Stellingen

behorende bij het proefschrift

The Application of Robust Control Theory Concepts to Mechanical Servo Systems

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20 september 1994

1. Wetenschap en technologie, of in andere woorden theorie en praktijk, zijn als goede vrienden: ze zijn duidelijk verschillend, desondanks onafscheidelijk, samen tot meer in staat dan elk afzonderlijk.

2. De exacte wetenschapper die geïnteresseerd is in de praktijk constateert vaker verschillen dan overeenkomsten tussen theorie en praktijk; ter bevordering van de technische wetenschap zou hij zich desondanks meer moeten concentreren op de overeenkomsten tussen theorie en praktijk en minder op de verschillen.

3. Een goede wiskundige kan een echt probleem oplossen; een echte wiskundige kan alleen een goed probleem oplossen.

4. Eenieder die ontevreden is met een geïmproviseerde, ad hoc oplossing voor een technisch probleem dient zich te realiseren dat een goed gefundeerde oplossing meestal slechts mogelijk is dankzij ontwikkeling, onderhoud en juiste toepassing van gereedschappen.

5. Als Nederlandse vertaling van 'system performance' wordt wel gesproken van 'de prestaties van het systeem'. Dit is echter principieel fout: er is sprake van 'het prestatieniveau van het systeem'.

6. Ieder regelaarontwerp bestaat uit een zo goed mogelijke afweging van conflicterende eisen. In een 'standard plant' structuur worden de *eisen* bepaald door de keuze van relevante signalen en hun relatie met het te regelen systeem en wordt de *afweging* bepaald door de keuze van weegfuncties; het toegepaste optimaliseringsalgorithme bepaalt vervolgens nog slechts de *aard* en *kwaliteit* van de afweging.

7. Modelvorming omwille van regelaarontwerp bestaat niet alleen uit de modelvorming van het te regelen systeem, maar ook uit de modelvorming van externe invloeden hierop en de modelvorming van het gewenste gedrag. In een 'standard plant' structuur [1] zijn deze drie modellen te herkennen als respectievelijk de 'plant', de ingangswegingen en de uitgangswegingen [2]. Het verdient aanbeveling om hier in het regeltechnisch onderwijs zeer gericht aandacht aan te besteden.

[1] S.P.Boyd, C.H.Barratt. *Linear controller design, limits of performance.* Prentice Hall Information and System Sciences Series, Englewood Cliffs, NJ, 1990.

[2] dit proefschrift.

8. Bij gebruik van de term 'robuustheid' met betrekking tot een regeling dient te worden aangegeven: 1. om welke eigenschap het gaat, 2. hoe robuustheid kan worden gekwantificeerd met betrekking tot deze eigenschap en 3. om welke invloeden het gaat. Aangezien deze drie punten alle een verscheidenheid aan mogelijkheden toelaten, die bovendien vaak conflicterend zijn, is het duidelijk dat de term 'robuuste regeling' zondermeer nietszeggend is.

9. Kunstmatige intelligentie bestaat niet en zal op afzienbare termijn niet bestaan. Onderzoek op deze gebieden is echter zeker niet zinloos maar men zou dienen te spreken van 'de modellering van intelligentie'. Of kunstmatige intelligentie ooit zal kunnen bestaan is onzeker, maar het mag niet verwacht worden dat zij het resultaat kan zijn van de op dit moment bestaande lijnen van onderzoek.

    - J.Kelly, *Artificial intelligence, a modern myth.* Ellis Horwood Series in Artificial Intelligence, New York, 1993.

10. De kans op verkoudheid is groter in een auto met (ingeschakelde) airconditioning dan in een (geopende) cabriolet: dit ervaringsfeit zou voor ontwerpers van airconditioning voor inbouw in auto's aanleiding moeten zijn hun ontwerpen opnieuw te evalueren.

11. Er zijn velen die wetenschap en geloof kunnen combineren: dit getuigt van de enorme creativiteit en flexibiliteit van de mens, daar hij in staat blijkt te zijn te zoeken naar waarheden en feiten, terwijl hij sommige daarvan systematisch verwerpt.

12. Een vermeende 'achterstand' van België op Nederland wordt gelogenstraft door het feit dat men daar al eeuwenlang 'bier brouwt zoals bier bedoeld is'.

13. De automobilist als sponsor van vele overheidsactiviteiten die niets met verbetering van de automobiliteit te maken hebben, is de enige sponsor die niet alleen toestaat dat er met hem gesolt wordt door degenen die hij sponsort maar tevens moet aanzien hoe door hem betaalde 'opvoedende televisiespotjes' zijn intelligentie beledigen.

14. Technici gaan prat op het feit dat er sprake is van een enorme toegevoegde waarde wanneer geïntegreerde schakelingen, ofwel 'chips', worden gewonnen uit zand; bescheidenheid past als men bedenkt dat een simpel stuk linnen, opgedroogde olie en een verscheidenheid aan pigmenten vele tientallen miljoenen guldens kan opleveren.

The Application of Robust Control Theory Concepts to Mechanical Servo Systems

Omslag: 'kubus met banden' Maurits Cornelis Escher (1957).

Twee schijnbaar onverenigbare concepten verenigd binnen één raamwerk.

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The Application of Robust Control Theory Concepts to Mechanical Servo Systems

PROEFSCHRIFT

ter verkrijging van de graad van doctor
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in het openbaar te verdedigen ten overstaan van een commissie,
door het College van Dekanen aangewezen,
op dinsdag 20 september 1994 te 13.30 uur,

door

Paul Frank LAMBRECHTS

werktuigkundig ingenieur

geboren te Roosendaal.

Dit proefschrift is goedgekeurd door de promotor

Prof. ir. O.H. Bosgra

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable.

There is another theory which states that this has already happened.

Douglas Adams, *The Restaurant at the End of the Universe, The Hitch Hiker's Guide to the Galaxy, Part 2.*

in herinnering aan mijn moeder
Juliana Lambrechts Elbertsen

Preface

The research project that resulted in this thesis started in 1987, and was based on my interest in robust control theory. Developments in this area were very fast: milestones are the paper of Zames (1981), which is generally recognised as the starting point of H_∞ control theory, the ONR/Honeywell workshop given by Doyle, Chu, Francis, Khargonekar, and Stein (1984), elaborating on the possibilities of H_2, H_∞ and μ, and the paper by Glover and Doyle in (1988), providing 'numerically friendly' state-space formulæ for the calculation of H_∞ optimal controllers. I was personally very much inspired by the course given by prof. Curtain and prof. Kaashoek for the Dutch graduate school of systems and control in 1988, based on 'a course in H_∞ control theory' by Francis (1987), and by the 'μ short course on theory and applications of robust multivariable control' given in Delft by Doyle, Packard, Balas and Glover (1990). The latter resulted in a good relation with John Doyle, Andy Packard and Gary Balas, which resulted in several helpful communications, especially at the CDC in 1991 and the ACC in 1993. It also resulted in finding in Samir Bennani a congenial spirit in promoting robust control in a reluctant world. I thank him for many interesting discussions, whether or not over a glass of fine malt whiskey.

Another main interest of mine was based on my M.Sc. work performed at Philips Research Laboratories in Eindhoven, where I learned many important practical insights in control design, and was introduced to the possibilities of state-of-the-art digital hardware for implementing complex controllers on relatively fast mechanical systems. The combination of these two interests can be seen as the basis for this thesis. The first four chapters are aimed at the combination of the general servomechanism problem with robust control theory. The design of servomechanisms is considered, with robustness properties in the sense of robust asymptotic tracking, robust stability and performance, and robustness guarantees for structured uncertainties. The fifth chapter considers the design of a robust servomechanism based on an actual three-degrees-of-freedom hydraulic positioning system.

I am grateful to many people that have contributed to the realization of this thesis, all in their own way. First of all I want to thank my promotor Okko Bosgra for allowing me to 'dive into the deep' and for teaching me to 'swim' (although I sometimes wished he did it the other way round). Next, there have been several students that either were part of my project or had projects that were closely related: Floor de Blok, Sven Smit, Jan Terlouw, Gert-Wim van der Linden, and Franky De Bruyne. I would like to thank my (former) room-mates: Zhu Fang, Henk Huisman, Gert van Schothorst, Dick de Roover, and Judi Soetjahjo for allowing me to talk blisters on their ears and for letting my 'work' pile up into an enormous heap of junk in which, I suspect, several new lifeforms have evolved. I would like to thank Johan Jansen for allowing me to use his computers when times were dire in the sense of PC availability, and Peter Valk for finally giving in to my desperate demands and buying one for me. Furthermore, I thank him for his financial support which allowed me to regularly visit prof.dr.ir. de Koninck, without whom this thesis would never have been completed.

I am grateful to all other (temporary) members of the Mechanical Engineering Systems and Control Group for their help and support, of whom I want to mention Sjoerd Dijkstra (because he is generally a nice guy), Ton van der Weiden (because he is sometimes a nice guy), Paul van den Hof (for lots of small things), Cor Kremers (for catering and the construction of a perfect computer table), Els Arkesteijn (for secretarial support), Piet Ruinard (for librarial support), and Fred den Hoedt (for being a good neighbour). I also want to mention Ben Wenneker for his PC hardware support, Leo Beckers for his PC software support (and talking blisters on *my* ears for a change), and Paul Breedveld for the brainstorming session leading to the cover design.

Three persons have been of special importance to this thesis, because of their efforts, but also because they represent three totally different viewpoints. I thank Piet Teerhuis for the design of the three-degrees-of-freedom hydraulic positioning system and his help in constructing a model for it. I thank Rens de Keijzer for installing it and for sharing his enormous practical experience in controlling such systems. I thank Carsten Scherer for his efforts in upgrading the mathematical quality of this thesis to an acceptable level.

Finally, I want to thank my father for his loving and enthousiastic support in hard times, and for his continuous confidence in my ability to bring this to a good end.

Paul Lambrechts
Delft, August 1994

Contents

Summary

The output regulation and tracking problem, also known as the servomechanism problem, is a standard subject in both classical and modern control theory. For instance with the help of state-space methods, it is possible to construct a description of a set of signals that is 'persistent': i.e. not decaying to zero when time goes to infinity. Using linear time-invariant models, such a set may consist of polynomial and sinusoidal functions of time. Then, the control problem is to construct a controller for a given system such that one or more system outputs asymptotically 'track' the specified persistent signals, that may therefore be interpreted as 'reference signals'. It is known that such a controller has certain structural properties: the controller must contain a dynamical model, a 'servocompensator', such that the combination of controller and system contains an 'internal model' of the set of reference signals.

However, the control objective thus specified is not sufficient to obtain a realistic controller: the asymptotic tracking property must be combined with other objectives, like speed of response, disturbance attenuation and robustness properties. Usually, this can be done by application of a given control design method on a system model, extended with a servo compensator that is later incorporated into the controller. This approach works well with the 'linear quadratic optimal' control methods that were mostly considered during the modern control period (1960-1980). Unfortunately, difficulties arise when it is attempted to apply 'robust' control methods, as under development since 1980. It appears that the extension of the system to be controlled with a servo compensator is in violation with certain standard assumptions that are made to facilitate the application of these methods.

In this thesis, an approach is suggested to solve these difficulties in a general sense: given any linear time-invariant model of a set of persistent signals for which the output regulation and tracking problem is solvable, the given approach allows any existing robust control method to be used to find an appropriate controller. To determine solvability of this output regulation and tracking problem, a new necessary and sufficient condition is suggested and related to earlier results. From this condition, the construction of an appropriate servo compensator is set up that appears to be of minimal order for the given set of persistent signals. The correct formulation of a standard control configuration is considered such that physical interpretation of disturbance inputs, weight functions, and control objective outputs is possible, under the guarantee that the tracking objective is obtained. Robust control methods, like H_2 and H_∞ optimization, and μ analysis and synthesis, can then be applied to obtain the desired closed-loop transfer from disturbances to objective functions.

Two important extensions of the presented approach are considered. First, the construction of a non-minimal servo compensator to obtain 'robust asymptotic tracking' is given. It is shown that in many cases the result can be seen as a special case of the given approach; furthermore, some possible disadvantages with respect to control objectives other than asymptotic tracking are discussed. Secondly, the two-degree-of-freedom problem formulation is given as an extension of the standard control configuration. This allows the simultaneous construction of feedforward and feedback controllers with guaranteed asymptotic tracking properties, by means of robust control methods.

A multivariable experimental control problem is considered, with much attention to the determination of an appropriate control configuration and the selection of the necessary weight functions. It provides a simple illustration of the integration of the asymptotic tracking property within a typical robust control problem. Several controllers are designed by means of H_∞ optimization; they are experimentally evaluated within a digital signal processor based implementation environment. Robustness properties with respect to stability, performance and asymptotic tracking are guaranteed by means of parametric uncertainty modelling and structured singular value analysis.

Chapter 1

Introduction

The main issue of this thesis is the design of controllers for servo systems, with an emphasis on mechanical servo systems. In control theory, this type of problem is known as the output regulation and tracking problem or the servomechanism problem. Many results are known in this area, both in classical control theory and in modern control theory. However, when looking at the latest developments in the area of robust control theory, the servomechanism problem appears to have certain properties that are hard to deal with. We will look at a solution of this from a practical point of view; application of many results from robust control theory on the servomechanism problem will be made possible.
The next section will place the servomechanism problem within linear control theory. After that, the general problem formulation for this thesis will be given, followed by an overview of contents.

1.1 The servomechanism problem in linear control theory

1.1.1 Developments in linear control theory

When considering control problems in practice, there are usually many objectives that are pursued. Basic for the scientific approach to control problems is the possibility to put those objectives in mathematical terminology, such that tools can be developed to define, analyze and solve them. Fortunately, this is possible for some of the more important requirements, like stability and performance.
However, there are several other objectives for which this is much harder: for

instance, it is very difficult to define a sensible trade-off between instrumentation costs and performance. Furthermore, as control design usually has to deal with trade-offs between several objectives, it is essential that the translation of objectives to mathematical representations is such that these representations are comparable with each other. In many cases it appears that this implies that mathematical representations of control objectives are *inaccurate* and *incomplete*.

In spite of this, the availability of mathematical tools to describe and analyze the system to be controlled and to synthesize an appropriate controller is so important, that it usually outweighs these disadvantages. This is one of the main reasons for the success of so-called 'classical' linear control theory that was mainly developed in the period 1930-1960: although in reality there are no linear systems, the ease with which it is possible to design controllers for single-input single-output systems still makes it a useful approach. There are many good textbooks that include the subject of classical control as well as more modern approaches: for instance Maciejowski (1989) and Boyd and Barratt (1991). Some references from the actual classical control period are James et al. (1947) and Truxal (1955).

Although application of classical control methods often produces acceptable results, the fact that control objectives are being tightened by constantly growing demands from society (safety, reliability, performance, economics, environment, etc.) is a major motivation for control engineers to use, and for system theoreticians to develop, more sophisticated mathematical tools. From a practical point of view, the time-domain, state-space methods for control of linear multivariable systems, whose development started around 1960 and which became known as 'modern' control methods, were often unsuccessful due to robustness problems. The developed theory depends importantly on the accuracy of the linear model of the system to be controlled. It does not provide sufficient possibilities to ensure acceptable behaviour of the closed-loop system when uncertainties, nonlinearities, time-dependence or other unmodelled behaviour is present. A selection from the enormous amount of literature available on modern control theory is Kalman (1960), Rosenbrock (1970), Wonham (1979), Luenberger (1966), MacFarlane and Karcanias (1976), Kwakernaak and Sivan (1972) and Anderson and Moore (1989).

In spite of the difficulties with modern control, it did form the basis of new developments starting around 1980 and leading to what is now usually referred to as 'robust' control. With the help of the singular values concept it became possible to analyze the developed state-space methods in the frequency domain, which showed their shortcomings in robustness in terms of classical control methods (Doyle and Stein 1981, Freudenberg and Looze 1988). Furthermore, it became clear that methods developed in operator theory can be used for controller synthesis, putting bounds on, or even minimizing bounds of, measures like the ∞-norm of the closed loop system, which can be used to determine the closed loop behaviour when uncertainties are present (Zames 1981, Doyle et al. 1984, Safonov and Doyle 1984, Safonov 1986, Francis 1987, Limebeer et al. 1988, Vidyasagar 1985). Later on, the operator theoretic approach was largely substituted by an algebraic Riccati equation approach, with the advantages of increased insight, decreased computational effort and decreased controller order (Glover and Doyle 1988, Khargonekar et al. 1988, Doyle et al. 1989, Zhou et al. 1993).

During the last decade, a standard control design structure is being developed to put these new insights into effect (Doyle et al. 1984, Boyd and Barratt 1991, Zhou et al. 1993):

- the system to be controlled can be described by means of a linear time-invariant model,
- the control objectives can be formulated using weights and signals retaining their physical interpretation,
- the effects of unmodelled behaviour can be added, leading to the possibility to precisely specify robustness demands as being one of the control objectives,

The development of both analysis and synthesis tools for this general framework is ongoing (Bernstein and Haddad 1989, Packard et al. 1991, Boyd et al. 1993, Vandenberghe and Boyd 1993, Packard et al. 1993, Gahinet and Apkarian 1993), but already has reached a level that generalizes those classical control results that allow controllers to be designed with an effective trade-off between performance and robustness.

1.1.2 The asymptotic tracking objective

An important limitation of the robust control paradigm is being dealt with in this thesis. It appears that the developed controller synthesis tools are not capable of handling the servomechanism problem, which is to obtain 'output regulation' or 'tracking'. The importance of this problem for control design was already apparent in classical control. It first appeared when it was found to be necessary to 'reset' a controlled system when a permanent (or long term) change of operating point or operating conditions was needed.

This becomes clear when we consider figure 1.1, in which G denotes a stable plant to be controlled and K is a proportional controller. The output signal y determines the physical quantity in G that has to be controlled in such a way, that the error signal e between y and the reference signal r is kept small, the input signal u is the signal that actuates the plant. Now consider a change in operating point,

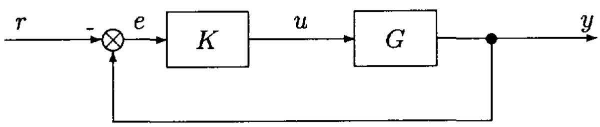

Fig. 1.1: Block-diagram of a feedback control system.

in the form of a change of r. The error signal will then be unequal to zero and the controller starts to actuate the plant in order to bring y closer to r. However, as e goes to zero, also $u = Ke$ must go to zero and y will tend to drop back to its former value. This implies that an equilibrium situation will result in which e and u are unequal to zero. To obtain $e = 0$ in the new operating point it is therefore necessary to add a constant value u_c to u, after which usually u is redefined: $u := u - u_c$. It was soon found that this resetting procedure could be automated by adding integral action to the controller: $e \neq 0$ then causes a continuously growing control input u until $u = u_c$ and $e = 0$.

The importance of this type of controller can be seen from its widespread use in industry: it is known as the PI-controller because of the combination of proportional and integral action. Further research on the properties and possible extensions of this controller has resulted in the introduction of several new concepts and some terminology. During the classical control period this research was mainly focussed on the concept of *system type* (James et al. 1947), which was put into a multivariable setting later on by several researchers (Sandell and Athans 1973, Hosoe and Ito 1974, Wolfe and Meditch 1977, Sebakhy 1984, Hara 1985). In accordance with the new insights developed in the modern control period the same problem was considered in a more general setting, and referred to as the *servomechanism problem* (Davison 1972, Bhattacharyya and Pearson 1972, Davison and Goldenberg 1975).

Even more important than the resetting procedure described earlier, is the ability of a controller with integral action to 'follow' or 'track' slowly varying reference signals. Perhaps the most obvious application of such systems is in the military (target tracking), but typical servomechanism problems can also be found in a large number of industries (aerospace, consumer electronics, automotive, nautical, etc.). The main difficulty in designing servomechanisms for such applications is the growing demand in speed of response. As stated before, the PI-controller is basically designed to track slowly varying signals: in fact the objective $e = 0$ is only reached asymptotically. If the servomechanism has to track reference signals relatively fast, the integral action has to be either 'speeded up' or, as we will show later, better adjusted to the form of these reference signals by means of an *internal model*. However, this implies that the controller has to introduce dynamical behaviour into the closed-loop system, that could impair other control design objectives like stability, noise attenuation and robustness. Therefore, there is a need for a control design environment that allows a sensible trade-off between the tracking objective and other design objectives, preferably the standard robust control design structure as mentioned before.

1.2 Robust control and the tracking objective

1.2.1 Robustness and performance of servomechanisms

In the previous section, robust control was introduced as the most recent large step in linear control theory, following the period of classical control (1930-1960) and the period of modern control (1960-1980). However, the robustness concept was already known in classical control. Gain and phase margins, defined on the Nyquist plot of the open-loop transfer function, were (and are) accepted measures of the robustness of the closed-loop system and are especially useful when uncertainties in the linear model of the system to be controlled must be considered. Furthermore, the need for robustness of controlled systems was also satisfied with respect to the output regulation and tracking problem. It was discovered that the asymptotic tracking property of a servomechanism is retained in spite of arbitrary plant uncertainties, as long as the closed loop system remains stable. This implies that a robust servomechanism can be designed by adding an appropriate internal model and designing a robustly stabilizing controller for the resulting structure.
These insights have been developed in many of the references given in the previous section, both in terms of (multivariable) system type (e.g. Sandell and Athans 1973) and in terms of the servomechanism problem (e.g. Davison and Goldenberg 1975). Unfortunately however, these developments occurred in the modern control era, such that usually the design of the stabilizing controller was performed using state-space methods. For this reason, no attention was given to robustness of the stabilizing controller other than the aforementioned result on robustness of

the asymptotic tracking property. Furthermore, as the performance of the servomechanism was mostly measured in terms of the asymptotic tracking property, other performance measures, like disturbance attenuation and speed of response, were largely neglected.

Although no general theoretical results are known in this area, it is clear that the asymptotic tracking objective may very well be in conflict with the robust stability objective and other performance objectives. This implies that in the design of a servomechanism, a trade-off is necessary between the asymptotic tracking property, other performance measures and robustness of the closed-loop system. As robustness of the asymptotic tracking property is clearly different from robust stability and performance of the closed-loop system, it is furthermore necessary to consider a trade-off between both forms of robustness. This thesis will consider the application of robust control methods for the specification of these trade-offs and the design of appropriate controllers.

1.2.2 Problem formulation: asymptotic tracking within the standard control design structure

The standard control design structure mentioned in the previous section is already a powerful platform for obtaining trade-offs between several design objectives. We will therefore develop a procedure to add the asymptotic tracking objective to this standard structure, with an attempt to minimize any effect on the trade-offs already present. This extension of the standard structure, or at least the use of robust control methods for tracking problems, has already been mentioned by several researchers (Xu and Mansour 1986, 1988, Wu and Mansour 1989, 1990, Sugie and Hara 1989, Abedor et al. 1991, Khargonekar et al. 1990, Liu and Mita 1991, Hosoe et al. 1992), but they were not able to find a structure in which the servomechanism problem can be specified *in its full extent*, i.e. with generality in the selection of reference signals and completeness in the possibility to trade-off robustness and performance objectives, and/or can be solved by *available controller synthesis methods*, i.e. methods for which stable numerical algorithms are readily available.

The problem that will be addressed in this thesis can therefore be split into two parts:

1. the setting up of a standard control design structure, in which trade-offs can be defined and performed between:
   - performance objectives, like disturbance attenuation and speed of re-

sponse,

- robustness objectives, like robust stability and performance in the face of plant uncertainties,
- the asymptotic tracking objective for a prespecified set of reference signals,
- robustness of the asymptotic tracking objective in the face of plant uncertainties,

2. the synthesis of controllers for this extended standard control design structure, using *any* method available for the standard control design structure *without* the asymptotic tracking objective.

Based on this, we will focus on the following four questions.

1. What are the advantages of robust control methods;
    - what is the system theoretical background of robust control,
    - what are the properties of robust control,
    - how does robust control compare with approaches from, 'classical' and 'modern' control.
2. What are the properties of the procedure presented in this thesis;
    - why is there a need for a special treatment of the output regulation and tracking problem when considering robust control methods,
    - what is the procedure to incorporate the tracking objective into the standard control design structure,
    - what are the properties of the resulting servomechanism.
3. How does the presented procedure compare with approaches available in literature;
    - what approaches for solving the output regulation and tracking problem using robust control methods are available in literature,
    - in what respects is the procedure presented in this thesis an improvement on available approaches.
4. What results can be obtained by applying the presented procedure to a realistic control problem;

- how can weight functions be set up to obtain a trade-off between various design objectives,
- what are the properties of the designed controllers,
- are the results obtained from actual implementation of the designed controllers in accordance with the expectations.

Some of these questions allow a clear and concise answer, others are too complex to be dealt with in a short formulation or are too general to be completely answered within the context of this thesis. For those, it is attempted to give directions in which further research and practical experience may lead to further developments.

1.3 Overview of contents

This thesis aims at making recently developed tools available for practical use. From this point of view, the complete line of thinking—if you like, philosophy—leading to the main results will be given. Because of this, there will be a large introductory part, discussing mostly known results and procedures, but in such a way that a line of thinking is set up that leads to the results in later chapters.

Especially the concept of systems is of great importance in this respect and will be the subject of the next chapter. Also the system and signal descriptions used in this thesis will be introduced here. We will consider a number of results based on the mathematical properties of these descriptions, that will provide the system theoretical basis for robust control and the most important tools for obtaining results later on.

Chapter 3 will introduce the general framework for linear controller design and analysis: the standard control design structure, and will provide procedures for designing mathematically optimal and sub-optimal controllers. Also the practical importance of mathematical optimality will be discussed here, as well as robustness issues and uncertainty descriptions.

The output regulation and tracking problem in combination with the design of controllers within the general framework will be the subject of chapter 4. The solution of this problem is the main result of this thesis and will appear to be possible using the control design procedures introduced in chapter 3. Connections with earlier results on the servomechanism problem will be discussed, as well as the extension to the two-degree-of-freedom tracking problem, the selection of weight functions and the combination with robust control.

Chapter 5 will consider an extensive example, based on an experimental three-degrees-of-freedom hydraulic positioning system. This example will show the complete design procedure based on the developed philosophy and the output regulation results. Implementation of designed controllers will be performed using state-of-the-art digital hardware (based on a Digital Signal Processor) and will show the practical applicability of the proposed procedure.

Finally, conclusions and recommendations for future research will be given in chapter 6.

Chapter 2

System descriptions and analysis tools

We will introduce the system concept from a very basic point of view, in order to prevent the common mistake that physical systems can be identified with their mathematical descriptions. This mistake usually appears when great effort is taken to find a 'best as possible' description for a given system, leading to complex models that are of high, or even infinite, order, highly non-linear, time-varying, etc. In using this approach one often finds that the actual issues of interest are quickly overlooked, and no mathematical tools are left to come to fundamental analyses and predictions. For this reason it is attempted in this thesis to use descriptions that are as simple as possible, but still demonstrate the phenomena under consideration.

After a discussion on the system concept in the next section, we will define and discuss several useful system descriptions in section 2.2. A more mathematically sound basis will be set up in section 2.3, in which function spaces are introduced as a means for signal and system classification. Section 2.4 will then define a fractional representation of a linear system, which will be used for the definition of poles and zeros of linear systems, leading to two more system descriptions, the definition of pole-zero cancellation and the definition of controllability and observability. Finally, section 2.5 will consider state-space methods for the calculation the 2-norm and the ∞-norm, defined in section 2.3.

2.1 The system concept

2.1.1 Systems in scientific research

In most areas of scientific research the concept of *systems* has been introduced in some way. Usually this is done by means of some kind of definition, with which an attempt is made to give a clear, preferably short, characterization. Such a characterization is often aimed at compatibility with the system concept of all these areas, which then results in something extremely vague and useless.
The problem is twofold. First the general system concept can not be defined in strictly mathematical terminology, which would be the only way to arrive at a complete and irrefutable definition. Secondly, the system concept has evolved in parallel in several scientific disciplines, causing—subtle—differences probably mainly due to the fact that the system concept is a very *human* concept. The second problem, of course, makes the system concept very subjective, but is at the same time the reason for its success, even necessity, in scientific research. It is the only way in which scientists are able to *select* their object of interest and set it apart from the complete universe in which we live. Where the whole of the universe is incomprehensible to man, the system concept allows him to focus on just a part of it, straightforwardly defining the concept of *environment* at the same time, as the complement of the system in the universe.
With this, the study of systems in general, using mathematical terms for their description, can be seen as the basis of *generalism* as the contrary of *specialism.* Although the specialistic approach to research seems to be most compatible with human nature, it also has led to the breaking down of science into more and more different disciplines and to increasing communication problems between them. Another logical consequence is the breaking down of the systems studied by these different disciplines into smaller and smaller parts. In contrast however, society and technology tend to pose questions to scientists of an increasingly general nature. One of the more recent developments in this area is the introduction of the term *mechatronics* which, obviously, emphasizes the need for a better integration of state of the art developments in the fields of mechanics and electronics.

2.1.2 System descriptions

Rather than giving a definition of the system concept, that is bound to be incomplete or at least unsuitable for some areas of science, we will therefore define some system *descriptions* that are useful within the context of this thesis. We will make use of the possibility to describe systems by means of variables and the relations between them, which is most fundamentally researched by Willems (1986,1988), who speaks of the behaviour of a system defined as a set of combinations of variables that are compatible with that system.

The problem with this concept is again twofold. First we have that the choice of the variables to be considered is up to the researcher; there is no guarantee that a system description based on variables is sufficiently complete for his purposes. Secondly, most systems of interest allow infinitely many of these combinations of variables, such that there is a need for a considerable 'data reduction'. Although the first problem will always remain, both problems can be largely accommodated by introducing the concept of behavioural equations; usually a set of mathematical relations between the variables under consideration, which can be used to test if certain combinations of variables may occur.

An example of such a set of relations is known as the state-space description or SSD:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{2.1}$$

The variables under consideration are given in the vectors x, u and y, clearly allowing a classification of variables into three groups. The state vector x consists of n internal variables and should be set up such that the model is sufficiently complete for the researchers purposes. The input vector u consists of q variables that can be considered independent from the behaviour of the system; they allow the environment to interact with the system. The output vector y consists of p variables that are of special interest to the researcher; they provide information on the system towards the environment. The state-space matrices A, B, C and D are assumed to be constant and to have real entries, as a matter of notation we have $A \in \mathrm{R}^{n \times n}$ the 'system matrix', $B \in \mathrm{R}^{n \times q}$ the 'input matrix', $C \in \mathrm{R}^{p \times n}$ the 'output matrix' and $D \in \mathrm{R}^{p \times q}$ the 'feedthrough matrix'.

Since we are particularly interested in dynamical systems, time is an important variable; it is accounted for by considering a set of first order differential equations, such that all aforementioned variables become functions of time, or more properly: signals. Relating the time dependent behaviour of physical quantities with signals is of course quite natural, causing the state-space description to be a very effective tool in studying systems. The introduction of an initial time t_0 and an initial state $x(t_0) = x_0$ is also natural; we are interested in the behaviour of the system, starting from a prespecified initial situation. In view of the previous remarks, the initial state can also be seen as a further possibility to reduce the number of combinations of variables or signals that are assumed to be allowable in the system behaviour. Finally, the assumption that the four state-space matrices A, B, C and D are constant implies that t_0 can be chosen arbitrarily and allows us to set $t_0 = 0$, this independence from the actual choice of initial time t_0 is referred to as the 'time invariance' property.

The considerations given above should however warn for irresponsible use of the state-space model, in making statements about an actual physical system. Another important point to be made here is that despite the simplicity of appearance of the state-space model, it is similar to any more complicated, non-linear, time dependent, etc., mathematical description in that all of these are only *descriptions* of an actual physical system, and therefore always incomplete and approximative. In this thesis we are interested in the possibility of describing essential physical phenomena in relation with control design problems; attempts to obtain more accurate models by allowing more complex descriptions often tend to obscure such phenomena, motivating the use of simple linear, time-invariant models like the state-space model.

2.2 Linear time invariant models

In the previous section the state-space description, or SSD, was already introduced as an example of a set of equations describing the behaviour of a system. The fact that this model is linear and time-invariant is well known and the result of the linearity of the equations and the constantness of the state-space matrices A, B, C and D. In this section several forms of linear time invariant models, or LTI-models for short, will be considered. We will also review the concepts of eigenvalues and -vectors and poles and zeros within the context of the system concept as introduced above and emphasize equivalent forms in various LTI models. This section is mainly based on Kailath (1980), Chen (1984), Maciejowski (1989) and Zhou et al. (1993).

2.2.1 The LFT form of the state-space model

With the differential equation form already given in equation 2.1, a block-diagram form of the state-space model can be given as in figure 2.1. From this figure it is clear that the state-space matrices can be seen as constant amplification factors and that the dynamical behaviour of the model is given by a feedback structure over a set of integrators, i.e. a square matrix with integrators on the diagonal.

This separation of simple known data, the constant state-space matrices, from the more complex part of the model, becomes even more clear if we consider the linear

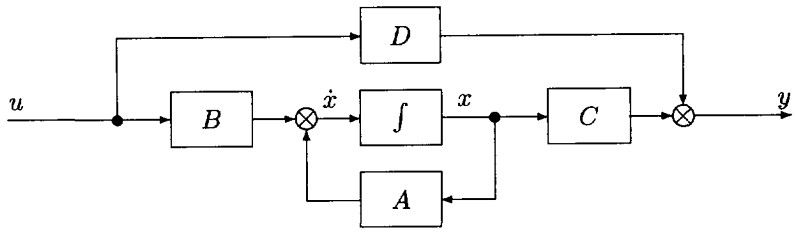

Fig. 2.1: Block-diagram form of the state-space model.

fractional transformation—or LFT—form of a state-space description. For this we define LFTs as follows:

Definition 2.2.1
Suppose a matrix M with entries in R *is partitioned as:*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad \in \mathrm{R}^{(p_1+p_2)\times(q_1+q_2)} \tag{2.2}$$

and let $\Delta_u \in \mathrm{R}^{q_1\times p_1}$ and $\Delta_l \in \mathrm{R}^{q_2\times p_2}$ be arbitrary. We will then define the upper *and* lower *LFTs as operators on Δ_u and Δ_l respectively:*

$$\begin{aligned} \mathcal{F}_u(M,\Delta_u) &:= M_{22} + M_{21}(I-\Delta_u M_{11})^{-1}\Delta_u M_{12} \\ \mathcal{F}_l(M,\Delta_l) &:= M_{11} + M_{12}(I-\Delta_l M_{22})^{-1}\Delta_l M_{21} \end{aligned} \tag{2.3}$$

Either LFT will be called well-defined *if the concerning inverse exists:*

$$\det(I-\Delta_u M_{11}) \neq 0, \quad \det(I-\Delta_l M_{22}) \neq 0 \tag{2.4}$$

The matrix M is referred to as the coefficient matrix of the LFT.

With this we can take:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \in \mathrm{R}^{(n+p)\times(n+q)} \tag{2.5}$$

and set Δ_u equal to the block of integrators. This then allows us to redraw figure 2.1 as figure 2.2. Note that in comparison with figure 2.1 the direction from inputs to outputs is reversed to be more compatible with the inherent matrix-vector multiplication:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{2.6}$$

When convenient, this notation will be used instead of the more conventional left-to-right notation.

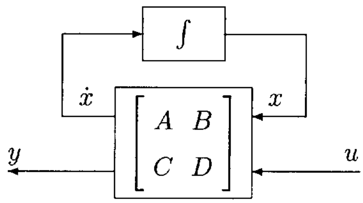

Fig. 2.2: LFT form of the state-space model.

2.2.2 Frequency domain analysis

The introduction of the LFT form of a state-space model seems rather trivial when we compare figures 2.1 and 2.2. However, it allows us to introduce an important system property if we consider the assumption that has to be made in definition 2.2.1. To be able to check if the LFT is well-defined we need to do an algebraic manipulation on a block of integrators, thus motivating the transformation of the problem into a domain in which this is possible. A well known approach to do this is to perform a Laplace-transformation on the state-space description, thus transferring the model from the time domain into the (complex) frequency domain. If, for the time being, we assume that all time dependent signals defined in equation 2.1 have a Laplace transform and that furthermore the initial conditions x_0 are zero, we can set up a transformed SSD as:

$$\begin{cases} sx(s) = Ax(s) + Bu(s) \\ y(s) = Cx(s) + Du(s) \end{cases} \tag{2.7}$$

in which $s \in \mathrm{C}$ denotes the Laplace operator.
With M given in equation 2.5, the LFT form of the transformed SSD then looks like:

$$\begin{aligned} \mathcal{F}_u(M, \tfrac{1}{s}I) &= D + C(I - \tfrac{1}{s}A)^{-1}\tfrac{1}{s}B \quad \Longleftrightarrow \\ \mathcal{F}_u(M, \tfrac{1}{s}I) &= D + C(sI - A)^{-1}B \end{aligned} \tag{2.8}$$

Clearly, the second part of this equation can also be obtained directly from equation 2.7 if we work out the relation between $u(s)$ and $y(s)$. This shows the equivalence with another popular form of describing LTI systems: the transfer function matrix description or TFMD:

$$y(s) = G(s)u(s) \quad G(s) := D + C(sI - A)^{-1}B \tag{2.9}$$

A generalisation of this form to real rational function in s will be discussed later.

2.2.3 The interconnectedness of LTI system descriptions

The condition we have to test to find out if the LFT-form of a state-space description is not well defined now becomes $\det(I - \frac{1}{s}IA) = 0$ or equivalently $\det(sI - A) = 0$. Note that this condition should hold for all $s \in \mathrm{C}$ such that it is easy to check that the LFT form of a state-space model is always well defined (for s sufficiently large and the elements of A bounded, $sI - A$ must become nonsingular).

Although we have that $\det(sI - A)$ is unequal to zero if considered as a polynomial with $s \in \mathrm{C}$, it is possible to find particular $s_i \in \mathrm{C}$ such that $\det(s_i I - A) = 0$. These values are clearly the eigenvalues of A: if $\lambda x_\lambda = A x_\lambda$ for some eigenvector x_λ we have $(\lambda I - A)x_\lambda = 0$ and we can set $s_i = \lambda$.

Hence we know that in the time domain $x(t) = ce^{\lambda t} \cdot x_\lambda$ is compatible with the behaviour of the system, with $c \in \mathrm{R}$ an arbitrary constant. Its Laplace transform $x(s) = \frac{c}{s-\lambda} x_\lambda$ must therefore be compatible with equation 2.7. This implies

$$\begin{aligned} s \cdot \tfrac{c}{s-\lambda} x_\lambda &= A \cdot \tfrac{c}{s-\lambda} x_\lambda + Bu(s) &\iff \\ \tfrac{cs}{s-\lambda} x_\lambda &= \tfrac{c\lambda}{s-\lambda} x_\lambda + Bu(s) &\iff \\ \tfrac{c(s-\lambda)}{s-\lambda} x_\lambda &= Bu(s) &\iff \\ c x_\lambda &= Bu(s) \end{aligned} \tag{2.10}$$

such that $Bu(s)$ must be constant while $u(t)$ was earlier said to be 0. A function with this property can be described mathematically by means of a generalised function; in practice we can think of $u(t)$ as an impulse at $t = 0$ that brings the system in a certain initial state, after which $u(t) = 0$ and an unforced response follows. From the Laplace transform of x given above it is furthermore clear that λ can be considered as a *pole* of x because $\lim_{s \to \lambda} \frac{c}{s-\lambda} x_\lambda = \infty$; usually λ is considered to be a pole of the *system* rather than of the *signal* because it is a system property that makes the signal compatible with the system.

The reason for this analysis is however not so much the introduction or definition of the concept of poles or the demonstration of the interconnectedness of all LTI system descriptions. It is to stress how easily signals and the systems they are compatible with can be confused, which calls for a more fundamental approach to systems and signals as will be done in the next sections.

2.3 Function spaces for systems and signals

As control objectives are usually stated in the form of desired properties of systems and signals, it is natural to introduce a mathematical classification of both systems and signals by means of well-chosen mathematical properties. In this, the term 'well-chosen' refers to the necessity of relating such properties to physical phenomena that are important for control design. On the other hand, these properties should be such that mathematical machinery can be employed to find relevant results. It appears that these demands can be complied with using function spaces.

For a fundamental and extensive introduction to the functional analysis background of this section, see Kreyszig (1978). The relation with system and control theory can be found in (among others) Francis (1987), Boyd and Barratt (1991) and Zhou et al. (1993).

2.3.1 Time domain function spaces

A function space is a set of functions having a common domain and range and usually a number of common properties.

For instance, consider a signal $x(t)$ defined for all time $-\infty < t < \infty$ and taking values in C^n; then x is a function $(-\infty, \infty) \longrightarrow \mathrm{C}^n$. An often useful property of such a function is the value of its square (Lebesgue) integral: $\int_{-\infty}^{\infty} \|x(t)\|^2 dt$. Here $\|x\|$ denotes $<x,x>^{1/2} = (x^*x)^{1/2}$, with x^* denoting the complex conjugate transpose of x, i.e. $\|x\|$ denotes the norm usually defined on C^n. In this way we can restrict the class of signals to those with

$$\int_{-\infty}^{\infty} \|x(t)\|^2 dt < \infty \tag{2.11}$$

and define a 'measure of size' for any such signal as:

$$\|x\|_{2,n} := \left\{ \int_{-\infty}^{\infty} \|x(t)\|^2 dt \right\}^{\frac{1}{2}} \tag{2.12}$$

Note that $\|x\|_{2,n}$ is a *norm* on the function space thus defined, because it has the properties:

- $\|x\|_{2,n} \geq 0$,
- $\|x\|_{2,n} = 0 \;\Leftrightarrow\; x(t) = 0, \quad$ (a.e.),
- $\|cx\|_{2,n} = |c| \cdot \|x\|_{2,n}, \quad c \in \mathrm{C}$,

- $\|x+y\|_{2,n} \leq \|x\|_{2,n} + \|y\|_{2,n}$.

This function space will be denoted as $L_{2,n}(-\infty,\infty)$ or in simplified form as $L_2(-\infty,\infty)$. It can be proven that $L_2(-\infty,\infty)$ is complete (Kreyszig 1978) and furthermore we have that equation 2.12 denotes an inner product: this implies $L_2(-\infty,\infty)$ is a Hilbert space.

Another example of a function space emerges if instead of the square integral of $x(t)$ we consider the *Root-Mean-Square (RMS) norm:*

$$\|x\|_{\mathrm{RMS}} := \left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)^* x(t) dt \right\}^{\frac{1}{2}} \tag{2.13}$$

This norm is a measure of the eventual average size of x and is especially useful when we consider signals that do not comply with equation 2.11, such as stochastic signals, for which the RMS-norm can also be defined as:

$$\|x\|_{\mathrm{RMS}} := \{\mathrm{E}\{x(t)^* x(t)\}\}^{\frac{1}{2}} \tag{2.14}$$

with 'E' denoting 'the expected value of' (note: we assume ergodicity and stationarity of stochastic signals).

Another important operation on stochastic signals is the determination of the autocorrelation matrix:

$$R_{xx}(\tau) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) x(t+\tau)^* dt = \mathrm{E}\{x(t)x(t+\tau)^*\} \tag{2.15}$$

The spectral density matrix can then be defined as:

$$S_{xx} := \mathcal{F}(R_{xx}) \tag{2.16}$$

with $\mathcal{F}$ denoting the Fourier transform. Now all signals having properties:

1) $x(t)$ bounded for all t,

2) $R_{xx}(\tau)$ exists and is bounded for all τ,

3) $S_{xx}(\omega)$ exists,

constitute the function space of all signals with bounded power (BP). The RMS-norm can then be seen as the *average power* of the signal x and can be related to $R_{xx}(\tau)$ and $S_{xx}(\omega)$ as follows:

$$\|x\|^2_{\mathrm{RMS}} = \mathrm{trace}\{R_{xx}(0)\} = \mathrm{trace}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega\right\} \tag{2.17}$$

2.3.2 Frequency domain function spaces

This last example shows that properties of signals in the time-domain (i.e. functions of t) can also be stated in the frequency-domain (i.e. functions of the complex variable $s = \lambda + j\omega$). A number of frequency-domain function spaces therefore appear to be very useful for purposes of control analysis and design and will be defined as follows:

- $\mathrm{R}[s]$ is the set of polynomials in the complex variable $s \in \mathrm{C}$ with coefficients in the field R of real numbers; if convenient, $\mathrm{R}[s]$ will be identified with $\mathrm{M}^{p\times q}\{\mathrm{R}[s]\}$, denoting matrices taking values in $\mathrm{C}^{p\times q}$ and having entries in $\mathrm{R}[s]$,

- $\mathrm{R}(s)$ is the field of fractions associated with $\mathrm{R}[s]$ and consists of real-rational functions in s; here too, $\mathrm{R}(s)$ will be identified with $\mathrm{M}^{p\times q}\{\mathrm{R}(s)\}$, denoting matrices taking values in $\mathrm{C}^{p\times q}$ and having entries in $\mathrm{R}(s)$,

- L_2 is the set of functions $x(j\omega)$ defined for all frequencies $-\infty < \omega < \infty$, taking values in C^n and being square (Lebesgue) integrable:

$$\|x\|_2 := \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)^* x(j\omega) d\omega\right]^{1/2} < \infty \tag{2.18}$$

  as this function space is complete (Kreyszig 1978) and as equation 2.18 constitutes an inner product, L_2 is a Hilbert space with $\|x\|_2$ denoting its norm,

- H_2 is the space of functions $x(s)$ which are analytic in Re $s > 0$ (the complex right half plane), take values in C^n and satisfy the square-integrability condition:

$$\|x\|_2 := \left[\sup_{\xi>0}\frac{1}{2\pi}\int_{-\infty}^{\infty} x(\xi + j\omega)^* x(\xi + j\omega) d\omega\right]^{1/2} < \infty \tag{2.19}$$

  H_2 may be seen as a closed subspace of L_2 via 'boundary value identification' (see Francis 1987, p.11),

- L_∞ is the set of functions $X(j\omega)$ defined for all frequencies $-\infty < \omega < \infty$, taking values in $\mathrm{C}^{p\times q}$ and for which the largest singular value $\bar{\sigma}(X) := \{\bar{\lambda}(X^*X)\}^{1/2}$ is essentially bounded ($\bar{\lambda}$ denotes largest eigenvalue); although $\bar{\sigma}(X)$ does not constitute an inner product, it can still be used to define a norm: L_∞ is a Banach space with norm

$$\|X\|_\infty := \operatorname*{ess\,sup}_{\omega} \bar{\sigma}(X(j\omega)) \tag{2.20}$$

- H_∞ is the set of functions $X(s)$ which are analytic in Re $s > 0$, take values in $C^{p\times q}$ and are also bounded in Re $s > 0$:

$$\|X\|_\infty := \sup_{\text{Re } s>0} \bar{\sigma}(X(s)) < \infty \tag{2.21}$$

  H_∞ may be seen as a closed subspace of L_∞.

From these spaces a number of important subspaces can be derived:

- RL_2 is the intersection of $R(s)$ with L_2, it consists of n-dimensional vectors each entry of which is real-rational, strictly proper and without poles on the imaginary axis,
- RH_2 is the intersection of $R(s)$ with H_2 or of RL_2 with H_2, it consists of n-dimensional vectors each entry of which is real-rational, strictly proper and stable,
- $H_2^\perp$ is the orthogonal complement of H_2 in L_2,
- $RH_2^\perp$ is the orthogonal complement of RH_2 in RL_2 (strictly proper, no stable poles),
- RL_∞ is the intersection of $R(s)$ with L_∞, it consists of $(p \times q)$-dimensional matrices each entry of which is real-rational, proper and without poles on the imaginary axis,
- RH_∞ is the intersection of $R(s)$ with H_∞ or of RL_∞ with H_∞, it consists of $(p \times q)$-dimensional matrices each entry of which is real-rational, proper and stable.

Note that the '2-norm' notation $\|x\|_2$ is the same for all '2-spaces' and that similarly $\|X\|_\infty$ is used for all '∞-spaces'. Furthermore note that dependency on the dimension of the range space is deleted.
Finally, as a matter of notation, we introduce the function space R_{ss}:

- R_{ss} is the set of proper, real-rational functions: it consists of all transfer functions that may arise from state-space models. R_{ss} will be identified with $M^{p\times q}\{R_{ss}\}$, denoting $p \times q$ matrices with entries in R_{ss}.

Note that R_{ss} is a subspace of $R(s)$ and that it includes RL_∞.

2.3.3 Relations between function spaces

An important connection between time-domain Hilbert spaces and frequency domain Hilbert spaces is provided by the Plancherel / Paley-Wiener theorem (Francis 1987):

Theorem 2.3.1
The Fourier transform is a Hilbert space isomorphism from $\mathrm{L}_2(-\infty,\infty)$ *onto* L_2. *It maps* $\mathrm{L}_2[0,\infty)$ *onto* H_2 *and* $\mathrm{L}_2(-\infty,0]$ *onto* $\mathrm{H}_2^{\perp}$.

Here *Hilbert space isomorphism* means that the Fourier transform is a linear surjection which is continuous, norm-preserving, injective and has a continuous inverse. This implies that the 2-norm of a signal in the time domain is equal to the 2-norm of the Fourier transformed signal in the frequency domain, which motivates the notation $\|x\|_2$ for all previously defined Hilbert spaces. The spaces $\mathrm{L}_2(-\infty,0]$ and $\mathrm{L}_2[0,\infty)$ denote two complementary subspaces in $\mathrm{L}_2(-\infty,\infty)$, the interval-argument denotes the part of the domain on which their elements are not necessarily essentially equal to zero.
The importance of the 2-norm for control analysis and design is its physical interpretation as the energy incorporated in a given signal in $\mathrm{L}_2(-\infty,\infty)$. The aforementioned equivalence with the 2-norm in the frequency domain gives reason to consider the ∞-norm, because it can be seen as the *induced* 2-norm according to the following theorem (Francis 1987).

Theorem 2.3.2
Given $G \in \mathrm{L}_\infty$, *then* $Gx \in \mathrm{L}_2, \ \forall\, x \in \mathrm{L}_2$ *and:*

$$\|G\|_\infty = \sup_{x\in\mathrm{L}_2} \frac{\|Gx\|_2}{\|x\|_2} \tag{2.22}$$

Given $G \in \mathrm{H}_\infty$, *then* $Gx \in \mathrm{H}_2, \ \forall\, x \in \mathrm{H}_2$ *and:*

$$\|G\|_\infty = \sup_{x\in\mathrm{H}_2} \frac{\|Gx\|_2}{\|x\|_2} \tag{2.23}$$

With this theorem it becomes possible to see '2-spaces' as spaces containing signals, and '∞-spaces' as spaces containing transfer function matrices describing the relation between independent input signals u and output signals y:

$$y = Gu \tag{2.24}$$

2.3.4 System descriptions and function spaces

We now have the possibility to consider properties of a physical system by means of finding an appropriate description using function spaces. The 'classical' procedure of setting up a set of linear differential equations leading to a state-space model will always lead to an equivalent transfer function matrix description in $R(s)$ and often in RL_∞ (if there are no imaginary poles). Looking at properties of signals can also be very important; signals can be seen as a possibility to 'test' the system, and hence to find out its basic properties.

Theorem 2.3.2 is a clear example of this principle; applying an input signal in H_2 must always result in an output signal in H_2 if the system is to be in H_∞. Signals in RL_2 can be expressed as strictly proper real-rational functions in s such that they can be interpreted as the impulse response of an LTI system, or the unforced response to a set of initial conditions. If we consider a system in RH_∞ the unforced response $y(t)$ to any set of initial conditions must be exponentially decaying for $t \to \infty$: therefore $y(t)$ must be square integrable over the interval $[0, \infty)$ and can be set to zero for $t < 0$, i.e. $y(t) \in L_2[0, \infty)$, and $y(s) \in RH_2$.

Another important relation between signals and systems can be found if we consider the response of a scalar system $g \in RH_2$ to a stochastic input signal u with spectral density $S_{uu} = 1$ (i.e. u is a white stochastic process). In that case it can be found that $S_{yy}(\omega) = g(j\omega)^* g(j\omega)$ (Papoulis 1984, Priestley 1981), such that:

$$||y||^2_{\text{RMS}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(j\omega)^* g(j\omega) d\omega = ||g||_2^2 \tag{2.25}$$

This implies that g can be considered as the frequency domain model of a signal that may be represented in the time domain as the impulse response *or* white noise response of a system with TFMD g, and that the RMS-norm of this signal is equal to its 2-norm, both in frequency domain and in time domain.

These are just a few examples of manipulations with signals and systems within the framework of function spaces that can be used for analysis and design. The next sections will go into more detail on some other important tools related to the use of function spaces, their interconnections and their relation with physical properties.

2.4 System properties related to poles and zeros

In the previous section it already appeared to be possible to classify systems and signals by restricting their descriptions to be in a certain function space. For instance, it appeared that all LTI state-space models have an associated transfer function matrix description in $\mathrm{R}(s)$ and that RL_∞ consists of all LTI systems that are proper and have no poles on the imaginary axis. More importantly, for both the 2-spaces and the ∞-spaces a norm was defined to be able to relate elements within a specific space with each other. However, these possibilities for classification are in general not sufficient to adequately describe the behaviour of the system under consideration and more specific statements are required.

To be able to do this, we will restrict ourselves to system and signal descriptions with a frequency domain description in $\mathrm{R}(s)$. This implies that we assume that properties of systems and signals we are interested in can be described sufficiently accurate by means of linear, time-invariant and finite-dimensional models. The advantage of this is that we can then make use of the concept of poles and zeros to further specify system behaviour.

The next subsection will introduce polynomial coprime fractions of linear systems, which will be one of the main tools for the derivation of the main result in chapter 4. Based on such a factorization it is possible to give a formal definition of poles and zeros, which will be done in subsection 2.4.2. After that we will consider system descriptions with internal variables in subsection 2.4.3 and the concepts of controllability, observability and pole-zero cancellation in subsection 2.4.4. Finally, this will lead to the definition of invariant and decoupling zeros in subsection 2.4.5

As already indicated, the exposition given in this section is aimed at introducing the tools and notations we need in later chapters. It is mainly based on Chen (1984), in which a much more elaborate discussion on these subjects can be found. Some other references are Kailath (1980), Maciejowski (1989), Callier and Desoer (1991) and Zhou et al. (1993).

2.4.1 Polynomial coprime fractions of linear systems

Considering the scalar case we have, as stated in the previous section, that $\mathrm{R}(s)$ is the field of fractions of elements of $\mathrm{R}[s]$, i.e. any element of $\mathrm{R}(s)$ can be written as the fraction of two elements of $\mathrm{R}[s]$. This notion can be generalized to matrices with entries in $\mathrm{R}(s)$ by describing them as a *coprime factorization* or *coprime fraction* over $\mathrm{R}[s]$:

Definition 2.4.1 (right (left) coprime fractions)
Suppose $G(s) \in \mathrm{R}(s)$.
An ordered pair (D_G, N_G) (or $(\tilde{D}_G, \tilde{N}_G)$) where D_G, N_G $(\tilde{D}_G, \tilde{N}_G) \in \mathrm{R}[s]$ is a right (left) coprime fraction—denoted as RCF or LCF—of G if:

1. *$D_G(\tilde{D}_G)$ is square and nonsingular (as a matrix over $R(s)$),*
2. *$G = N_G D_G^{-1} \quad (= \tilde{D}_G^{-1}\tilde{N}_G)$,*
3. *D_G and N_G ($\tilde{D}_G$ and $\tilde{N}_G$) are right (left) coprime.*

Here, right and left coprimeness can be specified by means of the right and left Bezout equation:

Definition 2.4.2 (coprimeness)
Given $D_G, N_G \in \mathrm{R}[s]$, then D_G and N_G are said to be right coprime *if there exist polynomial matrices $V_1, V_2 \in \mathrm{R}[s]$ such that the right Bezout equation is satisfied:*

$$V_1 D_G + V_2 N_G = [V_1\ V_2]\begin{bmatrix} D_G \\ N_G \end{bmatrix} = I \tag{2.26}$$

Given $\tilde{D}_G, \tilde{N}_G \in \mathrm{R}[s]$, then $\tilde{D}_G$ and $\tilde{N}_G$ are said to be left coprime *if there exist polynomial matrices $U_1, U_2 \in \mathrm{R}[s]$ such that the left Bezout equation is satisfied:*

$$\tilde{D}_G U_1 + \tilde{N}_G U_2 = [\tilde{D}_G\ \tilde{N}_G]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = I \tag{2.27}$$

Note that right coprimeness of (D_G, N_G) is equivalent to left invertibility of $[D_G' N_G']'$, i.e. $[D_G(p)' N_G(p)']'$ has full column rank for all $p \in \mathrm{C}$, and that left coprimeness of $(\tilde{D}_G, \tilde{N}_G)$ is equivalent to right invertibility of $[\tilde{D}_G\ \tilde{N}_G]$, i.e. $[\tilde{D}_G(p)\ \tilde{N}_G(p)]$ has full row rank for all $p \in \mathrm{C}$. Also note that coprime fractions of matrices in $\mathrm{R}(s)$ are not unique: they are unique modulo multiplication by a *unimodular* polynomial matrix, i.e. a nonsingular polynomial matrix with a polynomial inverse. By means of multiplication by a unimodular matrix it is possible to construct D_G such that it is *column-reduced* or $\tilde{D}_G$ such that it is *row-reduced*:

Definition 2.4.3 (column/row-reduced polynomial matrices)
Suppose $D \in \mathrm{R}[s]$ is square and nonsingular.

- *D is called* column-reduced *if the degree of its determinant is equal to the sum of the degrees of its columns.*
- *D is called* row-reduced *if the degree of its determinant is equal to the sum of the degrees of its rows.*

The reason for restricting coprime fractions in this sense is that reducedness ensures properness of D^{-1}.

2.4.2 Definition of poles and zeros

Poles were already introduced in section 2.2, showing that for a given system the eigenvalues of the system matrix of an SSD are equal to the poles of an equivalent TFMD. In this subsection we will give a definition of poles and zeros of a transfer function matrix, which is based on polynomial coprime fractions. For this consider:

$$G(s) = N_G(s)D_G(s)^{-1} = \tilde{D}_G(s)^{-1}\tilde{N}_G(s) \tag{2.28}$$

then:

Definition 2.4.4 (poles of a transfer function matrix)
The poles of a transfer function matrix $G(s) \in \mathrm{R}(s)$ *are those values* $p \in \mathrm{C}$ *such that* $\mathrm{rank}(D_G(p)) < \mathrm{rank}(D_G(s))$ *and* $\mathrm{rank}(\tilde{D}_G(p)) < \mathrm{rank}(\tilde{D}_G(s))$ *or—because* $D_G(s)$ *and* $\tilde{D}_G(s)$ *are square and nonsingular—such that* $\det(D_G(p)) = 0$ *and* $\det(\tilde{D}_G(p)) = 0$*).*
An output direction of a pole p *of* $G(s)$ *is a constant vector* $x_p \in \mathrm{C}\backslash\{0\}$ *such that* $\tilde{D}_G(p)x_p = 0$ *and* $x_p = N_G(p)\tilde{x}_p$ *for some vector* $\tilde{x}_p \in \mathrm{C}\backslash\{0\}$ *such that* $D_G(p)\tilde{x}_p = 0$.
The multiplicity of a pole p *of* $G(s)$ *in the direction* x_p *is the largest integer* $q \in \mathrm{N}$ *for which* $\lim_{s\to p}\,(s-p)^{-q}\tilde{D}_G(s)x_p$ *is bounded.*

Definition 2.4.5 (zeros of a transfer function matrix)
The zeros—or transmission zeros—of a transfer function matrix $G(s) \in \mathrm{R}(s)$ *are those values* $z \in \mathrm{C}$ *such that* $\mathrm{rank}(N_G(z)) < \mathrm{rank}(N_G(s))$ *and* $\mathrm{rank}(\tilde{N}_G(z)) < \mathrm{rank}(\tilde{N}_G(s))$.
An output direction of a zero z *of* $G(s)$ *is a constant vector* $x_z \in \mathrm{C}\backslash\{0\}$ *such that* $x_z' N_G(z) = 0$ *while* $x_z' N_G(s)$ *is a non-zero polynomial vector*
The multiplicity of a zero z *of* $G(s)$ *in the direction* x_z *is the largest integer* $q \in \mathrm{N}$ *for which* $\lim_{s\to z}\,(s-z)^{-q}x_z' N_G(s)$ *is bounded.*

These definitions are mainly based on Callier and Desoer (1982) and MacFarlane and Karcanias (1976). Note that $\mathrm{rank}(X(s))$ with s the Laplace variable denotes the rank of X over $\mathrm{R}(s)$, while for any given constant $c \in \mathrm{C}$, $\mathrm{rank}(X(c))$ denotes the rank of $X(c)$ over C.

2.4.3 System descriptions with internal variables

From the definition of poles and zeros it is immediately clear that they are completely determined by respectively D_G and N_G. For this reason it is useful to consider the polynomial fractions of a transfer function matrix as a separate system description if we want to analyze system properties related with poles and zeros. This system description is also known as the Matrix Fraction Description (MFD) and is especially useful to introduce the concept of internal variables as follows:

$$y(s) = G(s)u(s) = N_G(s)D_G^{-1}(s)u(s) \tag{2.29}$$

Now define:

$$\xi(s) = D_G^{-1}(s)u(s) \tag{2.30}$$

such that:

$$\begin{aligned} D_G(s)\xi(s) &= u(s) \\ y(s) &= N_G(s)\xi(s) \end{aligned} \tag{2.31}$$

A more general form of this equation is known as Rosenbrock's polynomial matrix description or PMD (Rosenbrock 1970):

$$\begin{gathered} \left\{ \begin{array}{rcl} T(s)\xi(s) + U(s)u(s) & = & 0 \\ V(s)\xi(s) + W(s)u(s) & = & y(s) \end{array} \right. \\ \Longleftrightarrow \begin{bmatrix} T(s) & U(s) \\ V(s) & W(s) \end{bmatrix} \begin{bmatrix} \xi(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} 0 \\ y(s) \end{bmatrix} \end{gathered} \tag{2.32}$$

We will use the following notation for PMDs:

$$\Sigma(s) := \begin{bmatrix} T(s) & U(s) \\ V(s) & W(s) \end{bmatrix} \tag{2.33}$$

Comparison with equation 2.7 shows that the Laplace transformed state-space model fits this description with $\xi = x$, $T(s) = sI - A$, $U(s) = -B$, $V(s) = C$ and $W(s) = D$. Furthermore, we can calculate a TFMD from a PMD as follows:

$$G(s) = W(s) - V(s)T(s)^{-1}U(s) \tag{2.34}$$

In section 2.4.1 it was already noted that polynomial fractions of a transfer function matrix are not unique. This obviously also implies that MFDs are not unique, in the sense that there are infinitely many MFDs that all represent the same TFMD. This non-uniqueness can be interpreted as freedom in the choice of the internal variables vector ξ. For any unimodular polynomial transformation matrix $\Theta \in R[s]$, we can define $\tilde{\xi} := \Theta\xi$ such that equation 2.30 becomes:

$$\tilde{\xi}(s) = \Theta(s)D_G^{-1}(s)u(s) \tag{2.35}$$

and equation 2.31 can be written as:

$$\begin{aligned} D_G(s)\Theta^{-1}(s)\tilde{\xi}(s) &= u(s) \\ y(s) &= N_G(s)\Theta^{-1}(s)\tilde{\xi}(s) \end{aligned} \tag{2.36}$$

Note that $\Theta^{-1} \in R[s]$ and that the pair $(N_G\Theta^{-1}, D_G\Theta^{-1})$ is right coprime if, and only if, the pair (N_G, D_G) is.

A similar type of non-uniqueness can be defined for PMDs. Again the main mechanism is provided by freedom in the choice of ξ, although the more general structure of the PMD complicates the transformation. We will only allow transformations such that the transformed vector of internal variables $\tilde{\xi}$ has the same dimension as ξ and that the order of $\det(T(s))$ is left constant. A transformation with these properties and the property that the transformed PMD corresponds with the same TFMD as the original one, is called a transformation of *Strict System Equivalence* or SSE (Rosenbrock 1970). All such transformations can be found from the following equation:

$$\begin{bmatrix} \Theta_1 & 0 \\ \Theta_2 & I_p \end{bmatrix} \begin{bmatrix} T & U \\ V & W \end{bmatrix} = \begin{bmatrix} \tilde{T} & \tilde{U} \\ \tilde{V} & \tilde{W} \end{bmatrix} \begin{bmatrix} \Theta_3 & \Theta_4 \\ 0 & I_q \end{bmatrix} \tag{2.37}$$

in which the Θ_i are polynomial matrices, with Θ_1 and Θ_3 unimodular. With n defined as the dimension of ξ (i.e. the dimension of T), any SSE transformation can be generated by the following elementary operations:

- the multiplication of any one of the first n rows of Σ with a constant,
- the addition of a polynomial multiple of any one of the first n rows of Σ to any other row,
- the interchanging of any two among the first n rows of Σ,
- the application of any of the above operations on the (first n) columns of Σ.

2.4.4 Controllability, observability and cancellations

The problem with the system descriptions introduced in the previous subsection is that there is no guarantee that values of s for which any of the polynomial matrices mentioned looses rank, will be a pole or zero of the corresponding transfer function matrix. A loss of rank occurring in T (equation 2.32) or $(sI - A)$ (equation 2.7) but not in D_G is of special importance, because in that case there is a singularity in the system description that either cannot be influenced by the control signals u (uncontrollable) or whose effect cannot be observed by means of measurement signals y (unobservable) or both. In this case there may be four causes:

1. the system under consideration is ill-designed; a certain dynamical effect cannot be controlled,
2. the system description is not set up accurately,
3. the dynamical effect is of no importance to the researcher's goal,
4. the singularity does not represent a physical phenomenon but is the result of mathematical manipulations.

The first two are straightforward and must be dealt with, the last two call for caution, but may be acceptable as long as a physical interpretation remains possible.

A possibility to characterize controllability and observability of PMDs is given by the following definition:

Definition 2.4.6 (controllability and observability of PMDs)
Given a PMD of an LTI system as in equation 2.32. Then:

1. *the PMD is controllable if and only if $T(s)$ and $U(s)$ are left coprime,*
2. *the PMD is observable if and only if $T(s)$ and $V(s)$ are right coprime.*

Note that this immediately implies that an MFD as given in equation 2.29 is always controllable, but only observable if D_G and N_G are right coprime; similarly, a system description given as a *left* fraction is always observable, but only controllable if $\tilde{D}_G$ and $\tilde{N}_G$ are left coprime.
When considering controllability and observability of SSDs, we may—as mentioned before—take the Laplace transformed SSD as a special case of a PMD. Application of definition 2.4.6 and equations 2.26 and 2.27 then results in the more commonly used controllability and observability tests for state-space models (the Popov-Belevitch-Hautus tests, see Hautus 1969).

Theorem 2.4.7 (controllability and observability of SSDs)
Given an SSD of an LTI system as in equation 2.7. Then:

1. *the SSD is controllable if and only if*

$$\text{rank}\;[\lambda I - A \;\vdots\; B] = n \quad \forall \lambda \in \mathrm{C},$$

2. *the SSD is observable if and only if*

$$\text{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathrm{C}.$$

An important cause for the occurrence of uncontrollable or unobservable system descriptions is the possibility of *cancellation* of poles, when forming the product of two transfer function matrices $G_1(s)$ and $G_2(s)$. For this we will use the following proposition (Chen 1984, Anderson and Gevers 1981):

Proposition 2.4.8 (cancellation of poles)
Given $G_1, G_2 \in \mathrm{R}(s)$ *and the coprime fractions* $G_1 = \tilde{D}_1^{-1}\tilde{N}_1 = N_1 D_1^{-1}$ *and* $G_2 = \tilde{D}_2^{-1}\tilde{N}_2 = N_2 D_2^{-1}$:

1. *there is no cancellation of any of the poles of* G_1 *in forming the product* G_1G_2 *if and only if any one of the following three pairs of matrices are left coprime:* D_1 *and* N_2; $\tilde{D}_2 D_1$ *and* $\tilde{N}_2$; $\tilde{D}_1$ *and* $\tilde{N}_1 N_2$,

2. *there is no cancellation of any of the poles of* G_2 *in forming the product* G_1G_2 *if and only if any one of the following three pairs of matrices are right coprime:* $\tilde{D}_2$ *and* $\tilde{N}_1$; $\tilde{D}_2 D_1$ *and* N_1; D_2 *and* $\tilde{N}_1 N_2$.

It should be noted that it is possible to distinguish two forms of cancellation. First, we may have that either condition is violated because G_1 or G_2 does not have full rank. Secondly, we may have that either condition is violated because $\text{rank}(N_1(p)) < \text{rank}(N_1(s))$ with p being any pole of G_2 or $\text{rank}(N_2(p)) < \text{rank}(N_2(s))$ with p being any pole of G_1. Clearly only in the second case it is appropriate to talk about a *pole-zero cancellation.*

Although controllability and observability are usually desired properties of the system description, it is useful to make a distinction between stable and unstable poles, i.e. poles with a negative real part (λ in the open complex left half plane C^-) and poles with a non-negative real part (λ in the closed complex right half plane C^+). When a system description is uncontrollable or unobservable due to merely stable poles, it is called stabilizable and detectable: these properties are necessary and sufficient for the existence of a stabilizing controller.

2.4.5 Invariant zeros

Poles that disappear when a system description with internal variables (SSD, MFD or PMD) is transformed into a TFMD can be assumed to have been cancelled by appropriate zeros, when forming the TFMD. Such zeros are referred to as *decoupling zeros*, for the obvious reason that they decouple the dynamical behaviour represented by an unobservable or uncontrollable pole from the behaviour of the TFMD. An uncontrollable pole will give rise to an input-decoupling zero and an unobservable pole will result in an output-decoupling zero: note that it is possible that a zero is both input- and output-decoupling.

The combination of these zeros with possible transmission zeros that *do* appear in the TFMD, is also known as the set of *invariant zeros*, because neither of them can be relocated in the complex plane by means of feedback control. The set of invariant zeros can be found from a PMD by means of the following theorem (Rosenbrock 1970, Callier and Desoer 1982):

Theorem 2.4.9 (invariant zeros)
The set of invariant zeros of a PMD is given by those values z_i for which:

$$\operatorname{rank}\begin{bmatrix} T(z_i) & U(z_i) \\ V(z_i) & W(z_i) \end{bmatrix} < \operatorname{rank}\begin{bmatrix} T(s) & U(s) \\ V(s) & W(s) \end{bmatrix} \tag{2.38}$$

Such a z_i is a decoupling zero if and only if $\det T(z_i) = 0$.
Loss of row rank of $[T(z_i)\ U(z_i)]$ *implies that z_i is an input-decoupling zero; loss of column rank of* $\begin{bmatrix} T(z_i) \\ V(z_i) \end{bmatrix}$ *implies that z_i is an output-decoupling zero.*

It is easy to find similar conditions for SSDs and MFDs by interpreting them as special cases of a PMD. With this theorem, definition 2.4.2 and definition 2.4.6, we have that a PMD is controllable if and only if it does not have any input-decoupling zeros, and that a PMD is observable if and only if it does not have any output-decoupling zeros. Removal of decoupling zeros from a PMD (or SSD) is possible without affecting the corresponding TFMD: for this reason a controllable and observable PMD (or SSD) is called *minimal.*

One of the consequences of theorem 2.4.9 is of particular importance for the development in chapter 4 and will be considered here in some detail.

For this take a PMD with the following structure:

$$\Sigma_s := \left[\begin{array}{cc} T_s & U_s \\ V_s & W_s \end{array}\right] = \left[\begin{array}{cc|c} T_{11} & 0 & U_1 \\ 0 & T_{22} & U_2 \\ \hline 0 & V_2 & W \end{array}\right] \tag{2.39}$$

Hence, $\left[\begin{array}{c} T_s(z_i) \\ V_s(z_i) \end{array}\right]$ looses column rank if z_i is taken such that $\det(T_{11}(z_i)) = 0$, i.e. all such z_i are output decoupling zeros and the dynamical behaviour represented by T_{11} is unobservable.

Now consider SSE transformations with the corresponding partitioning:

$$\left[\begin{array}{cc|c} \Theta_{111} & \Theta_{112} & 0 \\ \Theta_{121} & \Theta_{122} & 0 \\ \hline \Theta_{21} & \Theta_{22} & I \end{array}\right] \left[\begin{array}{cc|c} T_{11} & 0 & U_1 \\ 0 & T_{22} & U_2 \\ \hline 0 & V_2 & W \end{array}\right] = \left[\begin{array}{cc|c} \tilde{T}_{11} & \tilde{T}_{12} & \tilde{U}_1 \\ \tilde{T}_{21} & \tilde{T}_{22} & \tilde{U}_2 \\ \hline \tilde{V}_1 & \tilde{V}_2 & \tilde{W} \end{array}\right] \left[\begin{array}{cc|c} \Theta_{311} & \Theta_{312} & \Theta_{41} \\ \Theta_{321} & \Theta_{322} & \Theta_{42} \\ \hline 0 & 0 & I \end{array}\right] \tag{2.40}$$

We will restrict these transformations to those that leave T_{11} unchanged ($\tilde{T}_{11} = T_{11}$); we then have $\Theta_{111} = \Theta_{311} = I$. Next, take only the first column:

$$\left[\begin{array}{c} I \\ \Theta_{121} \\ \Theta_{21} \end{array}\right] T_{11} = \left[\begin{array}{cc} T_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \\ \hline \tilde{V}_1 & \tilde{V}_2 \end{array}\right] \left[\begin{array}{c} I \\ \Theta_{321} \end{array}\right] \tag{2.41}$$

This implies that $\tilde{T}_{12} = 0$ or $\Theta_{321} = 0$; in both cases we can state that there must exist polynomial matrices $\Theta_{121}, \Theta_{21}$ and Θ_{321} such that:

$$\left[\begin{array}{c} \Theta_{121} \\ \Theta_{21} \end{array}\right] T_{11} = \left[\begin{array}{cc} \tilde{T}_{21} & \tilde{T}_{22} \\ \hline \tilde{V}_1 & \tilde{V}_2 \end{array}\right] \left[\begin{array}{c} I \\ \Theta_{321} \end{array}\right] \tag{2.42}$$

Hence, a necessary condition for a PMD:

$$\tilde{\Sigma}_s := \left[\begin{array}{cc|c} T_{11} & \tilde{T}_{12} & \tilde{U}_1 \\ \tilde{T}_{21} & \tilde{T}_{22} & \tilde{U}_2 \\ \hline \tilde{V}_1 & \tilde{V}_2 & \tilde{W} \end{array}\right] \tag{2.43}$$

to be strictly system equivalent with the PMD Σ_s in equation 2.39 is given by equation 2.42.

2.5 State-space calculation of norms

In this section a concise review is given on the possibility to calculate both the ∞-norm and the 2-norm by means of state-space methods. For a more extensive treatment see, for instance, Doyle et al. (1984), Francis (1987), Doyle et al. (1989), Zhou et al. (1993)

2.5.1 The ∞-norm

Calculation of $||G||_\infty$ can be done by the procedure already mentioned in section 2.3, which involves the calculation of a singular value decomposition over the imaginary axis. Although we will see in later chapters that this procedure may provide important information on robustness and performance of closed loop systems, it would be convenient to have a possibility to calculate the ∞-norm more directly. For this we will first look at transfer function matrices that are strictly proper and have a minimal state-space realization denoted as $[A, B, C, 0]$. Now define the *Hamiltonian matrix*:

$$H := \begin{bmatrix} A & BB' \\ -C'C & -A' \end{bmatrix} \tag{2.44}$$

We then have:

Theorem 2.5.1 (∞-norm calculation)
$||G||_\infty < 1$ iff H has no eigenvalues on the imaginary axis.

Without going into detail to prove this theorem (see references given above) it is interesting to note that H is the system matrix of the state-space description of $(I - G^\sim G)^{-1}$ with $G^\sim(s) := G'(-s)$. If $||G||_\infty < 1$ then $I - G(j\omega)^* G(j\omega) > 0$, $\forall\omega$, and hence $(I - G^\sim G)^{-1} \in \mathrm{RL}_\infty$. If $||G||_\infty \geq 1$ then $\bar{\sigma}(G(j\omega)) = 1$ for some ω, i.e. 1 is an eigenvalue of $G(j\omega)^* G(j\omega)$, such that $I - G(j\omega)^* G(j\omega)$ is singular. By calculating the eigenvalues of H we can thus determine if $||G||_\infty$ is smaller or larger than 1. By introducing a parameter γ we can therefore set up an iterative procedure to find the actual ∞-norm: select γ, determine if $||\gamma^{-1}G||_\infty < 1$, if so decrease γ, if not increase γ. Extension of this procedure to proper, but not strictly proper transfer matrices ($D \neq 0$) is readily possible, but complicates the calculation of the Hamiltonian matrix (see Boyd et al. 1988, Doyle et al. 1989, Bruinsma and Steinbuch 1990).

2.5.2 The 2-norm

Calculation of the 2-norm by means of the integral relation of equation 2.18 is clearly quite complex and also calls for an alternative, computationally less intensive procedure. When we consider signals in RH_2, it appears that this is possible using a state-space description of a system that is able to generate the signal of which we want to calculate the 2-norm. Note that we can consider a frequency domain signal representation of a signal in RH_2 as a TFMD of a system with one input and n outputs. By setting the single input signal to 1 in the frequency domain, i.e. a white noise signal or an impulsive signal with intensity 1 in the time domain, we will get the desired signal at the output.

Given a state-space realization of a signal $g(s) \in RH_2$ as $[A, B, C, 0]$, the time domain impulse response can be found as $g(t) = Ce^{At}B$, $\forall t \geq 0$. With this we can write the 2-norm of g as:

$$\|g\|_2 = \left\{ \int_{-\infty}^{\infty} g(t)^* g(t) dt \right\}^{\frac{1}{2}} = \left\{ \int_0^{\infty} B' e^{A't} C' C e^{At} B dt \right\}^{\frac{1}{2}} \tag{2.45}$$

This equation can be generalized to the determination of the 2-norm of strictly proper transfer function matrices, by taking the trace of the right hand side:

$$\|G\|_2 = \left\{ \text{trace} \left(B' \int_0^{\infty} e^{A't} C' C e^{At} dt B \right) \right\}^{\frac{1}{2}} \tag{2.46}$$

The integral in this equation determines the total amount of energy in the system output, starting from a given initial state with no input. It therefore determines how well the internal system behaviour can be observed from the outputs and is called the *observability Gramian*. Using $\text{trace}(PQ) = \text{trace}(QP)$ we can find a dual expression that determines the amount of energy that must be applied to the system through the input signals to obtain a certain energy content of the system state, and is called the *controllability Gramian*:

$$\begin{aligned} X_o &:= \textstyle\int_0^{\infty} e^{A't} C' C e^{At} dt \\ X_c &:= \textstyle\int_0^{\infty} e^{A't} B B' e^{At} dt \end{aligned} \tag{2.47}$$

These Gramians can easily be calculated, as they are the unique solutions of the Lyapunov equations:

$$\begin{aligned} A'X_o + X_o A + C'C &= 0 \\ AX_c + X_c A' + BB' &= 0 \end{aligned} \tag{2.48}$$

which can be calculated without much computational effort. The procedure for calculating the 2-norm of a signal description in RH_2 or a strictly proper TFMD in RH_∞ is therefore as follows:

- determine a (minimal, stable) state space realization $[A, B, C, 0]$ of the frequency domain description $G(s)$,
- calculate either X_o or X_c from equation 2.48,
- $\|G\|_2 = \{\text{trace}\,(B'X_oB)\}^{\frac{1}{2}} = \{\text{trace}\,(CX_cC')\}^{\frac{1}{2}}$,
- the 2-norm in the time domain is equal to the 2-norm in the frequency domain by theorem 2.3.1.

Numerical implementation of the calculation of the ∞-norm by means of the Hamiltonian matrix, and of the 2-norm by means of Lyapunov equations is, for instance, available in PC MatLab (Moler et al. 1987).

Chapter 3

Linear controller design and analysis

3.1 Introduction

3.1.1 Classical control methods

Although examples of technical solutions involving control can already be found in ancient China, it may be stated that a fundamental approach to control design has only started this century. The main paradigm up to 1960 is the frequency domain approach involving analysis using poles and zeros of transfer function descriptions and graphical methods to obtain a quantification of the behaviour of the system under consideration. Controller design is mainly feedback design and aimed at obtaining an open-loop behaviour of the cascade connection of plant and controller that guarantees desired closed-loop behaviour.

The main method of design is graphical, especially the Nyquist diagram or equivalently the Nichols chart can be used to relate certain properties of the open-loop behaviour to that of the closed loop. Stability of the closed-loop system can easily be checked by means of the Nyquist stability theorem. Performance can be assessed by checking the cross-over frequency: the closed-loop bandwidth will be close to the frequency at which the open-loop system has an amplification factor of 1. Robustness against uncertainties in the system description can be checked by means of phase and amplitude margins. The controller structure can usually be kept very simple: PID control, sometimes extended with lead-lag and notch filters.

The relative simplicity and effectiveness of these tools are the main cause of their widespread use in industry. In fact, the only practical reason for considering more modern techniques instead of this 'classical' design procedure is in the inherent difficulty with graphical design procedures, that multi-input multi-output systems can only be handled in very simple cases.

3.1.2 The modern control era

This has been the main reason for developing 'modern' control theory in the period from 1960 to 1980. In contrast with classical control theory, analysis and design is based on time-domain descriptions, i.e. on making use of state-space models. This implies that the approach is multivariable and allows single-input, single-output systems to be seen as a special case.

For analysis purposes, the use of state-space descriptions leads to the introduction of the controllability and observability concepts and a deeper understanding of poles and zeros. For design, the complete description of a system's internal behaviour allows the application of controllers that make use of it to force closed-loop behaviour that can be pre-specified in a greater extent than what was possible with classical control methods (for instance pole-placement and linear quadratic gaussian optimal control (LQG)).

Using such controllers, however, appeared to be very unsatisfying when the state-space model is not a completely accurate description of the system to be controlled. As this is usually the case, modern control theory is not very often applied in practice: only when accurate state-space descriptions are available, Kalman filters and/or linear quadratic optimal controllers can be found.

3.1.3 Robust control developments

In the late seventies, researchers found the mechanisms behind the unsatisfactory results of modern control theory and defined the need for robustness of multivariable controllers in a fashion similar to classical control. This lead to a renewed interest in frequency-domain methods, but now transfer function matrices were considered by means of state-space realizations.

For robustness and performance analysis of multivariable closed-loop systems the singular values of the closed-loop transfer function matrix, evaluated along the imaginary axis, can be used as a measure of magnitude leading to Bode-like plots. For controller design, several results from operator theory allowed a different approach to finding 'optimal' controllers.

More attention was directed at the inherent trade-offs in any controller design: instead of attempting to find a single optimal controller, it appeared useful to first determine classes of 'allowable' controllers. The concept of operator norms can then be used to define selection criteria for choosing an appropriate controller from this set: especially the ∞-norm can be used to bound the amplification factor of certain prespecified transfer functions, thus allowing both robustness and performance demands on the closed-loop system to be met.

Usually there is no analytical solution to find controllers that minimize the ∞-norm. An iterative procedure can be used to tighten one or more of the boundary conditions until a solution can no longer be found (γ-iteration). As an arbitrarily close approximation of the optimal solution can be found with this procedure, the resulting controller is sometimes referred to as 'the H_∞-optimal controller'.

3.1.4 Overview

The next section will define the standard control design structure already mentioned in chapter 1, which will be used as a general framework for the solution of linear control problems. The main reason for the introduction of this framework is that it provides an interface between practical control problems and theoretical results.

The practical importance of setting up of the standard plant for a specific control problem is that it involves not only the derivation of an accurate linear model of the plant to be controlled, but also the determination of weight functions to model disturbance behaviour, the specification of desired closed loop behaviour and the description of the influence of uncertainties. Thus, the standard plant can be used to determine many aspects of the actual control problem, after which theoretical results derived for this general description can be used to synthesize a controller that takes all these aspects into account.

One of these controller synthesis procedures is in fact the frequency domain approach of LQG control and will be discussed in section 3.3. It will appear that this approach is equivalent to the problem of minimizing the 2-norm of the closed-loop transfer function from disturbances to error signals, resulting in an important connection between time-domain and frequency-domain design techniques.

Next the synthesis procedure for H_∞ optimal controllers will be considered in section 3.4: here the ∞-norm of the closed-loop transfer function from disturbances to error signals will be minimized.

After this, the problem of designing robust controllers will be discussed in section 3.5: it will be shown that a property of the standard plant approach is that

robustness of a controlled system depends on the way the standard plant has been set up rather than on the selection of the controller synthesis procedure.
This will result in the need for the incorporation of detailed uncertainty descriptions, which will be the subject of section 3.6. Here the structured singular value is introduced and its role in robustness analysis and robust controller synthesis will be considered.
Finally, section 3.7 will consider state-space descriptions with uncertain entries in the coefficient matrices. A procedure for the automated transformation of such a model into the standard control design structure will be discussed, providing the possibility to set up highly detailed uncertainty models.

This chapter is of an introductory nature, although some considerations are not readily found in literature. Apart from section 3.7, which is based on Lambrechts et al. (1993), it is mainly based on Doyle et al. (1984), Francis (1987), Doyle et al. (1990), Boyd and Barratt (1991), Zhou et al. (1993).

3.2 The general framework

3.2.1 The standard control design structure

In classical and modern control, many useful control design frameworks are used. An important property of all these frameworks is the central position of the 'plant', the model of the physical system to be controlled. This plant is always seen as a given entity: it has to approximate the behaviour of the actual plant as close as possible, especially in view of the interaction it may have with the controller to be designed. Usually this implies that the plant only describes the transfer from actuator inputs to measured outputs. Any other information, like performance objectives, acting disturbances and robustness demands, is available as 'side-information' or is assumed to be contained in the measured outputs.
As a consequence of this, specific design objectives are usually added to the problem as extra signals and models: for instance, measurement noise, models to describe the behaviour of disturbances, actuator and sensor models, etc. For the design of a controller this often implies that separate compensators or filters are employed to try to meet with all these separate design objectives. Hence, the structure of the controller itself becomes complex and it is hard to find a controller that simultaneously meets all specifications.

To deal with this, we need a situation in which the controller can be synthesized in one stroke, while optimizing all control objectives. For this to be possible, we must have that the plant, together with all these objectives is specified in a single structure for which a single controller can be found. Because the original plant is usually a significant part of this new single structure, it is called the 'standard plant'. The standard plant is set up such that both its input signals and its output signals can be divided into two groups (e.g. Boyd and Barratt 1991):

Definition 3.2.1 (standard plant input signals)
The inputs of the standard plant are divided into two vector signals:

- *the* **control input vector**, *denoted u, consists of inputs to the standard plant that can be manipulated by the controller; it is the signal vector created by the controller,*
- *the* **disturbance vector**, *denoted w, consists of all other input signals to the standard plant.*

Definition 3.2.2 (standard plant output signals)
The outputs of the standard plant are divided into two vector signals:

- *the* **measurement signal vector**, *denoted y, consists of output signals that are accessible to the controller; it is the input signal vector to the controller,*
- *the* **control objectives vector**, *denoted z, consists of all other output signals of the standard plant; it usually contains (weighted versions of) u and y.*

A pictorial representation of the structure including a controller is given in figure 3.1. Here P denotes the standard plant and K denotes the controller. We

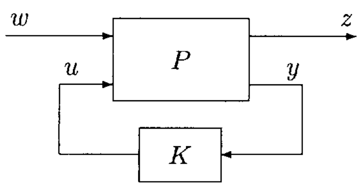

Fig. 3.1: Standard control design structure

will assume that P and K are proper real-rational transfer function matrices such that they can be represented by means of a state-space realization: $P, K \in \mathrm{R}_{\mathrm{ss}}$.

Now consider a partitioning of P according to the partitioning of input signals and output signals:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad \in \mathrm{R}_{\mathrm{ss}}^{(p_1+p_2)\times(q_1+q_2)} \tag{3.1}$$

and note that we have dimension q_1 for w, q_2 for u, p_1 for z and p_2 for y. According to the definition of LFTs in section 2.2 (definition 2.2.1), P can be seen as the coefficient matrix of a lower LFT, operating on K, with the only difference that all matrices now have entries in R_{ss}:

$$\mathcal{F}_l(P, K) := P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21} \tag{3.2}$$

As a matter of notation, we will define the closed loop transfer matrix from w to z as:

$$T_{wz} := \mathcal{F}_l(P, K) \tag{3.3}$$

A state-space realization of P with a partitioning according to equation 3.1 can be given as:

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right] \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{3.4}$$

Note that this equation is given in the same form as equation 2.6, such that the LFT form of the state-space model of P is easily set up; we could consider the closed-loop system of figure 3.1 as a combination of an upper and a lower LFT.

3.2.2 Internal stability of the closed-loop system

When considering the combination of standard plant P and controller K given in figure 3.1 as a lower LFT, the first thing to check is whether this LFT is well-defined. According to definition 2.2.1, we can do this by verifying: $\det(I - KP_{22}) \neq 0$. However, in this case we have that both K and P are elements of R_{ss}, such that $\det(I - KP_{22})$ is a function of s. We will therefore define a more restrictive property as follows:

Definition 3.2.3 (well-posedness)
The configuration in figure 3.1 is said to be well-posed *if* $(I - KP_{22})^{-1}$ *is a proper, real-rational transfer function, i.e. an element of* R_{ss}.

Note that this condition can be checked by verifying whether:

$$\det(I - K(j\infty)P(j\infty)) \neq 0.$$

Unfortunately, well-posedness of the closed-loop configuration of figure 3.1 is not sufficient for internal stability. To check internal stability we introduce two auxiliary signals, which results in figure 3.2. Hence, internal stability may be defined

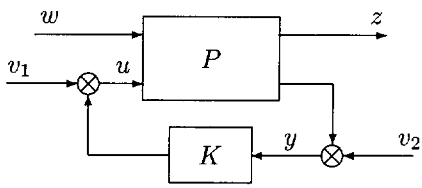

Fig. 3.2: Internal stability test configuration

as:

Definition 3.2.4 (internal stability)
The configuration in figure 3.2 is said to be internally stable if all nine transfer function matrices from input signals w, v_1 and v_2 to output signals z, u and y are stable and proper, i.e. are elements of RH_∞.

Now let us introduce a state-space description of the standard plant with a partitioning as in equation 3.4 and including the effect of auxiliary signals v_1 and v_2:

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u + B_2 v_1 \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u + v_2 \end{cases} \tag{3.5}$$

A standard result for the partial system:

$$\begin{cases} \dot{x} = Ax + B_2 u \\ y = C_2 x + D_{22} u \end{cases} \tag{3.6}$$

is that there exists a proper controller K achieving internal stability if and only if (A, B_2) is stabilizable and (A, C_2) is detectable. Note that the SSD in equation 3.6 is in general not minimal, i.e. there may be uncontrollable and/or unobservable poles as long as they are stable. This implies that an internally stabilizing controller for equation 3.6 must also be internally stabilizing for equation 3.5.

3.3 Optimal controller synthesis: LQG and H_2

In classical control the actual controller synthesis consists of tuning standard elements, like PID-controllers, lead-lag filters and notch filters. Design parameters are usually directly related to physical properties of the plant: 'rules of thumb' and graphical methods are used to adjust them. In modern control a first attempt is made to find a mathematical basis for controller synthesis. Control design is considered to be the search for a controller that is optimal in some sense. A basic trade-off between several design objectives is required, to make sure that this optimization is non-trivial. Furthermore, it is desirable that the optimization process is sufficiently simple, to be able to solve realistic problems.

First we will consider some preliminary results on the properties of solutions to the algebraic Riccati equation. After that, we will review the method known as Linear Quadratic Gaussian control (LQG), which is the most established result from modern control that meets the aforementioned demands. Finally, generalization of this method will be discussed, resulting in the minimization of the 2-norm of the closed-loop transfer function and therefore known as H_2-optimal control.

3.3.1 The algebraic Riccati equation

A central role in the application of the controller synthesis methods discussed in this section: LQG and H_2, as well as in the following section: H_∞, is played by the *algebraic Riccati equation*, or *ARE* in short. Let A, Q and R be real valued $n \times n$ matrices with Q and R symmetric, i.e. $Q' = Q$ and $R' = R$; then an ARE is the following matrix equation:

$$A'X + XA + XRX + Q = 0 \tag{3.7}$$

Algorithms for finding solutions X to this equation are readily available. They are usually based on the possibility to associate the ARE with the following $2n \times 2n$ Hamiltonian matrix:

$$H = \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix} \tag{3.8}$$

For an extensive discussion on the properties of such solutions see for instance Zhou et al. (1993).

In this chapter we will only need to consider a *stabilizing solution*, i.e. a solution X such that the eigenvalues of $A + RX$ are all in C^-. For such a solution we have the following theorem:

Theorem 3.3.1
There exists at most one stabilizing solution to the algebraic Riccati equation 3.7.

This implies that if we are able to find a stabilizing solution, it must be the *unique* stabilizing solution.
The question whether there exists such a unique stabilizing solution may be answered by examining the associated Hamiltonian matrix. Let $\mathbf{X}_-(H)$ denote the stable subspace of H; we then have:

Theorem 3.3.2
Given the ARE of equation 3.7 and its associated Hamiltonian matrix given in equation 3.8, then there exists a unique stabilizing solution if:

- *H has no eigenvalues on the imaginary axis,*
- *$\mathbf{X}_-(H)$ and $\mathrm{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$ are complementary.*

Furthermore, this solution is symmetric: $X = X'$.

In some cases we can also make use of the following result:

Theorem 3.3.3
Given the ARE of equation 3.7, then there exists a unique stabilizing solution if $R \leq 0$, $Q \geq 0$, (A, R) stabilizable and (A, Q) detectable. Furthermore, this solution is symmetric and positive semi-definite: $X = X' \geq 0$.

This result will be useful when considering the LQG and H_2 controller synthesis methods.

3.3.2 LQG controller synthesis

Although many researchers have given extensive descriptions of LQG methods (Anderson and Moore 1971, 1989, Kwakernaak and Sivan 1972, etc.), we will review the basic procedure to introduce some notation and stress the relation with the robust control approach.

LQG problem definition

LQG is based on a state-space description of the plant with two auxiliary signals:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + v_1(t) \\ y(t) &= Cx(t) + Du(t) + v_2(t) \end{cases} \tag{3.9}$$

The auxiliary signals $v_1(t)$ and $v_2(t)$ are assumed to be zero-mean Gaussian stochastic processes which are uncorrelated in time (white noise) and have constant spectral density matrices V_1 and V_2 respectively (see equations 2.15 and 2.16):

$$\begin{aligned} V_1 &:= \mathcal{F}(\lim_{t\to\infty} \mathrm{E}\{v_1(t)v_1(t+\tau)'\}) \geq 0 \\ V_2 &:= \mathcal{F}(\lim_{t\to\infty} \mathrm{E}\{v_2(t)v_2(t+\tau)'\}) > 0 \end{aligned} \tag{3.10}$$

The restriction that V_2 must be non-singular can be avoided, but there are two reasons not to pursue this:

- calculations and formulæ will become much more complicated without adding any insight,
- the proposition that *all* measurement signals are subject to *some* measurement noise is in agreement with practical experience.

Another assumption that will be made here, is that v_1 and v_2 are uncorrelated with each other:

$$\lim_{t\to\infty} \mathrm{E}\{v_1(t)v_2'(t+\tau)\} = 0 \quad \forall\tau \tag{3.11}$$

This restriction can also be removed, resulting in slightly more complicated calculations (e.g. Kwakernaak and Sivan, 1972).
Finally we will set $D = 0$: this assumption will appear to be possible without loss of generality.

Usually, the control objective that we want to optimize to obtain the desired performance, is a linear combination of the state-vector x:

$$\zeta(t) = Ex(t) \tag{3.12}$$

of which we want a minimal spectral density, when given the occurrence of disturbances v_1 and v_2. Furthermore, to obtain a feasible solution, we want to make sure that the spectral density of the control input vector u remains within acceptable levels, when our controller (to be designed) is in place.
This implies the need for a trade-off: when no controller is used ζ will be 'large', but u will be 0. Making ζ 'smaller' by means of a controller will result in a 'growing' u, which may become unacceptable. To determine an optimal trade-off between these two effects we need a 'cost-function', that should be minimized to obtain an optimal controller. The cost-function, usually considered for this, can be given as:

$$J := \lim_{T\to\infty} \frac{1}{T}\mathrm{E}\left\{\int_0^T (\zeta' Q_1 \zeta + u' Q_2 u)dt\right\} \tag{3.13}$$

which, under the assumption that the resulting closed loop system is stable, such that ζ and u are stationary random processes, can be simplified to:

$$J = \mathrm{E}\left\{\zeta' Q_1 \zeta + u' Q_2 u\right\} \tag{3.14}$$

The actual trade-off between allowable spectral density of ζ and allowable spectral density of u can be prescribed by means of Hermitian weight matrices Q_1 and Q_2 ($Q_1 = Q_1'$, $Q_2 = Q_2'$). Usually these matrices are chosen to be diagonal, such that a weight can be appended to each of the elements of vectors ζ and u and a simple physical interpretation of the effect of the weights remains possible. By increasing the weight belonging to one specific element, the optimization procedure will result in a decrease of that element's spectral density, at the cost of an increase in spectral density of other elements.
Note that the condition that all elements of vectors ζ and u are to be minimized, implies that all their spectral densities must be weighted in such a way, that an increase in spectral density implies an increase in cost. For this to be the case, we must have $Q_1 > 0$ and $Q_2 > 0$; usually it is stated that $Q_1 \geq 0$ is allowed, but this simply implies that a linear combination of ζ does not have an effect on the cost function and that E can be reselected such that the dimension of ζ and Q_1 decreases and $Q_1 > 0$ results. Solutions for the case $Q_2 \geq 0$ can also be derived: the reasons for avoiding this in this thesis are equivalent (dual) to the reasons for avoiding non-singularity of V_2 given above. Not only will the calculations be more difficult, it is also impracticable to *not* restrict the spectral density of any input: this would imply that such an input was allowed to have unbounded spectral density.

Solution of the LQG problem

The solution of the LQG problem is based on the separation principle: it can be derived that the optimal LQG controller can be found in two steps. First, a deterministic linear quadratic optimization problem is solved to obtain an optimal state-feedback regulator. Next, an optimal state estimator is constructed, which minimizes the expected estimation error: $\mathrm{E}\{(x - \hat{x})'(x - \hat{x})\}$; such an estimator is also known as a *Kalman filter*. Under the given assumptions: a linear model, a quadratic cost function and Gaussian stochastic processes v_1 and v_2, the optimal LQG-controller is then given by the optimal state feedback regulator acting on the optimal state estimate $\hat{x}$, rather than on the actual state x.

The internally stabilizing optimal state feedback regulator K_R can be found from:

$$K_R = Q_2^{-1} B' X_R \tag{3.15}$$

in which X_R is the stabilizing solution of the ARE:

$$A'X_R + X_R A - X_R B Q_2^{-1} B' X_R + E'Q_1 E = 0 \tag{3.16}$$

Note that $Q_1 \geq 0$ and $Q_2 > 0$ implies that the mild condition: $(A, B, E'Q_1E)$ is stabilizable and detectable, is sufficient for the existence of such a stabilizing solution, and furthermore implies that $X_R = X_R' \geq 0$ (theorem 3.3.3). The resulting feedback structure is given in figure 3.3.

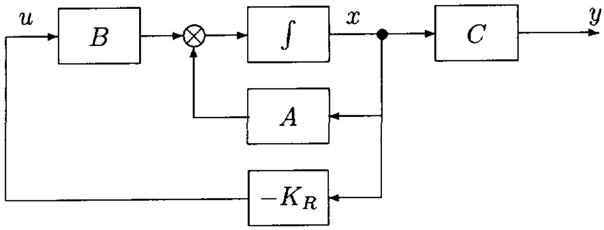

Fig. 3.3: Optimal state feedback structure

The optimal state-estimator mainly consists of a replica of the linear system model, submitted to the same control inputs. Minimization of the expected error $\mathrm{E}\{(x-\hat{x})'(x-\hat{x})\}$ is done by updating the estimated state $\hat{x}$, based on the error between measurement signals y and estimated measurement signals $\hat{y}$. The constant output injection matrix K_E determines the behaviour of the resulting filter and can be chosen optimal in the sense of a trade-off between the effect of system noise v_1 and measurement noise v_2. The calculation of an internally stabilizing K_E is dual to that of K_R and can be found from:

$$K_E = X_E C' V_2^{-1} \tag{3.17}$$

in which X_E is the stabilizing solution of the ARE:

$$X_E A' + A X_E - X_E C' V_2^{-1} C X_E + V_1 = 0 \tag{3.18}$$

Similar to the optimal regulator case we now have $V_1 \geq 0$ and $V_2 > 0$: (A, V_1, C) stabilizable and detectable is sufficient for the existence of a stabilizing solution, and $X_E = X_E' \geq 0$ (theorem 3.3.3). The resulting optimal state estimator is given in figure 3.4.

The dynamical behaviour of the LQG-optimal controlled system can now be con-

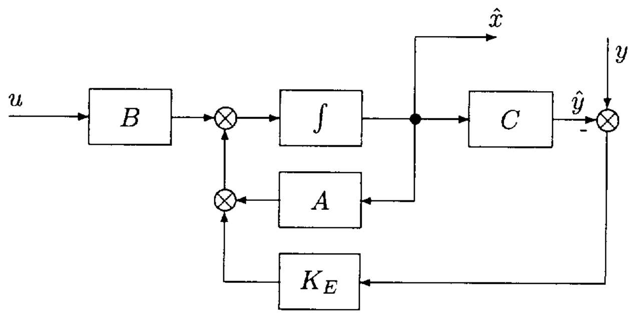

Fig. 3.4: Optimal state estimator structure

structed:

$$\begin{array}{llcl} \text{plant}: & \dot{x}(t) & = & Ax(t) + Bu(t) + v_1(t) \\ & y(t) & = & Cx(t) \quad + v_2(t) \\ \text{controller}: & \dot{\hat{x}}(t) & = & A\hat{x}(t) - K_E C\hat{x}(t) + Bu(t) + K_E y(t) \\ & u(t) & = & -K_R\hat{x}(t) \end{array} \tag{3.19}$$

$$\Downarrow$$

$$\begin{cases} \dot{x}(t) = Ax(t) - BK_R\hat{x} + v_1(t) \\ \dot{\hat{x}}(t) = K_E Cx(t) + (A - K_E C - BK_R)\hat{x} + K_E v_2(t) \end{cases}$$

With $e := x - \hat{x}$ another description of the resulting system is found:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BK_R & BK_R \\ 0 & A - K_E C \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_1(t) - K_E v_2(t) \end{bmatrix} \tag{3.20}$$

From this it can be concluded that the eigenvalues of the LQG-optimal controlled system are the union of those of the optimal state feedback regulator and the optimal state estimator. This implies that under the given assumptions the resulting closed-loop system is internally stable.

The state-space description of the complete LQG-compensator can now be given as:

$$\begin{cases} \dot{\hat{x}}(t) = (A - K_E C - BK_R)\hat{x}(t) + K_E y(t) \\ u(t) = -K_R\hat{x}(t) \end{cases} \tag{3.21}$$

Figure 3.5 shows a block diagram of the resulting closed-loop system. Check that any effect of a nonzero feedthrough matrix: $D \neq 0$, can be counteracted by incorporating this D into the compensator. As D has no effect on u or z the optimal LQG-cost achieved with $D = 0$ will still be obtained.

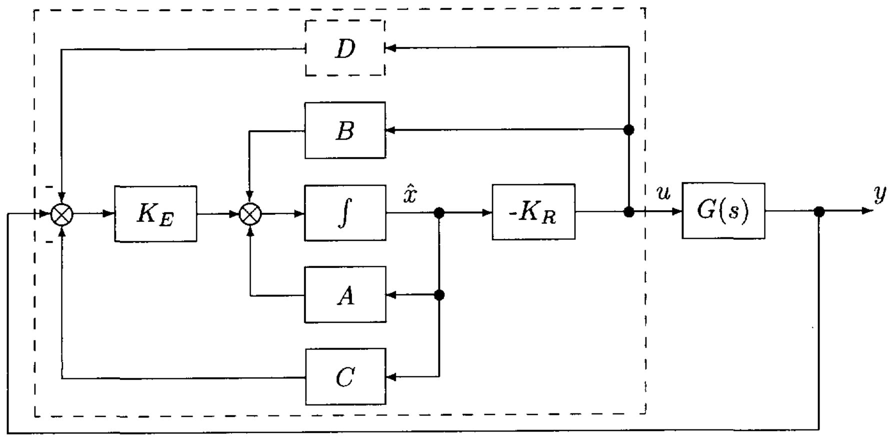

Fig. 3.5: The complete LQG compensator structure

3.3.3 H_2 controller synthesis

LQG in the standard control design structure

In the previous subsection, the LQG-problem was considered in a way that is compatible with most of the literature on linear quadratic methods in optimal control (e.g. Anderson and Moore 1971, 1989, Kwakernaak and Sivan 1972). To place the LQG-problem into proper perspective with respect to robust control methods, we will now restate it within the general framework discussed in section 3.2. This implies that we need to define a control objectives vector z such that minimization of z is equal to minimization of the cost function in equation 3.13. Furthermore, we have to define a disturbance vector w in such a way that it reflects the effect of system noise vector v_1 and measurement noise vector v_2. Based on Boyd and Barratt (1991), the standard plant, which is equivalent to the formulation of the LQG-problem can thus be set up as in figure 3.6. A state-space description of this standard plant with a partitioning as in equation 3.4 can then easily be found as:

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|cc:c} A & V_1^{\frac{1}{2}} & 0 & B \\ \hline Q_1^{\frac{1}{2}}E & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_2^{\frac{1}{2}} \\ \hdashline C & 0 & V_2^{\frac{1}{2}} & 0 \end{array}\right] \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{3.22}$$

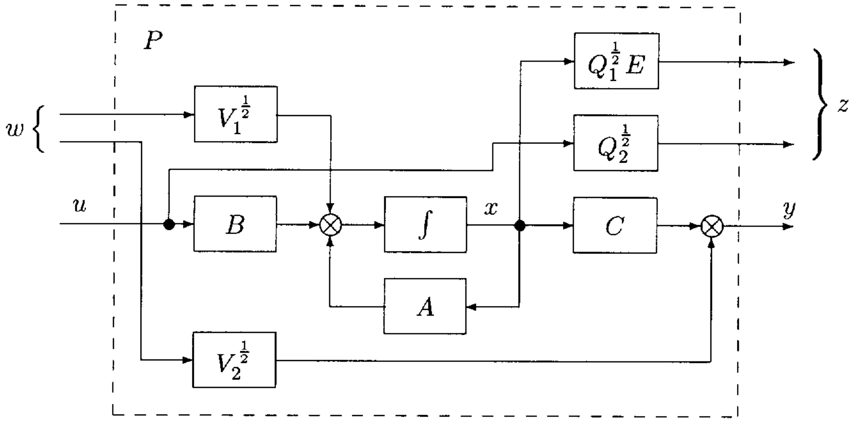

Fig. 3.6: The standard plant formulation of the LQG-problem

Note that with the definition of z as:

$$z = \begin{bmatrix} Q_1^{\frac{1}{2}} E x \\ Q_2^{\frac{1}{2}} u \end{bmatrix} \tag{3.23}$$

the LQG cost as given in equation 3.14 is exactly the RMS-norm of z (equation 2.14). Next, we have that the square root of the spectral density matrices V_1 and V_2 are incorporated in the standard plant such that we can assume that w is a white stochastic process with spectral density matrix $W = I$. With equation 2.25 and the extension of the definition of the 2-norm to matrix functions in equation 2.46, this implies that the RMS-norm of z is equal to the 2-norm of the transfer function from w to z. We can therefore solve the LQG-problem by finding a controller K that uses control inputs u and measurements y, such that the 2-norm of the closed-loop $||T_{wz}||_2$ is minimal.

The H_2 problem definition

The importance of setting up the LQG-problem within the general framework is that it is now possible to generalize it to a larger class of problems. This general problem is usually referred to as the 'H_2-problem' and it is based on the standard plant description:

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{3.24}$$

(see for instance Doyle et al. 1989 and Zhou et al. 1993).
With this we have the following definition:

Definition 3.3.4 (the H_2-problem)
Given the standard plant of equation 3.24 and the standard control design structure of figure 3.1, find a proper, real-rational controller $K \in RL_\infty$, *that achieves internal stability of the closed-loop system and minimizes the 2-norm of the transfer function matrix* T_{wz} *from* w *to* z.

Extension to the general case with $D_{11} \neq 0$ is possible, but will not be elaborated on in this thesis. However, note that $D_{11} \neq 0$ implies that the open loop system is not in RL_2, which usually makes it impossible to find a controller such that T_{wz} is in RH_2.

The following assumptions are made to ensure solvability of the H_2 problem:

A1 (A, B_2) is stabilizable and (A, C_2) is detectable,

A2 D_{12} has full column rank and D_{21} has full row rank,

A3 $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω: the SSD of the open loop transfer from u to z may not have any transmission zeros or input-decoupling zeros on the imaginary axis (see theorem 2.4.9),

A4 $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω: the SSD of the open loop transfer from w to y may not have any transmission zeros or output-decoupling zeros on the imaginary axis (again theorem 2.4.9).

The first assumption is necessary and sufficient for the existence of an internally stabilizing controller for the standard plant. The second assumption is on the selection of appropriate weight functions, such that the possibility to minimize $||T_{wz}||_2$ is ensured. This assumption is a generalization of the assumptions $Q_2 > 0$ and $V_2 > 0$ that were made for the LQG-problem in the previous section. The third and fourth assumption are introduced for technical reasons and will be discussed later.

Solution of the H_2 problem

A complete *derivation* of the solution of the H_2 problem under these assumptions is rather involved and beyond the scope of this thesis: for this refer to for instance

Doyle et al. (1989) and Zhou et al. (1993). As in the solution of the LQG-problem, it appears that a separation structure is applicable, leading to the design of a regulator and an estimator based on two algebraic Riccati equations:

$$\begin{aligned} &X_R(A - B_2(D_{12}'D_{12})^{-1}D_{12}C_1) + (A - B_2(D_{12}'D_{12})^{-1}D_{12}C_1)'X_R \\ &\quad - X_RB_2(D_{12}'D_{12})^{-1}B_2'X_R + C_1'(I - D_{12}(D_{12}'D_{12})^{-1}D_{12}')C_1 = 0 \\ &(A - B_1D_{21}(D_{21}D_{21}')^{-1}C_2)X_E + X_E(A - B_1D_{21}(D_{21}D_{21}')^{-1}C_2)' \\ &\quad - X_EC_2'(D_{21}D_{21}')^{-1}C_2X_E + B_1(I - D_{21}'(D_{21}D_{21}')^{-1}D_{21})B_1' = 0 \end{aligned} \tag{3.25}$$

From the stabilizing solutions X_R and X_E it is again possible to calculate the constant state-feedback matrix K_R and output injection matrix K_E:

$$\begin{aligned} K_R &:= (D_{12}'D_{12})^{-1}(B_2'X_R + D_{12}'C_1) \\ K_E &:= (X_EC_2' + B_1D_{21}')(D_{21}D_{21}')^{-1} \end{aligned} \tag{3.26}$$

It can be verified that under assumptions A1 through A4 the conditions of theorem 3.3.3 are met: there exist stabilizing, positive semi-definite solutions $X_R = X_R' \geq 0$ and $X_E = X_E' \geq 0$, resulting in an internally stabilizing controller.

The resulting compensator structure is then completely analogous to that of the LQG-problem in figure 3.5 and is given in figure 3.7. Note that with the LQG standard plant in equation 3.22 we have that $(D_{12}'D_{12})^{-1} = Q_2$ and $(D_{21}D_{21}')^{-1} = V_2$. Furthermore, we have $D_{12}'C_1 = 0$ and $B_1D_{21}' = 0$, such that it is an easy exercise to substitute this in equations 3.25 and 3.26 to find the results in equations 3.15, 3.16, 3.17 and 3.18. Also note, that a state-space description of the H_2-compensator can be found, equivalent to that of the LQG-compensator given in equation 3.21:

$$\begin{cases} \dot{\hat{x}}(t) = (A - K_EC_2 - B_2K_R + K_ED_{22}K_R)\hat{x}(t) + K_Ey(t) \\ u(t) = -K_R\hat{x}(t) \end{cases} \tag{3.27}$$

We thus have the possibility to find the H_2 optimal controller for the general plant of equation 3.24 by solving the two AREs given in equation 3.25 and substituting the result in equation 3.26. In the next section we will consider the control problem that is, to this date, the most important for the design of robust linear controllers: the H_∞-problem.

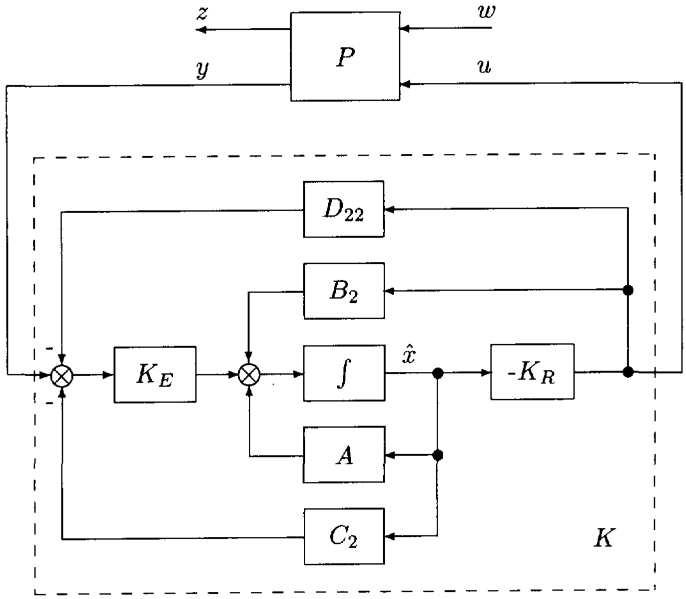

Fig. 3.7: The H_2 optimal compensator structure

3.4 H_∞ controller synthesis

3.4.1 H_∞ problem definition

The H_∞-problem can be defined as follows:

Definition 3.4.1 (the H_∞-problem)
Given the standard plant of equation 3.4 and the standard control configuration of figure 3.1. Find a proper, real-rational controller $K \in R_{ss}$*, that achieves internal stability of the closed-loop system and minimizes the* ∞*-norm of the transfer function matrix* T_{wz} *from* w *to* z*.*

Note that we have from section 2.3, theorem 2.3.2, that the ∞-norm is the induced 2-norm: the ∞-norm determines the maximum amplification of a signal in H_2 by a system in H_∞ in the sense of the 2-norm. Hence, in comparison with the H_2-problem in which we assume that the disturbance input w is white noise with spectral density I such that we minimize the *average* 2-norm, we now aim to minimize the 2-norm of the control objectives vector z under the influence of the *worst case* disturbance input with unit 2-norm.

Because this is a very hard problem to solve in the general case, a slightly simpler problem is also considered:

Definition 3.4.2 (the sub-optimal H_∞-problem)
Given the standard plant of equation 3.4 and the standard control configuration of figure 3.1. Furthermore, given a real positive scalar constant $\gamma > 0$. Find a proper, real-rational controller $K \in R_{ss}$, that achieves internal stability of the closed-loop system and obtains $\|T_{wz}\|_\infty < \gamma$.

3.4.2 Solution of the H_∞-problem

The solution of the sub-optimal H_∞-problem appears to be very similar to the solution of the H_2-problem in some respects:

- there is a separation structure, allowing the output feedback problem to be split into a state-feedback problem and an estimator design problem,
- each of these sub-problems leads to an algebraic Riccati equation, the solution of which can be used to obtain a constant state-feedback matrix K_R and a constant estimator state-injection matrix K_E,
- a sub-optimal H_∞ controller has a structure, similar to that of an optimal H_2 controller as given in figure 3.7.

There are, however, some important discrepancies also:

- there are infinitely many sub-optimal H_2 and H_∞ controllers, but, in general, there are also infinitely many *optimal* H_∞ controllers, whereas there is only a single H_2-optimal controller,
- in general it is very hard to solve the optimal H_∞-problem directly, while, as stated before, the sub-optimal H_∞-problem is similar to the H_2-problem and therefore readily solvable; the optimum can be approximated by means of an iterative procedure on γ.

For all practical purposes it is sufficient to calculate a sub-optimal H_∞ controller by means of this so-called *γ-iteration.* The *optimal* H_∞-norm is given as: $\gamma_o := \inf_K \|T_{wz}\|_\infty$ such that for any $\epsilon > 0$ there exists a *sub-optimal* H_∞ controller for $\gamma = \gamma_o + \epsilon$. By means of this sub-optimal solution and γ-iteration it is possible to make ϵ *arbitrarily small*: for this reason, the result of the γ-iteration procedure is usually considered to be *the* solution of the H_∞-problem.

We will now set up the calculation of the sub-optimal H_∞ controller and give a parametrization of all sub-optimal H_∞ controllers that obtain $\|T_{wz}\|_\infty < \gamma$. The assumptions that have to be made for solvability of this problem are the same as those for the H_2-problem and, as mentioned before, the solution of the sub-optimal H_∞-problem is again based on two algebraic Riccati equations. However, the complexity of these equations has grown in such a way that we first introduce some more notation:

$$\begin{aligned} R &:= D_{1*}'D_{1*} - \begin{bmatrix} \gamma^2 I_{q_1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{where} \qquad D_{1*} := [D_{11}\ D_{12}] \\ \tilde{R} &:= D_{*1}D_{*1}' - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{where} \qquad D_{*1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \end{aligned} \tag{3.28}$$

Then, under the assumption that both R and $\tilde{R}$ are non-singular, the two Riccati equations for the design of a regulator and an estimator are given as:

$$\begin{aligned} & X_R(A - BR^{-1}D_{1*}'C_1) + (A - BR^{-1}D_{1*}'C_1)'X_R \\ & \qquad - X_RBR^{-1}B'X_R + C_1'(I - D_{1*}R^{-1}D_{1*}')C_1 = 0 \\ & (A - B_1D_{*1}'\tilde{R}^{-1}C)X_E + X_E(A - B_1D_{*1}'\tilde{R}^{-1}C)' \\ & \qquad - X_EC'\tilde{R}^{-1}CX_E + B_1(I - D_{*1}'\tilde{R}^{-1}D_{*1})B_1' = 0 \end{aligned} \tag{3.29}$$

As was done in the solution of the H_2-problem, we can now calculate the constant state-feedback matrix K_R and output injection matrix K_E from the stabilizing solutions X_R and X_E:

$$\begin{aligned} K_R &:= R^{-1}(B'X_R + D_{1*}'C_1) \\ K_E &:= (X_EC' + B_1D_{*1}')\tilde{R}^{-1} \end{aligned} \tag{3.30}$$

The connection with the solution of the H_2-problem becomes even more clear if we consider the case $D_{11} = 0$ such that:

$$R = \begin{bmatrix} -\gamma^2 I_{q_1} & 0 \\ 0 & D_{12}'D_{12} \end{bmatrix}, \qquad \tilde{R} = \begin{bmatrix} -\gamma^2 I_{p_1} & 0 \\ 0 & D_{21}D_{21}' \end{bmatrix} \tag{3.31}$$

Now let $\gamma \to \infty$ such that:

$R^{-1} \to \begin{bmatrix} 0 & 0 \\ 0 & (D_{12}'D_{12})^{-1} \end{bmatrix}$ and $\tilde{R}^{-1} \to \begin{bmatrix} 0 & 0 \\ 0 & (D_{21}D_{21}')^{-1} \end{bmatrix}$.

Substitution in equation 3.29 then results in equation 3.25 and substitution in equation 3.30 results in:

$$\begin{bmatrix} K_{R1} \\ K_{R2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (D_{12}'D_{12})^{-1} \end{bmatrix} (B'X_R + D_{1*}'C_1)$$
$$[K_{E1} \;\; K_{E2}] = (X_E C' + B_1 D_{*1}') \begin{bmatrix} 0 & 0 \\ 0 & (D_{21}D_{21}')^{-1} \end{bmatrix} \tag{3.32}$$

This implies that K_R and K_E in equation 3.26 are equal to K_{R2} and K_{E2} in equation 3.32 and that the compensator structure is given by figure 3.7. It appears that for $\gamma \to \infty$ the solution of the AREs in equation 3.29 converge to the H_2 optimal values; hence, the matrices K_R and K_E of the H_∞-problem converge to those of the H_2-problem.

For the H_∞-problem the case $D_{11} \neq 0$ is not trivial: now T_{wz} is allowed to be in RH_∞ and may therefore have a feedthrough term. Unfortunately, this significantly complicates the calculation of not only X_R, X_E, K_R and K_E, but also of the compensator structure. This can be alleviated by allowing two extra assumptions:

A5 $D_{12} = \begin{bmatrix} 0 \\ I_{q2} \end{bmatrix}$ and $D_{21} = [0 \;\; I_{p2}]$,

A6 $D_{22} = 0$.

Assumption A5 appears to be possible without loss of generality and can be obtained by scaling of u and y and unitary transformation of w and z (Glover and Doyle 1988, Zhou et al. 1993). Assumption A6 can easily be removed later on. The reason for introducing these assumptions is that they suggest a partitioning of matrices K_R, K_E and D as follows:

$$\left[\begin{array}{c|c} & K_R' \\ \hline K_E' & D \end{array}\right] = \left[\begin{array}{c|ccc} & K_{R11}' & K_{R12}' & K_{R2}' \\ \hline K_{E11}' & D_{1111} & D_{1112} & 0 \\ K_{E12}' & D_{1121} & D_{1122} & I_{q2} \\ K_{E2}' & 0 & I_{p2} & 0 \end{array}\right] \tag{3.33}$$

With this, we can set up the following theorem (Glover and Doyle 1988):

Theorem 3.4.3 (solution of the sub-optimal H_∞-problem)
Suppose γ is given and the standard plant P satisfies assumptions A1 through A6.

(a) There exists an internally stabilizing controller $K \in R_{ss}$ such that $\|T_{wz}\|_\infty < \gamma$ if and only if:

(i) $\gamma > \max\left(\bar{\sigma}[D_{1111}\ D_{1112}], \bar{\sigma}[D'_{1111}\ D'_{1121}]\right)$;

(ii) *there exists a stabilizing and positive semi-definite solution* X_R *to the regulator ARE;*

(iii) *there exists a stabilizing and positive semi-definite solution* X_E *to the estimator ARE;*

(iv) $\rho(X_R X_E) < \gamma^2$, *with* $\rho(X)$ *denoting the spectral radius of* X.

(b) *Given that the conditions of part (a) are satisfied, then* all *internally stabilizing controllers* $K \in \mathrm{R}_{\mathrm{ss}}$ *satisfying* $\|T_{wz}\|_\infty < \gamma$ *are given by:*

$$K = \mathcal{F}_l(K_a, \Phi) \quad \textit{for arbitrary} \quad \Phi \in \mathrm{RH}_\infty \quad \textit{such that} \quad \|\Phi\|_\infty < \gamma$$

where:

$$K_a := \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right], \tag{3.34}$$

$$\hat{D}_{11} := -D_{1121}D'_{1111}(\gamma^2 I - D_{1111}D'_{1111})^{-1}D_{1112} - D_{1122}, \tag{3.35}$$

$\hat{D}_{12} \in \mathrm{R}^{q_2 \times q_2}$ *and* $\hat{D}_{21} \in \mathrm{R}^{p_2 \times p_2}$ *are any matrices satisfying:*

$$\begin{aligned} \hat{D}_{12}\hat{D}'_{12} &= I - D_{1121}(\gamma^2 I - D_{1111}D'_{1111})^{-1}D'_{1121}, \\ \hat{D}'_{21}\hat{D}_{21} &= I - D'_{1112}(\gamma^2 I - D_{1111}D'_{1111})^{-1}D_{1112}. \end{aligned} \tag{3.36}$$

Furthermore:

$$\begin{aligned} \hat{B}_2 &:= Z(B_2 + K_{E12})\hat{D}_{12}, \\ \hat{C}_2 &:= -\hat{D}_{21}(C_2 + K_{R12}), \\ \hat{B}_1 &:= -ZK_{E2} + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11}, \\ \hat{C}_1 &:= K_{R2} + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2, \\ \hat{A} &:= A + BK_R + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2, \end{aligned} \tag{3.37}$$

where:

$$Z := (I - \gamma^{-2}X_E X_R)^{-1}. \tag{3.38}$$

An important discrepancy with the optimal H_2-problem is that existence of an internally stabilizing sub-optimal H_∞ controller for a given value of γ is not guaranteed by the assumptions A1 through A6. It should be checked whether there exists a stabilizing solution to both AREs, for instance by means of theorem 3.3.2, and it should be checked whether this solution is positive semi-definite. In spite of this, the existence of a sub-optimal H_∞ controller can easily be established by attempting to construct the stabilizing solutions X_R and X_E by means of numerical methods. If this construction is successful, equation 3.30 and part (b) of the theorem provide all sub-optimal H_∞ controllers for the given value of γ, if not, the given value of γ is smaller than the optimal value and no controllers exist. This entire procedure, often even including γ-iteration, is readily available in software packages like PC-MatLab (Moler et al. 1987, Balas et al. 1993) and MATRIX_X (Gupta 1991).

All H_∞-sub-optimal controllers are parametrized by means of an LFT acting on a single matrix-valued parameter Φ, that is only restricted by the demand that it is in RH_∞ and has ∞-norm less than γ. The compensator structure for any such choice of Φ can thus be given as a combined upper and lower LFT as given in figure 3.8. An obvious choice of Φ would be $\Phi = 0$, which simplifies the compensator structure significantly and results in the so-called *central controller.* As indicated earlier, we have that the H_∞-sub-optimal controller for $\gamma \rightarrow \infty$ approaches the H_2-optimal controller: with $\Phi = 0$ and $D_{11} = 0$ the compensator structure of figure 3.8 reverts to that of figure 3.7. For bounded γ it has been shown by Glover and Mustafa (1989), that the central controller can be seen as an appropriate trade-off between the H_2 and H_∞ objectives, thus giving an extra motivation for choosing $\Phi = 0$. Furthermore, it should be noted that $\Phi \neq 0$ usually, but not necessarily, implies an increase of the order of the compensator.

As mentioned before, assumption A6 ($D_{22} = 0$) can now be removed, which appears to be as simple as what was done in the LQG and H_2 cases. Suppose K is a controller for P with D_{22} set to zero, then the effect of a nonzero D_{22} can be completely compensated by adding D_{22} to this compensator according to figure 3.9. Now $u = Ky - KD_{22}u$ such that $u = (I + KD_{22})^{-1}Ky$ and we can define the resulting controller as $\tilde{K} := (I + KD_{22})^{-1}K$.

This implies that only assumptions A1 through A4 on the standard plant P are left. Assumption A1 is clearly basic for the existence of any stabilizing controller and can not be removed. Assumption A2 (or A5 without loss of generality) can be removed, which may lead to singular control problems and requires more complex calculations (Stoorvogel and Trentelman 1990, Scherer 1992). Removal of

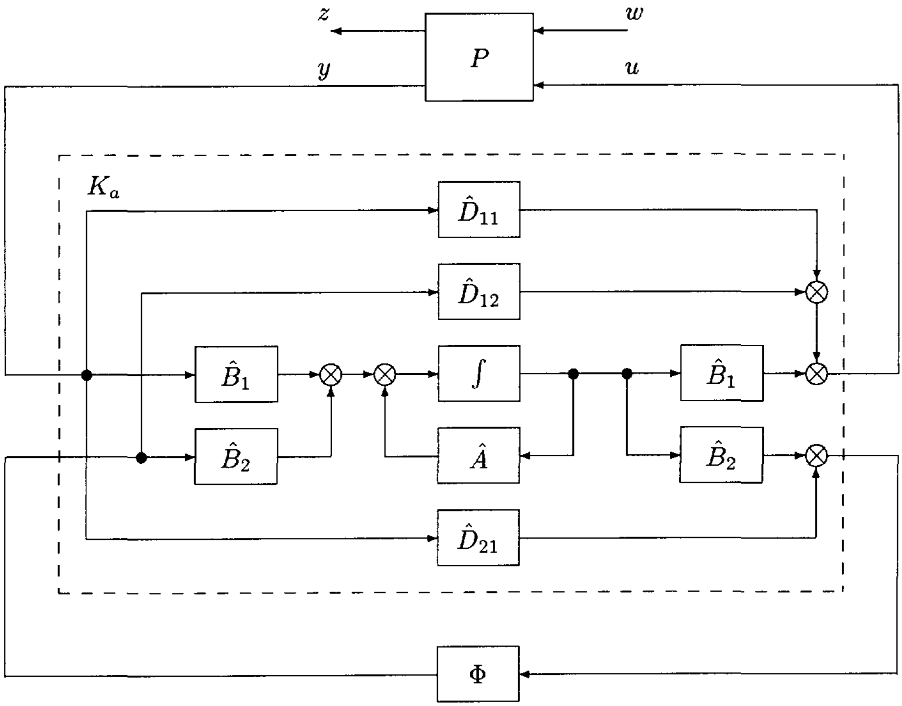

Fig. 3.8: The H_∞ sub-optimal compensator structure

assumption A3 and A4 is also possible, again leading to significantly more complex calculations (Scherer 1990, 1992). Chapter 4 will consider a different approach to dealing with these assumptions. It will be shown that the procedure developed in this section under the assumptions A1 through A4 can be used to solve the output regulation and tracking problem, although it will appear that this may lead to violation of assumptions A3 and A4.

3.5 The robustness issue

3.5.1 Definition of the robust controller

Apart from the tuning approach of classical control, where the controller parameters are set using the immediate response of the physical system to be controlled, control design is always based on a mathematical model of the plant. This implies that unexpected and sometimes even dangerous results can occur when a controller, that works perfectly well on its design model, is implemented on the actual plant. The reason for this is that, as already discussed in section 2.1, the

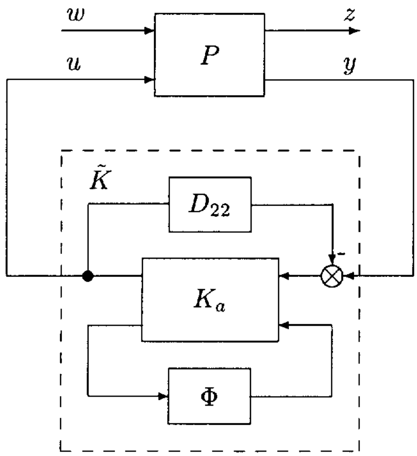

Fig. 3.9: Adding the effect of D_{22} to the compensator structure

plant model will never be a perfect representation of the physical system. The causes for discrepancies between physical system and model can be distinguished as follows:

- parts of known linear behaviour, like high frequency dynamics, are not included in the model to keep it manageable and/or to make a certain design method applicable,
- some physical quantities in the system, like mass, stiffness, size, etc., are not precisely known, or vary from case to case (e.g. when a controller is to be designed for a mass product),
- some physical quantities in the system are changing with time,
- some physical quantities in the system are changing as a function of operational conditions or other external influences (non-linearity).

In practical situations, usually all of these possibilities occur in some form at the same time and are generally described as *plant uncertainties.*
Basically there are two approaches to this problem. The first is to put a lot of effort in the modelling process, to obtain a model that gives an 'as good as possible' representation of the physical plant. This then leads to a complex model, usually non-linear, time-dependent, etc., for which it is hard to design a controller based on predetermined control objectives. It usually involves an iterative procedure of design, simulation and tuning, but in the end, the control engineer still needs to

make a 'leap of faith' with regard to the performance of the final controller on the actual plant.

The second approach is based on the practical experience that, especially using classical control methods, it is possible to find controllers based on simple models, that perform well on plants with significant uncertainties. It is to leave the concept of 'there is a model, albeit complex, that matches the physical plant' and instead look for a *set* of simple (i.e. LTI) models, which is assumed to *contain* a 'good enough' description of the physical plant (at a certain time, in a certain operating point etc.). The controller design should then be based on this set, rather than on a single model, such that the performance of the controller is sufficient for all members of the set. A controller with this property is called a *robust* controller:

Definition 3.5.1
Given a set of models, then a controller is said to be a robust controller, if it achieves a number of prespecified design objectives when applied to any member of this set.

However, also in the case of a robust controller the control engineer requires a leap of faith. Most of robust control theory is based on sets of LTI models and at least one of these models must be a 'good enough' description of the physical plant. Furthermore, the effects of time-dependence and non-linearity are usually considered to be perturbations of a nominal LTI model in the model set: they cause changes from one member of the model set to another, and in general it must be assumed that these changes do not upset the performance of the controller.

In spite of this, the possibility to explicitly define a set of models for which analysis and synthesis of controllers can be done to provide *guaranteed* closed-loop stability and performance properties is a powerful concept. It allows the designer to put emphasis on those areas of the control problem that are known to contain uncertainties and to design a robust controller with better *average* performance than a controller designed for the nominal plant.

3.5.2 The robustness objective in the general framework

Thusfar the standard control design structure defined in section 3.2 has only been used to specify two control objectives:

- internal stability;
- disturbance attenuation.

When taking into account that the disturbance attenuation objective is represented by a *vector* of error signals (z), this already can be a sensible control problem, in the sense that a trade-off can be defined between several objectives. A clear example of this is the LQG problem, in which error minimization is traded-off against actuator effort.

The extension of the standard control design structure to include also the robustness objective, would now add a further trade-off to the control design problem. It would then be possible to design a robust controller in the sense of definition 3.5.1, which depends on a trade-off between disturbance attenuation objectives on the one hand, and the 'size' of the set for which internal stability and a certain amount of disturbance attenuation is obtained on the other.

A very simple example will now be used to show how the incorporation of the robustness objective into the general framework can be performed. For this consider the first order system transfer function:

$$G(s) = \frac{c}{s+1} \tag{3.39}$$

in which the gain parameter c is uncertain, but assumed to be in a known interval: $c \in [c_1, c_2]$. We then have:

$$\begin{array}{ll} c = c_o + \delta s_o & \delta \in [-1, 1] \\ & c_o := \frac{c_1 + c_2}{2} \quad s_o := \frac{c_2 - c_1}{2} \end{array} \tag{3.40}$$

and $G(s)$ can be written as:

$$G(s) = \frac{c_o}{s+1} + \frac{\delta s_o}{s+1} \tag{3.41}$$

Now this can be represented by means of an upper LFT as given in figure 3.10 (note the right-to-left notation).

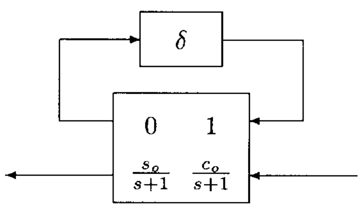

Fig. 3.10: An uncertain transfer function in LFT form

The extension of this example to the general case thus suggests the extension of the

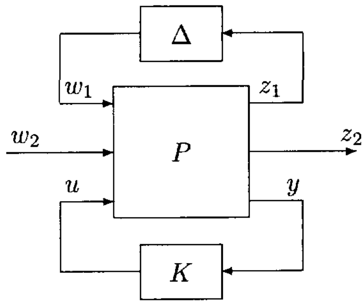

Fig. 3.11: The standard control structure with uncertainties

standard control structure to that of figure 3.11. Here the 'uncertainty matrix' Δ is an unknown matrix valued parameter in some bounded set $\mathbf{\Delta}$; we will assume for some given real scalar constant γ:

$$\mathbf{\Delta} := \{\Delta : \Delta \in \mathrm{RH}_\infty, \|\Delta\|_\infty \leq \gamma\} \tag{3.42}$$

The interconnection of P with Δ is given by two new vector valued input and output signals: w_1 and z_1. The 'original' disturbance input vector w and objectives vector z are renamed to w_2 and z_2.
This control structure clearly suggests a further partitioning of the standard plant description as follows:

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \\ y \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right] \begin{bmatrix} x \\ w_1 \\ w_2 \\ u \end{bmatrix} \tag{3.43}$$

Note that for a given uncertainty matrix $\Delta \in \mathbf{\Delta}$ and a given controller K we have:

$$T_{w_2 z_2} = \mathcal{F}_u(\mathcal{F}_l(P, K), \Delta) \tag{3.44}$$

For robustness analysis purposes (K known) we will denote:

$$M := \mathcal{F}_l(P, K) \tag{3.45}$$

as M is again the coefficient matrix of an LFT.

With this we have the following definition:

Definition 3.5.2 (the robust control problem)
Given the standard plant of equation 3.43, the uncertainty matrix of equation 3.42 and the standard control design structure of figure 3.11. Find a proper, real-rational controller $K \in \mathrm{R}_{ss}$*, that achieves internal stability of the closed-loop system and minimizes the 2-norm or the* ∞*-norm of the transfer function matrix* $T_{w_2 z_2}$ *from* w_2 *to* z_2*, when perturbed by the 'worst case'* Δ.

In this definition, the term 'worst case Δ' means: 'that $\Delta \in \mathbf{\Delta}$ for which the norm of $T_{w_2 z_2}$ is maximal' (i.e. in the closed-loop situation of equation 3.44). For robust control problems, usually the minimization of the ∞-norm is considered: the reason for this is the 'small gain theorem' discussed in the next subsection.

3.5.3 The small gain theorem

Before looking at the complete problem formulation of definition 3.5.2, we will first consider the case that w_2 and z_2 are absent. In that case the performance objectives drop out of the control problem and we are left with a 'robust stabilization problem'. Now suppose we have an internally stabilizing controller K for the nominal plant ($\Delta = 0$) and define M as in equation 3.45 ($z_1 = Mw_1$ with $M \in \mathrm{RH}_\infty$). Figure 3.11 then reduces to figure 3.12.

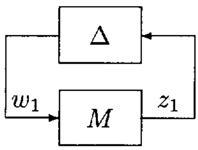

Fig. 3.12: Feedback structure for the small gain theorem

For this situation we have the following theorem:

Theorem 3.5.3 (small gain theorem)
Given $M \in \mathrm{RH}_\infty$ *and* $\mathbf{\Delta}$ *according to equation 3.42, then the closed-loop system of figure 3.12 is internally stable for all* $\Delta \in \mathbf{\Delta}$ *if and only if* $\|M\|_\infty < \gamma^{-1}$.

For a complete proof see for instance Doyle et al. (1984) and Zhou et al. (1993). It is based on the fact that $\Delta, M \in \mathrm{RH}_\infty$, such that instability can only occur if $\det(I - \Delta(j\omega)M(j\omega)) = 0$ for some $\omega \in \mathrm{R}$ and some $\Delta \in \mathbf{\Delta}$. Sufficiency then immediately follows from $\|\Delta M\|_\infty < 1 \quad \forall \Delta \in \mathbf{\Delta}$. Necessity can be proven by assuming $\|M\|_\infty \geq \gamma^{-1}$ and constructing a $\Delta \in \mathbf{\Delta}$ such that $\det(I - \Delta(j\omega)M(j\omega)) = 0$.

The main importance of this theorem is that we now have an equivalence between robust stability and performance in the sense of the ∞-norm. For analysis purposes we can guarantee robust stability for $\Delta \in \mathbf{\Delta}$ (equation 3.42) by checking if the ∞-norm of the known closed-loop transfer function from w_1 to z_1 is smaller than γ^{-1}: $\|M\|_\infty = \|T_{w_1 z_1}\|_\infty < \gamma^{-1}$. Furthermore, we can use H_∞ controller synthesis to find a controller which *achieves* robustness for $\Delta \in \mathbf{\Delta}$. A sub-optimal H_∞ controller can be used to guarantee robust stability when γ is precisely known. Iteration on γ can be used to maximize robustness margins in the sense that the minimal ∞-norm of $T_{w_1 z_1}$ will determine the maximal bound on $\|\Delta\|_\infty$ for which robust stability is (still) obtained.

3.6 Analysis and design using the structured singular value

To motivate the introduction of the structured singular value we will first look at the robust performance objective for the standard control configuration in figure 3.11. After that, the definition of the structured singular value will be given, followed by its calculation using upper and lower bounds and a short discussion on its use for analysis and design.

3.6.1 Robust performance

By means of the small gain theorem we have established a link between ∞-norm minimization and robust stabilization. However, this is not sufficient to solve the robust control problem as given in definition 3.5.2 and depicted in figure 3.11. To achieve performance objectives we have to minimize the 2-norm or ∞-norm of $T_{w_2 z_2}$ for the worst case Δ, while simultaneously considering the ∞-norm of $T_{w_1 z_1}$ to ensure robust stability.
A possible approach is to combine w_1 and w_2 to w and z_1 and z_2 to z, after which an H_∞ controller can be synthesised that minimizes $\|T_{wz}\|_\infty$ (i.e. $\|M\|_\infty$). By the small gain theorem this is equivalent to the extended robust stabilization problem given in figure 3.13.
Here the extended uncertainty matrix Δ_e is defined as:

$$\Delta_e := \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \tag{3.46}$$

in which we can set $\Delta_{11} := \Delta$.

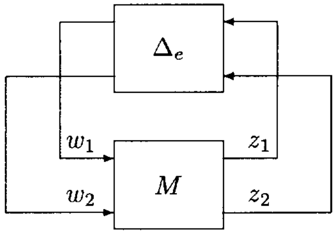

Fig. 3.13: Extended robust stabilization problem

The problem with this approach is that we now allow the disturbance input vector w_2 to be related with uncertainty output vector z_1 via Δ_{21}, although there is no physical interpretation for this relation. Similarly, the uncertainty input vector w_1 not only depends on uncertainty output vector z_1, but also on performance objectives vector z_2 via Δ_{12}. The calculation of the H_∞ controller that minimizes $\|M\|_\infty$ will therefore take uncertainties into account that do not occur in the actual plant. Obviously the performance of the resulting controller, measured as $\|T_{w_2 z_2}\|_\infty$ will only deteriorate because of this, as will the *actual* robust stability margin $\{\|T_{w_1 z_1}\|_\infty\}^{-1}$.

On the other hand, if $\|M\|_\infty < \gamma$ is obtained, we do have the *guarantee* that the controller achieves robust stability, and that $\|T_{w_2 z_2}\|_\infty < \gamma$ for all $\Delta \in RH_\infty$ with $\|\Delta\|_\infty \leq \gamma^{-1}$. We can therefore consider this approach as a sub-optimal solution to the robust control problem, similar to the solution of the sub-optimal H_∞-problem (definition 3.4.2). However, in contrast with the H_∞-problem, we may have that the resulting controller is arbitrarily far from optimality or, in other words, the resulting controller may be arbitrarily *conservative.*

To remedy this situation it is clear that we need to set $\Delta_{12} = \Delta_{21} = 0$ and try to find a test determining the stability of the configuration in figure 3.13 when Δ_e has this special structure. In order to do this we can define the set $\boldsymbol{\Delta}_e$ as follows:

$$\boldsymbol{\Delta}_e := \{\text{diag}(\Delta, \Delta_p) : \Delta, \Delta_p \in RH_\infty, \|\Delta\|_\infty \leq \gamma, \|\Delta_p\|_\infty \leq \gamma\} \tag{3.47}$$

Δ_p is added to transform the performance objective into a stability objective and is therefore known as the *performance block.* Note that the block-diagonal structure of Δ_e ensures that $\|\Delta_e\|_\infty \leq \gamma$.

As with the small gain theorem, we can test stability by determining whether or not $\det\{I - \Delta_e(j\omega)M(j\omega)\} = 0$ for any $\Delta_e \in \boldsymbol{\Delta}_e$. However, it is clearly impracticable to do this test directly and we would like to have a 'measure of size' for the known matrix M, that can be used to determine stability. Similar to how the ∞-norm is based on the largest *singular* value $\bar{\sigma}$ and determines stability for 'unstructured' Δ, this new measure of size will be based on the *structured singular value* μ.

3.6.2 Definition of the structured singular value

Besides the robust performance problem considered in the previous subsection, we will see later on that there are many other cases leading to a block-diagonal structure of the uncertainty matrix Δ. For this reason, it is useful to first define a more general set of uncertainty matrices, which will offer a general framework in which many forms of structured plant uncertainty can be specified.

$$\begin{aligned} \boldsymbol{\Delta}_s := \; & \{\mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}, \Delta_1, \ldots, \Delta_f) : \\ & \delta_i, \Delta_i \in \mathrm{RH}_\infty, \|\delta_i\|_\infty \leq \gamma, \|\Delta_i\|_\infty \leq \gamma\} \end{aligned} \tag{3.48}$$

Here $\delta_i I_{k_i}, i = 1 \ldots r$ denote *repeated scalar* blocks and $\Delta_i, i = 1 \ldots f$ denote *full* blocks with dimensions $k_{r+i} \times k_{r+i}$. The assumption that all Δ_i are square can be removed, but will not be considered to reduce complexity of notation. Note that for $\Delta \in \boldsymbol{\Delta}_s$ to be compatible with the standard control design structure with uncertainties given in figure 3.11, we must have $\sum_{i=1}^{r+f} k_i = q_1 = p_1$ with q_1 and p_1 the dimension of w_1 and z_1 respectively. Also note that, due to the block-diagonal structure, we have $\|\Delta\|_\infty < \gamma$ for all $\Delta \in \boldsymbol{\Delta}_s$. Hence, we now have a 'robust stabilization problem with structured uncertainties', of which the robust performance problem, discussed in the previous subsection, is a special case.
Sometimes it is convenient to assume that γ is incorporated in the standard plant, such that we can set $\gamma = 1$. As all elements of $\boldsymbol{\Delta}_s$ are then within the unit ball, we will use the notation:

$$\mathbf{B}\boldsymbol{\Delta}_s := \{\Delta \in \boldsymbol{\Delta}_s : \|\Delta\|_\infty \leq 1\} \tag{3.49}$$

A measure of size of M, similar to the largest singular value, that determines robust stability when Δ has structure ($\Delta \in \boldsymbol{\Delta}_s$) can now be defined as:

Definition 3.6.1 (the structured singular value)
Given the configuration of figure 3.12 with $M \in \mathrm{RH}_\infty$.
The structured singular value $\mu_{\boldsymbol{\Delta}_s}(M(j\omega))$ *is then defined as:*

$$\mu_{\boldsymbol{\Delta}_s}(M(j\omega)) := \frac{1}{\min(\bar{\sigma}(\Delta(j\omega)) : \Delta \in \boldsymbol{\Delta}_s, \det(I - \Delta(j\omega)M(j\omega)) = 0)} \tag{3.50}$$

unless no $\Delta \in \boldsymbol{\Delta}_s$ *makes* $I - \Delta(j\omega)M(j\omega)$ *singular in which case* $\mu_{\boldsymbol{\Delta}}(M(j\omega)) := 0$.

Note that μ is a function of the uncertainty structure $\mathbf{\Delta}_s$, the matrix M, and frequency ω.
Similar to the definition of the ∞-norm as the supremum over ω of the largest singular value of M we now define:

$$||M||_\mu := \sup_\omega \; \mu_{\mathbf{\Delta}_s}(M(j\omega)) \tag{3.51}$$

However, since μ does not satisfy the triangle inequality, $||M||_\mu$ is not a norm (we do have $\mu(cM) = |c|\mu(M) \;\; \forall c \in \mathrm{C}$). We use it to illustrate the similarity between ∞-norm and 'μ-norm' with respect to the small gain theorem with structured uncertainties:

Theorem 3.6.2 (small gain theorem with structured uncertainties)
Given $M \in \mathrm{RH}_\infty$ and $\mathbf{\Delta}_s$ according to equation 3.48, then the closed-loop system of figure 3.12 is internally stable for all $\Delta \in \mathbf{\Delta}_s$ if and only if $||M||_\mu < \gamma^{-1}$.

The proof of this theorem follows immediately from the definition of μ and the proof of theorem 3.5.3.
Clearly, μ is a generalization of the largest singular value; with $\mathbf{\Delta}$ according to equation 3.42 we have: $\mu_{\mathbf{\Delta}}(M(j\omega)) = \bar{\sigma}(M(j\omega))$. $\mathbf{\Delta}$ can therefore be interpreted as an extreme set within $\mathbf{\Delta}_s$: the unstructured case. Another extreme set can be defined as:

$$\mathbf{\Delta}_i := \{\delta I : \delta \in \mathrm{RH}_\infty, ||\delta||_\infty < \gamma\} \tag{3.52}$$

For this 'highly structured' set we have $\mu_{\mathbf{\Delta}_i}(M(j\omega)) = \rho(M(j\omega))$ with $\rho(M)$ denoting the spectral radius of M. From definition 3.48 and equations 3.52 and 3.42 we then have:

$$\mathbf{\Delta}_i \subset \mathbf{\Delta}_s \subset \mathbf{\Delta} \tag{3.53}$$

such that we can consider ρ and $\bar{\sigma}$ as (conservative) lower and upper bounds on μ:

$$\rho(M(j\omega)) \leq \mu_{\mathbf{\Delta}_s}(M(j\omega)) \leq \bar{\sigma}(M(j\omega)) \tag{3.54}$$

3.6.3 Calculation of the structured singular value

With equations 3.53 and 3.54 we can calculate a lower and an upper bound on μ by means of relatively simple eigenvalue calculations at all values of ω that are of interest. However, these bounds can be arbitrarily conservative, and therefore far apart. It appears that the gap between lower and upper bound can be reduced significantly by making use of the known special structure of $\mathbf{\Delta}_s$. For this, we will define two subsets of $C^{p_1 \times q_1}$ (with $p_1 = q_1$), that have a special relation with $\mathbf{\Delta}_s$:

$$\begin{aligned} \mathbf{U}_{\mathbf{\Delta}_s} &:= \{U \in \mathbf{\Delta}_s \cap C^{p_1 \times p_1} : U^* U = I_{p_1}\} \\ \mathbf{D}_{\mathbf{\Delta}_s} &:= \{\mathrm{diag}(D_1, \ldots, D_r, d_1 I_{k_{r+1}}, \ldots, d_f I_{k_{r+f}}) : \ D_i \in C^{k_i \times k_i}, \\ & \qquad D_i = D_i^* > 0, \quad d_i \in R, d_i > 0\} \end{aligned} \tag{3.55}$$

We then have (Doyle 1982):

Theorem 3.6.3
For all $U \in \mathbf{U}_{\mathbf{\Delta}_s}$ *and all* $D \in \mathbf{D}_{\mathbf{\Delta}_s}$

$$\mu_{\mathbf{\Delta}_s}(MU) = \mu_{\mathbf{\Delta}_s}(UM) = \mu_{\mathbf{\Delta}_s}(M) = \mu_{\mathbf{\Delta}_s}(DMD^{-1}) \tag{3.56}$$

Proof:
From equation 3.55 we have $U^* \in \mathbf{U}_{\mathbf{\Delta}_s}$, $U\Delta \in \mathbf{\Delta}_s$, $\Delta U \in \mathbf{\Delta}_s$, $\bar{\sigma}(U\Delta) = \bar{\sigma}(\Delta U) = \bar{\sigma}(\Delta)$. Now we have: $\det(I - M\Delta) = \det(I - MUU^*\Delta) = \det(I - U^*UM\Delta)$, which proves the first part of equation 3.56. For the last part, note that $D^{-1}\Delta D = \Delta$, $\forall \Delta \in \mathbf{\Delta}_s$ and that $\det(I - MD^{-1}\Delta D) = \det(I - DMD^{-1}\Delta)$.
□

This theorem implies that μ is independent of the special forms of unitary transformation and scaling of transfer function matrix M as given in equation 3.56. Substitution in equation 3.54 then results in:

$$\rho(UM) \le \mu_{\mathbf{\Delta}_s}(UM) = \mu_{\mathbf{\Delta}_s}(M) = \mu_{\mathbf{\Delta}_s}(DMD^{-1}) \le \bar{\sigma}(DMD^{-1}) \tag{3.57}$$

which suggests that a search over allowable U and D will result in tighter bounds on μ:

$$\max_{U \in \mathbf{U}_{\mathbf{\Delta}_s}} \rho(UM) \le \mu_{\mathbf{\Delta}_s}(M) \le \inf_{D \in \mathbf{D}_{\mathbf{\Delta}_s}} \bar{\sigma}(DMD^{-1}) \tag{3.58}$$

In fact, Doyle (1982) proved:

$$\max_{U \in \mathbf{U}_{\mathbf{\Delta}_s}} \rho(UM) = \mu_{\mathbf{\Delta}_s}(M) \tag{3.59}$$

although this does not lead to a straightforward calculation of μ: $\rho(UM)$ can have multiple local maxima.
The upper bound, on the other hand, appears to be a convex optimization problem (Safonov and Doyle 1984, Tsing 1990), such that there is only one local minimum, which is the global minimum. However, there are only a few cases in which this

minimum of the upper bound is equal to μ: it can be proven that equality occurs when $2r + f \leq 3$, i.e. the uncertainty structure $\mathbf{\Delta}_s$ consists of less than three full blocks, less than two repeated blocks or less than the combination of 1 repeated block and two full blocks (see Doyle et al 1990, Zhou et al. 1993).
Finally, note that the determination of μ as a function of frequency ω by means of the upper bounds provided by equation 3.58, implies that matrices U and D also become functions of ω.

3.6.4 Application of the structured singular value

It has already been discussed that the design of a robust controller should take both robust stability and robust performance into account. It was argued that calculation of an H_∞-optimal controller would necessarily lead to conservative results, due to the fact that the special uncertainty structure of this problem is not considered. This is an important motivation for the definition of the structured singular value: calculation of μ makes it possible to come to non-conservative statements.
Perhaps even more important is that the definition of μ, or rather the definition of the uncertainty set $\mathbf{\Delta}_s$, also allows structuring of the uncertainty matrix for other reasons. When setting up the standard plant, usually several subsystems and weight functions are combined. In each of these subsystems, uncertainties may occur for any of the reasons mentioned in section 3.5, leading to an uncertainty matrix for every subsystem. Combining all subsystems into the standard plant to set up the standard control design structure of figure 3.11, then automatically implies that all separate uncertainty matrices are combined into Δ. Correct ordering of all uncertainty inputs and outputs immediately results in a block-diagonal structure of Δ, possibly extended with a performance block Δ_p.
This implies that careful modelling of the system to be controlled not only leads to an accurate and detailed description in the form of the standard plant P, but also to a highly structured uncertainty matrix Δ. Discarding this structure by performing singular value analysis and H_∞ controller synthesis will therefore lead to conservative results, which can be remedied by using μ.

Although direct calculation of μ is not possible, it appears in many practical cases that the upper and lower bounds defined in the previous subsection are close together. Furthermore, the upper bound of $\mu(M)$ is usually much smaller than $\bar{\sigma}(M)$, such that also $\|M\|_\mu$ is much smaller than $\|M\|_\infty$. Hence, for analysis of robust stability and performance (using a performance block) of a given controller, the calculation of bounds on μ is much less conservative than the calculation of the ∞-norm. This is an important reason to allow the expense of considerably more numerical calculation time to obtain μ instead of $\bar{\sigma}$. Furthermore, development of numerical methods is ongoing to reduce this calculation time, to improve numerical stability and to further tighten the bounds, for instance in the case that there are *real repeated* δ_i (Young et al. 1991).

When considering the robust controller synthesis problem, the calculation of μ can be seen as part of an iterative procedure known as D-K iteration:

1. set up the standard plant P according to figure 3.11 and combine w_1 and w_2 to w, as well as z_1 and z_2 to z, i.e. extend the uncertainty matrix with a performance block,

2. calculate an H_∞-(sub)optimal controller K minimizing $\|T_{wz}\|_\infty$,

3. minimize $\bar{\sigma}(DT_{wz}D^{-1})$ over $D \in \mathbf{D}_{\mathbf{\Delta}_s}$, pointwise across frequency; compare $\sup_\omega \left\{\bar{\sigma}(DT_{wz}D^{-1})\right\}$ (the upper bound for $\|T_{wz}\|_\mu$) with its previous value and stop if they are close (first time use $\|T_{wz}\|_\infty$),

4. determine an invertible transfer function matrix $D_1 \in RH_\infty$, such that $D_1(j\omega) \in \mathbf{D}_{\mathbf{\Delta}_s}$, and such that $D_1(j\omega) \approx D(\omega)$ and define: $\tilde{P} := \begin{bmatrix} D_1 & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} D_1^{-1} & 0 \\ 0 & I \end{bmatrix}$,

5. calculate an H_∞-(sub)optimal controller $\tilde{K}$, such that $\|\mathcal{F}_l(\tilde{P},\tilde{K})\|_\infty$ is minimal and determine: $T_{wz} = \mathcal{F}_l(P,\tilde{K})$, i.e. the new controller $\tilde{K}$ applied to the original plant P,

6. goto step 3.

This procedure can be performed until no significant changes occur in $\|T_{wz}\|_\mu$ and in D (see step 3). However, there is no guarantee that a global optimum will be found: this joint optimization of D and K is not convex. Another important disadvantage is that D is calculated as a complex-valued matrix at each frequency point of interest: this implies that transfer function matrix D_1 must be fitted through these points, resulting in a trade-off between accuracy and order (which also determines the order of K). Nevertheless, several design examples have shown that an essential improvement of robustness and performance can be obtained (Balas and Doyle 1989, Skogestad et al. 1988, Smith et al. 1987). The final controller has an order equal to that of the plant including weight functions and

D-scalings (i.e. D_1 and D_1^{-1}).

It should be noted that developments in this area are ongoing. New theoretical and numerical results in convex optimization theory and Linear Matrix Inequalities or LMIs may allow further generalization of the concept of structured singular values, and may provide a more fundamental approach to robust control design problems (Balakrishnan et al. 1992, Boyd et al. 1993, El Ghaoui et al. 1992, Kaminer et al. 1993, Packard et al. 1991, 1992, Vandenberghe and Boyd 1993).

3.7 Parametric uncertainty modelling for structured singular value calculation

3.7.1 The real-repeated uncertainty structure

In the previous section the set of structured uncertainty matrices $\mathbf{\Delta}_s$ was introduced to solve the robust performance problem and to enable detailed modelling of uncertainties in the standard plant. Unfortunately, it is not always straightforward to transform a known uncertainty into the block-diagonal structure of $\mathbf{\Delta}_s$. Note that Δ in figure 3.11 is basically an unknown transfer function matrix and that $\mathbf{\Delta}_s$ is a set of transfer matrix descriptions in the frequency domain. However, uncertainties are often known in the time domain as uncertain parameters or coefficients in differential equations. This leads to state-space descriptions with coefficient matrices that are functions of the uncertain parameters (usually with a physical meaning).

This section is based on Lambrechts et al. (1993) and will provide a procedure for transforming such an uncertain state-space description into the standard control structure. The resulting uncertainty matrix will have a real-repeated structure, which still fits the general structure of $\mathbf{\Delta}_s$, but with $f = 0$ and with δ_i restricted to R:

$$\mathbf{\Delta}_{rr} := \{\mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}) : \delta_i \in \mathrm{R}, |\delta_i| \leq \gamma\} \tag{3.60}$$

The development of numerical methods to calculate tight bounds on μ when Δ or one of the diagonal sub-blocks of Δ is real-repeated (Young et al. 1991), motivates the effort of such detailed uncertainty modelling. However, it should be noted that there is no controller synthesis procedure that makes full use of real-repeated uncertainty matrices.

In the next subsection we will assume that a state-space description of a standard plant according to figure 3.1 is given (i.e. equation 3.4), in which (some of) the entries of the coefficient matrices are real rational functions in a number of varying parameters. From this description we will obtain a standard control structure with uncertainties as given in figure 3.11, with $\Delta \in \mathbf{\Delta}_{rr}$. This generalizes earlier results in parametric uncertainty modelling as given by Morton and McAfoos (1985) and Steinbuch et al. (1991, 1992).

3.7.2 Transforming the standard plant with uncertainties to an LFT

Consider a vector $\theta = (\theta_1, ..., \theta_r) \in \mathrm{R}^r$ containing r bounded scalar parameters. Let the model of the perturbed standard plant be given as a state-space description in which the entries of the matrices depend on the parameter vector θ:

$$\begin{bmatrix} \dot{x} \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{c|c} A(\theta) & B(\theta) \\ \hline C(\theta) & D(\theta) \end{array}\right] \begin{bmatrix} x \\ \hline w \\ u \end{bmatrix} \tag{3.61}$$

Now define the $(n+p) \times (n+q)$ matrix $S(\theta)$ as:

$$S(\theta) := \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} \tag{3.62}$$

and note that $S(\theta)$ is the coefficient matrix of the LFT form of the state-space description of the standard plant (see figure 2.2). Now we would like to extract the dependency of S on θ by constructing an LFT as follows:

$$S(\theta) = \mathcal{F}_u(M, \Delta) = M_{22} + M_{21}(I - \Delta M_{11})^{-1}\Delta M_{12} \tag{3.63}$$

$\Delta \in \mathbf{\Delta}_{rr}$ (equation 3.60) should be such that all variations in θ are accounted for; in fact each δ_i will be a normalized version of the corresponding θ_i, such that $\Delta \in \mathbf{B\Delta}_{rr}$, the unit ball in $\mathbf{\Delta}_{rr}$. Furthermore, the matrices $M_{22}, M_{21}, M_{11}, M_{12}$ must be independent of variations in θ and partitioned according to the partitioning of S.

This then allows us to draw the LFT form of the uncertain state-space description as a combination of an LFT on an n-dimensional block of integrators $(\frac{1}{s}I_n)$ and an LFT on Δ (see figure 3.14). If we consider only the non-trivial case that $\delta_i \neq 0, \quad i = 1 \dots r$ we can define $\phi_i := 1/\delta_i$ and rewrite equation 3.63 as:

$$S(\theta) = \mathcal{F}_u(M, \Delta) = M_{22} + M_{21} \left\{ \begin{bmatrix} \phi_1 I_{k_1} & & 0 \\ & \ddots & \\ 0 & & \phi_r I_{k_r} \end{bmatrix} - M_{11} \right\}^{-1} M_{12} \tag{3.64}$$

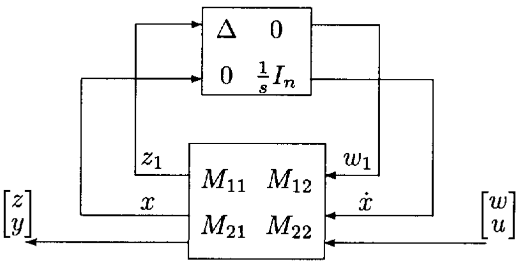

Fig. 3.14: LFT form of an uncertain state-space description of a standard plant

This shows that the problem of finding an LFT form of the state-space model of an uncertain standard plant can be seen as an ND-realization problem (Bose 1982).

Using a constructive algorithm we can now prove the following theorem:

Theorem 3.7.1
A transformation of a state-space model with parametric uncertainty to an LFT exists if the entries of the state-space matrices are bounded and can be given as real-rational functions in the parameters.

The algorithm proving this theorem is mainly based on the important property of LFTs that linear interconnections of LFTs can always be written as one single LFT on a single block-diagonal variable Δ. Therefore we can write the varying entries in a state-space model as individual LFTs, after which they can be collected in a single LFT.
However, minimality of the obtained LFT can not be guaranteed since it is not straightforward to generalize the 1D concepts of controllability and observability to ND-systems (Roesser 1975).

3.7.3 A procedure for the transformation

The algorithm consists of eight steps:

1. **Scaling the varying parameters**
   Lower and upper bound vectors for the parameter vector θ can be determined, denoted respectively as $\underline{\theta}$ and $\bar{\theta}$: Now define $\theta_o := (\underline{\theta} + \bar{\theta})/2$, $s_o := (\bar{\theta} - \underline{\theta})/2$, $\delta = (\delta_1 \ldots \delta_r)$, $\delta_i \in [-1, +1]$, such that $\theta_i = \theta_{oi} + s_{oi}\delta_i$ for $i = 1 \cdots r$. Substitution of this result in equation 3.61 then gives scaled polynomial expressions for all varying numerators and denominators.

2. **Individual varying terms as LFTs**
   The varying parts of a numerator or denominator consist of a number of terms (monomials) that can be written as separate LFTs acting on the δ_i.

3. **Numerators of varying entries**
   Using the fact that two parallel LFTs form again an LFT, the sum of all terms in each numerator can again be written as an LFT.

4. **Denominators of varying entries**
   To obtain an LFT of the *inverse* of a polynomial, set up an LFT for the denominator as was done for the numerators in the previous step, subtract 1 and put the result in the feedback path (see figure 3.15). The obvious fact that the entries of the nominal model must be bounded guarantees well definedness of this feedback structure.

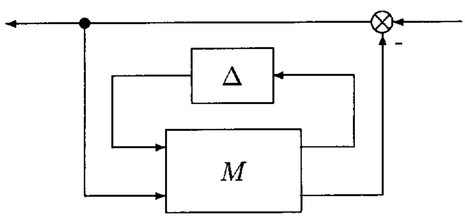

Fig. 3.15: An LFT in the feedback path

5. **Combining numerators and denominators of individual entries**
   Cascade connection of the LFTs of each numerator-denominator pair found in the previous steps leads to a single LFT for each varying entry.

6. **Combining all varying entries**
   LFTs for the A, B, C and D matrices can be set up separately and can be rewritten as one single LFT with $\Delta = \text{diag}(\Delta_A, \Delta_B, \Delta_C, \Delta_D)$.

7. **Transformation to the real-repeated blockstructure**
   Δ can be rearranged into the real-valued repeated scalar block structure of equation 3.60 by interchanging rows and columns of the LFT.

8. **Reducing the dimension of Δ**
   The resulting LFT can be set in state-space form in which the uncertainty inputs can be appended to u and the uncertainty outputs can be appended

to y. Any uncontrollable and/or unobservable parts of this state-space model can be removed using a standard reduction technique, thus reducing the dimensions of the 'block of integrators'. The same procedure can be used to reduce the size of any of the real-repeated blocks in Δ.

Rewrite the LFT by considering x as an uncertainty input and $\dot{x}$ as an uncertainty output; this implies that the block of integrators is appended to Δ. Next, separate the uncertainty block $\delta_1 I$ from Δ and consider its uncertainty inputs as 'pseudo-states' and its uncertainty outputs as 'pseudo-derivatives' (in fact δ_1 should be replaced by $\frac{1}{\phi_1}$ as was done in equation 3.64). Removing the parts that are uncontrollable and/or unobservable when considering all other inputs and outputs will then reduce the size of $\delta_1 I$. This procedure can be repeated for all other real-repeated uncertainty blocks.

We now have obtained an LFT description of the uncertain standard plant, which is equivalent to the state-space description of equation 3.61. These steps have been implemented within the environment of PC MatLab (Moler et al. 1987), such that the entire procedure can be performed interactively (Terlouw and Lambrechts 1992). It should be noted that the obtained real-repeated uncertainty matrix may be combined (block-diagonally) with any uncertainty matrix that was constructed to model other uncertainties than those represented by θ.

Chapter 4

The output regulation and tracking problem

4.1 Introduction

In the previous chapter we set up a general framework for linear controller design in which several important design objectives and trade-offs can be specified. A standard control design structure was introduced and several controller synthesis procedures were considered. LQG or, more generally, H_2 synthesis appeared to be useful when internal stability and a trade-off between several performance objectives is needed. H_∞ synthesis can be used as an alternative procedure to obtain stability and performance, with more emphasis on performance in the frequency domain. More importantly, it appeared that H_∞ synthesis can be used to guarantee robustness of the closed loop system by means of the small gain theorem (theorem 3.5.3). Finally the structured singular value was introduced, to prevent extremely conservative results when robust performance and robustness against structured uncertainties is to be considered.

Apart from these methods for analysis and synthesis of controllers for the standard control design structure, several other developments are ongoing, that are also based on this general framework. For instance, there is the mixed H_2-H_∞ problem of finding a controller that achieves performance by minimizing the 2-norm of $T_{w_2 z_2}$, while obtaining robustness by bounding the ∞-norm of $T_{w_1 z_1}$ (Bernstein and Haddad 1988, 1989). Another example is the incorporation of *gain-scheduling* by defining one or more uncertain parameters that act on the controller as well as

on the standard plant (Packard et al. 1992, 1993).
This motivates our attempt to also include the formulation of the output regulation and tracking problem into the general framework, such that all available and developing tools for analysis and synthesis of controllers can be used to find solutions. The general idea then is to set up a standard control design structure that combines trade-offs between all relevant control objectives:

- internal stability;
- robust stability;
- performance in the sense of minimization of z;
- robust performance in the sense of minimization of z;
- performance in the sense of output regulation and tracking;
- robust performance in the sense of output regulation and tracking.

By making sure that the resulting standard control design structure complies with assumptions A1 through A4 (section 3.3), it is then possible to synthesize a controller by solving an H_2 or H_∞ problem.

The next section will review the results with respect to the output regulation and tracking problem as available in literature, using the simple asymptotic tracking problem considered in section 1.1. After that we will set up a modelling procedure to find an appropriate model for the expected persistent reference signals. Section 4.4 will then provide solvability conditions and a complete constructive procedure for obtaining a solution. We will next consider some special cases of this approach, leading to the robust output regulation and tracking solution in section 4.5 and the two-degrees-of-freedom solution in section 4.6. Some discussion on the selection of appropriate weights and the possibilities for defining trade-offs between several control objectives in section 4.7 will conclude this chapter.

4.2 Analysis of the asymptotic tracking problem

The output regulation problem has been addressed by many authors during the modern control era. Based on geometric control theory, the *Internal Model Principle* was developed by Wonham and coworkers (Wonham 1979, Francis and Wonham 1975, Sebakhy and Wonham 1976). This resulted in necessary and sufficient conditions on the structural properties of a controller to solve the asymptotic tracking and disturbance rejection problem (see also Francis 1977, Schumacher 1983, Gonzales and Antsaklis 1989, 1991). Simultaneously, an algebraic approach was developed by Davison and coworkers (Davison 1972, Bhattacharyya and Pearson 1972, Davison and Goldenberg 1975), providing an easy to construct *servo compensator* for a given set of persistent reference or disturbance signals. Later, we will see that the presence of an internal model is a property of a dynamical system description, while a servo compensator can be defined as follows:

Definition 4.2.1 servo compensator
A servo compensator is a dynamical model included in a controller for the purpose of achieving asymptotic tracking.

Several authors, such as Bengtsson (1977), Cheng and Pearson (1978) and Francis (1977), adopted a frequency domain approach to analyze and solve the problem. From that, it appeared possible to find a parametrization of all controllers that achieve the tracking objective (Francis and Vidyasagar 1983, Sugie and Vidyasagar 1989). However, it appeared to be not straightforward to use H_2 or H_∞ synthesis methods to select a single (sub-)optimal controller from this set of controllers.

To make clear why the output regulation and tracking problem needs special treatment we will reconsider the simple classical feedback control problem discussed in section 1.1. Restating this problem within the standard control design structure results in figure 4.1, in which P and K are assumed to be LTI models. Clearly

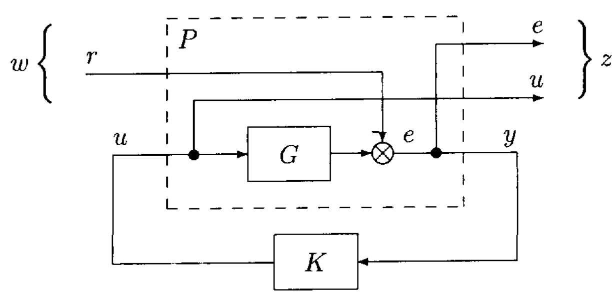

Fig. 4.1: The classical feedback control system in the general framework

the reference signal r can be viewed as disturbance input w. Similarly, the error signal e together with the control input u act as performance objectives vector z that should be minimized. Note that u must be added as a performance objective to obtain a trade-off between error signal minimization and actuator effort, and that this is sufficient to satisfy assumption A2 in section 3.3.

To perform the minimization of z, we could either solve the H_2-problem or the H_∞-problem. Minimizing the 2-norm appears as the optimal solution when w may be assumed to be white noise with spectral density 1 and z should be minimized in the sense of its average spectral density. When considering the H_∞-problem, we want to minimize $\|T_{wz}\|_\infty$ with $T_{wz} \in \mathrm{RH}_\infty$: as the ∞-norm is the induced 2-norm (theorem 2.3.2), this implies that we are looking for the disturbance w in H_2 for which $\|z\|_2$ is maximal and that we want to find a controller that minimizes it.

The problem with both approaches is that there is no guarantee that the resulting controller has certain tracking properties. This can be demonstrated by considering the simple case of a change in operating point, which implies that r has the appearance of a step-function. A step function may be modelled in the frequency domain as the impulse response of a system with TFMD $R := \frac{1}{s}$. The error response of the closed-loop system on this step-function may thus be modelled as the impulse response of $T_{re}R$. In general, this impulse response will not asymptotically decay to zero, due to the unstable pole in R, which implies that there is a static error in e. Obviously, this can only be remedied by assuring that the unstable pole in R is cancelled by an appropriate zero in T_{re}: any internally stabilizing controller will then guarantee that $T_{re}R \in \mathrm{RH}_2$, such that $e(t) \to 0$ for $t \to \infty$. To be able to use robust control theory for the design of those controllers it is interesting to somehow extend the general control design structure to guarantee that $T_{re}R \in \mathrm{RH}_2$. We will discuss several possible approaches to this problem.

The first approach is depicted in figure 4.2 and simply tries to force the desired behaviour onto T_{re} by applying a real-rational weight function $W \in \mathrm{R}(s)$ to the error signals e: $z_1 := We$. By choosing $W = R = \frac{1}{s}$ and solving the H_2- or H_∞-problem for the resulting standard plant P, we would get that $WT_{re} \in \mathrm{RH}_2$ and, under the condition that R is scalar (or scalar times I), also that $T_{re}R \in \mathrm{RH}_2$. Unfortunately however, we have from figure 4.2 that P is no longer detectable from y. This implies that assumption A1 (section 3.3) is violated, and that no H_2 or H_∞ controller can be found.

A practical solution to this problem is to conclude that asymptotic tracking is too much to ask for and to allow a 'small' static deviation of e by selecting $W = \frac{1}{(s+\epsilon)}$ such that $W(0)$ is large but bounded. Using H_∞ synthesis then guarantees that the static deviation of e when r is a unit step function is smaller than $\gamma W(0)^{-1}$ with $\gamma = \|T_{wz}\|_\infty$.

A more fundamental approach is presented by Liu and Mita (1991), who define a

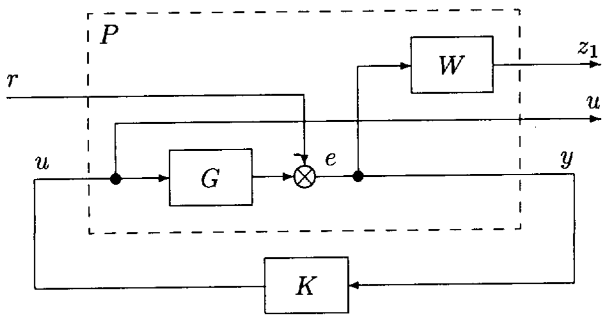

Fig. 4.2: The tracking objective as a weight on e

generalisation of the H_∞-problem to solutions that are not stabilizing, but allow imaginary hidden modes (i.e. assumption A1 is partially removed). The obvious problem with this approach is the possibility of constructing unstable control systems unless weight functions are chosen with extreme care.

The second approach does not require an extra weight function and is therefore directly based on the structure in figure 4.1. It considers the output regulation objective as a boundary constraint on the closed-loop transfer function T_{re} at the frequency point of interest. In this case the demand $T_{re}R \in RH_2$ is clearly met if and only if $T_{re} \in RH_\infty$ and $T_{re}(0) = 0$. An H_2 or H_∞ controller can be derived using the operator theoretic approach to (sub-)optimal control (see Francis 1987) in which the boundary constraint is incorporated. Existence of solutions can then be established using frequency-domain interpolation theory (Hara and Sugie 1989, Sugie and Hara 1989, Cevik and Schumacher 1993). However, the actual calculation of a controller is very hard and may lead to high order controllers. It is based on selection of the (sub-)optimal controller from the parametrization of all output regulating controllers, in which a servo compensator should be present, to guarantee the presence of the appropriate internal model in the closed loop system (Francis and Vidyasagar 1983, Sugie and Vidyasagar 1989).

The third approach directly starts with this parametrization and tries to find a (sub-)optimal controller using the operator theoretic approach. In this approach, the H_∞-problem is replaced by a model matching problem, in which transmission zeros on the imaginary axis appear. These transmission zeros prevent solution of this problem: the usual assumption that no such zeros occur is equivalent to asumptions A3 and A4. A technique known as *$j\omega$-axis shifting* is then applied to solve a slightly perturbed H_∞-problem, after which the controller for the original problem is recovered by a reverse transformation (Xu and Mansour 1986, 1988, Wu and Mansour 1989, 1990). As in the previous approach, this method leads to involved calculations and, due to the use of the operator theoretic approach for H_∞ controller design, to high order controllers (this may be remedied by using perturbation techniques in combination with state-space techniques). Furthermore, there is no clarity on how to incorporate trade-offs with other control design objectives.

The fourth approach also presumes the necessity of an internal model. However, in this approach the internal model is added to the standard plant, in order to incorporate it later in the controller as a servo compensator. The resulting control configuration is given in figure 4.3 and is similar to the use of a PI-controller. S

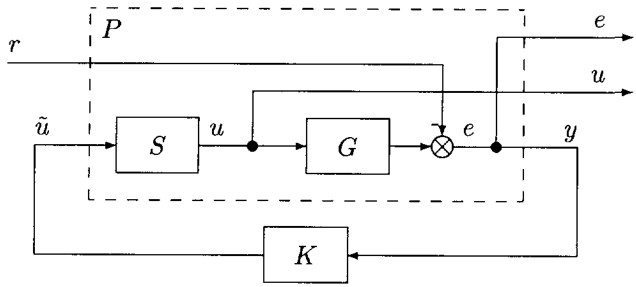

Fig. 4.3: The application of the internal model principle

denotes the internal model: in our example $S = \frac{1}{s}$, and $\tilde{u}$ is taken as the new control input. After determination of a controller K it is easy to set up a controller for the original problem: $\tilde{K} := SK$.

The fact that this controller achieves the tracking objective is purely due to the presence of the internal model, which causes the appropriate zero in T_{re} (in our case $T_{re}(0) = 0$). For this reason, the transfer function T_{re} is only specified at one point (in the frequency domain), such that other restrictions on the frequency response of T_{re} must be added by means of a separate stable weight function. The relation between internal model and this separate weight function is not clear, hence also the relation between internal model and other performance objectives (specified by means of weight functions) is not straightforward.

An example of this approach can be found in Abedor et al. (1991, 1994), with the only difference that the internal model is taken at the output of the plant rather than at the input.

The fifth approach is based on a problem formulation that combines that of the first and fourth approach: we add a weight function $W = \frac{1}{s}$ and an internal model $S = \frac{1}{s}$. The advantage above the fourth method is that the weight function may be used not only to specify the desire to achieve asymptotic tracking, but also to define restrictions on the frequency response of T_{re} to obtain a certain attenuation of signals that are not compatible with the internal model.
The problem mentioned before, that this resulting system is no longer detectable from y can be dealt with elegantly by using the *Riccati inequality approach* as can be found in (Khargonekar et al. 1990, Scherer 1992). The disadvantage of this Riccati inequality approach is again the more difficult calculation of a controller (although much simpler than with the operator theoretic approach).
Another approach to solve the detectability problem is given by Hosoe et al. (1992): they suggest to take the internal state of the unstable weight function as extra measurement signals, such that detectability is (trivially) ensured. This can, of course, only be done if the weight function becomes part of the final controller, which is in accordance with the internal model principle. Note that this implies that the internal model is taken at the output of the plant, rather than at the input.

It should be noted here, that this classification into five approaches could be done differently. All these approaches consider the same problem and have more similarities than perhaps suggested. The reason for introducing them this way, is to show that the approach suggested in this thesis can be seen as a separate approach, and to be able to compare this approach with those available in literature.

The basic idea of the approach, that will be developed in all detail in the following sections, is as follows. First set up a model of the expected persistent signals, which will be called the *Reference Signal Generator* or RSG and denoted as R. Next, instead of deriving weight functions, boundary constraints or internal models from this RSG, we immediately incorporate it into the standard plant as depicted in figure 4.4, with u_r denoting an impulsive input. Setting up solvability conditions will then automatically result in the necessity of an internal model in the closed loop, the construction of an appropriate servo compensator, and the need for a special treatment of the control objectives related with the actuator effort.
As a preview of the results derived in the following sections, the standard control configuration for our simple example is given by figure 4.5. A minimal realization of the obtained standard plant will appear to be easily modified to comply with

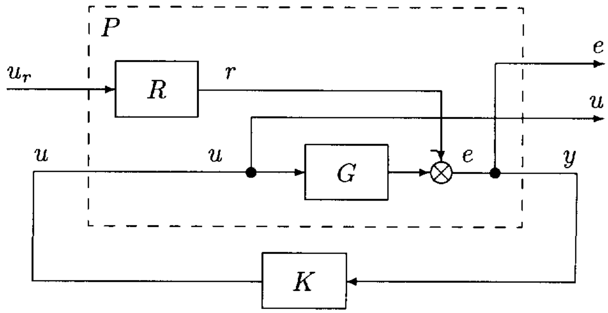

Fig. 4.4: Incorporation of the reference signal model

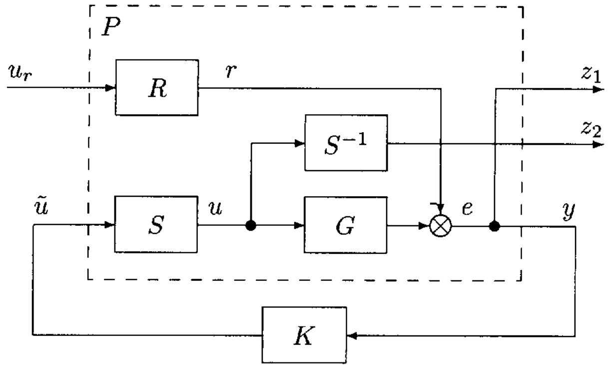

Fig. 4.5: Standard control configuration for the simple tracking problem

assumptions A1 through A4, such that an H_2 or H_∞ optimal controller can be synthesized. The combination of this controller with the appropriate servo compensator can then be proven to obtain the tracking objective.
The first time this particular structure of the standard plant was suggested, was in Lambrechts and Bosgra (1991). The most important features of this approach are the following:

- the standard plant can easily be modified to comply with assumptions A1 through A4, such that the standard synthesis procedures mentioned in chapter 3 can be used,

- the incorporation of an RSG implies the possibility to not only specify the *form* of the reference signals, but also their *direction*: they do not have to act on all error signals,

- the RSG may be combined with a stable weight function to specify other restrictions on the frequency response of T_{re}: the tracking objective may be combined with other disturbance attenuation objectives,
- any other control objectives and stable weight functions can be added to the standard plant without affecting the tracking property of the resulting controller.

The importance of these features will become more clear in the following sections. In section 6.3 we will summarize the differences between our approach and the ones considered in literature.

4.3 Description of persistent signals

In this section we will review the procedure of setting up an appropriate LTI model for the description of the reference signals that may be expected. This procedure is mainly based on Johnson (1971), who uses it to model both persistent external disturbances and reference signals. We will only discuss the effect of persistent reference signals, as it is well known that the effect of persistent external disturbances is basically the same. The incorporation of these disturbances into the problem formulation is therefore straightforward and does not have any consequences for the procedure developed in this chapter.
We assume that for a given physical system we have an LTI model G, in which all relevant output signals are known and have a physical meaning. We will denote the vector of output signals as y_G with dimension p: $y_G \in \mathrm{R}^p$. Furthermore, we assume that control design specifications are available, such that we can set up an inventory of allowable persistent reference signals for which we want to achieve asymptotic tracking of y_G. Such an inventory should consist of a finite number of signals: $r_i(t),\ i = 1 \cdots \nu$ that each take the form: $r_i(t) = y_i f_i(t),\ i = 1 \cdots \nu$. Here $f_i(t)$ is a scalar real-valued function of time and $y_i \in \mathrm{R}^p$ is a normalized constant vector determining the way in which $f_i(t)$ appears in $r_i(t)$.
This inventory of reference signals will form the basis for the construction of a *set* of reference signals: $\mathcal{R}$. We will assume that this set at least contains all linear combinations of $r_i(t),\ i = 1 \cdots \nu$ as follows:

$$r(t) = \sum_{i=1}^{\nu} a_i r_i(t) = \sum_{i=1}^{\nu} a_i \cdot y_i f_i(t), \qquad a_i \in \mathrm{R} \tag{4.1}$$

with a_i denoting a real amplitude-scaling factor that may be chosen arbitrarily. Next, to be able to find an LTI description of $\mathcal{R}$, we will assume that each of the functions $f_i(t)$, $i = 1 \cdots \nu$ may be found as a solution of a linear homogeneous differential equation with given constant real coefficients α_{ij} and given initial conditions:

$$(\frac{d}{dt})^{n_i} f_i(t) + \alpha_{i1}(\frac{d}{dt})^{n_i-1} f_i(t) + \cdots + \alpha_{i(n_i-1)} \frac{d}{dt} f_i(t) + \alpha_{in_i} = 0 \tag{4.2}$$

As a standard result (e.g. Chen 1984) we then have that we can set up a state-space description of this equation as:

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t), \qquad x_i(0) = x_{i0} \\ f_i(t) &= c_i x_i(t) \end{aligned} \tag{4.3}$$

with initial conditions x_{i0} given and:

$$\begin{aligned} A_i &:= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_{n_i} & -\alpha_{n_i-1} & -\alpha_{n_i-2} & \cdots & -\alpha_1 \end{bmatrix} \\ c_i &:= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned} \tag{4.4}$$

The shape and size of the function $f_i(t)$ is now determined by the order of the SSD (n_i), the coefficients α_{ij}, $j = 1 \cdots n_i$ and the initial conditions vector x_{i0}. Whereas n_i and α_{ij} can usually be fixed, it is in general not possible to precisely determine the initial condition vector x_{i0}. We will therefore include in the set $\mathcal{R}$ all reference signals that may be the result of an initial condition that is an arbitrary element of an n_i-dimensional linear vector space: $x_{i0} \in \mathrm{R}^{n_i}$, or one of its subspaces. Hence, we will consider x_{i0} to be the result of:

$$x_{i0} = B_i u_i \tag{4.5}$$

in which the columns of B_i form a basis for a subspace of R^{n_i} and u_i is an arbitrary vector of appropriate dimension.

To obtain an interpretation of B_i and u_i, we can Laplace transform equation 4.3 into the frequency-domain, using the fact that $\mathcal{L}(\dot{x}(t)) = s\mathcal{L}(x(t)) - x(0)$ (see e.g. Boyce and DiPrima 1965):

$$\begin{aligned} s x_i(s) &= A_i x_i(s) + B_i u_i \\ f_i(s) &= c_i x_i(s) \end{aligned} \tag{4.6}$$

Hence, the arbitrary constant initial condition generating vector u_i may be interpreted as an impulsive input, and B_i acts as the appropriate input matrix. Because of this, only a choice of B_i such that (A_i, B_i) is controllable makes sense. Although it is possible to select $B_i = I_{n_i}$ without changing the asymptotic tracking objective as such, it is sensible to reduce the dimension of u_i as much as possible: later we will see that u_i acts as an external disturbance input such that other design objectives may be affected.

With this, we are able to set up a state-space description of the basic reference signals r_i:

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) \qquad x_i(0) = 0 \\ r_i(t) &= C_i x_i(t) \end{aligned} \tag{4.7}$$

in which $C_i := a_i y_i c_i$ and $u_i(t)$ is an impulsive input.
The complete reference signal $r(t)$ can then be found from equation 4.1 as:

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u_r \qquad x_r(0) = 0 \\ r(t) &= C_r x_r(t) \end{aligned} \tag{4.8}$$

with:

$$A_r := \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & A_\nu \end{bmatrix} \qquad B_r := \begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & B_\nu \end{bmatrix}$$

$$C_r := \begin{bmatrix} C_1 & C_2 & \dots & C_\nu \end{bmatrix} \tag{4.9}$$

$$x_r := \begin{bmatrix} x_1(t) \\ \vdots \\ x_\nu(t) \end{bmatrix} \qquad u_r := \begin{bmatrix} u_1(t) \\ \vdots \\ u_\nu(t) \end{bmatrix}$$

With this, we will consider $\mathcal{R}$ to be the set of all reference signals $r(t)$ that may be generated by equations 4.8 and 4.9 with u_r an arbitrary impulsive input. The LTI system determined by (A_r, B_r, C_r) will therefore be referred to as *the Reference Signal Generator* or RSG; if necessary, we can obtain a minimal realization of the RSG by means of standard numerical methods. Its frequency-domain representation will be denoted as $R(s)$; the transfer function matrix description from u_r to r. We will have obtained asymptotic tracking for any reference signal $r(t) \in \mathcal{R}$, if $(y_G(t) - r(t)) \to 0$ for $t \to \infty$.

To obtain a sensible problem formulation we will now restrict our choice of possible reference signals somewhat further. As mentioned before, we are interested in persistent reference signals, i.e. signals that are not going to zero as time goes to infinity. Although stable dynamics, represented by open left half plane eigenvalues of A_r, are allowable in the RSG, they will not present any problems in solving the tracking problem and can be assumed to be absent. Open right half plane eigenvalues of A_r are however not allowed: they would present an unrealistic situation as the plant outputs should be able to track a signal that is unbounded in all its derivatives. Only eigenvalues on the imaginary axis are therefore considered: this implies that we want to be able to track reference signals that are combinations of step-functions, polynomial functions and sinusoids.

4.4 The asymptotic tracking problem in the general framework

In the next subsection we will set up the standard control configuration for the tracking problem and give a formal problem formulation. After that we will consider solvability and the possibility of applying robust control methods to find optimal controllers.

4.4.1 Problem formulation

Our aim is to set up a control configuration that contains all aspects of the general solution we want to obtain, but is still as simple as possible. This standard control configuration must contain the reference signal generator discussed in the previous section plus an extra output weight to make the problem solvable. However, we will not include a servo compensator on beforehand, but instead let the necessity of such a servo compensator follow from the solvability conditions. The resulting standard control design structure for the tracking problem is then given in figure 4.6 (compare with figure 4.5). $G \in \mathrm{R}_{ss}$ is the model of the physical plant, $R \in \mathrm{R}_{ss}$ is the RSG, $W \in \mathrm{RH}_\infty$ is a stable weight function and $K \in \mathrm{R}_{ss}$ is the controller to be designed. The signal vector u_r is a unit impulse in an arbitrary direction in the input space of R, but it may just as well be interpreted as an arbitrary white noise signal with unit intensity, as both signal forms have the same representation in the frequency domain. Signal vectors z_1 and z_2 are objective functions to be minimized, u is the control input and y is the measured

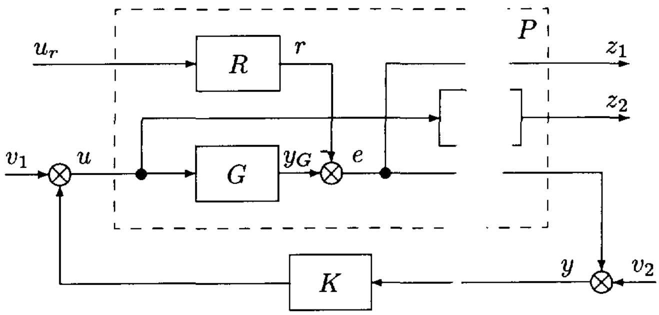

Fig. 4.6: Standard control configuration for the tracking problem

output. The auxiliary signal vectors v_1 and v_2 are added to be able to check internal stability of the closed loop system in accordance with definition 3.2.4. Hence, the standard plant for our problem formulation is defined as:

$$P := \left[\begin{array}{c|c} -R & G \\ 0 & W \\ \hline -R & G \end{array}\right] \tag{4.10}$$

Note that the usual assumption that only error signals are measured implies that the final controller K must be a so-called one-degree-of-freedom controller. The extension to the more general two-degree-of-freedom problem will be discussed in section 4.6.

The fact that u_r should be considered as either an impulsive or a white noise input would suggest the use of H_2 optimal control to find a solution. However, the following proposition will state a frequency-domain equivalent of the output regulation and tracking objective, which is basically a time-domain objective. With that it will appear possible to also consider H_∞ methods for finding a controller, implying that u_r could also be viewed as a function in H_2.

Proposition 4.4.1 (tracking equivalence)
Consider the tracking control configuration in figure 4.6, and let $G, K, R \in \mathrm{R}_{ss}$ and $W \in \mathrm{RH}_\infty$ be given, R an appropriate RSG with poles on the imaginary axis and K some internally stabilizing controller for G.
Furthermore, let the error signal $e(t)$ be defined as $e(t) := y_G(t) - r(t)$.
Then the following statements are equivalent.

1. *Given any initial condition of the plant, the output $y_G(t)$ asymptotically*

tracks any reference signal $r(t)$ that is an impulse response of R. In other words: $\lim_{t\to\infty} e(t) = 0$.

2. *The transfer function $(I - G(s)K(s))^{-1}R(s)$ is in RH_∞.*

Proof:
We have from figure 4.6 that the transfer function from u_r to e is given by $(I - GK)^{-1}R$. Furthermore, $(I - GK)^{-1}R \in \mathrm{R}(s)$ because $G, K, R \in \mathrm{R}_{ss}$, and $(I - GK)^{-1}R$ is proper because R is proper and K is internally stabilizing.
Now suppose the first condition holds. $\lim_{t\to\infty} e(t) = 0$ then implies that none of the poles of $(I - GK)^{-1}R$ can be unstable. So $(I - GK)^{-1}R$ must be real-rational, proper and stable and therefore the second condition must hold.
Next, suppose the second condition holds. Then any error signal e that is the result of any allowable reference signal $r = Ru_r$ can be found as the response of $(I - GK)^{-1}R$ on impulsive input u_r and a set of initial conditions. As $(I - GK)^{-1}R$ is real-rational, proper and stable we have that any such response must be exponentially decaying to zero: hence, the first condition must be satisfied. □

This proposition thus allows us to achieve the desired tracking objective in the time-domain, by ensuring that a specific transfer function matrix in the frequency-domain is an element of RH_∞. In the sequel we will therefore consider the problem in the frequency-domain.

From section 3.2 and proposition 4.4.1 it can now be easily verified that the tracking objective would be achieved if one could find any internally stabilizing controller $K \in \mathrm{R}_{ss}$. When considering H_2 or H_∞ methods to obtain a controller, the following two design goals would be pursued:

- internal stability of the closed loop system,
- minimization of the transfer from disturbance input u_r to the error signals z_1 and the weighted control inputs z_2.

However, there are two problems in finding and interpreting such a controller:

- even if the model of the physical system G is stabilizable and detectable, the standard plant of figure 4.6 is in general not internally stabilizable; the RSG is uncontrollable from the control inputs u,
- as internal stability of the closed-loop system is sufficient to obtain the tracking objective, it is not clear what the effect of minimization of the transfer

from u_r to z is, especially when other control design objectives are added to the plant.

The first problem can be dealt with by defining a form of stability that is slightly less restrictive than internal stability: tracking stability.

Definition 4.4.2 (tracking stability)
Consider the tracking control configuration in figure 4.6, and let $G, K, R \in \mathrm{R_{ss}}$ and $W \in \mathrm{RH}_\infty$ be given, with R an appropriate RSG with poles on the imaginary axis. Then a controller K is said to achieve tracking stability *of the closed loop system if:*

- *the three transfer function matrices from u_r, v_1 and v_2 to y are in* RH_∞,
- *the two transfer function matrices from v_1 and v_2 to u are in* RH_∞.

Note that, in comparison with the definition of internal stability, we now allow the transfer function from u_r to u—and with that in general also z_2—to be unstable. In the next subsection we will derive a necessary and sufficient solvability condition for the existence of controllers that achieve tracking stability. After that, we will show that, under mild assumptions on the weight function W, we can use standard optimization techniques to find an 'optimal' controller.

The second problem is to give a correct interpretation of the 'optimality' of the resulting controller, especially when we extend the control configuration to more general problems. This will appear to be a problem of carefully defining extra disturbance inputs, control objective outputs and weight functions, and will be the subject of section 4.7.

4.4.2 Solvability of the tracking stability problem

First we will consider a necessary and sufficient condition for the existence of a controller that achieves tracking stability in the following theorem:

Theorem 4.4.3 solvability of the tracking stability problem
Consider the tracking control configuration in figure 4.6, and let $G, R \in \mathrm{R_{ss}}$ and $W \in \mathrm{RH}_\infty$ be given, with R an appropriate RSG with poles on the imaginary axis. Furthermore, let $G = N_G D_G^{-1}$ and $R = N_R D_R^{-1}$ be right coprime fractions. Then there exists a proper, real rational controller K that achieves tracking stability if and only if there exist polynomial matrices $Q, L \in \mathrm{R}[s]$ such that

$$N_G Q + L D_R = N_R \tag{4.11}$$

Proof:

First we will prove necessity for which we only need to consider the closed loop transfer function from u_r to y.

From figure 4.6 and equation 4.10 it is clear that the open loop transfer from u_r and u to y can be found as:

$$y = [-R \;\; G] \begin{bmatrix} u_r \\ u \end{bmatrix} \tag{4.12}$$

We will prove that the condition of equation 4.11 is necessary for the existence of a controller K that uses measurements y and control inputs u and that stabilizes the closed loop transfer function from u_r to y.

First take right coprime fractions of both R and G:

$$\begin{aligned} G &=: N_G D_G^{-1} \\ R &=: N_R D_R^{-1} \end{aligned} \tag{4.13}$$

We can then set up a PMD for $[-R \;\; G\,]$ (index 'ol' stands for 'open loop'):

$$\Sigma_{\text{ol}} := \left[\begin{array}{cc|cc} D_R & 0 & I & 0 \\ 0 & D_G & 0 & I \\ \hline N_R & -N_G & 0 & 0 \end{array}\right] \tag{4.14}$$

Note that the poles represented by D_R are uncontrollable from the control inputs u and therefore invariant under feedback.

Now any controller K can be given as a polynomial fraction: $K = N_K D_K^{-1}$ such that the system matrix for the closed loop system becomes:

$$\Sigma_{\text{cl}} := \left[\begin{array}{ccc|c} D_R & 0 & 0 & I \\ 0 & D_G & -N_K & 0 \\ N_R & -N_G & D_K & 0 \\ \hline N_R & -N_G & 0 & 0 \end{array}\right] \tag{4.15}$$

Obviously, the imaginary poles represented by the zeros of D_R are controllable from u_r and will result in an unstable closed-loop system, unless they turn out to be unobservable in y. We may therefore state that, given any allowable controller, all zeros of D_R must appear to be output decoupling zeros.

From subsection 2.4.5 we then have that Σ_{cl} must be strictly system equivalent with Σ_s defined in equation 2.39 with $T_{11} := D_R$. Hence, we have from equation 2.42 that there must exist polynomial matrices Q_1, Q_2, L_1, L_2 and L_3 of appropriate dimensions such that:

$$\left[\begin{array}{ccc} 0 & D_G & -N_K \\ N_R & -N_G & D_K \\ \hline N_R & -N_G & 0 \end{array}\right] \begin{bmatrix} I \\ Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} D_R \tag{4.16}$$

The last row of this equation is independent of the choice of controller: with $Q := Q_1$ and $L := L_3$ this immediately shows necessity of equation 4.11.

The proof of sufficiency starts with the assumption that condition 4.11 holds and will result in the basis for the construction of all controllers later on.
So suppose we have some $L, Q \in \mathbb{R}[s]$ such that $N_G Q + L D_R = N_R$. With $G = N_G D_G^{-1}$ we then have $G D_G Q = N_R - L D_R$ and with D_R invertible this results in:

$$G D_G Q D_R^{-1} = R - L \tag{4.17}$$

Now define:

$$M := D_G Q D_R^{-1} =: \tilde{D}_M^{-1} \tilde{N}_M \tag{4.18}$$

with $(\tilde{D}_M, \tilde{N}_M)$ a left coprime fraction and $\tilde{D}_M^{-1}$ proper (this can be ensured by making $\tilde{D}_M$ row-reduced: see section 2.4).
Then the first important fact to prove is that there is no cancellation of any of the poles of M (i.e. $\tilde{D}_M^{-1}$) in forming the product GM. According to proposition 2.4.8, this can be verified by checking whether $(\tilde{D}_M D_G, N_G)$ is right coprime, i.e. we must check whether $\begin{bmatrix} \tilde{D}_M(p) D_G(p) \\ N_G(p) \end{bmatrix}$ has full column rank for all $p \in \mathbb{C}$.
Now consider:

$$GM = N_G D_G^{-1} D_G Q D_R^{-1} = N_G Q D_R^{-1} \tag{4.19}$$

Due to right coprimeness of (D_R, N_R) and equation 4.11 we have right coprimeness of $(D_R, N_G Q + L D_R)$. Hence, with equation 2.26, there exist matrices $V_1, V_2 \in \mathbb{R}[s]$ such that:

$$[V_1 \; V_2] \begin{bmatrix} D_R \\ N_G Q + L D_R \end{bmatrix} = I \iff [V_1 + V_2 L \; V_2] \begin{bmatrix} D_R \\ N_G Q \end{bmatrix} = I \tag{4.20}$$

This implies that $(D_R, N_G Q)$ is right coprime.

Next, suppose that $[(\tilde{D}_M(p)D_G(p))' \; N_G(p)']'$ does not have full column rank for all $p \in$ C: then, there must exist a $p \in$ C for which there exists a non-zero complex-valued vector x_p such that:

$$\begin{bmatrix} \tilde{D}_M(p)D_G(p) \\ N_G(p) \end{bmatrix} x_p = 0 \tag{4.21}$$

Now, $N_G(p)x_p = 0$ implies $\tilde{x}_p := D_G(p)x_p \neq 0$ (due to right coprimeness of (D_G, N_G)), i.e. we have $\tilde{D}_M(p)\tilde{x}_p = 0$. With $M = \tilde{D}_M^{-1}\tilde{N}_M$ this implies that $\tilde{x}_p$ must be an output direction of pole p of M (definition 2.4.4). To comply with equation 4.18, there must therefore exist some non-zero vector $\tilde{\tilde{x}}_p$, such that $\tilde{x}_p = D_G(p)Q(p)\tilde{\tilde{x}}_p$ and $D_R(p)\tilde{\tilde{x}}_p = 0$. Hence, $D_G(p)Q(p)\tilde{\tilde{x}}_p = \tilde{x}_p = D_G(p)x_p$ and with $\tilde{x}_p \neq 0$: $x_p = Q(p)\tilde{\tilde{x}}_p$. Right coprimeness of (D_R, N_GQ) and $D_R(p)\tilde{\tilde{x}}_p = 0$ now imply that $N_G(p)Q(p)\tilde{\tilde{x}}_p \neq 0$, i.e. $N_G(p)x_p \neq 0$, which is in contradiction with equation 4.21. Hence, there cannot be any cancellations of poles of M in forming the product GM

Next, take any internally stabilizing controller, which we will denote as $\tilde{K}$, for the product $G\tilde{D}_M^{-1}$. From the definition of internal stability (definition 3.2.4) we then must have:

$$\left.\begin{array}{rcl} S_1 & := & (I - \tilde{K}G\tilde{D}_M^{-1})^{-1} \\ S_2 & := & (I - \tilde{K}G\tilde{D}_M^{-1})^{-1}\tilde{K}, \\ S_3 & := & (I - G\tilde{D}_M^{-1}\tilde{K})^{-1} \\ S_4 & := & (I - G\tilde{D}_M^{-1}\tilde{K})^{-1}G\tilde{D}_M^{-1} \end{array}\right\} \in \mathrm{RH}_\infty \tag{4.22}$$

With this we will prove that for any such $\tilde{K}$ the controller $K := \tilde{D}_M^{-1}\tilde{K}$ achieves tracking stability.

To do this we will construct all transfer functions mentioned in definition 4.4.2 that can be found from examination of figure 4.6:

$$\begin{array}{rcl} u_r \rightarrow y: & -R - GK(I-GK)^{-1}R & = & -(I-GK)^{-1}R \\ v_1 \rightarrow y: & G + GK(I-GK)^{-1}G & = & (I-GK)^{-1}G \\ v_2 \rightarrow y: & I + GK(I-GK)^{-1} & = & (I-GK)^{-1} \\ v_1 \rightarrow u: & I + K(I-GK)^{-1}G & = & (I-KG)^{-1} \\ v_2 \rightarrow u: & & & K(I-GK)^{-1} \end{array} \tag{4.23}$$

First note that properness of these transfer functions is guaranteed by properness of R, G and K. For stability we will consider each transfer function separately:

$u_r \longrightarrow y$:

$$\begin{aligned}(I-GK)^{-1}R &= (I-G\tilde{D}_M^{-1}\tilde{K})^{-1}\{GM+L\}\\ &= (I-G\tilde{D}_M^{-1}\tilde{K})^{-1}\{G\tilde{D}_M^{-1}\tilde{N}_M+L\}\\ &= S_4\tilde{N}_M+S_3L\end{aligned}$$

Hence $(I-GK)^{-1}R \in \mathrm{RH}_\infty$.

$v_1 \longrightarrow y$:

$$\begin{aligned}(I-GK)^{-1}G &= (I-G\tilde{D}_M^{-1}\tilde{K})^{-1}G\\ &= (I-G\tilde{D}_M^{-1}\tilde{K})^{-1}G\tilde{D}_M^{-1}\tilde{D}_M\\ &= S_4\tilde{D}_M\end{aligned}$$

Hence $(I-GK)^{-1}G \in \mathrm{RH}_\infty$.

$v_2 \longrightarrow y$:

$$\begin{aligned}(I-GK)^{-1} &= (I-G\tilde{D}_M^{-1}\tilde{K})^{-1}\\ &= S_3\end{aligned}$$

Hence $(I-GK)^{-1} \in \mathrm{RH}_\infty$.

$v_1 \longrightarrow u$:

$$\begin{aligned}(I-KG)^{-1} &= (I-\tilde{D}_M^{-1}\tilde{K}G)^{-1}\\ &= \tilde{D}_M^{-1}(I-\tilde{K}G\tilde{D}_M^{-1})^{-1}\tilde{D}_M\\ &= \tilde{D}_M^{-1}S_1\tilde{D}_M\end{aligned}$$

So all unstable poles are poles of $\tilde{D}_M^{-1}$.
As we have established that none of these poles are cancelled in forming the product $G\tilde{D}_M^{-1}$, we have: $(I-KG)^{-1} \in \mathrm{RH}_\infty$ if $G\cdot(I-KG)^{-1} = (I-GK)^{-1}G \in \mathrm{RH}_\infty$, which is the transfer function matrix from v_1 to y already considered above.
Hence $(I-KG)^{-1} \in \mathrm{RH}_\infty$.

$v_2 \longrightarrow u$:

$$\begin{aligned}K(I-GK)^{-1} &= \tilde{D}_M^{-1}\tilde{K}(I-G\tilde{D}_M^{-1}\tilde{K})^{-1}\\ &= \tilde{D}_M^{-1}(I-G\tilde{D}_M^{-1}\tilde{K})^{-1}\tilde{K}\\ &= \tilde{D}_M^{-1}S_2\end{aligned}$$

Again all unstable poles are poles of $\tilde{D}_M^{-1}$:

$$K(I-GK)^{-1} \in \mathrm{RH}_\infty \ \text{ if } \ GK(I-GK)^{-1} \in \mathrm{RH}_\infty:$$

$$\begin{aligned}GK(I-GK)^{-1} &= G\tilde{D}_M^{-1}\tilde{K}(I-G\tilde{D}_M^{-1}\tilde{K})^{-1}\\ &= (I-G\tilde{D}_M^{-1}\tilde{K})^{-1}G\tilde{D}_M^{-1}\tilde{K}\\ &= S_4\end{aligned}$$

Hence $K(I-GK)^{-1} \in \mathrm{RH}_\infty$.

This now establishes that the controller $K = \tilde{D}_M^{-1}\tilde{K}$ indeed achieves tracking stability, proving sufficiency of equation 4.11 □

From this proof it is clear that $\tilde{D}_M^{-1}$ becomes an essential part of the tracking stability controller and constitutes the appropriate servo compensator (see definition 4.2.1).

An earlier solvability condition which is equivalent to equation 4.11 is given by Bengtsson (1977) in the sense that there exist polynomial matrices $X, Y \in \mathrm{R}[s]$ such that:

$$\tilde{N}_G X + Y\tilde{D}_S = I \tag{4.24}$$

in which $\tilde{D}_S$ is found from:

$$\tilde{D}_S^{-1}\tilde{N}_S := \tilde{D}_G\tilde{D}_R^{-1} \tag{4.25}$$

and the servo compensator is given by $X\tilde{D}_S^{-1}$. The derivation of this condition is based on the a priori knowledge that an internal model must be present in the product GK to solve the asymptotic tracking problem. For this purpose an internal model is characterized as follows:

Definition 4.4.4 characterization of an internal model
Given transfer function matrices G and R. G contains an internal model of R if $\tilde{D}_R$ is a right divisor of $\tilde{D}_G$, i.e. $\tilde{D}_G\tilde{D}_R^{-1} \in \mathrm{R}[s]$.

Next, it is proven that the product of G with any appropriate servo compensator must contain an internal model of R, and that indeed $S := X\tilde{D}_S^{-1}$ has this property. An internally stabilizing controller $\tilde{K}$ for this product GS will then solve the asymptotic tracking problem. Note that S becomes a part of the controller: the product $K := S\tilde{K}$ constitutes the actual controller for G.
The advantage of condition 4.11 is that it is directly given in terms of MFDs of R and G: there is no need to define S or M a priori. The fact that $\tilde{D}_M^{-1}$ is an appropriate servo compensator results from the fact that $K = \tilde{D}_M^{-1}\tilde{K}$ achieves tracking stability. Note that the internal model principle must be satisfied: $G\tilde{D}_M^{-1}$ must contain an internal model of R. Furthermore, note that the proof of theorem 4.4.3 is relatively straightforward and fully self contained: only basic linear system theoretic tools are used (PMDs, coprime MFDs, SSE transformation, loop transformations and properties of poles and zeros).
Next, we can use the aforementioned characterization of an internal model to prove the following corollary:

Corollary 4.4.5 minimality of the servo compensator
Given any servo compensator S and controller $\tilde{K}$ such that the tracking stability problem considered in theorem 4.4.3 is solved by controller $K := S\tilde{K}$. Then the dynamic order of S must at least be equal to that of $\tilde{D}_M^{-1}$ as defined by equations 4.11 and 4.18.

Proof:
Suppose S is the TFMD of an arbitrary compensator and consider the tracking control configuration of figure 4.6 with G substituted by the product GS. Next, denote an LCF of GS as: $\tilde{D}_{GS}^{-1}\tilde{N}_{GS}$. Then GS contains an internal model of R if and only if $\tilde{D}_{GS}\tilde{D}_R^{-1} \in \mathrm{R}[s]$. This implies that the order of GS must at least be equal to the order of R.
Now consider $GM = N_G Q D_R^{-1}$ as given by equation 4.19. With $(D_R, N_G Q)$ right coprime this implies that the order of GM is equal to that of R. Furthermore, as $(\tilde{D}_M D_G, N_G)$ is right coprime, we have that there are no cancellations of poles of M in forming the product GM. Hence, M must be a servo compensator of minimal order. Finally, as $\tilde{D}_M^{-1}$ has an order equal to that of M, $\tilde{D}_M^{-1}$ must be a servo compensator of minimal order. □

Remark 4.4.6
We have not established *uniqueness* of the minimal servo compensator: there is no guarantee that solutions Q and L to equation 4.11 are unique and, furthermore, also $\tilde{D}_M$ as part of a left coprime fraction is not unique (see section 2.4). Intuitively, it is not expected that the resulting freedom in constructing $\tilde{D}_M$ has an important effect on the behaviour of the closed-loop system. However, this could be a subject of further investigation.

4.4.3 H$_2$ and H$_\infty$ optimal solutions of the tracking stability problem

Given the standard control configuration for the tracking problem in figure 4.6, we have from theorem 4.4.3 that whenever the tracking problem is solvable we are able to find polynomial matrices Q and L satisfying condition 4.11. This then allows us to construct a *modified standard plant* as follows:

1. take left coprime fractions $G =: N_G D_G^{-1}$ and $R =: N_R D_R^{-1}$,

2. define $M := D_G Q D_R^{-1}$,

3. take left coprime fraction $M =: \tilde{D}_M^{-1}\tilde{N}_M$,

4. define the modified standard plant $\tilde{P}$ as:

$$\tilde{P} := \left[\begin{array}{c|c} -R & G\tilde{D}_M^{-1} \\ 0 & W\tilde{D}_M^{-1} \\ \hline -R & G\tilde{D}_M^{-1} \end{array}\right] \tag{4.26}$$

   (compare with equation 4.10),

5. define a new vector of control inputs $\tilde{u}$.

We thus arrive at the control configuration given in figure 4.7, in which a new vector of auxiliary signals v_3 is added to check internal stability.

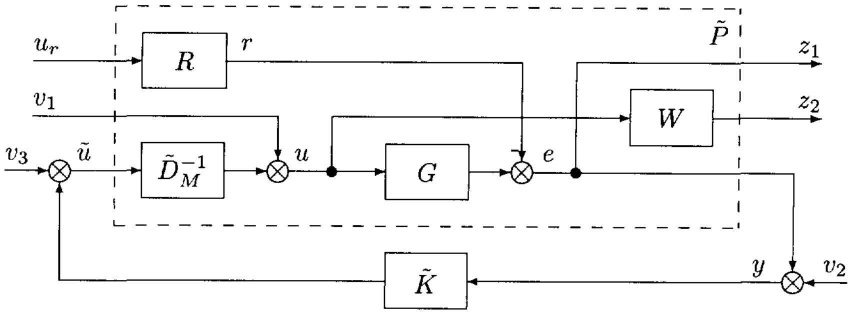

Fig. 4.7: Modified control configuration for the tracking problem

From the proof of sufficiency of theorem 4.4.3 we immediately have that any internally stabilizing controller $\tilde{K}$ for the product $G\tilde{D}_M^{-1}$ will provide a controller $K := \tilde{D}_M^{-1}\tilde{K}$ that achieves tracking stability for the original problem, i.e. in the standard control configuration for the tracking problem given in figure 4.6. This provides us with a large class of controllers that achieve tracking stability: we will therefore restrict our search for an *optimal* controller to controllers that may be constructed this way.

Hence, we will consider a controller that achieves tracking stability for the original problem in figure 4.6 to be H_2 or H_∞ optimal if it is constructed from an H_2 or H_∞ optimal controller for the modified control configuration of figure 4.7. This implies that we are looking for an internally stabilizing controller $\tilde{K}$, that minimizes either the 2-norm or the ∞-norm of the closed-loop transfer function matrix from u_r to z. However, if we want to be able to apply the standard robust control methods discussed in chapter 3 to find such a controller, the assumptions A1 through A4 given in section 3.3 must be satisfied. We will therefore consider some slight changes to both the original problem formulation and the associated modified control configuration.

First, to satisfy assumption A1, we should have that any internally stabilizing controller $\tilde{K}$ for $G\tilde{D}_M^{-1}$ is also internally stabilizing for the modified standard plant $\tilde{P}$ (see section 3.2). For this we have to check stability of the transfer functions from all external signals to z_1 and z_2, which is ensured by the property of tracking stability, except for that from u_r to z_2 (see definitions 3.2.4 and 4.4.2). This transfer function can be found from figure 4.7 and using equation 4.17, 4.18 and 4.22 as:

$$\begin{aligned}
&-W\tilde{D}_M^{-1}\tilde{K}(I - G\tilde{D}_M^{-1}\tilde{K})^{-1}R &= \\
&-W\tilde{D}_M^{-1}\tilde{K}(I - G\tilde{D}_M^{-1}\tilde{K})^{-1}\{G\tilde{D}_M^{-1}\tilde{N}_M + L\} &= \\
&-W\tilde{D}_M^{-1}(I - \tilde{K}G\tilde{D}_M^{-1})^{-1}\tilde{K}\{G\tilde{D}_M^{-1}\tilde{N}_M + L\} &= \\
&-W\tilde{D}_M^{-1}\{(I - \tilde{K}G\tilde{D}_M^{-1})^{-1}\tilde{N}_M - \tilde{N}_M + (I - \tilde{K}G\tilde{D}_M^{-1})^{-1}\tilde{K}L\} &= \\
&-W\tilde{D}_M^{-1}\{S_1\tilde{N}_M - \tilde{N}_M + S_2L\}
\end{aligned} \tag{4.27}$$

We can therefore ensure stabilizability of $\tilde{P}$ by selecting W as follows:

$$W := \tilde{W}\tilde{D}_M, \qquad \tilde{W} \in \mathrm{RH}_\infty \tag{4.28}$$

This choice can be interpreted as the desire to minimize the actuator effort as represented by $\tilde{u}$ rather than u. We must allow u to contain persistent signals as represented by the zeros of $\tilde{D}_M$, to be able to compensate the corresponding allowable reference signal generated by R. In view of this, this choice of W is not a restriction, but a necessary structural property for solvability of the tracking problem.

Remark 4.4.7

Assuming that W has been chosen according to equation 4.28, it can be verified that all real rational controllers K achieving tracking stability with servo compensator $\tilde{D}_M^{-1}$ and for which $\tilde{K} := \tilde{D}_M K$ is proper, can be found by parametrizing all internally stabilizing controllers $\tilde{K}$ for $\tilde{P}$ and taking $K := \tilde{D}_M^{-1}\tilde{K}$. We can guarantee that the resulting controller K is proper, by making sure that $\tilde{D}_M$ is row-reduced (definition 2.4.3). If there is any K for which $\tilde{K}$ is not proper, we may approximate it by means of a proper $\tilde{K}$.

Next, to satisfy assumptions A2 through A4, we must extend our problem formulation with an extra disturbance input w_1 and with a proper and stable weight function W_1. If this extension would only be motivated for the technical reason of complying with assumptions A2 through A4, it would seem restrictive with respect to the optimality of H_2 or H_∞ controllers. However, in practical situations w_1 is an essential part of the problem formulation as it can be used to account for measurement noise. Hence, in the sequel we will extend the standard control configuration for the tracking problem given in figure 4.6 by including w_1 and W_1 and we will consider the resulting structure given in figure 4.8 as the new, most basic, standard control configuration for the tracking problem. As a matter of notation, we will refer to this configuration as configuration I. Standard plant P_{I}

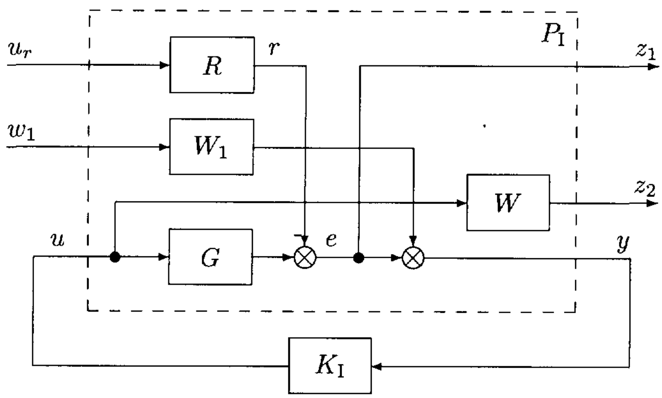

Fig. 4.8: Standard control configuration for the tracking problem with measurement noise: configuration I

can be given as:

$$P_{\mathrm{I}} := \left[\begin{array}{cc|c} -R & 0 & G \\ 0 & 0 & W \\ \hline -R & W_1 & G \end{array}\right] \qquad (4.29)$$

This extension of the original problem formulation will, of course, also result in an extension of the modified control configuration of figure 4.7. The resulting structure is given in figure 4.9 and will be referred to as configuration II. Standard

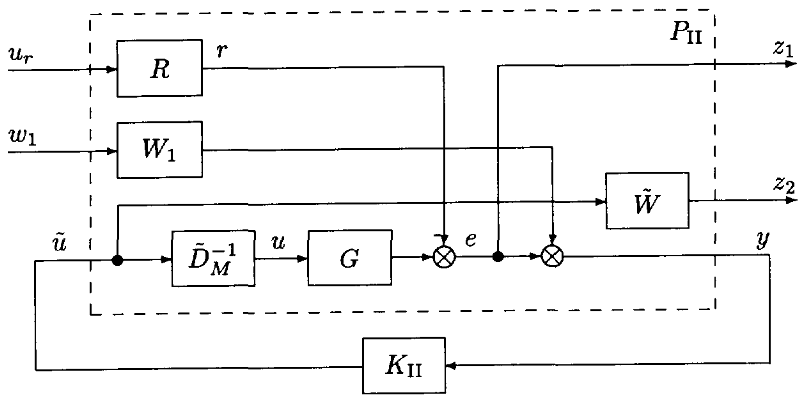

Fig. 4.9: Modified control configuration for the tracking problem with measurement noise: configuration II

plant P_{II} can be given as:

$$P_{\text{II}} := \left[\begin{array}{cc|c} -R & 0 & G\tilde{D}_M^{-1} \\ 0 & 0 & \tilde{W} \\ \hline -R & W_1 & G\tilde{D}_M^{-1} \end{array}\right] \tag{4.30}$$

Note that in comparison with figure 4.7 we have made use of our choice of W: $W := \tilde{W}\tilde{D}_M$, to simplify this configuration: W_1 and $\tilde{W}$ can now easily be chosen such that the assumptions A1 through A4 are satisfied.

Hence, configuration I may be considered as the minimal configuration for which the combination of an asymptotic tracking problem and an H_2 or H_∞ optimal control problem makes sense and can be solved by the standard robust control methods discussed in chapter 3. Further extension of this configuration to include other known disturbance inputs and/or robustness objectives may readily be performed: the example given in chapter 5 will illustrate this.

Remark 4.4.8

Although internal stabilizability of standard plant P_{II} has already been established by means of the tracking stability property and the specific choice of W, it is now possible to rewrite configuration II into a form such that this result is obtained immediately. From equation 4.17 we have:

$$R = GD_GQD_R^{-1} + L = GM + L = G\tilde{D}_M^{-1}\tilde{N}_M + L \tag{4.31}$$

Substitution into configuration II results in figure 4.10 with standard plant $\tilde{P}_{\text{II}}$

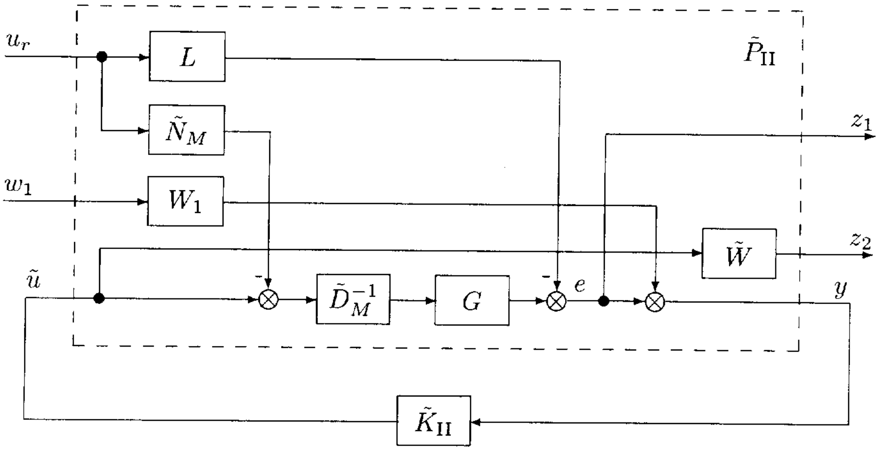

Fig. 4.10: Configuration II after substitution of R with $GM + L$

defined as:

$$\tilde{P}_{II} := \left[\begin{array}{cc|c} -GM - L & 0 & G\tilde{D}_M^{-1} \\ 0 & 0 & \tilde{W} \\ \hline -GM - L & W_1 & G\tilde{D}_M^{-1} \end{array}\right] \tag{4.32}$$

Although L and $\tilde{N}_M$ are polynomial matrices, properness is ensured by properness of R. Furthermore, L, $\tilde{N}_M$, W_1 and $\tilde{W}$ do not introduce unstable poles, so that stabilizability of $\tilde{P}_{II}$ immediately follows.

Remark 4.4.9

As we have proven that any internally stabilizing controller for $G\tilde{D}_M^{-1}$ will obtain the asymptotic tracking objective, it is not necessary to consider u_r as an external disturbance input for the H_2 or H_∞ problem: after we have constructed $\tilde{D}_M^{-1}$ we may premultiply R with any finite real constant, including zero, without affecting the asymptotic tracking properties of the resulting controller. However, it is suggested that the incorporation of R, possibly premultiplied with a non-zero real constant as an extra parameter, may be very useful to specify the closed-loop behaviour with respect to reference signals that are *close* to the ones specified in the RSG. For instance, if asymptotic tracking of step-signals is desired, it is possible to specify a minimal attenuation of slowly varying sinusoid signals by considering the ∞-norm of the closed-loop transfer from u_r to z_1. An example of this approach is given in chapter 5.

The main result of this section may now be given by means of the following theorem:

Theorem 4.4.10 H_2 or H_∞ optimal solution of the tracking stability problem
Given configuration I in figure 4.8, with $G, R \in R_{ss}$ *and* $W \in RH_\infty$*, and with* R *an appropriate RSG with poles on the imaginary axis. Furthermore, let condition 4.11 be satisfied and let configuration II be set up as in figure 4.9 such that assumptions A1 through A4 are satisfied.*
Then:

- *any internally stabilizing controller* K_{II} *for the modified standard plant* P_{II} *will give rise to a controller* $K_{\mathrm{I}} := \tilde{D}_M^{-1} K_{\mathrm{II}}$ *for the original standard plant* P_{I} *that achieves tracking stability,*
- *there exists an* H_2 *or* H_∞ *optimal controller* K_{II} *for the modified standard plant* P_{II} *that minimizes* $\|T_{wz}\|_2$ *or* $\|T_{wz}\|_\infty$ *respectively; such a controller will give rise to a controller* $K_{\mathrm{I}} := \tilde{D}_M^{-1} K_{\mathrm{II}}$ *for the original standard plant* P_{I} *that achieves tracking stability and results in a transfer function* T_{wz} *that is equal to that obtained in configuration II.*

The proof of this theorem immediately results from the development given in this section.

4.5 Extensions of the servo compensator

This section will explore two cases in which the applied servo compensator is extended with respect to the minimal servo compensator derived in the previous section. The first one will appear to be of importance when considering the implementation of a servo compensator. The second is the result of the robust servomechanism problem as it is usually considered in literature; we will show that the procedure given here can be seen as a generalization of this problem.

4.5.1 The extended tracking problem

For the solution of an H_2 or H_∞ control problem it is convenient to set up a state-space model of the modified standard plant of figure 4.9. This makes it necessary to find a state-space description of the servo compensator: $\tilde{D}_M^{-1} =: [A, B, C, D]_M$, such that its outputs can easily be connected to the inputs of the state-space model of the system to be controlled: G. Now we have that in the implementation of the final controller the servo compensator, obviously, is incorporated in the controller and should be fitted in digital or analog hardware. This then implies that after implementing the controller all signals within the servo compensator are easily available. Especially when considering digital implementation, for instance in a Digital Signal Processor, all states of the servo compensator, as well as the calculated control inputs, are represented by internal memory locations of the processor. Hence, we may assume that all servo compensator states are directly accessible by the controller to be designed (for configuration II given in figure 4.9). This accessibility can be expressed by replacing the servo compensator's input matrix B_M and feedthrough matrix D_M as follows:

$$B_M := [I_n \ 0], \quad D_M := [0 \ I_q] \tag{4.33}$$

with n the dimension of A_M and q the number of control inputs of G. This procedure has two advantages:

1. a larger class of controllers is parametrized; also controllers for which $\tilde{D}_M K$ is non-proper are possible and it is conjectured that for this case indeed *all* controllers can be found,
2. if we redefine $\tilde{W}$ such that all input signals of the extended servo compensator are weighted and available in z_2 we have that the amplitude of all servo compensator states and outputs can be checked and influenced separately.

However, the disadvantage is that the effect of all weight functions as given in figure 4.9 is modified such that a new interpretation of these weight functions (not only $\tilde{W}$) is necessary.

4.5.2 The robust tracking problem

Whereas the previous subsection discussed an extension of the servo compensator without changing its order, it is clear that if we use this extended structure and *do* add extra dynamics to it, the set of signals for which the tracking objective is achieved can only increase, while still including the original set. A relevant example of this appears when we consider the solution to the robust tracking problem:

1. start with any reference signal generator R describing a set of persistent signals,
2. take the smallest scalar polynomial $\alpha_R \in \mathrm{R}[s]$ such that $\alpha_R I_p \tilde{D}_R^{-1} \in \mathrm{R}[s]$ with p the number of outputs of G: note that $\frac{1}{\alpha_R} I_p$ contains an internal model of R (see definition 4.4.4),

3. take $S := \frac{1}{\alpha_R} I_q$ as servo compensator with q the number of control inputs of G,

4. find an internally stabilizing controller $\tilde{K}$ for the product GS, and take $K := S\tilde{K}$ as the final controller.

Clearly we now have that any directional information in the servo compensator is discarded; any signal that can be obtained by taking any set of initial conditions for the RSG and arbitrarily adding the resulting state trajectories, can now be generated by the servo compensator in any direction of the input space of G. This independence of directional information also accounts for the well known robustness properties of this type of servo compensator; if the directional properties of G change, due to some parameter variation or added dynamics, there is no influence on the tracking objective as long as the closed loop system is stable. Furthermore, because S contains an internal model of R, the internal model property cannot be lost due to any changes in G. The original derivation of this result is due to Davison and Goldenberg (1975), who introduced it as the solution to the 'robust servomechanism problem'; a complete derivation and discussion can, for instance, be found in Desoer and Wang (1980).

In our approach to the tracking problem, it would be desirable that the aforementioned robustness properties would result from the correct definition of the RSG. Fortunately, this is possible by taking $R := \frac{1}{\alpha_R} I_p$ as RSG, rather than directly taking $S := \frac{1}{\alpha_R} I_q$ as servo compensator. An RCF of R is then given by $D_R := \alpha_R I_p$, $N_R := I_p$, and from equation 4.18 we have:

$$M := D_G Q \alpha_R^{-1} = \alpha_R^{-1} D_G Q \tag{4.34}$$

Hence, we can take $\tilde{D}_M := \alpha_R I_q$, $\tilde{N}_M := D_G Q$, which results in the desired servo compensator. To analyze whether this is allowable we have to check the solvability condition of equation 4.11:

$$N_G Q + L\alpha_R = I_p \tag{4.35}$$

From this we immediately get the well known solvability conditions for the robust asymptotic tracking problem:

- G must have full row rank (as a real-rational matrix) ($\text{rank}(N_G(s)) = p$),

- G may not have any zeros corresponding to poles of the RSG ($\text{rank}(N_G(z_i)) = p$ for all z_i such that $\alpha_R(z_i) = 0$).

When considering the approach suggested in this thesis with respect to the known solution to the robust servomechanism problem, a few remarks can now be made.

Remark 4.5.1

From equation 4.34 we have that the servo compensator $S := \frac{1}{\alpha_R} I_q$ can only be minimal if $\alpha_R I_q$ and $D_G Q$ are left coprime. As $\alpha_R I_p$ and Q are right coprime due to equation 4.35 (which is a Bezout equation: see equation 2.26), non-minimality cannot occur as a result of a zero of Q. Hence, it can only occur when G already contains one or more poles of S or when the number of control inputs is larger than the number of error signals ($q > p$). In these cases, the surplus of poles can be removed from the servo compensator by taking an LCF of $\alpha_R^{-1} D_G Q$. However, the resulting minimal servo compensator will obviously not be robust against uncertainties that affect the location of the poles of G. Hence, we can consider S as a non-minimal servo compensator, extended from a minimal servo compensator, for the purpose of obtaining robustness.

Remark 4.5.2

When considering any given servo compensator S, we can parametrize all RSGs that may be associated with S as:

$$R := GSQ_1 + Q_2 \tag{4.36}$$

with $Q_1, Q_2 \in \mathrm{R}[s]$ free parameters as long as R remains proper (see equations 4.17 and 4.18). Note that with $S =: N_S D_S^{-1}$ a right coprime fraction, (D_S, Q_1) does not have to be coprime; in that case the considered servo compensator is non-minimal.

Remark 4.5.3

Equation 4.36 shows that for an arbitrary servo compensator the expected persistent reference signals are 'filtered' by G: as there is no physical ground to assume this, our approach of *first* defining the RSG and *then* constructing an appropriate servo compensator seems more natural. The free parameters Q_1 and Q_2, that are implicitly introduced by determining the servo compensator directly, can be seen as extra weight functions in the standard plant description. The influence of this issue on the final result remains largely unconsidered in literature, because it does not affect the tracking objective as such. Within the standard plant setting of figure 4.9 it is clear that they may very well affect the result with respect to the other control objectives as reflected in z_1 and z_2 and the weight functions $\tilde{W}$ and W_1.

Remark 4.5.4

From the point of view of the tracking objective the robust servo compensator seems very useful, but the introduction of extra unstable dynamics in the standard plant will usually have a negative effect on other control objectives. Furthermore, if more information is known on the structural properties of for instance parameter variations for which robust tracking is to be achieved, a 'complete' robust servo compensator is usually not necessary (Grasselli and Longhi 1991).

Our approach enables us to ensure robustness of the asymptotic tracking property as well as robust stability and performance in the face of such structured uncertainties, by means of the robust control methods presented in chapter 3, i.e. uncertainty modelling and structured singular value analysis and design. An example of the application of these methods to obtain robustness in spite of the use of a minimal order servo compensator can be found in section 5.4.

4.6 The two-degree-of-freedom tracking problem

In the previous sections the output regulation and tracking problem was considered to be purely a feedback problem. We assumed that all measurement signals available to the controller are error signals, resulting from the difference between reference signals and the actual plant outputs:

$$y := y_G - r \tag{4.37}$$

This enabled us to make use of proposition 4.4.1 and definition 4.4.2 to state that the tracking objective is obtained by any controller that achieves tracking stability. In general, this specific choice of measurement signals may be very restrictive, especially when we take into account that reference signals are usually generated by the same device that calculates the control action. We may therefore assume that these reference signals are directly available to the controller, and that feedforward control could be used to improve the results.

The classical feedback control system as given in figure 1.1 would then be extended to contain a feedforward controller, resulting in the configuration of figure 4.11. As the controller to be designed now clearly consists of two separate blocks, this configuration is usually referred to as the 'two-degree-of-freedom' control structure. The advantages of feedforward control are widely recognized in literature and in practice:

- as long as K_1 is internally stabilizing, the resulting control structure will remain stable for any choice of $K_2 \in \mathrm{RH}_\infty$,

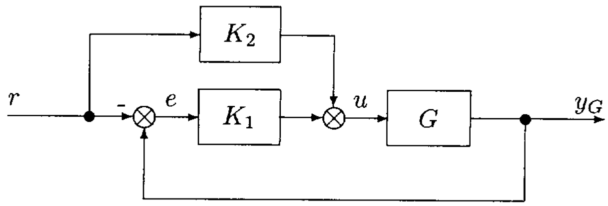

Fig. 4.11: The classical feedback control system extended with feedforward controller

- the reference signals are usually free of disturbances and clearly unaffected by plant uncertainties,
- the reference signals are usually not only known in *form* (the dynamical behaviour of the RSG), but also in *size* (the initial conditions of the RSG).

These advantages allow the safe use of most of the available actuator power to directly compensate the effects of the reference signals, after which the feedback controller K_1 uses the actual measurement signals: $y_G - r$, to account for disturbances and plant uncertainties.

As with the 'one-degree-of-freedom' problem considered in the previous sections, it would now be desirable to be able to synthesize both controllers K_1 and K_2 in a single calculation. Preferrably, it would again be possible to establish a trade-off between several design objectives, using the standard control design structure and robust control methods. In this section a suggestion will be offered, that enables us to do this and is based on the results from the previous sections.
For this consider the configuration in figure 4.12 with the standard plant P_{It} defined as:

$$P_{\mathrm{It}} := \left[\begin{array}{ccc|c} -R & 0 & 0 & G \\ 0 & 0 & 0 & W \\ \hline R & 0 & W_2 & 0 \\ -R & W_1 & 0 & G \end{array}\right] \tag{4.38}$$

In comparison with figure 4.8 we now have that not only the error signals, but also the reference signals are available as measurements. Similar to the interpretation of w_1 as measurement noise, disturbance input w_2 may be used to account for discrepancies between the actual reference signals and the reference signals as they are available to the controller. Obviously, these discrepancies are relatively

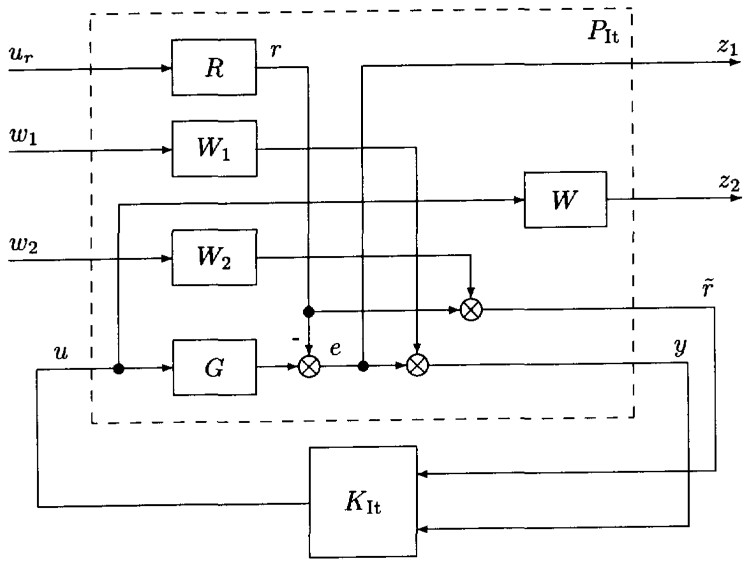

Fig. 4.12: Two-degree-of-freedom control configuration

small, which may be reflected by choosing W_2 accordingly: in practical situations W_2 may be related to equipment noise or quantization noise.

As in the one-degree-of-freedom case, it is impossible to find an internally stabilizing controller for the configuration of figure 4.12. For this reason we will again consider a modified configuration for which internally stabilizing controllers can be found that give rise to controllers that achieve tracking stability in the original configuration. This configuration is given in figure 4.13 with the standard plant $P_{\rm IIt}$ defined as:

$$P_{\rm IIt} := \left[\begin{array}{ccc|c} -R & 0 & 0 & G\tilde{D}_M^{-1} \\ 0 & 0 & 0 & \tilde{W} \\ \hline -R & W_1 & 0 & G\tilde{D}_M^{-1} \\ -R & 0 & W_2 & G\tilde{D}_M^{-1} \end{array}\right] \tag{4.39}$$

In comparison with figure 4.9 we have added a *second* model of the physical system: G. This second model is driven by the actual control inputs and its output signals are compared with the reference signals, just like the actual plant outputs. The essential difference between the actual measurement signal y and the estimated measurement signal $\hat{y}$ is twofold:

- w_1 and W_1 model measurement noise that occurs due to actual sensor noise; w_2 and W_2 reflect noise inside the controller (equipment noise or quantization

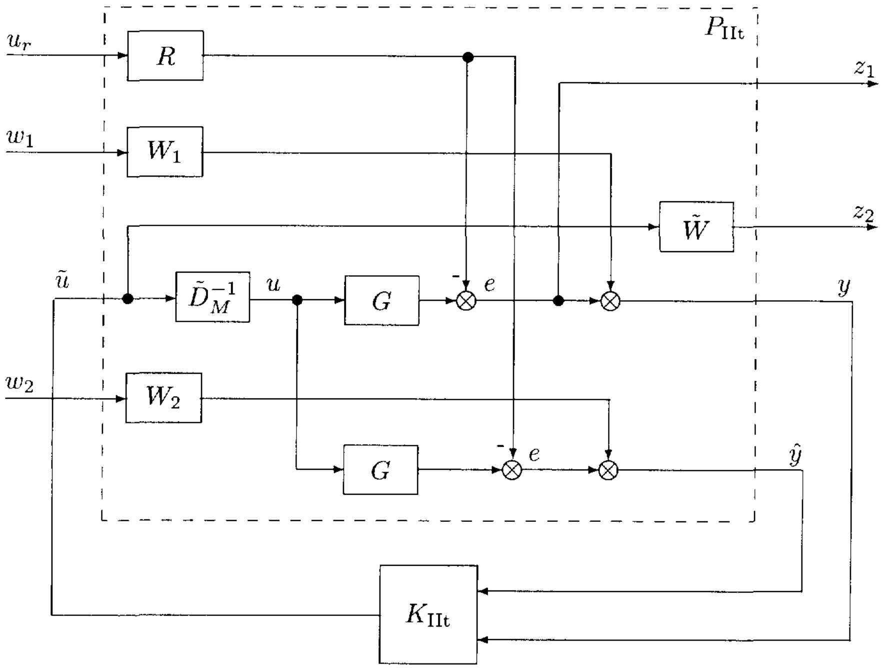

Fig. 4.13: Modified two-degree-of-freedom control configuration

noise) which is usually very small,

- the estimated measurement signal $\hat{y}$ is not influenced by any parameter variations or other uncertainties occurring in the plant.

With the considerations given in subsection 4.4.3, it can be verified that P_{IIt} again complies with all necessary assumptions for application of standard H_2 or H_∞ methods: an internally stabilizing controller K_{IIt} can be designed that achieves an optimal value for either $\|T_{wz}\|_2$ or $\|T_{wz}\|_\infty$. Obviously, once we have found an appropriate controller K_{IIt}, we should not only incorporate the servocompensator, but also the second model of the plant into the controller to obtain a controller K_{It} for the original problem given in figure 4.12. With K_{IIt} partitioned into K_{IIt1} acting on y and K_{IIt2} acting on $\hat{y}$ we get:

$$\begin{aligned} u &= \tilde{D}_M^{-1} K_{\text{IIt1}} y + \tilde{D}_M^{-1} K_{\text{IIt2}} (Gu - \tilde{r}) \\ &= \tilde{D}_M^{-1} K_{\text{IIt1}} y - \tilde{D}_M^{-1} K_{\text{IIt2}} \tilde{r} + \tilde{D}_M^{-1} K_{\text{IIt2}} G u \\ &= (I - \tilde{D}_M^{-1} K_{\text{IIt2}} G)^{-1} \tilde{D}_M^{-1} \cdot \left[\begin{array}{cc} K_{\text{IIt1}} & -K_{\text{IIt2}} \end{array} \right] \left[\begin{array}{c} y \\ \tilde{r} \end{array} \right] \end{aligned} \tag{4.40}$$

Hence, the controller K_{It} for the original problem can be found as:

$$K_{\mathrm{It}} := (I - \tilde{D}_M^{-1} K_{\mathrm{IIt2}} G)^{-1} \tilde{D}_M^{-1} \cdot [\ K_{\mathrm{IIt1}} \quad - K_{\mathrm{IIt2}}\ ] \tag{4.41}$$

Although practical experience with this approach to the two-degree-of-freedom problem is still lacking, it is possible to make some observations.

- In comparison with the one-degree-of-freedom modified standard plant P_{II}, the two-degree-of-freedom modified standard plant P_{IIt} has its order increased with the order of G and W_2. As H_2 and H_∞ methods result in controllers of the order of the standard plant, K_{IIt} will have the order of P_{IIt}. To construct K_{It} we must add the second model of G and the servocompensator $\tilde{D}_M^{-1}$ to K_{IIt}: the order of K_{It} is then given by the order of all weight functions plus two times the order of the servocompensator plus three times the order of G. As the order of G appears only once in K_{I}, we thus have that the order of the two-degree-of-freedom controller K_{It} is increased with respect to that of a comparable one-degree-of-freedom controller K_{I} by two times the order of G plus the order of W_2.

- The feedforward part of controller K_{It} can be found by taking a minimal state-space realization for K_{It}, and determining the part that is uncontrollable from y (i.e. it is only controllable from r). Denoting this part as K_2 and denoting the part of K_{It} that is left after removing K_2 as K_1, we can set up a configuration similar to that of figure 4.11 as given in figure 4.14. Intuitively it may be expected that the upper partition of K_1 is small, such

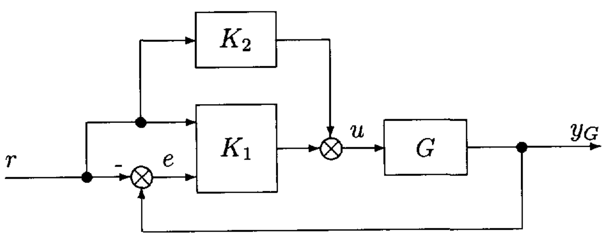

Fig. 4.14: Control configuration with feedback and feedforward parts

that we actually have feedback of the error signal.

- The standard plant P_{IIt} does not account for plant uncertainties or disturbances acting directly on the plant (note: neither does P_{II}; it is not necessary for solving a standard H_2- or H_∞-problem). This implies that there is no need for feedback of noisy measurement signals y, as estimated signals $\hat{y}$ are more accurate (if W_2 is small). The resulting controller K_{It} will therefore only consist of a feedforward controller. Note that we are thus forced to explicitly define robustness demands and system disturbances, if we want to design a realistic controller.

- As the estimated measurement signal $\hat{y}$ is hardly influenced by noise, high gains may be expected in K_{IIt2}: with
  $$u = K_{\mathrm{It2}}\,\tilde{r} = -(I - \tilde{D}_M^{-1}K_{\mathrm{IIt2}}G)^{-1}\tilde{D}_M^{-1}K_{\mathrm{IIt2}}\,\tilde{r}$$
  it is clear that K_{It2} approaches G^{-1} as K_{IIt2} becomes large. Considering that K_{IIt2} must be an internally stabilizing controller for $G\tilde{D}_M^{-1}$ we can use similar arguments as in the sufficiency proof of theorem 4.4.3 to find that K_{It2} is proper and stable: K_{It2} is a proper and stable approximation of G^{-1}.

4.7 General issues in weight function selection

In the previous sections it has become clear that the selection of weight functions is a very important step in the practical application of any control design procedure based on mathematical optimization. An 'optimal' controller based on inappropriately chosen weight functions is obviously not optimal for the actual control design problem at hand. In fact, as there is no general procedure to choose weight functions in some optimal sense, it may be argued that it is impossible to find an optimal controller for any practical design problem.

An often heard complaint among control engineers with an interest in robust control methods is that there seems to be a lack of interest among control theoreticians to address this difficult and important weight function selection problem. However, it should be understood that the selection of weight functions is highly related to the specific control problem at hand, such that it is very hard to set up general guidelines.

The next subsection will give a general discussion on the use of weight functions in the standard control design structure, after which some considerations with respect to the tracking problem will be made.

4.7.1 Weight functions in the standard control design structure

When considering the selection of weight functions for application of robust control methods, there is one approach that is well established in literature, known as *loop-shaping*. Some key references are: Doyle and Stein 1981, Doyle et al. 1984, Freudenberg 1988, 1990, Safonov and Chiang 1988. Its primary idea is to bring the control objectives into the form of a 'mixed-sensitivity' problem as given in figure 4.15. From this figure it is possible to define four closed-loop transfer function

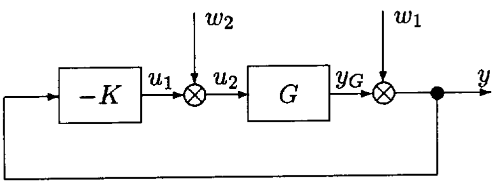

Fig. 4.15: The mixed sensitivity control configuration

matrices:

$$\begin{aligned} S_o &:= T_{w_1 y} &&= (I+GK)^{-1} \\ T_o &:= T_{w_1 y_G} &&= (I+GK)^{-1}GK \\ S_i &:= T_{w_2 u_2} &&= (I+KG)^{-1} \\ T_i &:= T_{w_2 u_1} &&= (I+KG)^{-1}KG \end{aligned} \tag{4.42}$$

S_o and S_i are called the output and input sensitivity matrix respectively, T_o and T_i are called the output and input complementary sensitivity matrix respectively. The properties of the loop-shaping approach can be briefly summarized as follows:

- the sensitivity matrix determines closed loop-performance and should be kept small,
- the complementary sensitivity matrix determines actuator effort and robustness and should also be kept small,
- the fundamental trade-off between robustness and performance is characterized by:

$$S_o + T_o = S_i + T_i = I \tag{4.43}$$

- control objectives are determined by restrictions on the bode plots of the singular values of S and T, either at the input or at the output: these restrictions determine desired loop-shapes,
- transfer function fits for the upper bounds of the desired loop-shapes determine the *inverse* of the weight function on the appropriate output signal:

$\|W_S S\|_\infty < 1$ and $\|W_T T\|_\infty < 1$ guarantee that S and T are below their respective restrictions,

- other than in the scalar case, there is an important difference between (complementary) sensitivity at the input or at the output: for this reason only well conditioned plants ($\bar{\sigma}_G \sim \underline{\sigma}_G$) can be properly dealt with,
- whether H_2 or H_∞ optimization is applied, controllers will always appear to perform exact cancellation of the plant's poles: this is extremely dangerous with respect to robustness against plant uncertainties,
- to remedy this, the four blocks problem of minizing the 2-norm or ∞-norm of $\begin{bmatrix} S & KS \\ SG & T \end{bmatrix}$ may be considered: however, the nice trade-off interpretation of equation 4.43 is then lost.

The obvious drawbacks of this approach have made most researchers decide to abandon it, although there are examples in which good results are obtained.
Nevertheless, experience with the loop-shaping approach gives rise to several guidelines concerning weight function selection in the more general standard plant control design structure.

- Make sure that all standard plant inputs and outputs are *scaled* appropriately:
  - disturbance inputs: normalize their range to 1; if frequency-domain information is available (e.g. spectral density), determine a transfer function fit and use its inverse as a weight function,
  - control inputs: normalize their range to 1; incorporate actuator dynamics into the standard plant,
  - objective functions: normalize their largest allowable deviation under influence of disturbances and uncertainties to 1; usually this allowable deviation is frequency dependent and gives rise to (the inverse of) another weight function,
  - measurement outputs: use relative (error) measurements rather than absolute measurements and normalize their range to 1; incorporate sensor dynamics into the standard plant.
- Each weight function should be chosen such that it is 'normalizing' for the appropriate closed-loop transfer function: determine the desired frequency-domain shape of each closed-loop transfer function (from w_i to z_i) and use

its inverse as a weight function on either the input or the output. In general, conflicting demands are to be expected; it is up to the designer to find a compromise.

- Usually it is not desirable to apply standard γ-iteration for the synthesis of an H_∞ sub-optimal controller. Most control objectives, especially those concerning robustness, give rise to a *bound* on a closed-loop transfer function, and in most cases there is only one closed loop transfer function for which actual *minimization* is required. γ-iteration should then be replaced by the adjustment of only the appropriate weight function.

Clearly these guidelines do not provide a complete procedure for finding correct weight functions. As mentioned before, the structure of the standard plant should be set up in accordance with the actual problem at hand, and is mainly the responsibility of the control designer. The main difference with earlier 'classical' and 'modern' methods for control design is the necessity to not only set up an appropriate model of the physical system to be controlled, but also to 'model' control objectives when setting up the standard plant. In this respect, it may be very useful to associate the setting up of the standard plant with three separate modelling concepts:

- the modelling of the physical system to be controlled, which is normally considered as being the *only* modelling concept (like in this thesis, the resulting model is usually denoted as G),
- the modelling of the external influences on the physical system, such as disturbances and uncertainties: this results in weight functions on the standard plant's disturbance input vector w (the RSG falls in this category),
- the modelling of the desired behaviour, which is clearly linked with the loop-shaping approach: this results in weight functions on the standard plant's control objectives vector z.

Note that according to this classification, the use of loop-shaping for disturbance attenuation is an *indirect* approach: this is the fundamental cause for the problems with the loop-shaping approach as indicated earlier.

4.7.2 Weight functions for the tracking problem

When considering a tracking problem it is usually helpful to first consider several control designs with u_r or R set to zero. The guidelines offered in the previous subsection can then be used to analyze the other control objectives and to set up correct scalings and weight functions. Although it should be clear that the tracking objective may have an important effect on the influence of these weight functions, it is usually possible to maintain their original interpretation and to adjust them accordingly.

Next to these 'original' weight functions, the addition of the tracking objective introduces several new elements to the standard plant: the RSG, the servo compensator $\tilde{D}_M^{-1}$ and the redefinition of the actuator effort weight function. The setting up of the RSG as discussed in section 4.3 is usually not very difficult. However, the solution of equation 4.11 for Q and L would involve manipulation of polynomial equations, which may be numerically unattractive. This implies that it is not straightforward to find $\tilde{D}_M^{-1}$ from equation 4.18. It is expected that in many practical situations it is not necessary to explicitly solve equations 4.11 and 4.18; the problem considered in the next chapter is an example of this.

Once the servo compensator has been set up, it is useful to consider the extended tracking problem as discussed in section 4.5: it provides more freedom in the design of the controller without increasing its order or complexity. The actuator effort weight function ($\tilde{W}$ in figure 4.9) should be augmented accordingly and then allows separate weights on servo compensator inputs, determining the trade-off between internal variable saturation and speed of response, and original plant inputs, determining the allowable actuator effort (in fact, this part of $\tilde{W}$ can be chosen equal to the original actuator effort weight function W in figure 4.6).

In the next chapter an example is considered, which demonstrates the applicability of several of the control design methods discussed in chapter 3 in combination with our approach to the output regulation and tracking problem as developed in this chapter. The selection of appropriate weight functions will be discussed in some detail and will illustrate the application of the considerations given in this section.

Chapter 5

Application to a three-degrees-of-freedom hydraulic positioning system

In this chapter we will consider the application of several robust control methods discussed in the previous chapters on an experimental set-up available at the laboratory of the Systems and Control Group of the Mechanical Engineering and Marine Technology department of the Delft University of Technology. This set-up is designed to be 'well-suited' for control design experiments: it is well instrumented and non-linear behaviour, especially the effect of dry friction, is minimized, although still present. On the other hand, it is designed to reflect control problems that may actually occur in practice and are difficult to solve with classical control theory: it is multivariable with three actuator inputs and six measurement signals and there is a significant amount of interaction.

We will consider the tracking problem for this three-degrees-of-freedom positioning system in the horizontal plane, or 3DOF system for short. The use of hydraulic actuators implies that integral action appears in the plant model, such that tracking of constant reference signals is possible without adding a servo compensator. Nevertheless, we will apply the procedure of chapter 4 to obtain a controller that achieves the tracking objective with H_2 or H_∞ methods to illustrate some of the advantages of the presented approach. As argued before, in any example designed to reflect an actual control problem, the tracking objective will only be one of many objectives that are to be pursued. Furthermore, we will pay attention to other important aspects, like modelling, weight function selection, implementation and evaluation. The application of the approach developed in the previous chapter can be found in subsection 5.3.2

A detailed description of the experimental set-up and the control implementation environment will be given in section 5.1. After that, we will set up a non-linear and linear model of the system in section 5.2. Section 5.3 will then consider the standard plant approach to define control objectives and will experimentally evaluate several control designs. Finally, in section 5.4, we will consider the robustness of the controlled system against an important form of parameter uncertainty and use H_∞ controller synthesis to obtain a robust controller.

5.1 The experimental set-up and implementation environment

5.1.1 The three-degrees-of-freedom hydraulic positioning system

A photograph of the mechanical part of the 3DOF system is given in figure 5.1. This mechanical part is based on a very stable horizontal table made of steel. It supports a steel block of dimensions 304 x 304 x 80 mm and weighing 48 kg. The steel block is allowed to slide over the table by means of an active air-bearing; it may therefore be assumed that there is no physical contact between table and block, and that the block may move without friction. The resulting configuration clearly has three degrees of freedom in the horizontal plane: two translations and one rotation.

Our goal is to control the movement and positioning of the block with respect to the table, for which we will use three hydraulic actuators, clearly visible in figure 5.1 and attached to table and block by means of rotary joints according to the schematic diagram of figure 5.2. This diagram shows the 3DOF system in its nominal position, in which there is a simple relation between hydraulic actuator lengths and block position: a change of block position in x-direction can be obtained by a change in length of actuator 1, a change in y-direction results from a simultaneous change in length of actuators 2 and 3, and a change in ϕ-direction can be obtained by simultaneous action of all three actuators.

The hydraulic system operates at a constant supply pressure p_s of 70 bar. The

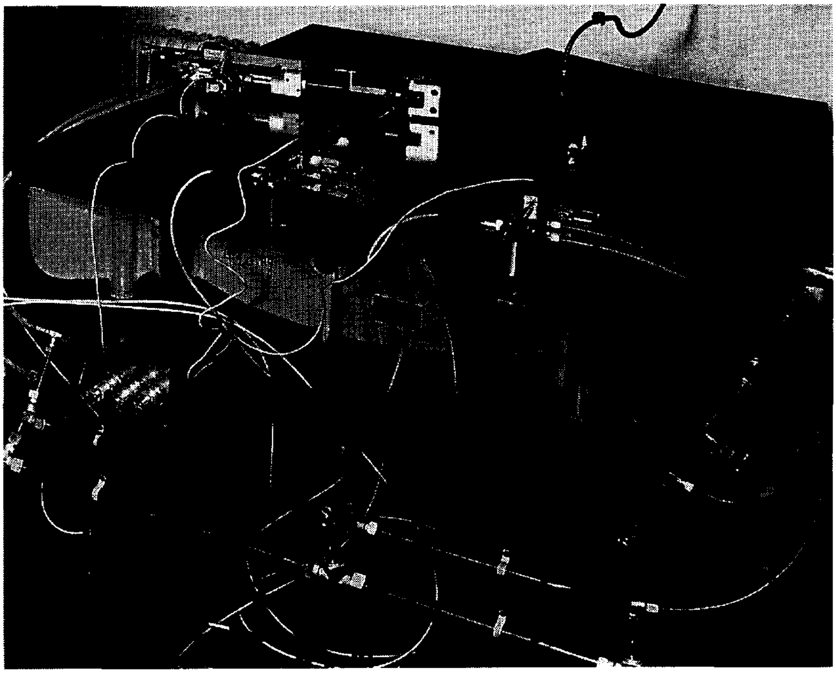

Fig. 5.1: The three-degrees-of-freedom hydraulic positioning system

hydraulic actuators are of the asymmetrical type with an area ratio of 0.56. They are provided with hydrostatic bearings to minimize friction. The actuator motion is controlled by an electro-hydraulic servo valve, producing an output flow proportional to the electrical input signal u_i. This output flow is towards the first actuator compartment, i.e. the compartment with the largest effective area. The compartment with the smallest effective area is constantly connected with the supply pressure. For experimental purposes, the first compartment of each actuator is connected with a variable ineffective volume v_o of nominally 1.5 litres: due to this, the hydraulic frequency of the set-up is artificially brought down from about 40 Hz to about 4 Hz (the hydraulic frequency is the eigenfrequency of the resonant mode caused by the flexibility of a trapped volume of fluid under a piston, and affected by piston area and supported mass, it is usually the first, i.e slowest, resonant mode of a hydraulic system).

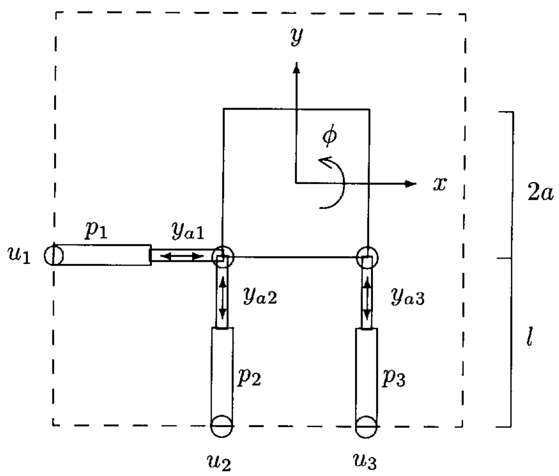

Fig. 5.2: Schematic representation of the 3DOF system in its nominal position

Each hydraulic actuator is fitted with two sensors for control purposes. A low friction linear potentiometer measures the displacement y_{ai} of the piston of each actuator with respect to the nominal position: this enables us to derive the position (x, y, ϕ) of the moving block. Secondly, a piezoresistive pressure transducer is used to measure the absolute pressure p_i in the first compartment of each actuator. Knowing that the pressure in the second compartment is equal to the supply pressure and knowing the effective area ratio, this enables us to derive a good estimate of the forces acting on the moving block.

All electrical signals to and from the set-up are conditioned for direct use by a digital controller; signals are scaled within a range of -10 to $+10$ Volts and, if necessary, filtered to prevent aliasing.

5.1.2 The control implementation environment

From the theory discussed in the previous chapters, it is clear that controllers designed by means of robust control methods are usually quite complex; they are multivariable and at least of the order of the standard plant. This implies that it is virtually impossible to implement such a controller by means of analog devices and that a digital implementation environment is necessary. Furthermore, as the developed theory is based on continuous time considerations, it is necessary to use high sampling rates and high quantization accuracy for a digital controller to be a sufficient approximation of a continuous time controller. Fortunately, with the recent development of high performance dedicated digital hardware like the

Digital Signal Processor or DSP, it appears to be possible to accommodate these demands.

Our intention to design and implement controllers using robust control methods also makes it very desirable to have an implementation environment that allows controllers calculated with PC MatLab (see Moler et al. 1987) or similar high level matrix calculation tools to be implemented quickly and efficiently without having to write low-level programming code. One of the few DSP-based commercially available solutions for this is produced by dSPACE GmbH as the 'DSP-CITpro Control Implementation Tool' (Hanselmann 1989). The hardware is supplied as add-on cards for the IBM compatible PC and is built around a DSP of the Texas Instruments TMS320 line.

The main processor board (dSPACE type nr. DS1001) that is used for the hydraulic positioning system contains a fixed-point 40MHz TMS320C25 DSP with a 100 ns cycle time and a 16 × 16 bit hardware multiplier for single cycle multiplication and accumulation. Communication with this processor is possible from the PC (acting as host system) via the 16-bit AT bus of the PC. Programs can be downloaded to the DSP and during program execution 4K words of 16-bit true dual-port RAM is available for monitoring key variables (simultaneous DSP- and host-access). Further memory available to the DSP is 64K words of program memory and 59K words of data memory, both accessible with zero wait states and with DSP- and host-access arbitration.

The interaction with the experimental set-up is performed by means of A/D and D/A converters: two DS2001 boards containing 5 A/D converters each and one DS2101 board with 5 D/A converters. Communication between processor board and interface boards is performed via the PHS-bus (Peripheral High Speed bus), a 32-bit synchronous I/O bus with 13.3 MB/s peak transfer speed. This allows the digital controller to run completely independent of the PC. The DS2001 A/D boards each contain 5 fully parallel 16-bit A/D converters with 5 μs conversion time, 14-bit linearity (typical) and sample and hold circuits (tracking and hold). A/D conversions can be started separately or simultaneously and ADC ready may be signalled via interrupt or flag. The DS2101 board contains 5 12-bit D/A converters with 3 μs full scale settling time to 0.01%.

Equally essential to the usefulness of this control implementation environment is the implementation software package IMPAC, consisting of the Implementation Expert module IMPEX combined with the high level programming language DSPL. IMPEX is a menu-driven programming tool, independent of specific target hardware, allowing the setting up of any linear time-invariant controller. The controller parameters should be available in state-space form and given in an ASCII-file according to a prespecified format. Utilities to interface with PC MatLab are available to automatically create this file, such that any control design algorithm implemented in MatLab can be used to create a state-space controller and prepare it for IMPEX.

In general, such a controller will be continuous time and in an arbitrary state-space realization; IMPEX provides tools to convert this into a description suitable for implementation in the digital fixed-point TMS320C25 processor. First the controller may be discretized either step-invariant, ramp-invariant or bilinear, according to specific requirements. Next a transformation may be performed to reduce the number of controller parameters, the number of calculations and—most importantly—the coefficient sensitivity, for instance with respect to quantization effects. Thirdly it is possible to perform automatic or user-specified scaling of variables (input, output and state variables) to user-defined ranges; because the standard-plant approach usually makes sure that inputs and outputs are correctly scaled, this is especially important for (internal) state variables when using a fixed-point processor. The final step is automatic code generation based on this discretized, transformed and scaled state-space model, first setting up the high level language code DSPL, followed by the compilation into TMS320C25 target processor assembly source code. After that, assembling and downloading of object code will complete the automated implementation procedure (see Hanselmann 1987).

Disadvantages of such a highly automated implementation procedure are, of course, the restrictions on the usable hardware (dSPACE products) and on the implementable controllers (linear time-invariant). However, it is possible to extend IMPEX with templates and drivers for user defined interface hardware or even completely different TMS320C25-based processor boards. Furthermore, IMPEX provides very well documented ASCII-files of the automatically generated DSPL-code and assembly source code. These files can be used as shell-files, that allow a programmer to add non-linear relations, limitations, gain-scheduling, start up sequences, etc. Clearly this implies a large programming effort of the user, although in most cases it is sufficient to make changes to the DSPL-code, which can be seen as a high-level programming language that was tailor-made for every TMS320 processor. Only if time optimality is necessary, it may be useful to get into programming assembly code: to give an impression, a 12th order, 9 inputs, 3 outputs state-space controller for the hydraulic positioning system was automatically implemented with a calculation time of about 75 μs, giving a processor load of only 7.5% at the more than sufficient sampling rate of 1kHz.

This then sums up the most essential parts of the implementation environment used to obtain the results in the following sections. It is noted here that developments in the area of hardware and software are extremely fast and have already surpassed the possibilities of this implementation environment in several respects.

- Developments in hardware may be illustrated by the introduction of the TMS320C30 and TMS320C40 to replace the TMS320C25. Single-cycle fixed-point calculation is replaced by single-cycle floating-point calculation, calculation speed is doubled, high-speed communication links are integrated to allow multiple processor solutions, etc.

- The floating-point capabilities of the latest DSPs allow the use of high level programming languages like C: this allows the development of flexible and user-friendly software environments for fully automated implementation of complex non-linear controllers. An example of this is the collaboration between dSPACE and The MathWorks to develop fully automated implementation of non-linear controllers defined in the block-oriented simulation environment of MatLab/SimuLink (Moler et al. 1987).

An important consequence of these developments is a sharp decrease in costs of high-speed calculation time: it may be expected that DSPs will allow increasingly complex controllers to be used for relatively low cost applications, like for instance consumer electronics. For most industrial installations, the development of DSPs implies that bounds on the complexity of controllers are no longer related to available calculation time.

5.2 Modelling the 3DOF system

As mentioned in section 2.1 the modelling process should provide us with a model, that is an accurate description of the behaviour of the 3DOF system insofar as it is of importance for our goals. In this case we want to be able to design a robust LTI controller for the 3DOF system, which achieves a bandwidth that is higher than the slowest open loop resonant modes of the setup (about 10 Hz, whereas the slowest resonant mode is about 4 Hz: the hydraulic frequency). This implies that we need to have an LTI model, describing the significant behaviour, plus an evaluation of sources of uncertainty or inaccuracy with respect to this model. Furthermore, for control purposes it is necessary to precisely define the interface between physical system and controller, i.e. the control input signals and the measurement signals, plus other input and output signals that may be used, to accurately define the *scope* of the system, or the *system boundary*. The behaviour of the system may then be defined by making use of behavioural equations that give a description of the relations between these input and output signals and internal variables to be defined. These behavioural equations are usually a combination of algebraic and differential equations, that are based on known physical relations: in our case balance equations and Newton's second law.

To simplify the derivation of these equations, it is a usual approach to subdivide the complete system to be controlled into a number of subsystems, each with their own boundary and their own set of input and output signals, which may contain those of the entire system and its internal variables. However, it should be noted that only control input signals and measurement output signals have a physical ground to determine whether they are inputs or outputs, all other signals should better be considered as *interface signals*, i.e. signals that determine a (sub-)system's boundary without being predetermined to be either input or output signals.

With this in mind it is always possible to label interface signals as either input or output in such a way that their relation is causal and the resulting models are proper. Note that there do not exist physical systems which have a non-causal relation between (control) inputs and (measurement) outputs, although it is sometimes possible to model them as such over a restricted frequency range.

The following subsection will discuss the construction of a non-linear model, which was used for simulation to obtain a first evaluation of designed controllers, and to obtain an impression of the inaccuracies that must be introduced when deriving a linear model. The linearization of this non-linear model will be the subject of a separate subsection, and will provide us with a linear model to be used for actual controller design.

5.2.1 Construction of a non-linear model

It is easy to recognise that the 3DOF system consists of four main subsystems: the three hydraulic actuators and the steel block sliding on the table. As the three hydraulic actuators are similar, we will start by separately considering two subsystems:

- a model of one of the three hydraulic actuators will be set up with the electrical signal towards the servo valve, the shaft's position and its velocity as inputs and the generated force as output,

- a model of the moving block will be set up with actuator forces as inputs and the position and velocity of the block as outputs.

The selection of input and output signals is such, that no causality or properness problems will occur. The non-linear models derived for these subsystems will then be linked to obtain a non-linear model of the entire set-up.

Modelling a single hydraulic actuator

All three actuators are essentially the same and are mainly composed of two distinct parts (i.e. sub-subsystems): the electro-hydraulic servo valve, which converts an applied current into a proportional flow, and the hydraulic cylinder, in which this flow is converted into a proportional motion of the piston shaft.

The electro-hydraulic servo valve In figure 5.3 a schematic representation of the Dowty series 4553 two-stage electro-hydraulic servo valve is given. This

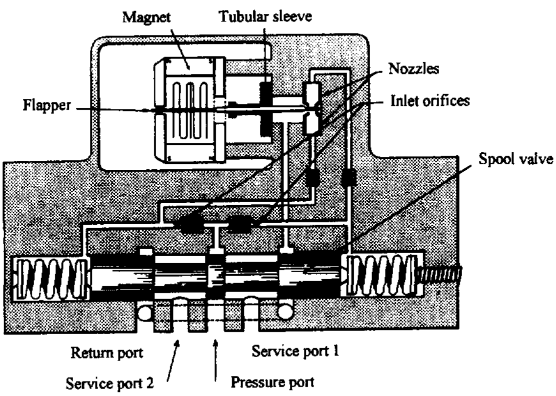

Fig. 5.3: The electro-hydraulic servo valve

servo valve consists of an electric torque motor and two stages of hydraulic power amplification. The first stage (pilot stage) of the valve consists of a symmetric flapper-nozzle system, fed by the constant supply pressure via two restrictions. As an electrical signal is applied to the motor, a force is developed upon the flapper, causing it to pivot and to move between the nozzles. This motion causes the orifice between the flapper and one of the nozzle tips to open as the other closes, resulting in a differential pressure between the nozzle chambers. The pressure difference, acting upon the nozzle projected areas, results in a feedback force upon the flapper to balance the motor force. The pressure differential also acts upon the second stage spool, resulting in a displacement and, hence, output flow proportional to the electrical signal input. When applying this type of valve to symmetrical hydraulic actuators there are two output flows; one to each side of the cylinder. Since in this case we use it to control an asymmetrical actuator, only the output flow towards the first compartment is needed (service port 1), as the second compartment is constantly connected to the supply pressure.

According to the manufacturer, the bandwidth of the applied servo valve lies between 130 and 200 Hz, depending on signal amplitude, temperature and operating pressure, which in any case is much more than that of the fastest controlled system we will consider, which is about 10 Hz. For this reason we may neglect the dynamical behaviour of the servo valve and make use of a steady state approximation. The main spool displacement x_s is taken to be directly proportional to the control current, which in turn is proportional to the voltage u, generated by the controller: at all instants $x_s = c_u u$ for some constant c_u. The flow φ to the actuator's first compartment is then proportional to the product of u with the square root of the pressure drop over the resulting orifice:

$$\begin{aligned} \varphi &= c_\varphi u \sqrt{1 - \tfrac{p}{p_s}} \quad \text{when} \quad u > 0 \\ \varphi &= c_\varphi u \sqrt{\tfrac{p}{p_s}} \quad \text{when} \quad u \leq 0 \end{aligned} \tag{5.1}$$

for some constant c_φ. The pressure drop has been made dimensionless by dividing it by p_s.

The hydraulic cylinder A diagram of the driving part of the actuator is given in figure 5.4. Its behaviour is governed by two balance equations: a balance of

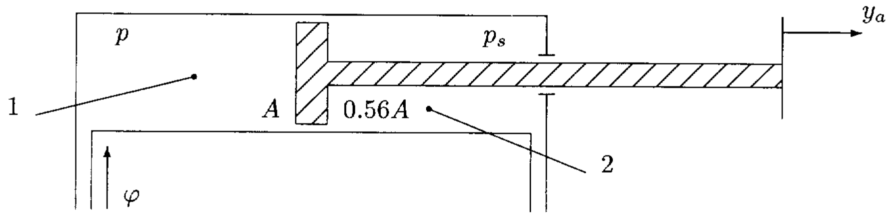

Fig. 5.4: The hydraulic cylinder

force and a balance of mass for compartment 1. Three types of forces acting on the hydraulic actuator will be considered:

1. the driving force F_d, caused by the oil pressure difference over the piston and the active area at each side of the piston:

$$F_d := Ap - 0.56Ap_s \tag{5.2}$$

   with A denoting the active area of compartment 1,

2. the frictional force F_f, caused by viscous friction proportional to the actuator speed $\dot{y}_a$:

$$F_f := -f\dot{y}_a \tag{5.3}$$

   in which friction coefficient f will be determined from measurements,

3. the external force F_e, acting on the moving block.

Note that we do not consider two effects that may also be of importance:

1. dynamical forces caused by the piston mass are neglected, as they are very small in relation with the dynamical behaviour of the moving block,

2. dry friction forces are neglected, as the design of the set-up succeeded in minimizing them.

These assumptions will be the cause of some of the differences between the behaviour of the non-linear model and that of the actual set-up.
We thus obtain the following equation for the force acting on the moving block:

$$F_e = Ap - 0.56Ap_s - f\dot{y}_a \tag{5.4}$$

The balance of mass equation will result from the following four effects:

1. the input flow from the servo valve towards compartment 1: φ,

2. the change of the active volume of compartment 1: $A\dot{y}_a$,

3. the compressibility of the total volume of compartment 1: $\frac{v}{E}\dot{p}$, with v the sum of ineffective and effective volume: $v := v_o + Ay_a$, and E the compressibility coefficient or *bulk modulus* for the hydraulic fluid,

4. the leakage flow over the piston towards compartment 1: $c_l(p_s - p)$, with constant c_l to be determined from measurements.

We thus obtain the following balance of mass equation:

$$\varphi - A\dot{y}_a - \frac{v}{E}\dot{p} + c_l(p_s - p) = 0 \tag{5.5}$$

Modelling the moving block

Figure 5.5 shows a possible position of the moving block in relation to its nominal position. It allows us to set up the kinematical relation between actuator displacements y_{ai}, $i = 1 \cdots 3$, as measured by the linear potentiometers, and the position of the moving block. We will make use of an inertial frame X_0, Y_0 and a moving

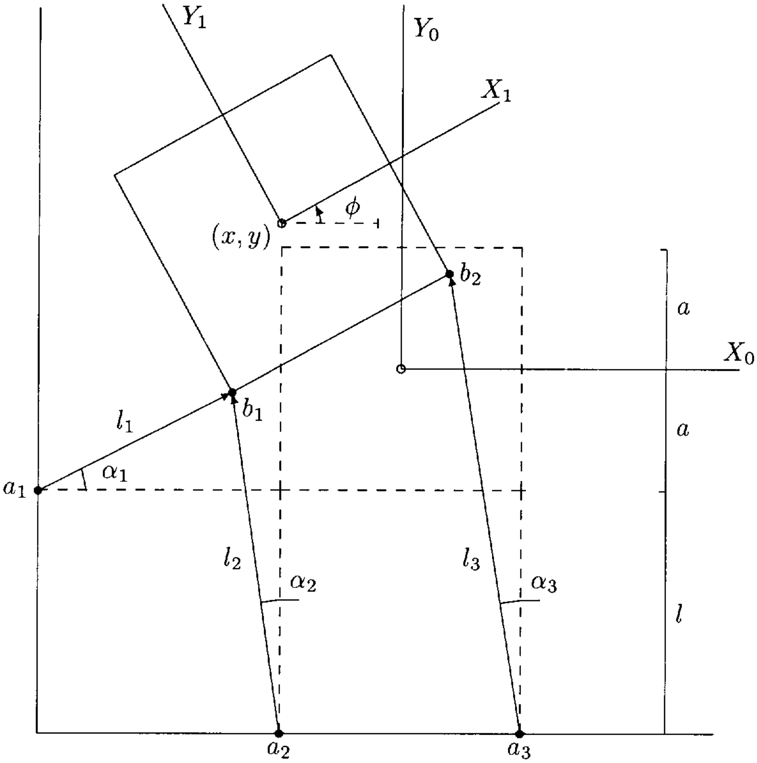

Fig. 5.5: Moving block kinematics

frame X_1, Y_1, which is rigidly attached to the moving block and has its origin at the centre of gravity of the block. Hence, we can determine the location of any point p on the moving block relative to the inertial frame: $(x_p, y_p)_0$, as a function of its coordinates in the X_1, Y_1 frame: $(x_p, y_p)_1$, the position of the block's centre of gravity in the inertial frame: (x, y), and the angle ϕ:

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix}_0 = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix}_1 \tag{5.6}$$

It is clear that we can take (x, y, ϕ) as the set of generalized coordinates describing the three degrees of freedom of the system.
Next, we will assume that the nominal length of all three actuators is the same and is denoted as l; the side of the moving block is given as $2a$. The coordinates of the fixed points a_i, $i = 1 \cdots 3$ are then given as:

$$a_1 = (-l-a \;,\; -a)_0, \quad a_2 = (-a \;,\; -l-a)_0, \quad a_3 = (a \;,\; -l-a)_0 \tag{5.7}$$

and the coordinates of joints b_1 and b_2 can then be found as:

$$\begin{aligned} b_1 &= (-a \;,\; -a)_1 &= (x - a\cos\phi + a\sin\phi \;,\; y - a\sin\phi - a\cos\phi)_0 \\ b_2 &= (a \;,\; -a)_1 &= (x + a\cos\phi + a\sin\phi \;,\; y + a\sin\phi - a\cos\phi)_0 \end{aligned} \tag{5.8}$$

Now we can interpret the inertial frame X_0, Y_0 as a two-dimensional linear vector space and define the following *actuator vectors*:

$$\begin{aligned} l_1 &:= b_1 - a_1 \\ l_2 &:= b_1 - a_2 \\ l_3 &:= b_2 - a_3 \end{aligned} \tag{5.9}$$

With the two elements of a vector l_i with respect to the inertial frame denoted as x_{l_i} and y_{l_i} the actual length of each actuator as a function of coordinates (x, y, ϕ) can be determined as:

$$|l_i| = \sqrt{l_i' l_i} = \sqrt{x_{l_i}^2 + y_{l_i}^2}, \quad i = 1 \cdots 3 \tag{5.10}$$

Furthermore, the angles α_i, $i = 1 \cdots 3$ as given in figure 5.5 are determined by:

$$\begin{aligned} \alpha_1 &= \arctan \tfrac{y_{l_1}}{x_{l_1}} \\ \alpha_2 &= \arctan \tfrac{-x_{l_2}}{y_{l_2}} \\ \alpha_3 &= \arctan \tfrac{-x_{l_3}}{y_{l_3}} \end{aligned} \tag{5.11}$$

The actuator displacements, as measured by the linear potentiometers, y_{ai}, $i = 1 \cdots 3$ can then be found as the difference between actual actuator lengths and nominal actuator length:

$$y_{ai} := |l_i| - l, \quad i = 1 \cdots 3 \tag{5.12}$$

To determine the dynamical behaviour of the moving block, we must add the effect of external forces. We will assume that the only external forces are those generated by the actuators (defined in equation 5.4). Furthermore, we assume that the rotary joints at positions a_i, b_i are frictionless: the external forces act along the actuator vectors l_i. Applying Newton's second law on the three generalized coordinates (x, y, ϕ) and using actuator vector angles α_i, we then find three second order differential equations:

$$\begin{aligned}
\ddot{x} &= \tfrac{1}{M}(F_{e1}\cos\alpha_1 - F_{e2}\sin\alpha_2 - F_{e3}\sin\alpha_3) \\
\ddot{y} &= \tfrac{1}{M}(F_{e1}\sin\alpha_1 + F_{e2}\cos\alpha_2 + F_{e3}\cos\alpha_3) \qquad (5.13) \\
\ddot{\phi} &= \tfrac{a\sqrt{2}}{J}(F_{e1}\cos(\alpha_1 - \phi + \tfrac{\pi}{4}) - F_{e2}\sin(\alpha_2 - \phi + \tfrac{\pi}{4}) + F_{e3}\cos(\alpha_3 - \phi + \tfrac{\pi}{4}))
\end{aligned}$$

with $J := \frac{2}{3}Ma^2$ and M the mass of the moving block.

Combining the subsystems

We can now set up equations 5.1, 5.4 and 5.5 for each of the three actuators and combine them with equations 5.7 through 5.13 to find a set of equations describing the entire non-linear behaviour of the 3DOF system.

The parameters used in this non-linear model were derived from manufacturer specifications and measurements, and are given in table 5.1. The difference be-

p_s	$70 \cdot 10^5$	$\frac{\mathrm{N}}{\mathrm{m}^2}$
c_φ	$1.18 \cdot 10^{-7}$	$\frac{\mathrm{m}^3}{\mathrm{Vs}}$
A	$2.54 \cdot 10^{-4}$	m^2
f_1	$1.5 \cdot 10^2$	$\frac{\mathrm{Ns}}{\mathrm{m}}$
$f_{2,3}$	$4.5 \cdot 10^2$	$\frac{\mathrm{Ns}}{\mathrm{m}}$
v_o	$1.5 \cdot 10^{-3}$	m^3
E	$1.3 \cdot 10^9$	$\frac{\mathrm{N}}{\mathrm{m}^2}$
c_{l1}	$2.6 \cdot 10^{-13}$	$\frac{\mathrm{m}^5}{\mathrm{Ns}}$
$c_{l2,l3}$	$1.3 \cdot 10^{-13}$	$\frac{\mathrm{m}^5}{\mathrm{Ns}}$
l	0.49	m
a	0.152	m
M	48	kg

Table 5.1: parameters of the 3DOF system

tween actuator 1 and actuators 2 and 3 in friction coefficient f and leakage flow coefficient c_l are caused by wear of actuator 1, that was extensively used in earlier experiments: it illustrates the need for robust controllers, although the effect of variations of f and c_l on the dynamical behaviour of the set-up is relatively small.

5.2.2 Derivation of a linear model

To be able to use robust control methods to design controllers, we need a linear time-invariant representation of the 3DOF system. We will obtain this by linearization of the set of equations given in the previous subsection in the nominal position of the moving block with zero velocities and nominal actuator pressures: $x, \dot{x}, y, \dot{y}, \phi, \dot{\phi} = 0$ and $p_{1,2,3} = 0.56p_s$. Furthermore we will select appropriate scalings for input, output and state variables, according to the guidelines of section 4.7.

First we will introduce the new pressure variable p_n:

$$p_n := \frac{p - 0.56p_s}{0.56p_s} \tag{5.14}$$

Note that under allowable operating conditions $p \in [0, p_s]$ such that $p_n \in [-1, 0.786]$ and that in the nominal situation we have $p_n = 0$. Equation 5.4 describing the external force generated by each of the three actuators may then be written as:

$$F_{ei} = 0.56Ap_s \cdot p_{ni} - f_i \cdot \dot{y}_{ai}, \qquad i = 1 \cdots 3 \tag{5.15}$$

To introduce a new flow variable φ_n, we calculate the nominal flow from equation 5.5 as:

$$\varphi_o := -c_l(p_s - p) = -0.44c_lp_s \tag{5.16}$$

and use the rated flow of $5\frac{l}{min}$ or $8.33 \cdot 10^{-5}\frac{\mathrm{m}^3}{\mathrm{s}}$ as specified by the manufacturer to obtain:

$$\varphi_n := \frac{\varphi + 0.44c_lp_s}{8.33 \cdot 10^{-5}} \tag{5.17}$$

With the nominal flow (leakage flow) being less than 1 percent of the rated flow we assume that under normal conditions we have $\varphi_n \in [-1, 1]$.

As the nominal flow of equation 5.16 must be obtained by applying a nominal control voltage u_o to the electro-hydraulic servo valve, we have from equation 5.1:

$$u_o := \frac{\varphi_o}{c_\varphi\sqrt{0.56}} = \frac{-0.44c_lp_s}{c_\varphi\sqrt{0.56}} \tag{5.18}$$

With the control voltage ranging from -10 V to 10 V, we may thus introduce a new control input variable as:

$$u_n := \frac{u - u_o}{10} \tag{5.19}$$

After linearization we then have $\varphi_n = u_n$.
Next, we can linearize equation 5.5 and express it in the new variables p_n and u_n:

$$\begin{aligned} &8.33 \cdot 10^{-5} \cdot u_{ni} - A \cdot \dot{y}_{ai} - \frac{v_o}{E} 0.56 p_s \cdot \dot{p}_{ni} - 0.56 c_{li} p_s \cdot p_{ni} \quad \Longleftrightarrow \\ &\dot{p}_{ni} = -\frac{AE}{0.56 v_o p_s} \cdot \dot{y}_{ai} - \frac{E c_{li}}{v_o} \cdot p_{ni} + \frac{8.33 \cdot 10^{-5} E}{0.56 \cdot v_o p_s} \cdot u_{ni}, \quad i = 1 \cdots 3 \end{aligned} \tag{5.20}$$

Actuator displacements y_{ai} may be substituted by block coordinates (x, y, ϕ) using the following linearized relations:

$$\begin{aligned} y_{a1} &= x + a\phi \\ y_{a2} &= y - a\phi \\ y_{a3} &= y + a\phi \end{aligned} \tag{5.21}$$

Under the assumption that angles α_i and ϕ remain small, we can also linearize equation 5.13 and substitute equation 5.21:

$$\begin{aligned} \ddot{x} &= -\frac{f_1}{M}\dot{x} - \frac{af_1}{M}\dot{\phi} + \frac{0.56Ap_s}{M}p_{n1} \\ \ddot{y} &= -\frac{(f_2+f_3)}{M}\dot{y} + \frac{a(f_2-f_3)}{M}\dot{\phi} + \frac{0.56Ap_s}{M}p_{n2} + \frac{0.56Ap_s}{M}p_{n3} \\ \ddot{\phi} &= -\frac{af_1}{J}\dot{x} + \frac{a(f_2-f_3)}{J}\dot{y} - \frac{a^2(f_1+f_2+f_3)}{J}\dot{\phi} \\ &\quad + \frac{0.56aAp_s}{J}p_{n1} - \frac{0.56aAp_s}{J}p_{n2} + \frac{0.56aAp_s}{J}p_{n3} \end{aligned} \tag{5.22}$$

A state vector x_G and an input vector u_G can then be defined as follows:

$$\begin{aligned} x_G &= (x\ y\ \phi\ \dot{x}\ \dot{y}\ \dot{\phi}\ p_{n1}\ p_{n2}\ p_{n3})' \\ u_G &= (u_{n1}\ u_{n2}\ u_{n3})' \end{aligned} \tag{5.23}$$

such that the state equation for the entire 3DOF system can be derived from equations 5.20 and 5.22:

$$\dot{x}_G = A_G x_G + B_G u_G \tag{5.24}$$

with: $A_G =$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{f_1}{M} & 0 & -\frac{af_1}{M} & \frac{0.56Ap_s}{M} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{(f_2+f_3)}{M} & \frac{a(f_2-f_3)}{M} & 0 & \frac{0.56Ap_s}{M} & \frac{0.56Ap_s}{M} \\ 0 & 0 & 0 & -\frac{af_1}{J} & \frac{a(f_2-f_3)}{J} & -\frac{a^2(f_1+f_2+f_3)}{J} & \frac{0.56aAp_s}{J} & -\frac{0.56aAp_s}{J} & \frac{0.56aAp_s}{J} \\ 0 & 0 & 0 & -\frac{AE}{0.56v_op_s} & 0 & -\frac{aAE}{0.56v_op_s} & -\frac{Ec_{l1}}{v_o} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{AE}{0.56v_op_s} & \frac{aAE}{0.56v_op_s} & 0 & -\frac{Ec_{l2}}{v_o} & 0 \\ 0 & 0 & 0 & 0 & -\frac{AE}{0.56v_op_s} & -\frac{aAE}{0.56v_op_s} & 0 & 0 & -\frac{Ec_{l3}}{v_o} \end{pmatrix}$$

and:

$$
B_G = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{8.33 \cdot 10^{-5} E}{0.56 \cdot v_o p_s} & 0 & 0 \\
0 & \frac{8.33 \cdot 10^{-5} E}{0.56 \cdot v_o p_s} & 0 \\
0 & 0 & \frac{8.33 \cdot 10^{-5} E}{0.56 \cdot v_o p_s}
\end{pmatrix}
$$

The measurement outputs consist of actuator pressures p_i and actuator displacements y_{ai}. For p_i we already introduced the variable p_{ni} ranging between -1 and 1 under normal operating conditions: therefore no scaling is necessary. For the scaling of y_{ai} we use half the stroke of the actuator: 75 mm. With equation 5.21 we thus get:

$$
\begin{aligned}
y_{an1} &= \frac{1}{0.075}\, x + \frac{a}{0.075}\, \phi \\
y_{an2} &= \frac{1}{0.075}\, y - \frac{a}{0.075}\, \phi \\
y_{an3} &= \frac{1}{0.075}\, y + \frac{a}{0.075}\, \phi
\end{aligned} \tag{5.25}
$$

The output vector can then be defined as:

$$
y_G = (p_{n1}\ p_{n2}\ p_{n3}\ y_{an1}\ y_{an2}\ y_{an3})' \tag{5.26}
$$

such that the output equation results as:

$$
y_G = C_G x_G + D_G u_G \tag{5.27}
$$

with:

$$
C_G = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\frac{1}{0.075} & 0 & \frac{a}{0.075} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{0.075} & -\frac{a}{0.075} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{0.075} & \frac{a}{0.075} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

and $D_G = 0$.

To get an impression of the dynamical behaviour of the resulting model we may look at the eigenvalues of A_G representing the poles of the open-loop system:

$$\begin{aligned}
p_{G1} &= 0 \\
p_{G2} &= 0 \\
p_{G3} &= 0 \\
p_{G4} &= -16.2448 + 73.4859i \\
p_{G5} &= -16.2448 - 73.4859i \\
p_{G6} &= -1.8929 + 26.7620i \\
p_{G7} &= -1.8929 - 26.7620i \\
p_{G8} &= -9.4313 + 47.3593i \\
p_{G9} &= -9.4313 - 47.3593i
\end{aligned} \tag{5.28}$$

Note that the integral action of the three hydraulic actuators results in three poles in zero. This can also be concluded from the bode plot of the open loop transfer functions from actuator inputs to actuator displacements as given in figure 5.6; also note the important effect of the slowest resonant mode (p_{G6} and p_{G7}) on the transfer function of the first actuator.

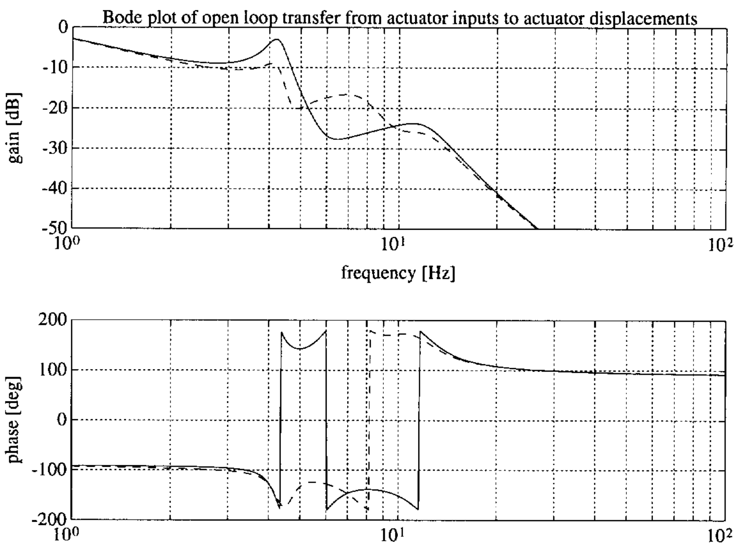

Fig. 5.6: Bode plot of the open loop transfer functions from actuator inputs to actuator displacements: actuator 1 solid, actuators 2 and 3 dashed

5.3 Controller design and implementation

In this section we will set up a standard plant description for the 3DOF system by selecting appropriate signals and weight functions. We will then discuss the application of H_∞ suboptimal controller synthesis on this standard plant and consider the resulting controllers. The procedure of chapter 4 will be applied to obtain a controller that achieves tracking of constant reference signals. A tuning procedure will be presented to improve performance in the sense of speed of response by adjusting the available weight functions. The resulting reference tracking and disturbance attenuating controller will be implemented and experimentally evaluated.

5.3.1 A standard plant description for the 3DOF system

As we want to solve a tracking problem for the 3DOF system, the basic structure of the standard plant description should be in accordance with configuration I as given in section 4.4 (figure 4.8). Hence, we need to construct an appropriate Reference Signal Generator and consider (weighted) tracking errors and actuator inputs as control objectives. Furthermore, we will make sure that measurement signals consist of tracking errors rather than absolute measurements: this would otherwise lead to the more complex two-degree-of-freedom problem as discussed in section 4.6.

In this case we want to obtain tracking of constant reference signals, i.e. asymptotic tracking of step signals, which can be described in the frequency domain as $r(s) = \frac{\kappa}{\tau s}$. The parameters κ and τ determine the expected size of the reference signals: τ determines the frequency at which the amplitude of r is equal to κ. Based on the physical boundaries of the set-up: 75 mm in x and y directions and $\frac{\pi}{4}$ rad in ϕ direction, we will take $\kappa_x, \kappa_y = 0.075$ for the x and y reference, and $\kappa_\phi = \frac{\pi}{4}$ for the ϕ reference. We will start with a conservative value for τ: $\tau = \frac{1000}{2\pi}$, which implies a frequency of 0.001 Hz.

Above this frequency, we will not aim at minimizing the error signals, but for safety reasons we will assume that they may occur; we will attempt to make sure that they do not lead to excessive actuator effort. This requirement may be incorporated in the description of the reference signals by adding the parameter κ as a constant:

$$r(s) \;:=\; \frac{\kappa}{\tau s} + \kappa \;=\; \kappa \cdot \frac{\tau s + 1}{\tau s} \tag{5.29}$$

The RSG generating independent reference signals of this form for each of the three degrees of freedom may thus be given as:

$$R := \begin{bmatrix} K_x \cdot \frac{\tau s+1}{\tau s} & 0 & 0 \\ 0 & K_y \cdot \frac{\tau s+1}{\tau s} & 0 \\ 0 & 0 & K_\phi \cdot \frac{\tau s+1}{\tau s} \end{bmatrix} \tag{5.30}$$

To give an impression of the frequency-domain behaviour of the RSG, a Bode-magnitude plot of the diagonal elements is given in figure 5.7.

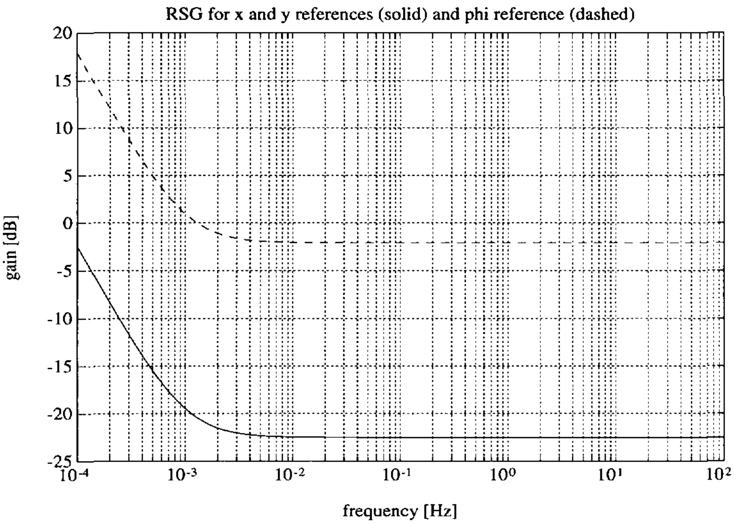

Fig. 5.7: Bode-magnitude plot of RSG with $\tau = \frac{1000}{2\pi}$: x and y reference solid, ϕ reference dashed

Next, according to the structure of configuration I (figure 4.8), the position errors obtained by subtracting r from $[x\ y\ \phi]'$ must be made available to the controller as measurement signals. Using equation 5.25 we can find x, y and ϕ directly from the actuator displacement measurements as:

$$\begin{bmatrix} x \\ y \\ \phi \end{bmatrix} = 0.075 \cdot \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2a} & \frac{1}{2a} \end{bmatrix} \begin{bmatrix} y_{an1} \\ y_{an2} \\ y_{an3} \end{bmatrix} \tag{5.31}$$

Hence, it is convenient to define an alternative output equation for the SSD of the

3DOF system: $\tilde{G} = [A_G, B_G, \tilde{C}_G, \tilde{D}_G]$ with:

$$\tilde{C}_G = \begin{pmatrix} I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & I_3 \end{pmatrix}$$

and $\tilde{D}_G = 0$, thus defining an alternative measurement output vector:

$$y_{\tilde{G}} := (x \;\; y \;\; \phi \;\; p_{n1} \; p_{n2} \; p_{n3})' \tag{5.32}$$

Note that we consider the part of the controller (to be designed) that implements equation 5.31 to be a part of the physical system, such that we have the actual position coordinates of the block available as measurements. Also note that in this case there is no necessity to scale these 'virtual' measurements according to the guidelines of section 4.7.

With this, we are able to set up the standard plant structure for the 3DOF tracking problem as given in figure 5.8, in which disturbance input signals w_e and w_p

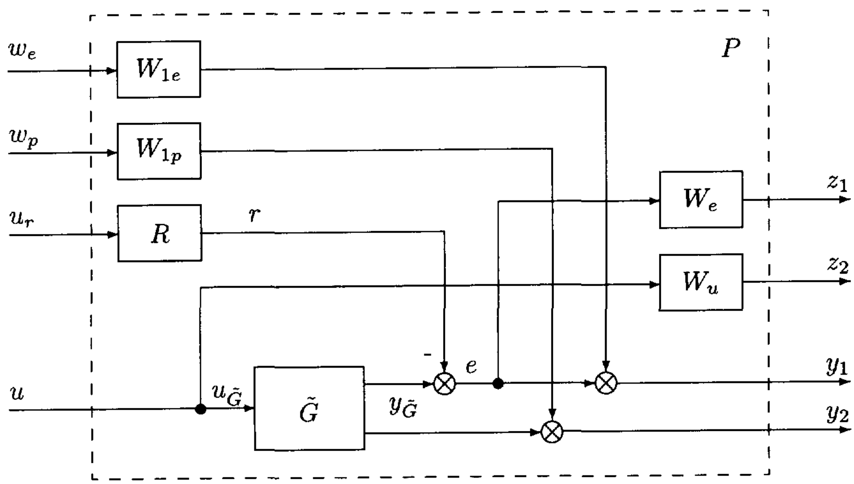

Fig. 5.8: Standard plant structure for the 3DOF tracking problem

and weight functions W_{1e} and W_{1p} are introduced to reflect the effect of measurement noise on the position error signals and the pressure signals. The apparent partitioning of measurement signals into a part that is affected by r and a part that is not, illustrates an important difference with configuration I. In general, the use of absolute pressure measurements (y_2) will result in a two-degree-of-freedom problem as discussed in section 4.6. Hence, to prevent this we should define reference signals for the pressure measurements that are compatible with those for the position measurements. Fortunately, in this particular case, such compatible reference signals for the pressure measurements are all zero at all times: in any desired position of the moving block the normalized pressure measurement signal should be zero. It is possible to incorporate these zero reference signals into the standard plant of figure 5.8, resulting in a structure in accordance with configuration I, but this would only make the problem more complex (three additional 'disturbance' inputs would be introduced). Hence, in the sequel we will consider the standard plant of figure 5.8 as an alternative for configuration I.

Next we will consider the selection of appropriate values for weight functions W_{1e}, W_{1p}, W_e and W_u. For this, we will assume that the entire disturbance input vector $w = (w_e' \ w_p' \ u_r')'$ is normalized, such that its 2-norm, or RMS-norm, is equal to one. Furthermore, we will assume that the weight functions must be constructed such that the control objectives are achieved when the control objectives vector also has a 2-norm of one. This has the advantage that any controller with which the closed-loop transfer function from w to z has ∞-norm less than one, must achieve the desired control objectives.
The measurement noise on the applied displacement sensors will be set to 0.1 mm, resulting in errors of 0.1 mm in the x and y directions and about 0.001 rad in ϕ direction:

$$W_{1e} := \begin{bmatrix} 0.0001 & 0 & 0 \\ 0 & 0.0001 & 0 \\ 0 & 0 & 0.001 \end{bmatrix} \tag{5.33}$$

Pressure measurement inaccuracies are expected to be about 1% of full range, we select: $W_{1p} := 0.01 \cdot I_3$.
Weight functions W_u and W_e should be set up to reflect the control objectives. As the range of the control input vector has already been scaled to one we choose $W_u := I_3$; selecting W_e is however a bit more complicated. Obviously, we want the position error to be small at low frequencies, but we also know that we must allow the position error to be large at higher frequencies. This implies that we need a frequency dependent weight function for the position error.
At high frequencies the system will not respond to any reference inputs: the expected amplitude of the error will be equal to the expected amplitude of r. We already stated that the expected amplitude of r at higher frequencies will be limited to 75 mm in x and y directions and $\frac{\pi}{4}$ rad in ϕ direction. Our choice for W_e will therefore be such that high frequency reference signals within these boundaries result in a weighted error amplitude of one. At low frequencies we

want to obtain a considerable reduction of the error: we will aim at better than 1% of the open-loop error (equivalent to -40 dB).
To realize this a first order weight function is taken for each error signal, such that we get the following form for W_e:

$$W_e := \frac{1}{\gamma} \cdot \frac{s + 100 \cdot 2\pi \cdot \beta}{s + 2\pi \cdot \beta} \cdot \begin{bmatrix} \frac{1}{0.075} & 0 & 0 \\ 0 & \frac{1}{0.075} & 0 \\ 0 & 0 & \frac{4}{\pi} \end{bmatrix} \tag{5.34}$$

The parameters β and γ may be used to further specify the effect of W_e: β determines the cross-over frequency of the weight function and γ determines its over-all amplification factor. Because of this, γ is the obvious choice to act as the parameter to be minimized in the H_∞ controller synthesis procedure.
A Bode-magnitude plot of W_e with a conservative choice for β: $\beta = 0.01$, is given in figure 5.9. The goal of obtaining 40 dB attenuation for reference signals with a

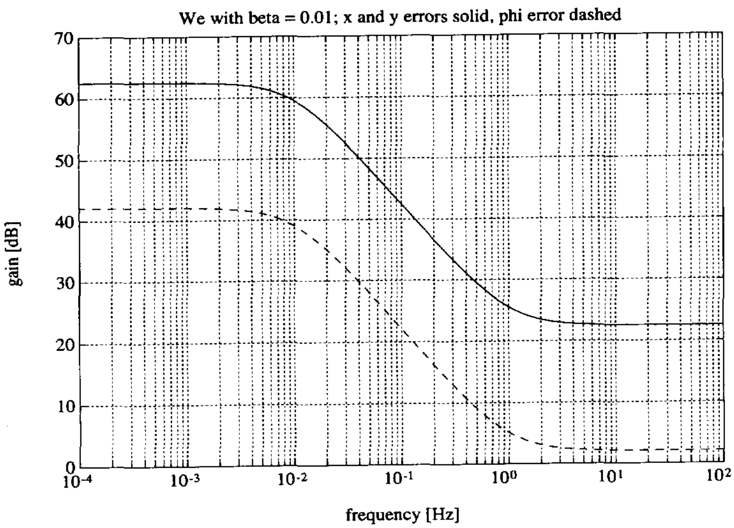

Fig. 5.9: Bode-magnitude plot of W_e with $\beta = 0.01$ and $\gamma = 1$; solid for x and y errors, dashed for ϕ error

frequency below 0.01 Hz will be achieved if we are able to find an H_∞ controller for $\gamma \leq 1$.

We are now able to set up the state-space description of the entire standard plant P, based on that of $\tilde{G}$, R and W_e. The SSD of $\tilde{G}$ was already given as $\{A_G, B_G, C_{\tilde{G}}, D_{\tilde{G}}\}$ and state-space realizations of R and W_e, given as $\{A_R, B_R, C_R, D_R\}$ and $\{A_e, B_e, C_e, D_e\}$ are easy to find. The state space matrices of P are then given as:

$$A_P = \begin{bmatrix} A_R & 0 & 0 \\ -B_eC_R & A_e & B_eC_{\tilde{G}1} \\ 0 & 0 & A_G \end{bmatrix} \qquad B_P = \left[\begin{array}{ccc|c} 0 & 0 & B_R & 0 \\ 0 & 0 & -B_eD_R & B_eD_{\tilde{G}1} \\ 0 & 0 & 0 & B_G \end{array}\right]$$

$$C_P = \left[\begin{array}{ccc} -D_eC_R & C_e & D_eC_{\tilde{G}1} \\ 0 & 0 & 0 \\ \hline -C_R & 0 & C_{\tilde{G}1} \\ 0 & 0 & C_{\tilde{G}2} \end{array}\right] \quad D_P = \left[\begin{array}{ccc|c} 0 & 0 & -D_eD_R & D_eD_{\tilde{G}1} \\ 0 & 0 & 0 & W_u \\ \hline W_{1e} & 0 & -D_R & D_{\tilde{G}1} \\ 0 & W_{1p} & 0 & D_{\tilde{G}2} \end{array}\right]$$

5.3.2 Construction of the modified standard plant for the 3DOF system

It may now be verified that the standard plant we obtained in the previous subsection is not suitable for application of standard H_∞ synthesis: in particular assumptions A1 and A3 are simultaneously violated due to the unstable dynamics in R. Although an ad hoc approach for solving this problem is possible in this relatively simple case, we will use it as an example for the application of the procedure discussed in chapter 4.

First we will check the solvability of the problem with the help of theorem 4.4.3. To do this we will not actually calculate the polynomial matrices Q and L that solve equation 4.11, but merely show that these exist. For this, it is sufficient to look at solvability of the problem when only position error measurements are available, i.e. only the first three elements of $y_{\tilde{G}}$ (equation 5.32) are used. The resulting plant model will be denoted as $\tilde{G}_1$, having an SSD: $[A_G, B_G, C_{\tilde{G}1}, D_{\tilde{G}1}]$. Now consider an RCF of $\tilde{G}_1$ given as: $[N_{\tilde{G}1}, D_{\tilde{G}1}]$. Clearly both matrices are square (three inputs and three outputs). Furthermore it can easily be verified that $N_{\tilde{G}1}$ has full rank, and that $\tilde{G}_1$ does not have any transmission zeros. This implies that $N_{\tilde{G}1}(z)$ does not loose rank for any $z \in \mathrm{C}$ (definition 2.4.5), such that $\det(N_{\tilde{G}1})$ is nonzero and not a function of s. Hence, $N_{\tilde{G}1}$ must be unimodular and have a polynomial inverse (Chen 1984).

Next, from equation 5.30 it is easy to verify that an RCF of R may be given as

$[D_R, N_R]$ with $D_R := \tau s \cdot I_3$ and

$$N_R := \begin{bmatrix} \kappa_x(\tau s+1) & 0 & 0 \\ 0 & \kappa_y(\tau s+1) & 0 \\ 0 & 0 & \kappa_\phi(\tau s+1) \end{bmatrix} \tag{5.35}$$

We may therefore set L equal to zero and define $Q := N_{\tilde{G}1}^{-1} N_R$, such that according to theorem 4.4.3 there must exist a solution for the tracking problem, both for $\tilde{G}_1$ and $\tilde{G}$ (the addition of extra measurement signals can never influence the solvability).

The next step is to construct the necessary servocompensator $\tilde{D}_M^{-1}$ as defined in equation 4.18. Again we do not need to explicitly construct Q for this particular case: $\tilde{G}$ already contains the necessary internal model, which can be verified by checking that $D_{\tilde{G}}(s) = 0$ for $s = 0$. Hence, $D_{\tilde{G}} Q D_R^{-1}$ must be polynomial and have full rank: there exists an LCF of M with $\tilde{D}_M = I_3$.

Finally, we can set up the required 'modified control configuration for the tracking problem' in accordance with figure 4.9 (configuration II). With $\tilde{D}_M = I_3$ it is easily verified that in this case configurations I and II are equal: a state-space description of P_{II} is equal to that of P as given in the previous subsection. The determination of a minimal realization of P_{II} can be performed numerically. The resulting standard plant complies with assumptions A1 through A4 and can therefore be dealt with using the standard H_∞ procedure as implemented, for instance, in PC MatLab (see Moler et al. 1987, Balas et al. 1993). This will be the subject of the following subsection.

5.3.3 Application of H_∞ synthesis, implementation and tuning

Before being able to apply the procedure for the calculation of sub-optimal H_∞ controllers presented in section 3.4 to the previously defined standard plant, it should be noted that there is a small but conceptually important difference between the formulation of the sub-optimal H_∞-problem in definition 3.4.2 and the problem that is considered here. First, we have $\|T_{wz}\|_\infty < 1$ rather than $\|T_{wz}\|_\infty < \gamma$: the iteration will be performed on the parameter γ that was already incorporated into the standard plant by defining W_e to be a function of γ. More importantly, we take W_u to be constant, which implies that only $\|T_{wz_1}\|_\infty$ is to be minimized while maintaining a constant bound on $\|T_{wz_2}\|_\infty$, i.e. we want to minimize the weighted positioning errors for a given constant level of actuator effort. The level of performance, as specified in subsection 5.3.1, that we are trying to obtain by this procedure, will be achieved if we find a controller such that both $\|T_{wz}\|_\infty < 1$ and $\gamma \leq 1$.

The result of this will be a controller for which the closed-loop transfer function T_{re} from references r to positioning errors e is determined by the combination of R and W_e. In accordance with the discussion given in section 4.7 and under the assumption that $\|T_{wz}\|_\infty \leq 1$, we can determine an upper bound for $\bar{\sigma}(T_{re})$ as follows:

$$\begin{aligned} \bar{\sigma}(T_{re}(\omega)) &= \bar{\sigma}(W_e^{-1}(\omega)W_e(\omega)T_{re}(\omega)R(\omega)R^{-1}(\omega)) \\ &= \bar{\sigma}(W_e^{-1}(\omega)T_{u_r z_1}(\omega)R^{-1}(\omega)) \\ &\leq \bar{\sigma}(W_e^{-1}(\omega))\bar{\sigma}(T_{u_r z_1}(\omega))\bar{\sigma}(R^{-1}(\omega)) \\ &\leq \bar{\sigma}(W_e^{-1}(\omega))\bar{\sigma}(R^{-1}(\omega)) \qquad \forall\, \omega \in \mathrm{R} \end{aligned} \tag{5.36}$$

Note that this inequality can be set up for any submatrix or element of T_{wz}.

In the case of this example, we find $\gamma = 4.071$ with a tolerance of .001, such that the *optimal* value for γ is given as: $\gamma_o \in [4.070, 4.071]$. This implies that we are not able to comply with the specifications that were incorporated in the standard plant, for which we should be able to find $\gamma \leq 1$. To verify this, we will consider the upper bound for each of the three diagonal elements of T_{re} as can be determined from inequality 5.36. In figure 5.10 the Bode-magnitude plot for each of these three diagonal elements is given, together with the upper bound as desired and the upper bound as actually obtained. It should be noted that tracking of constant reference signals is apparently obtained, as the diagonal elements of T_{re} have a transmission zero at $\omega = 0$: in fact $\bar{\sigma}(T_{re}(0)) = 1.12 \cdot 10^{-12}$, which is within the numerical accuracy of being zero.

To get an impression of the time-domain behaviour of the closed-loop system, figure 5.11 shows the unit step responses of the moving block in its three coordinates. Obviously, we have obtained an extremely low performance controller, which may be expected from the conservative choices that were made for parameters τ and β. However, for this reason it is suitable for a first implementation on the experimental set-up to test the proper operation of the equipment. Measured step responses that were rescaled to match those of figure 5.11 are given in figure 5.12. It should be mentioned that achievement of the tracking objective can only be verified on a much larger time scale.

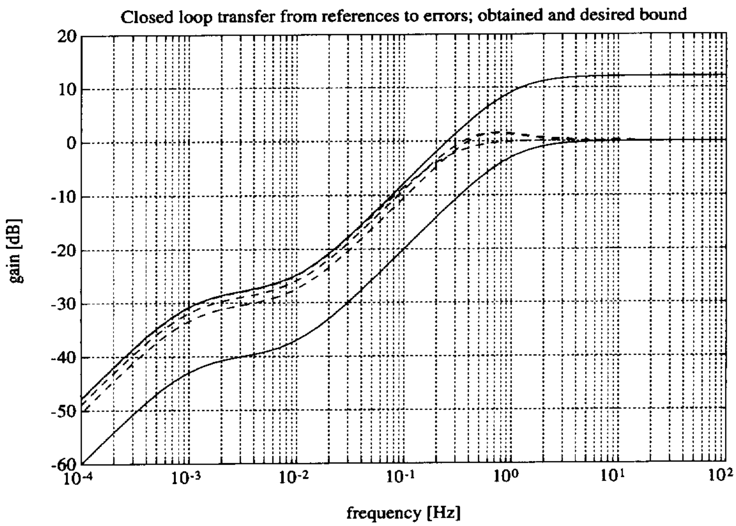

Fig. 5.10: Bode-magnitude plot of diagonal elements of T_{re} (dashed) and their desired and obtained upper bound (solid): first controller

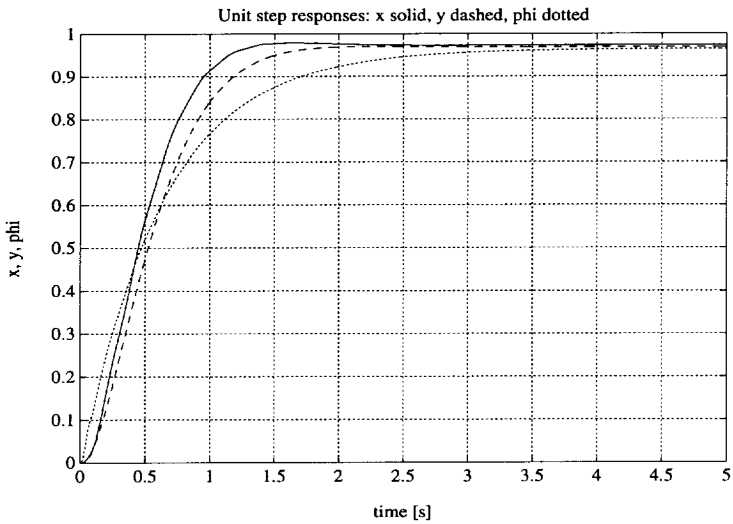

Fig. 5.11: Simulated step responses to unit steps on x (solid) y (dashed) and ϕ (dotted) references: first controller

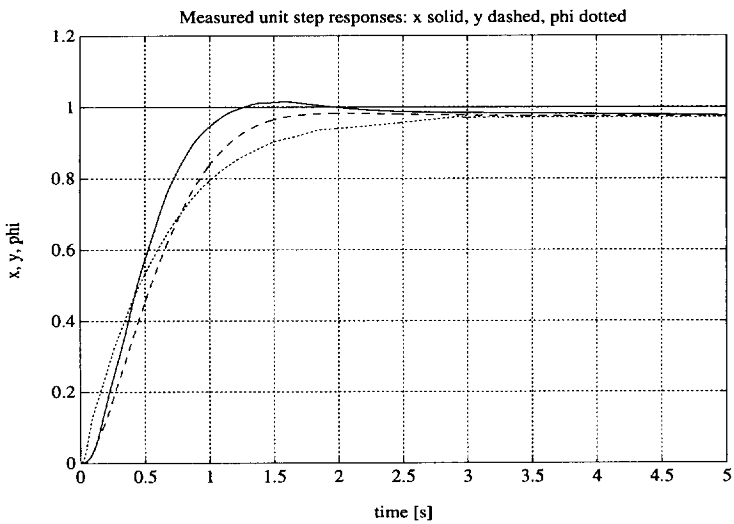

Fig. 5.12: Measured step responses to unit steps, on x (solid) y (dashed) and ϕ (dotted) references: first controller (rescaled: actual step sizes were 40 mm and 0.4 rad)

The main purpose of the previous excercise is not to obtain a high performance controller in one stroke, but to set up an environment that allows a *tuning* procedure for finding one. For this it is important that we have set up the standard plant in such a way that selected weight functions and parameters allow a physical interpretation, or directly reflect important design objectives. Adjustment of these weight functions and parameters can now be done in an iterative way, and allow the designer to precisely specify the trade-off between the separate control objectives. To summarize the effects of previously introduced weight functions and parameters, we review them in order of appearance:

- τ determines the frequency up to which tracking is pursued; decreasing τ implies a higher frequency and puts more emphasis on the tracking objective at the cost of higher actuator effort and decreased measurement noise reduction: the asymptotic tracking property is guaranteed by the presence of integral action, but τ determines the speed of response in the sense of how fast the error diminishes to an acceptable level,

- κ stands for the expected size of the high frequency part of the reference signals; decreasing κ implies that reference signals are not expected to have a significant high frequency part, resulting in more emphasis on the low frequency part, but demanding greater care in the generation of reference signals,

- W_{1e} and W_{1p} reflect expected measurement noise levels: decreasing their size results in a higher overall performance, but obviously also a higher sensitivity to actual measurement noise,

- W_u determines allowable actuator effort: a decrease allows higher performance levels, but it should be chosen such that actuator effort stays within the physical limits of the actuators and the set-up (power levels) under expected operating conditions; furthermore, an increase is usually advantageous for stability robustness of the closed-loop system,

- β sets the desired cross-over frequency for the closed-loop system: increasing β results in an increased bandwidth of the closed-loop system, at the cost of an increase in actuator effort, measurement noise sensitivity and, in most cases, stability robustness,

- γ determines the overall amplification factor of $T_{u_r z_1}$: it is the most obvious parameter for iteration in the H_∞ synthesis procedure, as $\gamma \leq 1$ implies that all other objectives are satisfied.

In view of these considerations, it was decided to reduce κ for all three reference signals to $\kappa_x = \kappa_y = 0.004$ and $\kappa_\phi = \frac{\pi}{75}$. This allows us to increase both τ and β to obtain faster step responses: it appeared that $\tau = \frac{1}{2\pi}$ and $\beta = 1$ are possible without unacceptable increase of actuator effort. With W_{1e}, W_{1p} and W_u unchanged, a γ of 1.606 was found with a tolerance of 0.001. Although $\gamma > 1$, which implies that the error signals are slightly larger than specified, the resulting controller was found to be acceptable. The significant improvement in performance obtained with this controller can clearly be observed in both frequency-domain and time-domain: figure 5.13 shows the Bode-magnitude plot for the three diagonal elements of T_{re} with their obtained upper bound (compare with figure 5.10) and figure 5.14 gives the associated unit step responses (compare with figure 5.11).

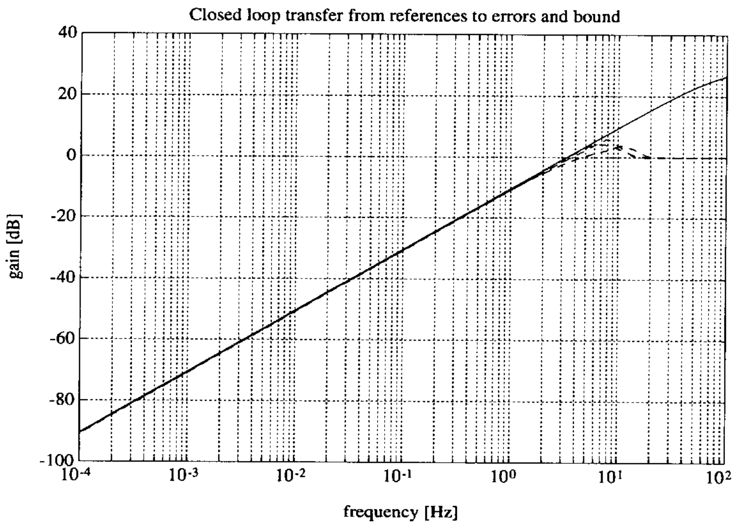

Fig. 5.13: Bode-magnitude plot of diagonal elements of T_{re} (dashed) and their obtained upper bound (solid): second controller

Implementation of this controller on the 3DOF system resulted in the measured step responses given in figure 5.15: they were rescaled to match the unit step responses in figure 5.14. Obviously, the measured responses are significantly different from those expected: there is more overshoot and less damping. This suggests that there is a robustness problem with the obtained controller, which will be analyzed and solved in the following section.

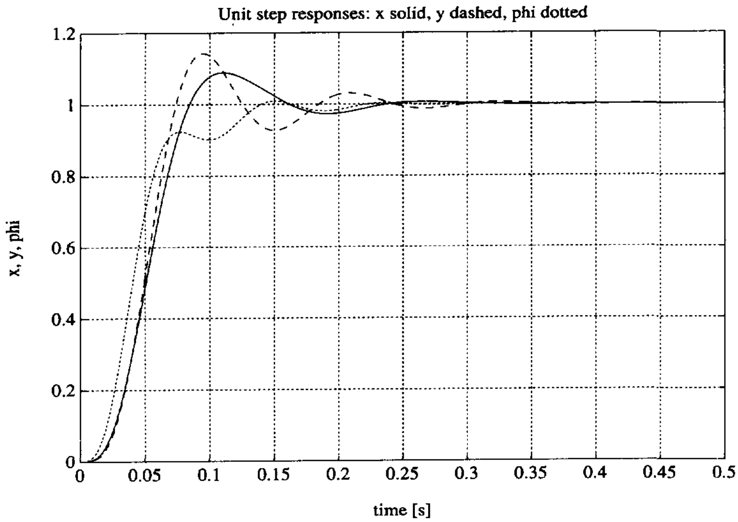

Fig. 5.14: Simulated step responses to unit steps on x (solid) y (dashed) and ϕ (dotted) references: second controller

5.4 Robustness analysis and design

In the previous section we found significant differences between the simulated behaviour of the controlled system and the actual behaviour, as measured on the experimental set-up of the 3DOF system. Although some small differences may be expected and should be accepted when using linear control theory for the design of a controller for a clearly non-linear set-up, we will show that the behaviour of the controlled system can be improved significantly by means of robustness analysis and design. The first step will be the modelling of the relevant uncertainties and the setting up of the 'standard control structure with uncertainties' as defined in section 3.5 and given in figure 3.11. In doing so, it will appear to be possible to define a highly structured uncertainty model by means of the procedure in section 3.7. Next, we will analyse the robustness of the controller developed in the previous section by means of the structured singular value, as discussed in section 3.6. Finally, we will construct an H_∞ sub-optimal controller for the standard plant with uncertainties. In spite of a possibly large amount of conservatism, this controller will appear to be robust with hardly any decrease in nominal performance as compared to the controller developed in the previous

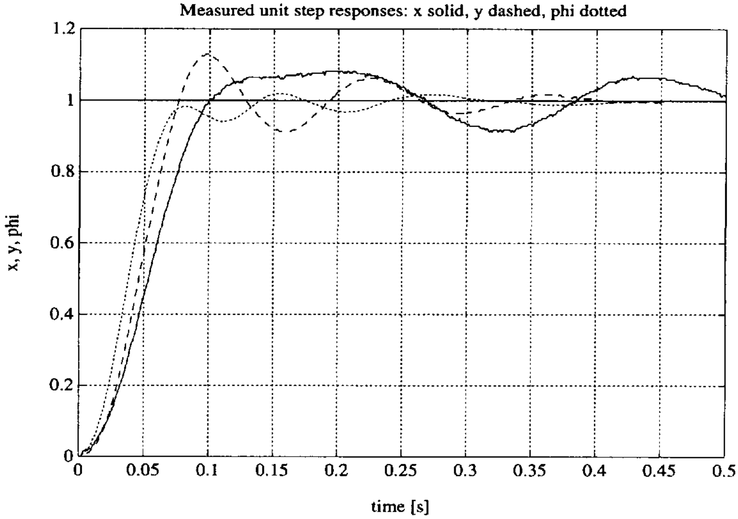

Fig. 5.15: Measured step responses to unit steps, on x (solid) y (dashed) and ϕ (dotted) references: second controller (rescaled: actual step sizes were 3 mm and 0.02 rad)

section.

5.4.1 Uncertainty modelling for the 3DOF system

When considering uncertainties that may be of importance for the actual behaviour of the 3DOF system, all four separate causes mentioned in section 3.5 could be investigated:

- neglected linear behaviour can be found in the use of a steady-state approximation for the servo valve and, for instance, the assumption that the actuator rods are infinitely stiff,
- for several physical quantities, like hydraulic compressibility coefficient and rated flow, manufacturer's specifications were used, that are only accurate up to a certain tolerance,
- examples of time varying parameters are viscous friction and leakage, which can be seen from the differences between actuator 1 and actuators 2 and 3 (equation 5.1),
- main non-linearities are geometric non-linearity, dry friction and the effect of the piston area ratio of 0.56 (it should be 0.5 for linearity).

However, from experimental measurements it was found that the most important effect on the dynamical behaviour of the 3DOF system consists of variations in the frequencies of its resonant modes. These variations can be explained by variations in the compressibility coefficient E and the neglected flexibility of the hydraulic hoses that were used. As these quantities can not easily be changed, we will consider a different variation with similar effects: a variation in ineffective volume. The statement that this variation has similar effects can be verified from the state-space matrices A_G and B_G given in section 5.2: v_o appears in the same entries as E. By considering variations in v_o we may set up a robust control problem for a well-defined parameter variation, which can also be performed on the actual set-up without too much effort.

For analysis purposes we will consider variations in v_o of 0.5 litres, which is significant with respect to the nominal v_o of 1.5 litres. Furthermore we will assume that variations in v_o may occur independently for each actuator: $v_{oi} \in [1.0, 2.0]$, $i = 1 \cdots 3$. The effect of this on the dynamical behaviour of the open loop system is illustrated in figure 5.16, in which the Bode-magnitude plot is given of the open loop transfer functions from actuator inputs to actuator displacements in the nominal case $v_{oi} = 1.5$ litres, the 'minimal' case $v_{oi} = 1.0$ litres and the 'maximal' case $v_{oi} = 2.0$ litres.

It is now straightforward to set up the state-space matrices for the uncertainty model of the 3DOF system, or rather the *set* of models, parametrized by the vector $\theta = [v_{o1}\ v_{o2}\ v_{o3}]'$, as:

$$G_u(\theta) := [A_G(\theta), B_G(\theta), \tilde{C}_G, \tilde{D}_G] \tag{5.37}$$

Next, we can apply the parametric uncertainty modelling procedure given in section 3.7 to find:

$$G_u(\theta) = \mathcal{F}_u(M, \Delta) = M_{22} + M_{21}(I - \Delta M_{11})^{-1}\Delta M_{12} \tag{5.38}$$

It appears that we can take $\Delta = \text{diag}(\delta_1, \delta_2, \delta_3)$, with $\delta_i \in [-1, 1]$ representing the variation of v_{oi}. Note that we thus find that $\Delta \in \mathbf{\Delta}_{rr}$ as defined in equation 3.60. In comparison with $G_u(\theta)$, we have that M is a transfer function matrix with three extra inputs and three extra outputs representing the effect of the uncertainties. Substitution of M for $\tilde{G}$ in figure 5.8 then provides us with the 'standard control structure with uncertainties' given in figure 5.17 (compare with figure 3.11). With the state-space realization of M, which follows directly from the procedure of section 3.7, and with SSDs of the frequency dependent weight functions, it is possible to construct a state-space model for the entire standard plant P_{u} as was done for P in section 5.3.

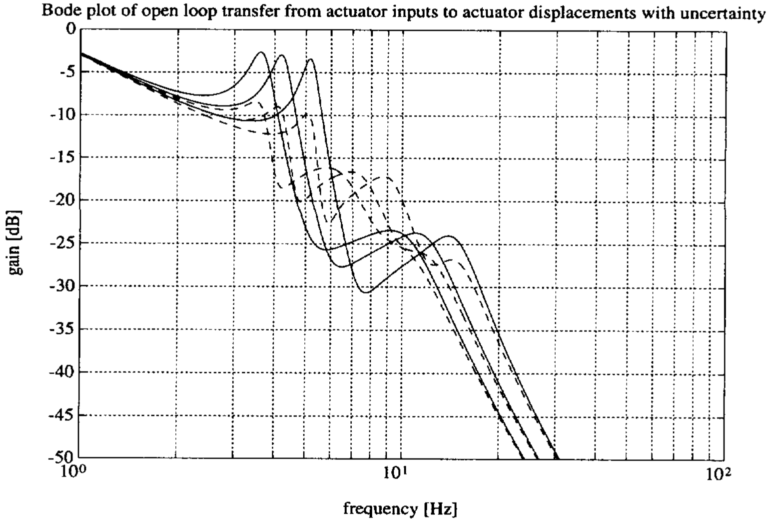

Fig. 5.16: Bode-magnitude plot of the open loop transfer functions from actuator inputs to actuator displacements with $v_{oi} = 1.0, 1.5, 2.0$ litres: actuator 1 solid, actuators 2 and 3 dashed

5.4.2 Robust stability analysis

To be able to check whether the controller from the previous section is robust against the specified variations in v_o, we will incorporate it into the standard plant P_u and we will only consider the three uncertainty inputs and outputs w_Δ and z_Δ (w_e, w_p, u_r, z_1 and z_2 are discarded). The resulting interconnection structure for robust stability analysis will be denoted as N_{rs}: note that in this case we have $N_{rs} = \mathcal{F}_l(M, K)$. Hence we have that N_{rs} determines the closed loop stability of the 3DOF system as a function of uncertainty matrix Δ, according to the feedback configuration of figure 5.18. With the small gain theorem (theorem 3.5.3) and the definition of the structured singular value (definition 3.50), we thus have that robust stability of the closed loop system is guaranteed if $||N_{rs}||_\infty < 1$, or if *and only if* $||N_{rs}||_\mu < 1$ (see equation 3.51).

Using the μ-Tools toolbox for PC MatLab (see Moler et al. 1987, Balas et al. 1993), both $\bar{\sigma}(N_{rs})$ and an upper bound of ('real') $\mu_\Delta(N_{rs})$ were calculated and plotted over the relevant frequency range in figure 5.19. From this we can see that neither bound is smaller than 1 (i.e. 0 dB), which implies that no robust stability can be guaranteed for the 3DOF system with the second controller and variations in v_o between 1.0 and 2.0 litres. Furthermore, it can be seen that the largest singular

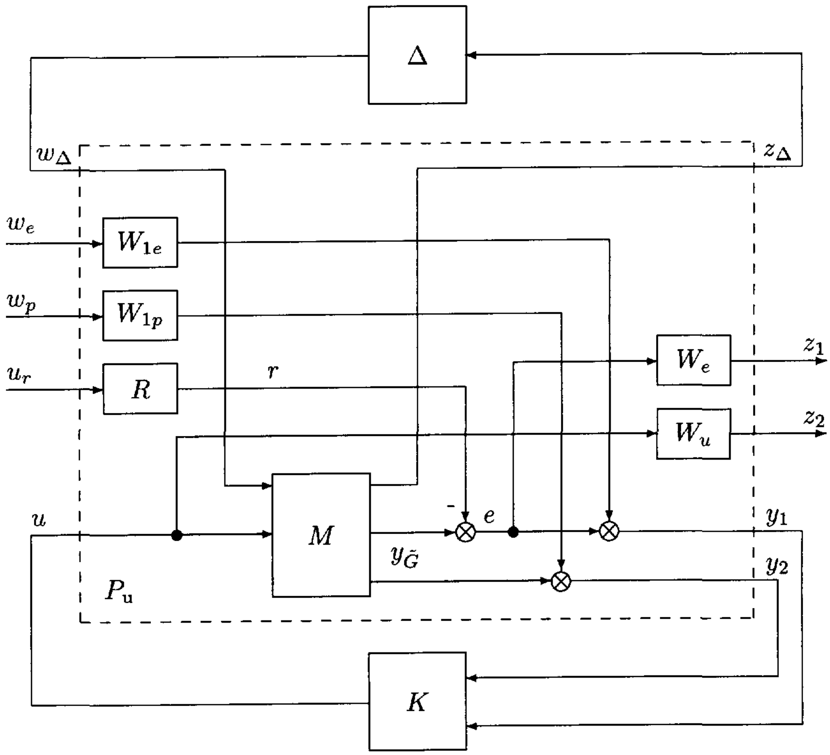

Fig. 5.17: The standard control structure with uncertainties for the 3DOF system

value is indeed an upper bound for μ and is therefore conservative. The spiky appearance of the μ-plot is typical for problems with purely real perturbations; this also causes the lower bound to be zero, or rather incalculable, at most points. This implies that also the calculated upper bound for μ may be conservative: to show that this is not the case, figure 5.20 gives the simulated unit step responses of the 3DOF system with all ineffective volumes set to 2.0 litres. Clearly this system is indeed unstable, and we may conclude that the used controller does not achieve robust stability with respect to variations in v_o from 1.0 to 2.0 litres.

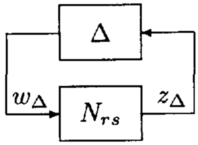

Fig. 5.18: Feedback structure for robustness analysis of the 3DOF system

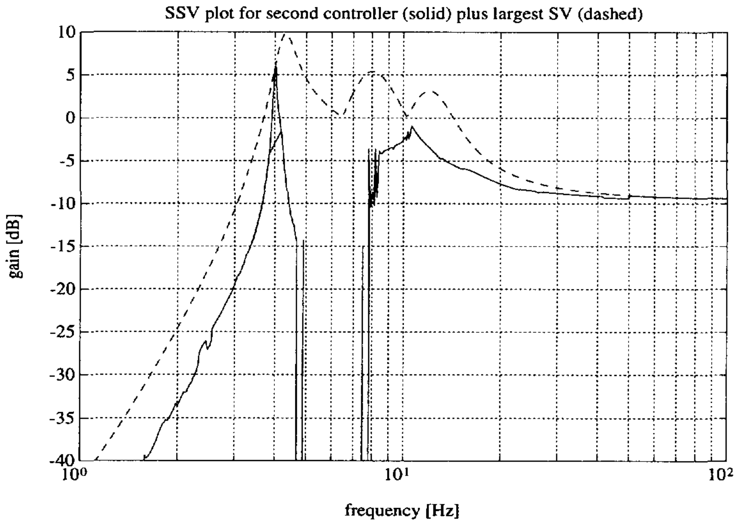

Fig. 5.19: Structured singular value plot (solid) and largest singular value plot (dashed) for robust stability analysis of 3DOF system with second controller

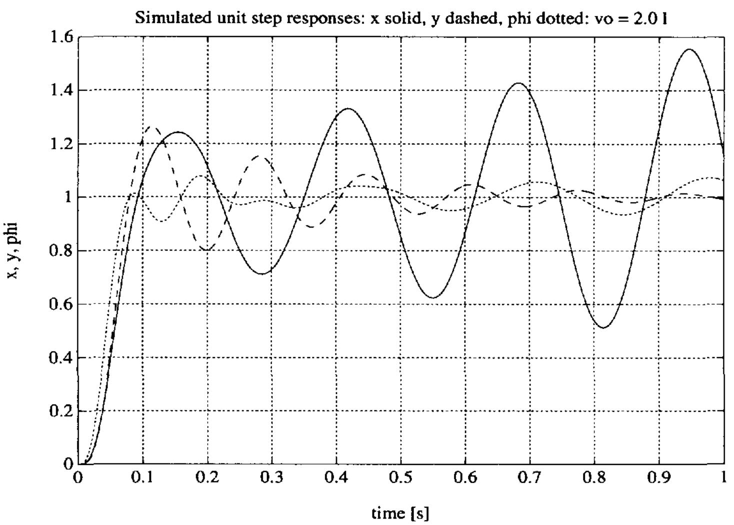

Fig. 5.20: Simulated step responses to unit steps on x (solid) y (dashed) and ϕ (dotted) references: second controller with perturbed plant ($v_o = 2.0$ litres)

5.4.3 Robust controller design using H_∞ synthesis

To obtain a robust controller for the 3DOF system we could now apply the D-K iteration procedure suggested in subsection 3.6.4 on the standard plant in figure 5.17. Instead however, we will use an approach that is often sufficient in practical situations and that makes it unnecessary to construct transfer function fits for D-scalings. In spite of the simplicity of the approach and the resulting 'danger for conservatism', it will appear to be possible to significantly improve robustness in the case of the 3DOF system. However, it should be noted that no guarantees are obtained with respect to robust stability: whether robust stability is obtained must be verified afterwards using μ-analysis.

The basic idea is that, with the definition of the uncertainty model for the 3DOF system, we have introduced three extra input and output signals: w_Δ and z_Δ respectively, that could be used in a similar fashion as the external signals already defined. To do this we add one extra weight function to the standard plant P_u of figure 5.17 to obtain $\tilde{P}_\mathrm{u}$ as given in figure 5.21. For the new weight function

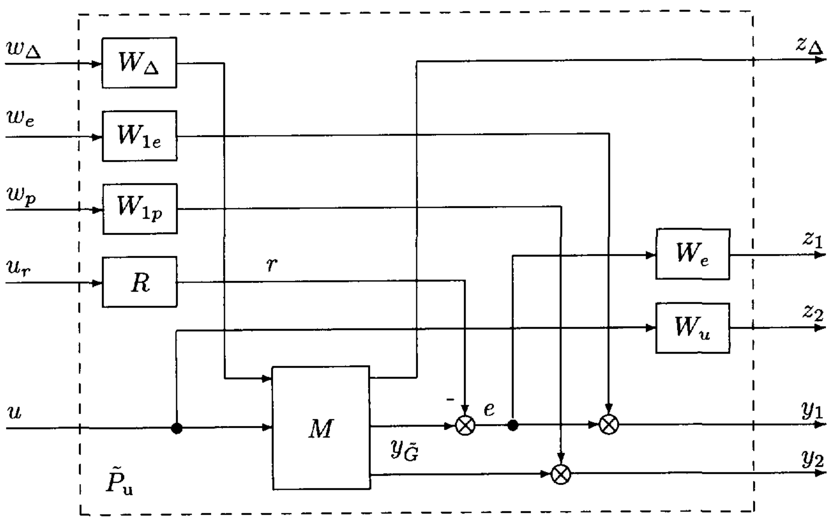

Fig. 5.21: Standard plant for robust controller design for the 3DOF system

W_Δ we choose $W_\Delta = c_\Delta I$ with c_Δ some small real constant. Just like the weight functions W_{1e}, W_{1p}, W_e and W_u together with the RSG R determine the trade-off between several performance objectives, we can use this W_Δ to find a trade-off between performance and robustness. It is noted once again that this trade-off may be conservative in the sense that better performance could be obtained with equal robustness, or better robustness with equal performance.

With all other weight function equal to those selected in the previous subsection, it appeared that a choice of $c_\Delta = 10^{-5}$ provides us with an acceptable result. A γ of 1.6595 was found with a tolerance of 0.001, which implies that only a small deterioration of the nominal performance is necessary, when compared to the previous controller with $\gamma = 1.606$. This can clearly be observed from the simulated unit step responses given in figure 5.22, that may be compared to those of figure 5.14.

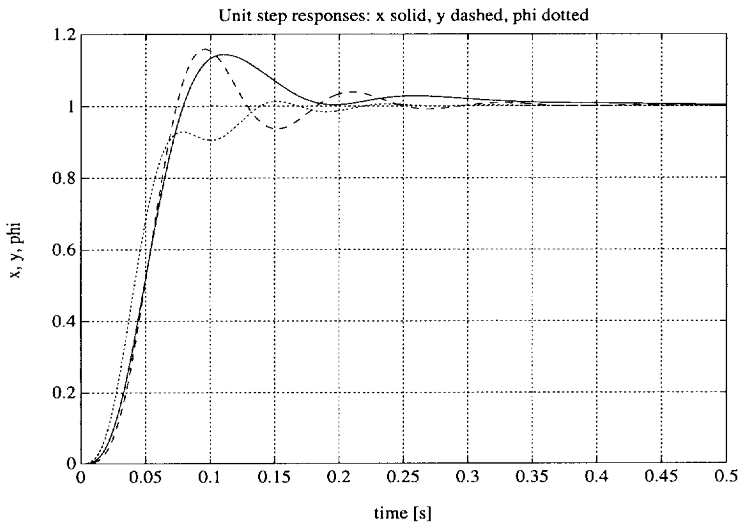

Fig. 5.22: Simulated step responses to unit steps on x (solid) y (dashed) and ϕ (dotted) references: third controller

To verify the improvement in robustness, figure 5.23 shows both $\bar{\sigma}(N_{\mathrm{rs}})$ and an upper bound of $\mu_\Delta(N_{\mathrm{rs}})$ as was done in the previous subsection in figure 5.19. Clearly, we now have $||N_{\mathrm{rs}}||_\mu < 1$ and we even have $||N_{\mathrm{rs}}||_\infty < 1$. This is of course without the effect of weight function W_Δ: c_Δ is set to 1 such that $W_\Delta = I$. Hence we may conclude that this controller does achieve robust stability with respect to the specified variations in v_o. This makes it interesting to also consider to what extent we have also obtained robust *performance*. This can be done according to section 3.6, by setting up the extended robust stability problem as depicted in figure 3.13 and calculating μ for the resulting mixed real-complex problem with a 9×6 performance block connecting outputs z_1 and z_2 to inputs w_e, w_p and

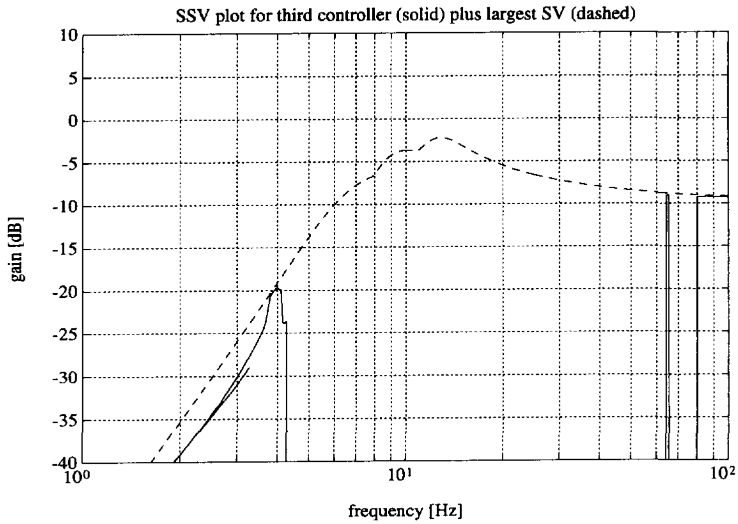

Fig. 5.23: Structured singular value plot (solid) and largest singular value plot (dashed) for robust stability analysis of 3DOF system with third controller

u_r. As we denoted the interconnection structure for robust stability analysis as N_{rs}, we will denote the structure for robust performance analysis as N_{rp} and calculate $\mu(N_{\text{rp}})$ over the appropriate frequency range: the result of this is given in figure 5.24. Note that this plot evaluates μ for the *weighted* standard plant, but with c_Δ set to 1: hence, $\mu = 1$ (0 dB) denotes the desired nominal performance. It is therefore to be expected that μ is larger than 1, as plant perturbations will usually result in performance deterioration. The amount of this deterioration is reflected by $\|N_{\text{rp}}\|_\mu$, in this case 1.29 (2.18 dB): the maximal gain from disturbance inputs to performance objectives is increased by 29% for a worst case parameter perturbation which is 29% smaller than the specified maximum (i.e. 71% of 0.5 litres).

To illustrate this performance deterioration in the time domain, figure 5.25 gives the simulated unit step responses when the controller is applied on the 3DOF system with $v_o = 2.0$ litres for all three actuators. When compared to the nominal responses given in figure 5.22 and the result of the previous controller given in figure 5.20, it is clear that we have obtained a controller which is not only robustly stable but also retains good performance.

Implementation of this controller on the actual experimental set-up will show whether this claim is correct and will allow us to verify the argument that the bad performance of the previous controller was due to lack of robustness against variations in the resonant modes. As was done in figure 5.15, the measured step responses were rescaled to unit step responses, to be able to compare them directly with figure 5.22: the result is given in figure 5.26. When compared to the simulated step responses for the nominal system (figure 5.22) there are still some clear differences: most notably the amount of overshoot in y-direction. It appeared that this was caused by an error in modelled viscous friction coefficient for actuators two and three, which was due to wear. In spite of this, we can still conclude that the general closed loop behaviour of the 3DOF system is significantly improved, by forcing robustness against variations in the frequencies of the resonant modes.

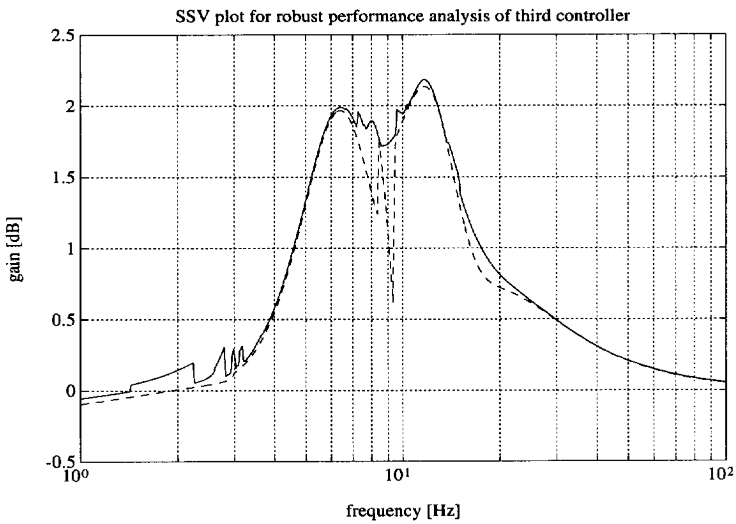

Fig. 5.24: Structured singular value plot for robust performance analysis of 3DOF system with third controller

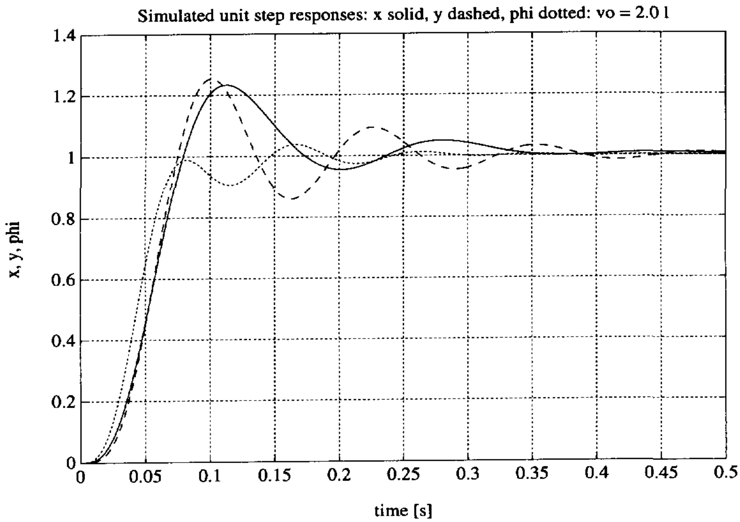

Fig. 5.25: Simulated step responses to unit steps on x (solid) y (dashed) and ϕ (dotted) references: third controller with perturbed plant ($v_o = 2.0$ litres)

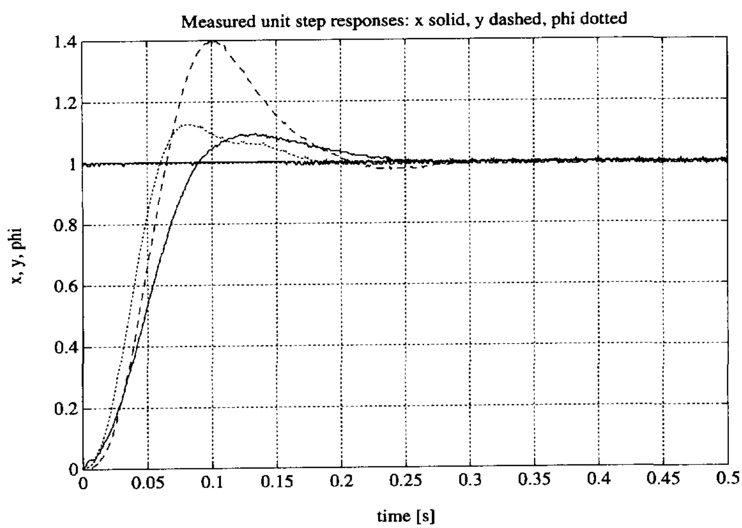

Fig. 5.26: Measured step responses to unit steps, on x (solid) y (dashed) and ϕ (dotted) references: third controller (rescaled: actual step sizes were 3 mm and 0.02 rad)

Chapter 6

Conclusions

This thesis provides a complete procedure for incorporating the output regulation and tracking problem, also known as the servomechanism problem, into the general framework of the recently developed robust control paradigm. The motivation for doing this is based on the important advantages of robust control methods over both the 'classical' control approach and the 'modern' control approach and will be briefly reviewed in section 6.1. The fact that the output regulation and tracking problem constitutes important difficulties when attempting to apply robust control methods is recognized by many researchers; properties of approaches and solutions developed in literature will be considered in comparison with the procedure developed in this thesis in section 6.3, after a review of the properties of the proposed procedure in section 6.2. The specific results of application on a hydraulic positioning system with three degrees of freedom will be considered in section 6.4. Finally, section 6.5 will give recommendations with respect to possible extensions of the presented approach.

6.1 Advantages of robust control methods

The development of control theory should always be aimed at the solution of problems that may occur in practice. Whereas 'classical' control methods provide solutions for a great many problems still encountered to date, the transfer to 'modern' control methods was marked by the growing need for a more general approach. In many cases demands for higher performance necessitated a multivariable model-based control theory, to be able to deal with the problem of interaction between several important process variables. The time-domain framework and state-space methods of this period prompted great interest in the modelling of the system to be controlled; especially when it appeared that the applied methods are very sensitive to modelling errors. In spite of this it was found that in many relevant cases it is not possible to find models that are sufficiently accurate: in general this is caused by non-linearities, time-dependencies and neglected dynamical behaviour. The result of this is that there are only a very few examples of successful long-term application of the developed theory.

A better understanding of the causes for the failure of application of modern control theory was developed when frequency-domain properties of resulting controllers and closed-loop systems were considered. This led to the introduction of singular values analysis and multivariable frequency-domain methods for controller synthesis: H_2 and H_∞ methods. More importantly, it led to the introduction of a general framework for control system design: the standard control design structure or standard plant framework. This framework provides a basis for the formulation of the control problem such that the application of the newly developed methods can be done by means of a standard procedure. On the other hand, it allows sufficient flexibility in setting up the control problem, such that concepts from both classical and modern control theory can be included. The modelling of the system to be controlled is usually done by means of multivariable state-space methods. The setting up of performance specifications, like disturbance attenuation and actuator effort minimization, and robustness specifications, based on either unstructured or structured plant uncertainties, is usually done by means of frequency dependent weight functions.

The advantage of this approach is that the actual controller synthesis may be performed in a single well-defined numerical optimization by means of several available methods: H_2 and H_∞ synthesis, mixed H_2/H_∞ methods (under development) and *D-K* iteration (under development). The inherent trade-off between the specified design objectives is not primarily dependent on the selected synthesis method, but on the selection of appropriate weight functions. Disturbance inputs are characterized by (frequency-dependent) input weights and desired closed loop behaviour is given by (frequency dependent) output weights. This may be seen as one of the main advantages for practical application of controller synthesis methods based on the standard control design structure: these weight functions usually allow a simple physical interpretation and have sufficient flexibility to accommodate commonly stated demands.

The selection of controller synthesis method is of importance for the *level* at which the underlying trade-off is performed: the applied method should as much as possible be in accordance with the type of problem under consideration. Because of this, H_2 methods are appropriate when disturbances and objective functions are of a statistical nature, whereas H_∞ methods consider absolute bounds on signals, which is more appropriate for robustness objectives (via the small gain theorem) and the suppression of undesirable dynamical behaviour. Being unable to achieve the highest possible level of trade-off is generally indicated as conservatism of the applied method: mixed H_2/H_∞ methods and D-K iteration are clear attempts to reduce conservatism. This constitutes another important reason to define control problems, including servomechanism problems, in terms of the standard control design structure: new developments in the reduction of conservatism are immediately applicable.

6.2 Properties of the presented procedure

In view of the advantages of application of robust control methods within a standard plant framework, it is sensible to attempt to include the known solutions to the linear servomechanism problem in this approach, such that output regulation and tracking objectives can also be dealt with. This would allow the solution of linear multivariable control problems in which design objectives are a mixture of performance specifications like disturbance attenuation and actuator effort minimization, robustness specifications based on either unstructured or structured plant uncertainties and asymptotic tracking objectives for polynomial or sinusoidal persistent reference signals. Unfortunately, it appears that it is not possible to do this without violating an assumption that is necessary for the application of standard robust control methods: the internal stabilizability of the standard plant.

Usually this problem appears in the occurrence of invariant zeros on the imaginary axis, which are not allowed when H_2 or H_∞ methods are to be applied. As this problem seems to be of a technical nature, it is often considered to generalize the used methods such that solutions may be found that allow imaginary hidden modes in the closed-loop standard plant, as long as they are the result of weight functions rather than actual plant dynamics. This implies that the difficulties with the application of robust control methods on output regulation and tracking problems are seen as a flaw in the applied synthesis procedures. The approach considered in this thesis is different: difficulties with output regulation and tracking problems are seen as the result of incorrect problem formulation, i.e. the incorrect incorporation of the tracking problem into the standard control design structure.

For this reason, this thesis provides a procedure for setting up a standard plant that contains the output regulation and tracking objective in such a way that standard controller synthesis methods can be applied. The procedure consists of a number of steps that can be outlined as follows:

1. given a linear model of the system to be controlled, set up a linear model of the expected persistent signals: a Reference Signal Generator (section 4.3),
2. construct the 'standard control configuration for the tracking problem' as given in figure 4.8 (configuration I),
3. consider solvability by means of theorem 4.4.3 and construct the appropriate servo compensator ($\tilde{D}_M^{-1}$),
4. construct the 'modified control configuration for the tracking problem' as given in figure 4.9 (configuration II),
5. define the input signals of the servo compensator as new control inputs: consider these new inputs as a measure for the actuator effort (see equation 4.28),
6. add any other disturbance inputs, control objectives and weight functions.

The resulting standard plant should then comply with the necessary assumptions for application of standard H_2 and H_∞ methods.
The most important properties of this procedure may be summarized as follows:

- the tracking objective is defined by the RSG: the RSG may be any linear time-invariant model with imaginary poles: any combination of sinusoids and polynomials can be incorporated,
- the necessity of adding a servo compensator, based on the RSG, to the controller results from the solvability condition for the tracking stability problem (theorem 4.4.3),
- a 'minimal' servo compensator is constructed, as suitable dynamics of the physical system may also be used to obtain the tracking objective: the 3DOF system is a clear example of this,
- if the RSG is constructed with the minimal order needed for the definition of the occurring persistent signals, the servo compensator will also be of minimal order: extension of the RSG to the 'robust' problem will prevent the need to solve polynomial equation 4.11 and determine $\tilde{D}_M$ from equation 4.18, both of which are numerically difficult,

- the effect of weight functions for other control objectives than output regulation and tracking is affected by the RSG and necessary modifications to the standard plant: tuning of weight functions may be required, but is not complicated as their physical interpretation remains valid,

- by considering the extended tracking problem (section 4.5) it is possible to separately influence all state variables of the servo compensator: this allows a trade-off between amplitudes of these state variables and speed of response with respect to the tracking objective,

- the 'robust servomechanism' solution will be found if the RSG does not have any directional information (all reference signals may occur in all combinations on any plant output), unless the plant has more inputs than outputs or the plant already contains (one of the) poles of the RSG: in those cases the servo compensator can be extended to obtain the robust servomechanism solution,

- if structural information on plant perturbations is available (uncertainty model), it is possible that robust asymptotic tracking is obtained in spite of the use of a minimal servo compensator; this may be verified by means of structured singular value analysis: the robust servomechanism solution may therefore be considered as a conservative approach, as it provides robustness for unstructured plant perturbations,

- a two-degree-of-freedom control configuration can be set up with the same approach and terminology; it allows the design of a combined feedforward and feedback controller by means of robust control methods; the feedforward part is a proper and stable approximation of the inverse of the system to be controlled,

- once the modified and extended control configuration has been set up correctly, the necessary assumptions for application of standard H_2, H_∞, or related synthesis methods are trivially complied with,

- in general the resulting controller has the order of the standard plant including servo compensator and weight functions, plus the order of the servo compensator that is explicitly added to it; the two-degree-of-freedom controller has an order that is increased with two times the order of the physical plant model (G) plus the order of weight function W_2.

6.3 Comparison with available approaches

Several authors have recognised the possible advantages of application of robust control methods on the output regulation and tracking problem. For this reason, a number of approaches have been proposed to deal with the inherent difficulty of incorporating the output regulation and tracking problem into the standard control design structure, or at least of the application of H_2 or H_∞ methods on servomechanism problems. An extensive discussion on the properties of these approaches is given in section 4.2. We conclude that in comparison with the procedure developed in this thesis, these approaches all have one or more of the following disadvantages:

- decreased generality in the control design structure: most approaches make use of a framework that may be seen as a special case of the standard control structure, such as mixed-sensitivity methods and structures in which external disturbances only consist of persistent reference signals; other external disturbances and/or the effects of plant uncertainty are usually not considered,
- decreased generality in the considered output regulation and tracking problem: all approaches available in literature are restricted to the robust servomechanism problem and in many cases only the scalar (SISO) problem and asymptotic tracking of step signals is considered; none of them considers the construction of a minimal servo compensator which is shown to be more general than the robust servomechanism problem (section 4.5),
- decreased insight in the physical reality of the problem: all approaches available in literature consider symptomatic effects of the output regulation and tracking problem, like unstable weight functions, the necessity of an internal model or the occurrence of unstable invariant zeros, while the problem is in reality caused by the specific behaviour of certain external input signals, which is most naturally dealt with by incorporating the RSG into the standard plant,
- insufficient experience with practical applications: if an example is considered, it is usually simplified and mostly of academic importance,
- increased calculational effort and insufficient numerical implementation in comparison with standard H_2 and H_∞ methods: some approaches make use of operator theory to find results, leading to involved calculations and high

order controllers, others are based on extensions of the standard algebraic Riccati equation method or on Riccati inequalities, but also with an increase in calculational effort and numerical problems.

With respect to the last item, it should be noted that the procedure developed in this thesis also leads to an increase in calculational effort when compared to standard H_2 and H_∞ methods: the incorporation of the RSG into the standard plant leads to an increase in order and the determination of a minimal realization requires an extra calculation (for which stable numerical methods are readily available).
In light of these disadvantages, the further development of any of these approaches for the single purpose of incorporating the output regulation and tracking problem into the robust control paradigm should be reconsidered. However, it must be noted that some of these approaches use this problem to illustrate the need for extending existing (standard) controller synthesis methods. They show directions in which theory may be developed to also include completely different problems like the design of non-linear and/or time-varying controllers and the synthesis of reduced order output feedback controllers. Especially methods based on Linear Matrix Inequalities (LMIs) are promising in this area; the procedure presented in this thesis is not in conflict with these developments.

6.4 Conclusions with respect to the 3DOF system example

The 3DOF system example was treated as a robust control problem in which the asymptotic tracking objective is merely one of several control objectives: actuator effort minimization, disturbance attenuation, speed of response and robustness. It is therefore not only of use to demonstrate the procedure proposed in this thesis for the design of a multivariable servomechanism, but is also an example of the application of robust control theory. Special attention was given to the construction of the standard plant and the selection of weight functions. The resulting servomechanism is quite simple, as the servo compensator appears to be a unit matrix. It is notable that this formalizes the practical experience that the integral action inherent in hydraulic actuators may be seen as an internal model for achieving asymptotic tracking of step signals. In spite of the simplicity of the servo compensator, the procedure was executed in such a way that extension to more complex situations is straightforward, especially when considering the problem of robust asymptotic tracking.

Three controllers are considered. The first controller demonstrates an important advantage of the robust control approach in general. The direct link between weight functions and physical properties allows the designer to specify a very conservative controller which can be safely implemented and used as a starting point for an iterative tuning approach; this property is a further example of the relation between robust control and 'classical' control. On the other hand, the used controller synthesis method is fully model-based and multivariable and generalizes methods from 'modern' control. Sensible construction of the standard plant thus enables the combination of a number of 'classical' and 'modern' control concepts.

The second controller is the result of a trade-off between speed of response and actuator effort, with the asymptotic tracking objective as an extra condition. It appears that it is possible to obtain a significant improvement in speed of response when considering the transfer from reference inputs to plant outputs. However, actual implementation of the resulting controller shows unexpected and undesired behaviour: there is more overshoot and less damping.

To deal with this problem a third controller is constructed, based on an analysis of possible modelling errors and uncertainties. It was established that sensitivity to variations in hydraulic frequency could very well explain the unwanted behaviour of the closed-loop system with the second controller. For this reason an uncertainty model was set up, based on three uncertain parameters reflecting changes in hydraulic frequency of each of the three actuators. The resulting controller is robust against the specified uncertainties without notable decrease in performance and appeared to perform very well on the actual set-up.

6.5 Recommendations for future research

The solution of the output regulation and tracking problem as presented in this thesis gives rise to several questions, which may lead to further developments in the future. From both a theoretical and a practical point of view, we have considered the importance of adding tracking objectives to control design problems, in the sense of their influence on other control design objectives. Especially for multivariable problems it should be considered that the addition of a servo compensator to construct a servomechanism, implies the addition of unstable dynamics to the control problem and therefore a reduction of obtainable robustness and performance. This aspect of servomechanism design is usually disregarded in literature, leading to the impression that a servo compensator of 'maximal order' to obtain 'robust asymptotic tracking' is always the right way to proceed. Instead, there should be a careful selection of 'expected' or 'allowable' persistent signals to be represented by the RSG, such that the servo compensator order is minimal. Furthermore, robustness of a controller with respect to the tracking objective and other objectives is often possible without application of a servo compensator of maximal order. Although the procedure presented in this thesis provides the theoretical possibility to do this, it should be noted that the construction of a servo compensator of minimal order by means of theorem 4.4.3 involves the solution of polynomial equations, which is numerically hard to do. It may be possible to construct a minimal servo compensator with less numerical effort when starting with alternative solvability conditions, such as those using state-space methods as given by, for instance, Francis (1977) and Schumacher (1983).

Another, related, practical problem for which a more fundamental approach may provide a solution, is unwanted high order of the final controller. Although the development of Digital Signal Processors and related hardware provides an enormous increase in real-time calculation power, it is often necessary to limit this order: usually for economical reasons. Especially for the two-degree-of-freedom problem, we may have that the final controller is too complex to implement. This could be seen as an extra motivation for the development of reliable controller order reduction methods or fixed order controller design for the standard control configuration. Other than the necessity of retaining the servo compensator in the controller, the addition of the output regulation and tracking objective does not impair the application of such methods.

It has been argued that the application of robust control methods on the servomechanism problem provides a number of advantages over 'classical' and 'modern' approaches. However, it is desirable to further evaluate these advantages by means of experience with more examples of practical application. We have, for instance, that the two-degree-of-freedom solution is only given as a theoretical concept: this approach should be compared with known examples of combined feedforward and feedback control. Furthermore, when considering the 3DOF system example, we did not compare the results that were obtained by means of the proposed procedure with results that might have been obtained by means of 'classical' or 'modern' methods. The reason for this is that application of, for instance, simple PID or LQG controllers on a practical set-up is not realistic: for an objective evaluation of the potential improvements that may be obtained, the proposed procedure should be applied on a system for which a realistic control system is already 'optimally tuned', and in practical use.

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Notation

General notation

N	set of nonnegative integers	25
R	field of real numbers	19
$R[s]$	ring of polynomials	19
$R(s)$	field of real-rational functions	19
R as prefix	real-rational functions in...	20
R_{ss}	set of proper real-rational functions	20
C	field of complex numbers	16
C^+	closed complex right half plane	29
C^-	open complex left half plane	29
H_2	Hardy space	19
H_∞	Hardy space	20
$L_2(-\infty, \infty)$	time-domain Lebesgue space	18
$L_2[0, \infty)$	time-domain Lebesgue space	21
$L_2(-\infty, 0]$	time-domain Lebesgue space	21
L_2	frequency-domain Lebesgue space	19
L_∞	frequency-domain Lebesgue space	20
$\|x\|$	inner product norm of $x \in C^n$	17
$\|x\|_{RMS}$	RMS-norm of a signal x	18
$\|x\|_2$	2-norm of $x \in L_2$	17
$\|x\|_\infty$	∞-norm of $x \in L_\infty$	20
$\|x\|_\mu$	'μ-norm' of $x \in RH_\infty$	67
$\dot{x}$	derivative of vector x	12
$\bar{x}$	largest x	19
$\underline{x}$	smallest x	73
$\hat{x}$	estimate of vector x	45
$\hat{X}$	state-space matrix of controller parametrization	56

X^{-1}	inverse of matrix X	14
X'	transpose of matrix X	17
X^*	complex conjugate transpose of matrix X	17
$X^\sim(s)$	$X'(-s)$	32
$\mathrm{X}^\perp$	the orthogonal complement of X	20
$\mathrm{E}\{X\}$	the expected value of X	18
$\mathrm{trace}\{X\}$	the sum of the diagonal elements of a square matrix X	18
$\det(X)$	the determinant of a square matrix X	16
$\mathrm{diag}(X_i)$	the (block-)diagonal matrix with diagonal elements X_i	65
$\mathrm{Im}(X)$	the image of X	43
$\mathbf{X}_-(H)$	the subspace spanned by the stable eigenvectors of H	43
$\mathbf{\Delta}$	set of unstructured uncertainty matrices	62
$\mathbf{\Delta}_e$	set of uncertainty matrices extended with a performance block	65
$\mathbf{\Delta}_i$	set of uncertainty matrices with one repeated scalar block	67
$\mathbf{\Delta}_{rr}$	set of real-repeated uncertainty matrices	71
$\mathbf{\Delta}_s$	general set of structured uncertainty matrices	66
$\mathbf{B\Delta}$	unit ball in $\mathbf{\Delta}$	66
$\mathbf{U}_{\mathbf{\Delta}_s}$	set of structured unitary matrices in $\mathbf{\Delta}_s$	68
$\mathbf{D}_{\mathbf{\Delta}_s}$	set of structured scaling matrices	68

Constants, variables and functions

α_i	angles in moving block kinematics ($i = 1, 2, 3$)	128
α_{ij}	real coefficient of a linear differential equation	85
β	parameter in 3DOF system's weight function W_e	138
γ	(estimate of) ∞-norm	32
	or parameter in 3DOF system's weight function W_e	138
γ_o	∞-norm of optimal solution to the H_∞-problem	53
Δ	variable matrix of an LFT; uncertainty matrix	62
Δ_X	uncertainty matrix corresponding to uncertain matrix X	74
Δ_l	variable matrix of a lower LFT	14
Δ_p	performance block	65
Δ_u	variable matrix of an upper LFT	14
δ	(normalized) uncertain scalar number or function	61
ϵ	small perturbation	53
ζ	LQG control objective	44
Θ	transformation matrix	27
θ	vector of uncertain parameters in a state-space description	72

θ_o	nominal value of θ	73
κ	parameter in 3DOF system's RSG	134
λ	eigenvalue	16
μ	structured singular value	66
ν	number of allowable persistent signals	84
ξ	internal variable (vector)	26
π	circle ratio	18
ρ	spectral radius: $\lvert\bar{\lambda}\rvert$	56
Σ	matrix description of a PMD	26
Σ_{ol}	matrix description of open loop PMD	91
Σ_{cl}	matrix description of closed loop PMD	91
Σ_{s}	PMD with specific structure	31
σ	singular value	19
τ	time	18
	or parameter in 3DOF system's RSG	134
Φ	transfer function matrix for controller parametrization	56
ϕ	inverse of δ	72
	or position coordinate of 3DOF system	119
φ	flow to hydraulic actuator's first compartment	125
φ_n	normalized flow to hydraulic actuator's first compartment	130
φ_o	nominal flow to hydraulic actuator's first compartment	130
ω	frequency (radians/second)	18
A	system matrix of a state-space description	12
	or active area of hydraulic cylinder	126
a	half the side of the moving block	128
a_i	amplitude scaling factor	84
	or fixed point in inertial frame	128
B	input matrix of a state-space description	12
b_i	joint in moving frame	128
C	output matrix of a state-space description	12
c	scalar constant	16
c_o	nominal value of scalar constant	61
D	feedthrough matrix of a state-space description	12
	or scaling matrix in $\mathbf{D}_{\mathbf{\Delta}_s}$	68
D_G	denominator term of a (right coprime) MFD of G	24
$\tilde{D}_G$	denominator term of a (left coprime) MFD of G	24
D_{1*}	$[D_{11}\ D_{12}]$	54

D_{*1}	$[D'_{11}\ D'_{21}]'$	54
d	differential operator	17
E	output matrix for LQG objective	44
	or hydraulic compressibility coefficient	126
e	error signal (vector)	4/47
	or ground number of exponential function	16
F_d	driving force (hydraulic actuator)	126
F_e	external force (hydraulic actuator)	126
F_f	frictional force (hydraulic actuator)	126
$\mathcal{F}$	the Fourier transform of	18
$\mathcal{F}_u(M,\Delta)$	the upper LFT operation on Δ with coefficient matrix M	14
$\mathcal{F}_l(M,\Delta)$	the lower LFT operation on Δ with coefficient matrix M	14
f	number of full uncertainty blocks in $\mathbf{\Delta}_s$	66
	or viscous friction force coefficient (hydraulic actuator)	126
$f_i(t)$	scalar basis function of an allowable persistent reference signal	84
$G(s)$	transfer function matrix	15
$G(t)$	impulse response of an LTI system	33
$g(s)$	vector-valued transfer function	33
$g(t)$	impulse response of an vector-valued LTI system	33
H	Hamiltonian matrix	32
I_n	unit matrix of dimension n	14
J	LQG cost-function	44
j	$\sqrt{-1}$	19
K	controller	39
$\tilde{K}$	controller with added D_{22} matrix	57
	or controller for the modified standard plant	81/93
K_R	constant state feedback regulator matrix	45
K_E	constant state injection estimator matrix	46
K_a	coefficient matrix for controller parametrization	56
K_{I}	controller in configuration I (standard problem)	99
K_{II}	controller in configuration II (modified problem)	100
K_{It}	controller in the two-degree-of-freedom configuration	110
K_{IIt}	controller in the modified two-degree-of-freedom configuration	109
k_i	dimension of the ith diagonal block in $\mathbf{\Delta}_s$	66
L	polynomial matrix in tracking stability solvability condition	90
$\mathcal{L}$	the Laplace transform of	85
l	nominal length of hydraulic actuators	128

l_i	actuator vector	128
M	coefficient matrix of an LFT	14/62
	or auxiliary TFMD in the construction of an internal model	92
	or mass of moving block of the 3DOF system	129
N_G	numerator term of a (right coprime) MFD of G	24
$\tilde{N}_G$	numerator term of a (left coprime) MFD of G	24
N_{rp}	interconnection structure for robust performance analysis	154
N_{rs}	interconnection structure for robust stability analysis	149
n	dimension of state vector	12
P	(standard) plant	39
$\tilde{P}$	modified standard plant	97
P_{I}	standard plant in configuration I (standard problem)	99
P_{II}	standard plant in configuration II (modified problem)	100
P_{It}	standard plant for the two-degree-of-freedom problem	107
P_{IIt}	modified standard plant for the two-degree-of-freedom problem	108
p	dimension of (measurement) output vector,	12
	or a pole of a transfer function matrix	25
p_i	matrix partition row dimension ($i \in \text{N}$)	14
	or hydraulic actuator first compartment pressure ($i = 1, 2, 3$)	119
p_n	normalized hydraulic actuator pressure	130
p_s	hydraulic supply pressure	117
Q	polynomial matrix in tracking stability solvability condition	90
Q_1	LQG performance weight	44
Q_2	LQG actuator effort weight	44
q	dimension of (control) input vector	12
	or the multiplicity of a pole p or zero z	25
q_i	matrix partition column dimension ($i \in \text{N}$)	14
$\mathcal{R}$	set of persistent reference signals	84
R	weight matrix for H_∞ regulator algebraic Riccati equation	54
	or Reference Signal Generator for the tracking problem	82/86
$\tilde{R}$	weight matrix for H_∞ estimator algebraic Riccati equation	54
$R_{xx}(\tau)$	autocorrelation matrix of signal x	18
r	reference signal (vector)	4/84
	or number of repeated scalar uncertainty blocks in $\boldsymbol{\Delta}_s$	66
S	coefficient matrix of the LFT form of a state-space description	72
	or internal model / servo compensator	81
	or sensitivity matrix	112

S_i	stable transfer function matrix ($i = 1, 2, 3, 4$)	93
$S_{xx}(\omega)$	spectral density matrix of signal x	18
s	Laplace transform variable	15
s_i	a particular constant value of s	16
s_o	scaling factor for uncertain parameter	61
T	time	18
	or system polynomial matrix of a PMD	26
	or complementary sensitivity matrix	112
T_{wz}	closed loop transfer function matrix from w to z	40
t	time	12
t_0	initial time	12
U	unitary matrix in $\mathbf{U}_{\mathbf{\Delta}_s}$	68
	or input polynomial matrix of a PMD	26
$[U_1' \; U_2']'$	right inverse of a left coprime fractional representation	24
u	control input (vector)	12/39
u_c	constant value for control input (vector)	4
u_i	electrical input signal for hydraulic actuators ($i = 1, 2, 3$)	118
u_n	normalized electrical input signal for hydraulic actuators	130
u_o	nominal electrical input signal for hydraulic actuators	130
u_r	input (vector) of RSG	86
$\tilde{u}$	input (vector) of internal model	81
V	output polynomial matrix of a PMD	26
V_i	spectral density matrix of v_i ($i = 1, 2$)	44
$[V_1 \; V_2]$	left inverse of a right coprime fractional representation	24
v	volume (hydraulic actuator)	126
v_1	auxiliary input (vector)	41
v_2	auxiliary input (vector)	41
v_o	ineffective volume of hydraulic actuator	118
W	feedthrough polynomial matrix of a PMD	26
	or weight function	79
w	disturbance vector	39
X_0, Y_0	inertial frame	127
X_1, Y_1	moving frame	127
X_c	controllability gramian	33
X_o	observability gramian	33
X_R	solution of regulator algebraic Riccati equation	46
X_E	solution of estimator algebraic Riccati equation	46

x	state vector	12
	or position coordinate of 3DOF system	119
x_0	initial state vector	12
x_λ	eigenvector belonging to λ	16
x_p	output direction of pole p	25
x_r	state vector of RSG	86
x_s	spool displacement of electro-hydraulic servo valve	125
x_z	output direction of zero z	25
y	measurement signal (vector)	12/39
	or position coordinate of 3DOF system	119
y_{ai}	actuator displacement measurement ($i = 1, 2, 3$)	119
y_{ani}	normalized actuator displacement measurement ($i = 1, 2, 3$)	132
y_i	constant vector determining how $f_i(t)$ appears in $r_i(t)$	84
Z	matrix in calculation of H_∞ sub-optimal controller	56
z	control objectives vector	39
	or a transmission zero of a transfer function matrix	25

Abbreviations

3DOF	Three-Degrees-Of-Freedom positioning system	116
ARE	Algebraic Riccati Equation	42
DSP	Digital Signal Processor	120
LCF	Left Coprime Fraction	24
LFT	Linear Fractional Transformation	14
LTI	Linear Time Invariant	13
MFD	Matrix Fraction Description	26
PID	Proportional-Integral-Differential	35
PMD	Polynomial Matrix Description	26
RCF	Right Coprime Fraction	24
RMS	Root-Mean-Square	18
RSG	Reference Signal Generator	82/86
SSD	State-Space Description	12
SSE	Strict System Equivalence	27
TFMD	Transfer Function Matrix Description	15

Samenvatting

Het uitgangsreguleer- en volgprobleem, ook bekend als het servoprobleem, is een standaard onderwerp van zowel de klassieke als de moderne regeltheorie. Vooral met behulp van toestandsmodellen is het mogelijk een beschrijving op te bouwen van een verzameling van 'persistente' signalen, ofwel signalen die niet naar nul gaan als de tijd naar oneindig gaat. Indien lineaire, tijd-invariante modellen gebruikt worden, zou een dergelijke verzameling kunnen bestaan uit combinaties van polynomen en sinusvormige functies. Het regelprobleem is dan een regelaar te ontwerpen voor een gegeven systeem zodat één of meer systeemuitgangen de gespecificeerde persistente signalen asymptotisch volgen: deze persistente signalen kunnen dan dus gezien worden als 'referentiesignalen'. Het is bekend dat een dergelijke regelaar bepaalde structuureigenschappen heeft: de regelaar dient een dynamisch model, een 'servocompensator', te bevatten, zodat de combinatie van regelaar met te regelen systeem een 'intern model' bevat van de verzameling van referentiesignalen.
De aldus gespecificeerde regeldoelstelling is echter niet voldoende om een realistische regelaar te ontwerpen: het asymptotisch volggedrag moet gecombineerd worden met andere regelaardoelstellingen, zoals responsiesnelheid, storingsonderdrukking en robuustheid. Normaal gesproken kan dit worden gedaan door een gegeven regelaarontwerpmethode toe te passen op een systeemmodel dat is uitgebreid met een servocompensator, die vervolgens wordt toegevoegd aan de regelaar. Dit blijkt goed te werken bij gebruik van 'lineair kwadratisch optimale' regelaarontwerpmethoden, zoals die vooral worden toegepast in de moderne regeltechniek gedurende de periode 1960-1980. Helaas ontstaan er echter problemen als wordt getracht 'robuuste' regelaarontwerpmethoden toe te passen, zoals die in ontwikkeling zijn sinds 1980. Het blijkt dat de uitbreiding van het te regelen systeem met een servocompensator in tegenspraak is met bepaalde standaardaannamen die gedaan worden om toepassing van deze methoden mogelijk te maken.
In dit proefschrift wordt een procedure voorgesteld waarmee deze problemen

opgelost kunnen worden in een zeer algemene zin: gegeven een willekeurig tijd-invariant model van een verzameling van persistente signalen waarvoor een oplossing van het asymptotisch volgprobleem bestaat, is het hiermee mogelijk iedere bestaande robuuste regelaarontwerpmethode toe te passen om een geschikte regelaar te vinden. Om oplosbaarheid te bepalen, wordt een nieuwe noodzakelijke en voldoende voorwaarde voorgesteld en gerelateerd aan eerdere resultaten. Uitgaande van deze oplosbaarheidsvoorwaarde wordt de constructie van een geschikte servocompensator besproken, die tevens van minimale orde voor de gegeven verzameling van persistente signalen blijkt te zijn. De juiste formulering van een standaard regelaarontwerpconfiguratie wordt gegeven, zodat een fysische interpretatie van stoorsignalen, weegfuncties en beoordelingsgrootheden mogelijk is, terwijl de volgeigenschap gegarandeerd blijft. Robuuste regelaarontwerpmethoden, zoals H_2 en H_∞ optimalisering, kunnen dan worden toegepast om een gewenste gesloten-lus overdracht van storingen naar beoordelingsgrootheden te verwezenlijken.

Twee belangrijke uitbreidingen van de voorgestelde procedure worden beschouwd. Ten eerste wordt de constructie van een niet minimale servocompensator gegeven, waarmee het mogelijk is 'robuust asymptotisch volggedrag' te realiseren. Er wordt aangetoond dat het resultaat in veel gevallen kan worden beschouwd als een speciaal geval van de voorgestelde procedure; bovendien worden een paar mogelijke nadelen met betrekking tot andere regelaardoelstellingen dan volggedrag besproken. Ten tweede wordt de twee-graden-van-vrijheid probleemformulering gegeven als een uitbreiding van de standaard regelaarontwerpconfiguratie. Deze maakt het mogelijk om tegelijkertijd zowel een voorwaartskoppelende als een terugkoppelende regelaar met gegarandeerd volggedrag te ontwerpen met behulp van robuuste regelaarontwerpmethoden.

Een multivariabel experimenteel voorbeeld wordt besproken met veel aandacht voor de bepaling van een geschikte regelaarontwerpconfiguratie en de keuze van de noodzakelijke weegfuncties. Dit levert een eenvoudig voorbeeld op van de integratie van het asymptotisch volgprobleem in een typisch robuust regelprobleem. Verscheidene regelaars worden ontworpen met behulp van H_∞ optimalisering; deze worden vervolgens experimenteel geëvalueerd met behulp van een op een digitale signaalprocessor gebaseerde implementatieomgeving. Robuustheidseigenschappen met betrekking tot stabiliteit, prestatieniveau, en volggedrag worden gegarandeerd met behulp van parametrische onzekerheidsmodellering en gestructureerde singuliere waarden analyse.

Biografie/Biography

Paul Frank Lambrechts is geboren op 19 mei 1963 in Roosendaal. Hij begon zijn studie aan de faculteit werktuigbouwkunde van de technische universiteit Delft in 1981 en slaagde 'met lof' in 1987 in de vakgroep meet- en regeltechniek. Zijn afstudeerwerk over het regelen van het bandloopmechanisme van een digitale videorecorder, uitgevoerd aan het Philips natuurkundig laboratorium in Eindhoven, werd beloond met de jaarlijkse prijs voor het beste afstudeerwerk door de afdeling voor werktuig- en scheepsbouw van het koninklijk instituut van ingenieurs (KIVI). Sinds 1988 werkte hij als assistent in opleiding van de eerdergenoemde vakgroep meet- en regeltechniek, aan de toepassing van robuuste regeltheorie op mechanische volgsystemen met behulp van moderne digitale electronica. Deze periode werd onderbroken door een periode van 14 maanden militaire dienst, als reserve officier academisch gevormd (ROAG) aan het Prins Maurits laboratorium van TNO (1992-1993). Momenteel is hij werkzaam bij het Nationaal Lucht- en Ruimtevaartlaboratorium NLR op het gebied van 'slimme actuatoren'.

Paul Frank Lambrechts was born on May 19, 1963 in Roosendaal, The Netherlands. He started studying mechanical engineering at the Delft university of technology in 1981 and graduated in 1987 in the systems and control group. His M.Sc. thesis on the control of the tape transportation system of a digital video recorder, performed at the Philips research laboratories in Eindhoven, was awarded that years prize for best graduation work by the mechanical engineering and marine technology department of the Dutch royal institute of engineers (KIVI). Since 1988, he worked as a research assistant at the Delft mechanical engineering systems and control group on the application of robust control theory on mechanical servosystems using state-of-the-art digital control hardware. This period was interupted by a 14 month period of military service at the Dutch defense research institute PML-TNO (1992-1993). Presently he is employed at the Dutch National Aerospace Laboratory NLR, and working in the field of 'smart actuators'.