ADMINISTRAÇÃO - GERAL DO PORTO DE LISBOA

Hydro-morphological study Tagus Estuary

Part 1
Wave penetration model

Volume 1.2
Annex A
Mathematical model

June 1982 / P473

PORT AND WATERWAY ENGINEERS

hydronamic bv

MATHEMATICAL BACKGROUND OF THE HYDRONAMIC WAVE PENETRATION MODEL

Contents | Page

MATHEMATICAL BACKGROUND OF THE HYDRONAMIC WAVE PENETRATION MODEL

Introduction

The transformation of waves coming from the deep sea and approaching the shore depends on the local bathymetry, the type of bottom and the type of waves. The Hydronamic Wave Penetration Model consists of a number of programs in order to calculate these transformations. The main program of the model is the program REFDIF. The input for this program is the bottom topography of the nearshore area and the wave characteristics of one component of the wave climate. The bottom-topography is entered as a depth-matrix. For each component of the wave climate the program is runned and calculates along wave-rays the various wave-height determining parameters.
These are parameters for refraction, shoaling, friction, percolation and breaking. The results of these calculations are printed out and stored on computer disc for further processing.
Seperate programs can plot these data as ray-diagrams.
In order to tackle the caustic problem a seperate program will be runned to process the output-data according to the method of Bouws
A contour plotting program then transforms the results of the Bouws-program to charts which can be interpreted visually.
In the next chapters of this reports the mathematical background of these programs is discussed in more detail.

Refraction and shoaling

The phase-speed (celerity) of a small amplitude wave can be expressed by:

$$c^2 = \frac{g}{k} \tanh (kh) \tag{1}$$

This is the so-called dispersion formula for linear waves, in which second order effects are neglected. k is the wave number $(2\pi/L)$ h is the local waterdepth and g the acceleration of gravity. Consequently, the celerity has a varation with depth and the situation is a natural one for the application of Snell's law, which relates the bending of rays to the speed change:

$$\frac{\sin \phi}{\sin \phi_o} = \frac{c}{c_o} \tag{2}$$

in which ϕ is the angle between the wave-crest and the depth contour. The index o indicates deep-water conditions.
The energy of a linear wave per unit of crest width is given by

$$E = \frac{1}{8} \rho g H^2 \tag{3}$$

and is transported to the coast with the group-celerity:

$$c_g = \frac{1}{2} \left(1 + \frac{2\ kh}{\sinh 2kh}\right) c = nc \tag{4}$$

Consequently the rate at which energy is transported to the coast is:

$$P = \frac{1}{8} \rho g H^2 c_g \tag{5}$$

If a section of crest of width s between two rays of orthogonals is considered, the average rate of energy-transmission between the orthogonals is:

$$Ps = \frac{1}{8} \rho\, g H^2 \tag{6}$$

It is assumed in classic refraction theory that the energy contained between a pair of orthogonals in a long-crested system will remain between these orthogonals. On this assumption it is possible to follow orthogonals and trace the changes in wave-height.
If the orthogonals spread, the wave height must become less, since the same energy is spread over a larger area.
If they contract, the heights grow as more energy is concentrated in less space. According to this assumption crossing of wave-rays is not possible. However, when executing the actual computation, wave-rays will cross frequently. The problem to determine the wave-height in such crossings can be solved by applying the Bouws-method, which will be discussed later. At any rate, if the assumption is reasonable good, then:

$$P_s = P_{s_o} \tag{7}$$

$$\text{or: } \frac{1}{8}\rho g H^2 C_g s = \frac{1}{8}\rho g H_o^2 C_{go} S_o \tag{8}$$

which means that the wave-height in shoal water is the height in deep water, multiplied by two factors:

$$H = H_o \left(\frac{C_{go}}{C_g}\right)^{\frac{1}{2}} \left(\frac{S_o}{S}\right)^{\frac{1}{2}} \tag{9}$$

The factor $(S_o/s)^{\frac{1}{2}} = K_r$ is called the refraction coefficient; the other factor $(C_{g_o}/C_g)^{\frac{1}{2}} = K_s$ is called the shoaling coefficient.
Elaborating the formula for the shoaling-coefficient, applying relation (4), one gets:

$$K_s = \left(\frac{k}{k_o}\right)^{\frac{1}{2}} \left(1 + \frac{2kh}{\sinh 2kh}\right)^{-\frac{1}{2}} \tag{10}$$

The shoaling coefficient is only a function of deep water data and the local waterdepth. Unfortunately it is not possible to derivate such a simple formula for the refraction coefficient, because the refraction coefficient depends on the shape of the depth contours, which are crossed.

In general $K_r=(S_o/s)^{\frac{1}{2}}$, in which s is the distance between two adjacent wave-rays. However, because in the computational process, only one ray is computed at a time, it is not possible to determine the distance directly. But a relation between the ray-distance s and the curvature of the wave ray (μ) exists:

$$\mu = \frac{1}{s} \frac{\partial s}{\partial n} \qquad (11)$$

in which is n the distance along the ray. Formula (11) allows sequential calculation of the value of s along the ray, and thus sequential calculation of K_r.

The value of μ can be calculated with the second order derivative of the trajectory, which can be expressed as a product of the celerity and the step-distance. The second order differential equations are solved by numercial techniques.
For a more detailed derivation of the above mentioned formules can be referred to Kinsman (1965).

Friction and percolation

The dissipation of energy by bottom friction and/or percolation can bring about significant loss of wave energy with a possible reduction of wave-height, particularly for high waves of long period which are propagated into a shallow region of very gentle bottom slope. The rapid attenuation of energy by bottom friction for waves of long period can be explained qualitatively as due to the fact that the long waves effectively "feel" bottom sooner than the short period waves and consequently are subject to frictional dissipation over a greater distance. In a complex wave group this selctive attenuation could produce, under certain conditions, a shift in the peak of the energy-spectrum towards lower periods as the waves travel towards shore.
The method used in the Hydronamic Wave Penetration Model is the method described by Bretschneider, although some more recent coefficients for friction have been applied (Treloar & Abernethy, 1978).

Putnam & Johnson (1949) have shown for sinusoïdal waves of small steepness that the amount of energy. D_f dissipated per unit area at the bottom per unit time (averaged over a wavelength) is given by:

$$D_f = \frac{4}{3}\pi^2 \frac{\partial f H^3}{T^3 \left(\sinh \frac{2\pi h}{L}\right)^3} \qquad (12)$$

f is a dimensionless parameter representing the friction factor for the bottom.

Putnam (1949) has examined the oscillatory percolation of water through a permeable sea bed, associated with sinusoidal waves of small amplitude.

The amount of energy dissipated in this way is given by:

$$D_p = \frac{\pi g^2}{\nu} \frac{\rho p H^2}{L\ (\cosh \frac{2\pi h}{L})^2} \tag{13}$$

in which p is the permeability of the bottom and ν is the kinematic viscosity

For steady state conditions, the rate at which the total energy-flux is altered per unit distance along one of the wave-rays is:

$$\frac{\partial}{\partial n}\ (Ec_g s) = -(D_f + D_p)s \tag{14}$$

For practical application a coefficient with the same properties as the refraction and shoaling coefficients is required, thus a coefficient K which allows the following equation

$$H = K.K_g.K_r.H_o \tag{15}$$

Equation (15) can be entered in (12) and (13). Entering (12) and (13) in (14) and some mathematics give

$$\frac{dK}{dn} + F_1 K^2 + F_2 K = 0 \tag{16}$$

where

$$F_1 = K_r \frac{f H_o}{T^4} \Phi_f \tag{17}$$

$$\Phi_f = \frac{64\pi^3}{3g^2} \left(\frac{K_3}{\sinh \frac{2h\pi}{L}} \right)^3 \tag{18}$$

$$F_2 = \frac{P}{\nu . T^3} \Phi_p \tag{19}$$

$$\Phi p = \frac{64\pi^3}{g} \left(\frac{K_s^{\ 2}}{\sinh \frac{4\pi h}{L}} \right) \tag{20}$$

Equation (16) is put in a linearform by deviding by K^2. The result is

$$\frac{dK^{-1}}{dn} - F_2 K^{-1} = F_1 \qquad (21)$$

In the special case of no bottom friction $F_1=0$ and the solution for K will be denoted by K_p in this case, is simply

$$K_p = \exp\left(-\int_{n_o}^{n} F_2 dn\right) \qquad (22)$$

where the condition $K_p \approx 1$ at deep water position n_o has been used to evaluate the constant of integration. This solution is an integrating factor for the general equation (16), for if the latter equation is multiplied with K_{ps} it is found upon collecting terms that

$$\frac{\partial}{\partial n}\left(K^{-1} K_p\right) = F_1 K_p \qquad (23)$$

or:

$$K = K_p \left\{1 + \int_{n_o}^{n} F_1 K_p dn\right\}^{-1} \qquad (24)$$

where the conditions K=1 at the deep water position n_o is again implied. Because a seperate expression K_f for the friction-coefficient is required, equation (24) is devided by K_p, and thus:

$$K_f = \left\{1 + \int_{n_o}^{n} F_1 K_p dn\right\}^{-1} \qquad (25)$$

During the computational process the values of F_1 and F_2 can be calculated along the wave ray, and consequently the integrations in eq. (22) and (25) can be carried out along the ray, giving percolation and friction coefficients in any point.

Breaking

At a certain relation between wave-height and water-depth the wave will break. This relation is defined by

$$\gamma = \frac{H}{h} \tag{26}$$

Several investigations do not agree upon the exact relation between waterdepth and breakerheight. For a solitay wave γ=0.78, as can be derived theoretically. A linear small-amplitude wave cannot break theoretically. Mostly a value of approx. 0.8 is assumed for monochromatic waves. For Rayleigh-distributed waves Bijker & Svasek (1969) found a γ of 0.5 for the breakerpoint of the significant wave. Gerritsen (1979) found for the H_{rms} of waves breaking on a reef a value of 0.65.
Recent studies indicate that γ may be a function of the Iribarren-number

$$Ir = \tan\alpha / \sqrt{H/L_o} \tag{27}$$

in which α is the bottom slope. Until final results area available the choosen value of γ can be entered in the program as an input variable

During the breaking process a part of the wave energy is dissipated. The remaining energy will continue its way to the shore. Until today only a few research has been made on this phenomenon. It is known that the period may change somewhat by this process, but no exact quantities are available. In the model is assumed that the period is not changed. The wave-height is decreased until a value of ε times the original wave-height. Thus, if a wave breaks q times the wave-height will be (neglecting refraction, friction etc.)

$$H = \varepsilon^q H_o \tag{28}$$

The problem of caustics

The equations describing refraction of water waves are based on the geometrical-optics approximation, in which waves rays are calculated independently of each other. For this reason the model reacts oversensitively to small variations in depths, as well in incident wave frequency and direction. This leads to an uncertainty in the interpretation of the location of individual rays, particularly in cases of large travel distances through regions with smooth but irregular bottom topography. An alternative formulation is to say that this uncertainty reduces the spatial resolution.
Bouws & Battjes have developed a method to overcome this problem by considering the wave rays as realisations of a partially random process. The wave energy within each square in the studied area is estimated from the propagation time of wave energy along the rays crossing the square. The size of the square determines the spatial resolution.
This viewpoint is worked out by Bouws & Battjes quantitatively by adopting a Monte -Carlo method, in which a number of dense sets of rays is generated, each for a specific initial T-ϕ-combination

On deep water the rays have a uniform spacing s_o. Thus, the total incident wave energy, which is in reality continuous along the incident wave front, is discretized into equal lumps $\delta P_o = E_o S_o$. Each lump is to associated with a ray. In other words, each ray is considered as an energycarrier with a total power δP_o The local energy transport velocity is C_g. The energy density along the ray (energy per unit length of ray) is therefore equal to $\delta P_o / c_g$.
A length of ray between $n=n_1$ and $n=n_2$ then represents an energy equal to:

$$\int_{n_1}^{n_2} \frac{\delta Po}{c_g} \, dn = \delta \, Po \int_{n_1}^{n_2} c_g^{-1} \, dn \qquad (29)$$

The latter integral is the time it takes a particle of energy to travel the given path with speed c_g. Now, the total wave energy in a square is estimated as:

$$E_{sq} = \delta P_o \sum_{i=1}^{m} \int_{n_{i_1}}^{n_{i_2}} c_g^{-1} \, dn \tag{30}$$

The energy in a deep-water square can be calculated in the same way and is called E_{sq_o}. In this way the refraction-coefficient for every square becomes:

$$K_B = \sqrt{\frac{E_{sq}}{E_{sq_o}}} \tag{31}$$

Actual Computation

The area to be studied is devided into a large number of squares. A normal ray-refraction calculation is carried out. Along the wave rays the values of K_p, K_f and ε^q are calculated in discrete points. After termination of the ray calculation for every square the value of E_{sq} is calculated, using the data from the former program and the geometry of the wave rays.
Then, in every square the value of K_B is calculated. With the use of this value the wave height in every square can be determined. Finally the wave-heights are presented as wave-height charts for a given H-T-ϕ combination.

A combination of all the wave-height chart allows to determine the (directional) spectrum in any point of the studied area.

List of Symbols

c	- wave celerity	m/s
cg	- wave group celerity	m/s
D_f	- energy dissipation due to friction	kgm/s
D_p	- energy dissipation due to percolation	kgm/s
E	- wave energy	kgm/s^2
f	- friction factor	-
g	- acceleration of gravity	m/s^2
H	- wave height	m
h	- water-depth	m
k	- wave number ($2\pi/L$)	-
K_r	- refraction coefficient	-
K_s	- shoaling coefficient	-
K_f	- friction coefficient	-
K_p	- percolation coefficient	-
K_B	- refraction/shoaling coeff acc. Bouws	-
L	- wave length	m
n	- ratio group velocity to wave velocity	-
n	- coödinate along a wave ray	m
P	- wave power	kgm/s
p	- permeability	m^2
q	- number of times a wave is broken	-
s	- distance between wave rays	m
T	- wave period	s
x	- bottom slope	-
γ	- breaker-index	-
ε	- energy dissipation factor at breaking	-
μ	- curvature of a ray	rad^{-1}
ν	- kinematic viscosity	m^2/s
ρ	- density of sea water	kg/m^2
ϕ	- direction of waves	rad

References

Kinsman, B.; Wind waves, Prentice-Hall, Inc, New Jersey (1965)

Brettschneider, C.L. & R.O. Reid; Modification of wave-height, due to bottom friction, percolation and refraction, Beach Erosion Board technical memorandum 45, (1954)

Putnam, J.A. & J.W. Johnson; The dissipation of wave energy by bottom friction, Trans. Amer. Geophys. Union, vol 30, (1949)

Putnam, J.A.; Loss of wave energy due to percolation in a permeable sea bottom, Trans. Amer. Geophys. Union vol 30 (1949)

Battjes, J.A; Beschouwingen over de interpretatie van refractieberekeningen volgens Bouws, Delft University of Technology (1979 Unpublished)

Battjes, J.A. & E. Bouws; A Monte-Carlo approach to the computation of refraction of water waves; to be published (1981)

Bijker E.W. & J.N. Svasek; Two methods for determining of morphological changes induced by coastal structures, PIANC-parts, (1969)

Gerritsen ,F; Energy dissipation in breaking waves, POAC-Trondheim(1979)

Treloar, P.D. & C.L. Abernethy; Determination of a bed friction factor for Botany Bay, Australia, Coastal Engineering, Vol 2 (1978)