse groepen van belang om de existentie te onderzoeken van groepen (eventueel gedeeltelijk geordend) met voorgeschreven automorfismengroep.

IV

Stelt G een quasigroep voor en zijn L, C en R de verzamelingen van de resp. links-, centraal- en rechts-associatieve elementen van G, dan zijn L, C en R semigroepen.

V

Er bestaan commutatieve quasigroepen met elementen, die centraalassociatief, doch niet linksassociatief zijn.

G. N. Garrison, Quasi-groups, Ann. of Math. 41 (1940) p. 479.

VI

De door McCoy bewezen stelling, dat een commutatieve ring R regulair is, dan en slechts dan, als elk ideaal in R met zijn radikaal samenvalt, kan worden gegeneraliseerd voor niet-commutatieve ringen.

N. H. McCoy, Rings and Ideals, The Mathematical Association of America, Baltimore (1948) p. 148.

A. P. J. van der Walt, Bydrae tot die nie-kommutatiewe ideaalteorie, Proefschrift, Potchefstroom (1963) p. 23.

VII

De restklassenring van de gehele getallen modulo n is een zg. von Neumann-ring dan en slechts dan als de Möbiusfunctie μ voldoet aan de voorwaarde $\mu(n) \neq 0, n > 1$.

VIII

In zijn "Partially ordered algebraic systems" (Oxford 1963; p. 19, (f)) spreekt L. Fuchs de volgende stelling uit: „Zij de ondergroep C op grond van zijn in G geïnduceerde gedeeltelijke ordening een gerichte groep, dan is C dan en slechts dan convex, als uit $c \in C$ en $c^{-1} \leqq x \leqq c$ volgt, dat $x \in C$". De voorwaarde, dat C gericht zou moeten zijn, is overbodig.

IX

$u_0 + u_1 + u_2 \ldots$ zij een reeks van reële getallen, waarvan de partiële sommen s_n voldoen aan $s_n = 0(n^t)$ voor $n \rightarrow \infty$ (t constant, $0 < t < 1$), terwijl $s_m - s_n \rightarrow c$ als $m \rightarrow \infty$, $n \rightarrow \infty$ en $m/n \rightarrow \lambda$. (λ en c zijn constanten met λ positief). Als k een constante is met $k > t$ en $s_n^{(k)}$ de n-de Cesàro-som van de orde k voorstelt, dan geldt $s_m^{(k)} - s_n^{(k)} \rightarrow c$ als m en $n \rightarrow \infty$, terwijl $m/n \rightarrow \lambda$.

X

De beschouwing, die A. Dinghas wijdt aan het complexe getal $z = \infty$, is aanvechtbaar.

A. Dinghas, Vorlesungen über Funktionentheorie, Springer-Verlag (1961) § 4, p. 8.

XI

Gezien het grote aantal studiemislukkingen onder eerstejaarsstudenten aan Zuid-Afrikaanse universiteiten, verdient het, met het oog op een verbetering van het studierendement, aanbeveling om een onderzoek in te stellen naar de toelatingseisen, en wel in dié zin, dat het jaarlijkse aantal afstuderenden niet vermindert.

XII

Het bijzondere beleggingspatroon in Zuid-Afrika kan in grote mate worden toegeschreven aan geschiedkundige factoren.

XIII

Van medisch standpunt bekeken is het gevaar van een bacteriologische oorlogsvoering veel groter dan dat van een aanval met chemische wapens.