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Multivariate Askey–Wilson functions and overlap coefficients

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Abstract We study certain overlap coefficients appearing in representation theory of the quantum algebra $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$. The overlap coefficients can be identified as products of Askey–Wilson functions, leading to an algebraic interpretation of the multivariate Askey–Wilson functions introduced by Geronimo and Iliev [1]. We use the underlying coalgebra structure to derive q-difference equations satisfied by the multivariate Askey–Wilson functions.

1 Introduction

The Askey–Wilson functions are q-hypergeometric functions generalising the Askey–Wilson polynomials [2]. The latter are the polynomials in $x + x^{-1}$ that are eigenfunctions of the Askey–Wilson q-difference operator

$$A(x)(T_q - 1) + A(x^{-1})(T_{q^{-1}} - 1), \tag{1.1}$$

where $(T_q f)(x) = f(qx)$ and

$$A(x) = \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}.$$

Ismail and Rahman [3] obtained explicit eigenfunctions, not necessarily polynomials, of the Askey–Wilson q-difference operator. The Askey–Wilson function is a specific nonpolynomial eigenfunction which appears as the kernel in an integral transform due to Koelink and Stokman [4] describing the spectral properties of the Askey–Wilson q-difference operator. Unitarity of the integral transform implies orthogonality relations for the Askey–Wilson functions first obtained by Suslov [5,6]. In [7] it is shown that the Askey–Wilson functions appear as spherical functions on the SU(1, 1) quantum group; other representation theoretic interpretations are e.g. in double affine Hecke algebras [8] and as $6j$-symbols [9].

Gasper and Rahman [10] introduced multivariate extensions of the Askey–Wilson polynomials, defined as nested products of univariate Askey–Wilson polynomials, generalising Tratnik's [11] multivariate Wilson polynomials. It was then shown by Iliev [12] that the multivariate Askey–Wilson polynomials in N variables are eigenfunctions of N independent q-difference operators that can be considered as extensions of the Askey–Wilson q-difference operator (1.1). Moreover, by using a symmetry property the multivariate -Wilson polynomials were shown to be bispectral. These multivariate polynomials also naturally appear in representation theory, see e.g. [13–15]. Geronimo and Iliev [1] extended the results from [12] to the level of Askey–Wilson functions using analytic continuation, leading to q-difference equations for multivariate Askey–Wilson functions. In this paper we give a representation theoretic interpretation of these multivariate functions and their q-difference equations.

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The organisation of the paper is as follows. In Sect. 2 we introduce the quantum algebra $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ and the representation of the algebra we will use. In Sect. 3 we consider eigenfunctions of two different algebra elements and compute their overlap coefficients. These coefficients are given in terms of a q-hypergeometric integral which can be identified as a univariate Askey–Wilson function. By writing the action of the algebra on the eigenfunctions as operators acting in the spectral variables, we construct q-difference operators for which the overlap coefficients are eigenfunctions. The q-difference operators are shown to be the Askey–Wilson q-difference operators. The construction of the Askey–Wilson function is similar to Stokman's construction in [16], but the construction of the related q-difference operators is different. Furthermore, the construction immediately gives a symmetry property of the overlap coefficients which leads to bispectrality of the Askey–Wilson functions. We briefly also consider simpler versions of the overlap coefficients, which we use to derive bilateral summation formulas involving ${}_2\varphi_1$-functions. In Sect. 4 we extend the results from Sect. 3 to a multivariate setting using the coalgebra structure of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$. Similar to the construction of multivariate Askey–Wilson polynomials in [15] this results in an interpretation of the multivariate Askey–Wilson functions as overlap coefficients, and leads to a construction of q-difference equations for these functions.

1.1 Notations

Throughout the paper $q \in (0, 1)$ is fixed. We use standard notations for q-shifted factorials, θ-functions and q-hypergeometric functions as in [17]. In particular, q-shifted factorials and theta-functions are defined by

$$(x; q)_n = \prod_{j=0}^{n-1}(1 - xq^j), \quad x \in \mathbb{C},\ n \in \mathbb{Z}_{\geq 0} \cup \{\infty\},$$
$$\theta(x; q) = (x, q/x; q)_\infty, \quad x \in \mathbb{C}^\times,$$

from which it follows that

$$(qx; q)_\infty = \frac{1}{1-x}(x; q)_\infty, \qquad \theta(qx; q) = -\frac{1}{x}\theta(x; q). \tag{1.2}$$

We use the standard shorthand notations

$$(x_1, x_2, \ldots, x_k; q)_n = \prod_{j=1}^{k}(x_j; q)_n,$$
$$\theta(x_1, x_2, \ldots, x_k; q) = \prod_{j=1}^{k}\theta(x_j; q).$$

Moreover, $\pm$-symbols in exponents inside q-shifted factorials or theta functions means taking products over all possible combinations of $+$ and $-$ signs, e.g.

$$(xy^{\pm 1}z^{\pm 1}; q)_\infty = (xyz, xyz^{-1}, xy^{-1}z, xy^{-1}z^{-1}; q)_\infty.$$

2 The quantum algebra

The quantum algebra $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ is the unital, associative, complex algebra generated by K, K^{-1}, E, and F, subject to the relations

$$KK^{-1} = 1 = K^{-1}K,$$
$$KE = qEK, \quad KF = q^{-1}FK,$$
$$EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}}.$$

$\mathcal{U}_q$ has a comultiplication $\Delta : \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$ defined on the generators by

$$\begin{aligned} \Delta(K) &= K \otimes K, & \Delta(E) &= K \otimes E + E \otimes K^{-1}, \\ \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, & \Delta(F) &= K \otimes F + F \otimes K^{-1}. \end{aligned} \tag{2.1}$$

We equip $\mathcal{U}_q$ with the $*$-structure $* : \mathcal{U}_q \to \mathcal{U}_q$ defined on the generators by

$$K^* = K, \quad E^* = -F, \quad F^* = -E, \quad (K^{-1})^* = K^{-1},$$

which corresponds to the real form $\mathfrak{su}(1,1)$ of $\mathfrak{sl}_2(\mathbb{C})$.

2.1 Twisted primitive elements

For $s, u \in \mathbb{C}^\times$ we define two twisted primitive elements $Y_{s,u}$ and $\widetilde{Y}_{s,u}$ by

$$\begin{aligned} Y_{s,u} &= uq^{\frac{1}{2}} EK - u^{-1}q^{-\frac{1}{2}} FK + \mu_s(K^2 - 1), \\ \widetilde{Y}_{s,u} &= uq^{\frac{1}{2}} FK^{-1} - u^{-1}q^{-\frac{1}{2}} EK^{-1} + \mu_s(K^{-2} - 1), \end{aligned} \tag{2.2}$$

where

$$\mu_s = \frac{s + s^{-1}}{q^{-1} - q}.$$

We have

$$(Y_{s,u})^* = Y_{\bar{s},\bar{u}^{-1}} \quad \text{and} \quad (\widetilde{Y}_{s,u})^* = \widetilde{Y}_{\bar{s},\bar{u}^{-1}}.$$

From (2.1) it follows that

$$\begin{aligned} \Delta(Y_{s,u}) &= K^2 \otimes Y_{s,u} + Y_{s,u} \otimes 1, \\ \Delta(\widetilde{Y}_{s,u}) &= \widetilde{Y}_{s,u} \otimes K^{-2} + 1 \otimes \widetilde{Y}_{s,u}, \end{aligned} \tag{2.3}$$

so that $Y_{s,u}$ and $\widetilde{Y}_{s,u}$ belong to a left, respectively right, coideal of $\mathcal{U}_q$.

2.2 Representations

Let $\mathcal{M}$ be the space of meromorphic functions on $\mathbb{C}^\times$. For $\lambda, \varepsilon \in \mathbb{R}$ we define a representation $\pi = \pi_{\lambda,\varepsilon}$ of $\mathcal{U}_q$ on $\mathcal{M}$ by

$$\begin{aligned} \big[\pi_{\lambda,\varepsilon}(K)\big](z) &= q^\varepsilon f(qz), \\ [\pi_{\lambda,\varepsilon}(E)f](z) &= z\frac{q^{-\frac{1}{2}-i\lambda-\varepsilon} f(z/q) - q^{\frac{1}{2}+i\lambda+\varepsilon} f(qz)}{q^{-1} - q}, \\ [\pi_{\lambda,\varepsilon}(F)f](z) &= z^{-1}\frac{q^{-\frac{1}{2}-i\lambda+\varepsilon} f(qz) - q^{\frac{1}{2}+i\lambda-\varepsilon} f(z/q)}{q^{-1} - q}. \end{aligned} \tag{2.4}$$

We define an inner product by

$$\langle f, g \rangle = \frac{1}{2\pi i} \int_{\mathbb{T}} f(z) g^\star(z) \frac{dz}{z},$$

with $g^\star(z) = \overline{g(\overline{z}^{-1})}$, and where the unit circle $\mathbb{T}$ has positive orientation. Let $\mathcal{U}_q^1$ be the subspace of $\mathcal{U}_q$ spanned by $1, K, K^{-1}, E$ and F. Suppose that $f, g \in \mathcal{M}$ are analytic on the annulus $\{q \le |z| \le q^{-1}\}$, then

$$\langle \pi(X) f, g \rangle = \langle f, \pi(X^*) g \rangle, \qquad X \in \mathcal{U}_q^1, \tag{2.5}$$

which follows from Cauchy's theorem to shift the path of integration. For $X = X_1 \cdots X_k$ with $X_i \in \mathcal{U}_q^1$ the same property holds for functions f, g that are analytic on the annulus $\{q^k \le |z| \le q^{-k}\}$.

The following result, which is proved by direct verification, will be useful later on.

Lemma 2.1 *The assignment*

$$\vartheta(K) = K^{-1}, \quad \vartheta(E) = F, \quad \vartheta(F) = E$$

extends to an involutive algebra isomorphism and coalgebra anti-isomorphism $\vartheta : \mathcal{U}_q \to \mathcal{U}_q$ *satisfying*

- $\vartheta(Y_{s,u}) = \widetilde{Y}_{s,u}$;
- $\pi_{\lambda,\varepsilon}(\vartheta(X)) = r \circ \pi_{\lambda,-\varepsilon}(X) \circ r$, $X \in \mathcal{U}_q$, *where* $r : \mathcal{M} \to \mathcal{M}$ *is the reflection operator defined by* $[rf](z) = f(1/z)$.

To end this section let us introduce some convenient notation. The functions we study later on will depend on (a subset of) the parameters s, u, t, v, λ and ε coming from the twisted primitive elements $Y_{s,u}$ and $\widetilde{Y}_{t,v}$, and the representation $\pi_{\lambda,\varepsilon}$. To simplify notation we let α be the ordered 6-tuple

$$\alpha = (s, u, t, v, \lambda, \varepsilon).$$

On such 6-tuples we define an involution ϑ, which corresponds to the $\mathcal{U}_q$-involution ϑ, by

$$\alpha^{\vartheta} = (t, v, s, u, \lambda, -\varepsilon).$$

If $f = f_\alpha$ is a function depending on α, then we denote by f^{ϑ} the same function with α replaced by α^{ϑ}; $f^{\vartheta} = f_{\alpha^{\vartheta}}$. We sometimes use the notation

$$\bar{\alpha} = (\bar{s}, \bar{u}, \bar{t}, \bar{v}, \lambda, \varepsilon).$$

Note that $(\bar{\alpha})^{\vartheta} = \overline{\alpha^{\vartheta}}$.

3 Overlap coefficients and univariate Askey–Wilson functions

In this section we consider eigenfunctions of $\pi(Y_{s,u})$ and $\pi(\widetilde{Y}_{t,v})$ and we identify the overlap coefficients between the eigenfunctions with Askey–Wilson functions. We show that $\pi(\widetilde{Y}_{t,v})$ acts on the eigenfunctions of $\pi(Y_{s,u})$ as a q-difference operator in the spectral variable, which leads to a q-difference equation satisfied by the overlap coefficients.

3.1 Eigenfunctions

We consider eigenfunctions of $\pi(Y_{s,u})$. From the actions of the $\mathcal{U}_q$-generators (2.4) and the definition of $Y_{s,u}$ (2.2) it follows that, for $f \in \mathcal{M}$,

$$\begin{aligned}[]
[\pi(Y_{s,u})f](z) = {} & q^{2\varepsilon} \frac{s + s^{-1} - uzq^{1+i\lambda} - u^{-1}z^{-1}q^{-1-i\lambda}}{q^{-1} - q} f(q^2 z) \\
& + \frac{uzq^{-i\lambda} + u^{-1}z^{-1}q^{i\lambda} - s - s^{-1}}{q^{-1} - q} f(z).
\end{aligned} \tag{3.1}$$

The eigenvalue equation $\pi(Y_{s,u})f = \mu f$ now becomes a first-order q^2-difference equation, for which the eigenfunctions can be determined in terms of q^2-shifted factorials.

Lemma 3.1 *Define* $f_x = f_{x,\alpha}$ *by*

$$f_x(z) = \frac{(sq^{1-2i\lambda}x^{\pm 1}, uszq^{1+i\lambda}; q^2)_\infty \theta(q^{2\varepsilon - i\lambda}uz/s; q^2)}{(uzq^{-i\lambda}x^{\pm 1}, sq^{1-i\lambda}/uz; q^2)_\infty}, \qquad x \in \mathbb{C}^\times,$$

then $\pi(Y_{s,u})f_x = (\mu_x - \mu_s)f_x$.

The z-independent factor $(sq^{1-2i\lambda}x^{\pm 1}; q^2)_\infty$ is of course not needed for f_x to be an eigenfunction and is only inserted for convenience later on. Note that $f_{x,\alpha}$ only depends on the four parameters $s, u, \lambda, \varepsilon$ of the 6-tuple α.

Proof The eigenvalue equation $\pi(Y_{s,u})f = (\mu_x - \mu_s)f$ is equivalent to

$$f(q^2z) = \left(-\frac{sq^{i\lambda-2\varepsilon}}{uz}\right)\frac{(1-xuzq^{-i\lambda})(1-uzq^{-i\lambda}/x)}{(1-suzq^{1+i\lambda})(1-sq^{-1-i\lambda}/uz)}f(z).$$

From (1.2) it follows that f_x is a solution. □

Next we compute the action of $\pi(K^{-1})$ on the $\pi(Y_{s,u})$-eigenfunctions f_x. It will be convenient to use the notation $f_{x,s}(z)$ for $f_{x,\alpha}(z)$ to stress its dependence on the parameter s.

Lemma 3.2 *The function $f_{x,s}$ satisfies*

$$\begin{aligned}\pi(K^{-1})f_{x,s} &= a^-(x,s)f_{xq,s/q} + a^-(x^{-1},s)f_{x/q,s/q}\\ &= a^+(x,s)f_{xq,sq} + a^+(x^{-1},s)f_{x/q,sq},\end{aligned}$$

with $a^\pm(x,s) = a^\pm_\alpha(x)$ given by

$$a^-(x,s) = -\frac{xq^\varepsilon}{s(1-x^2)}, \qquad a^+(x,s) = \frac{q^{-\varepsilon}(1-sxq^{1-2i\lambda})(1-sxq^{1+2i\lambda})}{1-x^2}.$$

Proof We consider the function

$$g_{x,s}(z) = \frac{f_{x,s}(z)}{(sq^{1-2i\lambda}x^{\pm1};q^2)_\infty} = \frac{(uszq^{1+i\lambda};q^2)_\infty\theta(q^{2\varepsilon-i\lambda}uz/s;q^2)}{(uzq^{-i\lambda}x^{\pm1}, sq^{1-i\lambda}/uz;q^2)_\infty}.$$

This function satisfies

$$\begin{aligned}g_{xq,s/q}(z) &= -\frac{sq^{1+i\lambda-2\varepsilon}(1-uzxq^{-1-i\lambda})}{uz(1-sq^{-i\lambda}/uz)}g_{x,s}(z/q),\\ g_{xq,sq}(z) &= \frac{1-uzxq^{-1-i\lambda}}{1-suzq^{i\lambda}}g_{x,s}(z/q).\end{aligned}$$

From $g_{x,s} = g_{x^{-1},s}$ we obtain similar expressions for $g_{x/q,s/q}$ and $g_{x/q,sq}$ by replacing x by x^{-1}. From a direct calculation it then follows that

$$\begin{aligned}q^{1+2i\lambda-2\varepsilon}g_{x,s}(z/q) &= \frac{1-xq^{1+2i\lambda}/s}{1-x^2}g_{xq,s/q}(z) + \frac{1-q^{1+2i\lambda}/xs}{1-x^{-2}}g_{x/q,s/q}(z),\\ g_{x,s}(z/q) &= \frac{1-xsq^{1+2i\lambda}}{1-x^2}g_{xq,sq}(z) + \frac{1-sq^{1+2i\lambda}/x}{1-x^{-2}}g_{x/q,sq}(z).\end{aligned}$$

Note that

$$\begin{aligned}g_{xq,s/q} &= -\frac{xq^{1+2i\lambda}/s}{1-xq^{1+2i\lambda}/s}\frac{f_{xq,s/q}}{(sq^{1-2i\lambda}x^{\pm1};q^2)_\infty},\\ g_{xq,sq} &= (1-sxq^{1-2i\lambda})\frac{f_{xq,sq}}{(sq^{1-2i\lambda}x^{\pm1};q^2)_\infty},\end{aligned}$$

then the result follows from $f_x = f_{x^{-1}}$. □

Combining the two actions of $\pi(K^{-1})$ on $f_{x,s}$ it follows that $\pi(K^{-2})$ acts as a q^2-difference operator in x on $f_{x,s}$.

Corollary 3.3 *The function f_x satisfies*

$$\pi(K^{-2})f_x = A(x)f_{xq^2} + B(x)f_x + A(x^{-1})f_{x/q^2},$$

where $A = A_\alpha$ and $B = B_\alpha$ are given by

$$\begin{aligned}A(x) &= -\frac{x(1-sxq^{1-2i\lambda})(1-sxq^{1+2i\lambda})}{s(1-x^2)(1-q^2x^2)},\\ B(x) &= -A(x) - A(1/x).\end{aligned}$$

Proof This follows from Lemma 3.2 using

$$A(x) = a^-(xq, sq)a^+(x, s),$$
$$B(x) = a^-(1/xq, sq)a^+(x, s) + a^-(xq, sq)a^+(1/x, s),$$

and $a^-(1/xq, sq) = -a^-(xq, sq)$. □

Applying Lemma 3.2 twice it follows that $\pi(K^{-2})$ can also be realized as a difference operator acting in x and s. This result will be useful in Sect. 4 where we consider difference operators in a multivariate setting.

Corollary 3.4 *The function $f_{x,s}$ satisfies*

$$\begin{aligned}\pi(K^{-2})f_{x,s} &= A^-(x)f_{xq^2,s/q^2} + B^-(x)f_{x,s/q^2} + A^-(x^{-1})f_{x/q^2,s/q^2}\\ &= A^+(x)f_{xq^2,sq^2} + B^+(x)f_{x,sq^2} + A^+(x^{-1})f_{x/q^2,sq^2},\end{aligned}$$

where $A^\pm = A^\pm_\alpha$ and $B^\pm = B^\pm_\alpha$ are given by

$$A^-(x) = \frac{x^2q^{2+2\varepsilon}}{s^2(1-x^2)(1-q^2x^2)},$$
$$B^-(x) = \frac{q^{2\varepsilon}}{s^2(1-x^2/q^2)(1-1/x^2q^2)},$$

and

$$A^+(x) = \frac{q^{-2\varepsilon}(1-sxq^{1-2i\lambda})(1-sxq^{3-2i\lambda})(1-sxq^{1+2i\lambda})(1-sxq^{3+2i\lambda})}{(1-x^2)(1-q^2x^2)},$$
$$B^+(x) = \frac{q^{-2\varepsilon-1}(q^{-1}+q)(1-sxq^{1-2i\lambda})(1-sq^{1-2i\lambda}/x)(1-sxq^{1+2i\lambda})(1-sq^{1+2i\lambda}/x)}{(1-x^2/q^2)(1-1/x^2q^2)}.$$

Proof This follows from Lemma 3.2 using

$$A^\pm(x) = a^\pm(x, s)a^\pm(xq, sq^{\pm1})$$

and

$$B^\pm(x) = a^\pm(x, s)a^\pm(1/xq, sq^{\pm1}) + a^\pm(1/x, s)a^\pm(x/q, sq^{\pm1}).$$

Writing this out and simplifying gives the expressions given in the lemma. □

Eigenfunctions of $\pi(\widetilde{Y}_{t,v})$ can be obtained from the eigenfunctions of $\pi(Y_{s,u})$. By applying Lemma 2.1 to (3.1), or by direct verification, it follows that

$$\begin{aligned}[\pi(\widetilde{Y}_{t,v})f](z) &= q^{-2\varepsilon}\frac{t+t^{-1}-v^{-1}zq^{-1-i\lambda}-vz^{-1}q^{1+i\lambda}}{q^{-1}-q}f(z/q^2)\\ &\quad+\frac{v^{-1}zq^{i\lambda}+vz^{-1}q^{-i\lambda}-t-t^{-1}}{q^{-1}-q}f(z),\end{aligned}$$

which is essentially equation (3.1) with (s, u, ε, z) replaced by $(t, v, -\varepsilon, z^{-1})$. Using Lemma 2.1 it follows that $f_x^{r,\vartheta} = rf_{x,\alpha^\vartheta}$, where r is the reflection operator, is an eigenfunction of $\pi(\widetilde{Y}_{t,v})$:

$$\pi_{\lambda,\varepsilon}(\widetilde{Y}_{t,v})(rf_{x,\alpha^\vartheta}) = (r\circ\pi_{\lambda,-\varepsilon}(Y_{t,v^{-1}}))f_{x,\alpha^\vartheta} = (\mu_x-\mu_t)rf_{x,\alpha^\vartheta}.$$

Lemma 3.5 *The function*

$$f_x^{r,\vartheta}(z) = \frac{(tq^{1-2i\lambda}x^{\pm1}, tvq^{1+i\lambda}/z; q^2)_\infty\theta(vq^{-2\varepsilon-i\lambda}/tz; q^2)}{(vq^{-i\lambda}x^{\pm1}/z, tzq^{1-i\lambda}/v; q^2)_\infty}$$

satisfies $\pi(\widetilde{Y}_{t,v})f_x^{r,\vartheta} = (\mu_x-\mu_t)f_x^{r,\vartheta}$.

Slightly abusing notation we will omit r in our notation and write $f_x^\vartheta = f_x^{r,\vartheta}$.

3.2 Overlap coefficients: Askey–Wilson functions

Let us write $\bar{f}_x^{\vartheta} = r f_{x,\bar{\alpha}^{\vartheta}}$, then $\bar{f}_x^{\vartheta}$ is an eigenfunction of $\pi(\widetilde{Y}_{\bar{t},\bar{v}})$ for eigenvalue $\mu_x - \mu_{\bar{t}}$.

Definition 3.6 Let $0 < |ux|, |u/x|, |vy|, |v/y| \le q^2$ and $0 < |s/u|, |t/v| \le q$. We define $\Phi(x, y) = \Phi_\alpha(x, y)$ to be the overlap coefficient between f_x and $\bar{f}_{\bar{y}}^{\vartheta}$, i.e.

$$\Phi(x, y) = \left\langle f_x, \bar{f}_{\bar{y}}^{\vartheta} \right\rangle.$$

The following symmetry property of Φ is immediate from the definition.

Proposition 3.7 $\Phi_\alpha(x, y) = \overline{\Phi_{\bar{\alpha}^{\vartheta}}(\bar{y}, \bar{x})}$

First we will derive difference equations for Φ. We show that we can write the action of $\widetilde{Y}_{\bar{t},\bar{v}^{-1}}$ on f_x as a difference operator in x. Then the symmetry property of Proposition 3.7 immediately leads to a corresponding difference operator in y. Since we already know the actions of $Y_{s,u}$ and K^{-2} on f_x, it suffices to express $\widetilde{Y}_{t,v^{-1}}$ in terms of $Y_{s,u}$ and K^{-2}:

$$\begin{aligned} \widetilde{Y}_{t,v^{-1}} &= \frac{qu/v - v/uq}{q^{-2} - q^2} K^{-2} Y_{s,u} + \frac{vq/u - u/vq}{q^{-2} - q^2} Y_{s,u} K^{-2} \\ &\quad + \frac{(q^{-1} + q)(t + t^{-1}) - (v/u + u/v)(s + s^{-1})}{q^{-2} - q^2} (K^{-2} - 1). \end{aligned} \tag{3.2}$$

For $v = 1$ this is [15, Lemma 4.3] and the proof, which we have included in the appendix, is the same. It will be convenient to introduce parameters a, b, c, d corresponding to the 6-tuple α by

$$(a, b, c, d) = (sq^{1-2i\lambda}, sq^{1+2i\lambda}, vtq/u, vq/ut), \tag{3.3}$$

Note that

$$(a^{\vartheta}, b^{\vartheta}, c^{\vartheta}, d^{\vartheta}) = (tq^{1-2i\lambda}, tq^{1+2i\lambda}, usq/v, uq/vs). \tag{3.4}$$

Proposition 3.8 *The function* f_x *satisfies*

$$\pi(\widetilde{Y}_{t,v^{-1}}) f_x = \mathcal{A}(x) f_{xq^2} + \mathcal{B}(x) f_x + \mathcal{A}(x^{-1}) f_{x/q^2}, \tag{3.5}$$

with $\mathcal{A} = \mathcal{A}_\alpha$ *and* $\mathcal{B} = \mathcal{B}_\alpha$ *given by*

$$\begin{aligned} \mathcal{A}(x) &= \frac{1}{q^{-1} - q} \frac{(1 - ax)(1 - bx)(1 - cx)(1 - dx)}{(q^2/d^{\vartheta})(1 - x^2)(1 - q^2 x^2)}, \\ \mathcal{B}(x) &= \frac{q^2/d^{\vartheta} + d^{\vartheta}/q^2}{q^{-1} - q} - \mu_t - \mathcal{A}(x) - \mathcal{A}(x^{-1}). \end{aligned}$$

Proof From (3.2), Lemma 3.1 and Corollary 3.3 it follows that $\pi(\widetilde{Y}_{t,v^{-1}})$ acts as (3.5) on f_x with

$$\begin{aligned} \mathcal{A}(x) &= \frac{A(x)}{q^{-2} - q^2} \Big((uq/v - v/uq)(\mu_x - \mu_s) + (vq/u - u/vq)(\mu_{xq^2} - \mu_s) \\ &\quad + (q^{-1} + q)(t + t^{-1}) - (v/u + u/v)(s + s^{-1}) \Big) \\ &= -\frac{u}{vxq} \frac{(1 - vqx/ut)(1 - vqtx/u)}{q^{-1} - q} A(x) \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}(x) &= \frac{B(x)}{q^{-2} - q^2} \Big((uq/v - v/uq + vq/u - u/vq)(\mu_x - \mu_s) \\ &\quad + \frac{B(x) - 1}{q^{-2} - q^2} \Big((q^{-1} + q)(t + t^{-1}) - (v/u + u/v)(s + s^{-1}) \Big) \\ &= \frac{(t + t^{-1})(B(x) - 1)}{q^{-1} - q} + \frac{(u/v + v/u)((s + s^{-1}) - (x + x^{-1})B(x))}{q^{-2} - q^2}. \end{aligned}$$

A calculation shows that

$$\mathcal{B}(x) = \frac{svq/u + u/svq - t - 1/t}{q^{-1} - q} - \mathcal{A}(x) - \mathcal{A}(x^{-1}).$$

Rewriting this in terms of the parameters (3.3) proves the proposition. □

Now we are ready to show that the overlap coefficient Φ is an eigenfunction of the Askey–Wilson difference operator (1.1).

Theorem 3.9 *The overlap coefficient* $\Phi(x, y)$ *satisfies*

$$(\mu_y - \mu_t)\Phi(x, y) = \mathcal{A}(x)\Phi(q^2x, y) + \mathcal{B}(x)\Phi(x, y) + \mathcal{A}(x^{-1})\Phi(x/q^2, y),$$
$$(\mu_x - \mu_s)\Phi(x, y) = \mathcal{A}^\vartheta(y)\Phi(x, q^2y) + \mathcal{B}^\vartheta(y)\Phi(x, y) + \mathcal{A}^\vartheta(y^{-1})\Phi(x, y/q^2),$$

with $\mathcal{A}(x)$ *and* $\mathcal{B}(x)$ *from Proposition 3.8.*

Proof Using Lemma 3.5 we obtain

$$(\mu_y - \mu_t)\langle f_x, \bar{f}^\vartheta_{\bar{y}}\rangle = \langle f_x, \pi(\widetilde{Y}_{\bar{t},\bar{v}})\bar{f}^\vartheta_{\bar{y}}\rangle = \langle \pi(\widetilde{Y}_{t,v^{-1}})f_x, \bar{f}^\vartheta_{\bar{y}}\rangle.$$

Note that this is allowed, since the given conditions ensure that the integrand is analytic on the annulus $\{q^2 \le |z| \le q^{-2}\}$. By Proposition 3.8 this is the first stated q^2-difference equation. The second q^2-difference equation follows from the first using the symmetry from Proposition 3.7 and observing that $\overline{\mathcal{A}_{\bar{\alpha}^\vartheta}(\bar{y})} = \mathcal{A}^\vartheta(y)$ and $\overline{\mathcal{B}_{\bar{\alpha}^\vartheta}(\bar{y})} = \mathcal{B}^\vartheta(y)$. □

Explicitly $\Phi(x, y)$ is given by the q-hypergeometric integral

$$\Phi_\alpha(x, y) = (sq^{1-2i\lambda}x^{\pm1}, tq^{1+2i\lambda}y^{\pm1}; q^2)_\infty \times \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(uszq^{1+i\lambda}, tvzq^{1-i\lambda}; q^2)_\infty \theta(uzq^{2\varepsilon-i\lambda}/s, vzq^{-2\varepsilon+i\lambda}/t; q^2)}{(uzq^{-i\lambda}x^{\pm1}, sq^{1-i\lambda}/uz, vzq^{i\lambda}y^{\pm1}, tq^{1+i\lambda}/vz; q^2)_\infty}\frac{dz}{z}, \tag{3.6}$$

where $\mathcal{C} = \mathbb{T}$. We can get rid of the conditions on s, u, t, v, x, y from Definition 3.6 by replacing the contour $\mathbb{T}$ by a deformation $\mathcal{C}$ of $\mathbb{T}$ such that the poles $u^{-1}sq^{1-i\lambda}q^{\mathbb{Z}_{\ge0}}$, $v^{-1}tq^{1+i\lambda}q^{\mathbb{Z}_{\ge0}}$ are inside $\mathcal{C}$, and the poles $u^{-1}q^{i\lambda}x^{\pm1}q^{-\mathbb{Z}_{\ge0}}$, $v^{-1}q^{-i\lambda}y^{\pm1}q^{-\mathbb{Z}_{\ge0}}$ are outside $\mathcal{C}$.

We show that Φ is a multiple of an Askey–Wilson function [4]. For convenience we assume $s, t, u, v \in \mathbb{R}^\times$. In this case we have $a = \overline{b}$ for the Askey–Wilson parameters (3.3). Let us now introduce the Askey–Wilson function. Assume that A, B, C, D are parameters satisfying $A = \overline{B}$ and $C, D \in \mathbb{R}$. Define dual parameters $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ by

$$\widetilde{A} = \sqrt{ABCD/q}, \quad \widetilde{B} = AB/\widetilde{A}, \quad \widetilde{C} = AC/\widetilde{A}, \quad \widetilde{D} = AD/\widetilde{A}. \tag{3.7}$$

The Askey–Wilson function with parameters A, B, C, D is defined by

$$\psi_\gamma(x; A, B, C, D\,|\,q) = \frac{(AB, AC; q)_\infty}{(q/AD; q)_\infty}\, {}_4\varphi_3\left(\begin{matrix} Ax, A/x, \widetilde{A}\gamma, \widetilde{A}/\gamma \\ AB, AC, AD \end{matrix}; q, q\right) + \frac{(Ax^{\pm1}, \widetilde{A}\gamma^{\pm1}, qB/D, qC/D; q)_\infty}{(qx^{\pm1}/D, q\gamma^{\pm1}/\widetilde{D}, AD/q; q)_\infty}\, {}_4\varphi_3\left(\begin{matrix} qx/D, q/Dx, q\gamma/\widetilde{D}, q/\widetilde{D}\gamma \\ qB/D, qC/D, q^2/AD \end{matrix}; q, q\right). \tag{3.8}$$

The Askey–Wilson function (3.8) is normalized slightly different compared to [4]. Using Bailey's transformation [17, (III.36)] $\psi_\gamma(x)$ can also be written as a multiple of a very-well-poised ${}_8\varphi_7$-series,

$$\psi_\gamma(x; A, B, C, D\,|\,q) = \frac{(AB, AC, BC, Aq/D, qA\gamma x^{\pm1}/\widetilde{D}; q)_\infty}{(\widetilde{A}\widetilde{B}\widetilde{C}, q\gamma/\widetilde{D}, \widetilde{A}q/\widetilde{D}, qx^{\pm1}/D; q)_\infty} \times {}_8W_7\left(\widetilde{A}\widetilde{B}\widetilde{C}\gamma/q; Ax, A/x, \widetilde{A}\gamma, \widetilde{B}\gamma, \widetilde{C}\gamma; q, q/\widetilde{D}\gamma\right),$$

for $|\widetilde{D}\gamma| > q$. In [18, Section 5.5] it is shown that $\psi_\gamma(x)$ can be expressed as a q-hypergeometric integral,

$$\psi_\gamma(x) = \frac{(q, Ax^{\pm1}, \widetilde{A}\gamma^{\pm1}; q)_\infty}{\theta(1/\nu, q/AD\nu; q)}\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(ABz, ACz; q)_\infty \theta(AD\nu z, z/\nu; q)}{(Azx^{\pm1}, \widetilde{A}z\gamma^{\pm1}, 1/z, q/ADz; q)_\infty}\frac{dz}{z}, \tag{3.9}$$

where $\mathcal{C}$ is a deformation of the unit circle with positive orientation, such that the poles $q^{\mathbb{Z}_{\ge0}}$ and $(AD/q)q^{\mathbb{Z}_{\ge0}}$ are inside $\mathcal{C}$ and the poles $(x^{\pm1}/A)q^{-\mathbb{Z}_{\ge0}}$ and $(\gamma^{\pm1}/\widetilde{A})q^{-\mathbb{Z}_{\ge0}}$ are outside $\mathcal{C}$, and $\nu \in \mathbb{C}^\times$ can be chosen arbitrarily.

Theorem 3.10 *The overlap coefficient* $\Phi(x, y)$ *can be expressed as an Askey–Wilson function by*

$$\Phi(x, y) = \frac{(b^{\vartheta} y^{\pm 1}; q^2)_\infty \theta(1/\kappa, ad\kappa; q^2)}{(q^2, q^2 y^{\pm 1}/d^{\vartheta}; q^2)_\infty} \psi_y(x; a, b, c, d \,|\, q^2),$$

where

$$(a, b, c, d, \kappa) = (sq^{1-2i\lambda}, sq^{1+2i\lambda}, tvq/u, qv/ut, q^{-1-2\varepsilon+2i\lambda}). \tag{3.10}$$

Proof In the integral (3.6) for Φ we substitute $z \mapsto (sq^{1-i\lambda}/u)w$ then

$$\begin{aligned}\Phi_\alpha(x, y) &= (sq^{1-2i\lambda}x^{\pm 1}, tq^{1+2i\lambda}y^{\pm 1})_\infty \\ &\quad \times \frac{1}{2\pi i} \int_{(u/sq)\mathbb{T}} \frac{(s^2q^2w, stvq^{2-2i\lambda}w/u; q^2)_\infty \theta(wq^{1+2\varepsilon-2i\lambda}, svq^{1-2\varepsilon}w/ut)}{(sq^{1-2i\lambda}wx^{\pm 1}, svqwy^{\pm 1}/u, 1/w, utq^{2i\lambda}/svw; q^2)_\infty} \frac{dw}{w}.\end{aligned}$$

Then the result follows from comparing this to the integral representation (3.9) of the Askey–Wilson function with

$$(A, B, C, D, \nu) = (sq^{1-2i\lambda}, sq^{1+2i\lambda}, tvq/u, qv/ut, q^{-1-2\varepsilon+2i\lambda}) = (a, b, c, d, \kappa),$$

and using $\widetilde{A} = suq/v = q^2/d^{\vartheta}$. □

Remark 3.11 (i) The representation parameter ε appears only in the parameters κ, so we see from the expression for Φ in Theorem 3.10 that ε only appears in the multiplicative constant in front of the Askey–Wilson function.

(ii) The duality property of the Askey–Wilson function as given in [4, (3.4)] states that the Askey–Wilson function is invariant under interchanging the variables x and γ up to an involution on the parameters A, B, C, D;

$$\psi_\gamma(x; A, B, C, D \,|\, q) = \psi_x(\gamma; \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \,|\, q). \tag{3.11}$$

The symmetry property of Proposition 3.7 is very similar, but it is not the same identity. Note that we have

$$(a^{\vartheta}, b^{\vartheta}, c^{\vartheta}, d^{\vartheta}) = (ac/\tilde{a}, bc/\tilde{a}, ab/\tilde{a}, q^2/\tilde{a}) \tag{3.12}$$

where $\tilde{a} = \sqrt{abcd/q^2}$. To obtain the duality (3.11) from the identity in Proposition 3.7 we need to apply the following symmetries of the Askey–Wilson function:

$$\begin{aligned}\psi_\gamma(x; A, B, C, D \,|\, q) &= \psi_\gamma(x; A, C, B, D \,|\, q) \\ &= \frac{(Ax^{\pm 1}, \widetilde{A}\gamma^{\pm 1}; q)_\infty}{(qx^{\pm 1}/D, q\gamma^{\pm 1}/\widetilde{D}; q)_\infty} \psi_\gamma(x; q/D, B, C, A \,|\, q) \\ &= \frac{(\widetilde{C}\gamma^{\pm 1}; q)_\infty}{(q\gamma^{\pm 1}/\widetilde{D}; q)_\infty} \psi_\gamma(x; B, A, C, D \,|\, q).\end{aligned} \tag{3.13}$$

The first two identities are immediate from (3.8), the third identity is proved in [18, Proposition 5.27].

3.3 Overlap coefficients: Little q-Jacobi functions

We can also calculate overlap coefficients between the eigenfunctions f_x of $Y_{s,u}$ and eigenfunctions of K. The eigenfunction of K are the functions $e_n : \mathcal{M} \to \mathbb{C}$ given by

$$e_n(z) = z^n, \qquad n \in \mathbb{Z}.$$

Using (2.4) it is clear that $\pi(K)e_n = q^{n+\varepsilon}e_n$.

Definition 3.12 Let $0 < |ux|, |u/x| \leq q^2$ and $0 < |s/u| \leq q$. For $n \in \mathbb{Z}$ we define $\phi(x, n) = \phi_\alpha(x, n)$ to be the overlap coefficient between f_x and e_n, i.e.

$$\phi(x; n) = \langle f_x, e_n \rangle.$$

The overlap coefficient is given explicitly by

$$\phi(x,n) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(sq^{1-2i\lambda}x^{\pm 1}, uszq^{1+i\lambda}; q^2)_\infty \theta(q^{2\varepsilon - i\lambda}uz/s; q^2)}{(uzq^{-i\lambda}x^{\pm 1}, sq^{1-i\lambda}/uz; q^2)_\infty} z^{-n-1}\, dz. \tag{3.14}$$

Note that this is just the Fourier coefficient of f_x; $f_x(e^{i\theta}) = \sum_n \phi(x,n)e^{in\theta}$. The next result shows that $\phi(x,n)$ is a multiple of a ${}_2\varphi_1$-function, which can be recognised as a little q-Jacobi function, which is defined by

$$\varphi_\gamma(w; A, B; q) = {}_2\varphi_1\left(\begin{matrix} A\gamma, A/\gamma \\ AB \end{matrix}; q, -w\right), \qquad w \in \mathbb{C}\backslash[1,\infty).$$

Here we use the one-valued analytic continuation of the ${}_2\varphi_1$-function, see [17, §4.3].

Proposition 3.13 *$\phi(x,n)$ is given in terms of a ${}_2\varphi_1$-function by*

$$\phi(x,n) = \tau^n \frac{(ab; q^2)_\infty \theta(1/\kappa; q^2)}{(q^2; q^2)_\infty} {}_2\varphi_1\left(\begin{matrix} ax, a/x \\ ab \end{matrix}; q^2, \kappa q^{2-2n}\right),$$

with $(a, b, \kappa, \tau) = (sq^{1-2i\lambda}, sq^{1+2i\lambda}, q^{-1+2i\lambda-2\varepsilon}, uq^{i\lambda-1}/s)$.

Proof We use the following integral representation of the ${}_2\varphi_1$-function, see [19, Section 7],

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(t_1 z; q)_\infty \theta(\mu z/t_2; q)}{(t_2/z, t_3 z, t_4 z; q)_\infty}\frac{dz}{z} = \frac{(t_1 t_2; q)_\infty \theta(\mu; q)}{(q, t_2 t_3, t_2 t_4; q)_\infty} {}_2\varphi_1\left(\begin{matrix} t_2 t_3, t_2 t_4 \\ t_1 t_2 \end{matrix}; q, \frac{q}{\mu}\right),$$

where $\mathcal{C}$ is a deformation of the positively oriented unit circle including the poles $t_2 q^{\mathbb{Z}_{\geq 0}}$ and excluding the poles $t_3^{-1}q^{-\mathbb{Z}_{\geq 0}}$ and $t_4^{-1}q^{-\mathbb{Z}_{\geq 0}}$. In (3.14) we substitute $z \mapsto (u^{-1}q^{i\lambda})z$, then we recognize the above integral representation with q replaced by q^2 and

$$(t_1, t_2, t_3, t_4, \mu) = (sq^{1+2i\lambda}, sq^{1-2i\lambda}, x, x^{-1}, q^{1-2i\lambda+2\varepsilon+2n}).$$

The result then follows from using θ-function identities $\theta(x;q) = \theta(q/x;q)$ and $\theta(q^k x; q) = (-x)^{-k}q^{-\frac{1}{2}k(k-1)}\theta(x;q)$, $k \in \mathbb{Z}$. □

The Fourier expansion $f_x = \sum_n \phi(x,n)e_n$ leads to the following identity, which is a special case of a generating function from [20, Lemma 3.3].

Corollary 3.14 *Under the conditions of Definition 3.12,*

$$\sum_{n\in\mathbb{Z}} {}_2\varphi_1\left(\begin{matrix} ax, a/x \\ ab \end{matrix}; q^2, \kappa q^{2-2n}\right) t^{-n} = \frac{(q^2, ax^{\pm 1}, ab/t; q^2)_\infty \theta(1/t\kappa; q^2)}{(ab, ax^{\pm 1}/t, t; q^2)_\infty \theta(1/\kappa; q^2)},$$

with a, b, κ as in Proposition 3.13 and $t = 1/\tau z = sq^{1-i\lambda}/uz$.

Let us also consider the overlap coefficient between f_x^ϑ and $re_n = e_{-n}$,

$$\phi^\vartheta(x,n) = \langle f_x^\vartheta, e_{-n}\rangle = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(tq^{1-2i\lambda}x^{\pm 1}, tvq^{1+i\lambda}/z; q^2)_\infty \theta(vq^{-2\varepsilon - i\lambda}/tz; q^2)}{(vq^{-i\lambda}x^{\pm 1}/z, ztq^{1-i\lambda}/v; q^2)_\infty} z^{n-1}\, dz,$$

Using the substitution $z \mapsto z^{-1}$ we immediately see that

$$\phi^\vartheta(x,n) = \phi_{\alpha^\vartheta}(x,n),$$

(as was already implied by the notation). So ϕ^ϑ can be expressed as a ${}_2\varphi_1$-function by

$$\begin{aligned}\phi^\vartheta(x,n) &= (\tau^\vartheta)^n \frac{(a^\vartheta b^\vartheta; q^2)_\infty \theta(1/\kappa^\vartheta; q^2)}{(q^2;q^2)_\infty} {}_2\varphi_1\left(\begin{matrix} a^\vartheta x, a^\vartheta/x \\ a^\vartheta b^\vartheta \end{matrix}; q^2, \kappa^\vartheta q^{2-2n}\right) \\ &= (\tau^\vartheta)^n \frac{(cq^2/d; q^2)_\infty \theta(\bar{\kappa}q^2; q^2)}{(q^2;q^2)_\infty} {}_2\varphi_1\left(\begin{matrix} acx/\tilde{a}, ac/x\tilde{a} \\ cq^2/d \end{matrix}; q^2, q^{-2n}/\bar{\kappa}\right)\end{aligned}$$

using (3.12) and $\kappa^{\vartheta} = 1/q^2\bar{\kappa}$. Parseval's identity $\langle f_x, \bar{f}^{\vartheta}_{\bar{y}}\rangle = \sum_n \langle f_x, e_n\rangle\langle e_n, \bar{f}^{\vartheta}_{\bar{y}}\rangle$, which is written in terms of the overlap coefficients as

$$\sum_n \phi_\alpha(x,n)\overline{\phi_{\bar{\alpha}^\vartheta}(\bar{y},-n)} = \Phi_\alpha(x,y),$$

then leads to the following summation formula for ${}_2\varphi_1$-functions.

Corollary 3.15 *Under the conditions of Definition 3.6 the following summation formula holds:*

$$\begin{aligned}\sum_{n\in\mathbb{Z}} {}_2\varphi_1\left(\begin{matrix} ax, a/x \\ ab \end{matrix}; q^2, \kappa q^{2-2n}\right) {}_2\varphi_1\left(\begin{matrix} acy/\tilde{a}, ac/y\tilde{a} \\ cq^2/d \end{matrix}; q^2, \frac{q^{2n}}{\kappa}\right)\left(\frac{q^2}{ad}\right)^n \\ = \frac{(q^2, ac, bcy^{\pm1}/\tilde{a})_\infty \theta(ad\kappa;q^2)}{(q^2/ad, cq^2/d, \tilde{a}y^{\pm1};q^2)_\infty\theta(1/\kappa;q^2)} {}_4\varphi_3\left(\begin{matrix} ax, a/x, \tilde{a}y, \tilde{a}/y \\ ab, ac, ad \end{matrix}; q^2, q^2\right) \\ + \frac{(q^2, q^2b/d, ax^{\pm1})_\infty\theta(ad\kappa;q^2)}{(ab, ad/q^2, q^2x^{\pm1}/d)_\infty\theta(1/\kappa;q^2)} {}_4\varphi_3\left(\begin{matrix} q^2x/d, q^2/dx, bcy/\tilde{a}, bc/y\tilde{a} \\ q^2b/d, q^2c/d, q^4/ad \end{matrix}; q^2, q^2\right).\end{aligned}$$

where the parameters a, b, c, d, κ *are given by* (3.10) *and* $\tilde{a} = \sqrt{abcd/q^2}$.

4 Overlap coefficients and multivariate Askey–Wilson functions

In this section we extend the results from Sect. 3 to a multivariate setting using the coalgebra structure of $\mathcal{U}_q$. We obtain multivariate Askey–Wilson functions as overlap coefficients for representations of $\mathcal{U}_q^{\otimes N}$. We also show that the multivariate Askey–Wilson functions are simultaneous eigenfunctions of N commuting difference operators coming from commuting elements in $\mathcal{U}_q^{\otimes N}$. Recall from Remark 3.11 that the representation parameter ε is not important for studying q-difference equations for the Askey–Wilson functions, therefore we choose $\varepsilon = 0$ for all representations $\pi_{\lambda,\varepsilon}$ in this section.

Let $N \in \mathbb{Z}_{\geq 2}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$. We consider the representation $\pi_{\boldsymbol{\lambda}}$ of $\mathcal{U}_q^{\otimes N}$ on $\mathcal{M}^{\otimes N}$ given by

$$\pi_{\boldsymbol{\lambda}} = \pi_{\lambda_1} \otimes \cdots \otimes \pi_{\lambda_N},$$

where $\pi_{\lambda_j} = \pi_{\lambda_j,0}$. We use the following notation for iterated coproducts: we define Δ^0 to be the identity on $\mathcal{U}_q$, and for $n \geq 1$ we define $\Delta^n : \mathcal{U}_q \to \mathcal{U}_q^{\otimes(n+1)}$ by

$$\Delta^n = (\Delta \otimes 1^{\otimes(n-1)}) \circ \Delta^{n-1},$$

with the convention $A \otimes B^0 = A$.

We consider the following coproducts of the twisted-primitive elements $Y_{s,u}$ and $\widetilde{Y}_{t,v}$, $s, u, t, v \in \mathbb{C}^\times$: for $j = 1, \ldots, N$,

$$\begin{aligned}\mathsf{Y}^{(j)}_{s,u} &= 1^{\otimes(N-j)} \otimes \Delta^{j-1}(Y_{s,u}),\\ \widetilde{\mathsf{Y}}^{(j)}_{t,v} &= \Delta^{j-1}(\widetilde{Y}_{t,v}) \otimes 1^{\otimes(N-j)}.\end{aligned}$$

These elements commute, see [15, Lemma 5.1]: for $j, j' = 1, \ldots, N$,

$$\mathsf{Y}^{(j)}_{s,u}\mathsf{Y}^{(j')}_{s,u} = \mathsf{Y}^{(j')}_{s,u}\mathsf{Y}^{(j)}_{s,u}, \qquad \widetilde{\mathsf{Y}}^{(j)}_{t,v}\widetilde{\mathsf{Y}}^{(j')}_{t,v} = \widetilde{\mathsf{Y}}^{(j')}_{t,v}\widetilde{\mathsf{Y}}^{(j)}_{t,v}.$$

4.1 Eigenfunctions

Using the eigenfunction f_x of $\pi_{\lambda,0}(Y_{s,u})$ from Lemma 3.1 we obtain simultaneous eigenfunctions of $\pi_{\boldsymbol{\lambda}}(\mathsf{Y}^{(j)}_{s,u})$, $j = 1, \ldots, N$. We use the notation $f_x = f_{x,s,u,\lambda}$ to stress dependence on the parameters s, u and λ.

Lemma 4.1 *(i) For* $\boldsymbol{x}, z \in (\mathbb{C}^\times)^N$ *let* $f_{\boldsymbol{x}} = f_{\boldsymbol{x},\boldsymbol{\lambda},s,u} \in \mathcal{M}^{\otimes N}$ *be given by*

$$f_{\boldsymbol{x}}(z) = \prod_{j=1}^{N} f_{x_j,x_{j+1},u,\lambda_j}(z_j),$$

with $x_{N+1} = s$*, then*

$$\pi_{\boldsymbol{\lambda}}(\mathsf{Y}^{(j)}_{s,u}) f_{\boldsymbol{x}} = (\mu_{x_{N-j+1}} - \mu_s) f_{\boldsymbol{x}}, \qquad j = 1, \ldots, N.$$

(ii) For $\boldsymbol{y}, \boldsymbol{z} \in (\mathbb{C}^\times)^N$ *let* $f^\vartheta_{\boldsymbol{y}} = f^\vartheta_{\boldsymbol{y},\boldsymbol{\lambda},t,v} \in \mathcal{M}^{\otimes N}$ *be given by*

$$f^\vartheta_{\boldsymbol{y}}(z) = \prod_{j=1}^{N} f_{y_j,y_{j-1},v,\lambda_j}(1/z_j),$$

with $y_0 = t$*, then*

$$\pi_{\boldsymbol{\lambda}}(\widetilde{\mathsf{Y}}^{(j)}_{t,v}) f^\vartheta_{\boldsymbol{y}} = (\mu_{y_j} - \mu_t) f^\vartheta_{\boldsymbol{y}}, \qquad j = 1, \ldots, N.$$

Proof The proof for (ii) follows by induction using the identities

$$\Delta^j(\widetilde{Y}_{t,v}) = \Delta^{j-1}(1) \otimes \widetilde{Y}_{t,v} + \Delta^{j-1}(\widetilde{Y}_{t,v}) \otimes K^{-2},$$
$$\widetilde{Y}_{t,v} + (\mu_y - \mu_t)K^{-2} = \widetilde{Y}_{y,v} + (\mu_y - \mu_t)1.$$

See also the proof of [15, Proposition 5.5] for details. The proof of (i) follows after applying the involution ϑ and using Lemma 2.1. □

Remark 4.2 The function $f_{\boldsymbol{x}}$ is an 'eigenfunction' of the following $\boldsymbol{x}$-dependent operators: For $i = 0, \ldots, N-1$ and $j = 1, \ldots, N-i$ we define

$$\mathsf{Y}^{(i,j)}_{\boldsymbol{x},u} = 1^{\otimes(N-i-j)} \otimes \Delta^{j-1}(Y_{x_{N-i+1},u}) \otimes 1^{\otimes i},$$

then $\mathsf{Y}^{(i,j)}_{\boldsymbol{x},u}\mathsf{Y}^{(i,j')}_{\boldsymbol{x},u} = \mathsf{Y}^{(i,j')}_{\boldsymbol{x},u}\mathsf{Y}^{(i,j)}_{\boldsymbol{x},u}$ for $j, j' = 1, \ldots, N-i$, and

$$\pi_{\boldsymbol{\lambda}}(\mathsf{Y}^{(i,j)}_{\boldsymbol{x},u}) f_{\boldsymbol{x}} = (\mu_{x_{N-j-i+1}} - \mu_{x_{N-i+1}}) f_{\boldsymbol{x}}.$$

Note that the $\boldsymbol{x}$-dependent element $\mathsf{Y}^{(i,j)}_{\boldsymbol{x},u}$ depends only on x_{N-i+1}. In particular $\mathsf{Y}^{(0,j)}_{\boldsymbol{x},u} = \mathsf{Y}^{(j)}_{s,u}$ depends on $x_{N+1} = s$, i.e. it is independent of $\boldsymbol{x}$.

We define for $j = 1, \ldots, N$,

$$\mathsf{K}^{-2,(j)} = (K^{-2})^{\otimes j} \otimes 1^{\otimes(N-j)} \in \mathcal{U}_q^{\otimes N}.$$

We can express the action of $\mathsf{K}^{-2,(j)}$ on $f_{\boldsymbol{x}}$ as a q-difference operator in $\boldsymbol{x}$. We first need notation for q-difference operators. For $i = 1, \ldots, N$ we define

$$[T_i f](\boldsymbol{x}) = f(x_1, \ldots, x_{i-1}, x_i q^2, x_{i+1}, \ldots, x_N),$$

and for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_j) \in \{-1, 0, 1\}^j$ we write

$$T_{\boldsymbol{\nu}} = T_1^{\nu_1} \cdots T_j^{\nu_j}.$$

We will use the q-difference equations from Corollaries 3.3 and 3.4 (with $\varepsilon = 0$). To stress dependence on the parameters s and λ, let us write $A(x) = A_{s,\lambda}(x)$ and $B(x) = B_{s,\lambda}(x)$ for the coefficients of the difference equation in Corollary 3.3, and similarly $A^\pm_{s,\lambda}(x)$ and $B^\pm_{s,\lambda}(x)$ for the coefficients in Corollary 3.4. Now for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_j) \in \{-1, 0, 1\}^j$ we define

$$A_{\boldsymbol{\nu},i}(x_i, x_{i+1}) = \begin{cases} A^-_{x_{i+1},\lambda_i}(x_i), & \nu_{i+1} = -1, \\ A_{x_{i+1},\lambda_i}(x_i), & \nu_{i+1} = 0, \\ A^+_{x_{i+1},\lambda_i}(x_i), & \nu_{i+1} = 1, \end{cases}$$

$$B_{\boldsymbol{\nu},i}(x_i,x_{i+1})=\begin{cases}B^-_{x_{i+1},\lambda_i}(x_i), & \nu_{i+1}=-1,\\ B_{x_{i+1},\lambda_i}(x_i), & \nu_{i+1}=0,\\ B^+_{x_{i+1},\lambda_i}(x_i), & \nu_{i+1}=1.\end{cases}$$

Here we set $\nu_{j+1}=0$ for $\boldsymbol{\nu}\in\{-1,0,1\}^j$.

Proposition 4.3 *For* $j=1,\dots,N$,

$$\pi(\mathsf{K}^{-2,(j)})f_{\boldsymbol{x}}=\sum_{\boldsymbol{\nu}\in\{-1,0,1\}^j}C^{(j)}_{\boldsymbol{\nu}}(\boldsymbol{x})T_{\boldsymbol{\nu}}f_{\boldsymbol{x}},$$

where

$$C^{(j)}_{\boldsymbol{\nu}}(\boldsymbol{x})=\prod_{i=1}^{j}C^{(j)}_{\boldsymbol{\nu},i}(\boldsymbol{x})$$

with

$$C^{(j)}_{\boldsymbol{\nu},i}(\boldsymbol{x})=\begin{cases}A_{\boldsymbol{\nu},i}(x_i^{\nu_i},x_{i+1}), & \nu_i\neq 0,\\ B_{\boldsymbol{\nu},i}(x_i,x_{i+1}), & \nu_i=0.\end{cases}$$

Proof Using Corollary 3.3 to act with $\pi(K^{-2})$ in the j-th factor of $f_{\boldsymbol{x}}$ we see that $\pi_{\boldsymbol{\lambda}}(\mathsf{K}^{-2,(j)})f_{\boldsymbol{x}}$ is equal to

$$\left[\bigotimes_{i=1}^{j-1}\pi_{\lambda_i}(K^{-2})\otimes\Big(A_{x_{j+1},\lambda_j}(x_j)T_j+B_{x_{j+1},\lambda_j}(x_j)\mathrm{Id}+A_{x_{j+1},\lambda_j}(x_j^{-1})T_j^{-1}\Big)\right]f_{\boldsymbol{x}}$$
$$=\left[\bigotimes_{i=1}^{j-1}\pi_{\lambda_i}(K^{-2})\otimes\sum_{\nu_j\in\{-1,0,1\}}C^{(j)}_{\boldsymbol{\nu},j}T_j^{\nu_j}\right]f_{\boldsymbol{x}}.$$

Next act with $\pi(K^{-2})$ in the $(j-1)$-th factor of f_x in each term of this sum as follows: apply the $+$-version of Corollary 3.4 if T_j is applied in the jth-factor of $f_{\boldsymbol{x}}$ and apply the $-$-version if T_j^{-1} is applied in the j-th factor; otherwise apply Corollary 3.3. Then we see that $\pi(\mathsf{K}^{-2,(j)})f_{\boldsymbol{x}}$ is equal to

$$\left[\bigotimes_{i=1}^{j-2}\pi_{\lambda_i}(K^{-2})\otimes\sum_{\nu_{j-1},\nu_j\in\{-1,0,1\}}C^{(j)}_{\boldsymbol{\nu},j-1}(\boldsymbol{x})C^{(j)}_{\boldsymbol{\nu},j}(\boldsymbol{x})T_{j-1}^{\nu_{j-1}}T_j^{\nu_j}\right]f_{\boldsymbol{x}}.$$

Continuing in this way gives the result. □

Next we determine how $\pi(\widetilde{\mathsf{Y}}^{(j)}_{t,v})$ acts on $f_{\boldsymbol{x}}$ as a q-difference operator in $\boldsymbol{x}$. We need the elements $\mathsf{Y}^{(N-j,j)}_{\boldsymbol{x},u}$, $j=1,\dots,N$, from Remark 4.2. Since Δ is an algebra homomorphism it follows from the definitions of $\mathsf{Y}^{(N-j,j)}_{\boldsymbol{x},u}$, $\widetilde{\mathsf{Y}}^{(j)}_{t,v}$, $\mathsf{K}^{-2,(j)}$ and (3.2) that for $t,v\in\mathbb{C}^\times$

$$\begin{aligned}\widetilde{\mathsf{Y}}^{(j)}_{t,v^{-1}}&=\frac{qu/v-v/uq}{q^{-2}-q^2}\mathsf{K}^{-2,(j)}\mathsf{Y}^{(N-j,j)}_{\boldsymbol{x},u}+\frac{vq/u-u/vq}{q^{-2}-q^2}\mathsf{Y}^{(N-j,j)}_{\boldsymbol{x},u}\mathsf{K}^{-2,(j)}\\&\quad+\frac{(q^{-1}+q)(t+t^{-1})-(v/u+u/v)(x_{j+1}+x_{j+1}^{-1})}{q^{-2}-q^2}(\mathsf{K}^{-2,(j)}-1),\end{aligned}\tag{4.1}$$

for arbitrary $\boldsymbol{x}\in(\mathbb{C}^\times)^N$ and $u\in\mathbb{C}^\times$.

Proposition 4.4 *For* $j=1,\dots,N$,

$$\pi_{\boldsymbol{\lambda}}(\widetilde{\mathsf{Y}}^{(j)}_{t,v^{-1}})f_{\boldsymbol{x}}=\sum_{\boldsymbol{\nu}\in\{-1,0,1\}^j}C^{(j)}_{\boldsymbol{\nu}}(\boldsymbol{x})T_{\boldsymbol{\nu}}f_{\boldsymbol{x}}+\left(\frac{(u/v+v/u)(x_{j+1}+x_{j+1}^{-1})}{q^{-2}-q^2}-\mu_t\right)f_{\boldsymbol{x}},$$

where

$$\mathcal{C}_{\boldsymbol{\nu}}^{(j)}(\boldsymbol{x}) = C_{\boldsymbol{\nu}}^{(j)}(\boldsymbol{x}) \times \begin{cases} -\dfrac{ux_1^{-\nu_1}}{vq}\dfrac{(1-vqx_1^{\nu_1}/ut)(1-vqtx_1^{\nu_1}/u)}{q^{-1}-q}, & \nu_1 \neq 0, \\ \dfrac{(u/v+v/u)(x_1+x_1^{-1})+(q^{-1}+q)(t+t^{-1})}{q^{-2}-q^2}, & \nu_1 = 0, \end{cases}$$

with $C_{\boldsymbol{\nu}}^{(j)}$ *from Proposition 4.3.*

Proof Note that $\pi_{\boldsymbol{\lambda}}(\mathrm{Y}_{\boldsymbol{x},u}^{(N-j,j)})$ acts on $f_{\boldsymbol{x}}$ as multiplication by $\mu_{x_1} - \mu_{x_{j+1}}$, and $\pi_{\boldsymbol{\lambda}}(K^{-2,(j)})$ acts as a q-difference operator in $x_1, \ldots, x_j$ on $f_{\boldsymbol{x}}$. From (4.1), Proposition 4.3 and Remark 4.2, we see that $\pi(\widetilde{\mathrm{Y}}_{t,v^{-1}}^{(j)}) f_{\boldsymbol{x}}$ is a q-difference operator as stated in the proposition with coefficients

$$\mathcal{C}_{\boldsymbol{\nu}}^{(j)}(\boldsymbol{x}) = \frac{C_{\boldsymbol{\nu}}^{(j)}(\boldsymbol{x})}{q^{-2}-q^2}\Big((qu/v - v/uq)(\mu_{x_1} - \mu_{x_{j+1}}) + (vq/u - u/vq)(\mu_{q^{2\nu_1}x_1} - \mu_{x_{j+1}}) + (q^{-1}+q)(t+t^{-1}) - (v/u+u/v)(x_{j+1}+x_{j+1}^{-1})\Big).$$

Simplifying gives the result. □

4.2 Overlap coefficients: multivariate Askey–Wilson functions

We are now ready to define the overlap coefficient $\Phi(x, y)$ similar as in Definition 3.6. We define an $(N+4)$-tuple $\boldsymbol{\alpha}$ by

$$\boldsymbol{\alpha} = (s, u, t, v, \lambda_1, \ldots, \lambda_N).$$

We define a pairing depending on $\boldsymbol{\alpha}$ and $\boldsymbol{x}, \boldsymbol{y} \in (\mathbb{C}^\times)^N$ by

$$\langle f, g \rangle = \frac{1}{(2\pi i)^N} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_N} f(z) g^\star(z) \frac{dz_N}{z_N} \cdots \frac{dz_1}{z_1},$$

where $g^\star(z) = \overline{g(\bar{z}_1^{-1}, \ldots, \bar{z}_N^{-1})}$, and $\mathcal{C}_j$ is a deformation of the positively oriented unit circle such that the sequences $u^{-1}x_{j+1}q^{1-i\lambda_j}q^{\mathbb{Z}_{\geq 0}}$, $v^{-1}y_{j-1}q^{1+i\lambda_j}q^{\mathbb{Z}_{\geq 0}}$ are inside $q^2\mathcal{C}_j$, and the sequences $u^{-1}q^{i\lambda_j}x_j^{\pm 1}q^{-\mathbb{Z}_{\geq 0}}$, $v^{-1}q^{-i\lambda_j}y_j^{\pm 1}q^{-\mathbb{Z}_{\geq 0}}$ are outside $q^{-2}\mathcal{C}_j$. Here $x_{N+1} = s$ and $y_0 = t$. Assume that $f(z)$ and $g(z)$ are analytic in z_j on $\{z \in q^\theta \mathcal{C}_j \mid -2 \leq \theta \leq 2\}$ for $j = 1, \ldots, N$. Then from applying Cauchy's theorem it follows that this pairing satisfies $\langle \pi_{\boldsymbol{\lambda}}(X_1X_2)f, g\rangle = \langle f, \pi_{\boldsymbol{\lambda}}(X_2^*X_1^*)g\rangle$ for $X_1, X_2 \in (\mathcal{U}_q^1)^{\otimes N}$.

Definition 4.5 For $\boldsymbol{x}, \boldsymbol{y} \in (\mathbb{C}^\times)^N$ we define

$$\Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = \langle f_{\boldsymbol{x}}, \bar{f}_{\bar{\boldsymbol{y}}}^{\vartheta} \rangle.$$

It immediately follows from (3.6) that $\Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y})$ can be written as a product of the overlap coefficients $\Phi(x_j, y_j)$ which are essentially Askey–Wilson functions, so $\Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y})$ can be considered as a multivariate Askey–Wilson function. For this multivariate function we have a symmetry property and difference equations similar to Proposition 3.7 and Theorem 3.9.

Theorem 4.6 *The overlap coefficient* $\Phi(\boldsymbol{x}, \boldsymbol{y})$ *satisfies*

(i) $\Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{j=1}^{N} \Phi_{x_{j+1},u,y_{j-1},v,\lambda,0}(x_j, y_j)$;

(ii) $\overline{\Phi_{\bar{\boldsymbol{\alpha}}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})} = \Phi_{\boldsymbol{\alpha}^\vartheta}(\hat{\boldsymbol{y}}, \hat{\boldsymbol{x}})$, *with*

$$\hat{\boldsymbol{x}} = (x_N, \ldots, x_1), \quad \hat{\boldsymbol{y}} = (y_N, \ldots, y_1), \quad \boldsymbol{\alpha}^\vartheta = (t, v, s, u, \lambda_N, \ldots, \lambda_1);$$

(iii) For $j = 1, \dots, N$,

$$\mu_{y_j} \Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{\nu} \in \{-1,0,1\}^j} \mathcal{C}_{\boldsymbol{\nu}}^{(j)}(\boldsymbol{x}) [T_{\boldsymbol{\nu}} \Phi_{\boldsymbol{\alpha}}(\,\cdot\,, \boldsymbol{y})](\boldsymbol{x}) + \frac{(u/v + v/u)(x_{j+1} + x_{j+1}^{-1})}{q^{-2} - q^2} \Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}), \quad (4.2)$$

n with $\mathcal{C}_{\boldsymbol{\nu}}^{(j)}$ *from Proposition 4.4.*

Proof Identity (ii) follows from the first identity and Proposition 3.7, or directly from writing $\Phi_{\boldsymbol{\alpha}}$ explicitly as an integral. For identity (iii) we assume that the $\boldsymbol{\alpha}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ are chosen such that $f_{\boldsymbol{x}}(z)$ and $f_{\boldsymbol{y}}^{\vartheta}(z)$ are analytic in z_j on $\{z \in q^{\theta} \mathcal{C}_j \mid -2 \leq \theta \leq 2\}$ for $j = 1, \dots, N$. Then the q-difference equations follow from

$$\langle \pi_{\boldsymbol{\lambda}}(\widetilde{\mathsf{Y}}_{t,v}^{(j)}) f_{\boldsymbol{x}}, \bar{f}_{\bar{\boldsymbol{y}}}^{\vartheta} \rangle = \langle f_{\boldsymbol{x}}, \pi_{\boldsymbol{\lambda}}(\widetilde{\mathsf{Y}}_{\bar{t},\bar{v}^{-1}}^{(j)}) \bar{f}_{\bar{\boldsymbol{y}}}^{\vartheta} \rangle,$$

using Lemma 4.1 and Proposition 4.4. The conditions on $\boldsymbol{\alpha}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ can be removed again by continuity. □

Clearly, combining identities (ii) and (iii) from Theorem 4.6 gives a difference equation in $\boldsymbol{y}$ for $\Phi(\boldsymbol{x}, \boldsymbol{y})$.

Remark 4.7 The Askey–Wilson algebra encodes the bispectral properties of the Askey–Wilson polynomials. The elements $Y_{t,v}$ and $Y_{s,u}$, together with the Casimir element Ω, generate a copy of the Askey–Wilson algebra in $\mathcal{U}_q$, see [21]. Similarly, for $N = 2$ the elements $\widetilde{\mathsf{Y}}_{t,v}^{(j)}$, $\mathsf{Y}_{s,u}^{(j)}$, $j = 1, 2$, and $\Delta(\Omega)$, generate a copy of a rank 2 Askey–Wilson algebra [22] in $\mathcal{U}_q^{\otimes 2}$. It seems likely that $\widetilde{\mathsf{Y}}_{t,v}^{(j)}$, $\mathsf{Y}_{s,u}^{(j)}$, $j = 1, \dots, N$, together with appropriate coproducts of Ω generate a copy of the rank N Askey–Wilson algebra [23].

To end the section, let us summarize the results we have obtained in terms of multivariate Askey–Wilson functions. We set

$$x_{N+1} = s, \quad y_0 = t, \quad \alpha_0 = v/u, \quad \alpha_j = q^{2i\lambda_j} \text{ for } j = 1, \dots, N,$$

and write $\boldsymbol{\alpha} = (y_0, \alpha_0, \alpha_1, \dots, \alpha_N, x_{N+1})$. The multivariate Askey–Wilson functions are given by

$$\Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = \Theta_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) \prod_{j=1}^{N} \psi_{y_j}(x_j; q x_{j+1} \alpha_j, q x_{j+1}/\alpha_j, q \alpha_0 y_{j-1}, q \alpha_0 / y_{j-1} | q^2),$$

with

$$\Theta_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{j=1}^{N} \frac{(q \alpha_j y_{j-1} y_j^{\pm 1}; q^2)_\infty \theta(q/\alpha_j, q \alpha_0 x_{j+1}/y_{j-1}; q^2)}{(q^2, q x_{j+1} y_j^{\pm 1}/\alpha_0; q^2)_\infty}.$$

Symmetry property: Φ satisfies

$$\overline{\Phi_{\bar{\boldsymbol{\alpha}}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})} = \Phi_{\hat{\boldsymbol{\alpha}}}(\hat{\boldsymbol{y}}, \hat{\boldsymbol{x}}),$$

where $\hat{\boldsymbol{\alpha}} = (x_{N+1}, \alpha_0^{-1}, \alpha_N, \dots, \alpha_1, y_0)$.

q-Difference equations: Φ satisfies

$$\frac{y_j + y_j^{-1}}{q^{-1} - q} \Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{\nu} \in \{-1,0,1\}^j} \mathcal{C}_{\boldsymbol{\nu}}^{(j)}(\boldsymbol{x}) [T_{\boldsymbol{\nu}} \Phi_{\boldsymbol{\alpha}}(\,\cdot\,, \boldsymbol{y})](\boldsymbol{x}) + \frac{(\alpha_0 + \alpha_0^{-1})(x_{j+1} + x_{j+1}^{-1})}{q^{-2} - q^2} \Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}),$$

where the coefficients $\mathcal{C}_{\boldsymbol{\nu}}^{(j)}(\boldsymbol{x})$ are given explicitly by

$$\mathcal{C}_{\boldsymbol{\nu}}^{(j)}(\boldsymbol{x}) = \prod_{i=1}^{j} C_{\boldsymbol{\nu},i}^{(j)}(\boldsymbol{x}) \times \begin{cases} -\dfrac{x_1^{-\nu_1}}{\alpha_0 q} \dfrac{(1 - q \alpha_0 x_1^{\nu_1} y_0^{\pm 1})}{q^{-1} - q}, & \nu_1 \neq 0, \\[2ex] \dfrac{(\alpha_0 + \alpha_0^{-1})(x_1 + x_1^{-1}) + (q^{-1} + q)(t + t^{-1})}{q^{-2} - q^2}, & \nu_1 = 0, \end{cases}$$

with

$$C^{(j)}_{\boldsymbol{\nu},i}(\boldsymbol{x}) = \begin{cases} \dfrac{x_i^{2\nu_i}}{x_{i+1}^2(1-x_i^{2\nu_i})(1-q^2x_i^{2\nu_i})}, & \nu_i \neq 0,\ \nu_{i+1}=-1, \\ -\dfrac{x_i^{\nu_i}(1-qx_i^{\nu_i}x_{i+1}\alpha_i^{\pm 1})}{x_{i+1}(1-x_i^{2\nu_i})(1-q^2x_i^{2\nu_i})}, & \nu_i \neq 0,\ \nu_{i+1}=0, \\ \dfrac{(1-qx_i^{\nu_i}x_{i+1}\alpha_i^{\pm 1})(1-q^3x_i^{\nu_i}x_{i+1}\alpha_i^{\pm 1})}{(1-x_i^{2\nu_i})(1-q^2x_i^{2\nu_i})}, & \nu_i \neq 0,\ \nu_{i+1}=1, \\ \dfrac{1}{x_{i+1}^2(1-x_i^{\pm 2}/q^2)}, & \nu_i = 0,\ \nu_{i+1}=-1, \\ \dfrac{x_i(1-qx_ix_{i+1}\alpha_i^{\pm 1})}{x_{i+1}(1-x_i^2)(1-q^2x_i^2)} + \dfrac{(1-qx_i^{-1}x_{i+1}\alpha_i^{\pm 1})}{x_ix_{i+1}(1-x_i^{-2})(1-q^2x_i^{-2})}, & \nu_i = 0,\ \nu_{i+1}=0, \\ \dfrac{(1+q^{-2})(1-q\alpha_i^{\pm 1}x_{i+1}x_i^{\pm 1})}{(1-x_i^{\pm 2}/q^2)}, & \nu_i = 0,\ \nu_{i+1}=1. \end{cases}$$

Recall here that we use the convention $\nu_{j+1} = 0$ for $\boldsymbol{\nu} \in \{-1, 0, 1\}^j$.

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5 Appendix

We prove (3.2):

$$\widetilde{Y}_{t,v^{-1}} = \frac{qu/v - v/uq}{q^{-2}-q^2}K^{-2}Y_{s,u} + \frac{vq/u - u/vq}{q^{-2}-q^2}Y_{s,u}K^{-2} + \frac{(q^{-1}+q)(t+t^{-1}) - (v/u+u/v)(s+s^{-1})}{q^{-2}-q^2}(K^{-2}-1).$$

The proof runs along the same lines as in [15, Lemma 4.3]. We define $S, T \in \mathcal{U}_q$ by

$$S = K^{-2}Y_{s,u} + \mu_s(K^{-2}-1), \qquad T = \frac{K^{-2}Y_{s,u} - Y_{s,u}K^{-2}}{q^{-1}-q},$$

then by the definition of $Y_{s,u}$ (2.2) it follows that

$$S = uq^{-\frac{3}{2}}EK^{-1} - u^{-1}q^{\frac{3}{2}}FK^{-1}, \qquad T = uq^{-\frac{1}{2}}EK^{-1} + q^{\frac{1}{2}}u^{-1}FK^{-1}.$$

Using the definition (2.2) of $\widetilde{Y}_{t,v^{-1}}$ we obtain

$$\widetilde{Y}_{t,v^{-1}} = \frac{u}{v}\frac{q^{-1}T - S}{q+q^{-1}} - \frac{v}{u}\frac{S+qT}{q+q^{-1}} + \mu_t(K^{-2}-1),$$

then expressing S and T in terms of $Y_{s,u}$ and K^{-2} gives the desired expression.

References

1. Geronimo, J.S.; Iliev, P.: Multivariable Askey-Wilson function and bispectrality. Ramanujan J. **24**(3), 273–287 (2011)
2. Askey, R., Wilson, J.: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Am. Math. Soc. **54**(319) (1985)
3. Ismail, M.E.H.; Rahman, M.: The associated Askey-Wilson polynomials. Trans. Am. Math. Soc. **328**(1), 201–237 (1991)
4. Koelink, E.; Stokman, J.V.: The Askey-Wilson function transform. Int. Math. Res. Not. **22**, 1203–1227 (2001)
5. Suslov, S.K.: Some orthogonal very well poised ${}_8\phi_7$-functions. J. Phys. A **30**(16), 5877–5885 (1997)
6. Suslov, S.K.: Some orthogonal very-well-poised ${}_8\phi_7$-functions that generalize Askey-Wilson polynomials. Ramanujan J. **5**(2), 183–218 (2001)
7. Koelink, E.; Stokman, J.V.: Fourier transforms on the quantum $SU(1,1)$ group, with an appendix by Mizan Rahman. Publ. Res. Inst. Math. Sci. **37**(4), 621–715 (2001)
8. Stokman, J.V.: Difference Fourier transforms for nonreduced root systems. Selecta Math. (N.S.) **9**(3), 409–494 (2003)
9. Groenevelt, W.: Wilson function transforms related to Racah coefficients. Acta Appl. Math. **91**(2), 133–191 (2006)
10. Gasper, G., Rahman, M.: Some systems of multivariable orthogonal Askey-Wilson polynomials, Theory and applications of special functions, 209–219, Dev. Math., vol. 13, Springer, New York, (2005).
11. Tratnik, M.V.: Multivariable Wilson polynomials. J. Math. Phys. **30**(9), 2001–2011 (1989)
12. Iliev, P.: Bispectral commuting difference operators for multivariable Askey-Wilson polynomials. Trans. Am. Math. Soc. **363**(3), 1577–1598 (2011)
13. de Bie, H.; de Clercq, H.: The q-Bannai-Ito algebra and multivariate $(-q)$-Racah and Bannai-Ito polynomials. J. Lond. Math. Soc. (2) **103**(1), 71–126 (2021)
14. Genest, V.X.; Iliev, P.; Vinet, L.: Coupling coefficients of $su_q(1,1)$ and multivariate q-Racah polynomials. Nuclear Phys. B **927**, 97–123 (2018)
15. Groenevelt, W.: A quantum algebra approach to multivariate Askey-Wilson polynomials. Int. Math. Res. Not. IMRN **5**, 3224–3266 (2021)
16. Stokman, J.V.: Askey-Wilson functions and quantum groups, Theory and applications of special functions, 411–442, Dev. Math., vol. 13 Springer, New York, (2005)
17. Gasper, G.; Rahman, M.: Basic Hypergeometric Series. Encyclopedia of Mathematics and its Applications, vol. 96, 2nd edn Cambridge University Press, Cambridge (2004)
18. van de Bult, F.J.; Rains, E.M.; Stokman, J.V.: Properties of generalized univariate hypergeometric functions. Commun. Math. Phys. **275**(1), 37–95 (2007)
19. van de Bult, F.J.; Rains, E.M.: Basic hypergeometric functions as limits of elliptic hypergeometric functions. SIGMA Symmetry Integr. Geom. Methods Appl. **5**, 059 (2009)
20. Koelink, E.; Rosengren, H.: Transmutation kernels for the little q-Jacobi function transform, Conference on Special Functions (Tempe, AZ, 2000). Rocky Mt. J. Math. **32**(2), 703–738 (2002)
21. Granovskiĭ, Ya..I..; Zhedanov, A.S.: Linear covariance algebra for $SL_q(2)$. J. Phys. A **26**(7), L357–L359 (1993)
22. Groenevelt, W.; Wagenaar, C.: An Askey-Wilson algebra of rank 2. SIGMA Symmetry Integr. Geom. Methods Appl. **19**, 008 (2023)
23. Crampé, N.; Frappat, L.; Poulain d'Andecy, L.; Ragoucy, E.: The higher-rank Askey-Wilson algebra and its braid group automorphisms. SIGMA Symmetry Integr. Geom. Methods Appl. **19**, 077 (2023)