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Canonical Local Heights on Low Genus Curves

Author:
Quinten DONKER

Supervisors:
Dr. R.S. DE JONG
Dr. B. JANSSENS

Examination Committee:
Prof. Dr. D.C. Gijswijt
Dr. B. Janssens
Dr. R.S. de Jong

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Abstract

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by Quinten DONKER

In this thesis, we construct explicit formulas for canonical local height function on low genus curves. We do this based on an abstract definition of canonical local heights. In the case of elliptic curves over number fields there is a full 'formulaire' by Néron and Tate. We present a new proof of the formula for the canonical local height at a non-archimedean place, if the curve has multiplicative reduction at that place. We do this by evaluating the normalized tropical Riemann theta function on the Jacobian of the reduction graph.

In the case of a Jacobian of a genus 2 curve with semistable reduction, we can again evaluate the normalized tropical Riemann theta function, to obtain formulas for the *Néron correction term*, at non-archimedean places. This yields seven formulas, one for every semistable reduction type of the curve. These formulas are dependent on the *tropical theta characteristic*, which is generally hard to find. Under certain assumptions on the tropical theta characteristic, these formulas resemble formulas for canonical local heights on the Kummer surface. This leads us to explicitly calculate the intersection multiplicity of a point on the Jacobian, with the Theta divisor, by evaluating functions on the Kummer surface. This suggests that our assumptions may be correct, and should bring us closer to understanding the tropical theta characteristic.

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Introduction

Solutions to polynomial equations have been a subject of study for mathematicians since the Ancient Greeks. Even today, the area of *Diophantine geometry*, which studies solutions to these polynomials by means of methods in algebraic geometry, is one of the most rich theories in mathematics. Diophantine geometry also offers a great source of open problems in mathematics.

It turns out that determining the solutions to these *Diophantine* equations is generally very difficult. The solution to *Hilbert's tenth problem* shows there cannot exist a general algorithm that determines whether a polynomial equation with integer coefficients has an integer solution. Even a single, deceptively straightforward equation, can present a monumental mathematical challenge. The most famous example of this is Fermat's Last Theorem. It took centuries before it was finally, and famously, proven by Andrew Wiles in 1995 that the seemingly simple equation

$$x^n + y^n = z^n$$

has no (non-trivial) integer solutions (x, y, z) for any integer $n > 2$.

Luckily, there are also numerous meaningful and important results that we *can* give. A lot of these results are based on measuring the arithmetic complexity of possible solutions to an equation. If we want to determine solutions in a number field K, i.e. a finite extension of the field of rational numbers $\mathbb{Q}$, we can use the concept of heights to study arithmetic complexity. These heights roughly measure the 'size' of a solution. They reflect both the arithmetic of the solution and the geometry of the equation. This leads us to the framework of algebraic geometry. We study varieties V over fields K, which roughly speaking define the set of zeroes of a polynomial equation, and points P on those varieties, corresponding to the solutions to the given equation. To use the theory of heights, we let K be a number field.

Many fundamental theorems in Diophantine geometry are based on the fact that there are only finitely many points of bounded heights. Using this, we can prove e.g. the Mordell-Weil theorem that states that the group of K-rational points on an abelian variety is finitely generated [Weil, 62] and Faltings' theorem that states a projective curve of genus $g \geq 2$ has only finitely many K-rational points [Faltings, 21]. In the case that the set of K-rational points on a variety is not finite, heights can also be used to define a 'counting function' that still offers insight into the arithmetic of the variety.

Heights can be defined in a large number of ways. There is a way of associating a height h_D with every divisor D on a variety, due to André Weil. This height measures the 'size' of a point from the perspective of that divisor. On abelian varieties A these heights have almost-quadratic properties and are only unique only up to a bounded functions. We can make a unique choice among these functions, that does define

a quadratic form, called the *canonical height* $\hat{h}_D$. This canonical height is used for example to construct the *canonical regulator* of an abelian variety. This canonical regulator is a constant in the Birch and Swinnerton-Dyer (BSD) Conjecture, which is one of the unsolved Millennium Prize Problems [13].

It is important to be able to calculate canonical heights explicitly, for example to calculate canonical regulators and gather evidence for the BSD Conjecture. The definition of canonical heights requires evaluating a limit, which can be computationally expensive. An alternative method to calculating canonical heights $\hat{h}_D$, comes from the decomposition of these heights into local factors $\hat{\lambda}_{D,v}$, one for every place v of K. The resulting *canonical local heights* vanish for all but finitely many places. The calculation of the canonical height thus reduces to the calculation of finitely many canonical local heights.

Finding explicit formulas for canonical local heights not only leads to efficient ways of calculating the global height, but also offers fundamental insight into the finer structure of the abelian variety. For example, the *canonical* local height of a point can be formulated in terms of intersection multiplicities on a model of the curve.

The goal of this thesis is to find explicit formulas for these canonical local heights. We restrict our attention to the case that the abelian variety is either an elliptic curve, or the Jacobian of a curve of genus 2.

For elliptic curves, there are well-known explicit formulas, due to Néron [47] and Tate [58]. We present these results, with a new proof for some cases, in Chapter 3. For the case of genus 2 Jacobians, there is no complete 'formulaire', as is the case for elliptic curves. There are some results distributed through literature. For example, Uchida [60] provides formulas at archimedean places and Müller and Stoll [43] give explicit formulas at non-archimedean places for certain semistable reduction types, via calculations on the Kummer surface. We also present these formulas in Chapter 4. Furthermore there are algorithms to approximate the canonical local height via arithmetic intersection theory [42] [61] and via division polynomials [40].

Our approach is the following. In Chapter 1, we introduce the theory of curves and Jacobian varieties. In particular we define the Θ-divisor. We also define the *Néron model* of the Jacobian J of a curve C, and show that is can be calculated from the *reduction graph* Γ_v of the C at a place $v \in M_K$, if C has semistable reduction. In Chapter 2, we give a brief introduction of (global) heights and give a extensive exposition on the theory of local heights. We define (canonical) local heights abstractly, which allows us to give a more general theory than is usual in literature.

Throughout this thesis, we use a result of Néron [47] that states the canonical local height, associated to the Θ-divisor at non-archimedean places $v \in M_K$ can be written as the intersection of Θ and a point on the Néron model plus a correction term γ_v, that factors via the *component group* of the Néron model. In the case of semistable reduction, we can use a result of De Jong and Shokrieh [18] to calculate γ_v as the *tropical Riemann theta function*. This function essentially gives the squared length of a vector in a real torus, whose dimension is bounded by the genus of the curve. This makes explicit calculations for genus 1 and 2 possible. This method is dependent on finding the *tropical theta characteristic*. It is not known in literature how this can be done.

In Chapter 3 the case of elliptic curves, we use the tropical Riemann theta function to

find a new calculation of the canonical local height function in the case of multiplicative reduction. We find that

$$\gamma_v(i \bmod n) = \frac{n}{2} B_2(\frac{i}{n}),$$

where $B_2(T) = T^2 - T + \frac{1}{6}$ is the second Bernoulli polynomial. This agrees with the classic formula as given by Néron and Tate.

In Chapter 4 we can calculate formulas for γ_v on Jacobians of genus 2 curves, in an entirely new way via the tropical Riemann theta function. These results can be found in Section 4.4. In the case of genus 2 we compare these calculations to the results of Müller and Stoll [43], where the calculations are done on the *Kummer surface*, a surface in $\mathbb{P}^3$ related to the Jacobian of a curve. In this case a correction function is calculated as the resistance on the reduction graph. Not only is this correction function heuristically similar to $-\gamma_v$, they actually coincide in four of the seven semistable reduction types, and differ by an integer in the other cases.

As calculations on the Kummer surface are very explicit, this comparison allows us to draw conclusions on the intersection multiplicity on the Néron model of a genus 2 Jacobian. These results are dependent on an assumption (4.5.12) about the tropical theta characteristic. We find under this assumption that the intersection multiplicity

$$i_v(\Theta, \cdot) = \lambda_v + c,$$

see Proposition 4.5.13. Here λ_v is a function that is easily calculated on the Kummer surface, and c is an integer that is relatively easy to find. Under the same assumption, we can connect the inner product on the Jacobian of a reduction graph, with the shortest path between vertices on this graph. We find that

$$s(x, y) = -2\langle w, k \rangle,$$

where s denotes the length of the shortest path between the vertices x and y, and w and k represent a path between x and y and the tropical theta characteristic, respectively. These results suggest that our assumption is correct, and should bring us closer to understanding the tropical theta characteristic.

Chapter 1

Curves and Jacobians

In this chapter we construct the objects of study in this thesis, curves and their Jacobians. In Section 1.1 we classify curves by their genus and give complete descriptions of curves of genus 0 and 1. For genus ≥ 2 we restrict our attention to hyperelliptic curves, which can be given by a simple Weierstrass equation. These hyperelliptic curves are not suitable for the theory of canonical heights, we will embark on in Chapter 2, as we need the structure of an abelian variety. In Section 1.2, we construct the Jacobian of such a curve, which is a self-dual abelian variety and can be seen as a linearization of the curve. This variety is suited for the theory of canonical heights. Next, in Section 1.3, we construct models of varieties, which give insight in the reduction of the variety. A special model is the Néron model, which has a certain group structure and is essential for the construction of canonical local heights. Finally we will look at the skeleton of the Jacobian of a curve, which can be constructed from the reduction graph of the curve. This skeleton is the basis for Theorem 2.5.4, which we will use in Chapters 3 and 4 to calculate canonical local heights explicitly.

In this chapter K denotes an arbitrary field, unless explicitly stated otherwise. If we work with the algebraic closure of K, we always denote this by $\overline{K}$. The general example to keep in mind is that K is a number field (i.e. a finite extension of $\mathbb{Q}$), or the completion of such a number field at an absolute value.

Our main objects of study are varieties over K, which are integral, separated schemes of finite type over $\mathrm{Spec}(K)$, which we generally denote by X. We also assume all varieties to be connected. If we work over $\overline{K}$, this coincides with the classical definition of varieties, as defined in Hartshorne [Hartshorne, 26, Chapter I]. We assume basic knowledge about varieties as in Chapter I and II of loc. cit. Particular types of varieties of interest are curves, which we generally denote by C, which are varieties of dimension one, and abelian varieties, generally denoted by A, which are varieties that admit a group structure, given by algebraic morphisms. In Section 1.3 we will also use scheme terminology, as defined in [Hartshorne, 26, Chapter II].

We will freely switch between the notions of Weil and Cartier divisors on varieties and use the word 'divisor' for both terms, if we work on a smooth variety. In all other cases, the term 'divisor' will mean a Cartier divisor. On smooth varieties we hence identify the Picard and divisor class group. If X is a variety defined over K, we say a divisor $D \in \mathrm{Div}(X) := \mathrm{Div}_{\overline{K}}(X)$ is defined over K, if D is invariant under the action of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$.

Notation 1.0.1. *If we work with divisors on a smooth curve C, prime divisors are simply points on C. We denote the prime divisor given by the point $P \in C$ by (P), to avoid*

confusion. A general divisor on a smooth curve is thus given by a finite sum

$$D = \sum_{P \in C} n_P(P), \quad n_P \neq 0 \text{ for finitely many } P.$$

When working in the Picard group, we denote by $[D]$ the class containing D. The support of a divisor is denoted $|D|$. If D is effective, it is denoted $D \geq 0$. Two divisors D and E being linearly equivalent is denoted by $D \sim E$.

On projective curves C, we use that all principal divisors have degree 0, so we can define the degree-zero Picard group

$$\mathrm{Pic}^0(C) := \ker(\deg) = \{[D] \in \mathrm{Pic}(X) : \deg(D) = 0\}. \tag{1.1}$$

1.1 Curves and Genus

Curves are varieties of dimension one. In this sense, they are the easiest varieties to study after varieties of dimension zero, which are just finite sets of points. Curves still offer a rich theory. Famously, the Riemann-Roch Theorem gives insight into the structure of divisors and rational functions on curves and defines the genus of a curve, acting as a good measure for its complexity. We will introduce the Riemann-Roch Theorem in this section and study curves of genus 0, 1 and ≥ 2, respectively. The main reference for this section is [Hindry and Silverman, 27, Section A.4].

1.1.1 The Riemann-Roch Theorem

Let X be a projective variety, defined over K, and let $D \in \mathrm{Div}(X)$ be a (Cartier) divisor. We can associate to D a subspace of $\overline{K}(X)$ by

$$L(D) = \{f \in \overline{K}(X)^\times \mid D + \mathrm{div}(f) \geq 0\} \cup \{0\}.$$

This space is in fact a $\overline{K}$-vector space and it is finite dimensional (see e.g. [Hartshorne, 26, Theorem III.5.2] for a more general statement). In this case, we denote its dimension by $\ell(D)$. If K is a perfect field (e.g. if K is number field), $L(D)$ has a basis consisting of K-rational functions. Via these spaces, we can construct maps from projective varieties to projective space that are associated with a divisor.

Definition 1.1.1. *Let $D \in \mathrm{Div}(X)$ be a divisor on X. Choose a basis $f_0, \ldots, f_n$ of $L(D)$. The rational map associated with D, denoted by φ_D, is the map*

$$\varphi_D : X \dashrightarrow \mathbb{P}^{\ell(D)-1}, \quad x \mapsto (f_0(x) : \cdots : f_n(x)).$$

The map clearly depends on the chosen basis and is hence only uniquely defined up to an automorphism of $\mathbb{P}^{\ell(D)-1}$. Furthermore, this map is only defined outside of the poles of the individual f_i's and the common zeros of the f_i's. Clearly, if the f_i have common zeros or common poles, then the *set of base points* of D, by which we mean the intersection of the supports of all divisors in $\{E \in \mathrm{Div}(X) \mid E \geq 0 \text{ and } E \sim D\}$, is non-empty. It can be easily shown that φ_D is a morphism outside of the base points of D. If a divisor has no base points, we say it is *base point free*.

Definition 1.1.2. *A divisor D on a projective variety X is* very ample *if the associated rational map $\varphi_D : X \dashrightarrow \mathbb{P}^n$ is a closed immersion (i.e. X maps isomorphically onto its*

image $\varphi_D(X)$). A divisor D is said to be ample *if a positive multiple nD is very ample.*

Proposition 1.1.3. *Any divisor can be written as the difference of two ample divisors.*

For a proof, see [Hindry and Silverman, 27, Theorem A.3.2.3].

Now we move our attention to curves, and their *genus*. The genus of a curve is a nonnegative integer g, and if the curve C is projective, non-singular and defined over $\mathbb{C}$, it is equal to the number of holes in the Riemann surface $C(\mathbb{C})$. Over general fields, the genus of a curve can be defined via the famous Riemann-Roch Theorem. To state the theorem, we need a canonical divisor. A canonical divisor on a smooth curve C is the divisor of a non-zero differential 1-form. This determines a unique divisor class in $\mathrm{Pic}(C)$.

Theorem 1.1.4 (Riemann-Roch)**.** *Let C/K be a smooth projective curve and let $K_C \in \mathrm{Div}(C)$ be a canonical divisor. Then there is a non-negative integer $g(C)$, called the genus of C, such that for all divisors $D \in \mathrm{Div}(C)$,*

$$\ell(D) - \ell(K_C - D) = \deg(D) - g(C) + 1.$$

The genus may be calculated as the dimension over K of the first cohomology group of the structure sheaf $\mathcal{O}_C$ of C:

$$g(C) = \dim_K H^1(C, \mathcal{O}_C). \tag{1.2}$$

For general varieties, there are separate notions for the arithmetic and geometric genus, but for a smooth projective curve, these notions coincide and we may talk unambiguously about *the* genus of a curve (see [Hartshorne, 26, Remark III.7.12.2]).

In the remainder of this thesis, we will not be concerned with calculating the genus of a curve, but rather take the genus as a given.

1.1.2 Curves of genus 0 and 1

Curves of genus 0 are easy to understand, using the following theorem.

Theorem 1.1.5. *Let C/K be a smooth projective curve of genus 0.*

(a) The curve C is isomorphic over K to a conic in $\mathbb{P}^2$.

(b) The curve C is isomorphic over K to $\mathbb{P}^1$ if and only if it has a K-rational point.

If K is algebraically closed, then any smooth projective curve C of genus 0 is isomorphic to $\mathbb{P}^1$.

Curves of genus 1 offer a richer theory. If such a curve admits a rational point, there is a group structure on the set of rational points on the curve, and we call it an *elliptic curve*. Elliptic curves and their arithmetic are well-studied, see for example [Silverman, 56].

Definition 1.1.6. *An elliptic curve, defined over K, is a pair (E, O), where E is a smooth projective curve of genus 1, defined over K, and $O \in E(K)$ is a rational point.*

Proposition 1.1.7. *Let E/K be an elliptic curve. Then there exist constants $a_1, \ldots a_6 \in K$, such that there is an isomorphism of E over K to the smooth plane cubic given by*

the projective closure Weierstrass equation

$$E' : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{1.3}$$

and the point O is mapped to the point $(0 : 1 : 0) \in E'$.

From the Riemann-Roch Theorem it can be proven that $\ell(n()) = n$. The space $L(6(O))$ is thus 6-dimensional, but contains the seven functions $1, x, x^2, x^3, y, xy$ and y^2, for some non-constant functions $x \in L(2(O))$ and $y \in L(3(O))$. There must thus be a non-trivial relation between these functions, which gives the Weierstrass equation.

If $\mathrm{char}(K) \neq 2, 3$, we can complete the square in y and the cube in x to reduce the Weierstrass equation to the form $y^2 = x^3 + Ax + B$, for some $A, B \in K$.

The rational points of an elliptic curve form an abelian group. The group law can be given very explicitly by a 'chord-and-tangent' operation if we have a Weierstrass equation for the elliptic curve, see for example [Silverman, 56, Section III.2]. The identity element is the point at infinity O. The negation $[-1] : E \to E$ is given by the involution $(x, y) \mapsto (x, -y - a_1x - a_3)$.

It is particularly interesting that this group law is algebraic and intrinsic, in the sense that it is given by rational functions and only depends on the abstract curve E and the choice $O \in E(K)$. That is, the group law makes E into an abelian variety.

Theorem 1.1.8. *Let E/K be an elliptic curve. Then there is a group structure on E that makes E into an abelian variety. Furthermore, the group $\mathrm{Pic}^0(E)$ can also be made into an abelian variety, and the map*

$$\kappa : E \to \mathrm{Pic}^0(E), \quad P \mapsto [(P) - (O)],$$

is an isomorphism of abelian varieties.

We will later see that $\mathrm{Pic}^0(E)$ is the Jacobian of E, which is an abelian variety. That is, an elliptic curve is isomorphic to its own Jacobian.

1.1.3 Hyperelliptic Curves

There is (significantly) more diversity in curves of genus $g \geq 2$. In this thesis we will focus on a particular family of curves, known as *hyperelliptic curves*.

Definition 1.1.9. *A smooth, projective curve C/K of genus $g \geq 2$ is called a* hyperelliptic curve *if there exists a morphism $C \to \mathbb{P}^1$ of degree 2.*

Note that elliptic curves also have a double covering of the projective line, via the morphism $x : E \to \mathbb{P}^1$, given by $(x, y) \mapsto [x : 1]$, but in most literature, and in the remainder of this thesis, only curves of genus $g \geq 2$ are considered hyperelliptic.

Similar to Weierstrass equations for elliptic curves, we can construct explicit equations that define hyperelliptic curves. Suppose that $\mathrm{char}(K) \neq 2$[1]. The degree two morphism implies that the function field of the hyperelliptic curve C is a quadratic extension of the function field $K(\mathbb{P}^1) = K(x)$. That is, $K(C)$ is of the form $K(x, y)$,

[1] In Chapter 4 we work with hyperelliptic curves over number fields, which have characteristic 0, so this assumption is harmless for our purposes.

and x, y are related via an equation of the form

$$y^2 = F(x), \tag{1.4}$$

for some $F(x) \in K[x]$. This equation now gives an affine model for C. If the equation has double root α, we can replace y by $(x-\alpha)y$ and divide the double root out. That is, we may assume F has distinct roots. A complete smooth model is now obtained by gluing this affine curve to the affine curve defined by

$$v^2 = u^d F(u^{-1}), \tag{1.5}$$

where $d = \deg(F)$ if $\deg(F)$ is even, and $d = \deg(F)+1$ if it is odd. The gluing is done via the map $(u, v) = (x^{-1}, yx^{-d/2})$. We call the affine equation $y^2 = F(x)$ a Weierstrass equation for C.

If $\deg(F)$ is odd, Equation 1.5 has a unique solution for $u = 0$, and if $\deg(F)$ is even, it has two solutions. These solutions correspond to the points "at infinity" of the curve C, as given by the affine Weierstrass equation 1.4. In the prior case we denote the single point at infinity by ∞, in the latter case we denote two points at infinity by ∞^+ and ∞^-. In the case of an elliptic curve E/K with Weierstrass equation as in Equation 1.3, corresponding to $\deg(F) = 3$, the point at infinity ∞ coincides with the identity element $O \in E(K)$.

By the Riemann-Hurwitz formula (see [Hindry and Silverman, 27, Theorem A.4.2.5]) we get

$$2g(C) - 2 = \deg(C \to \mathbb{P}^1)(2g(\mathbb{P}^1) - 2) + \sum_{P \in C(\overline{K})} (e_P - 1),$$

where e_P is the ramification index at the point P of the map $C \to \mathbb{P}^1$. This map is ramified exactly at all roots of F, and if $\deg(F)$ is odd also at ∞. The ramification indices at these points are all equal to 2, so that $\sum_{P \in C}(e_P - 1) = d$. Hence $2g(C) - 2 = d - 4$, and

$$g(C) = \frac{d}{2} - 1 = \lceil \deg(F)/2 \rceil - 1.$$

We see that there are two possibilities for $\deg(F)$ once $g(C)$ is fixed. Suppose $\deg(F) = 2g+2$, and that F has a rational root $\alpha \in K$. The transformation

$$x \mapsto 1/(x' - \alpha), \quad y \mapsto y'/(x' - \alpha)^{g+1} \tag{1.6}$$

is then an isomorphism over K from the curve defined by Equation 1.4 to a degree 5 equation. For example if we do this transformation to the equation $y^2 = x^6 - 1$, with root $x = 1$, we get the substitutions $x = \frac{1+x'}{x}$ and $y = \frac{y'}{x'^3}$, yielding the equation

$$y'^2 = (1 + x')^6 - x'^6 = 6x'^5 + 15x'^4 + 20x'^3 + 15x'^2 + 6x' + 1.$$

For this reason, when working with a hyperelliptic curve of genus g, we will take a Weierstrass equation of degree $2g + 1$.

There is no analogue to an addition morphism on hyperelliptic curves (see Theorem 1.2.1). However, there does exist a hyperelliptic involution given by

$$i : C \to C, \quad i(x, y) = (x, -y), \tag{1.7}$$

which is the analogue of the map $[-1]$ on an ellitpic curve.

On genus 2 curves, the additional assumption of having a degree 2 map to $\mathbb{P}^1$ is not restrictive.

Proposition 1.1.10. *Every smooth, projective curve of genus 2 is hyperelliptic.*

Proof. Let C be a curve of genus 2 and let K_C be a canonical divisor on C. Note that $L(0)$ consists of only constant functions, so $\ell(0) = 1$. Applying Riemann-Roch to both the zero divisor $D = 0$ and the canonical divisor $D = K_C$ yields $\deg(K_C) = \ell(K_C) = 2$. Thus the divisor K_C gives a rational map $\varphi_{K_C} : C \dashrightarrow \mathbb{P}^1$. This map is actually a morphism, as C is a projective curve ([Hartshorne, 26, Proposition I.6.8]).

Now the map φ_{K_C} satisfies that pullbacks of hyperplanes H on $\mathbb{P}^1$ land in $|K_C|$ (see e.g. [Hartshorne, 26, Theorem II.7.1]). We thus get

$$\deg(\varphi_{K_C}) \deg(H) = \deg(\varphi^*(H)) = \deg(K_C) = 2.$$

As $\deg(H) = 1$, we get $\deg(\varphi_{K_C}) = 2$, hence the divisor K_C defines a map of degree 2, and C is hyperelliptic. □

Not all curves of genus $g \geq 3$ are hyperelliptic. Curves of genus g are parametrized by a variety $\mathcal{M}_g$, called the moduli space of curves of genus g, and for $g \geq 2$ one can show that this variety has dimension $3g - 3$. The moduli space of *hyperelliptic* curves of genus g is a subvariety of $\mathcal{M}_g$ of dimension $2g-1$. Note that the dimensions coincide for $g = 2$, exactly what we have shown.

1.2 Jacobian Varieties

The 'chord-and-tangent' operation on an elliptic curve is unique to elliptic curves. It cannot be translated to curves of genus larger than 1. In fact, there is no (algebraic) group structure possible at all on smooth projective curves of genus unequal to 1.

Theorem 1.2.1. *Suppose C is a smooth projective curve and an abelian variety. Then the genus of C is 1.*

This can be proven e.g. by noting the canonical divisor of such a curve must be the zero divisor, as the group structure trivializes the (co)tangent bundle, and by Riemann-Roch, $\deg(K_C) = 2g(C) - 2 \neq 0$, for $g(C) \neq 1$.

In Chapter 2, we will define *canonical heights* on abelian varieties. These canonical heights are hence not defined on hyperelliptic curves. The closest we can get to an abelian structure on a curve of genus $g \geq 2$ is by embedding the curve into a variety of dimension g, that does admit a group structure. This variety is called the *Jacobian of C*. It can be seen as a linearization of the curve. Canonical heights can then be defined on this Jacobian.

Theorem 1.2.2. *Let C/K be a smooth projective curve of genus ≥ 1. There exists an abelian variety $\mathrm{Jac}(C)$ of dimension g, defined over K, called the Jacobian of C, and a closed immersion $\Phi : C \hookrightarrow \mathrm{Jac}(C)$, called the Abel-Jacobi map, with the following properties:*

(a) *Extend Φ linearly to divisors on C. Then Φ induces a group isomorphism between $\mathrm{Pic}^0(C)$ and $\mathrm{Jac}(C)$.*

(b) *Define*

$$\Theta_\Phi = \underbrace{\Phi(C) + \cdots + \Phi(C)}_{g-1 \text{ times}}.$$

Then Θ_Φ is an irreducible ample divisor on $\mathrm{Jac}(C)$.

For the full proof, see [Hindry and Silverman, 27, Section A.8.1]. To simplify notation, we often write $J(C)$ for $\mathrm{Jac}(C)$, or even simply J, if it is clear from context what C is.

From now on, we will always use the identification of $\mathrm{Jac}(C)$ with $\mathrm{Pic}^0(C)$, as in Theorem 1.2.2(a), as a definition of the Jacobian and take for granted that this defines an abelian variety. Then we can make the injection $\Phi : C \hookrightarrow \mathrm{Jac}(C)$ explicit by taking a divisor D of degree 1 and defining

$$\Phi = \Phi_D : C \hookrightarrow \mathrm{Pic}^0(C) \cong \mathrm{Jac}(C), \quad P \mapsto [(P) - D]. \tag{1.8}$$

If D is defined over K, then Φ_D and Θ_{Φ_D} are defined over K. If we take a Weierstrass form for a hyperelliptic curve C, with a unique point ∞ at infinity, we will usually choose $D = (\infty)$. We then set

$$\Theta := \Theta_{\Phi_\infty}. \tag{1.9}$$

If C has two points ∞^+ and ∞^- at infinity, we define $\Theta^\pm = \Theta_{\Phi_{\infty^+}} + \Theta_{\Phi_{\infty^-}}$.

Under the identification of $\mathrm{Jac}(C)$ with $\mathrm{Pic}^0(C)$, the embedding Φ, and therefore also the divisor Θ_Φ, are dependent on the choice of the divisor D, but are unique up to translation.

Working directly with $J = \mathrm{Jac}(C)$ as $\mathrm{Pic}^0(C)$ can be quite difficult. For example, it is not immediately clear what the function field on J should look like. This will be very relevant in Chapter 4, when we will construct local height functions on Jacobians. Luckily, we can use a different perspective, coming from the symmetric powers of the curve C.

We define the *r-th symmetric power* of a curve C to be

$$\mathrm{Sym}^r C = (C \times \cdots \times C)/S_r,$$

where S_r is the symmetric group on r elements, acting on the r copies of C in the obvious way. The set $\mathrm{Sym}^r C$ is naturally identified with effective divisors of degree r on C, and can be given the structure of a variety. If D is a degree 1 divisor as in Equation 1.8, then we can give a map

$$\pi_r : \mathrm{Sym}^r C \to J = \mathrm{Pic}^0(C), \quad (P_1, \ldots, P_r) \mapsto [(P_1) + \cdots + (P_r) - rD]. \tag{1.10}$$

As noted before, we usually set $D = (\infty)$. Now the map $\pi = \pi_g$ has some useful properties. First of all, it is surjective, which is to say that any degree 0 divisor on C is linearly equivalent to a divisor of the form $\sum_{i=1}^g (P_i) - g(\infty)$. Here $P_i = \infty$ is allowed. Next, define the set

$$D_g = \{(P_1, \ldots, P_g) : P_i \neq \infty, P_i \neq i(P_j), \forall i \neq j\} \subseteq \mathrm{Sym}^g C.$$

The map π is injective on D_g. This fact can be proven by showing that $L(D) = K$ for divisors $\sum_{i=1}^{g}(P_i)$, with $(P_1, \ldots, P_g) \in D_g$. Both these facts are proven in [Mumford, 44, Section IIIa.2] (where the base field is assumed to be $\mathbb{C}$, but the arguments work for general K).

From Theorem 1.2.2(b), we can see that the Theta divisor is defined by

$$\Theta = \{[(P_1) + \cdots (P_{g-1}) - (g-1)(\infty))] \in \mathrm{Pic}^0(C) : P_1, \ldots, P_{g-1} \in C\} \subseteq J. \quad (1.11)$$

No point in D_g maps to a point in Θ. Conversely, if $P = (P_1, \ldots, P_g)$ is not in D_g, then either $P_i = \infty$, for some i, in which case $\pi(P)$ is a divisor of the form in Equation 1.11, or $P_i = i(P_j)$ for some $i \neq j$. In the latter case, let $f = 1/(x - x(P_i))$. Then $\mathrm{div}(f) = -(P_i) - (P_j) + 2(\infty)$, so that the divisor $\pi(P)$ is linearly equivalent to the same divisor, where $(P_i) + (P_j)$ is replaced with $2(\infty)$. That is, $\pi(P)$ is of the form in Equation 1.11. We conclude that the set D_g can be naturally identified with the set $J \backslash \Theta$.

In fact, we can realize the set $J \backslash \Theta$ as affine variety in $\mathbb{A}^{2g}$, via Mumford coordinates, see [Mumford, 44, Section IIIa.1]. Furthermore, any divisor in J is linearly equivalent to a unique divisor of the form $\sum_{i=1}^{r}(P_i) - r(\infty)$, with $0 \leq r \leq g$ and no points P_i equal to ∞ or an involution of another point. That is, defining

$$D_0 = \{\infty\} \quad \text{and} \quad D_r = \{(P_1, \ldots, P_r) : P_i \neq \infty, P_i \neq i(P_j), \forall i \neq j\},$$

we get a stratification of J as a disjoint union

$$J = \pi_1(D_0) \cup \pi_1(D_1) \cup \pi_2(D_2) \cdots \cup \pi(D_g).$$

We can now explain addition on J as taking the sum of two divisors and using the procedure in [Mumford, 44, Section IIIa.2] to reduce the result to one of the D_i. Note that in practice it may be hard to carry out this process.

1.2.1 The divisor Θ

The divisor Θ is in a certain sense a natural choice for a divisor on J to study, due to its properties. First of all, Θ is a symmetric divisor, by which we mean that $[-1]^*\Theta$ is linearly equivalent to Θ. As we will see in Lemma 2.3.1, symmetry is used to define the canonical height associated with a divisor. Furthermore Θ defines a *principal polarization* which we define in this section. This is a requirement to use Theorem 2.5.4 to calculate local height functions. Moreover, if the genus of the curve C is 2, $\Theta \subseteq J = \mathrm{Jac}(C)$ is isomorphic to C itself.

Proposition 1.2.3. *Let C be a hyperelliptic curve of genus g. Then the divisor Θ is symmetric. That is, $[-1]^*\Theta \sim \Theta$.*

Proof. We use the fact that $[-1]^*\Theta_\Phi = \tau_\kappa^*\Theta_\Phi$, with $\kappa = \Phi(K_C)$, see [Hindry and Silverman, 27, Theorem A.8.2.1(i)].

Let $x : C \to \mathbb{P}^1$ be the morphism of degree 2 mapping each point on C to its x-coordinate, w.r.t. the given Weierstrass equation. By the Hurwitz formula ([Hindry

and Silverman, 27, Proposition A.2.2.8]) we find that for $\mathrm{char}(K) \neq 2$, we get

$$K_C \sim x^* K_{\mathbb{P}^1} + \sum_{P \in C} (e_P(f) - 1)(P).$$

The map x is ramified above all roots of $F(x)$ and at ∞, all of ramification index 2. A canonical divisor on $\mathbb{P}^1$ is given by $-2([1:0])$, so that $x^*(K_{\mathbb{P}^1}) = -4(\infty)$. Note that $\mathrm{div}(y^2) = \sum_{P:F(x(P))=0}(P) - (2g+1)(\infty)$. We get

$$K_C \sim -4(\infty) + \sum_{P:F(x(P))=0} (P) + (\infty) - \mathrm{div}(y^2) = (2g-2)(\infty).$$

That is, $(2g-2)(\infty)$ is a canonical divisor on C.

Now we get that $\kappa = \Phi(K_C) = \Phi((2g-2)(\infty)) = 0$, the zero divisor, which is the identity element of J. That is, τ_κ^* is the identity map, and

$$[-1]^*\Theta = \tau_\kappa^*\Theta = \Theta. \qquad \square$$

Given an abelian variety A, we can construct its *dual abelian variety* $\hat{A}$ by giving the set

$$\mathrm{Pic}^0(A) = \{c \in \mathrm{Pic}(A) : \tau_a^*(c) = c \text{ for all } a \in A\} \subseteq \mathrm{Pic}(A)$$

the structure of abelian variety. This set is the connected component of the zero divisor. Note that in the case that A is an abelian variety and a curve (so A is an elliptic curve) this definition of Pic^0 agrees with our earlier definition in Equation (1.1). Given a $c \in \mathrm{Pic}(A)$, there is a map $\Phi_c : A \to \hat{A}$, given by $a \mapsto \tau_a^* c - c$. A class $c \in \mathrm{Pic}(A)$ is said to give a *principal polarization* if the map Φ_c is an isomorphism. In this case the abelian variety A is *self-dual*.

The divisor Θ gives such a principal polarization, which makes the Jacobian of a curve self-dual, see [Hindry and Silverman, 27, Corollary A.8.2.3]. Duals come up regularly in algebraic geometry and the fact that Jacobians are self-dual makes this study often easier and more explicit. In this thesis, we will not directly use duals, but rather make use of results, like Theorem 2.5.4, that only hold for principal polarizations.

Note that elliptic curves are isomorphic to their Jacobian. By construction we have $\Theta = (O)$, where O is the identity element of the elliptic curve. The fact that $\Theta = (O)$ gives a principal polarization is exactly the content of Theorem 1.1.8.

1.2.2 Jacobians over $\mathbb{C}$

Jacobians over the complex numbers can be constructed very explicitly. For Jacobians of curves of genus 1, this construction is the classical uniformization of elliptic curves. We will give a brief summary of the construction of the Jacobian of a hyperelliptic curve, without proofs. A full treatment can be found in e.g. [Hindry and Silverman, 27, Section A.6]. We use this explicit description in Sections 3.1 and 4.2 to construct explicit formulas for canonical local heights at archimedean places.

Let C be a smooth projective hyperelliptic curve of genus $g \geq 1$ over $\mathbb{C}$, given by a Weierstrass equation of the form

$$y^2 = x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_0,$$

with all $\lambda_i \in \mathbb{C}$. Then C is a compact Riemann surface of genus g. The space of regular 1-forms $\Omega^1[C]$ is g-dimensional with a basis given by

$$\omega_1 = \frac{dx}{y},\ \omega_2 = \frac{x\,dx}{y}, \ldots,\ \omega_g = \frac{x^{g-1}\,dx}{y}.$$

The homology group $H_1(C, \mathbb{Z})$ is free on $2g$ generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ that can be chosen to not intersect the point ∞ and such that

$$\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0, \quad \alpha_i \cdot \beta_j = \delta_{i,j}, \quad \forall i, j,$$

where $\delta_{i,j}$ is Kronecker's delta, see [Mumford, 44, Section IIIa.5].

We define the matrices

$$\Omega_1 = \left(\int_{\alpha_i} \omega_j\right)_{1\leq i,j\leq g} \quad \text{and} \quad \Omega_2 = \left(\int_{\beta_i} \omega_j\right)_{1\leq i,j\leq g}.$$

The choice of intersections of the symplectic basis implies the Riemann period relations:

$$\Omega_1\Omega_2^T = \Omega_2\Omega_1^T, \quad \text{and} \quad -\sqrt{-1}(\overline{\Omega_1}\Omega_2^T - \overline{\Omega_2}\Omega_1^T) > 0,$$

where $M > 0$ denotes that a matrix is positive definite. The matrix Ω_1 is invertible. To see this, suppose $\Omega_1^T X = 0$. Then $\overline{X}^T(-\sqrt{-1}(\overline{\Omega_1}\Omega_2^T - \overline{\Omega_2}\Omega_1^T))X = 0$, from which we find $X = 0$. Set $\tau = \Omega_1^{-1}\Omega_2$. Then the Riemann relations say that τ is symmetric and the imaginary part $\mathrm{Im}(\tau)$ is positive definite.

Theorem 1.2.4. *Let $\Lambda_\tau = \mathbb{Z}^g + \tau\mathbb{Z}^g$. Then Λ is a lattice in $\mathbb{C}^g$. The g-dimensional torus $\mathbb{C}^g/\Lambda_\tau$ is an abelian variety over $\mathbb{C}$ and is isomorphic to $J = \mathrm{Jac}(C)$.*

We always use this identification of the Jacobian of a complex curve over $\mathbb{C}$ with the torus in Theorem 1.2.4. The Abel-Jacobi map, relative to a fixed basepoint $a \in C$, is now given by the map

$$\Phi_a : C \hookrightarrow J = \mathbb{C}^g/\Lambda_\tau, \quad b \mapsto (\int_a^b \omega_1, \ldots, \int_a^b \omega_g) \bmod \Lambda_\tau. \tag{1.12}$$

Now that we have a description of the variety $\mathrm{Jac}(C)/\mathbb{C}$, we also want to have some elements of the function field to work with. To this end, we introduce on the lattice $\Lambda = \mathbb{Z}^g + \tau\mathbb{Z}^g$, the function $\theta_\tau \begin{bmatrix} a \\ b \end{bmatrix} : \mathbb{C}^g \to \mathbb{C}$, called the *theta function with characteristics* $a, b \in \mathbb{R}^g$, given by

$$\theta_\tau \begin{bmatrix} a \\ b \end{bmatrix}(z) = \sum_{n\in\mathbb{Z}^g} \exp(\pi i(n+a)^T\tau(n+a) + 2\pi i(n+a)^T(z+b)),$$

for $z \in \mathbb{C}^g$ and $a, b \in \mathbb{R}^g$. This function is a *theta function* in the sense that it is entire and satisfies a functional equation of the form

$$\theta_\tau \begin{bmatrix} a \\ b \end{bmatrix} (z+\lambda) = \exp(g_\lambda(z))\theta_\tau \begin{bmatrix} a \\ b \end{bmatrix} (z), \quad \forall z \in \mathbb{C}^g, \lambda \in \Lambda_\tau, \tag{1.13}$$

for an affine function $g_\lambda : \mathbb{C}^g \to \mathbb{C}$. By a Theorem of Poincaré, for any effective divisor D on a complex torus $\mathbb{C}^g/\Lambda$, there exists a theta function that vanishes on the pullback $\overline{D}$ of D to $\mathbb{C}^g$. Two theta functions define the same divisor if and only if their quotient is a *trivial theta function*, which is a theta function of the form $\exp(f(z))$, with f a polynomial of degree 2. For $a = \delta = [\frac{1}{2}, \ldots, \frac{1}{2}]^T$ and $b = \delta' = [\frac{g}{2}, \frac{g-1}{2}, \ldots, \frac{1}{2}]^T$, the theta function $\theta_\tau \begin{bmatrix} \delta \\ \delta' \end{bmatrix} (z)$ has Θ as its divisor, see [Mumford, 44, Theorem 3.5.3].

To talk about the function field on $\mathbb{C}^g/\Lambda_\tau$, we first need to define a slight variation of the theta function with characteristics, called the hyperelliptic σ-function, for which we need some more notation. For $j = 1, 2, \ldots, g$, we define a differential form with a pole only at ∞

$$\eta_j = \frac{1}{2y} \sum_{k=j}^{2g-j} (k+1-j)\lambda_{k+1+j} x^k \, dx,$$

and matrices

$$E_1 = \left(\int_{\alpha_i} \eta_j \right)_{1 \le i,j \le g}, \quad E_2 = \left(\int_{\beta_i} \eta_j \right)_{1 \le i,j \le g},$$

where the α_i and β_i form a symplectic basis of $H_1(X, \mathbb{Z})$ as before.

Definition 1.2.5. *The hyperelliptic σ-function on $\mathbb{C}^g$, w.r.t. a lattice $\Lambda\tau \subseteq \mathbb{C}^g$ is the function*

$$\sigma(z) = c \exp\left(\frac{1}{2} z^T E_1 \Omega_1^{-1} z \right) \theta - \tau \begin{bmatrix} \delta \\ \delta' \end{bmatrix} (\Omega_1^{-1} z).$$

The constant c in the definition is chosen such that the Taylor expansion around $u = 0$ has coefficients in $\mathbb{Q}[\lambda_0, \ldots, \lambda_{2g}]$ and

$$\sigma(u) = \det\left((u_{i+j-1})_{1 \le i,j \le l} \right) + (\text{terms of degree } \ge l+2), \tag{1.14}$$

where we write $g = 2l - 1$, or $g = l$. An explicit expression of c can be found in [Buchstaber, Enolskiĭ, and Leĭkin, 11, Definition 2.1].

As σ is defined by multiplying $\theta \begin{bmatrix} \delta \\ \delta' \end{bmatrix}$ by a trivial theta function, the hyperelliptic σ-function also has Θ as divisor.

The hyperelliptic σ-function is not periodic w.r.t. Λ (as it is entire and non-constant), but it does satisfy a relatively nice translational relation. In order to state it, we define the $\mathbb{C}$-valued form $L(z, w)$ on $\mathbb{C}^g \times \mathbb{C}^g$ by

$$L(z, w) = \frac{1}{2\pi i} z^T (E_1 w_1 + E_2 w_2), \tag{1.15}$$

where we decompose a vector w into $w_1, w_2 \in \mathbb{R}^g$, via $w = \Omega_1 w_1 + \Omega_2 w_2$. This form is $\mathbb{C}$-linear in the first variable, and $\mathbb{R}$-linear in the second. Furthermore, for lattice

points $l \in \Lambda$, we define a characteristic

$$\chi(l) = \exp\left(2\pi i\left(l_1^T\delta' - l_2^T\delta + \frac{1}{2}l_1^T l_2\right)\right) \in \{\pm 1\}.$$

Proposition 1.2.6 (Translation relation, [Uchida, 60, Proposition 2.2]). *For any $u \in \mathbb{C}^g$ and any $l \in \Lambda$, we have*

$$\sigma(z+l) = \chi(l)\exp\left(2\pi i L\left(z + \frac{1}{2}l, l\right)\right)\sigma(z).$$

The logarithmic derivatives of σ give us our first elements of the function field on $\mathbb{C}^g/\Lambda$.

Definition 1.2.7. *The hyperelliptic $\wp$-functions are the meromorphic functions on $\mathbb{C}^g$ given by*

$$\wp_{ij}(z) = -\frac{\partial^2}{\partial z_i \partial z_j}\log\sigma(z), \quad \wp_{ijk}(z) = -\frac{\partial^3}{\partial z_i \partial z_j \partial z_k}\log\sigma(z), \quad \forall i,j,k \in \{1,2,\dots,g\}.$$

The hyperelliptic $\wp$-functions do not depend on the order of their indices and are periodic with respect to Λ. This is clear from the translational relation in Proposition 1.2.6:

$$\begin{aligned}
\wp_{ij}(z+l) &= -\frac{\partial^2}{\partial z_i \partial z_j}\log\sigma(z+l) \\
&= -\frac{\partial^2}{\partial z_i \partial z_j} 2\pi i L\left(z + \frac{1}{2}l, l\right) - \frac{\partial^2}{\partial z_i \partial z_j}\log\sigma(z) \\
&= -\frac{\partial^2}{\partial z_i \partial z_j}(z + \frac{l}{2})^T(E_1 l_1 + E_2 l_2) + \wp_{ij}(z) \\
&= \wp_{ij}(z),
\end{aligned}$$

where the second derivative of $(z + \frac{l}{2})^T(E_1 l_1 + E_2 l_2)$ vanishes, as it is linear in z.

That is, $\wp_{ij}, \wp_{ijk} \in \mathbb{C}(J)$. We will use this functions to describe the division polynomials on J in Section 4.1.2.

1.3 Models

To construct local heights on abelian varieties in later chapters, we will make distinctions based on the type of reduction at a certain place of the variety. Reducing a variety at, e.g. a non-archimedean place on $\mathbb{Q}$, corresponding to a prime number $p \in \mathbb{Z}$, roughly corresponds to reducing the defining equations of the variety modulo p. A good way to get a grip on the type of reduction is to construct *models* for these varieties. In particular a good understanding of the *Néron model* of an abelian variety, constructed in Section 1.3.2, will be crucial to understand local heights, as can be seen in Theorem 2.5.1.

In this section, we denote by R a Dedekind domain (a Noetherian, integrally closed domain of dimension one), by $K = \mathrm{Frac}(R)$ its field of fractions and by $S = \mathrm{Spec}(R)$

the spectrum of R. The main examples to keep in mind are either that K is a number field, with $R = \mathcal{O}_K$ its ring of integers, or R is a discrete valuation ring (then S has only two elements). The latter case is especially important in the case that $K = K_v$ is the completion of a number field at an absolute value (see Section 2.1) and $R = R_v$ the corresponding valuation ring.

The scheme S consists of a dense point $\eta \in S$, which is called the generic point. For a morphism $f : \mathcal{X} \to S$, we call the fiber of f over η, $\mathcal{X}_\eta := X \times_S \operatorname{Spec}(k(\eta))$, the *generic fiber* of f. Fibers over closed points $p \in S$ are called *special fibers*.

Definition 1.3.1. *Let K, R, and S be as above. Let X/K be a variety. A* model *for X over S is a separated scheme of finite type $\mathcal{X} \to S$, together with a K-isomorphism $\mathcal{X}_\eta \to X$.*

For example, the scheme S is a model for $\operatorname{Spec}(K)$ over itself and the scheme $\mathbb{P}^n_{\mathbb{Z}}$ is a model for $\mathbb{P}^n_{\mathbb{Q}}$ over $\operatorname{Spec}(\mathbb{Z})$. If $E/\mathbb{Q}$ is an elliptic curve given by a Weierstrass equation $E : y^2 = x^3 + Ax + B$, $A, B \in \mathbb{Z}$, we may look at the closed subscheme $\mathcal{E} \subseteq \mathbb{P}^2_{\mathbb{Z}}$ of $\mathbb{P}^2_{\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$ given by the equation

$$\mathcal{E} : y^2 = x^3 + Ax + B.$$

The generic fiber is simply the elliptic curve $E/\mathbb{Q}$. The special fibers at a prime $p \in \mathbb{Z}$ are the (possibly singular) curves over $\mathbb{F}_p$ given by

$$\mathcal{E}_p : y^2 = x^3 + \tilde{A}x + \tilde{B}, \quad \tilde{A} = A, \tilde{B} = B \bmod p.$$

We see that taking special fibers makes the notion of reducing a variety modulo a prime precise.

Intuitively, we would like to say X has good reduction at $p \in S$ if the special fiber $\mathcal{X}_p$ of a model $\mathcal{X} \to S$ is smooth. The fiber $\mathcal{X}_p$ is the same as the one obtained by forming the model $\mathcal{X} \times_S \operatorname{Spec}(R_p) \to \operatorname{Spec}(R_p)$ for X over the localized ring R_p and then taking the unique special fiber. This leads to the following definition.

Definition 1.3.2. *We say a smooth projective variety X over K has good reduction at the closed point $p \in S$, if there exists a projective model of X over $\operatorname{Spec}(R_p)$, whose special fiber is smooth. We say X has bad reduction at p if no such model exists.*

1.3.1 The Minimal Proper Regular Model

Models of varieties are not unique. There are also many models that do not provide relevant information about the variety. If we add some assumptions on the structure of a model, we can give some interesting results. In this section we restrict our attention to smooth projective curves C.

Proposition 1.3.3 ([Silverman, 57, Corollary Iv.4.4])**.** *Let K, R and S be as before. Let C/K be a smooth projective curve and let $\mathcal{C} \to S$ be a model for C, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the largest subscheme of $\mathcal{C}$ such that the map $\mathcal{C}^0 \to S$ is a smooth morphism.*

If $\mathcal{C}$ is proper over S, then every K-rational point $\operatorname{Spec}(K) \to C$ of C extends to a section $S \to \mathcal{C}$. In other words, $C(K) = \mathcal{C}(R)$. If in addition $\mathcal{C}$ is regular, then

$$C(K) = \mathcal{C}(R) = \mathcal{C}^0(R).$$

Note that all *projective* models (meaning all fibers of the model are projective varieties) are proper.

Now the question is if such proper, regular models actually exist. For smooth projective curves they do.

Theorem 1.3.4. *Let K, R and S be as before. Let C/K be a smooth projective curve of genus g.*

(a) There exists a proper, regular model $\mathcal{C} \to S$ for C.

(b) Assume that $g \geq 1$. Then there exists a proper, regular model $\mathcal{C}^{min} \to S$ for C, with the following minimality property:

Let $\mathcal{C} \to S$ be any other proper regular model for C. Then the R-birational map

$$\mathcal{C} \dashrightarrow \mathcal{C}^{min},$$

induced from the isomorphism of the generic fibers of $\mathcal{C}$ and $\mathcal{C}^{min}$, is an R-morphism. We call $\mathcal{C}^{min} \to S$ the minimal proper regular (mpr) model *for C. It is unique up to unique R-isomorphism.*

The first result is due to Abhyankar and Lipman, and the second due to Lichtenbaum and Shafarevich. Proofs are provided by [Artin, 2] and [Shafarevich, 54, Lectures 6-8].

For (hyper)elliptic curves C/K we can easily construct an explicit model of the curve via the Weierstrass equation

$$C : y^2 = F(x),$$

with $F \in R[x]$, by taking the closed subscheme of $\mathbb{P}^2_R$ defined by the equation. We call this model the Weierstrass model $\mathcal{W}$ for C. This model is proper, as it is projective, but will in general not be regular. Singular points on special fibers will generally result in singular points of $\mathcal{W}$. Sections coming from points $P \in C(K)$ may go through these singular points. If we take the largest smooth subscheme $\mathcal{W}^0$ of $\mathcal{W}$, we get a regular model for C, but we lose properness (in general), so $\mathcal{W}^0(R) \neq C(K)$.

Suppose now R is a discrete valuation ring, and K is its field of fractions. We say a Weierstrass curve for C/K as above is *minimal*, if the valuation of its discriminant is minimized, subject to the condition that the coefficients are integral. The resulting model is then called the *minimal Weierstrass model*.

1.3.2 The Néron Model

When considering models for abelian varieties, another property we can ask for is if the group law on the abelian variety can be extended to the model.

Definition 1.3.5. *Let A/K be an abelian variety. A model $\mathcal{N} \to S$ for A is called a Néron model for A if it is smooth, separated and of finite type and it satisfies the Néron Mapping Property:*

Let $\mathcal{X}/R$ be a smooth R-scheme with generic fiber X/K. Then every morphism $X \to A$ over K can be extended uniquely to a morphism $\mathcal{X} \to \mathcal{N}$.

In other words, the natural map

$$\mathrm{Hom}_S(\mathcal{X},\mathcal{N}) \to \mathrm{Hom}_K(X,A)$$

given by extending the base from S to K is a bijection.

The Néron model is a group scheme. This essentially means there are S-morphisms that constitute a group structure, see [Silverman, 57, Section IV.3] for a full definition. The group structure arises from the Néron mapping property: the addition map $A \times A \to A$ extends uniquely to a morphism $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$.

The group scheme $\mathcal{N}$ comes equipped with an identity element $\sigma_0 \in \mathcal{N}(R)$. This identity element intersects each special fibers $\mathcal{N}_p$ in a point $\sigma_0(p)$, which lies in one of the connected components $\mathcal{N}_p^0$, which we call the identity component of $\mathcal{N}_p$. This component is naturally a subgroup of the special fiber $\mathcal{N}_p$. The quotient $\Phi_p^{\mathcal{N}} = \Phi_p = \mathcal{N}_p/\mathcal{N}_p^0$ forms a group, called *the group of connected components of the special fiber* $\mathcal{N}_p$ or simply the *component group*. This group is crucial in the theory of canonical local heights, as we will see in Theorem 2.5.1.

Due to the Néron Mapping Property, every point $P \in A(K)$ defines a section $\overline{P} : S \to \mathcal{N}$. This section maps the point $p \in S$ to one of the elements of Φ_v. We call the map sending a point $P \in A(K)$ to this element of Φ_v the *specialization map*, denoted

$$\mathrm{sp} : A(K) \to \Phi_v.$$

Theorem 1.3.6 (Existence of Néron models [Néron, 46], [Bosch, Lütkebohmert, and Raynaud, 10, Theorem 1.4.3])**.** *Every abelian variety over K admits a Néron model. It is unique up to unique isomorphism.*

Another (non-trivial!) fact proven in [Néron, 46] and [Bosch, Lütkebohmert, and Raynaud, 10, Corollary 7.2.2] is that if R is a discrete valuation ring with valuation v, then the Néron model over the completion R_v can be calculated by base changing the Néron model over R.

Remark 1.3.7. In Chapters 3 and 4 we will mostly be in the situation that we want to calculate properties of the Néron model $\mathcal{N}$ over $S = \mathrm{Spec}(\mathcal{O}_K)$, with $\mathcal{O}_K$ the ring of integers of a number field K. We previously mentioned that the special fiber $\mathcal{N}_p$ at a non-zero prime ideal $p \subseteq \mathcal{O}_K$ can be calculated by base changing to the localized ring R_p and taking the (unique) special fiber. Now R_p is a discrete valuation ring with valuation v, that corresponds to a place on K. The fact above now says we can base change once again to the completion $(R_p)_v$, which is the valuation ring of the completion K_v at the absolute value. The special fiber $\mathcal{N}_p$ is now the same as the special fiber of the Néron model over $(R_p)_v$.

For this reason, we will generally work over either a general discrete valuation ring R, or specifically over the valuation ring R_v of the completion K_v of a number field K at an absolute value v.

For general abelian varieties it can be very hard to deduce what the Néron model looks like. For the Jacobian of smooth projective curve of genus $g \geq 1$, we will be able to describe the component group Φ_p in Section 1.3.5. For elliptic curves we can make the entire Néron model explicit.

Theorem 1.3.8. *Let E/K be an elliptic curve and let $\mathcal{E} \to S$ be an mpr model for E. Let $\mathcal{E}^0$ be the largest subscheme of $\mathcal{E}$ that is smooth over R. Then $\mathcal{E}^0$ is a Néron model for E/K.*

That is, once we know what the mpr model of E/K looks like (which is unique up to unique isomorphism), we can remove all components that have multiplicity ≥ 2 and all singular points on the remaining components. There are 10 types of mpr models of elliptic curves, which are classified by Néron [Néron, 46] and Kodaira [Kodaira, 29]. We will encounter and use this classification in Section 3.2.

The identity component $\mathcal{N}_v^0$ of the Néron model is especially easy to describe for elliptic curves.

Proposition 1.3.9. *Let $\mathcal{W}^0$ be the largest subscheme of the minimal Weierstrass model for E/K that is smooth over R. Then $\mathcal{W}_v^0 \cong \mathcal{N}_v^0$. If $\mathcal{W}$ is already smooth over R, then $\mathcal{W}$ is a Néron model for E/K.*

The first assertion follows from Tate's algorithm, which we describe in Section 3.2. Both assertions are proven in [Silverman, 56, Chapter IV].

1.3.3 Intersections

We saw in previous sections that to understand the Néron model of an elliptic curve, it is important to study its mpr model. We are especially interested in the intersections of the components of the special fibers. In the case of Jacobians J of curves of genus $g > 1$, these intersections completely dictate what we can say about the Néron model of J. In this section we will briefly introduce the relevant properties of intersections that we will use in Sections 1.3.4, 1.3.5 and 2.5.

In this section R denotes a discrete valuation ring, with maximal ideal p, $K = \mathrm{Frac}(R)$ and $S = \mathrm{Spec}(R)$. See Remark 1.3.7 for why this suffices for the general case.

There are two types of irreducible divisors on a regular, proper model $\mathcal{C}$ over R of a curve C/K. A irreducible divisor on $\mathcal{C}$ can be completely contained in the special fiber $\mathcal{C}_p$, in which case it is a component of $\mathcal{C}_p$. We call such a divisor a *irreducible fibral divisor*. A linear combination of irreducible fibral divisors is called a fibral divisor. The group of fibral divisors is denoted $\mathrm{Div}_p(\mathcal{C}) \subseteq \mathrm{Div}(\mathcal{C})$. If an irreducible divisor is not a fibral divisor, we call it *horizontal*.

There now is a unique bilinear, symmetric intersection pairing

$$\mathrm{Div}(\mathcal{C}) \times \mathrm{Div}_p(\mathcal{C}) \to \mathbb{Z}, \quad (D, F) \mapsto D \cdot F,$$

that satisfies $D \cdot F = \sum_{x \in D \cap F} (D \cdot F)_x$, whenever $D \in \mathrm{Div}(\mathcal{C})$ and $F \in \mathrm{Div}_p(\mathcal{C})$ are distinct, and satisfies $\mathrm{div}(f) \cdot F = 0$, for all $f \in K(\mathcal{C})^\times$ and $F \in \mathrm{Div}_p(\mathcal{C})$. Here $(D \cdot F)_x$ denotes the local intersection index of D and F as defined in [Silverman, 57, Definition IV.7.1], which intuitively denotes the order of intersection at x of the two divisors.

This pairing has the following properties.

Proposition 1.3.10 ([Silverman, 57, Proposition 7.3])**.** *Let $\mathcal{C}$ be a regular, proper model over R of a curve C/K.*

(a) The special fiber $\mathcal{C}_p$ is connected.

(b) Let $F \in \mathrm{Div}_p(\mathcal{C})$ be a fibral divisor. Then $F^2 \leq 0$, and

$$F^2 = 0 \iff F \cdot F' = 0, \quad \forall F' \in \mathrm{Div}_p(\mathcal{C}) \iff F = n\mathcal{C}_v, \quad \text{for some } n \in \mathbb{Z}.$$

Proposition 1.3.11 ([Silverman, 57, Proposition 7.4])**.** *Assume the residue field $k = R/p$ is algebraically closed. Let $\mathcal{C}$ be a regular, proper model over R of a curve C/K.*

(a) (Adjunction formula) *There is a divisor $K_{\mathcal{C}} \in \mathrm{Div}(\mathcal{C})$, called a canonical divisor on $\mathcal{C}$ such that*

$$F^2 + K_{\mathcal{C}} \cdot F = 2p_a(F) - 2,$$

for every irreducible fibral divisor $F \in \mathrm{Div}_v(\mathcal{C})$.

(b) $p_a(\mathcal{C}_v) = p_a(C) = g(C)$.

Consider an abelian variety A defined over the field of fractions K of a discrete valuation ring R, with Néron model $\mathcal{N}$ over R. By the Néron Mapping Property, every rational point $P \in A(K)$ comes from a unique section $\overline{P} : S \to \mathcal{N}$ to the Néron model of A. Let Y be a prime divisor on A and denote by $\overline{Y}$ the Zariski closure of Y in $\mathcal{N}$, which is a prime divisor on $\mathcal{N}$. The divisor group on S is cyclic, generated by (p), for $p \in S$ the unique closed point. Assuming that Y does not contain the image of $\overline{P}$, we can write

$$\overline{P}^* Y = i_p(\overline{Y}, \overline{P})(p),$$

for some integer $i_p(\overline{Y}, \overline{P})$. In fact, this integer is uniquely determined by Y and P, so we can simply write $i_p(Y, P)$.

Definition 1.3.12. *Let $D \in \mathrm{Div}(A)$ be a divisor and write $D = \sum_Y n_Y Y$, with Y all prime divisors. Let $P \in A(K) \backslash |D|$ a rational point. The* intersection multiplicity *of D and P at p is the integer*

$$i(D, P) = i_p(D, P) = \sum_Y n_Y i_p(Y, P) \in \mathbb{Z}.$$

We will see in Theorem 2.5.1 that the intersection multiplicity defines a local height function, and is almost equal to a canonical local height function.

Remark 1.3.13. If we have a description of the closure $\overline{D}$ of a divisor D as Cartier divisor on $\mathcal{N}$, we can easily calculate $i(D, P)$ for any $P \in A(K)$. Suppose that $\overline{D}$ is represented by a pair (U, f), where U is open on $\mathcal{N}$ and $f \in K(\mathcal{N}) = K(A)$. Then for any $P \in A(K)$, such that f is defined at P and $f(P) \neq 0, \infty$, we have $i(D, P) = (v \circ f)(P)$, which is independent of the choice of pair, see [Lang, 32, Theorem 11.5.1].

1.3.4 Semistable Reduction of Curves

Fibers of (mpr) models of curves can take on many different forms. In this section we introduce the notion of semistable curves, and consider the case that the special fiber of a mpr model of a curve is a semistable curve. For calculations it is often helpful to assume this is the case. By the Semistable Reduction Theorem 1.3.15 this assumption is, in some sense, harmless. We then discuss how we can classify the possible semistable curves that can occur as a special fiber. We denote by C/K a

projective curve, of genus $g \geq 1$, and by $\mathcal{C}$ the mpr model of C. The notation of K, R, and S is as introduced in Section 1.3.

Definition 1.3.14. *We say a connected projective curve is semistable if all singularities are nodal and all of its rational components intersect other components at least two times. We say a smooth curve C (of genus $g \geq 1$) has semistable reduction at $p \in S$ if the special fiber $\mathcal{C}_p$ at p of the mpr model $\mathcal{C}$ for C is a semistable curve.*

Semistability is called so because of the following theorem, see Artin and Winters [Artin and Winters, 3].

Theorem 1.3.15 (Semistable Reduction Theorem)**.** *Let K be a number field and suppose C is defined over K. Then there exists a finite field extension L/K, such that C has semistable reduction at all places of L.*

By definition, the intersections of semistable curves are very well-behaved. This relates semistable curves to graphs.

By a graph, we mean a pair $\Gamma = (V, E)$, where V is a set of vertices and E is a set of edges. There is an (undirected) edge assignment map $\iota : E \to \{\{v, w\} : v, w \in V\}$, that associate an edge with the two vertices that it connects. We will allow two vertices to be connected by more than one edge (which in literature is often called a multigraph). The graph is finite if V and E are finite. A graph can have vertex (resp. edge) weights, by which we mean that there is a weight map $w : V$ (resp. E) $\to \mathbb{R}$.

Definition 1.3.16. *Let C be a semistable curve over K. The dual graph of C is the finite graph Γ with weighted vertices, constructed as follows:*

- *Γ has one vertex v for every irreducible component $\mathcal{C}_p^i$ of C,*
- *Γ has one edge e between two vertices for each intersection of the two corresponding components,*
- *the weight function $w : V \to \mathbb{Z}_{\geq 0}$ is given by setting $w(v)$ the genus $g(\mathcal{C}_p^i)$ of the (normalization of the) corresponding component.*

If C is a smooth curve with semistable reduction over a discrete valuation ring R, we call the dual graph Γ_p of the special fiber $\mathcal{C}_p$ the reduction graph of C.

The possible intersections on the special fiber are related to the genus of the curve, by the Adjunction Formula (Theorem 1.3.11(a)). This also has effect on the reduction graph. This severely restricts the shape of a reduction graph of a curve of a given genus, and will help us later in Theorem 3.2.1 and 4.3.1 to classify all possible reduction types of such curves.

Definition 1.3.17. *The genus of a finite graph $\Gamma = (V, E, w)$ with weighted vertices is*

$$g(\Gamma) = |E| - |V| + 1 + ws(\Gamma),$$

where $ws(\Gamma) = \sum_{v \in V} w(v)$ is the total weight of the graph.

Proposition 1.3.18. *The reduction graph of a curve C of genus g has genus g.*

Proof. Let $\Gamma_p = (V, E)$ be the reduction graph of C. Write $\mathcal{C}_p = \sum_{i=1}^{n} m_i \mathcal{C}_i$ for the decomposition of the special fiber of $\mathcal{C}$ into its irreducible components, where $n = |V|$.

As C has semistable reduction, all multiplicities m_i are equal to 1.

From the adjunction formula for the fibral divisor $F = \mathcal{C}_v$ we get

$$\mathcal{C}_p^2 + K_{\mathcal{C}} \cdot \mathcal{C}_p = 2p_a(\mathcal{C}_p) - 2.$$

Now we use $p_a(\mathcal{C}_p) = p_a(C) = g(C) = g$, $\mathcal{C}_p \cdot \mathcal{C}_p = 0$ from Theorem 1.3.11 and linearity of the intersection pairing to get

$$2g - 2 = K_{\mathcal{C}} \cdot \mathcal{C}_p = \sum_{i=1}^{n} K_{\mathcal{C}} \cdot \mathcal{C}_i = \sum_{i=1}^{n} (2p_a(\mathcal{C}_i) - 2 - \mathcal{C}_i^2).$$

The last step follows again from the adjunction formula.

For a irreducible component $\mathcal{C}_i$ we define the incident components multiset $I(\mathcal{C}_i)$ with every component $\mathcal{C}_j$, $j \neq i$ taken $|\mathcal{C}_i \cap \mathcal{C}_j|$ times and $\mathcal{C}_i$ itself taken $2(p_a(\mathcal{C}_i) - g(\mathcal{C}_i))$. That is, every node on $\mathcal{C}_i$ contributes $\mathcal{C}_i$ to $I(\mathcal{C}_i)$ twice. Note that $\#I(\mathcal{C}_i)$ is exactly equal to the valency of the vertex in the reduction graph corresponding to $\mathcal{C}_i$. Therefore also $\sum_{i=1}^{n} \#I(\mathcal{C}_i) = 2|E_\Gamma|$.

To calculate $\mathcal{C}_i^2$ we note that for any i, $\mathcal{C}_i \cdot \mathcal{C}_p = \mathcal{C}_i \cdot \sum_{j=1}^{n} \mathcal{C}_j = 0$, so that

$$\mathcal{C}_i^2 = -\sum_{j=1, j\neq i}^{n} \mathcal{C}_i \cdot \mathcal{C}_j = -\sum_{j=1, j\neq i}^{n} |\mathcal{C}_i \cap \mathcal{C}_j| = 2(p_a(\mathcal{C}_i) - g(\mathcal{C}_i)) - \#I(\mathcal{C}_i),$$

as semistability implies all components intersect transversally. This yields the formula

$$\begin{aligned} 2g - 2 &= \sum_{i=1}^{n} \Big(2p_a(\mathcal{C}_i) - 2 - 2(p_a(\mathcal{C}_i) - g(\mathcal{C}_i)) + \#I(\mathcal{C}_i)\Big) = \sum_{i=1}^{n} (2g(\mathcal{C}_i) - 2 + \#I(\mathcal{C}_i)) \\ &= 2\sum_{i=1}^{n} g(\mathcal{C}_i) - 2n + \sum_{i=1}^{n} \#I(\mathcal{C}_i) = 2ws(\Gamma_p) - 2|V| + 2|E| = 2g(\Gamma_p) - 2, \end{aligned}$$

so that $g = g(\Gamma_p)$, as wanted. □

1.3.5 The Néron model of a Jacobian

In the case that a curve C has semistable reduction, we can relate the component group of the Néron model of $\mathrm{Jac}(C)$ to the intersections on the special fiber of the mpr model of the curve.

Let C be a curve with mpr model $\mathcal{C}$ and assume the special fiber $\mathcal{C}_p$ is semistable. The intersections of the irreducible components $\{\mathcal{C}_i\}_{i\in I}$ of $\mathcal{C}_p$ can be encapsulated into a so-called intersection matrix A, defined by

$$A_{ij} = \mathcal{C}_i \cdot \mathcal{C}_j.$$

It must come as no surprise that this matrix can be calculated directly from the reduction graph Γ of C, as the graph is defined by the intersections of the irreducible components. Indeed A is equal to (-1) times the *Laplacian* matrix L of Γ, defined by

$L = D - A$, where the valency matrix D and adjacency matrix A are defined by

$$D_{ij} = \begin{cases} \mathrm{val}(v_i), & i = j \\ 0, & \text{else}, \end{cases} \qquad A_{ij} = \begin{cases} 2\#\{\text{self-loops at } v_i\}, & i = j, \\ \#\{\text{edges between } v_i \text{ and } v_j\}, & \text{else}. \end{cases}$$

Note that Proposition 1.3.10(b) implies that the sum of every row is 0.

Theorem 1.3.19 ([Raynaud, 50, Proposition 8.1.2]). *Let C be a curve of genus $g \geq 1$, with mpr model $\mathcal{C}$. Suppose C has semistable reduction at v, and let $\{\mathcal{C}_i\}_{i\in I}$ be the family of irreducible components of the special fiber $\mathcal{C}_p$. Let $A = (\mathcal{C}_i \cdot \mathcal{C}_j)_{i,j\in I}$ be the intersection matrix of the special fiber $\mathcal{C}_p$. Let J be the Jacobian of C. Define the maps*

$$\mathbb{Z}^I \xrightarrow{\alpha} \mathbb{Z}^I \xrightarrow{\beta} \mathbb{Z},$$

with α given by the matrix A and $\beta(a_1, \ldots, a_r) = \sum a_i$. Then the group of connected components $\Phi_p = \mathcal{J}_p/\mathcal{J}_p^0$ of the Néron model $\mathcal{J}$ of J is canonically isomorphic to the quotient $\ker\beta/\operatorname{im}\alpha$.

For the full proof, see [Bosch, Lütkebohmert, and Raynaud, 10, Theorem 9.6.1]. The theorem there is stated in greater generality, where semistability is not assumed. In our results in Chapter 4, we do only calculations in the case of semistable reduction, so our formulation of this theorem does not restrict us.

1.4 Skeleta of Jacobians

Reduction graphs are not only useful to gain insight into the structure of special fibers of Néron models. These graphs can be interpreted as so-called *tropical curves*. The area of tropical geometry is very rich and is closely related to classical algebraic geometry. Many fundamental results in algebraic geometry have a tropical analogue. Giving a full introduction to tropical geometry is beyond the scope of this thesis. Instead we briefly discuss only the specific topics that we need to give our desired results in later chapters. In this section, we denote by R a complete discrete valuation ring, with maximal ideal p and $K = \mathrm{Frac}(R)$.

Suppose C is a curve of genus ≥ 1 with semistable reduction at p. Let J be the Jacobian of C, with Néron model $\mathcal{J}$ over R. Using Raynaud's theory of non-archimedean uniformization (as explained e.g. in [Berkovich, 9]), we can associate with J a real torus $\Sigma(J)$, which is called the *canonical skeleton* of J. This skeleton is a tropical variety. The component group $\Phi_p^{\mathcal{J}}$ of J is now naturally embedded in $\Sigma(J)$, see [Faltings and Chai, 22, Corollary III.8.2]. In this section we construct the Jacobian of a graph, and show that the Jacobian of the reduction graph Γ_p is exactly the skeleton of J.

Let $\Gamma = (V, E)$ be a finite graph. We give the edges arbitrary orientation and denote the head and tail vertex of e by e^+ and e^-. Let A be either $\mathbb{Z}$ or R and denote by $H_1(\Gamma, A)$ the first homology group, with coefficients in A. We can construct this group by taking $C_0(\Gamma, A)$ and $C_1(\Gamma, A)$ the free A-module generated by respectively the vertex set V and the edge set E. Then we define the operator $d^* : C_1(\Gamma, \mathbb{R}) \to C_0(\Gamma, \mathbb{R})$ by

$$(d^*\alpha)(x) = \sum_{e\in E, e^+=x} \alpha(e) - \sum_{e\in E, e^-=vx} \alpha(e),$$

and let $H_1(\Gamma, A) \cong \ker(d^*)$. Note that the rank of this group is equal to the first Betti number of the graph, $b_1(\Gamma) = |E| - |V| + 1$.

We can identify chains $\alpha \in C_1(\Gamma, \mathbb{R})$ with a vector $\alpha \in \mathbb{R}^{|E|}$. Then we have an inner product on $C_1(\Gamma, A)$ via

$$\langle \alpha, \beta \rangle = \sum_{e \in E} \alpha(e)\beta(e) = \alpha^T \beta, \quad \alpha_1, \alpha_2 \in C_1(\Gamma, A). \tag{1.16}$$

The inner product restricts to a symmetric, positive pairing $\langle \cdot, \cdot \rangle : H_1(\Gamma, A) \to A$ for both $A = \mathbb{Z}$ and $A = \mathbb{R}$. We can now embed $H_1(\Gamma, \mathbb{Z})$ into both $H_1(\Gamma, \mathbb{Z})^* := \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{Z})$ and $H_1(\Gamma, \mathbb{R})^* := \mathrm{Hom}(H_1(\Gamma, \mathbb{R}), \mathbb{R})$ via

$$\mu : \alpha \mapsto \langle \alpha, \cdot \rangle.$$

The group $\mu(H_1(\Gamma, \mathbb{Z}))$ sits inside $H_1(\Gamma, \mathbb{R})^*$ as lattice.

Definition 1.4.1. *The Jacobian of a graph Γ is defined to be the real torus*

$$\mathrm{Jac}(\Gamma) = H_1(\Gamma, \mathbb{R})^* / \mu(H_1(\Gamma, \mathbb{Z})).$$

Now let $\alpha_1, \ldots, \alpha_n$ be a basis for $H_1(\Gamma, \mathbb{Z})$. Then it also defines a basis for $H_1(\Gamma, \mathbb{R})$, via which we can identify $H_1(\Gamma, \mathbb{R})$ with $\mathbb{R}^n$. Then also $H_1(\Gamma, \mathbb{R})^*$ can be identified with $\mathbb{R}^n$ via the dual basis $\alpha^1, \ldots, \alpha^n$ that satisfies $\alpha^i(\alpha_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta.

We can define the Gram matrix

$$\Omega = \Big(\langle \alpha_i, \alpha_j \rangle \Big)_{1 \le i,j \le n}$$

of the basis $\alpha_1, \ldots, \alpha_n$ with respect the inner product on $H_1(\Gamma, \mathbb{R})$. This is exactly the Gram matrix of the lattice $\mu(H_1(\Gamma, \mathbb{Z}))$. Via the identification above, we get that $\mathrm{Jac}(\Gamma)$ is isomorphic to $\mathbb{R}^n / \Omega\mathbb{Z}^n = \mathrm{Coker}(\Omega : \mathbb{Z}^n \to \mathbb{R}^n)$. The spanning vectors of the lattice thus are $v_i = (\langle \alpha_i, \alpha_j \rangle)_{1 \le j \le n}$, which are the columns of the Gram matrix. The induced inner product on $H_1(\Gamma, \mathbb{R}) \cong \mathbb{R}^n$ is then given by $\langle v, w \rangle = v^T \Omega w$, so that the induced inner product on $H_1(\Gamma, \mathbb{R})^* \cong \mathbb{R}^n$ is given by

$$\langle v, w \rangle = \langle v, w \rangle_\Omega = v^T \Omega^{-1} w.$$

Example 1.4.2. Suppose Γ is a cycle graph with n vertices, $x_n = x_0, \ldots, x_{n-1}$ ordered clockwise and n edges. We order the edges $e_1, \ldots, e_n$, such that $\iota(e_i) = (x_{i-1}, x_i)$. The homology groups satisfy $H_1(\Gamma, K) \cong K$ for $K = \mathbb{Z}, \mathbb{R}$, both generated by the cycle $\alpha = [e_1 + \cdots + e_n]$. That is, the torus $\mathrm{Jac}(\Gamma)$ is 1-dimensional, and hence a circle. We can write the cycle α as vector by $\alpha = (1, 1, \ldots, 1)^T$, so that the Gram matrix, and its inverse, are

$$\Omega = \big(\langle \alpha, \alpha \rangle \big) = (n), \quad \Omega^{-1} = (\frac{1}{n}).$$

The lattice in $H_1(\Gamma, \mathbb{R})^* \cong \mathbb{R}$ is now spanned by the vector $v = (n) \in \mathbb{R}$, so that $\mathrm{Jac}(\Gamma) \cong \mathbb{R}/n\mathbb{Z}$. The inner product on $H_1(\Gamma, \mathbb{R})^* \cong \mathbb{R}$ is given by $\langle v, w \rangle = v^T \Omega^{-1} w$. That is, for $v = (a), w = (b)$, with $a, b \in \mathbb{R}$, then $\langle v, w \rangle = \frac{ab}{n}$.

As we alluded to in the beginning of this section, the Jacobian of a reduction graph is a tropical variety.

Theorem 1.4.3. *Let C be a curve with semistable reduction graph Γ_p and let J be its Jacobian. There is a canonical isomorphism*

$$\mathrm{Jac}(\Gamma_p) \cong \Sigma(J).$$

Moreover, the isomorphism is compatible with the natural principal polarization on both sides.

As Γ_p is also the skeleton of the curve C, we can summarize this theorem as "The skeleton of the Jacobian is the Jacobian of the skeleton".

The inclusion $\Phi_p^{\mathcal{J}} \hookrightarrow \Sigma(J)$ can now also be made explicit in terms of the homology of the graph Γ.

Corollary 1.4.4. *Let C be a curve with semistable reduction graph Γ and let J be its Jacobian, with Néron model $\mathcal{J}$. Denote by $\Phi_p = \Phi_p^{\mathcal{J}}$ the component group of the special fiber $\mathcal{J}_p$. Then there is an isomorphism*

$$\Phi_p \cong H_1(\Gamma, \mathbb{Z})^* / \mu(H_1(\Gamma, \mathbb{Z})) \subseteq \mathrm{Jac}(\Gamma).$$

This is in essence a reformulation of Raynaud's formulation of the component group in Theorem 1.3.19. This result can also be found in [Baker and Rabinoff, 8, Corollary 2.4] and [Dokchitser et al., 19, Lemma 2.22]. If we identify $H_1(\Gamma, \mathbb{Z})^* \cong \mathbb{Z}^2$ via the dual basis of $\alpha_1, \ldots, \alpha_n$, we get that $\Phi_v \cong \mathbb{Z}^n / \Omega\mathbb{Z}^n = \mathrm{Coker}(\Omega : \mathbb{Z}^n \to \mathbb{Z}^n)$.

Now that we have a notion of Jacobians of graphs, we can also ask if there is an analogue of the Abel-Jacobi map, that maps a graph into its Jacobian. This is the case. Let x be a fixed vertex of Γ. We construct the tropical Abel-Jacobi map $\Phi_x : \Gamma \to \mathrm{Jac}(\Gamma)$ as follows. Let y be some other vertex, and let P be a path in Γ, from x to y. Then P defines a chain δ_P by setting $\delta_P(e)$ the amount of times e appears in P with the correct orientation, minus the amount of times it appears with the opposite orientation. This chain δ_P then defines an element $\mu(\delta_P) = \langle \delta_P, \cdot \rangle \in H_1(\Gamma, \mathbb{R})^*$. We then define

$$\Phi_x(y) := [\mu(\delta_P)] \in \mathrm{Jac}(\Gamma).$$

Remark 1.4.5. If we identify $H_1(\Gamma, \mathbb{R})^*$ via the dual basis of $\alpha_1, \ldots, \alpha_n$, we see that $\Phi_x(y)$ depends only on the vector $(\langle \delta_P, \alpha_i \rangle)_{1 \leq i \leq n} \in \mathbb{R}^n$. That is, if x, y are connected by a path P and x', y' by a path Q that satisfy $\langle \delta_P, \alpha_i \rangle = \langle \delta_Q, \alpha_i \rangle$, for all $1 \leq i \leq n$, then $\Phi_x(y) = \Phi_{x'}(y')$. Note in particular that $\Phi_x(y) = \Phi_y(x)$. We use this e.g. in Section 4.4 to 'shift' paths, so they start at a fixed vertex.

It is proven in [Baker and Faber, 7, Section 4] that this map depends on x, but not on the chosen path. Also the tropical Abel-Jacobi map collapses any segment that is not part of a cycle, and maps cycles to straight segments.

This tropical Abel-Jacobi map interacts well with the regular Abel-Jacobi map. Let $v \in M_K$ be a non-archimedean place. Let $P \in C(K_v)$, and let x_P be the vertex of the reduction graph Γ_v, corresponding to the component that $\mathrm{sp}(P)$ lies in. Then we

have that the square

$$\begin{array}{ccc} C(K_v) & \xrightarrow{\Phi_p} & \mathrm{Jac}(C)(K_v) \\ \downarrow \mathrm{sp} & & \downarrow \mathrm{sp} \\ \Gamma_v & \xrightarrow{\Phi_{x_P}} & \mathrm{Jac}(\Gamma_v) \end{array} \tag{1.17}$$

commutes, see [Baker and Rabinoff, 8, Theorem 1.3]. This has some favorable implications for us. We can represent any point $P \in \mathrm{Jac}(C)(K_v)$ on the Jacobian by $P = [(P_1) - (P_2)] = \Phi_{P_2}(P_1)$, for $P_1, P_2 \in C(K_v)$. Let x_1 and x_2 be the components of Γ_v that P_1 and P_2 respectively specialize to. Then

$$\mathrm{sp}(P) = \Phi_{x_2}(x_1) \in \Phi_p \subseteq \mathrm{Jac}(\Gamma_v). \tag{1.18}$$

We finish this section with defining a function on $\mathrm{Jac}(\Gamma)$, that will become very important when we consider local height functions in Section 2.5.

Definition 1.4.6 ([de Jong and Shokrieh, 18, Equation (4.4)]). *We define the normalized tropical Riemann theta function* $\|\Psi\| : H_1(\Gamma, \mathbb{R})^* \to \mathbb{R}$ *by*

$$\|\Psi\|(\nu) = \frac{1}{2} \min_{u \in \mu(H_1(\Gamma,\mathbb{Z}))} \langle \nu + u, \nu + u \rangle.$$

The inner product gives a notion of distance between elements in $H_1(\Gamma, \mathbb{R})^*$. The normalized tropical Riemann theta function thus is equal to half the distance between a vector ν and the nearest lattice point, squared. It is clear to see that $\|\Psi\|$ descends along the lattice Y to give a well-defined map $\|\Psi\| : \mathrm{Jac}(\Gamma) \to \mathbb{R}$.

It is useful to give a fundamental domain for Σ, on which the function $\|\Psi\|$ is easily evaluated. To do this, we define the Voronoi cell of a lattice point $u \in \mu(H_1(\Gamma, \mathbb{Z}))$, by

$$\mathrm{Vor}(u) = \{\nu \in H_1(\Gamma, \mathbb{R})^* : \langle \nu - u, \nu - u \rangle \leq \langle \nu - u', \nu - u' \rangle, \forall u' \in \mu(H_1(\Gamma, \mathbb{Z}))\}, \tag{1.19}$$

as the set of points that are at least as close to u as to any other lattice point. Clearly, each Voronoi cell contains at least one representative for each point in the torus Σ. On the Voronoi cell $\mathrm{Vor}(0)$, where the nearest lattice point is 0, the map $\|\Psi\|$ is simply given by

$$\|\Psi\|(\nu) = \frac{1}{2}\langle \nu, \nu \rangle, \quad \forall \nu \in \mathrm{Vor}(0).$$

Chapter 2

Heights

The height of a point on a variety is a great measure of the arithmetic complexity of that point. In this chapter we build up the theory of (canonical) heights on (abelian) varieties and show their decomposition into (canonical) local heights.

In Section 2.1 we review the theory of absolute values on number fields. We will later decompose heights into local factors, one for each of these absolute values. Section 2.2 introduces a basic notion of heights on $\mathbb{P}^n$ which is then extended to projective varieties. These heights are uniquely defined only up to $\mathcal{O}(1)$ and in Section 2.3 we will show that we can define a *canonical* height on abelian varieties that has desirable properties. In Section 2.4 we show that global heights, and in particular the canonical height, can be decomposed into local heights, which will be the basis for the remaining chapters.

2.1 Absolute values

In order to give a local decomposition of the global canonical height, we need to define what we mean by 'local'.

Definition 2.1.1. *Let K be a field. An* absolute value *on K is a function*

$$|\cdot| : K \to [0, \infty),$$

satisfying:

(1) $|x| = 0$ *if and only if* $x = 0$,

(2) $|xy| = |x||y|$, *for all* $x, y \in K$;,

(3) $|x + y| \leq |x| + |y|$, *for all* $x, y \in K$.

An absolute value is said to be nonarchimedean *if it also satisfies* $|x+y| \leq \max\{|x|, |y|\}$, *for all* $x, y \in K$, *otherwise it is said to be* archimedean.

Every field has a trivial absolute value, which sends the zero element to 0 and all other elements to 1. Two absolute values $|\cdot|_1$, $|\cdot|_2$ on K are called *equivalent* if there is a constant $a \in (0, \infty)$, such that $|x|_2 = |x|_1^a$, for all $x \in K$. Then equivalent absolute values define the same topology on K. It is easily checked that this is an equivalence relation. An equivalence class of non-trivial absolute values on K is called a *place* of K.

Absolute values on $\mathbb{Q}$.
We can classify all non-trivial absolute values on $\mathbb{Q}$. First of all we have the standard archimedean absolute value, given by

$$|x| = |x|_\infty = \max\{-x, x\},$$

which is the restriction to $\mathbb{Q}$ of the standard absolute value on $\mathbb{R}$. Furthermore we get an absolute value, for every prime number $p \in \mathbb{Z}$, as follows:

Let $\alpha \in \mathbb{Q}$. Write $\alpha = p^n \frac{a}{b}$, with $\gcd(a, b) = 1$, $p \nmid ab$ and $n \in \mathbb{Z}$. Then we define the p-adic absolute value of α as

$$|\alpha|_p = |p^n \frac{a}{b}|_p = p^{-n}.$$

This absolute value is non-archimedean. If $n \geq m$, then

$$|p^n \frac{a}{b} + p^m \frac{c}{d}|_p = |p^m (p^{n-m} \frac{a}{b} + \frac{c}{d})|_p \leq p^{-m}.$$

Note that for integers $n \in \mathbb{Z}$, we get $|n|_p \leq 1$, for any prime $p \in \mathbb{Z}$.

Any other non-trivial absolute value is equivalent to one of the absolute values above, which is a classic result due to Ostrowski [Ostrowski, 49]. We denote the set of these normalized non-trivial absolute values by $M_\mathbb{Q}$.

Absolute values on number fields.
For K a number field, i.e. a finite field extension of $\mathbb{Q}$, we then define the set M_K to be the set of all absolute values on K, whose restriction to $\mathbb{Q}$ is one of the absolute values in $M_\mathbb{Q}$. This defines a uniform normalization of all absolute values on K. We denote by M_K^∞ and M_K^0 the sets of resp. the archimedean and non-archimedean absolute values in M_K.

The set M_K^∞ contains an absolute value for every embedding of K into $\mathbb{R}$ and for every pair of embeddings into $\mathbb{C}$ (conjugate embeddings define the same absolute value). For such an embedding $\rho : K \hookrightarrow \mathbb{C}$, we define $|\cdot|_\rho = |\rho(\cdot)|$, where $|\cdot|$ denotes the standard absolute value on $\mathbb{C}$.

There is exactly one non-archimedean absolute value in M_K^0 for every non-zero prime ideal $\mathfrak{p}$ in the ring of integers $\mathcal{O}_K$, denoted $|\cdot|_\mathfrak{p}$. This absolute value satisfies that for $\alpha = \frac{x}{y} \in K^\times$, with $x, y \in \mathcal{O}_K$, we have $|\alpha|_\mathfrak{p} = 1$ if $x, y \notin \mathfrak{p}$. Every place is represented by an absolute value in M_K, see [Conrad, 15] for a proof of this fact and a full description of all absolute values. By abuse of notation we may call an element $v \in M_K$ a place, even though strictly speaking it is an absolute value.

We write the absolute value corresponding to a place $v \in M_K$ as $|\cdot|_v$. We write K_v for the topological completion of K, with respect to $|\cdot|_v$. For example, $\mathbb{Q}_v = \mathbb{R}$ if v is the archimedean absolute value and $\mathbb{Q}_v = \mathbb{Q}_p$, the field of p-adic numbers, if v is the p-adic absolute value. The completion K_v has only one place, the one induced by v. This place can then be uniquely extended to $\overline{K_v}$. If we fix an embedding $\overline{K} \hookrightarrow \overline{K_v}$, we can extend places $v \in M_K$ to a place $\overline{v}$ on the algebraic closure $\overline{K}$.

Let L/K be an extension of number fields. We say that $w \in M_L$ *divides* $v \in M_K$ and write $w|v$ if the restriction of w to K is equal to v. We define the *local degree* of a place $v \in M_K$ on a number field K that divides $v_0 \in M_\mathbb{Q}$ to be $n_v := [K_v : \mathbb{Q}_{v_0}]$.

One of the fundamentals reasons that we can build a theory of heights over number fields is that the product formula holds.

Theorem 2.1.2 (Product formula)**.** *Let K be a number field and let $x \in K^\times$. Then*

$$\prod_{v \in M_K} |x|_v^{n_v} = 1.$$

Proof. We first prove the formula over $\mathbb{Q}$. Then $n_v = 1$ for all $v \in M_\mathbb{Q}$. Let $x \in \mathbb{Q}^\times$. Then we can write $x = \pm \prod_p p^{n_p}$ as factorization over all primes in $\mathbb{Z}$, with $n_p \in \mathbb{Z}$. Note that only finitely many n_p are non-zero, so that the product below is a finite. Then

$$\prod_{v \in M_\mathbb{Q}} |x|_v^{n_v} = |x|_\infty \cdot \prod_p |x|_p = \prod_p p^{n_p} \cdot \prod_p p^{-n_p} = 1.$$

Now let K be an arbitrary number field and let $x \in K^\times$. We use [Lang, 33, Corollary II.1.2] to get $\prod_{v \in M_K, v|v_0} |x|_v^{n_v} = |N_\mathbb{Q}^K(x)|_{v_0}$, for $v_0 \in M_\mathbb{Q}$ and $N_\mathbb{Q}^K : K \to \mathbb{Q}$ the norm of the extension $K/\mathbb{Q}$. We get

$$\prod_{v \in M_K} |x|_v^{n_v} = \prod_{v_0 \in M_\mathbb{Q}} \prod_{v \in M_K, v|v_0} |x|_v^{n_v} = \prod_{v_0 \in M_\mathbb{Q}} |N_\mathbb{Q}^K(x)|_{v_0} = 1,$$

which finishes the proof. □

Let $v \in M_K^0$ be a non-archimedean place. Then the negative logarithm of the absolute value $|\cdot|_v$, is a valuation

$$-\log|\cdot|_v : K_v^\times \to \mathbb{R}.$$

Let $R_v = \{x \in K_v : -\log|x|_v \geq 0\} \subseteq K_v$ be the valuation ring. Then $\mathfrak{m}_v = \{x \in K_v : -\log|x|_v > 0\}$ is the unique maximal ideal of R_v. Let Nv denote the order of the residue field $k_v = R_v/\mathfrak{m}_v$.

We can now normalize the valuation $-\log|\cdot|_v$ to a discrete valuation $v(\cdot) : K_v^\times \twoheadrightarrow \mathbb{Z}$ that is surjective, which is defined by

$$v(x) := -\frac{n_v}{\log Nv} \log|x|_v. \tag{2.1}$$

For $v \in M_K^\infty$ an archimedean place, we define $v(x) = -\log|x|_v$ (which is not a valuation on $K_v^\times$). We also define the constant

$$q_v = \begin{cases} \log Nv, & v \in M_K^0, \\ n_v, & v \in M_K^\infty, \end{cases} \tag{2.2}$$

so that $v(x) = -\frac{n_v}{q_v} \log|x|_v$. Note that $q_v = 1$ for $v \in M_K^\infty$ coming from a real embedding $K \hookrightarrow \mathbb{R}$ and $q_v = 2$ for $v \in M_K^\infty$ coming from a pair of conjugate, strictly complex embeddings $K \hookrightarrow \mathbb{C}$. Later in this chapter we will work with logarithmic heights and hence with the logarithms of $|\cdot|_v$. Then we can rewrite the product formula as:

$$\sum_{v \in M_K} q_v v(x) = 0. \tag{2.3}$$

2.2 Global Heights

From now on, K always denotes a number field. Now that we have defined absolute values on K, we are ready to define heights. We start this definition of heights for points in $\mathbb{P}^n(K)$ and extend this to arbitrary projective varieties. Intuitively, this height measures 'arithmetic complexity', rather than the 'size' of a point. For example, the numbers 1 and 0.999 in $\mathbb{Q}$ are of similar size, as the (standard) absolute value of their difference is small. However, to write $0.999 = \frac{999}{1000}$ as a fraction of two integers, we need a (rather large) denominator of 1000, so that its arithmetic complexity is much bigger than that of 1. This intuition is the base of the definitions in this section.

2.2.1 Global Heights on $\mathbb{P}^n$

We can give an intrinsic definition of heights of points in $\mathbb{P}^n(\mathbb{Q})$. Let $P \in \mathbb{P}^n(\mathbb{Q})$. Write $P = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{Q})$ with homogeneous coordinates that are chosen such that $x_i \in \mathbb{Z}$ are coprime integers, we define the *multiplicative height* of P by

$$H(P) = \max\{|x_0|, \ldots, |x_n|\} \in \mathbb{Z},$$

where $|\cdot|$ denotes the standard absolute value on $\mathbb{Z}$.

For arbitrary number fields K, the ring of integers $\mathcal{O}_K$ is not necessarily a principal ideal domain, so there is no analogous notion of coprime homogeneous coordinates. To give a well-defined definition for heights on K, we need to take the contributions from all places of K into account.

Definition 2.2.1. *Let K be a number field and let $P = (x_0 : \ldots : x_n) \in \mathbb{P}^n(K)$. We define the multiplicative height of P, relative to K, as*

$$H_K(P) = \prod_{v \in M_K} \max\{|x_0|_v^{n_v}, \ldots, |x_n|_v^{n_v}\}.$$

The logarithmic height of P, relative to K, is

$$h_K(P) = \log H_K(P) = \sum_{v \in M_K} -q_v \min\{v(x_0), \ldots, v(x_n)\}.$$

For $P = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{Q})$, with $x_i \in \mathbb{Z}$ coprime, we see that the archimedean factor $\max\{|x_0|_\infty, \ldots, |x_n|_\infty\}$ in the product above is exactly $H(P)$. All non-archimedean factors $\max\{|x_0|_p, \ldots, |x_n|_p\}$ are equal to one, as the x_i are integer (i.e. $|x_i|_p \leq 1$) and coprime (i.e. at most one x_i has $|x_i|_p < 1$). Thus $H(P) = H_{\mathbb{Q}}(P)$ and the definitions agree for $K = \mathbb{Q}$.

It is more common to work with the logarithmic height, which is the standard height considered in literature. We will from now on work only with the logarithmic height and hence drop the term 'logarithmic' and simply talk about $h_K(P)$ as *the* height of P, relative to K.

Of course we wish for Definition 2.2.1 not to depend on the choice of homogeneous coordinates for P. We also wish that this definition does not depend on the choice of field over which we define P. Unfortunately, the latter is too much to ask for, but the definition does behave well with respect to field extensions.

Lemma 2.2.2. *Let K be a number field and $P \in \mathbb{P}^n(K)$ be a point.*

(a) The sum that defines $h_K(P)$ is finite and independent of the choice of homogeneous coordinates for P.

(b) Let L/K be a finite field extension. Then, for all $P \in \mathbb{P}^n(K)$,

$$h_L(P) = [L : K]h_K(P).$$

Proof. (a) Recall that there are finitely many archimedean places in M_K and there is one place v for every non-zero prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$. Let $P = (a_0 : \ldots : a_n) = (\frac{x_0}{y_0}, \ldots, \frac{x_n}{y_n})$, with all $x_i, y_i \in \mathcal{O}_K$. Then the x_i and y_i lie in only finitely many prime ideals as $\mathcal{O}_K$ is a Dedekind domain (see e.g. [Atiyah and MacDonald, 4, Theorem 9.5]). If $x_i, y_i \notin \mathfrak{p}$, we have $v(\alpha) = 0$, so that at most finitely many factors $\min\{v(x_0), \ldots, v(x_n)\}$ are unequal to 0. Thus the infinite sum is well-defined.

Let us again write $P = (a_0 : \ldots : a_n)$. Any other choice of homogeneous coordinates has the form $(ca_o : \ldots : ca_n)$ for some $c \in K^\times$. Using the product formula (Theorem 2.1.2) we get

$$\begin{aligned}\sum_{v \in M_K} -q_v \min\{v(ca_0), \ldots, v(ca_n)\} &= \sum_{v \in M_K} -q_v v(c) + \sum_{v \in M_K} -q_v \min\{v(a_0), \ldots, v(a_n)\} \\ &= \sum_{v \in M_K} -q_v \min\{v(a_0), \ldots, v(a_n)\}.\end{aligned}$$

(b) The proof depends on the *degree formula* $\sum_{w \in M_L, w|v} [L_w : K_v] = [L : K]$, and can be found in [Hindry and Silverman, 27, Lemma B.2.1(b)]. □

Note that we can always choose one of the homogeneous coordinates of $P \in \mathbb{P}^n(K)$ to be equal to 1, so that $h_K(P) \geq 0$.

We can now give a uniform definition for global height on an algebraic closure $\overline{\mathbb{Q}}$ of the rational numbers. Note that for a number field K, we have $\overline{K} = \overline{\mathbb{Q}}$.

Definition 2.2.3. *Let $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$. Let K be a number field over which we can define all coordinates of P. Then we define the (global) absolute height $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to [0, \infty)$ by*

$$h(P) = \frac{1}{[K : \mathbb{Q}]} h_K(P).$$

By Lemma 2.2.2(b) this definition does not depend on the chosen field of definition for P. This definition also allows us to define the height of an arbitrary element $\alpha \in \overline{\mathbb{Q}}$, by setting $h(\alpha) := h((\alpha : 1))$. Then $h(0.999) = 1000 \gg 1 = h(1)$.

Heights give a measure of arithmetic complexity of points (on projective space). The *Nortcott Property* (see [Hindry and Silverman, 27, Theorem B.2.3]) states that the set $\{P \in \mathbb{P}^n(K) \mid h(P) \leq N\}$ is finite for all $N \geq 0$. That is, there are only finitely many points with small complexity. Finiteness properties of heights, like the Northcott property, are crucial for many proofs in Diophantine geometry. To name a few, the Mordell-Weil theorem (the group of K-rational points on an abelian variety over K is finitely generated), Siegel's theorem (an affine curve over K of genus $g \geq 1$ has only finitely many integral points) and Faltings' theorem (a projective curve over K of

genus $g \geq 2$ has only finitely many K-rational points) are all based on this finiteness result.

2.2.2 Global Heights on Varieties

We wish to extend the definition of height on $\mathbb{P}^n(\overline{\mathbb{Q}})$ to work on arbitrary projective varieties, defined over a number field K. As projective varieties admit an embedding into projective space, we can give the height of a point on the variety by viewing it as point on the projective space, via this embedding. More generally, we can define heights of these points via any morphism to projective space.

Definition 2.2.4. *Let V be a projective variety, and let $\varphi : V \to \mathbb{P}^n(\overline{\mathbb{Q}})$ be any morphism. Then we define the height on V, relative to φ as*

$$h_\varphi : V(\overline{\mathbb{Q}}) \to [0, \infty), \quad h_\varphi(P) = h(\varphi(P)).$$

Example 2.2.5. Let C be a hyperelliptic curve defined over a number field K with a unique point at infinity ∞, given by a Weierstrass equation

$$C : y^2 = \lambda_{2g+1}x^{2g+1} + \cdots + \lambda_1 x + \lambda_0.$$

We can define a morphism $x : C \to \mathbb{P}^1(K)$, by projecting points onto their x-coordinate. That is, for $P = (x, y) \in C(K)\backslash\{\infty\}$, we have $x(P) = [x : 1] \in \mathbb{P}^1(K)$ and $x(\infty) = [1 : 0] \in \mathbb{P}^1(K)$.

Applying Definition 2.2.4 to this morphism gives us a height on $C(K)$, defined by

$$h_x(P) = h(x(P)) = h([x : 1]) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} -q_v \min\{v(x), 0\},$$

for $P = (x, y) \in C(K)\backslash\{\infty\}$ and $h_x(\infty) = 0$. We see that this (global) height naturally splits into the sum of 'local heights', one for every $v \in M_K$.

Definition 2.2.4 is obviously very dependent on the chosen morphism φ. We know from Definition 1.1.1 that we can associate a rational map to any divisor D. For base-point free divisors, we even get a morphism. We thus have a way of assigning height functions to base point-free divisors on V, by $h_{V,D} := h_{\varphi_D} = h \circ \varphi_D$. If D is not base point-free, we can write $D = E - F$ as the difference of two base-point free divisors (see Hindry, Silverman [Hindry and Silverman, 27, Theorem A.3.2.3]) and define $h_{V,D} = h_{V,E} - h_{V,F}$.

Divisors reflect the geometry of a variety. Associating a height function to each divisor (class) converts geometric statements into arithmetic statements in terms of height functions. This relation is exactly the content of the Height Machine, due to André Weil.

All height functions in the Height Machine are uniquely defined up to a bounded function. For two functions $h, h' : V(\overline{K}) \to \mathbb{R}$, we denote

$$h' = h + \mathcal{O}(1), \quad \text{or} \quad h'(P) = h(P) + \mathcal{O}(1), \quad \text{for all } P \in V(\overline{K})$$

if $|h'(P) - h(P)| \leq C$ for all $P \in V(\overline{K})$ and some constant $C \geq 0$, which does not depend on P .

Theorem 2.2.6 (Weil's Height Machine)**.** *Let V be a smooth projective variety defined over a number field K. Then there exists a map*

$$h_V : \mathrm{Div}(V) \to \{\textit{functions } V(\overline{K}) \to \mathbb{R}\},$$

that sends $D \in \mathrm{Div}(V)$ to a Weil height $h_{V,D}$*, associated with D, with the following properties:*

(a) (Normalization) *Let $H \subseteq \mathbb{P}^n$ be a hyperplane (i.e. the zero set of one non-zero linear form). Then $h_{\mathbb{P}^n,H} = h + \mathcal{O}(1)$, where h is the height on $\mathbb{P}^n(\overline{K})$ defined in Definition 2.2.3.*

(b) (Functoriality) *Let $\varphi : V \to W$ be a morphism and let $D \in \mathrm{Div}(W)$. Then $h_{V,\varphi^*(D)} = h_{W,D} \circ \varphi + \mathcal{O}(1)$.*

(c) (Additivity) *Let $D, E \in \mathrm{Div}(V)$. Then $h_{V,D+E} = h_{V,D} + h_{V,E} + \mathcal{O}(1)$.*

(d) (Linear Equivalence) *Let $f \in \overline{K}(V)$. Then $h_{V,\mathrm{div}(f)} = \mathcal{O}(1)$.*

(e) (Positivity) *Let $D \in \mathrm{Div}(V)$ be an effective, base-point free divisor. Then $h_{V,D}(P) \geq \mathcal{O}(1)$.*

(f) (Finiteness) *Let $D \in \mathrm{Div}(V)$ be ample. Then for every finite extension L/K and every constant B, the set*

$$\{P \in V(L) : h_{V,D}(P) \leq B\}$$

is finite.

(g) (Uniqueness) *The Weil heights $h_{V,D}$ are determined, up to $\mathcal{O}(1)$, by (a), (b) just for embeddings $V \hookrightarrow \mathbb{P}^n$, and (c).*

The map h_V, given by $h_V(D) = h_{V,D} = h_{\varphi_D}$ as discussed above, satisfies these conditions.

The Height Machine is said to be effective, in the sense that we can give explicit bound for the $\mathcal{O}(1)$ in terms of the coefficients of the defining equations of the V's, D's and φ's. However, in practice it is often difficult to give these bounds and they are generally quite large.

Example 2.2.7. Let us return to Example 2.2.5 and consider the divisor $2(\infty)$ on C. Let $D' = (0, y_1) + (0, y_2)$, where y_1, y_2 are the zeroes of $y^2 = \lambda_0$. Then D' has disjoint support from $2(\infty)$, is effective, and $D' - 2(\infty) = \mathrm{div}(x)$, which means $D' \in |2(\infty)|$. Thus $2(\infty)$ is a base-point free divisor.

We have the equality $L(2(\infty)) = K \oplus K \cdot x$, so the morphism $x : C \to \mathbb{P}^1$ is exactly $\varphi_{2\infty}$. Now we can use the additivity of Weil's Height Machine to find the height associated with the divisor (∞) for an $P \in C(K)\backslash\{\infty\}$:

$$h_{C,(\infty)}(P) = \frac{1}{2}h_x(P) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} -\frac{1}{2} q_v \min\{v(x(P)), 0\}. \tag{2.4}$$

The divisor (∞) is ample, so we find that the finiteness property from Theorem 2.2.6(f) holds for $h_{C,(\infty)}$. Also (∞) is effective, so we have positivity from Theorem 2.2.6(e), which is easy to see from Equation 2.4.

The height machine and the abelian structure on an abelian variety A combine well.

Abelian varieties come equipped with the *multiplication-by-m* maps $[m] : A \to A$ for $m \in \mathbb{Z}$, recursively defined by adding an element $|m|$ times to itself and, if m is negative, composing with the inverse map $[-1]$.

Corollary 2.2.8. *Let A be an abelian variety defined over a number field K and let $D \in \mathrm{Div}(A)$ be a divisor on A. Let $m \in \mathbb{Z}$ and let $P \in A(\overline{K})$. Then*

$$h_{A,D}([m]P) = \frac{m^2+m}{2} h_{A,D}(P) + \frac{m^2-m}{2} h_{A,D}(-P) + \mathcal{O}(1).$$

In particular, if D is a symmetric divisor (i.e. $[-1]^*D \sim D$), then

$$h_{A,D}([m]P) = m^2 h_{A,D}(P) + \mathcal{O}(1).$$

The proof relies on the Mumford formula (see [Hindry and Silverman, 27, Corollary A.7.2.5]),

$$[m]^*D \sim \frac{m^2+m}{2} D + \frac{m^2-m}{2} [-1]^*D, \tag{2.5}$$

and the properties (c) and (d) of the Height Machine.

2.3 Canonical Heights

The properties in Theorem 2.2.6 and Corollary 2.2.8 are very desirable properties to have, except that they only hold up to a factor $\mathcal{O}(1)$. In this section we will see that we can make a *canonical* choice for a height function on an abelian variety that satisfies all the properties of the height machine and for which all $\mathcal{O}(1)$ vanish.

This canonical height is an important tool to describe the structure of an abelian variety A/K. For example, if we have generators $P_1, \ldots, P_r$ of the (free, and finitely generated) Mordell-Weil group $A(K)/A(K)_{\text{tors}}$, we can calculate the *regulator* of A/K via the canonical height of these points. We do this in Section 3.6 for an elliptic curve. In particular, if the associated divisor defines a principal polarization, we get the *canonical* regulator. This canonical regulator is one of the constants appearing in the famous Birch and Swinnerton-Dyer Conjecture, one of the unsolved Millenium Prize Problems.

From now until the end of the chapter we only work with an abelian variety A. A canonical height function can also be defined on a certain class of non-abelian varieties, but they will not play a role in this thesis.

Lemma 2.3.1. *Let A be an abelian variety defined over a number field K. Let $D \in \mathrm{Div}(A)$ be a symmetric divisor, let $h_{A,D}$ be a Weil height, associated with D, and let $P \in A(\overline{K})$. Then the limit*

$$\hat{h}_{A,D}(P) = \lim_{n\to\infty} \frac{h_{A,D}([2^n]P)}{4^n}$$

exists. The resulting function $\hat{h}_{A,D} : A(\overline{K}) \to \mathbb{R}$ is called the canonical height, *associated with D.*

From Corollary 2.2.8 with $m = 2$, we can prove that, for all $P \in A(\overline{K})$, the sequence $(4^{-n} h_{A,D}([2^n]P))_{n\geq 1}$ is Cauchy, which proves the limit converges.

These canonical heights have useful properties, and do not depend on the chosen Weil height that we use to calculate the limit.

Theorem 2.3.2. *Let A be an abelian variety defined over a number field K. Let $D \in \mathrm{Div}(A)$ be a symmetric divisor. The canonical height $\hat{h}_{A,D} : A(\overline{K}) \to \mathbb{R}$, associated with D, satisfies the following properties:*

(a) *The canonical height satisfies the Height Machine. That is, let $h_{A,D}$ be any height, associated with D. Then $\hat{h}_{A,D} = h_{A,D} + \mathcal{O}(1)$.*

(b) (Quadraticity) *For all integers m, $\hat{h}_{A,D} \circ [m] = m^2 h_{A,D}$.*

(c) (Additivity) *For all $D, E \in \mathrm{Div}(A)$, we have $\hat{h}_{A,D+E} = \hat{h}_{A,D} + \hat{h}_{A,E}$.*

(d) (Functoriality) *Let $\varphi : B \to A$ be a homomorphism of abelian varieties over K. Then $\hat{h}_{B,\varphi^* D} = \hat{h}_{A,D} \circ \varphi$.*

(e) (Positivity) *Suppose D is ample. Then for all $P \in A(\overline{K})$, we have $\hat{h}_{A,D}(P) \geq 0$, with equality if and only if P is a point of finite order.*

(f) (Uniqueness) *The canonical height $\hat{h}_{A,D}$ depends only on the divisor class of the divisor D and is uniquely determined by (a) and (b) for any one integer $m \geq 2$.*

Proof. We only prove property (a), (b), (e) and (f). The other properties follow similarly from the properties of the Height Machine and can be found in [Hindry and Silverman, 27, Theorem B.5.6].

(a) Let $h'_{A,D}$ be the height function used to construct $\hat{h}_{A,D}$ in Lemma 2.3.1. Using the notation from that lemma, we find that if we take $m = 0$ and let $n \to \infty$, we get

$$|\hat{h}_{A,D}(P) - h'_{A,D}(P)| = \lim_{n\to\infty} |4^{-n} h'_{A,D}([2^n]P) - h'_{A,D}(P)| \leq \frac{C}{3},$$

which, together with the fact that $h_{A,D} = h'_{A,D} + \mathcal{O}(1)$, proves the statement (a).

(b) This follows directly from the definition and Corollary 2.2.8. Note that the multiplication-by-m maps commute. We calculate:

$$\begin{aligned} \hat{h}_{A,D}([m]P) &= \lim_{n\to\infty} 4^{-n} h_{A,D}([2^n m]P) \\ &= \lim_{n\to\infty} 4^{-n} \left(m^2 h_{A,D}([2^n]P) + C_n\right) \\ &= m^2 \hat{h}_{A,D}(P), \end{aligned}$$

as the sequence C_n is bounded.

(e) As D is ample, we can choose an associated height function with only nonnegative values by Theorem 2.2.6(e). Then $\hat{h}_{A,D}(P) \geq 0$ is obvious from the definition.

Now suppose that $P \in A(\overline{K})$ is a point of finite order n. Then the set generated by P is $\langle P \rangle = \{O, P, [2]P, \ldots, [n-1]P\}$, so that $\{[2^n]P \mid n \geq 0\} \subseteq \langle P \rangle$ is finite. It follows immediately that

$$\hat{h}_{A,D}(P) = \lim_{n\to\infty} 4^{-n} h_{A,D}([2^n]P) = 0.$$

Conversely suppose that $\hat{h}_{A,D}(P) = 0$. Then for any $n \geq 1$, $\hat{h}_{A,D}([2^n]P) = 4^n \hat{h}_{A,D}(P) = 0$. That is $\{[2^n]P : n \geq 0\} \subset \{Q \in A(\overline{K}) : \hat{h}_{A,D}(Q) \leq 0\}$. By Theorem 2.2.6(f) the

latter set is finite, as D is ample, so there must be some $n > m \geq 0$, such that $[2^n]P = [2^m]P$. Hence $[2^n - 2^m]P = O$ and P is a torsion point.

(f) To prove property (f), suppose that $\hat{h}$ and $\hat{h}'$ are both functions satisfying properties (a) and (b) for some integer $m \geq 2$. By property (a), $g = \hat{h} - \hat{h}'$ is a bounded function, say $|g(P)| \leq B$ for all $P \in A(\overline{K})$. We find from property (b) that $g \circ [m] = m^2 \cdot g$ and iterating this yields $g \circ [m^n] = m^{2n} \cdot g$ for all $n \geq 1$. Then

$$|g(P)| = \frac{|g([m^n]P)|}{m^{2n}} \leq \frac{B}{m^{2n}} \to_{n\to\infty} 0.$$

Hence $g(P) = 0$ for all $P \in A(\overline{K})$, so $h = \hat{h}$.

To prove that $\hat{h}_{A,D}$ only depends on the divisor class of D, note that for $f \in \overline{K}(A)$, any Weil height $h_{A,\mathrm{div}(f)}$ is bounded, so that the canonical height $\hat{h}_{A,\mathrm{div}(f)} = 0$. The statement now follows from additivity. □

These definitions and properties are all stated for symmetric divisors D, based on the property that $h_{A,D}([m]P) = m^2 h_{A,D}(P) + \mathcal{O}(1)$. For antisymmetric divisors D, we can do a similar procedure based on the property $h_{A,D}([m]P) = m h_{A,D}(P) + \mathcal{O}(1)$ (which can also be proven from the Mumford formula 2.5) by defining

$$\hat{h}_{A,D}(P) = \lim_{n\to\infty} \frac{h_{A,D}([2^n]P)}{2^n}.$$

Following the lines of the proof of Theorem 2.3.2, we can then prove this canonical height is linear.

For general divisors D, we can decompose $2D$ into a symmetric and antisymmetric part, by writing $2D = D^+ + D^- = (D + [-1]^*D) + (D - [-1]^*D)$. Then we can define

$$\hat{h}_{A,D}(P) = \frac{1}{2}\hat{h}_{A,2D}(P) = \frac{1}{2}(\hat{h}_{A,D^+}(P) + \hat{h}_{A,D^-}(P)).$$

That is, we can define a canonical height associated with any divisor D on an abelian variety A and it splits into a quadratic and a linear form. We henceforth focus only on symmetric divisors, as in Chapter 3 and 4 we work only with symmetric divisors.

2.4 Decomposition into Local Heights

Calculating the canonical height via its definition (Lemma 2.3.1) requires calculating many multiples of a point, and taking a limit, with uncertain speed of convergence. This is not ideal. An alternative methods is to use a decomposition of (canonical) heights into local factors, one for each place of K. These are generally easier to compute (once we have explicit formulas), and zero for almost all places. That is, the computation of canonical heights reduces to a calculation of finitely many terms.

In this section we introduce an abstract definition of local heights, associated with a divisor. We then show that local heights sum to a (global) Weil height. We similarly give an abstract definition of canonical local heights, which sum to the canonical height, up to a constant. The theory of local heights is also constructed in [Serre, 53, Chapter 6], where one place $v \in M_K$ is fixed, and in [Lang, 32, Chapter 10, Chapter 11]. Our exposition differs from these sources in our level of detail and generality. In

particular, we define canonical local heights abstractly and define the global height, associated to a local height (Definition 2.4.9) more generally than [Lang, 32, Section 10.4].

We saw in Example 2.2.7 that we have a natural decomposition of the global height $h_{C,\infty}$ into local factors, as follows:

$$h_{C,\infty}(P) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} -\frac{1}{2} q_v \min\{v(x(P)), 0\}, \quad \forall P \in C(K)\backslash\{\infty\}. \tag{2.6}$$

The local factor of the height at $v \in M_K$ for the point $P \in C(K)\backslash\{\infty\}$ is given by

$$\lambda_{C,\infty,v} := -\frac{1}{2} q_v \min\{v(x(P)), 0\}.$$

Intuitively, this local factor measures (minus) the logarithm of the v-adic distance between P and ∞ (where $P = \infty$ is not allowed). If the x-coordinate of P is v-adically large and hence the distance to the "point at infinity" ∞ is small, the value $|x(P)|_v$ is large. Then the value $v(x(P))$ is large and negative, and $\lambda_{C,\infty,v}$ is large. The fact that we take the minimum of $v(x(P))$ and 0 ensures that $\lambda_{C,\infty,v}$ is bounded away from ∞. The function x has a pole of order 2 at (∞), so measuring with x "overshoots" the v-adic distance between P and ∞ by a factor 2, which is counteracted with the factor $\frac{1}{2}$ in $\lambda_{C,\infty,v}$.

We can summarize this discussion by noting that the function x is a local equation for the divisor $-2(\infty)$, i.e. $\operatorname{div}(x) = -2(\infty) + (\text{zeros away from } \infty)$. Then, close to the support of the divisor $-2(\infty)$ (which is simply ∞), the local factor of $h_{C,-2(\infty)}$ looks like $v(x(P))$. Again, we take the maximum of this value and 0 to ensure λ is bounded away from ∞. To get back to the local factors $h_{C,\infty}$, we use additivity to multiply by $-\frac{1}{2}$.

We use this intuition to define local height functions on (arbitrary) varieties abstractly. We start with some definitions to make this discussion easier.

Remark 2.4.1. From the example above, we see that local heights, associated with a divisor D, on a variety V should be defined, both for all $P \in V(K)$ outside the support of D, and for all $v \in M_K$. In fact, global heights are given for all points defined over $\overline{K}$, so that local heights should be too. To make things easier, we work with the completion K_v at every place $v \in M_K$, and its algebraic closure $\overline{K_v}$. Note that v extends uniquely to $\overline{K_v}$. We write $V_D := V\backslash|D|$. Local heights, associated with D, are then given as functions with domain

$$\bigsqcup_{v \in M_K} V_D(\overline{K_v}).$$

We represent elements of this disjoint union by pairs (P, v), with $v \in M_K$, and $P \in V_D(\overline{K_v})$. Using such pairs, we can embed both $V_D(K) \times M_K$ and $V_D(\overline{K}) \times M_K$ in this set (by fixing an embedding $\overline{K} \hookrightarrow \overline{K_v}$). We use this embedding below as a standard.

An *M_K-constant* is a function $\varepsilon : M_K \to \mathbb{R}$, that satisfies $\varepsilon_v := \varepsilon(v) = 0$ for all but finitely many $v \in M_K$. Note that for a fixed $\alpha \in K^\times$, the function $v \mapsto v(\alpha)$ is an M_K-constant.

Let V/K be a variety. Let B be a subset of $\bigsqcup_{v\in M_K} V(\overline{K_v})$. We say that B is *affine M_K-bounded* if B is contained in $\bigsqcup_{v\in M_K} U(\overline{K_v})$ for some affine open $U \subseteq V$, with coordinate ring $K[f_1,\dots,f_n]$ and there exists an M_K-constant ε, such that for all $(x,v) \in B$, we have

$$|f_i(x)|_v \leq e^{-\frac{n_v}{q_v}\varepsilon(v)}, \quad \text{or equivalently} \quad v(f_i(x)) \geq \varepsilon(v), \tag{2.7}$$

for all i. If we consider only one place v, then Equation (2.7) is equivalent to saying B is bounded in U with respect to the v-adic topology on $\overline{K_v}$.

We say $B \subseteq \bigsqcup_{v\in M_K} V(\overline{K_v})$ is *M_K-bounded* if it is contained in the finite union of affine M_K-bounded subsets. Let $U \subseteq V$. Then we may abbreviate $\bigsqcup_{v\in M_K} U(\overline{K_v}) \subseteq \bigsqcup_{v\in M_K} V(\overline{K_v})$ being M_K-bounded by saying that U is an M_K-bounded subset of V.

Example 2.4.2. The space $\bigsqcup_{v\in M_K} \mathbb{P}^n(\overline{K_v})$ is M_K-bounded. As any projective variety V admits a closed embedding into projective space $\mathbb{P}^n$, this immediately implies that V is M_K-bounded as well.

We cover $\mathbb{P}^n$ by the (standard) open affines $U_i = \{x_i \neq 0\}$, in which we use the affine coordinates x_j/x_i. Now define

$$W_i = \{((x_0 : \dots : x_n), v) \in \bigsqcup_{v\in M_K} \mathbb{P}^n(\overline{K_v}) : |x_i|_v \geq |x_j|_v \text{ for all } j\} \subseteq \bigsqcup_{v\in M_K} U_i(\overline{K_v}).$$

Then we find that for all $(x,v) \in W_i$ and for all j, $|x_j/x_i|_v \leq 1$, so the sets W_i are affine M_K-bounded. The W_i cover $\bigsqcup_{v\in M_K} \mathbb{P}^n(\overline{K_v})$, so $\mathbb{P}^n$ is M_K-bounded.

Definition 2.4.3. *Let U be a subset of V. A function $\alpha : \bigsqcup_{v\in M_K} U(\overline{K_v}) \to \mathbb{R}$ is said to be* M_K-bounded *if there exist M_K-constants ε_1 and ε_2, such that*

$$\varepsilon_1(v) \leq \alpha(P,v) \leq \varepsilon_2(v),$$

for all $(P,v) \in \bigsqcup_{v\in M_K} U(\overline{K_v})$. Similarly we define α to be M_K-bounded from below and from above.

The function α is locally M_K-bounded *if it is M_K-bounded on all M_K-bounded subsets of U.*

From this definition we can easily see that for locally M_K-bounded functions α, the function $v \mapsto \alpha(P,v)$, for a fixed $P \in U$, is an M_K-constant.

A function α as above is said to be M_K-continuous, or simply continuous, if for each $v \in M_K$, the function

$$P \mapsto \alpha_v(P) := \alpha(P,v)$$

is continuous, with respect to the v-adic topology on U.

Example 2.4.4. Let V be a variety and let f be a regular function on V. Then the map

$$(P,v) \mapsto (v \circ f)(P)$$

is continuous and locally bounded from below. If f is also invertible on V, it is locally bounded.

To prove this is suffices to assume that V is an affine variety. Then f is a polynomial in the coordinates of V, so it is immediate from the definition that the map

$$(P, v) \mapsto \log |f(P)|_v = -\frac{q_v}{n_v}(v \circ f)(P)$$

is bounded from above on every bounded subset of $V \times M$, so the claim follows. If f is invertible, we use that $v \circ f^{-1} = -v \circ f$.

We are now ready to give a definition of local height functions. Afterwards we will prove these functions have desirable properties, and most importantly sum to a global height function.

Definition 2.4.5. *Let V be a variety and D a divisor on V. A real-valued function*

$$\lambda_D : \bigsqcup_{v \in M_K} V_D(\overline{K_v}) \to \mathbb{R}$$

is called a local height function*, associated with D, if the following condition holds:*

Let (U, f) be a pair representing D. Then there exists a locally M_K-bounded continuous function $\alpha := \alpha_U : \bigsqcup_{v \in M_K} U(\overline{K_v}) \to \mathbb{R}$, such that for all $P \in U_D$ and $v \in M_K$, we have

$$\lambda_D(P, v) = v(f(P)) + \alpha(P, v). \tag{2.8}$$

Note that λ_D is an M_K-continuous function. These functions are also known as *Weil functions*.

Example 2.4.6. For any $f \in K(V)$, the function $(P, v) \mapsto (v \circ f)(P)$ is a local height, associated with $\mathrm{div}(f)$.

Using the notation from the definition above, we have that for fixed $P \in V_D(\overline{K})$, both $v(f(P))$ and $\alpha(P, v)$ are M_K-constants, so that $\lambda_D(P, \cdot)$ is an M_K-constant as well. That is, $\lambda_D(P, v) = 0$ for all but finitely many $v \in M_K$.

Remark 2.4.7. Let $S \subseteq M_K$ be a finite set of places. Then $\lambda_D : \bigsqcup_{v \in M_K} V_D(\overline{K_v}) \to \mathbb{R}$ is a local height function, associated with D, if, for all pairs (U, f) representing D, the following conditions hold:

(i) For all $v \in S$, there is a continuous function $\alpha_v : U(\overline{K_v}) \to \mathbb{R}$ that is bounded w.r.t. the v-adic topology, such that

$$\lambda_D(P, v) = v(f(P)) + \alpha_v(P), \quad \forall P \in U_D(\overline{K_v}).$$

(ii) There exists a locally M_K-bounded continous function $\alpha := \alpha_U : \bigsqcup_{v \in M_K \setminus S} U(\overline{K_v}) \to \mathbb{R}$, such that for all $P \in U_D$ and $v \in M_K \setminus S$, we have

$$\lambda_D(P, v) = v(f(P)) + \alpha(P, v).$$

That is, if we restrict our attention to a finite set of places (such as the archimedean places of K) then we only need to check the difference between λ and $v \circ f$ is bounded in the v-adic topology, provided the condition from Equation 2.8 holds for all places outside S.

Suppose now we have two divisors $D, E \in \mathrm{Div}(V)$, represented on some open U by (U, f) and (U, g) respectively. Let λ_D and λ_E be two associated local height functions, given on U respectively by

$$\lambda_D(P, v) = v(f(P)) + \alpha_1(P, v), \quad \text{and} \quad \lambda_E(P, v) = v(g(P)) + \alpha_2(P, v).$$

The sum of these two functions is a priori only defined outside $|D| \cup |E|$, which may not coincide with $|D + E|$. However, (U, fg) represents the divisor $D + E$, so that $(v \circ (fg))(P)$ is well-defined outside $|D + E|$. Furthermore the sum $\alpha_1 + \alpha_2$ is also a locally M_K-bounded function on $\bigsqcup_{v \in M_K} U(\overline{K_v})$. That is all to say that the function

$$(P, v) \mapsto v(fg(P)) + \alpha_1(P, v) + \alpha_2(P, v)$$

defines a local height function associated with $D + E$ and restricts to $\lambda_D + \lambda_E$ outside $|D| \cup |E|$. It is also the unique local height function with this property (see e.g. [Lang, 32, Prop. 10.1.5]).

Suppose now $\varphi : V \to W$ is a morphism of varieties and $D \in \mathrm{Div}(W)$ is a divisor, such that $\varphi(V) \not\subseteq |D|$, represented by the set $\{(U, f)\}$. Then the divisor $\varphi^* D$ is represented by the set $\{(\varphi^{-1}U, f \circ \varphi)\}$. If λ_D is a local height function associated with D, one can now check that the function $\lambda_D \circ \varphi$ satisfies the condition of Definition 2.4.5 for the divisor $\varphi^* D$. This proves functoriality for local height functions.

Lemma 2.4.8. *Let V/K be a projective variety.*

(a) Let λ be a local height function associated with the zero divisor. Then λ is a M_K-bounded continuous function.

(b) If λ_D and λ'_D are two local height functions, associated with the same divisor D, then $\lambda_D - \lambda'_D$ is M_K-bounded.

Proof. Property (a) follows from the fact that the zero divisor is represented by $(V, 1)$ on all of V and V is M_K-bounded by Example 2.4.2. For the second statement, note that $-\lambda'_D$ is a local height function associated with $-D$. By the discussion above we can extend $\lambda_D - \lambda'_D$ to a local height function associated with $D - D = 0$, which is then M_K-bounded by the first statement. □

We have not proven yet that local height functions actually exist. Also, the terminology 'local height' is only justified if the local heights actually sum to a global height. For points defined over K, we know from Equation 2.6 what shape this formula should have. However, global heights are given on points defined over $\overline{K}$, and hence should be invariant under the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. We can define local heights on points defined over $\overline{K}$, by embedding $\overline{K} \hookrightarrow \overline{K_v}$. This choice is non-canonical, and it is not clear why the resulting formula should be $\mathrm{Gal}(\overline{K}/K)$-invariant, which it should be. To resolve this, we fix embeddings $\overline{K} \hookrightarrow \overline{K_v}$ and take the $\mathrm{Gal}(\overline{K}/K)$-orbit of a point $P \in V(\overline{K})$,

$$\mathrm{Orb}(P) = \mathrm{Orb}_{\overline{K}/K}(P) = \{\sigma(P) : \sigma \in \mathrm{Gal}(\overline{K}/K)\},$$

into account.

Definition 2.4.9. *Let V/K be a projective variety and let $D \in Div(V)$. Let $\lambda : \bigsqcup_{v \in M_K} V_D(\overline{K_v}) \to \mathbb{R}$ be a local height function, associated with D. We define the*

global height $h_\lambda : V_D(\overline{K}) \to \mathbb{R}$*, associated with* λ*, by*

$$h_\lambda(P) = \frac{1}{\#\operatorname{Orb}(P)} \sum_{Q \in \operatorname{Orb}(P)} \left(\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \lambda(Q, v) \right). \tag{2.9}$$

This sum is obviously invariant under the action of $\operatorname{Gal}(\overline{K}/K)$ and is well-defined as for fixed $P \in V_D(\overline{K})$, $\lambda(P, v)$ is non-zero only for finitely many $v \in M_K$, and $\operatorname{Orb}(P)$ is finite. It is not yet clear the definition is independent of the choice of K.

Lemma 2.4.10. *Let* L/K *be a finite Galois extension and let* $P \in V_D(L)$*. Then*

$$h_\lambda(P) = \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_K} q_w \lambda(P, w). \tag{2.10}$$

Proof. The extension L/K is Galois, so that $\operatorname{Orb}_{\overline{K}/K}(P) = \operatorname{Orb}_{L/K}(P)$. Write $G = \operatorname{Gal}(L/K)$ Let $K(P)$ the field of definition of P (that is, the smallest field K'/K such that $P \in V_D(K')$). Then $\#\operatorname{Orb}(P) = [K(P) : K]$, and $\operatorname{Stab}(P) = \{\sigma \in G : \sigma(P) = P\} = [L : K(P)]$. Then we get

$$\begin{aligned} h_\lambda(P) &= \frac{1}{[K(P):K]} \sum_{\sigma \in G} \frac{1}{[L:K(P)]} \left(\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \lambda(\sigma(P), v) \right) \\ &= \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_K} \sum_{\sigma \in G} q_v \lambda(\sigma(P), v). \end{aligned}$$

Now from [Lang, 34], we find that G acts transitively on the absolute values $w \in M_L$ dividing a fixed $v \in M_K$. This means that $\{|\sigma(x)|_v : \sigma \in G\} = \{|x|_w : w \mid v\}$, for any $x \in L$. By our definition of q_v (Equation 2.2), we then find that

$$\{q_v \lambda(\sigma(P), v) : \sigma \in G\} = \left\{ \frac{q_w n_w}{n_v} \lambda(P, w) : w \mid v \right\} = \left\{ \frac{q_w}{[L_w : K_v]} \lambda(P, w) : w \mid v \right\}.$$

The stabilizer of v under the action of G has size $[L_w : K_v]$, so that

$$\begin{aligned} h_\lambda(P) &= \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_K} \sum_{\sigma \in G} q_v \lambda(\sigma(P), v) \\ &= \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_K} \sum_{w \mid v} [L_w : K_v] \frac{q_w}{[L_w : K_v]} \lambda(P, w) \\ &= \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_K} q_w \lambda(P, w), \end{aligned}$$

which is exactly the desired result. □

Note that this lemma also gives argumentation for why we should include the coefficients q_v. The formula in Equation 2.10 is exactly the formula presented in [Lang, 32, Section 10.4], so our definition coincides with the definition in loc. cit. The advantage of using Definition 2.4.9 in practice is that it bypasses finding M_L and redefining local heights for every finite extension L/K, something that is necessary using Lang's formula.

Note that Equation 2.9 cannot define a Weil height, as it is a priori defined only outside the support of D. However, for all $P \in V_D(\overline{K})$, we can write

$$\lambda(P, v) = v(f(P)) + \alpha_U(P, v),$$

if (U, f) represents D and $P \in U_D$. Suppose K to be big enough to contain P. Then we see that by the product formula

$$\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \lambda(P, v) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v (v(f(P)) + \alpha_U(P, v)) \tag{2.11}$$

$$= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \alpha_U(P, v). \tag{2.12}$$

Now $\alpha(P, v)$ is defined on all of U, not just U_D. Hence setting

$$h_\lambda(P) = \begin{cases} \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \lambda(P, v), & P \in V_D, \\ \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \alpha_U(P, v), & P \in |D|, \text{ and } U \ni P, \end{cases} \tag{2.13}$$

gives a canonical way of summing a local height function to a function on all of $V(\overline{K})$. If (V, g) is any other representing pair of D, with $P \in V$, we get by writing out Equation 2.11 for (V, g) that $\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \alpha_U(P, v) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \alpha_V(P, v)$ on $(U \cap V) \backslash |D|$, and by continuity of α_U and α_V, we can extend the equality to $U \cap V$. Thus Equation 2.13 does not depend on the choice of $U \ni P$.

We will generally view h_λ as a function on $V(\overline{K})$ in this way.

We now get existence of local height functions, and the fact that Equation 2.13 actually defines a global height, from the following theorem.

Theorem 2.4.11. *Let $D \in \mathrm{Div}(V)$ be a divisor on a projective variety V over a number field K.*

(a) There exists a local height function

$$\lambda_D : \bigsqcup_{v \in M_K} V_D(\overline{K_v}) \to \mathbb{R}.$$

It is unique up to the addition of an M_K-bounded function.

(b) Let λ_D, λ'_D be two local height functions, associated with the same divisor D. Then $h_{\lambda_D} - h_{\lambda'_D}$ is bounded as a real-valued function.

(c) Let λ_D be a local height function, associated with D. Then h_{λ_D} defines a Weil height, in the sense of Theorem 2.2.6. In other words, if h_D is a global height, associated with D, then

$$h_D = h_{\lambda_D} + \mathcal{O}(1). \tag{2.14}$$

Proof. (a) We only provide a sketch of the proof, a full proof can be found in [Lang, 32, Section 10.3]. There exist positive divisors $E_1, \ldots, E_n, F_1, \ldots, F_m$ such that

$$\bigcap_{i=1}^{n} |E_i| = \bigcap_{j=1}^{m} |F_j| = \emptyset,$$

and for all i, j

$$D + E_i - F_j = \operatorname{div}(f_{ij})$$

for some rational functions f_{ij}. Then the function

$$(P, v) \mapsto \max_{1 \le j \le m} \min_{1 \le i \le n} v(f_{ij}(P))$$

defines a local height function, associated with D. Uniqueness up to a M_K-bounded function was proven in Lemma 2.4.8(b).

(b) From Lemma 2.4.8(b), we find an M_K-constant ε, such that

$$|\lambda_D(P, v) - \lambda'_D(P, v)| \le \varepsilon(v),$$

for all $(P, v) \in \bigsqcup_{v \in M_K} V_D(\overline{K_v})$. Then we find that

$$|h_{\lambda_D} - h_{\lambda'_D}| \le \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} q_v \varepsilon(v),$$

which is a finite constant.

(c) By the uniqueness of global height functions (Theorem 2.2.6(h)), we need to check normalization, functoriality for embeddings $V \to \mathbb{P}^n$ and additivity. Functoriality and additivity follow from (b) and the fact that local height functions satisfy functoriality and additivity.

To prove normalization, let H be a hyperplane in $\mathbb{P}^n$ defined by $t = \sum_{i=0}^n \alpha_i x_i = 0$. Let F_j, $j = 0, \ldots, n$ be the hyperplane in $\mathbb{P}^n$ corresponding to the vanishing of the coordinate function x_j. Then $\cap_{j=1}^n F_j = \emptyset$ and the rational functions $f_j = t/x_j$ satisfy $\operatorname{div}(f_j) = H - F_j$. By the proof of (a) we then find that

$$\lambda(P, v) = \max_{0 \le j \le n} v(f_j(P))$$

is a local height associated with H.

Suppose K is big enough to contain P. We calculate

$$\begin{aligned} h_\lambda(P) &= \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} q_v \max_{0 \le j \le n} v(f_j(P)) \\ &= \frac{1}{[K : \mathbb{Q}]} \left(\sum_{v \in M_K} q_v v(t(P)) + \sum_{v \in M_K} q_v \max_{0 \le j \le m} \{-v(x_j))\} \right) \\ &= \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} -q_v \min_{0 \le j \le m} \{v(x_j)\} = h(P), \end{aligned}$$

where $h(P)$ denotes the standard height of P, defined in Section 2.2.1. □

2.4.1 Canonical Local Height

It is now natural to ask whether it is possible to give local heights on an abelian variety A/K, that sum to the canonical height. And if so, if they also satisfy a similar quadratic property to Theorem 2.3.2(b). The latter is a bit too much to ask for. We

proved quadraticity by using the Mumford formula 2.5, and using that linearly equivalent divisors have the same associated height, up to $\mathcal{O}(1)$, which vanishes when taking the canonical height. We know, however, that local heights associated with principal divisors are not trivial, see Example 2.4.6. That is, a similar approach for local heights would not work. We can nevertheless decompose the canonical height into local functions that are almost quadratic. This leads us to the following definition, which is justified by the theorems that follow it.

Definition 2.4.12. *Let A/K be an abelian variety and $D \in \mathrm{Div}(A)$ a symmetric divisor. A function $\hat{\lambda}_D : \bigsqcup_{v\in M_K} A_D(\overline{K_v}) \to \mathbb{R}$ is called a canonical local height function associated with D, if the following properties hold:*

(i) *$\hat{\lambda}_D$ is a local height function associated with D.*

(ii) *Let $\varphi_2 = \varphi_{D,2}$ be a rational function on A, such that $[2]^*D = 4D + \mathrm{div}(\varphi_2)$. Then for all $(P, v) \in \bigsqcup_{v\in M_K} A_D(\overline{K_v})$, such that $[2]P \notin |D|$, we have*

$$\hat{\lambda}_{D,v}([2]P) = 4\hat{\lambda}_{D,v}(P) + v(\varphi_2(P)) + \varepsilon(v), \tag{2.15}$$

for some M_K-constant ε.

These functions are also called Néron functions, or Néron-Tate local height functions.

Note that the existence of a rational function φ_2 with these properties is guaranteed by Mumford's formula (2.5). We call such a rational function a *second division polynomial, associated with D*. In general, we define the nth division polynomial $\varphi_{D,n}$ to be a rational function, that satisfies $[n]^*D = n^2D + \mathrm{div}(\varphi_{D,n})$.

Remark 2.4.13. From the definition it is clear to see that canonical local heights are not unique, as we can choose the M_K-constant in Equation 2.15 freely. Below we will see that indeed canonical local heights are only unique up to the addition of an M_K-constant. If we fix a second division polynomial $\varphi_2 = \varphi_{D,2}$, we can define a canonical local height function $\hat{\lambda}_D = \hat{\lambda}_{D,\varphi_2}$, normalized with respect to φ_2, by requiring that $\hat{\lambda}$ satisfies

$$\hat{\lambda}_{D,v}([2]P) = 4\hat{\lambda}_{D,v}(P) + v(\varphi_2(P)), \tag{2.16}$$

instead of Equation 2.15 at every place $v \in M_K$. This canonical local height, normalized with respect to φ_2, is then unique.

We can also give a definition of canonical local height functions associated with *antisymmetric* divisors by replacing the 4 in Definition 2.4.12 by a 2. We can then write two times an arbitrary divisor as sum of a symmetric and an antisymmetric divisor to extend the definition of canonical local heights to arbitrary divisors.

Theorem 2.4.14 (Néron [47])**.** *Let A be an abelian variety defined over a number field K and let $D \in \mathrm{Div}(A)$. There exists a canonical local height function*

$$\hat{\lambda}_D : \bigsqcup_{v\in M_K} A_D(\overline{K_v}) \to \mathbb{R}$$

with the following properties:

(a) (Uniqueness) *$\hat{\lambda}_D$ is unique, up to the addition of an M_K-constant.*

(b) (Normalization) *If $D = \operatorname{div}(f)$ is a principal divisor, then*

$$\hat{\lambda}_D = v \circ f + \varepsilon_1.$$

(c) (Additivity) *For all $D_1, D_2 \in \operatorname{Div}(A)$,*

$$\hat{\lambda}_{D_1+D_2} = \hat{\lambda}_{D_1} + \hat{\lambda}_{D_2} + \varepsilon_2.$$

(d) (Functoriality) *Let $\pi : B \to A$ be a morphism of abelian varieties. If $\pi^*(D)$ is defined (i.e. $\pi(B) \not\subseteq |D|$), then*

$$\hat{\lambda}_{\pi^* D} = \hat{\lambda}_D \circ \pi + \varepsilon_3.$$

Here $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are M_K-constants.

The proof can be found in [Lang, 32, Chapter 11].

From now on we restrict our attention to symmetric divisors. We are now ready to show that canonical local heights sum to the global canonical height, up to a constant.

Theorem 2.4.15. *Let A be an abelian variety defined over K and let $D \in \operatorname{Div}(A)$ be a symmetric divisor. Let*

$$\hat{\lambda}_D : \bigsqcup_{v \in M_K} A_D(\overline{K_v}) \to \mathbb{R} \quad \text{and} \quad \hat{h}_D : A(\overline{K}) \to \mathbb{R}$$

be a canonical local height and the canonical (global) height, associated with D, respectively. Then for all $P \in A_D(\overline{K})$,

$$\hat{h}_D(P) = h_{\hat{\lambda}_D}(P) + c = \frac{1}{\#\operatorname{Orb}(P)} \sum_{P' \in \operatorname{Orb}(P)} \left(\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \hat{\lambda}_{D,v}(P) \right) + c, \quad (2.17)$$

for some constant $c \in \mathbb{R}$.

Again, the right-hand side of Equation 2.17 is only defined outside the support of D, but it can be uniquely extended to the entirety of $A(\overline{K})$, as in Equation 2.13.

Proof. We already saw in Theorem 2.4.11 that $h_{\hat{\lambda}_D}$ defines a global height. Now let $P \in A_D(\overline{K})$, such that $[2]P \notin |D|$. Then

$$\begin{aligned}
h_{\hat{\lambda}_D}([2]P) &= \frac{1}{\#\operatorname{Orb}(P)} \sum_{Q \in \operatorname{Orb}(P)} \left(\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \hat{\lambda}_{D,v}(Q) \right) \\
&= \frac{1}{\#\operatorname{Orb}(P)} \sum_{Q \in \operatorname{Orb}(P)} \left(\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \Big(4\hat{\lambda}_{D,v}(Q) + v(\varphi_2(Q)) + \varepsilon(v) \Big) \right) \\
&= \frac{4}{\#\operatorname{Orb}(P)} \sum_{Q \in \operatorname{Orb}(P)} \left(\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \hat{\lambda}_{D,v}(Q) \right) + \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \varepsilon(v) \\
&= 4h_{\hat{\lambda}_D}(P) + a.
\end{aligned}$$

The term $a := \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} q_v \varepsilon(v)$ is a constant, independent of P. By Theorem 2.3.2(f) we have proven that $h_{\hat{\lambda}_D} - \frac{a}{3}$ is the canonical height. □

Remark 2.4.16. Of course we want the constant c in Theorem 2.4.15 to be zero. We see that this is true whenever the M_K-constant in Equation 2.15 satisfies the product formula. This is the case, for example, if we take a canonical local height that is normalized with respect to a second division polynomial, see Remark 2.4.13.

In practice, we can easily find out what the constant c is, even if it is non-zero. We must have $\hat{h}_D([2]P) = 4\hat{h}(P)$, so that $h_{\hat{\lambda}_D}([2]P) + c = 4h_{\hat{\lambda}_D}(P) + 4c$ and

$$c = \frac{1}{3}(h_{\hat{\lambda}_D}([2]P) - 4h_{\hat{\lambda}_D}(P)),$$

for any $P \in A(\overline{K})$.

2.5 Local Heights as Intersection Multiplicities

Canonical local heights are not only useful for the computation of canonical (global) heights, but they are intrinsically interesting due to their geometric nature.

Let D be a symmetric divisor on an abelian variety A/K. In this section, we fix a place $v \in M_K$ and calculate $\hat{\lambda}_{D,v}$, the restriction of $\hat{\lambda}_D$ to $v \in M_K$. We restrict our domain to $A_D(K_v)$, the points of A_D defined over the completion of K at v.

The valuation ring $R_v = \{x \in K_v : v(x) \geq 0\}$ of K_v is a discrete valuation ring, with maximal ideal $\mathfrak{m}_v$. We can thus look at the Néron model $\mathcal{N}$ of A over $S = \operatorname{Spec}(R_v)$. See Section 1.3.2 for details. As R_v is a discrete valuation ring, S contains only two points, so we can speak about *the* special fiber $\mathcal{N}_v := \mathcal{N}_{\mathfrak{m}_v}$ of the Néron model. Note that this special fiber is the same as the special fiber above $\mathfrak{p} \in \mathcal{O}_K$ of the Néron model over $\mathcal{O}_K$, where $\mathfrak{p}$ is the prime ideal corresponding to v, see Remark 1.3.7.

Recall that the set $\Phi_v = \Phi_v^{\mathcal{N}}$ of connected components of the special fiber forms a group, and we have a specialization map $\operatorname{sp} : A(K_v) \to \Phi_v$. The following remarkable theorem by Néron relates the canonical local height to this specialization map.

Theorem 2.5.1 ([Néron, 47])**.** *Let A/K be an abelian variety and let $D \in \operatorname{Div}_K(A)$. Let $\hat{\lambda}_D$ be a canonical local height function, associated with D. Let $v \in M_K$ be a non-archimedean place and let $\mathcal{N}$ be the Néron model of A over R. Then there is a map $\gamma_v : \Phi_v \to \mathbb{R}$, such that for all $P \in A_D(K_v)$, we have*

$$\hat{\lambda}_{D,v}(P) = i_v(D, P) + \gamma_v(\operatorname{sp}(x)).$$

Here $i_v(D, P) \in \mathbb{Z}$ is the intersection multiplicity of P and D at v as defined in Definition 1.3.12.

By Remark 1.3.13, if the Zariski closure $\overline{D}$ of D in $\mathcal{N}$ is represented by a pair (U, f), with $f \in K(\mathcal{N}) = K(A)$, we get that $i_v(D, P) = (v \circ f)(P)$. The map γ_v thus gives the locally M_K-bounded term in Definition 2.4.5. For this reason we may call the function γ_v the *local correction term*. The fact that this correction term factors via the component group Φ_v comes from the Néron mapping property (see Definition 1.3.5). This ensures the intersection multiplicity is invariant under translations, which in

turn allows us to show the correction term is dependent only $\mathrm{sp}(P)$. See [Lang, 32, Theorem 5.1] for a full proof.

As canonical local heights are only defined up to an additive constant, there is no unique way to define the function γ_v. However, if we fix the value of the function to be rational at any element of Φ_v, the function is rational and we can get a bound on the denominator.

Theorem 2.5.2 ([Lang, 32, Theorem 11.5.2]). *Suppose that $\hat{\lambda}_D$ is normalized such that $\gamma_v(0) = 0$. Let N be the smallest integer, such that $N \cdot \Phi_v = 0$. Then*

$$\gamma_v(\Phi_v) \subseteq (1/N^2)\mathbb{Z}.$$

Note that in particular this ensures the local height itself is rational at non-archimedean places, if we normalize such that $\gamma_v(0) = 0$.

Remark 2.5.3. In the case that A has semistable reduction at all places, we can even prove $\gamma_v(\Phi_v) \subseteq (\frac{1}{2N})\mathbb{Z}$, see [de Jong and Shokrieh, 18, Theorem A(i)].

We see that it is important to know the structure of the component group Φ_v to calculate local heights. From the Theorems 1.3.8, 1.3.19 and 1.4.4, we can calculate the structure of the component group for semistable elliptic curves and Jacobians of (hyperelliptic) curves.

In the case that additionally D gives a principal polarization on the Jacobian J of a curve C, we can apply our knowledge about skeleta of Jacobians from Section 1.4. In particular the normalized tropical Riemann theta function plays an important role in the construction of local heights, as we alluded to.

Theorem 2.5.4 ([de Jong and Shokrieh, 18, Theorem A, B]). *Suppose the divisor D defines a principal polarization on J. Embed Φ_v into $\mathrm{Jac}(\Gamma)$ as a subgroup, using Theorem 1.4.4. Then there exists a unique element $\kappa \in \mathrm{Jac}(\Gamma)$, called the* tropical theta characteristic*, and a constant $r \in \mathbb{Q}$, such that $2\kappa = 0 \in \mathrm{Jac}(\Gamma)$ and*

$$\gamma_v = \|\Psi\| \circ \tau_\kappa + r$$

on $\Phi_v \subseteq \mathrm{Jac}(\Gamma)$. Here τ_κ denotes translation by $\kappa \in \mathrm{Jac}(\Gamma)$ and $\|\Psi\|$ is the normalized tropical Riemann theta function from Definition 1.4.6.

The tropical theta characteristic $\kappa \in \mathrm{Jac}(\Gamma)$ is a 2-torsion point. The 2-torsion points of a real lattice Λ inside the Voronoi cell $\mathrm{Vor}(0)$ are calculated in [Conway and Sloane, 16] to be exactly the zero vector and the *Voronoi vectors* of Λ. A vector $v \in \Lambda$ is called a Voronoi vector if the perpendicular bisector of the line segment between 0 and v, which is the set $\{x \in \mathbb{R}^n : \langle x, v\rangle = \frac{1}{2}\langle v, v\rangle\}$, has non-empty intersection with the Voronoi cell $\mathrm{Vor}(0)$.

Finding the tropical theta characteristic is a highly non-trivial task. A real torus of dimension n has exactly 2^n 2-torsion points, so there are only finitely many options for κ. In practice, formulas are calculated for all of these 2-torsion points. By using existing bounds for local height it can then be shown which of these formulas is actually correct, and hence what the tropical theta characteristic actually is. In Chapter 4, we take a similar approach, in that we present formulas for γ_v for all possible options for κ. We then show consequences of certain assumptions on κ, that give heuristic

argumentation for why these assumptions are reasonable.

If we impose that $\gamma_v(0) = 0$, as in Theorem 2.5.2, we get

$$r = -(\|\Psi\| \circ \tau_\kappa)(0) = -\|\Psi\|(\kappa) = -\frac{1}{2}\langle k, k\rangle, \tag{2.18}$$

for $k \in \mathrm{Vor}(0) \subseteq H_1(\Gamma, \mathbb{R})$ a representative of κ.

Remark 2.5.5. The normalization $\gamma_v(0) = 0$ is useful in the sense that γ_v is rational, with a clearly bounded denominator, but in light of summing to the canonical (global) height, it may not be desirable. Ideally we want the constant c in Theorem 2.4.15 to vanish. We already saw that this happens if we normalize with respect to a second division polynomial, see Remark 2.4.13.

Another way of normalization, that interacts well with Theorem 2.5.4, is set by requiring that the integral of the canonical local height over the skeleton is zero. In [de Jong and Shokrieh, 18, Definition 7.1] this is called the *normalized canonical local height*. For our purposes this means the value r in Theorem 2.5.4 is chosen to equal

$$r = -\int_{\mathrm{Jac}(\Gamma)} (\|\Psi\| \circ \tau_\kappa)\, d\mu_H = -\int_{\mathrm{Jac}(\Gamma)} \|\Psi\|\, d\mu_H,$$

where μ_H is the Haar measure on the torus $\mathrm{Jac}(\Gamma)$, which has the property $\int_{\mathrm{Jac}(\Gamma)} d\mu_H = 1$. The last equality comes from the fact that translation on the torus $\mathrm{Jac}(\Gamma)$ does not influence the value of the integral over the entire torus.

For archimedean places we can define a similar normalization, by asking that the integral of $\hat{\lambda}_{D,v}$ over the torus $A(\overline{K_v}) = A(\mathbb{C})$, with respect to the Haar measure on this torus, is zero. If we now choose a canonical local height, associated with D, that is normalized as above, then the constant c in Theorem 2.4.15 is calculated explicitly in [de Jong and Shokrieh, 18, Remark 9.2]. It turns out that $c = -g(C) \cdot h_D(D)$, where $h_D(D)$ is the canonical height of the divisor D itself, as discussed e.g. in [de Jong and Shokrieh, 17].

Chapter 3

Local Heights on Elliptic Curves

In the case that we work on an elliptic curve E, defined over a number field K, Néron [47] and Tate [58], [59] have given explicit formulas for canonical local height functions, associated with the divisor (O), where O is the identity element. In the case of non-archimedean places, these formulas are closed form expressions. The formulas for archimedean places are not closed form, but Tate has given a rapidly converging series, in the case that $K_v = \mathbb{R}$ and no point on the curve has x-coordinate equal to zero. This formula has been generalized by Silverman [Silverman, 55] to include the other cases as well. The full collection of these formulas is known as the 'Tate formulaire'.

These results are easily extended to arbitrary divisors. We can write an arbitrary divisor $D \in \mathrm{Div}(E)$ as sum $D = \sum_{P \in E(\overline{K})} n_P(P)$, where the P where only finitely many of the n_P are non-zero. All translations on elliptic curves are automorphisms and $P + (-P) = O$ in the group law, so that $\tau^*_{-P}(O) = (P)$ as equality of divisors. Thus we can write

$$D = \sum_{P \in E(K)} n_P(P) = \sum_{P \in E(K)} n_P \, \tau^*_{-P}(O).$$

Then by additivity and functoriality of canonical local height (Theorem 2.4.14(c), (d) we get

$$\hat{\lambda}_D(Q) = \sum_{P \in E(K)} n_P \hat{\lambda}_{\tau^*_{-P}O}(Q) + \varepsilon_v = \sum_{P \in E(K)} n_P \hat{\lambda}_O(Q - P) + \varepsilon_v,$$

with ε some M_K-constant.

As noted before, canonical local height functions are unique only up to an M_K-constant. In Equation 3.7, we will define a second division polynomial φ_2. The formulas stated below give the (unique!) canonical local height function, that is normalized with respect to φ_2, see Remark 2.4.13. This normalization corresponds to the classic formulas given by Tate, which are independent of a chosen Weierstrass form, which can be found e.g. in [Silverman, 57, Chapter VI].

Recall that a Weierstrass equation with discriminant Δ for an elliptic curve E/K is minimal at a non-archimedean place $v \in M_K$, if $v(\Delta)$ is minimized, subject to the condition that the coefficients are integral.

We can now state the main result of this chapter.

Theorem 3.0.1 (Tate Formulaire for Elliptic Curves)**.** *Let K be a number field and let E/K be an elliptic curve defined over a number field K, given by a Weierstrass equation*

$$f(x,y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0, \quad a_i \in K, \tag{3.1}$$

with discriminant Δ. Define $c_4 = (a_1^2 + 4a_2)^2 - 48a_4 - 24a_1a_3$. Let φ_2 and φ_3 be the second and third division polynomials, defined in Equations 3.7 and 3.8.

Let $v \in M_K$ be a place. Then the canonical local height, normalized w.r.t. φ_2, at v,

$$\hat{\lambda}_v = \hat{\lambda}_{O,v} : E(K_v)\backslash\{O\} \to \mathbb{R},$$

is given for $P \in E(K_v)\backslash\{O\}$ by the following formulas:

Case I. v is archimedean. Then there exist coefficients $c_n \in \mathbb{R}$, as defined in Theorem 3.1.3, such that

$$\hat{\lambda}_v(P) = \frac{1}{2}c_{-1} + \frac{1}{8}\sum_{n=0}^{\infty} 4^{-n}c_n. \tag{3.2}$$

Case II. v is non-archimedean. Assume the Weierstrass equation in 3.1 is minimal at v. Let $E_0(K_v) = \{P \in E(K_v) : P$ has non-singular reduction modulo $v\}$ (see Section 3.2 for a full definition). We separate the following (mutually exclusive) cases:

(a) $P \in E_0(K_v)$ (i.e. $v(f_x(P)) \leq 0$ or $v(f_y(P)) \leq 0$). Then

$$\hat{\lambda}_v(P) = -\frac{1}{2}\min\{v(x(P)), 0\} + \frac{1}{12}v(\Delta).$$

(b) E has multiplicative reduction at v (i.e. $n := v(\Delta) > 0$ and $v(c_4) = 0$) and $P \notin E_0(K_v)$. Then

$$\hat{\lambda}_v(P) = \frac{n}{2}B_2\left(\frac{i}{n}\right),$$

with $i = \min\{v((2y + a_1x + a_3)(P)), \frac{n}{2}\}$. Here $B_2(T) = T^2 - T + \frac{1}{6}$ is the second Bernoulli polynomial.

(c) E has additive reduction at v (i.e. $v(\Delta), v(c_4) > 0$) and $P \notin E_0(K_v)$. Then

$$\hat{\lambda}_v(P) = \begin{cases} -\frac{1}{3}v(\varphi_2(P)), & \text{if } v(\varphi_3(P)) \geq 3v(\varphi_2(P)) + \frac{1}{12}v(\Delta), \\ -\frac{1}{8}v(\varphi_3(P)), & \text{else.} \end{cases}$$

All cases of this theorem will be proven separately in the Sections 3.1, 3.3, 3.4 and 3.5. A full exposition with alternative proofs can also be found in [Silverman, 57, Chapter VI] and [Silverman, 55][1].

3.0.1 Division Polynomials

To apply the definition of canonical local height functions (Definition 2.4.12), and give a clear definition of our normalization, we need the second division polynomial.

[1]The formulas in Exercise VI.6.8 and VI.6.9 of the first source are off by an additive factor $\frac{1}{12}v(\Delta)$ (see the Errata). The normalization in the second source is different than ours and coincides with requiring that $\gamma_v(0) = 0$, as in Theorem 2.5.2.

This is a rational function $\varphi_2 := \varphi_{O,2} \in \overline{K}(E)$ that satisfies

$$[2]^*(O) = 4(O) + \operatorname{div}(\varphi_2). \tag{3.3}$$

It is only well-defined up to a constant.

We can construct this function explicitly, given a Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in K,$$

for the curve E/K. Define the associated values

$$b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3, \quad b_6 = a_3^2 + 4a_6, \tag{3.4}$$

and

$$b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2. \tag{3.5}$$

The discriminant of the equation can now be given by

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6.$$

We use the doubling formula

$$x([2]P) = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{(2y + a_1x + a_3)^2}, \tag{3.6}$$

for $P = (x, y) \in E(K)\backslash\{O\}$ to see that

$$[2]P = O \quad \iff \quad 2y + a_1x + a_3 = 0, \quad \forall P = (x, y) \in E(K)\backslash\{0\}.$$

That is, the polynomial $2y + a_1x + a_3$ satisfies $\operatorname{div}(2y + a_1x + a_3) = [2]^*O - 4O$. Note that this polynomial is in fact K-rational. To define our second division polynomial we divide by a factor $\Delta^{1/4}$, to get

$$\varphi_2 := \frac{2y + a_1x + a_3}{\Delta^{1/4}}. \tag{3.7}$$

By writing out the change-of-variable formulas (see [Silverman, 56, Table 3.1]), we can see that this function is independent of the choice of Weierstrass equation, and hence is intrinsic to the elliptic curve.

Similarly to this, we have the triplication formula

$$x([3]P) = \frac{f(x)}{(3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8)^2},$$

with f some polynomial in x. We define the third division polynomial

$$\varphi_3 = \frac{3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8}{\Delta^{2/3}}, \tag{3.8}$$

which is independent of the choice of Weierstrass equation and satisfies

$$\operatorname{div}(\varphi_3) = [3]^*(O) - 9(O). \tag{3.9}$$

The nth division polynomial φ_5 can be constructed similarly. For $n \geq 5$, there exist recursion relations, that make calculations easier, see e.g. [Silverman, 56, Exercise

3.7]. These satisfy $\operatorname{div}(\varphi_n) = [n]^*(O) - n^2(O)$.

Functoriality and normalization of canonical local heights (Theorem 2.4.14(b), (d)) now imply that

$$\hat{\lambda}_v([n]P) = n^2\hat{\lambda}_v(P) + v(\varphi_n(P)) + \varepsilon, \tag{3.10}$$

for some M_K-constant ε. If We require $\hat{\lambda}_v$ to be normalized with respect to φ_2, as we do in this chapter, the term ε vanishes.

3.1 Archimedean Places

Let $v \in M_K^\infty$ be an archimedean place. Then v comes from an embedding $\rho : K \hookrightarrow \mathbb{C}$, and we have $v(x) = -\log|\rho(x)|$, where $|\cdot|$ is the standard absolute value on $\mathbb{C}$. We have in this case that $K_v = \mathbb{R}$ or $K_v = \mathbb{C}$, and via ρ we can see K_v as subfield of $\mathbb{C}$. We can thus use the theory of elliptic curves over $\mathbb{C}$ to construct a canonical local height function, and evaluate this at points $P \in E(K_v)$, by using the embedding ρ.

The classical Uniformization Theorem says that any elliptic curve over $\mathbb{C}$ is isomorphic to $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subseteq \mathbb{C}$. Such an isomorphism is given as follows. Let α and β be closed paths on $E(\mathbb{C})$ forming a basis for $H_1(E, \mathbb{Z})$. Then the periods

$$\omega_1 = \int_\alpha \frac{dx}{y}, \quad \omega_2 = \int_\beta \frac{dx}{y}$$

are linearly independent over $\mathbb{R}$ and generate a lattice Λ. This lattice has a discriminant $\Delta(\Lambda)$ (see e.g. [Silverman, 56, p. VI.3.6]). Then the map

$$F : E(\mathbb{C}) \to \mathbb{C}/\Lambda, \quad F(P) = \int_O^P \frac{dx}{y} \pmod{\Lambda},$$

is a complex analytic isomorphism of Lie groups. This agrees with the construction in Theorem 1.2.4, and the fact that an elliptic curve is isomorphic to its own Jacobian, Theorem 1.1.8.

We can use this uniformization to give the canonical local height of an elliptic curve at an archimedean place $v \in M_K$. We use the Weierstrass σ-function

$$\sigma(z) = \sigma_\Lambda(z) = z \prod_{\omega \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2},$$

which is equal to the hyperelliptic σ-function defined in Definition 1.2.5 in the case $g = 1$. The quasi-period map associated with Λ is the map $\eta = \eta_\Lambda : \Lambda \to \mathbb{C}$, given by

$$\eta(\omega) = \frac{d}{dz}\log\sigma(z)\Big|_{u+\omega} - \frac{d}{dz}\log\sigma(z)\Big|_u, \quad \omega \in \Lambda, u \in \mathbb{C},$$

which is independent of $u \in \mathbb{C}$. This map can be extended linearly to a map $\eta : \mathbb{C} \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{C}$.

We can now make sense of the functions $\sigma_v, \eta_v : E(K_v) \to \mathbb{C}$, by embedding $E(K_v)$ into $E(\mathbb{C})$ via ρ, and using the isomorphism $F : E(\mathbb{C}) \to \mathbb{C}/\Lambda$. That is, we define $\sigma_v(P) := (\sigma \circ F \circ \rho)(P)$ and $\eta_v(P) := (\eta \circ F \circ \rho)(P)$.

Theorem 3.1.1. *Let E/K be an elliptic curve. Let $v \in M_K^\infty$ be an archimedean place,*

coming from an embedding $\rho : K \hookrightarrow \mathbb{C}$. Let Λ be a lattice in $\mathbb{C}$, such that $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$. Then the canonical local height function at v,

$$\hat{\lambda}_v : E(K_v)\backslash\{O\} \to \mathbb{R},$$

is given by the formula

$$\begin{aligned}\hat{\lambda}_v(P) &= -\log\left|e^{-\frac{1}{2}F(\rho(P))\eta_v(P)}\sigma_v(P)\Delta(\Lambda)^{\frac{1}{12}}\right| \\ &= \frac{1}{2}\operatorname{Re}\Big(F(\rho(P))\eta_v(P)\Big) - \log|\sigma_v(P)| - \frac{1}{12}\log|\Delta(\Lambda)|,\end{aligned}$$

for all $P \in E(\overline{K_v})\backslash\{O\}$.

We will prove the analogous statement for canonical local heights at archimedean places on Jacobians of curves of genus $g \geq 1$ (of which this is the special case $g = 1$) in Theorem 4.2.2. For a full proof we hence refer either to Theorem 4.2.2 and leave it to the reader to verify the statements coincide, or to [Silverman, 57, Section IV.3], where this formula is proven specifically for elliptic curves and it is shown this formula is normalized with respect to φ_2, as is desired.

This formula seems easy, but it requires the evaluation of the transcendental function σ, so in practice it can be hard to compute the local height via this formula, especially without the help of computer packages. Tate [59] has given an alternative formula, given by an easy to compute series. This formula only works in the case that the completion K_v at v is isomorphic to $\mathbb{R}$, and no rational point on the curve has x-coordinate equal to 0. Silverman [55] has extended this result to include the case $K_v \cong \mathbb{C}$, in which case there are always points with x-coordinate equal to 0, by the fundamental theorem of algebra. We prove Tate's formula and briefly show the extension by Silverman.

Theorem 3.1.2. *Let E/K be an elliptic curve defined by the Weierstrass equation*

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let $v \in M_K^\infty$ be a place, such that $K_v = \mathbb{R}$. Define b_2, b_4, b_6, b_8 as in Equations 3.4 and 3.5. Suppose that $E(K_v)$ contains no points with x-coordinate equal to 0. Define the functions

$$t = \frac{1}{x}, \quad z = 1 - b_4t^2 - 2b_6t^3 - b_8t^4,$$

which are by assumptions well-defined. Then the local height function at v is given by the converging sequence

$$\hat{\lambda}_v(P) = -\frac{1}{2}v(x(P)) - \frac{1}{8}\sum_{n=0}^{\infty}\frac{1}{4^n}v(z([2^n]P)) + \frac{1}{12}v(\Delta).$$

Proof. Rewriting the doubling formula 3.6, yields

$$t([2]P) = \frac{\Delta^{1/2}\varphi_2^2(P)}{x(P)^4z(P)}, \tag{3.11}$$

for $P \in E(\mathbb{R})\backslash\{O\}$. By assumption $|x(P)| \geq \varepsilon$ for some $\varepsilon > 0$ and all $P \in E(\mathbb{R})\backslash\{O\}$. Then t is bounded from above as a function, and so is z. Now also $t \circ [2]$ is bounded from above, and z has no zero in common with t and φ_2, so that z is also bounded

away from zero. That is, $v \circ z = -\log|z|$ is a bounded function, and

$$\mu(P) = \sum_{n=0}^{\infty} \frac{1}{4^n} v(z([2^n]P))$$

is an absolute convergent series. This series satisfies $\mu(2P) = 4\mu(P) - 4v(z(P))$ We also have from Equation 3.11 that

$$-v(x([2]P)) = v(t([2]P)) = \frac{1}{2}v(\Delta) + 2v(\varphi_2(P)) - 4v(x(P)) - v(z(P)).$$

Now define $\lambda(P) = -\frac{1}{2}v(x(P)) - \frac{1}{8}\mu(P) + \frac{1}{12}v(\Delta)$. Note that μ is continuous w.r.t. the v-adic topology, and $\mu(O) = 0$, as $z(O) = 1$. That is, $-\frac{1}{8}\mu(P) + \frac{1}{12}v(\Delta)$ is bounded in a v-adic neighbourhood of O. As there are only finitely many archimedean places, this means λ is a local height function, associated with O, at v, by Remark 2.4.7. Furthermore, λ satisfies

$$\begin{aligned}
\lambda([2]P) &= -\frac{1}{2}v(x([2]P)) - \frac{1}{8}\mu([2]P) + \frac{1}{12}v(\Delta) \\
&= v(\varphi_2(P)) - 2v(x(P)) - \frac{1}{2}v(z(P)) - \frac{1}{2}\mu(P) + \frac{1}{2}v(z(P)) + \frac{1}{3}v(\Delta) \\
&= 4\left(-\frac{1}{2}v(x(P)) - \frac{1}{8}\mu(P) + \frac{1}{12}v(\Delta)\right) + v(\varphi_2(P)) \\
&= 4\lambda(P) + v(\varphi_2(P)).
\end{aligned}$$

That is, λ is a canonical local height function, associated with O, normaized w.r.t. φ_2, which is what we are looking for. □

As the coefficients $v(z([2^n]P))$ in the series are bounded, we get an error of the order 4^{-N}, by stopping the sum at $n = N$.

In the case that there are points with x-coordinate equal to 0, e.g. when $K = \mathbb{C}$, we can get the problem that the coefficients $c_n := v(z([2^n]P))$ get too large, and the sum does not converge. To resolve this issue, we can switch over to another sequence c'_n, when this happens. This new sequence is then essentially the same sequence $c'_n = v(z'([2^n]P))$, where z' is calculated with the parameter $x' = x + 1$. We then continue with the sequence c'_n until that sequence gets too large, after which we return to the inital sequence c_n. This process results in a series that is convergent, and defines a local height function.

To state the new theorem, we introduce the following notation. Let t, w, z be the functions as before, and let t', w', z' be these functions after the substitution $x' = x+1$. The explicit formulas can be found in [Silverman, 55, Equation (14)]. Define the sets

$$U = \{Q \in E(K) : |t(Q)|_v \leq 2\}, \quad U' = \{Q \in E(K) : |t'(Q)|_v \leq 2\},$$

which satisfy $E(K) = U \cup U'$, see [Silverman, 55, Lemma 2.1].

Theorem 3.1.3. *Define a sequence of real numbers c_n, $n \geq -1$ and a sequence of Boolean values β_n, $n \geq -1$ as follows:*

$$c_{-1}, \beta_{-1} = \begin{cases} -v(t(P)), & 1, \quad \text{if } P \in U, \\ -v(t'(P)), & 1, \quad \text{if } P \notin U. \end{cases}$$

$$c_n, \beta_n = \begin{cases} v(z([2^n]P)), & 1, \ \textit{if } \beta_{n-1} = 1, \ [2^{n+1}]P \in U, \\ v(z([2^n]P) + w([2^n]P)), & 0, \ \textit{if } \beta_{n-1} = 1, \ [2^{n+1}]P \notin U, \\ v(z'([2^n]P)), & 1, \ \textit{if } \beta_{n-1} = 0, \ [2^{n+1}]P \in U', \\ v(z'([2^n]P) - w'([2^n]P)), & 0, \ \textit{if } \beta_{n-1} = 0, \ [2^{n+1}]P \notin U'. \end{cases}$$

Then the local height function at v is given by

$$\hat{\lambda}_v(P) = \frac{1}{2}c_{-1} + \frac{1}{8}\sum_{n=0}^{\infty} 4^{-n}c_n.$$

The proof can be found in [Silverman, 55, Theorem 2.2]. The error introduced by stopping the sum after N terms is again of the order 4^{-N}, as the coefficients c_n are bounded.

3.2 Reduction Types

From now on we work with non-archimedean places $v \in M_K^0$. We can classify semistable reduction types at v of elliptic curves by analyzing semistable reduction graphs, defined in Definition 1.3.16. By Proposition 1.3.18, we only have to classify graphs of genus 1.

Theorem 3.2.1. *Let $v \in M_K^0$ be a non-archimedean place. Let R_v be the valuation ring of the completion K_v, with maximal ideal $\mathfrak{m}_v$. Let E be an elliptic curve over K_v with minimal proper regular model $\mathcal{C}/R_v$ and assume E has semistable reduction at v. Then the special fiber $\mathcal{C}_v := \mathcal{C}_{\mathfrak{m}_v}$ has one of the following forms:*

Type $\mathbf{I}_0$. *$\mathcal{C}_v$ is a non-singular curve of genus 1 (i.e. E has good reduction at v).*

Type $\mathbf{I}_1$. *$\mathcal{C}_v$ is a curve with a node, whose normalization has genus 0.*

Type $\mathbf{I}_n$. *$\mathcal{C}_v$ consists of $n \geq 2$ curves of genus 0, arranged in the shape of an n-gon.*

With two genus 0 curves in the shape of a 2-gon we mean two non-singular curves with genus 0, that intersect transversally in two distinct points.

Proof. As noted, we will make these classifications based on the reduction graph $\Gamma = (V, E, w)$, with genus

$$g(\Gamma) = |E| - |V| + 1 + ws(\Gamma) = 1.$$

By Proposition 1.3.10(a) the special fiber $\mathcal{C}_v$ is connected, so we only consider connected graphs. From this we get the bound $|E| \geq |V| - 1$. Therefore we must have $ws(\Gamma) \leq 1$. We consider two cases.

$\boldsymbol{ws(\Gamma) = 1}$. We have $|E| = |V| - 1$, which means Γ is a path graph. If $|V| > 1$, there are two vertices of valency 1. At least one of them is of weight 0, which is not allowed, as C has semistable reduction. The only remaining option, which also defines a valid reduction graph, is the graph with one vertex of weight 1 and no edges. This corresponds to $\mathcal{C}_v$ being a non-singular curve of genus 1, which is Type I_0.

$\boldsymbol{ws(\Gamma) = 0}$. We have $|E| = |V|$ and all vertices have weight 0. The graph is connected, so it is constructed from a path graph with $|V|$ vertices by adding one extra edge. If $|V| = 1$, this can be done in only one way, resulting in the graph with one vertex and one edge, connecting the vertex with itself. This corresponds to $\mathcal{C}_v$ being a genus 0 curve, with one self-intersection, i.e. with a node. This is Type I_1. Suppose $|V| \geq 2$. To end up with a valid reduction graph, we must resolve the two outer vertices of the path graph with valency 1. This can only be done by connecting these two vertices by an edge. This results in a cycle graph. Now the corresponding fiber has $n = |V|$ components of genus 0 that intersect the other curves twice, which is Type I_n. □

These reduction types can also be studied classically, by looking at the reduction of the Weierstrass equation at a place $v \in M_K$. For this, we embed K into the completion K_v of K at v, and hence $E(K)$ into $E(K_v)$. We can then take a minimal Weierstrass equation, with coefficients a_i in the valuation ring R_v. We reduce these coefficients a_i modulo the maximal ideal $\mathfrak{m}_v$ of R_v to $\widetilde{a}_i \in k_v = R_v/\mathfrak{m}_v$. The resulting curve, defined over k_v, given by

$$\widetilde{E} : y^2 + \widetilde{a}_1 xy + \widetilde{a}_3 y = x^3 + \widetilde{a}_2 x^2 + \widetilde{a}_4 x + \widetilde{a}_6,$$

may be singular. Note that this curve is exactly the special fiber at $\mathfrak{m}_v$ of the minimal Weierstrass model of E over R_v. The chord-and-tangent operation makes the subset $\widetilde{E}_{ns}(k_v)$ of non-singular points on $\widetilde{E}$, defined over k_v, into a group.

If $P \in E(K_v)$ is given by homogeneous coordinates $P = (x : y : z)$, with $x, y, z \in R_v$ and at least one in $R^\times$, we get a reduced point

$$\widetilde{P} = (\widetilde{x} : \widetilde{y} : \widetilde{z}) \in \widetilde{E}(k_v).$$

This defines a reduction map

$$E(K_v) \to \widetilde{E}(k_v), \quad P \mapsto \widetilde{P}.$$

The subset

$$E_0(K_v) = \{P \in E(K_v) \mid \widetilde{P} \in \widetilde{E}_{ns}(k_v)\} \subseteq E(K_v) \tag{3.12}$$

is a subgroup, which can be checked by carefully analyzing the reduction of the group law (see e.g. [Silverman, 56, Proposition VII.2.1]). The group $E_0(K_v)$ called the group of points with smooth reduction. We can easily check if $P \in E_0(K_v)$ from the Weierstrass equation. The point $\widetilde{P}$ is non-singular if the partial derivatives, with respect to x and y, of the reduced Weierstrass equation $\widetilde{E}$ do not both vanish. This is equivalent to $v(f_x(P)) \leq 0$ or $v(f_y(P)) \leq 0$, where $f = y^2 + a_1 xy + a_3 y - x^3 - a_2 x^2 - a_4 x - a_6$ defines the original Weierstrass equation.

An elliptic curve E/K with reduction type I_n for $n \geq 1$ at a place v, is classicallty said to have *multiplicative reduction* at v. In this case the group of non-singular points $E_{ns}(\overline{k_v})$ is isomorphic to the multiplicative group $\overline{k_v}^\times$. The curve $\widetilde{E}(k_v)$ then has a node.

In all other cases of bad reduction (i.e. E does not have semistable reduction at v) we say that E has *additive reduction*. In this case the group $E_{ns}(\overline{K})$ is isomorphic to the additive group $\overline{K}^+$. The curve $\widetilde{E}(k_v)$ then has a cusp.

We can read off the reduction type of an elliptic curve from its minimal Weierstrass equation.

Proposition 3.2.2. *Let E/K be an elliptic curve and let $v \in M_K^0$. Let*

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

be a minimal Weierstrass equation for E at v and let Δ be the discriminant of this equation and let $c_4 = (a_1^2 + 4a_2)^2 - 48a_4 - 24a_1a_3$.

(a) E has good reduction at v if and only if $v(\Delta) = 0$ (i.e. $\Delta \in R_v^\times$).

(b) E has multiplicative reduction at v if and only if $v(\Delta) > 0$ and $v(c_4) = 0$ (i.e. $\Delta \in \mathfrak{m}_v$ and $c_4 \in R_v^\times$). The specific reduction type is I_n with $n = v(\Delta)$.

(c) E has additive reduction at v if and only if $v(\Delta), v(c_4) > 0$ (i.e. $\Delta, c_4 \in \mathfrak{m}_v$).

This proposition is classical and proven e.g. in [Silverman, 56, Proposition III.1.4] by analyzing the reduction of the Weierstrass equation. The fact that $n = v(\Delta)$ in Type I_n reduction is calculated in Tate's algorithm, which we discuss in more detail below.

For additive reduction, it is not yet clear what the special fiber looks like. Fortunately, there is a complete classification of all possibilities for special fibers of elliptic curves, due to Néron [46] and Kodaira [29]. The proof is essentially a combinatorial argument on the possible intersections of the irreducible components of the special fiber. The possible fibers can be seen in Figure 3.1. As the Néron model is the largest smooth subscheme of the minimal proper regular model (see Theorem 1.3.8), we can read off the component group Φ_v of the Néron model by looking at the irreducible components of multiplicity 1.

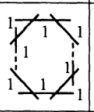
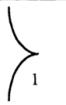
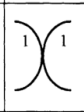
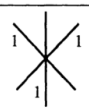
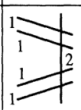
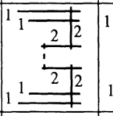
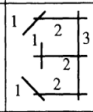
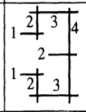
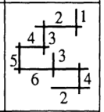
Kodaira symbol	I_0	I_n $(n \geq 1)$	II	III	IV	I_0^*	I_n^* $(n \geq 1)$	IV*	III*	II*
Special fiber $\tilde{\mathcal{C}}$ (The numbers indicate multiplicities)										
$m =$ number of irred. components	1	n	1	2	3	5	$5+n$	7	8	9
$E(K)/E_0(K)$ $\cong \tilde{\mathcal{E}}(k)/\tilde{\mathcal{E}}^0(k)$	(0)	$\frac{\mathbb{Z}}{n\mathbb{Z}}$	(0)	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$ (n even); $\frac{\mathbb{Z}}{4\mathbb{Z}}$ (n odd)	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	(0)

FIGURE 3.1: The Kodaira-Néron classification of special fibers of the minimal proper regular model of elliptic curves, with Kodaira symbols. The group $E(K)/E_0(K)$ here is calculated using Tate's algorithm and coincides with the component group Φ_v of the Néron model. Taken from [Silverman, 57, Table 4.1]

Given a (not necessarily minimal) Weierstrass equation, we can find the corresponding Kodaira symbol of the special fiber (and many more important quantities associated with the elliptic curve) via Tate's algorithm. The algorithm essentially performs a sequence of blow-ups to the singular points of the Weierstrass equation, until it is clear which special fiber arises. The algorithm is described and proven in [Silverman, 57, Algorithm IV.9.4].

The algorithm also computes a minimal Weierstrass equation for the elliptic curve, but it is inefficient if that is the only goal. Checking if a Weierstrass equation, with discriminant Δ is minimal is relatively easy. Any change of coordinates of a Weierstrass equation, resulting in a new integral Weierstrass equation, yields a new discriminant $\Delta' = u^{-12}\Delta$, for some $u \in K_v^\times$. That is, the valuation of the discriminant changes in multiples of 12. Thus if $0 \leq v(\Delta) < 12$, we can conclude the equation is minimal. If we cannot draw this conclusion, we can use an efficient algorithm due to Laska [35] to find a minimal equation.

3.3 Good reduction

In the case that an elliptic curve E/K has good reduction at a certain non-archimedean place $v \in M_K^0$, we know that the canonical local height at v is given by the intersection multiplicity of a point P with the divisor O, up to the addition of an M_K-constant. In the case of elliptic curves, we can extend this result, not only in the case of good reduction, but for any type of reduction, provided that P has smooth reduction at v, i.e. $P \in E_0(K_v)$.

Theorem 3.3.1. *Let E/K be an elliptic curve and let*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

be a minimal Weierstrass equation at v for E, with discriminant Δ. Then the canonical local height function at v,

$$\hat{\lambda}_v : E(K_v)\backslash\{O\} \to \mathbb{R},$$

is given by the formula

$$\hat{\lambda}_v(P) = -\frac{1}{2}\min\{v(x(P)), 0\} + \frac{1}{12}v(\Delta), \quad \textit{for all } P \in E_0(K_v).$$

Proof. Let $P \in E_0(K)$. Then P defines a divisor (P) on the generic fiber of the minimal Weierstrass model $\mathcal{W}/R$ of E/K. By definition of $E_0(K)$, the thickening $\overline{(P)}$ of this divisor lands in the largest smooth subscheme $\mathcal{W}^0$ of $\mathcal{W}$. By Proposition 1.3.9, we get that $\mathcal{W}_0^v \cong \mathcal{N}_v^0$, the identity component of the special fiber of the Néron model of E/K. Then the thickening $\overline{(P)}$ of the divisor (P) on the generic fiber of the Néron model, which coincides with the (unique) associated section $\overline{P} : S \to \mathcal{N}$, also lands in $\mathcal{N}_v^0$. By definition of the specialization map $\mathrm{sp} : E(K) \to \Phi_v$, we get that $\mathrm{sp}(P) = 0 \in \Phi_v$. By Theorem 2.5.1, we get

$$\hat{\lambda}_v(P) = i_v(O, P) + \varepsilon(v),$$

for some M_K-constant ε.

We calculate the intersection multiplicity $i_v(-2(O), P) = -2i_v(O, P)$, using Remark 1.3.13, by constructing the thickening $\overline{-2(O)} = -2\overline{(O)}$ of $-2(O)$ as Cartier divisor on $\mathcal{N}$.

Set P_1, P_2 the two points of vanishing on E of the function x. Note that the divisor of the functions $1, x \in K_v(E) = K_v(\mathcal{N})$ on $\mathcal{N}$ are given by

$$\mathrm{div}_{\mathcal{N}}(1) = 0, \quad \mathrm{div}_{\mathcal{N}}(x) = \overline{(P_1)} + \overline{(P_2)} - 2\overline{(O)} + F,$$

for some fibral divisor $F \subseteq \mathcal{N}_v$. Define

$$U = \mathcal{N} \backslash \overline{(O)} \quad \text{and} \quad V = \mathcal{N} \backslash \Big(|\overline{(P_1)}| \cup |\overline{(P_2)}| \cup |F| \Big).$$

From this we see that the pairs $(U, 1), (V, x)$ are pairs representing $-2\overline{(O)}$. Now this is not yet a full description of $-2\overline{(O)}$ as a Cartier divisor, as U and V do not cover $\mathcal{N}$. However, it suffices for our purposes, as the generic fiber $\mathcal{N}_\eta \cong E$ is included in $U \cup V$, since $F|_{\mathcal{N}_\eta} = 0$. Then for $P \in E(K_v)$, we can use the pair $(U, 1)$ to calculate $i_v(O, P) = -\frac{1}{2} i_v(-2O, P) = 0$ if $\overline{P} \cap \overline{(O)} = \emptyset$. This latter requirement is exactly saying that P does not reduce to O modulo v, which is equivalent to $v(x(P)) \geq 0$. If $P \in E(K_v)$ does reduce to O, so that $v(x(P)) < 0$, we use the set V to calculate $i_v(O, P) = v(x(P))$. Combining these results, we get

$$i_v(O, P) = -\frac{1}{2} i_v(-2O, P) = \begin{cases} 0, & v(x(P)) \geq 0, \\ -\frac{1}{2} v(x(P)), & v(x(P)) < 0 \end{cases} = -\frac{1}{2} \min \{ v(x(P)), 0 \}.$$

Now all that remains to do is to calculate the M_K-constant ε. This is done in the proof of [Silverman, 57, Theorem VI.4.1]. There it is calculated that $\varepsilon(v) = \frac{1}{12} v(\Delta)$, so that

$$\hat{\lambda}_v(P) = -\frac{1}{2} \min \{ v(x(P)), 0 \} + \frac{1}{12} v(\Delta)$$

defines a canonical local height function at v, normalized with respect to φ_2. We note additionally that $v(\Delta) = 0$ if E has good reduction at v, so that $\hat{\lambda}_v = i_v(O, P)$, which coincides with the normalization introduced in Remark 2.5.5. In the case of bad reduction, this coincides with the calculations $\gamma_v(0) = \frac{1}{12} v(\Delta)$ in Sections 3.4 and 3.5. □

Remark 3.3.2. If E has good reduction at v, then $E_0(K_v) = E(K_v)$ and $v(\Delta) = 0$, so

$$\hat{\lambda}_v(P) = -\frac{1}{2} \min \{ v(x(P)), 0 \}$$

defines a canonical local height function on the entirety of $E(K_v) \backslash \{O\}$.

Remark 3.3.3. Note that by definition of the identity component of the Néron model, the thickening of the divisor (O) lands in the identity component of the special fiber, and not in any other component. That is, if $P \notin E_0(K)$, the intersection multiplicity $i_v(O, P)$ is zero. Now by Theorem 2.5.1, we can say

$$\hat{\lambda}_v(P) = \gamma_v(\mathrm{sp}(P)), \quad \forall P \notin E_0(K). \tag{3.13}$$

3.4 Multiplicative Reduction

Let $v \in M_K^0$ again be a non-archimedean place, and suppose now that the elliptic curve has multiplicative reduction at v. Then the reduction type (see Theorem 3.2.1) is Type I_n, with $n = v(\Delta) \geq 1$. We know from the result of previous section that we already have formulas for $P \in E_0(K)$. Also from the discussion at the end of previous section, we know that $i_v(O, P) = 0$. Using Theorem 2.5.1, we only need to calculate the local correction term $\gamma_v : \Phi_v \to \mathbb{R}$ to compute the canonical local height. As (O)

gives a principal polarization on E, we can calculate this local correction term at v via Theorem 2.5.4.

This method is dependent on finding which of the (finitely many) 2-torsion points in the skeleton is the tropical theta characteristic κ. This is difficult to do by theoretical means. What this means in this case, is that we find two formulas for the canonical local height at v, that are equally likely to be correct based on Theorem 2.5.4. To find out which of these formulas is actually correct, and hence to find out what the theta characteristic is, we compare them to the formula for canonical local height, as calculated in e.g. [Silverman, 57, Theorem VI.4.2] and [Lang, 30, Theorem III.5.1], which both use the theory of the *Tate curve*. Although this method thus does not give one concrete formula, and is hence unsatisfactory in some way, we still present the calculations using this method in this section, as it is not yet present in literature and is easier to generalize to curves of higher genus.

To calculate the correction term using Theorem 2.5.4, we need to consider the Jacobian $\mathrm{Jac}(\Gamma)$ of the reduction graph $\Gamma = \Gamma_v$ of E at v, which is a cycle graph with n vertices, $x_n = x_0, \ldots, x_{n-1}$ ordered clockwise and n edges. The vertex x_0 represents the identity component of the special fiber of the Néron model. We order the edges $e_1, \ldots, e_n$, such that $\iota(e_i) = (x_{i-1}, x_i)$. The graph is pictured in Figure 3.2.

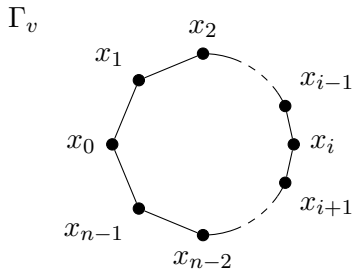


FIGURE 3.2: The reduction graph Γ_v of the special fiber of E, with multiplicative reduction of type I_n.

The Jacobian of this graph is calculated in Example 1.4.2. We get $\mathrm{Jac}(\Gamma) \cong \mathbb{R}/n\mathbb{Z}$. The Voronoi region $\mathrm{Vor}(0)$ is given by the interval $\mathrm{Vor}(0) = [-\frac{n}{2}, \frac{n}{2}] \subseteq \mathbb{R} \cong H_1(\Gamma, \mathbb{R})$.

The inner product on $H_1(\Gamma, \mathbb{R})^* \cong \mathbb{R}$ is given by $\langle v, w \rangle = v^T \Omega^{-1} w = v^T \left(\frac{1}{n}\right) w$.

We now use this to do the calculation that leads to a formula for the canonical local height, in the case of multiplicative reduction. We choose here the normalization, as discussed in Remark 2.5.5. We will see below that it coincides with the normalized with respect to the second division polynomial φ_2, that we use in the rest of the chapter.

Theorem 3.4.1. *Let E/K be an elliptic curve with v-minimal discriminant Δ. Suppose that E has multiplicative reduction at v of type I_n, with $n = v(\Delta)$. Then the canonical local height function at v,*

$$\hat{\lambda}_v : E(K_v) \backslash \{O\} \to \mathbb{R},$$

is given by the formula

$$\hat{\lambda}_v(P) = \frac{i(i-n)}{2n} + \frac{n}{12} = \frac{n}{2} B_2\left(\frac{i}{n}\right), \tag{3.14}$$

for $P \notin E_0(K_v)$, with $\mathrm{sp}(P) = i \in \mathbb{Z}/n\mathbb{Z} \cong \Phi_v$, and $0 < i < n$. Here $B_2(T) = T^2 - T + \frac{1}{6}$ is the second Bernoulli polynomial.

Proof. As noted in the beginning of this chapter, we have $\hat{\lambda}_v(P) = \gamma_v(\mathrm{sp}(P))$ for $P \notin E_0(K_v)$. We use Theorem 2.5.4 and the discussion above to calculate γ_v.

Let $P \in E(K_v)\backslash E_0(K_v)$ and suppose P maps to i mod $n \in \mathbb{Z}/n\mathbb{Z} \cong \Phi_v$, with $0 < i < n$. We assume that E has multiplicative reduction of type I_n^*, so that the component group Φ_v is cyclic of order n. We use the isomorphism of E with its Jacobian J, via the Abel-Jacobi map, based at O (see Theorem 1.1.8). Then using the commutative square 1.17, we have

$$(\Phi_{x_0} \circ \mathrm{sp}_E)(P) = (\mathrm{sp}_J \circ \Phi_O)(P) = [x_i - x_0] \in \mathrm{Jac}(\Gamma).$$

A path vector between path vector between the vertices x_i and x_0 is given by

$$\delta = \Big(\mathbf{1}_{j \le i}(j)\Big)_{1 \le j \le n} = (1, \dots, 1, 0, \dots, 0),$$

where $\mathbf{1}_S$ denotes the indicator function on S. Then the vector w representing δ in $H_1(\Gamma, \mathbb{R})^* \cong \mathbb{R}$ is $w = (\langle \delta, \alpha \rangle) = (i)$. The tropical theta characteristic $\kappa \in \mathrm{Jac}(\Gamma)$ is a 2-torsion point, so it is either equal to $0 \in \mathbb{R}/n\mathbb{Z}$ or $\frac{n}{2} \in \mathbb{R}/n\mathbb{Z}$. We will calculate two possible formulas $\hat{\lambda}_v^1$ and $\hat{\lambda}_v^2$ for the local height, assuming either of these options.

Case I. $\kappa = 0$. A representative for w in $\mathrm{Vor}(0)$ is given by

$$\widetilde{w} = (\langle \delta, \alpha \rangle) = (i), \quad \text{if } i \le \frac{n}{2}, \qquad \widetilde{w} = (i - n), \quad \text{if } i > \frac{n}{2}.$$

We get

$$\hat{\lambda}_v^1(P) = \|\Psi\|(w) = \begin{cases} \frac{1}{2}\langle i, i \rangle, & i \le \frac{n}{2}, \\ \frac{1}{2}\langle i - n, i - n \rangle, & i > \frac{n}{2} \end{cases} = \begin{cases} \frac{i^2}{2n}, & i \le \frac{n}{2}, \\ \frac{(i-n)^2}{2n}, & i > \frac{n}{2}. \end{cases}$$

To find r, we use the formula in Remark 2.5.5, and calculate

$$\begin{aligned} r = -\int_{\mathbb{R}/n\mathbb{Z}} \|\Psi\| \, d\mu_H &= -\int_0^{n/2} \frac{x^2}{2n} \frac{dx}{n} - \int_{n/2}^{n} \frac{(x-n)^2}{2n} \frac{dx}{n} \\ &= -\left[\frac{x^3}{6n^2}\right]_0^{n/2} - \left[\frac{(x-n)^3}{6n^2}\right]_{n/2}^{n} \\ &= -\frac{n}{48} - \frac{n}{48} = -\frac{n}{24}, \end{aligned}$$

so that

$$\hat{\lambda}_v^1(P) = \begin{cases} \frac{i^2}{2n} - \frac{n}{24}, & i \le \frac{n}{2}, \\ \frac{(i-n)^2}{2n} - \frac{n}{24}, & i > \frac{n}{2}. \end{cases}$$

Case II. $\kappa = \frac{n}{2}$. We evaluate $\|\Psi\|$ on a representative of $w + \kappa$ inside $\mathrm{Vor}(0)$. This representative is given by $\widetilde{w} = (i - \frac{n}{2})$ for all $0 < i < n$. We get

$$\hat{\lambda}_v^2(P) = \|\Psi\|(\widetilde{w}) + r = \frac{1}{2}\left\langle i - \frac{n}{2}, i - \frac{n}{2} \right\rangle + r = \frac{(i - \frac{n}{2})^2}{2n} + r.$$

Note that $\int_{\mathrm{Jac}(\Gamma)}(\|\Psi\|\circ\tau_\kappa)\,d\mu_H = \int_{\mathrm{Jac}(\Gamma)}\|\Psi\|\,d\mu_H$, so that $r = -\frac{n}{24}$ again. We find that

$$\hat{\lambda}_v^2(P) = \frac{(i-\frac{n}{2})^2}{2n} - \frac{n}{24} = \frac{i(i-n)}{2n} + \frac{n}{12} = \frac{n}{2}B_2(\frac{i}{n}).$$

Now based on Theorem 2.5.4, we cannot decide which of the formulas $\hat{\lambda}_v^1$ and $\hat{\lambda}_v^2$ should hold. Based on comparison with the formulas for canonical local height in the case of multiplicative reduction in [Silverman, 57, Theorem VI.4.2] and [Lang, 30, Theorem III.5.1], we find both that $\hat{\lambda}_v^2$ defines a canonical local height at v, and that the normalization of Remark 2.5.5 coincides with the normalization with respect to φ_2, that is used in loc cit., as well as in the rest of this chapter. □

The formula in Theorem 3.4.1 is not yet explicit, as it is dependent on the specialization map. We can manipulate the duplication formula to get this specialization map specific (modulo multiplication by -1) for the case of Type I_n reduction. Note that the local height formula is invariant under $P \mapsto -P$ and hence the substitution $i \mapsto n-i$, so that knowing the specialization map modulo multiplication by -1 is enough to get the formula (3.14) explicit.

Lemma 3.4.2. *Let E/K be an elliptic curve with Type I_n reduction at v. Then the specialization map, restricted to the set $E_0(K_v)^c$, is given, up to sign, by*

$$\mathrm{sp} : E_0(K)^c \to \Phi_v, \quad P \mapsto \pm\min\left\{v((2y+a_1x+a_3)(P)), \frac{n}{2}\right\}.$$

Proof. Assume for now that $\mathrm{sp}(P) = i \bmod n$, with $0 < i \le \frac{n}{2}$. If $0 < i < \frac{n}{2}$, we can substitute the local height function (3.14) in the duplication formula (3.10):

$$\frac{n}{2}B_2\left(\frac{2i}{n}\right) = 2nB_2\left(\frac{i}{n}\right) + v(\varphi_2(P)).$$

After some algebra, this yields

$$i = \frac{n}{4} + v(\varphi_2(P)) = \frac{v(\Delta)}{4} + v((2y+a_1x+a_3)(P)) - \frac{v(\Delta)}{4} = v((2y+a_1x+a_3)(P)).$$

Now if $i = \frac{n}{2}$, then $2P \in E_0(K)$ and we know from Theorem 3.3.1 that $\hat{\lambda}_v(2P) \ge \frac{v(\Delta)}{12} = \frac{n}{12}$. Substituting this in the duplication formula yields

$$\frac{n}{12} \le 2nB_2\left(\frac{1}{2}\right) + v((2y+a_1+a_3)(P)) - \frac{n}{4} = -\frac{5n}{12} + v(\varphi_2(P)),$$

so that

$$v(\varphi_2(P)) \ge \frac{n}{2} = i,$$

and $i = \min\{v(\varphi_2(P)), \frac{n}{2}\}$ holds in either case.

Finally we note that if $\frac{n}{2} < i < n$, then $0 < n-i < \frac{n}{2}$. Then $-\mathrm{sp}(P) = \mathrm{sp}(-P) = n-i$, so that

$$n - i = \min\left\{v(\varphi_2(P)), \frac{n}{2}\right\}, \quad \text{and} \quad i \equiv -\min\left\{v(\varphi_2(P)), \frac{n}{2}\right\} \in \mathbb{Z}/n\mathbb{Z}. \qquad \square$$

3.5 Additive reduction

In the case that an elliptic curve E/K has additive reduction at a non-archimedean place $v \in M_K$, we cannot use techniques from previous section, as we no longer have semistable reduction. It is possible to still get formulas in this case, by finitely extending fields until the elliptic curve has semistable reduction over that field. This is possible by the Semistable Reduction Theorem 1.3.15. Then the formulas from Sections 3.3 and 3.4 hold.

It is, however, not desirable to have to extend fields to get formulas. Based on the structure of the groups Φ_v that we know from Figure 3.1, we can manipulate the duplication and triplication formulas (Equation 3.10, for $n = 2, 3$) to get the local height functions.

Theorem 3.5.1. *Let E/K be an elliptic curve. Suppose that E has additive reduction at v. Then the canonical local height function at v,*

$$\hat{\lambda}_v : E(K_v)\backslash\{O\} \to \mathbb{R},$$

is given by the formula

$$\hat{\lambda}_v(P) = \begin{cases} -\frac{1}{3}v(\varphi_2(P)), & \text{if } v(\varphi_3(P)) \geq 3v(\varphi_2(P)) + \frac{1}{12}v(\Delta), \\ -\frac{1}{8}v(\varphi_3(P)), & \text{else,} \end{cases}$$

for $P \notin E_0(K_v)$.

Proof. E has additive reduction at v, so from Figure 3.1, we see that the reduction is of type II, III, IV, I_0^*, I_n^*, IV*, III*, II*. We see that $\Phi_v = 0$ for the types II and II*. That is, the special fiber $\mathcal{N}_v$ of the Néron model consists only of the identity component $\mathcal{N}_v^0$. As we showed in the proof of Theorem 3.3.1, $P \in E_0(K_v)$ implies the associated section $\overline{P}$ maps to the identity component $\mathcal{N}_v^0$. Thus for the types II and II* there is no point $P \notin E_0(K_v)$, and we have formulas for all $P \in E(K_v) = E_0(K_v)$ via Theorem 3.3.1, and we no longer need to consider these cases.

We use that $\hat{\lambda}_v(-P) = \hat{\lambda}_v(P)$, for all $P \in E(K_v)$. From the table, we see that for all additive reduction types, we have $\#\Phi_v \leq 4$, so that $\mathrm{sp}(nP) = 0$, and $nP \in E_0(K)$, for some $2 \leq n \leq 4$. We separate these three cases.

$\boldsymbol{2P \in E_0(K)}$. The reduction type is either III, I_n^*, or III*. The points P and $3P$ have the same image in Φ_v, so that $\hat{\lambda}_v(3P) = \hat{\lambda}_v(P)$. Filling this in the triplication formula (Equation 3.10 for $n = 3$), yields

$$\lambda(P) = 9\lambda(P) + v(\varphi_3(P)), \quad \text{so} \quad \lambda(P) = -\frac{1}{8}v(\varphi_3(P)). \tag{3.15}$$

By Theorem 3.3.1, we get that $\hat{\lambda}_v(2P) \geq \frac{1}{12}v(\Delta)$. Filling this in the duplication formula (Equation 3.10 for $n = 2$), yields

$$\frac{1}{12}v(\Delta) \leq 4\lambda(P) + v(\varphi_2(P)), \quad \text{so} \quad \lambda(P) \geq -\frac{1}{4}v(\varphi_2(P)) + \frac{1}{48}v(\Delta). \tag{3.16}$$

Combining Equations 3.15 and 3.16 gives us the bound

$$v(\varphi_3(P)) \leq 2v(\varphi_2(P)) - \frac{1}{6}v(\Delta). \tag{3.17}$$

$\boldsymbol{3P \in E_0(K)}$. The reduction type is either IV or IV*, and $\Phi_v \cong \mathbb{Z}/3\mathbb{Z}$. Now P and $-2P$ have the same images in Φ_v, so that $\hat{\lambda}_v(2P) = \hat{\lambda}_v(P)$. We also have $\hat{\lambda}_v(3P) \geq 0$ from Theorem 3.3.1. Then using the duplication and triplication formulas similarly to the previous case, we get

$$\lambda(P) = -\frac{1}{3}v(\varphi_2(P)), \quad \text{and} \quad \lambda(P) \geq -\frac{1}{9}v(\varphi_3(P)) + \frac{1}{108}v(\Delta). \tag{3.18}$$

Combining these gives the bound

$$3v(\varphi_2(P)) \leq v(\varphi_3(P)) - \frac{1}{12}v(\Delta). \tag{3.19}$$

Note now that the bounds in Equations 3.17 and 3.19 cannot both hold at the same time. If they were to hold at the same time, we get $3v(\varphi_2(P)) \leq 2v(\varphi_2(P)) - \frac{1}{4}v(\Delta)$, and so $v(\varphi_2(P)) + \frac{1}{4}v(\Delta) = v((2y + a_1x + a_3)(P)) \leq 0$. But by assumption $P \notin E_0(K)$, which means $v(f_y(P)) = v((2y + a_1x + a_3)(P)) > 0$, where f is the Weierstrass equation (see the definition of $E_0(K)$ in Section 3.2). We can hence separate these two cases by the bound 3.19.

$\boldsymbol{2P, 3P \notin E_0(K)}$. The reduction type is I_n^* with n odd, so $\Phi_v \cong \mathbb{Z}/4\mathbb{Z}$, and the image of P generates Φ_v. Then P and $-3P$ have the same image in Φ_v, so that $\hat{\lambda}_v(3P) = \hat{\lambda}_v(P)$ and we get

$$\lambda(P) = -\frac{1}{8}v(\varphi_3(P)), \tag{3.20}$$

as in the case $2P \in E_0(K)$. The inequality (3.17) is no longer true, but a consequence of Tate's algorithm is that reduction type I_n^* implies that

$$3v(\varphi_2(P)) = \frac{3}{4}n \geq \frac{1}{4}n = v(\varphi_3(P)) - \frac{1}{12}v(\Delta),$$

so that the equality in (3.19) is violated. Thus, we can still use the bound (3.19) to decide whether $\hat{\lambda}_v(P)$ is given by $-\frac{1}{3}v(\varphi_2(P))$ or $-\frac{1}{8}v(\varphi_3(P))$. □

Remark 3.5.2. The denominator 8 in the case that

$$\hat{\lambda}_v(P) = -\frac{1}{8}v(\varphi_3(P)) = -\frac{1}{8}v((3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8)(P)) + \frac{1}{12}v(\Delta)$$

seems to violate Theorem 2.5.2 for the Types III, III*, and I_n^* with n even. For these types, we have $2 \cdot \Phi_v = 0$, so that we expect a denominator of size 4, not 8. This proves, in a rather oblique way, that $v((3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8)(P))$ must be even in this case. Fortunately, we can also prove this fact using Tate's algorithm. For example, for Type III, we get that $v(b_2) \geq 0$, $v(b_4) \geq 1$, $v(b_6) \geq 2$, $v(b_8) = 2$, and $v(x) \geq 1$, so that

$$\begin{aligned} v((3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8)(P)) &= \min\{v(3x^4), v(b_2x^3), v(3b_4x^2), v(3b_6x), v(b_8)\} \\ &= v(b_8) = 2. \end{aligned}$$

For the other reduction types a similar calculation holds. This also proves that we cannot extend the result in Remark 2.5.3 to non-semistable reduction types.

3.6 Examples

In this section we illustrate the use of the formulas constructed in this chapter, by calculating canonical heights on an elliptic curve. We use the mathematical software `SageMath` [51], and specifically the functions described in [52], for all calculations below.

We calculate heights on the elliptic curve E_1, defined over $\mathbb{Q}$, given by the Weierstrass equation

$$E_1/\mathbb{Q} : y^2 + xy = x^3 + x^2 - 50x + 132. \tag{3.21}$$

The minimal discriminant of the equation is $\Delta = -2^4 \cdot 3^4 \cdot 1433$. The curve has no non-trivial torsion elements, and has rank 2, meaning its Mordell-Weil group $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z}^2$. Two of its generators are given by

$$P = (-8, 14) \quad \text{and} \quad Q = \left(\frac{29}{16}, \frac{401}{64}\right).$$

We calculate $\hat{h}(P)$ and $\hat{h}(Q)$, where $\hat{h} := \hat{h}_O$ is the canonical height, associated with the divisor (O). We do this by calculating the local heights $\hat{\lambda}_v := \hat{\lambda}_{O,v}$ of P and Q at all places $v \in M_{\mathbb{Q}}$.

From the discriminant, we see that $0 < v(\Delta) < 12$ for the places $v = 2, 3, 1433$, and $v(\Delta) = 0$ for all other non-archimedean places, so that Equation 3.21 defines a minimal Weierstrass equation at all non-archimedean places $v \in M_{\mathbb{Q}}^0$, and we have good reduction at all non-archimedean places $v \neq 2, 3, 1433$. From Tate's algorithm it follows that at $v = 2, 3, 1433$ there is multiplicative reduction. This can also be seen from the fact that $c_4 = 2425$, which is not divisible by these three primes. At $v = 2, 3$ the reduction type is Type I_4 and at $v = 1433$ it is Type I_1.

Note that $v(x(P)) \geq 0$ for all $v \in M_{\mathbb{Q}}^0$, and $v(x(Q)) \geq 0$ for all $v \in M_{\mathbb{Q}}^0 \backslash \{2\}$. That is, at all non-archimedean places with good reduction, we have $\hat{\lambda}_v(P) = \hat{\lambda}_v(Q) = 0$. Furthermore note that $q_p = \log |p|$ and $q_\infty = 1$, so that we get

$$\hat{\lambda}_v(R) = \log(2)\hat{\lambda}_2(R) + \log(3)\hat{\lambda}_3(R) + \log(1433)\hat{\lambda}_{1433}(R) + \hat{\lambda}_\infty(R), \tag{3.22}$$

for $R = P, Q$. We thus reduced the calculation of the canonical (global) height to the calculation of local heights at only four places. For the calculation of height of these places, we need to know if the points are in $E_0(\mathbb{Q}_v)$, which can be checked using the partial derivatives of the Weierstrass equation f. We get

$$f_x(P) = -112, f_y(P) = 20; \quad f_x(Q) = \frac{10953}{256}, f_y(Q) = \frac{459}{32}.$$

$\boldsymbol{v = 2}$**.** We see that $v(f_x(P)), v(f_y(P)) \geq 0$, but $v(f_x(Q)), v(f_y(Q)) < 0$, so that we have $P \notin E_0(\mathbb{Q}_2)$ and $Q \in E_0(K_v)$. We get from Theorem 3.4.1 and Lemma 3.4.2 that

$$\hat{\lambda}_2(P) = 2B_2\left(\frac{i}{4}\right), \quad \text{for } i = \min\{v(f_y(P)), 2\} = \min\{v(20), 2\} = 2,$$

so that $\hat{\lambda}_2(P) = 2B_2\left(\frac{1}{2}\right) = -\frac{1}{6}$. For Q, we use Theorem 3.3.1, to see that

$$\hat{\lambda}_2(Q) = -\frac{1}{2}\min\{v(x(Q)),0\} + \frac{1}{12}v(\Delta) = -\frac{1}{2}\min\{-4,0\} + \frac{1}{3} = \frac{7}{3}.$$

$v = 3$. We now have $v(f_x(R)), v(f_y(R)) \geq 0$ for $R = Q$, but not for $R = P$, so that we have $Q \notin E_0(\mathbb{Q}_2)$ and $P \in E_0(K_v)$. We calculate similarly to the case $v = 2$, that

$$\hat{\lambda}_3(P) = -\frac{1}{2}\min\{v(x(P)),0\} + \frac{1}{12}v(\Delta) = -\frac{1}{2}\min\{0,0\} + \frac{1}{3} = \frac{1}{3},$$
$$\hat{\lambda}_3(Q) = 2B_2\left(\frac{1}{4}\min\{v(459),2\}\right) = 2B_2\left(\frac{\min\{3,2\}}{4}\right) = 2B_2\left(\frac{1}{2}\right) = -\frac{1}{6}.$$

$v = 1433$. We have Type I_1 reduction, so that the special fiber of the Néron model has only one component, the identity component. That is, all $E(\mathbb{Q}_{1433}) = E_0(\mathbb{Q}_{1433})$, and we use Theorem 3.3.1 in any case. Note that $v(x(P)) = v(x(Q)) = 0$, so that we get

$$\hat{\lambda}_{1433}(P) = \hat{\lambda}_{1433}(Q) = \frac{1}{12}v(\Delta) = \frac{1}{12}.$$

$v = \infty$. We use Theorem 3.1.2, as no $\mathbb{Q}$-rational point on the curve with x-coordinate equal to 0. Denote $s(R) = \frac{1}{8}\sum_{n=0}^{\infty}\frac{1}{4^n}\log\left|z([2^n]R)\right|$ for $R \in E(\mathbb{Q}_\infty) = E(\mathbb{R})$. We can approximate this term by stopping the sum at $n = N$, which introduces an error of order $\mathcal{O}(4^{-N})$. We do this below for $N = 20$, although it should be noted that the the terms $n > 10$ in the sum did not change the first 10 decimals of the result. We get

$$\hat{\lambda}_\infty(P) = \frac{7}{6}\log(2) - \frac{1}{3}\log(3) - \frac{1}{12}\log(1433) + s(P),$$
$$\hat{\lambda}_\infty(Q) = -\frac{7}{3}\log(2) - \frac{1}{3}\log(3) + \frac{1}{2}\log(29) - \frac{1}{12}\log(1433) + s(Q).$$

with

$$s(P) \approx 0.2068941160\ldots, \quad s(Q) \approx 0.5998857943\ldots.$$

The canonical height. We are now ready to combine results and calculate the canonical heights of P and Q. We get from Equation 3.22 and the results above that

$$\hat{h}(P) = \log(2) + s(P) \approx 0.9000412965\ldots,$$
$$\hat{h}(Q) = -\frac{1}{2}\log(3) + \frac{1}{2}\log(29) + s(Q) \approx 1.7342275649\ldots.$$

We can calculate similarly that $\hat{h}(P+Q) = \hat{h}\left(\frac{123752}{24649}, -\frac{33890746}{3869893}\right) = 5.0594198986\ldots$. The canonical regulator of $E_1/\mathbb{Q}$ is the determinant of the matrix

$$\begin{pmatrix} 2\hat{h}(P) & \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q) \\ \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q) & 2\hat{h}(Q) \end{pmatrix},$$

which is thus equal to $0.3621515061\ldots$. This canonical regulator is a constant in the Birch and Swinnerton-Dyer Conjecture, which is one of the unsolved Millennium Prize Problems [13]. To gather evidence for the conjecture, it is thus useful to be able to calculate this canonical regulator.

Chapter 4

Local Heights on Genus 2 Jacobians

In the previous chapter we presented and proved the 'Tate formulaire', that gives a collection of local height functions on elliptic curves, for any reduction type. Our aim in this chapter is to give a similar formulaire for the Jacobian of a curve of genus 2. We construct canonical local heights, associated with the Θ-divisor, as defined in Equation 1.9. As Θ defines a principal polarization, we can use Theorem 2.5.4 to construct local height functions. Also, via these local heights we can construct the canonical height, associated with Θ. This canonical height is used e.g. in Mumford's Theorem (see [Hindry and Silverman, 27, Theorem B.6.5]) that estimates the so-called 'counting function', which calculates the number of rational solutions on a curve of genus g of bounded height.

In this chapter K denotes a number field, with its set of places M_K, and C denotes a smooth projective curve of genus 2, defined over K, given by a Weierstrass equation

$$C : y^2 = F(x) = \lambda_6 x^6 + \lambda_5 x^5 + \cdots + \lambda_1 x + \lambda_0, \quad \lambda_i \in K. \tag{4.1}$$

Note that such a curve is automatically hyperelliptic, by Proposition 1.1.10. Write $J = \mathrm{Jac}(C)$ for the Jacobian of C.

4.1 The divisor Θ

To construct canonical local height functions, associated with the Θ-divisor, via Definition 2.4.12, we first of all need to know what (non-canonical) local height functions look like. To apply Definition 2.4.5, we need to describe Θ as a Cartier divisor. We describe the function field of J in Section 4.1.1 and construct Θ as Cartier divisor. To then check if a local height function is canonical we need to check the duplication formula (2.15). This involves finding the second division polynomial, which we do in Section 4.1.2 and describing multiplication by 2 on the Jacobian explicitly, which we do in Section 4.1.3.

4.1.1 The function field $k(J)$

From the discussion after Equation 1.10, we know we can identify the (open) set $J \backslash \Theta$ with the (open) set

$$D_2 = \{(P_1, P_2) : P_i \neq \infty, P_1 \neq i(P_2)\} \subseteq \mathrm{Sym}^2 C.$$

That is, the varieties J and $\mathrm{Sym}^2 C$ are birationally equivalent. Now from the definition of function fields, birationally equivalent varieties have isomorphic function fields. That is, we can identify the function field $K(J)$ of J with the function field $K(\mathrm{Sym}^2 C)$. Now to give a rational function on $\mathrm{Sym}^2 C$ is to give a function on C^2 that is invariant under the action of S_2, the symmetric group on 2 elements. Via this construction, we always identify symmetric functions on C^2 with functions on J. We use the Weierstrass coordinates x, y on the first copy of C and u, v on the second copy.

Let $\pi : \mathrm{Sym}^2 C \to J$ be the map in Equation (1.10) and $\mathrm{p} : C^2 \to \mathrm{Sym}^2 C$ the natural projection. Note that we can describe the divisor $\Theta \subseteq J$ as the image under $(\pi \circ \mathrm{p})$ of $C \times \{\infty\}$ (or $\{\infty\} \times C$).

Lemma 4.1.1 ([Arledge and Grant, 1, Lemma 1]). *Let $f \in \overline{K}(J)$, and take $\mathfrak{f} = \mathrm{p}^* \pi^* f \in \overline{K}(C)(C)$ (where we consider functions in $\overline{K}(C^2)$ as functions on C with coefficients in $\overline{K}(C)$). Then*

$$\mathrm{ord}_\Theta(f) = \mathrm{ord}_\infty(\mathfrak{f}).$$

Consider the following functions on J:

$$\xi_1 = 1, \quad \xi_2 = x + u, \quad \xi_3 = xu, \quad \xi_4 = \frac{F_0(x,u) - 2yv}{(x-u)^2}, \tag{4.2}$$

with

$$F_0(x,u) = 2\lambda_0 + \lambda_1(x+u) + 2\lambda_2 xu + \lambda_3 xu(x+u) + 2\lambda_4(xu)^2 + \lambda_5(xu)^2(x+u).$$

Now using Lemma 4.1.1, it directly follows that

$$\mathrm{ord}_\Theta(\xi_1) = 0, \quad \mathrm{ord}_\Theta(\xi_2) = \mathrm{ord}_\Theta(\xi_3) = \mathrm{ord}_\infty(x) = -2,$$

and these functions are regular when $x, u, y, z \neq \infty$, so they have no other poles outside Θ. With a bit more work, we can also calculate using this lemma that

$$\begin{aligned}
\mathrm{ord}_\Theta(\xi_4) &= \mathrm{ord}_\Theta(F_0(x,u) - 2yv) - \mathrm{ord}_\Theta((x-u)^2) \\
&\geq \min\{\mathrm{ord}_\Theta(F_0), \mathrm{ord}_\Theta(2yv)\} - 2\,\mathrm{ord}_\Theta(x-u) \\
&\geq \min\{\mathrm{ord}_\infty(x^3), \mathrm{ord}_\infty(y)\} - 2\,\mathrm{ord}_\infty(x) \\
&\geq \min\{-6, -6\} - (-4) = -2.
\end{aligned}$$

The only problem for regularity outside Θ can occur when $x = u$, as then the denominator of ξ_4 vanishes. However, when $(x, y) = (u, v)$, we get that the numerator of ξ_4 is

$$F_0(x,u) - 2yv = F_0(x,x) - 2y^2 = 2(F(x) - y^2),$$

which vanishes with order 2, so that ξ_4 is regular. By analyzing the zeroes of ξ_i, we find that they are also linearly independent functions. We have now proven exactly that

$$\xi_1, \xi_2, \xi_3, \xi_4 \in L(2\Theta).$$

By Riemann-Roch for abelian varieties, we have $\ell(n\Theta) = n^2 \frac{\Theta \cdot \Theta}{2}$ (see [Hindry and Silverman, 27, Remark A.4.6.3.3]). Using the Adjunction Formula ([Hindry and Silverman, 27, Theorem A.4.6.2]) then yields $\Theta \cdot \Theta = 2$, so that $\ell(n\Theta) = n^2$, and $\ell(2\Theta) = 4$. That is, the set $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ actually forms a basis for $L(2\Theta)$.

The divisor -2Θ is base-point free ([Hindry and Silverman, 27, Theorem A.5.3.6]). That is, the divisors of the basis functions $\xi_1, \ldots, \xi_4$ share no common support outside Θ. We can then use these functions to describe -2Θ as a Cartier divisor.

Lemma 4.1.2. *The divisor -2Θ can be given as a Cartier divisor by the pairs*

$$(J\backslash\Theta, \xi_1), (U_2, \xi_2), (U_3, \xi_3), (U_4, \xi_4),$$

where the U_i are defined by $U_i = \{\xi_i \neq 0\}$. In particular $J = (J\backslash\Theta) \cup U_2 \cup U_3 \cup U_4$.

Now that we have a basis for $L(2\Theta)$ we can form the associated rational map (see Definition 1.1.1)

$$\xi := \varphi_{2\Theta} : J \to \mathbb{P}^3, \quad x \mapsto (\xi_1(x) : \xi_2(x) : \xi_3(x) : \xi_4(x)). \tag{4.3}$$

As 2Θ is base-point free, this map is a morphism. It is called the *Kummer map*. It is the analogue of the double cover $x : E \to \mathbb{P}^1$ of an elliptic curve. The image of this map in $\mathbb{P}^3$ is called the *Kummer surface*, denoted by KS. This surface can be given as the vanishing of the polynomial

$$\Xi(\xi_1, \ldots, \xi_4) = \Xi_2\xi_4^2 + \Xi_1\xi_4 + \Xi_0 = 0,$$

with Ξ_0, Ξ_1 and Ξ_2 polynomials in ξ_1, ξ_2, ξ_3 as in [Cassels and Flynn, 14, Equation (3.1.9)], if $\mathrm{char}(K) \neq 2$, and as in [Müller, 41] in general. The divisor 2Θ is not very ample and this map is not an embedding. If we denote by $i : J \to J$ the involution $P \mapsto -P$, then KS is isomorphic to the quotient variety $J/i(J)$, so that the Jacobian is a double cover for KS. Note that for a canonical local height function $\hat{\lambda}$ on J associated with the Θ-divisor, we have that $\hat{\lambda}(-P) = \hat{\lambda}(P)$, by the symmetry of Θ. That is, the canonical local height function descends to a function on KS.

By the Lefschetz embedding theorem ([Lang, 31, Theorem VI.6.1]) the divisor 3Θ is very ample. From Riemann-Roch we get that $\ell(3\Theta) = 9$, so the Jacobian can be embedded in $\mathbb{P}^8$. A basis for $L(4\Theta)$, of dimension 16, is constructed in [Cassels and Flynn, 14, Section 2.1], and gives an embedding of the Jacobian in $\mathbb{P}^{15}$.

The Jacobian of a curve is defined abstractly, which makes explicit calculations difficult. The embedding of the Jacobian in $\mathbb{P}^8$ requires 9 coordinate functions, and is hence also a rather difficult object to work with, due to its size. For this reason, the Kummer surface is used more regularly in literature. For example, in Section 4.5 we present a formula from [Müller and Stoll, 43] for local height functions, associated with Θ, that is proven using the Kummer surface.

4.1.2 Division Polynomials

In light of using the definition (2.4.12) to check if a function is a canonical local height function, associated with Θ, we must construct a second division polynomial. This polynomial, and in general the nth division polynomial, is studied e.g. in [Kanayama, 28] (on the Jacobian of a curve of genus 2) and [Uchida, 60] (for general hyperelliptic curves). In these papers, the division polynomials are constructed over $\mathbb{C}$, which can be generalized to number fields. In [Cantor, 12], a different approach is taken using Mumford coordinates, over a general field, with characteristic unequal to 2. As we work with number fields, we use the approach taken in the first two references.

We start our discussion of division polynomials for $\mathrm{Jac}(C)$ defined over $\mathbb{C}$. In this case, we can use the uniformization over $\mathbb{C}$, as discussed in Section 1.2.2. In particular, we constructed the hyperelliptic σ-function in Definition 1.2.5. From the discussion after this definition, we know that we have

$$\mathrm{div}(\sigma) = \kappa^{-1}\Theta,$$

where $\kappa : \mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\Lambda = \mathrm{Jac}(C)$ is the projection. Now we define the nth division polynomial φ_n by

$$\varphi_n(z) = \frac{\sigma(nz)}{\sigma(z)^{n^2}}, \quad z \in \mathbb{C}^2.$$

A priori this function is only defined on $\mathbb{C}^2$, but we will show in a moment that this function descends to a function on $\mathrm{Jac}(C)$. It is clear that this function has a simple zero whenever $nu \in \kappa^{-1}\Theta$, which is $z \in [n]^*(\kappa^{-1}\Theta)$ and that φ_n has a pole of order n^2 whenever $z \in \kappa^{-1}\Theta$.

Recall from Definition 1.2.7 the hyperelliptic $\wp$-functions

$$\wp_{ij}(z) = -\frac{\partial^2}{\partial z_i \partial z_j}\log\sigma(z), \quad \wp_{ijk}(z) = -\frac{\partial^3}{\partial z_i \partial z_j \partial z_k}\log\sigma(z), \quad \forall i,j,k \in \{1,2,\dots,g\}.$$

If we write $z = \Phi_\infty((x,y),(u,v)) \in \mathbb{C}^g/\Lambda$, where Φ_∞ is the Abel-Jacobi map as in Equation 1.12, then we can compute the functions $\wp_{ij}(z)$ and $\wp_{ijk}(z)$ in terms of x,y,u,v. This comes from classical computations by Baker [5] (see also [Grant, 25, Equation (1.4)]). We present the few of these that are relevant for Equation 4.6:

$$\begin{aligned}
\wp_{12}(z) &= -xu,\\
\wp_{22}(z) &= x+u,\\
\wp_{122}(z) &= -2\frac{yu-vx}{x-u},\\
\wp_{112}(z) &= 2\frac{yu^2-vx^2}{x-u},\\
\wp_{111}(z) &= 2\frac{v\psi(x,u)-y\psi(u,x)}{(x-u)^3}, \quad \text{where}\\
\psi(x,u) &= 4\lambda_0+\lambda_1(3x+u)+2\lambda_2 x(x+u)+\lambda_3 x^2(x+u)\\
&\quad +4\lambda_4 x^3 u+\lambda_5 x^3 u(3x+u).
\end{aligned} \tag{4.4}$$

Note that $\wp_{12}$ and $\wp_{22}$ coincide, up to sign, with the Kummer coordinates ξ_3 and ξ_2, from Equation 4.2, respectively. Also the other $\wp$-functions written out above coincide with the basis of 16 functions constructed in [Cassels and Flynn, 14, Section 2.1], that embed the Jacobian into $\mathbb{P}^{15}$.

Another classical formula from Baker [5] (see also [Arledge and Grant, 1] for a formula in greater generality) that encodes addition on the Jacobian is the following:

$$\frac{\sigma(z+w)\sigma(z-w)}{\sigma(z)^2\sigma(w)^2} = -\wp_{11}(z)+\wp_{11}(w)-\wp_{12}(z)\wp_{22}(w)+\wp_{22}(z)\wp_{12}(w) =: q(z,w).$$

We then get

$$\frac{\sigma(z+w)}{\sigma(z)^2\sigma(w)^2} = \frac{q(z,w)}{\sigma(z-w)}.$$

Now taking the limit $w \to z$ and using the Taylor expansion of σ (Equation 1.14) to see that $\partial\sigma(z-w)/\partial z_1|_{w=z} = 1$, we get

$$\frac{\sigma(2z)}{\sigma(z)^4} = \left.\frac{\partial q(z,w)}{\partial z_1}\right|_{z=w} = \wp_{12}(z)\wp_{122}(z) - \wp_{22}(z)\wp_{112}(z) - \wp_{111}(z), \tag{4.5}$$

so that

$$\varphi_2(z) = \wp_{12}(z)\wp_{122}(z) - \wp_{22}(z)\wp_{112}(z) - \wp_{111}(z). \tag{4.6}$$

Now the right-hand side of this equation is well-defined on J, so that $\varphi_2 \in \mathbb{C}(J)$. From the definition using σ-functions we now also can see that we have the desired relation

$$\operatorname{div}(\varphi_2) = [2]^*\Theta - 4\Theta. \tag{4.7}$$

If we now consider the Jacobian of a genus 2 curve, defined not over $\mathbb{C}$, but over a number field K, we see that the right-hand side of Equation (4.7) is a K-rational divisor. Moreover, if we use Equation 4.6 as a definition of φ_2, we see that $\varphi_2 \in K(J)$, as the $\wp$-functions are all K-rational, by their definition in Equation 4.4 and our description of $K(J)$ in Section 4.1. That is, the equality in (4.7) is an equality of K-rational divisors and φ_2 as given in Equation 4.6 is the second division polynomial on J.

4.1.3 Duplication

The third ingredient we need to compute canonical local heights via the definition is a way to calculate $2P$ for arbitrary points $P \in J(K)\backslash\Theta$. Theoretically this is very easy. The doubling of the element $[D] \in \operatorname{Pic}^0(C) = J$ is simply the class of the divisor $2D$. Also if we work with the completion of a number field at an archimedean place, we can use our knowledge about Jacobians over $\mathbb{C}$ from Section 1.2.2. However, if we work with functions in $K(J)$ as constructed in Section 4.1.1, we want elements in $J\backslash\Theta$ to be represented by a divisor of the form $D = (P_1) + (P_2) - (\infty^+) - (\infty^-)$. Then the divisor $2D$ is not of the correct form, so we need to reduce it to a divisor of the correct form, before we can explicitly evaluate a function on $2P = 2[D]$. This is not easy to do in practice.

Of course, the duplication map also induces a map that can be explicitly given by polynomials by embedding the Jacobian as a surface in $\mathbb{P}^8$. In fact, duplication commutes with negation, so that duplication is already well-defined on the Kummer surface KS of J. We get a map

$$\delta = (\delta_1, \delta_2, \delta_3, \delta_4) : KS \to KS, \tag{4.8}$$

such that

$$\xi(2P) = \delta(\xi(P)),$$

where ξ is the Kummer map from Equation 4.3. The polynomials δ_i are of degree 4. The full polynomials are rather large and can be found in [Flynn, 23, Appendix C] (taking up almost two pages). More generally, there are is a 4×4 matrix B consisting of biquadratic forms B_{ij}, that satisfy

$$B_{ij}(\xi(P), \xi(Q)) = (\xi_i(P+Q)\xi_j(P-Q) - \xi_j(P+Q)\xi_i(P-Q)).$$

This is the closest we can come to describing addition on KS, as we cannot distinguish $P+Q$ and $P-Q$ on KS. Explicit formulas for B can be found in [Flynn, 23,

Appendix B].

The methods described above either require the division and factorization of polynomials or use formulas that are rather long. Therefore, it is hard to 'guess' a function that possibly defines a canonical local height function using the defining equation, as is the structure of proof in [Silverman, 57, Chapter VI] for elliptic curves. We, for this reason, will only take this approach for archimedean places, and rely on other methods, described in Section 2.5, to find canonical local height functions at non-archimedean places.

4.2 Archimedean Places

To find a canonical local height function at an archimedean place $v \in M_K$, we can use the theory of Jacobians over $\mathbb{C}$, as in Section 1.2.2. This allows us to give local canonical heights at archimedean places not only on Jacobians of curves of genus 2, but in fact on Jacobians of hyperelliptic curves of arbitrary genus $g \geq 1$. Note that the case of $g = 1$ coincides with the formulas for elliptic curves in Section 3.1.

We follow the construction of $\hat{\lambda}_v$ as in Section 7 of [Uchida, 60]. We work with a Jacobian J of a hyperelliptic curve C of genus $g \geq 1$, that hence admits an isomorphism $J(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$, for some lattice Λ. We identify $J(\mathbb{C})$ as the complex torus in this way. There is a canonical projection $\kappa : \mathbb{C}^g \twoheadrightarrow \mathbb{C}^g/\Lambda = J(\mathbb{C})$.

In Section 1.2.2, we constructed the hyperelliptic σ-functions on $\mathbb{C}^g$ by

$$\sigma(z) = c \exp\left(\frac{1}{2} z^T E_1 \Omega_1^{-1} z\right) \theta_\tau \begin{bmatrix} \delta \\ \delta' \end{bmatrix} (\Omega_1^{-1} z),$$

with notation explained in the aforementioned section. The divisor of the hyperelliptic σ-function is the pull-back along κ of the Θ-divisor on J.

To ease notation, let us write $\boldsymbol{e}(z) := \exp(2\pi i z)$. Now we define the function $\Sigma(z)$ on $\mathbb{C}^g$ by

$$\Sigma(z) = \boldsymbol{e}\left(-\frac{1}{2} L(z, z)\right) \sigma(z),$$

where L is the $\mathbb{C}$-valued $\mathbb{R}$-bilinear form in Equation 1.15.

Proposition 4.2.1. *The function $|\Sigma(z)|$ on $\mathbb{C}^g$ is periodic with respect to Λ.*

Proof. This follows from the translation relation 1.2.6. For $z \in \mathbb{C}^g$ and $l \in \Lambda$, we get

$$\begin{aligned}
|\Sigma(z+l)| &= \left|\boldsymbol{e}\left(-\frac{1}{2} L(z+l, z+l)\right)\right| \left|\boldsymbol{e}\left(L\left(z + \frac{1}{2} l, l\right)\right) \sigma(z)\right| \\
&= \left|\boldsymbol{e}\left(-\frac{1}{2} L(z,z)\right)\right| \left|\boldsymbol{e}\left(-\frac{1}{2}(L(z,l) + L(l,z))\right)\right| \left|\boldsymbol{e}\left(-\frac{1}{2} L(l,l)\right)\right| \\
&\quad \left|\boldsymbol{e}\left(\frac{1}{2} L(l,l)\right)\right| \left|\boldsymbol{e}\left(L(z,l)\right)\right| |\sigma(z)| \\
&= \left|\boldsymbol{e}\left(\frac{1}{2}(L(z,l) - L(l,z))\right)\right| \left|\boldsymbol{e}\left(-\frac{1}{2} L(z,z)\right)\right| |\sigma(z)| \\
&= \left|\boldsymbol{e}\left(\frac{1}{2} E(z,l)\right)\right| |\Sigma(z)|,
\end{aligned}$$

where $E(z,l) = L(z,l) - L(l,z)$ is real-valued, so that

$$\left| \boldsymbol{e}\left(\frac{1}{2}E(z,l)\right)\right| = 1.$$

This proves the proposition. □

Theorem 4.2.2 ([Uchida, 60, Theorem 7.4])**.** *Let C be a hyperelliptic curve of genus $g \geq 1$, defined over a number field K, with Jacobian J that admits an isomorphism $J(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$. Let $v \in M_K^\infty$ be an archimedean place. Define a function $\hat{\lambda}_v : (J\backslash\Theta)(K_v) \to \mathbb{R}$ by*

$$\hat{\lambda}_v(P) = v(\Sigma(z)), \quad \forall P \in (J\backslash\Theta)(K_v),$$

where $z \in \mathbb{C}^g$ is such that $\kappa(z) = P$. Then $\hat{\lambda}_v$ is a local canonical height function, associated with Θ, at v.

Proof. By Proposition 4.2.1, $\hat{\lambda}_v$ is well-defined. We first prove $\hat{\lambda}_v$ is a local height function associated with Θ. Let (U, f) be a pair representing Θ as Cartier divisor. Define a function g on $\kappa^{-1}(U)$ by $g(w) = \sigma(w)/f(\kappa(w))$. Then g has no zeroes or poles, so g and $1/g$ are holomorphic on $\kappa^{-1}(U)$. Therefore $v(g(z))$ is continuous, and bounded on $\kappa^{-1}(U)$. By the definition, we have

$$\hat{\lambda}_v(P) = v(f(P)) + v(g(z)) - \pi \operatorname{Im} L(z,z),$$

which is now of the right form, by Remark 2.4.7. Thus $\hat{\lambda}_v$ is a local height function, associated with Θ.

Now to prove it is a *canonical* local height function, we calculate

$$\begin{aligned}
\hat{\lambda}_v([2]P) &= v\left(\boldsymbol{e}\left(-\frac{1}{2}L(2z,2z)\right)\sigma(2z)\right) \\
&= v\left(\boldsymbol{e}\left(-\frac{4}{2}L(z,z)\right)\sigma(z)^4\frac{\sigma(2z)}{\sigma(z)^4}\right) \\
&= 4v\left(\boldsymbol{e}\left(-\frac{1}{2}L(z,z)\right)\sigma(z)\right) + v\left(\frac{\sigma(2z)}{\sigma(z)^4}\right) \\
&= 4\hat{\lambda}_v(P) + v(\varphi_2(P)),
\end{aligned}$$

which finishes the proof. □

4.3 Reduction types

From now on we work with non-archimedean places $v \in M_K^0$. We let K_v be the completion of K at v, with R_v its valuation ring. Namikawa and Ueno [45] have given a complete geometrical classification of the special fibers of the mpr models of genus 2 curves over R_v. There turn out to be over a hundred reduction types. Among these there are seven semistable types. Just as we did in Theorem 3.2.1, we can classify all these semistable reduction types by analyzing semistable reduction graphs.

To give this classification, we first classify all *stable* reduction type. A curve is said to have stable reduction, if the special fiber of the mpr model has only nodes as

singularities and all of its rational components intersect other components at least three times (compared to Definition 1.3.14).

Theorem 4.3.1. *Let $v \in M_K^0$ be a non-archimedean place. Let R_v be the valuation ring of the completion K_v, with maximal ideal $\mathfrak{m}_v$. Let C be a curve of genus 2 over K_v with mpr model $\mathcal{C}/R_v$ and assume C has stable reduction at v. Then the special fiber $\mathcal{C}_v := \mathcal{C}_{\mathfrak{m}_v}$ has one of the following seven forms*

Type I. *$\mathcal{C}_v$ is a non-singular curve of genus 2.*

Type II. *$\mathcal{C}_v$ is an elliptic curve with a node.*

Type III. *$\mathcal{C}_v$ is a genus 0 curve with two nodes.*

Type IV. *$\mathcal{C}_v$ consists of two genus 0 curves, crossing transversally in three different points.*

Type V. *$\mathcal{C}_v$ consists of two elliptic curves crossing transversally in one point.*

Type VI. *$\mathcal{C}_v$ consists of a genus 0 curve with a node and an elliptic curve, crossing transversally in one point.*

Type VII. *$\mathcal{C}_v$ consists of two genus 0 curves with a node, crossing transversally in one point.*

The numbering of the types corresponds to the numbering in [Liu, 36].

Proof. Similarly to the proof of Theorem 3.2.1, we will make these classifications based on the reduction graph $\Gamma = (V, E, w)$, with genus

$$g(\Gamma) = |E| - |V| + 1 + ws(\Gamma) = 2.$$

We consider only stable reduction types, so the valence of all vertices with weight 0 must be at least 3. By Proposition 1.3.10(a) the special fiber $\mathcal{C}_v$ is connected, so we only consider connected graphs. From this we get the bound $|E| \geq |V| - 1$. Therefore we must have $ws(\Gamma) \leq 2$. We consider three cases.

$\boldsymbol{ws(\Gamma) = 2}$. We have $|E| = |V| - 1$, which means Γ is a path graph. In a path graph all vertices have valency ≤ 2, so there cannot be any weight 0 vertices. Then Γ consists of either one vertex of weight 2 and no edges, or two vertices of weight 1 connected by one edge. The first case corresponds to $\mathcal{C}_v$ being a non-singular genus 2 curve, which is Type I. The second corresponds to $\mathcal{C}_v$ consisting of two elliptic curves crossing in one point, which is Type V.

$\boldsymbol{ws(\Gamma) = 1}$. We have $|E| = |V|$. There is one vertex of weight 1, which has valency at least 1, and all others have weight 0, with valency at least 3. From this we can get a bound on the amount of vertices:

$$3(|V| - 1) + 1 = 3|V| - 2 \leq \sum_{v \in V} \mathrm{val}(v) = 2|E| = 2|V|.$$

That is, $|V| \leq 2$. If $|V| = 1$, we have one vertex of weight 1, with one edge connecting the vertex to itself. This corresponds to $\mathcal{C}_v$ being an elliptic curve with a node. This is Type II. If $|V| = 2$, we have two vertices, one of weight 1 and the other of weight 0, and two edges. One of the edges connects the vertices. In order for the weight

0 vertex to have valency 3, the other edge should connect the weight 0 vertex with itself. This graph corresponds to $\mathcal{C}_v$ consisting of a genus 0 component with a node and a genus 1 curve, crossing in one point. This is Type VI.

$\boldsymbol{ws(\Gamma) = 0}$. We have $|E| = |V| + 1$. All vertices have weight 0, so they must have valency at least 3. We get the bound:

$$3|V| \leq \sum_{v \in V} \mathrm{val}(v) = 2|E| = 2|V| + 2.$$

That is, $|V| \leq 2$. If $|V| = 1$, we have one vertex and two edges, that both connect the vertex to itself. This corresponds to $\mathcal{C}_v$ being a genus 0 curve with two nodes. This is Type III. If $|V| = 2$, we have three edges. At least one of the edges connects the two vertices. To get a valency of 3 for both edges, either all three edges connect the vertices to each other, or both vertices have an additional edge connecting the vertex to itself. The first case corresponds to $\mathcal{C}_v$ consisting of two genus 0 curves crossing in three different points. This is Type IV. The second case corresponds to $\mathcal{C}_v$ consisting of two genus 0 curves with a node, crossing in one point. This is Type VII. □

We can get all the semistable reduction types from these stable reduction types. To construct semistable reduction types that are not stable we need to consider reduction graphs Γ that have vertices v of weight 0 with valency (exactly) 2. Note that the valency cannot come from an edge that connects the vertex v to itself, as then we would get a disconnected graph. That is, the vertex v connects to two (possibly coinciding) vertices v^- and v^+ with a single edge. The graph is a refinement of the graph constructed from Γ by removing v and its two edges and instead connecting v^+ and v^- by a single edge.

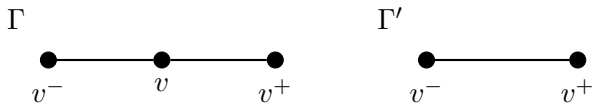


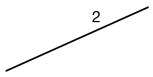
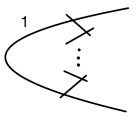
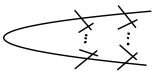
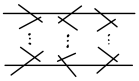
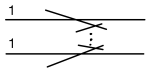
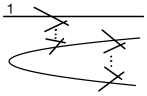
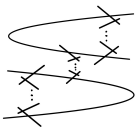
FIGURE 4.1: The graph Γ' is obtained from Γ by removing a vertex of valence 2.

By repeating this process, we can "remove" any vertices of weight 0 with valency 2 from a semistable reduction graph. The resulting graph is then a stable reduction graph. Conversely, any semistable reduction graph is a refinement of a stable reduction graph, which is to say that it is constructed from a stable reduction graph by 'adding vertices' on an edge.

Refining an edge with n vertices in a reduction graph corresponds in the special fiber $\mathcal{C}_v$ to replacing an intersection (or node) with a chain of n curves of genus 0 (which below we call an n-chain). We then get a classification of all semistable reduction types in Table 4.1. The notation is taken from [Namikawa and Ueno, 45].

There is an algorithm of Liu [39] that calculates the reduction type of genus 2 curve, and gives the corresponding stable reduction type that the curve has after field extension. This stable reduction type can also be read off from the so-called *Igusa invariants*, which are expressions in the coefficients of the Weierstrass equation, see [Liu, 37, Théorème 1]. This method is similar to Proposition 3.2.2. Finally, we note that the semistable reduction type can also be read off from the *cluster picture* of C,

TABLE 4.1: Classification of all semistable reduction types of genus 2. The numbers in the depiction of the special fiber denote the genus of the component, if it is ≥ 1.

Semistable reduction type	**Index range**	**Corresponding stable reduction type**	**Depiction of the special fiber $\mathcal{C}_v$**
$[I_{0-0-0}]$		Type I	
$[I_{n-0-0}]$	$n \geq 1$	Type II	
$[I_{n-p-0}]$	$n, p \geq 1$	Type III	
$[I_{n-p-q}]$	$n, p, q \geq 1$	Type IV	
$[I_0 - I_0 - m]$	$m \geq 1$	Type V	
$[I_n - I_0 - m]$	$n, m \geq 1$	Type VI	
$[I_n - I_p - m]$	$n, m, p \geq 1$	Type VII	

which is a description of the configuration of the roots of the Weierstrass equation. This is done in [Dokchitser et al., 19, Section 18].

4.4 Semistable reduction

Let $v \in M_K^0$ be a non-archimedean place of K. For the semistable reduction types found in the previous section, we can use Theorem 2.5.4, to calculate the correction term γ_v in Theorem 2.5.1. This approach is similar to that in Section 3.4. In that section, we could give an explicit expression for the canonical local height, in the case of semistable reduction, as we could describe the intersection multiplicity $i_v(O, P)$ explicitly (see Theorem 3.3.1 and Remark 3.3.3). We gave this description by using the connection between the minimal Weierstrass model of an elliptic curve and the Néron model and the fact that O is the identity element, which maps by definition to the identity component of the special fiber of the Néron model. For the divisor Θ on a genus 2 Jacobian, we do not have such theorems. The intersection multiplicity $i_v(\Theta, P)$ of a point with the Theta divisor, is thus much harder to find. The results in this section therefore only give the correction term γ_v and not a full formula for $\hat{\lambda}_v$. In Section 4.5 we will use these formulas to relate the intersection multiplicity to the Kummer map (Equation 4.3).

Calculating γ_v via the formula in Theorem 2.5.4 requires finding the tropical theta characteristic κ. As we noted before, this is hard to do. The characteristic κ is a 2-torsion point in the Jacobian of the reduction graph, so that there are only finitely many options. We thus can give multiple formula for γ_v by assuming κ is given by one of these finitely many options. One of these formulas is correct.

In this section, go through all semistable reduction types of C, and calculate γ_v. We use the normalization $\gamma_v(0) = 0$, which means we calculate the constant in Theorem 2.5.4 as in Equation 2.18. This will make for a better comparison with the results in Section 4.5.

Type $[I_{0-0-0}]$.
If C has reduction type $[I_{0-0-0}]$ at v, then C has good reduction at v, which implies that J has good reduction. Then γ_v is a constant, and our chosen normalization yields

$$\gamma_v = 0. \tag{4.9}$$

Type $[I_0 - I_0 - m]$.
If C has reduction type $[I_0 - I_0 - m]$ at v, for some $m \geq 1$, then the reduction graph Γ_v is a path graph. All cycles on this graph are trivial, so that the Jacobian of Γ_v and hence the component group Φ_v is trivial. The Jacobian J thus has good reduction at v and γ_v is again a constant, and with our normalization we again get

$$\gamma_v = 0. \tag{4.10}$$

Type $[I_{n-0-0}]$.
Next consider the case that C has reduction type $[I_{n-0-0}]$ at v, with $n \geq 1$. The reduction graph Γ_v is a cycle graph with n vertices. Note that this graph differs from the reduction graph of an elliptic curve with reduction type I_n only by the vertex weights. The vertex weights are not relevant in the definition of the Jacobian of Γ_v and the embedding of Φ_v in the Jacobian. That is, the calculations necessary to determine γ_v for J are exactly as those done in Section 3.4 for elliptic curves with reduction Type I_n. We can hence draw the same conclusion on the function γ_v as we did in that section, except that we now choose a different normalization. Also the arguments that we used to determine the tropical theta characteristic do not hold in this case. That is, the tropical theta characteristic is represented by either $k = 0$, or $k = \frac{n}{2}$ in $\mathrm{Jac}(\Gamma_v) \cong \mathbb{R}/n/Z$. This yields

$$\gamma_v(i \bmod n) = \frac{i(i-n)}{2n}, \quad \text{for } 0 \leq i < n, \quad \text{or} \quad \gamma_v(i \bmod n) = \begin{cases} \frac{i^2}{2n}, & 0 \leq i \leq \frac{n}{2}, \\ \frac{(i-n)^2}{2n}, & \frac{n}{2} < i < n. \end{cases} \tag{4.11}$$

Type $[I_n - I_0 - m]$.
The reduction graph Γ_v of a genus 2 curve C with reduction type $[I_n - I_0 - m]$ with $n, m \geq 1$ is a path with m edges attached to one of the vertices of a cycle graph with n vertices. Jacobians of graphs contract paths, so that the Jacobian of Γ_v is isomorphic to the Jacobian of a cycle graph with n vertices. That is, the function γ_v of J is calculated in the same way as for the reduction type $[I_{n-0-0}]$.

Type $[I_{n-p-0}]$.
Now suppose C has reduction type $[I_{n-p-0}]$, with $n, p \geq 1$. The reduction graph Γ_v

is a union of two cycle graphs with respectively n and p vertices, that are identified in one point. We label the vertices $x_0, \dots, x_n$, and $y_0, \dots, y_p$, where we identify $x_0 = x_n = y_0 = y_n$, as in Figure 4.3 on the left. We label the edge from x_i to x_{i+1} by e_i and the edge from y_j to y_{j+1} by e_{n+j}. The homology group $H_1(\Gamma, K) \cong K^2$ is generated by the cycles $\alpha_1 = [e_1 + \cdots + e_n]$ and $\alpha_2 = [e_{n+1} + \cdots + e_{n+p}]$. As vectors these are represented by $\alpha_1 = (\mathbf{1}_{i \le n}(i)) - (1, \dots, 1, 0, \dots, 0)$ and $\alpha_2 = (\mathbf{1}_{i > n}(i)) = (0, \dots, 0, 1, \dots, 1)$. Then the Gram matrix, and its inverse, are

$$\Omega = \big(\langle \alpha_i, \alpha_j \rangle\big)_{1 \le i,j \le 2} = \begin{pmatrix} n & 0 \\ 0 & p \end{pmatrix}, \quad \Omega^{-1} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{p} \end{pmatrix}.$$

We get

$$\mathrm{Jac}(\Gamma_v) = \mathbb{R}^2 / \Omega \mathbb{Z}^2,$$

so that the lattice in $H_1(\Gamma, \mathbb{R})^* \cong \mathbb{R}^2$ is spanned by the vectors $v_1 = (n, 0)^T$ and $v_2 = (0, p)^T$. Via Corollary 1.4.4, we get that the component group Φ_v is isomorphic to the cokernel of Ω, which is $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

The Voronoi region $\mathrm{Vor}(0)$ is the region $[-\frac{n}{2}, \frac{n}{2}] \times [-\frac{p}{2}, \frac{p}{2}] \subseteq \mathbb{R}^2$. We are now in the case of $p_{12} = 0$ in [Conway and Sloane, 16, Section 6], so that there are 4 pairs of Voronoi vectors, $\pm v_1, \pm v_2, \pm(v_1 + v_2), \pm(v_1 - v_2)$. Note that $\pm\frac{1}{2}(v_1 + v_2)$ is identified with $\pm\frac{1}{2}(v_1 - v_2)$ in the torus. The tropical theta characteristic $\kappa \in \mathrm{Jac}(\Gamma)$ is thus given by the class of either of the following four vectors:

$$k_1 = (0, 0), \quad k_2 = \left(\frac{n}{2}, 0\right), \quad k_3 = \left(0, \frac{p}{2}\right), \quad k_4 = \left(\frac{n}{2}, \frac{p}{2}\right).$$

We assume for the moment that κ is represented by $k_4 = (\frac{n}{2}, \frac{p}{2})$, so that our normalization requires via Equation 2.18 that

$$r = \frac{1}{2}\langle k_4, k_4 \rangle = -\frac{n}{8} - \frac{p}{8}.$$

Now let $P \in J(K_v)$ and write $P = [(P_1) - (P_2)]$, with $P_1, P_2 \in C(K_v)$. Without loss of generality (see Remark 1.4.5), we can assume that P_1 and P_2 specialize to the vertices x_i and y_j respectively, with $0 \le i < n$ and $0 \le j < p$. This means $\mathrm{sp}(P) = (i, j) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \cong \Phi_v$. Now a path vector from x_i to y_j is given by

$$\delta = \Big(\mathbf{1}_{n+1 \le k \le n+j}(k) - \mathbf{1}_{1 \le k \le i}(k)\Big)_{1 \le k \le n+p} = (-1, \dots, -1, 0, \dots, 0, 1, \dots, 1, 0, \dots, 0).$$

This path is represented in $H_1(\Gamma, \mathbb{R})^* \cong \mathbb{R}^2$ by the vector $w = \big(\langle \delta, \alpha_1 \rangle, \langle \delta, \alpha_2 \rangle\big)^T = (-i, j)^T$.

A representative for $w + \kappa$ inside $\mathrm{Vor}(0)$ is given by $\widetilde{w} = (-i + \frac{n}{2}, j - \frac{p}{2})$, so

$$\begin{aligned} \gamma_v(\mathrm{sp}(P)) = \gamma_v(i, j) &= (\|\Psi\| \circ \tau_\kappa)(w) + r = \|\Psi\|(\widetilde{w}) + r = \frac{1}{2}\langle \widetilde{w}, \widetilde{w} \rangle + r \\ &= \frac{1}{2}\widetilde{w}^T \Omega^{-1} \widetilde{w} + r \\ &= \frac{(i - \frac{n}{2})^2}{2n} + \frac{(j - \frac{p}{2})^2}{2p} - \frac{n}{8} - \frac{p}{8} \\ &= \frac{i(i-n)}{2n} + \frac{j(j-p)}{2p}. \end{aligned} \tag{4.12}$$

Note that this formula is invariant under both the substitutions $i \mapsto n-i$ and $j \mapsto p-j$. From this we see that the choice of a different path between x_i and y_j, does not influence the formula.

The formula resulting from the assumption that κ is represented by k_i for $1 \leq i \leq 3$ is calculated similarly. The difference in the calculation lies (only) in the fact that we need to find a representative of $w + k_i$ in $\mathrm{Vor}(0)$.

Type $[I_n - I_p - m]$.
The reduction graph Γ_v of a genus 2 curve C with reduction type $[I_n - I_p - m]$ with $n, p, m \geq 1$ consists of two cycle graphs with n and p vertices, connected by a bridge consisting of $m+1$ vertices. Jacobians of graphs contract bridges, so that the Jacobian of Γ_v is isomorphic to the Jacobian of a union of two cycle graphs with respectively n and p vertices, that are identified in one point. That is, the function γ_v of J is calculated in the same way as for the reduction type $[I_{n-p-0}]$.

Type $[I_{n-p-q}]$.
Now suppose C has reduction type $[I_{n-p-q}]$, with $n, p, q \geq 1$. We assume without loss of generality that $p, n \geq q$. Then the reduction graph Γ_v consists of two vertices, that are connected via three disjoint paths of n, p, q edges. We label the vertices $x_0, \ldots, x_n, y_0, \ldots, y_p, z_0, \ldots, z_q$, where we identify $x_0 = y_0 = z_0$ and $x_n = y_p = z_q$, as in Figure 4.4 on the left. We label the edge from x_i to x_{i+1} by e_i, the edge from y_j to y_{j+1} by e_{n+j}, and the edge from z_k to z_{k+1} by e_{n+p+j}. The homology group $H_1(\Gamma, K) \cong K^2$ is generated by the cycles $\alpha_1 = [e_1 + \cdots + e_n - e_{n+p+1} - \cdots - e_{n+p+q}]$ and $\alpha_2 = [-e_{n+1} - \cdots - e_{n+p} + e_{n+p+1} + \cdots + e_{n+p+q}]$. As vectors these are represented by $\alpha = (\mathbf{1}_{i \leq n}(i) - \mathbf{1}_{i > n+p}(i)) = (1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$ and $\beta = (\mathbf{1}_{i > n+p}(i) - \mathbf{1}_{n < i \leq n+p}(i)) = (0, \ldots, 0, -1, \ldots, -1, 1, \ldots, 1)$. Then the Gram matrix, and its inverse, are

$$\Omega = \big(\langle \alpha_i, \alpha_j \rangle\big)_{1 \leq i,j \leq 2} = \begin{pmatrix} n+p & -q \\ -q & p+q \end{pmatrix}, \quad \Omega^{-1} = \frac{1}{np + nq + pq} \begin{pmatrix} p+q & q \\ q & n+q \end{pmatrix}. \tag{4.13}$$

We now get

$$\mathrm{Jac}(\Gamma_v) = \mathbb{R}^2 / \Omega \mathbb{Z}^2,$$

so that the lattice in $H_1(\Gamma, \mathbb{R})^* \cong \mathbb{R}^2$ is spanned by the vectors $v_1 = (n+p, -q)^T$ and $v_2 = (-q, p+q)^T$. Via Corollary 1.4.4, we get that the component group Φ_v is isomorphic to the cokernel of Ω. The cokernel of a map given by such a matrix Ω can be calculated via the *Smith normal form*, which has the elementary divisors of Ω on its diagonal (see e.g. [Newman, 48]). The elementary divisors of an invertible 2×2 matrix are given by the greatest common divisor of the entries and the determinant of the matrix, divided by this greatest common divisor. That is,

$$\Phi_v \cong \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z},$$
$$d = \gcd(n+q, -q, p+q) = \gcd(n, p, q), \quad r = \frac{\det(\Omega)}{\gcd(n,p,q)} = \frac{np + nq + pq}{\gcd(n,p,q)}.$$

We are now in the case of $p_{12} \neq 0$ in [Conway and Sloane, 16, Section 6], so that there are 3 pairs of Voronoi vectors, $\pm v_1, \pm v_2, \pm(v_1 + v_2)$, which determine the Voronoi region $\mathrm{Vor}(0)$. The tropical theta characteristic $\kappa \in \mathrm{Jac}(\Gamma)$ is thus given by the class

of either of the following four vectors:

$$k_1 = (0,0), \quad k_2 = \left(\frac{n+p}{2}, -\frac{q}{2}\right), \quad k_3 = \left(-\frac{q}{2}, \frac{p+q}{2}\right), \quad k_4 = \left(\frac{n}{2}, \frac{p}{2}\right).$$

We assume for now that κ is represented by $k_4 = (\frac{n}{2}, \frac{p}{2})$. Then the constant in Theorem 2.5.4 is calculated by

$$r = \frac{1}{2}\langle k_4, k_4\rangle = -\frac{n}{8} - \frac{p}{8}.$$

Now let $P \in J(K_v)$ and write $P = [(P_1) - (P_2)]$, with $P_1, P_2 \in C(K_v)$. By Remark 1.4.5, we can assume without loss of generality that P_1 and P_2 specialize to the pair of vertices (x_i, y_j), (x_i, z_k), or (y_j, z_k), with $0 \leq i \leq n$, $0 \leq j \leq p$ and $0 \leq k \leq q$. Suppose that P_1 and P_2 specialize to the vertices x_i and y_j respectively, with $0 \leq i \leq n$ and $0 \leq j \leq p$. A path between x_i and y_j is given by

$$\delta = \left(\mathbf{1}_{n+1\leq k\leq n+j}(k) - \mathbf{1}_{1\leq k\leq i}(k)\right)_{1\leq k\leq n+p} = (-1,\ldots,-1,0,\ldots,0,1,\ldots,1,0,\ldots,0).$$

This path is represented in $H_1(\Gamma,\mathbb{R})^* \cong \mathbb{R}^2$ by the vector

$$w = \left(\langle\delta,\alpha_1\rangle, \langle\delta,\alpha_2\rangle\right)^T = (-i,-j)^T.$$

We get that a representative for $w+\kappa$ in $\mathrm{Vor}(0)$ is given by $\widetilde{w} = w+k_4 = \left(\frac{n}{2}-i, \frac{p}{2}-j\right)$. We calculate the formula $\gamma_v = \gamma_{v,k_4}$, depending on the assumption that κ is represented by k_4, to be

$$\begin{aligned}
\gamma_v(\mathrm{sp}(P)) &= (\|\Psi\| \circ \tau_\kappa)(w) + r = \|\Psi\|(\widetilde{w}) + r = \frac{1}{2}\langle\widetilde{w},\widetilde{w}\rangle + r \\
&= \frac{1}{2(np+nq+pq)} \begin{pmatrix} \frac{n}{2}-i & \frac{p}{2}-j \end{pmatrix} \begin{pmatrix} p+q & q \\ q & n+q \end{pmatrix} \begin{pmatrix} \frac{n}{2}-i \\ \frac{p}{2}-j \end{pmatrix} - \frac{n}{8} - \frac{p}{8} \\
&= \frac{i^2p + i^2q + j^2n + j^2q - inp - inq - ipq - jnq - jnp - jpq + 2ijq}{2(np+nq+pq)} \\
&= \frac{pi(i-n) + q(i+j)(i-n+j-p) + nj(j-p)}{2(np+nq+pq)}. \qquad (4.14)
\end{aligned}$$

From this expression it is clear to see that the formula is invariant under the map $i \mapsto n-i$, $j \mapsto p-j$.

The formulas for if P_1 and P_2 map to the pair of vertices (y_j, z_k) or (x_i, z_k) are analogous. Again, the formula resulting from another assumption on κ yields a similar formula, based on another representative of $w+k$ in $\mathrm{Vor}(0)$.

Remark 4.4.1. We can extend the results from this section to curves of higher genus. The formulas for γ_v are entirely dependent on the reduction graph Γ_v, but not on the vertex weights of this graph. We saw in Proposition 1.3.18 that the reduction graph of a genus g curve has genus g. The reduction graphs that are classified in Theorem 4.3.1 hence define reduction graphs for a curve of genus g if we adjust the vertex weights so that they sum to g. For these specific reduction types (that do not form a complete classification of all semistable reduction types) we get the same formulas for the correction term γ_v as in this section.

4.5 Calculations on the Kummer surface

We once again fix a non-archimedean place $v \in M_K^0$, and work over the completion K_v of K at v, with valuation ring R_v. We assume that the Weierstrass equation of C as in Equation 4.1 is integral, meaning the coefficients λ_i are elements of R_v. We furthermore assume the equation is minimal, meaning the valuation of the discriminant is minimal among all integral Weierstrass equations that define C. Criteria for such a Weierstrass being minimal are given in [Liu, 37].

In this section, we use calculations on the Kummer surface to construct local height functions on the Jacobian J of C.

First of all, we can use the Kummer coordinates (4.2) to give a local height function, w.r.t. the divisor Θ. This is exactly as done in the proof of Theorem 2.4.11(a). We have

$$\operatorname{div}\left(\frac{1}{\xi_i}\right) = 2\Theta - F_i,$$

with F_i effective divisors such that $\bigcap_{i=1}^4 |F_i| = \emptyset$, so the function

$$\lambda_v(P) := -\frac{1}{2}\min\{v(\xi_1(P)), v(\xi_2(P)), v(\xi_3(P)), v(\xi_4(P))\} \tag{4.15}$$

defines a local height function, w.r.t. $\frac{1}{2}(2\Theta) = \Theta$. We call this function the *standard local height* on the Kummer surface. This function gives the intersection multiplicity of the image of Θ and the image of P on the Kummer surface. Note that this function is very dependent on the used Weierstrass model.

Definition 4.5.1. *Let C be given by a fixed minimal Weierstrass equation at $v \in M_K^0$. Let $\hat{\lambda}_v$ be a canonical local height function on J at v, associated with Θ. We define the* local height correction function $\mu_v : J(\overline{K_v}) \to \mathbb{R}$ *by*

$$\mu_v(P) = \lambda_v(P) - \hat{\lambda}_v(P).$$

As λ defines an intersection between the theta divisor with points on the Jacobian, this function μ resembles (-1) times the Néron correction function from Theorem 2.5.1. However, it is not clear that intersections on the Kummer surface and on the Néron model should coincide. We will see in the remainder of this section that the resemblance is more than just a heuristic expectation.

Remark 4.5.2. As $\hat{\lambda}_v$ is only defined up to an additive constant, μ_v is as well. We adopt the same convention for normalization as in [Müller and Stoll, 43]. In this paper a Weierstrass equation of degree 6 is used, which means the Theta divisor $\Theta^{\pm}$ (see Section 1.2) is used. Then a function μ is constructed from the duplication formula 4.8. It is then later shown (in Lemma 8.2) that $2\lambda_v - \mu$ defines a canonical local height function, associated with $\Theta^{\pm}$. When transforming the degree 6 formula to a degree 5 formula, $\Theta^{\pm}$ corresponds to 2Θ, so that $\lambda_v - \frac{1}{2}\mu$ is a canonical local height, associated with Θ. Our function μ_v thus corresponds to $\frac{1}{2}\mu$ in [Müller and Stoll, 43].

As the correction function μ resembles the negation of the correction function γ_v from Theorem 2.5.1, it may be unsurprising that μ_v (at least in the case of semistable reduction) depends only on the component group Φ_v of the Jacobian.

Let $P \in J(K_v)$, with $P = [(P_1) - (P_2)]$, with $P_1, P_2 \in C(K_v)$. Then by the commutative square in Equation 1.17, The specialization of P, can be represented by the pair (x, y) of vertices in the reduction graph Γ_v of C that P_1 and P_2 specialize to. We can calculate μ_v in terms of the *effective resistance* between x_1 and x_2.

Effective resistance is a term used in electrical circuit theory that gives a measure of obstruction of current flow on an electrical network. We can interpret a reduction graph Γ_v as electrical network by assigning to all edges a weight, called *resistance*, of 1, and interpreting vertices as nodes, and edges as wires. The effective resistance is then a function $r := r_\Gamma : V^2 \to \mathbb{R}$. For a full exposition of the theory of effective resistance, see [Baker and Faber, 6]. For our purposes it suffices to show some calculation rules.

Lemma 4.5.3. *Let $\Gamma = (V, E)$ be a graph with resistance on the edges $r : E \to \mathbb{R}_{\geq 0}$. Then the following hold:*

(a) r is a symmetric non-negative function. Furthermore, $r(x, y) = 0$ if and only if $x = y$.

(b) If e is a bridge between vertices x and y, then $r(x, y) = r(e)$.

(c) (Series Law) *Let $x, y, z \in V$ and suppose Γ_1 and Γ_2 are subgraphs of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \{y\}$, $x \in \Gamma_1$, $z \in \Gamma_2$. Then*

$$r_\Gamma(x, z) = r_{\Gamma_1}(x, y) + r_{\Gamma_2}(y, z).$$

(d) (Parallel Law) *Let $x, y \in V$ and suppose Γ_1 and Γ_2 are subgraphs of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \{x, y\}$. Then*

$$\frac{1}{r_\Gamma(x, y)} = \frac{1}{r_{\Gamma_1}(x, y)} + \frac{1}{r_{\Gamma_2}(x, y)}.$$

Furthermore we can calculate the resistance using the Laplacian matrix (or the intersection matrix) of the graph (see Section 1.3.5).

Lemma 4.5.4. *Let L be the Laplacian of a graph Γ with weights on the edges, determined by the inverse of the resistance. Let L^+ the pseudoinverse of L and $v_{x,y}$ be the vector with a 1 at position x, -1 at position y and zero everywhere else. Then*

$$r(x, y) = v_{x,y}^T L^+ v_{x,y} = L^+_{xx} - 2L^+_{xy} + L^+_{yy}.$$

The effective resistance defines a notion of distance between vertices on a graph. This distance is always smaller or equal to the 'shortest path distance' on the graph. The difference between these two distances is given by the normalized tropical Riemann theta function.

Proposition 4.5.5 ([Godding, 24, Theorem 5])**.** *Let $P = [(P_1) - (P_2)] \in J(K_v)$ be a point, with $P_1, P_2 \in C(K_v)$ and write x, y for the vertices on the reduction graph Γ_v that P_1 and P_2 specialize to. Let $s(x, y)$ denote the length of the shortest path between x and y. Then*

$$r(x, y) = s(x, y) - 2\|\Psi\|(\mathrm{sp}(P)).$$

Theorem 4.5.6 ([Müller and Stoll, 43, Theorem 8.7]). *Suppose C has semistable reduction of type $[I_{n-p-q}]$, with $n, p, q \geq 0$. Let $P = [(P_1) - (P_2)] \in J(K_v)$ with $P_1, P_2 \in C(K_v)$, that specialize to the vertices x and y in Γ_v, respectively. Then we have*

$$\mu_v(P) = \frac{1}{2} r(x, y).$$

In the case that C has one of the semistable reduction types in the theorem above, there is a unique minimal Weierstrass model for C (e.g. by [Liu, 38, Theorem 3.12]), so that μ_v is uniquely defined.

The idea of the proof is as follows. In [Zhang, 63], a pairing between two points on a curve with semistable reduction is constructed that involves a Green's function associated with a measure on the reduction graph. It is then proven that a certain expression, using Zhang's pairing, defines a canonical local height function, w.r.t. 2Θ. Under certain conditions, that are satisfied when the reduction type is $[I_{n-p-q}]$ with $n, p, q \geq 0$, the correction function μ_v exactly ends up as $\frac{1}{2}$ times the Green's function, associated with the Dirac measure at Γ_1, evaluated in Γ_2. In [Zhang, 63] it is proven that this is exactly half of the resistance between Γ_1 and Γ_2.

Going through the reduction types for which we can use Theorem 4.5.6, we find the following results.

If C has semistable reduction of type $[I_{0-0-0}]$, then Γ_v consists of a single vertex x, and

$$\mu_v(P) = r(x, x) = 0, \tag{4.16}$$

for any $P \in J(K_v)$.

Corollary 4.5.7. *Suppose C has semistable reduction of type $[I_{n-0-0}]$, with $n \geq 1$ and reduction graph labeled as in Figure 4.2. Let $P = [(P_1) - (P_2)] \in J(K_v)$ with $P_1, P_2 \in C(K_v)$. Suppose that P_1 and P_2 specialize to the vertices x_0 and x_i. Then*

$$\mu_v(P) = \frac{i(n-i)}{2n}.$$

The reduction graph Γ of $\mathcal{C}_v$ is a cycle graph with n vertices, as in Figure 4.2. If P_1, P_2 specialize to x_j and x_k, we can see that by symmetry $r(x_j, x_k) = r(x_0, x_{|k-j|})$, so that our assumption is without loss of generality.

Proof. By Theorem 4.5.6,

$$\mu_v(P) = \frac{1}{2} r(x_0, x_i).$$

By the series law, we can replace the two paths from x_1 to x_n of lengths i and $n - i$ with edges of resistance i and $n - i$, respectively. The resulting graph is pictured as Γ' in Figure 4.2. Now the resistance is easily calculated from the parallel law:

$$\mu_v(P) = \frac{1}{2} r(x_0, x_i) = \frac{1}{\frac{2}{i} + \frac{2}{n-i}} = \frac{i(n-i)}{2n}. \qquad \square$$

Corollary 4.5.8. *Suppose C has semistable reduction of type $[I_{n-p-0}]$, with $n, p \geq 1$ and reduction graph labeled as in Figure 4.3. Let $P = [(P_1) - (P_2)] \in J(K_v)$ with*

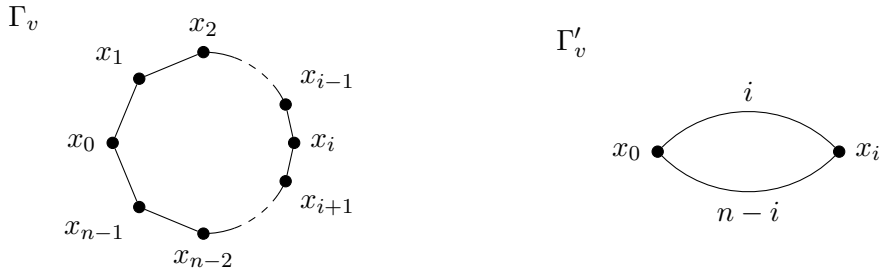


FIGURE 4.2: Left: The reduction graph Γ_v of the special fiber of C, with reduction type $[I_{n-0-0}]$, with unit resistance along every edge. Right: The graph Γ'_v with equal resistance between y_j and x_i as on Γ_v. The label of the edges denote the resistance.

$P_1, P_2 \in C(K_v)$. *Suppose that* P_1 *and* P_2 *specialize to the vertices* x_i *and* y_j. *Then*

$$\mu_v(P) = \frac{i(n-i)}{2n} + \frac{j(n-j)}{2n}.$$

Again, the assumption on the vertices that P_1 and P_2 is without loss of generality. For example if P_1 and P_2 specialize to x_k and x_l, with $k > l$, then $r(x_k, x_l) = r(x_0, x_{k-l}) = r(y_0, x_{k-l})$.

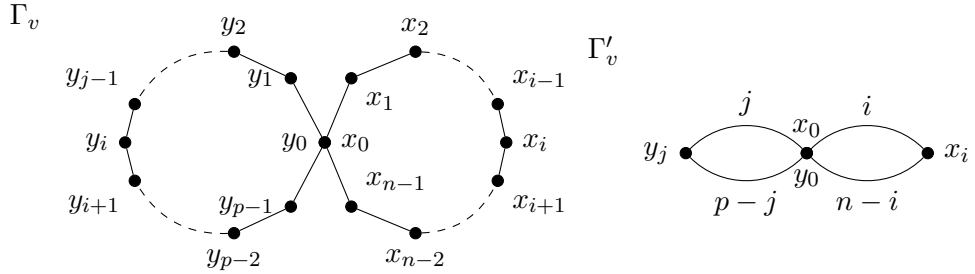


FIGURE 4.3: Left: The reduction graph Γ_v of the special fiber of C, with reduction type $[I_{n-p-0}]$, with unit resistance along every edge. Right: The graph Γ'_v with equal resistance between y_j and x_i as on Γ_v. The label of the edges indicate the resistance.

Proof. By Theorem 4.5.6,

$$\mu(P) = \frac{1}{2} r(x_i, y_j).$$

By the series law, the resistance can be decomposed

$$r(x_i, y_j) = r(x_i, x_0) + r(y_0, y_j).$$

Now the resistances on the right hand side are calculated on the cycle graphs Γ_1 and Γ_2, and are hence calculated as in Corollary 4.5.7. This yields

$$\mu(P) = \frac{1}{2} r(x_i, x_0) + \frac{1}{2} r(y_0, y_j) = \frac{i(n-i)}{2n} + \frac{j(p-j)}{2p}.$$ □

Suppose now that C has semistable reduction of type $[I_{n-p-q}]$, with $n, p, q \geq 1$ and reduction graph labeled as in Figure 4.4. Let $P = [(P_1) - (P_2)] \in J(K_v)$ with $P_1, P_2 \in C(K_v)$. Then, similar to the cases above, we can assume without loss of generality that P_1 and P_2 specialize to either a pair of vertices in Γ_v of the form (x_i, y_j), (x_i, z_k) or (y_j, z_k).

Corollary 4.5.9. *Suppose C has semistable reduction of type $[I_{n-p-q}]$, with $n, p, q \geq 1$. Let $P = [(P_1) - (P_2)] \in J(K_v)$ with $P_1, P_2 \in C(K_v)$. Suppose that P_1 and P_2 specialize to the vertices x_i and y_j. Then*

$$\mu_v(P) = \frac{pi(n-i) + q(i+j)(n-i+p-j) + nj(p-j)}{2(np+nq+pq)}. \tag{4.17}$$

The formulas for P_1 and P_2 mapping to either (x_i, z_k) or (y_j, z_k) are analogous.

Proof. We use Proposition 4.5.5. To evaluate $\|\Psi\|(\mathrm{sp}(P))$, we use the same calculations as in Section 4.4 for Type $[I_{n-p-q}]$, to find that we need to evaluate $\|\Psi\|$ on the vector $w = (-i, -j)^T$, and use the inner product defined by Ω^{-1} as in Equation 4.13. Note that here we do not need to translate with the tropical theta characteristic. We calculate

$$\begin{aligned}
\|\Psi\|(P) &= \frac{1}{2}\langle w, w\rangle = \frac{1}{2} w^T \Omega^{-1} w \\
&= \frac{1}{2(np+nq+pq)} (-i \quad -j) \begin{pmatrix} p+q & q \\ q & n+q \end{pmatrix} (-i \quad -j)^T \\
&= \frac{i^2 p + i^2 q + 2ijq + j^2 n + j^2 q}{2(np+nq+pq)}.
\end{aligned}$$

Suppose for now that $i + j \leq \frac{1}{2}(n+p)$, so that the shortest path between x_i and y_k has length $i + j$. Then we get

$$\begin{aligned}
\mu_v(P) &= \frac{1}{2} r(x_i, y_j) = \frac{1}{2} s(x_i, y_j) - \|\Psi\|(P) \\
&= \frac{1}{2}(i+j) - \frac{i^2 p + i^2 q + 2ijq + j^2 n + j^2 q}{2(np+nq+pq)} \\
&= \frac{-i^2 p - i^2 q - 2ijq - j^2 n - j^2 q + inp + inq + ipq + jnq + jnp + jpq}{2(np+nq+pq)} \\
&= \frac{pi(n-i) + q(i+j)(n-i+p-j) + nj(p-j)}{2(np+nq+pq)}.
\end{aligned}$$

This formula is invariant under the substitution $i \mapsto n - i$ and $j \mapsto p - j$, so that we get the same formula if $i + j > \frac{1}{2}(n+p)$, and the shortest path between x_i and y_k has length $(n-i) + (p-j)$.

Alternatively, we can use the series law to reduce the calculation of the effective resistance between x_i and y_j to a calculation on the graph with four vertices, pictured in Figure 4.4, on the right. Unfortunately, the series and parallel law cannot reduce this graph further, so we use Lemma 4.5.4. This then requires finding the pseudoinverse of a 4×4 matrix. Another method that calculates the effective resistance is using an electrical network approach, which goes beyond the parallel and series law, as in [Doyle and Snell, 20], where the resistance on a graph of this form is calculated in Section 1.3.1.

If P maps to (x_i, z_k) or (y_j, z_k), the resistance is calculated in the exact same way, by substituting the values n, p, q and i, j, k for each other in the appropriate way. □

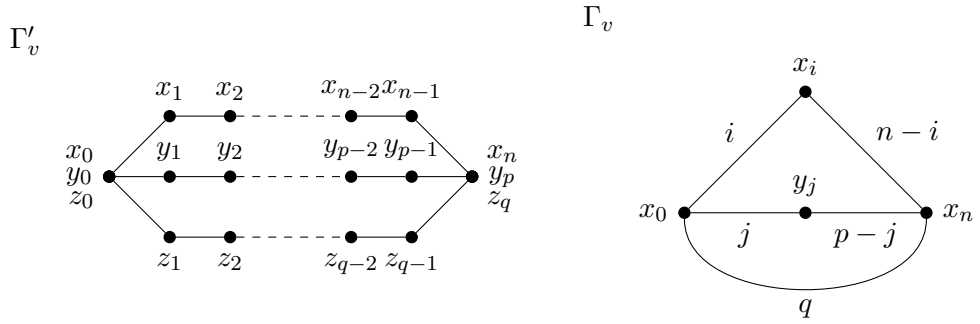


FIGURE 4.4: Left: The reduction graph Γ_v of the special fiber of C, with reduction type $[I_{n-p-q}]$, with unit resistance along every edge. Right: The graph Γ'_v with equal resistance between y_j and x_i as on Γ_v. The label of the edges indicate resistance.

In the case that C has semistable reduction type $[I_n - I_p - m]$, with $n, p \geq 0$ and $m \geq 1$, we cannot use the formula in Theorem 4.5.6 to calculate the function μ_v. This problem comes from the fact that, in this case, there is more than one minimal Weierstrass model for C and μ_v is inherently dependent on the chosen Weierstrass model.

For curves C with reduction type $[I_n - I_p - m]$, the special fiber of the mpr model consists of the special fibers of elliptic curves with reduction types I_n and I_p, connected by a chain of $m-1$ rational curves. The reduction graph consists of the reduction graphs of these elliptic curves, connected via a bridge consisting of $m+1$ vertices (see Figure 4.5). In this case, there are $m+1$ pairwise non-isomorphic minimal Weierstrass models for C (see [Liu, 38, Theorem 3.2]), that correspond to $m+1$ vertices $x_0 = z_0, z_1, \ldots, z_m = y_0$ that make up the bridge. We denote these Weierstrass models by $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_m$ and we denote the μ_v-function calculated w.r.t. the model $\mathcal{C}_l$ by μ_v^l. Explicit formulas for the Weierstrass equations defining $\mathcal{C}_l$ can be found in [Müller and Stoll, 43, Equation (10-1)].

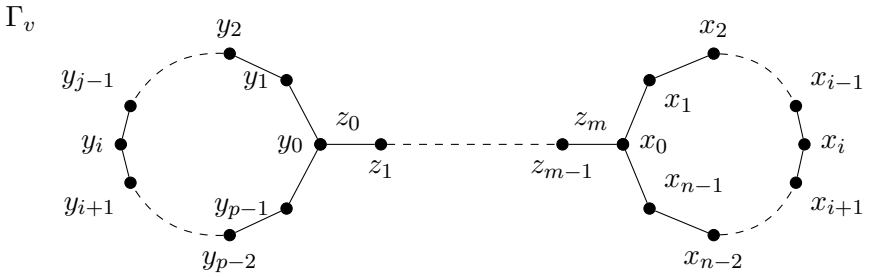


FIGURE 4.5: The reduction graph Γ_v of the special fiber of C, with reduction type $[I_n - I_p - m]$.

A simple path in a graph is a series of adjacent edges, where no vertices are repeated. We say that the vertex z_l *lies between* the vertices x and y in the reduction graph Γ_v, if there is a simple path from x to y that goes through z_l. Note that only z_0 lies between x_i and x_j, for $i \neq j$ and only z_m lies between y_i and y_j, for $i \neq j$. For any

$0 \leq i \leq n, 0 \leq j \leq p$, we have that all vertices z_l, $l = 0, \ldots, m$ lie between the vertices x_i and y_j. We can now state the following modification to Theorem 4.5.6.

Theorem 4.5.10 ([Müller and Stoll, 43, Proposition 10.5]). *Suppose C has semistable reduction of type $[I_n - I_p - m]$, with $n, p \geq 0$ and $m \geq 1$. Let $P = [(P_1) - (P_2)] \in J(K_v)$ with $P_1, P_2 \in C(K_v)$, that specialize to the vertices x and y in Γ_v, respectively. Let l_{min}, l_{max} be respectively the smallest and largest $l' \in \{0, \ldots, m\}$, such that the vertex $C_{l'}$ lies between x and y, and let $l_{min} \leq l \leq l_{max}$. Then we have*

$$\mu_v^l(P) = r(x, y) + l_{max} - l_{min}.$$

Corollary 4.5.11. *Suppose C has semistable reduction of type $[I_n - I_p - m]$, with $n, p \geq 0$ and $m \geq 1$. Let $P = [(P_1) - (P_2)] \in J(K_v)$ with $P_1, P_2 \in C(K_v)$, that specialize to the vertices x and y in Γ_v, respectively. Let l be as above. Then*

$$\mu_v^l(P) = \begin{cases} \frac{i(n-i)}{n} + 2(m-k), & \text{if } (x,y) = (x_i, z_k), \\ \frac{j(p-j)}{n} + 2k, & \text{if } (x,y) = (y_j, z_k), \\ \frac{i(n-i)}{n} + \frac{j(p-j)}{n} + 2m, & \text{if } (x,y) = (x_i, y_j), \\ 2(k-k'), & \text{if } (x,y) = (z_k, z_{k'}), \end{cases}$$

where $0 \leq i \leq n$, $0 \leq j \leq p$, $0 \leq k' \leq k \leq m$.

Proof. We can use Theorem 4.5.10 to calculate the value of $\mu_v^l(P)$. By the series law and Lemma 4.5.3(b), a bridge of length a between two vertices contributes a to the effective resistance between the vertices. Note that $l_{\text{max}} - l_{\text{min}}$ is the length of the largest bridge between x and y.

That is, to compute the effective resistance $r_\Gamma(x, y)$, we can construct a new graph Γ', by contract the bridge from z_0 to z_m in Γ_v to a single point and compute the $r_{\Gamma'}(x, y) + l_{\text{max}} - l_{\text{min}}$. The graph Γ' is the reduction graph of a curve with reduction type $[I_{n-p-0}]$. We can thus use the results from Corollary 4.5.7 (if n or p is zero), and Corollary 4.5.8 to compute $r_{\Gamma'}(x, y)$. To calculate μ_v^l using Theorem 4.5.10, we once again have to add $l_{\text{max}} - l_{\text{min}}$ to this value. The result of this corollary now comes from doing this procedure for the four given options. □

If we now compare the results of this section to the results in Section 4.4, we see a lot of similarities. Note that the results in that section are based on an assumption of what the tropical theta characteristic κ is. Hence, the following results are also based on this assumption .

Assumption 4.5.12. Let C be a curve of genus 2 and suppose C has semistable reduction. Let $\text{Jac}(\Gamma_v)$ be the Jacobian of the reduction graph Γ_v of C, and identify it with a real torus as in Section 4.4. Then we assume the tropical theta characteristic κ to be given by

$$k = \begin{cases} 0, & \text{if Type } C = [I_{0-0-0}], [I_0 - I_0 - m], \\ \frac{n}{2}, & \text{if Type } C = [I_{n-0-0}], [I_n - I_0 - m], \\ (\frac{n}{2}, \frac{p}{2})^T, & \text{if Type } C = [I_{n-p-0}], [I_{n-p-q}], [I_n - I_p - m], \end{cases}$$

with $n, p, q, m \geq 1$.

In general, κ is hard to determine theoretically, so there is no reason to assume this assumption is more likely than any other assumption on what κ should be. The results below should therefore not be taken as proven facts, but rather as argumentation for why the assumption is reasonable, and may be true.

As we noted before the correction function μ_v is heuristically very similar to the negation of the Néron correction term γ_v, as are the intersection multiplicity $i_v(\Theta, P)$ and the standard local height $\lambda(P)$. The following makes this precise.

Proposition 4.5.13. *Suppose Assumption 4.5.12 holds. Let C be a genus 2 curve over K, with Jacobian J and let Θ be the Theta-divisor on J. Let $v \in M_K^0$ be a non-archimedean place. Suppose C has semistable reduction at v.*

(a) Suppose C has reduction type $[I_{n-p-q}]$, with $n, p, q \geq 0$. Then there is an integer $c \in \mathbb{Z}$, such that

$$i_v(\Theta, \cdot) = \lambda_v + c.$$

(b) Suppose C has reduction type $[I_n - I_p - m]$, with $n, p \geq 0, m \geq 1$. Let l_{max} and l_{min} be as in Theorem 4.5.10. Let $l_{min} \leq l \leq l_{max}$ and let λ_v^l be the function λ_v calculated w.r.t. the minimal Weierstrass model $\mathcal{C}_l$. Then there is an integer $c \in \mathbb{Z}$, such that

$$i_v(\Theta, \cdot) = \lambda_v^l - 2(l_{max} - l_{min}) + c.$$

In both cases the integer c can be calculated as $c = i(\Theta, P) - \lambda_v^l(P)$, for some $P \in J(K_v)$, with $\mathrm{sp}(P) = 0 \in \Phi_v$.

Proof. (a) Combining Theorem 2.5.1 and Definition 4.5.1 we get that

$$i_v(\Theta, P) = \lambda_v(P) - \mu_v(P) - \gamma_v(\mathrm{sp}(P)) + c, \tag{4.18}$$

for all $P \in J(K_v)$ and some constant $c \in \mathbb{R}$. From our chosen normalization, we find $\mu_v(P) = \gamma_v(\mathrm{sp}(P)) = 0$, if $\mathrm{sp}(P) = 0$, so that

$$c = i_v(\Theta, P) - \lambda_v(P) \in \mathbb{Z},$$

for such P. Now by comparing Equations 4.9, 4.11, 4.12 and 4.14 to Equation 4.16 and Corollaries 4.5.7, 4.5.8 and 4.5.9, respectively, we see that $-\mu_v = \gamma_v$, under Assumption 4.5.12. Then Equation 4.18 becomes

$$i_v(\Theta, P) = \lambda_v(P) + c.$$

(b) Let $l_{\min} \leq l \leq l_{\max}$. We again get Equation 4.18, for $\lambda_v = \lambda_v^l$, and by the same reasoning $c = i_v(\Theta, P) - \lambda_v(P) \in \mathbb{Z}$, for $P \in J(K_v)$ with $\mathrm{sp}(P) = 0 \in \Phi_v$. From the discussions in Section 4.4 the formulas in Equations 4.9 4.11, and 4.12 hold for the reduction types $[I_0 - I_0 - m]$, $[I_n - I_0 - m]$, and $[I_n - I_p - m]$, respectively. Comparing these to the formulas in Corollary 4.5.11 yields $-\mu_v = \gamma_v - 2(l_{\max} - l_{\min})$, under Assumption 4.5.12, so that Equation 4.18 becomes

$$i_v(\Theta, P) = \lambda(P) - 2(l_{\max} - l_{\min}) + c. \qquad \square$$

There is another interesting conclusion we can make based on Assumption 4.5.12. We use Proposition 4.5.5 to relate the tropical theta characteristic to the shortest path function. Let $s_\Gamma : V(\Gamma)^2 \to \mathbb{Z}_{\geq 0}$ be the function that associates to each pair

(x, y) of vertices the length of the shortest path between x and y on Γ. For any graph Γ, let Γ' be the graph resulting from Γ by contracting all bridges to a single vertex. Then define $s'_\Gamma : V(\Gamma)^2 \to \mathbb{Z}_{\geq 0}$, by $s'_\Gamma(x, y) = s_{\Gamma'}(x, y)$.

Proposition 4.5.14. *Suppose Assumption 4.5.12 holds, and write k for the given representative of the tropical theta characteristic κ. Let $P = [(P_1) - (P_2)] \in J(K_v)$ with $P_1, P_2 \in C(K_v)$. Suppose that P_1 and P_2 specialize to the vertices x and y in the reduction graph Γ_v. Let $w \in \mathrm{Vor}(0) \subseteq R^n \cong H_1(\Gamma_v, \mathbb{R})$ be a vector representing a path from $\mathrm{sp}(P)$ in $\mathrm{Jac}(\Gamma_v)$, such that $w + k \in \mathrm{Vor}(0)$. Then*

$$s(x, y) = -2\langle w, k\rangle.$$

Proof. From Proposition 4.5.5 and $w \in \mathrm{Vor}(0)$, we get

$$r(x, y) = s(x, y) - 2\|\Psi\|(w) = s(x, y) - \langle w, w\rangle. \tag{4.19}$$

If Γ_v contains no bridges, we also have from the discussion in Proposition 4.5.13 and $w + k \in \mathrm{Vor}(0)$ that

$$r(x, y) = 2\mu_v(P) = -2\gamma_v(\mathrm{sp}(P)) = -2\|\Psi\|(w + k) = -\langle w + k, w + k\rangle + \langle k, k\rangle.$$

Combining these equations we find

$$s'(x, y) = s(x, y) = \langle w, w\rangle + \langle k, k\rangle - \langle w + k, w + k\rangle = -2\langle w, k\rangle.$$

If Γ_v does contain bridges, we find from the discussion in Proposition 4.5.13 that

$$r(x, y) = 2\mu_v(P) = -2\gamma_v(\mathrm{sp}(P)) + (l_{\max} - l_{\min}) = -\langle w+k, w+k\rangle + \langle k, k\rangle + (l_{\max} - l_{\min}).$$

Then again combining this with Equation 4.19 yields

$$s(x, y) - (l_{\max} - l_{\min}) = \langle w, w\rangle + \langle k, k\rangle - \langle w + k, w + k\rangle = -2\langle w, k\rangle.$$

We get our conclusion by noting that $(l_{\max} - l_{\min})$ is the length of the largest bridge between x and y, so that $s'(x, y) = s(x, y) - (l_{\max} - l_{\min})$. □

Under Assumption 4.5.12 we thus get that taking the inner product with a representative of the tropical theta characteristic calculates the length of the shortest path between two vertices.

These two results, based on Assumption 4.5.12, are desirable properties to have. There is often a profound elegance to the way mathematics is structured, so that it is in some sense likely for the assumption to be true. In any case, these results should bring us closer to understanding the tropical theta characteristic.

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