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PAPER

Surpassing the Carnot efficiency by extracting imperfect work

Nelly Huei Ying Ng[1,2,4], **Mischa Prebin Woods**[1,3] and **Stephanie Wehner**[1,2]

[1] QuTech, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands
[2] Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543, Singapore
[3] University College of London, Department of Physics & Astronomy, London WC1E 6BT, United Kingdom
[4] Author to whom any correspondence should be addressed.

E-mail: nell0002@e.ntu.edu.sg

Abstract

A suitable way of quantifying work for microscopic quantum systems has been constantly debated in the field of quantum thermodynamics. One natural approach is to measure the average increase in energy of an ancillary system, called the battery, after a work extraction protocol. The quality of energy extracted is usually argued to be good by quantifying higher moments of the energy distribution, or by restricting the amount of entropy to be low. This limits the amount of heat contribution to the energy extracted, but does not completely prevent it. We show that the definition of 'work' is crucial. If one allows for a definition of work that tolerates a non-negligible entropy increase in the battery, then a small scale heat engine can possibly exceed the Carnot efficiency. This can be done without using any additional resources such as coherence or correlations, and furthermore can be achieved even when one of the heat baths is finite in size.
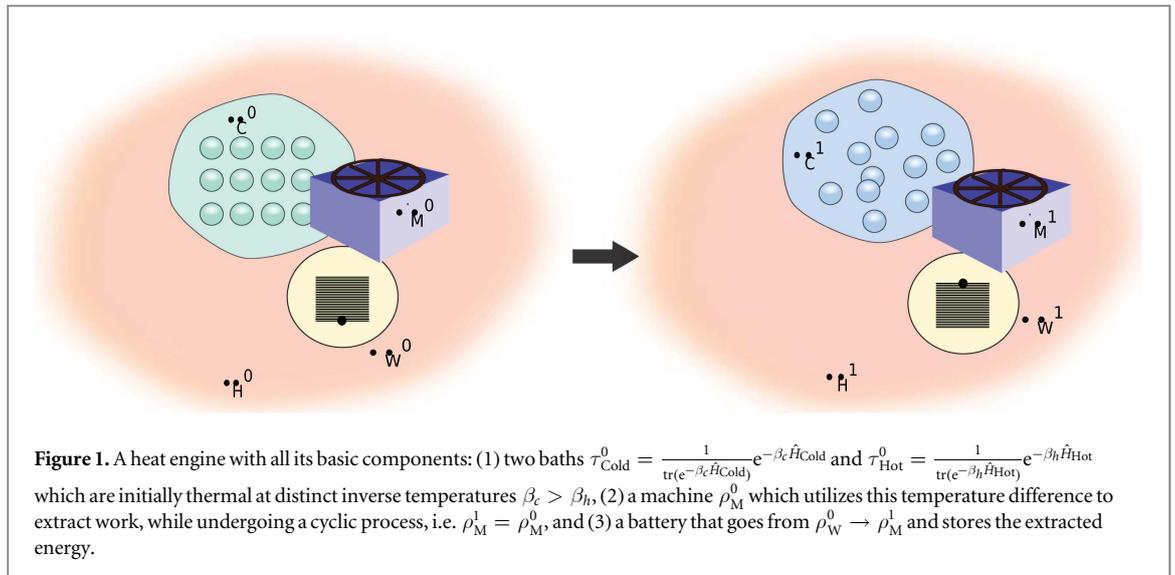
1. Introduction

Given resources where energy is only present in its most disordered form (heat), how efficiently can one convert such heat and store it as useful energy (work)? This question lies at the foundation of constructing heat engines, like the steam engine. Though nearly two centuries old, it remains one of central interest in physics, and can be applied in studying a large variety of systems, from naturally arising biological systems to intricately engineered ones. Classically it is known that a heat engine cannot perform at efficiencies higher than the Carnot efficiency (CE), which is given by

$$\eta_C = 1 - \frac{T_{\text{Cold}}}{T_{\text{Hot}}}, \tag{1}$$

T_{Cold}, T_{Hot} being the temperatures of the heat reservoirs at which the engine operates between. This fundamental limit on efficiency can be derived as a consequence of the second law of thermodynamics, which is regarded as one of the 'most perfect laws in physics' [1].

Recent advancements in the engineering and control of quantum systems have, however, pushed the applicability of conventional thermodynamics to its limits. In particular, instead of large scale machines that initially motivated the study of thermodynamics, we are now able to build nanoscale quantum machines. A quantum heat engine (QHE) is a machine that performs the task of work extraction when the involved systems are not only extremely small in size/particle numbers, but also subjected to the laws of quantum physics. Such studies are highly motivated by the prospects of designing small, energy efficient machines applicable to state-of-the-art devices, particularly those relevant for quantum computing and information processing. The question then arises: how efficient can these machines be?

Recently, a number of schemes for QHEs have been proposed and analyzed, involving systems such as ion traps, photocells, or optomechanical systems [2–10]. Some of these schemes lie outside the usual heat engine setting (see figure 1). For example, instead of using a hot and cold bath, the extended quantum heat engine (EQHE) has access to reservoirs which are not in a thermal state [3, 11, 12]. In this case, EQHE with high efficiencies (even surpassing η_C) have been proposed and demonstrated. Nevertheless, [13] has pointed out that the second law is, strictly speaking,

Figure 1. A heat engine with all its basic components: (1) two baths $\tau^0_{\text{Cold}} = \frac{1}{\text{tr}(e^{-\beta_c \hat{H}_{\text{Cold}})}} e^{-\beta_c \hat{H}_{\text{Cold}}}$ and $\tau^0_{\text{Hot}} = \frac{1}{\text{tr}(e^{-\beta_h \hat{H}_{\text{Hot}})}} e^{-\beta_h \hat{H}_{\text{Hot}}}$ which are initially thermal at distinct inverse temperatures $\beta_c > \beta_h$, (2) a machine ρ^0_M which utilizes this temperature difference to extract work, while undergoing a cyclic process, i.e. $\rho^1_M = \rho^0_M$, and (3) a battery that goes from $\rho^0_W \to \rho^1_M$ and stores the extracted energy.

never violated because one always has to invest extra work in order to create and replenish these non-thermal reservoirs. Nevertheless, the study of using such non-thermal reservoirs can still be of interest, since they potentially may boost other features of the heat engine, such as the rate of extracting work. However, in this manuscript we are focusing on the standard setting of a QHE, in which the baths are thermal, where in classical thermodynamics, it is proven that although CE can be approached, it can never be surpassed [14].

Even without additional resources such as those in EQHEs, QHEs are already radically different from classical engines, since energy fluctuations are much more prominent due to the small number of particles involved. The laws of thermodynamics for small quantum systems are more restrictive due to finite-size effects [14–19]. It is known that such second laws introduce additional restrictions on the performance of QHEs [14]. Specifically, not all QHEs can even achieve the CE. The maximal achievable efficiency depends not only on the temperatures, but also on the Hamiltonian structure of the baths involved. Furthermore, considering a probabilistic approach towards work extraction, [20] found that the achievement of CE is very unlikely, when considering energy fluctuations in the microregime.

Can we design a QHE that operates between genuinely thermal reservoirs and yet achieves a high efficiency[5]? To answer this, several protocols have been proposed and analyzed [9, 11, 17, 22–27], some showing QHEs that operate at the CE [9, 27, 28]. However, crucial to these results is the definition of work. In these approaches, the most common approach of quantifying work is to measure the average increase in energy of an ancillary system, sometimes referred to as the battery, after a certain work extraction protocol [9, 28–31]. Such a measure of work would be adequate, if the entropy of the battery, denoted as ΔS remains invariant. Indeed, already in classical thermodynamics, $\Delta S = 0$ is always assumed when deriving the upper bound on heat engine efficiency. However, all explicit QHE protocols to-date do not, and cannot satisfy such an assumption, since in the quantum nanoregime, fluctuations in work become highly non-negligible and hard to quantify/analyze. In this regime, work is almost always a random variable, characterized by a non-trivial probability distribution [32–35]. Attempts to keep this entropy increase arbitrarily small often use additional assumptions such as a catalyst/control system with an unphysical Hamiltonian [36, 37], or with infinite energy/coherence resources [38, 39]. If one does not make such assumptions, then one has to live with the fact that the energy extracted is tainted by heat, and be satisfied as long as the amount of heat contribution is simply upper bounded [9]. In the second approach, the quality of work extracted is usually argued to be good by quantifying higher moments of the energy distribution, or by restricting the amount of entropy to be low. Underlying all these results a fundamental concept remains hidden: how should work be quantified in the microscopic regime? A universally agreed upon definition of performing microscopic work is lacking, and this remains a constantly debated subject in the field of quantum thermodynamics [31, 33, 35, 40–42]. This is mainly why a complete picture describing the performance limits of a QHE remains unknown.

The goal of our paper is to show that average energy increase is not an adequate definition of work for microscopic quantum systems when considering heat engines, even when imposing further restrictions such as a limit on entropy increase. Specifically, we demonstrate that if one allows for a definition of work that tolerates a non-negligible entropy increase in the battery, then one can in fact exceed CE. Most importantly, this can already happen when (1) the cold bath only consists of 1 qubit, where finite-size effects further impede the possibility of

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[5] In this manuscript we concern ourselves with the main problem of efficiency, although there are other features of a heat engine such as power and constancy that are important as well. See [21] for a discussion about tradeoffs between these features.

thermodynamic state transitions, and (2) without using additional resources such as non-thermal reservoirs. The reason for being able to surpass CE stems from the fact that heat contributions have 'polluted' our definition of work extraction. We show that work can be divided into different categories: perfect and near-perfect work, where heat (entropy) contributions are negligible with respect to the energy gained; while imperfect work characterizes the case where heat contributions are comparable to the amount of energy gain. We find examples of extracting imperfect work where the CE is surpassed. This completes our picture of the understanding of work in QHEs, since we already know that by drawing perfect/near perfect work, no QHE can ever surpass CE [14].

2. General setting of a heat engine

The setup

Let us first describe a generic QHE, which is a setup that extracts work. A generic QHE comprises of four basic elements: two thermal baths at distinct temperatures T_{Hot} and T_{Cold} respectively, a machine that operates between the baths in a cyclic fashion, and a battery that stores energy for further usage (figure 1). The total Hamiltonian

$$\hat{H}_t = \hat{H}_{\text{Cold}} + \hat{H}_{\text{Hot}} + \hat{H}_{\text{M}} + \hat{H}_{\text{W}}, \tag{2}$$

is the sum of individual Hamiltonians, where indices Hot, Cold, M, W represent a hot bath (Hot), a cold bath (Cold), a machine (M), and a battery (W) respectively. Let us also consider an initial state $\rho^0_{\text{ColdHotMW}} = \tau^0_{\text{Cold}} \otimes \tau^0_{\text{Hot}} \otimes \rho^0_{\text{M}} \otimes \rho^0_{\text{W}}$. The state τ^0_{Hot} (τ^0_{Cold}) is the initial thermal state at temperature T_{Hot} (T_{Cold}), corresponding to the hot (cold) bath Hamiltonian $\hat{H}_{\text{Hot}}(\hat{H}_{\text{Cold}})$, and $T_{\text{Cold}} < T_{\text{Hot}}$. For notational convenience, we shall often work with inverse temperatures $\beta_h := 1/k_B T_{\text{Hot}}$ and $\beta_c := 1/k_B T_{\text{Cold}}$ where k_B is the Boltzmann constant. Given Hamiltonian $\hat{H}$ and temperature T, the thermal state is defined as $\tau = \frac{1}{\text{tr}(e^{-\beta \hat{H}})} e^{-\beta \hat{H}}$.
The initial machine (ρ^0_{M}, $\hat{H}_{\text{M}}$) can be chosen arbitrarily, as long as its final state is preserved (and therefore the machine acts like a catalyst).

In order to investigate the fundamental limits to the performance of QHEs, we adopt a thermodynamic resource theory approach [15, 43–45], where all unitaries U on the global system such that $[U, \hat{H}_{\text{ColdHotMW}}] = 0$ are allowed. Such operations conserve total energy, which is a requirement based on the first law of thermodynamics. If (τ^0_{Hot}, $\hat{H}_{\text{Hot}}$) and (ρ^0_{M}, $\hat{H}_{\text{M}}$) can be arbitrarily chosen, then any such unitary U, (τ^0_{Hot}, $\hat{H}_{\text{Hot}}$) and (ρ^0_{M}, $\hat{H}_{\text{M}}$) defines a *catalytic thermal operation* [16] which one can perform on the joint state ColdW. This implies that the cold bath is used as a non-thermal resource, relative to the hot bath. By catalytic thermal operations that act on the cold bath, using the hot bath as a thermal reservoir, and the machine as a catalyst, one can extract work and store it in the battery. The aim is to achieve a final *reduced* state $\rho^1_{\text{ColdMW}} = \text{tr}_{\text{Hot}}(\rho^1_{\text{ColdHotMW}})$, such that

$$\rho^1_{\text{ColdMW}} = \rho^1_{\text{ColdW}} \otimes \rho^1_{\text{M}}, \tag{3}$$

where $\rho^1_{\text{M}} = \rho^0_{\text{M}}$, and ρ^1_{Cold} is the final joint state of the cold bath and battery. For any bipartite state ρ_{AB}, we use the notation of reduced states $\rho_{\text{A}} := \text{tr}_{\text{B}}(\rho_{\text{AB}})$.

Finally, we need to describe the battery such that the state transformation $\rho^0_{\text{ColdHotMW}} \to \rho^1_{\text{ColdHotMW}}$ stores work in the battery. This is done as follows: consider the battery which has a Hamiltonian $\hat{H}_{\text{W}} := \sum_{i=1}^{n_{\text{W}}} E_i^{\text{W}} |E_i\rangle \langle E_i|_{\text{W}}$. For a parameter $\varepsilon \in [0, 1)$, we consider the initial and final battery states to be

$$\rho^0_{\text{W}} = |E_j\rangle \langle E_j|_{\text{W}}, \tag{4}$$

$$\rho^1_{\text{W}} = (1 - \varepsilon)|E_k\rangle \langle E_k|_{\text{W}} + \varepsilon |E_j\rangle \langle E_j|_{\text{W}} \tag{5}$$

respectively. This can be seen as a simple form of extracting work: going from a pure energy eigenstate to a higher energy eigenstate (except with failure probability ε). More general battery states may be in principle allowed, however this does not affect the main focus of our result, and therefore for simplicity of analysis we consider final battery states of the form in equation (5). The extracted work W_{ext} is defined as the energy difference $W_{\text{ext}} := E_k^{\text{W}} - E_j^{\text{W}}$, where we define $E_k^{\text{W}} > E_j^{\text{W}}$ such that $W_{\text{ext}} > 0$. The parameter ε corresponds to the failure probability of extracting work, usually chosen to be small. To summarize, we make the following minimal assumptions:

(A.1) Product state: There are no initial correlations between the cold bath, machine and battery, since each of the initial systems are brought independently into the process. This is an advantage of the setup, since if one assumed initial correlations, one would then have to use unknown resources to generate them in the first place.

(A.2) Perfect cyclicity: The machine undergoes a cyclic process, i.e. $\rho_M^0 = \rho_M^1$, and ρ_M^1 is also not correlated with ρ_{ColdW}^1. This is to ensure that the machine does not get compromised in the process: since if ρ_M^0 was initially correlated with some reference system R, then by monogamy of entanglement, correlations between ρ_M^1 and ρ_{ColdW}^1 would potentially destroy such correlations between the machine M with R.

(A.3) Isolated quantum system: The heat engine as a whole, is isolated from and does not interact with the world. This assumption ensures that all possible resources in a work extraction process are accounted for. Mathematically, this implies that the global Hamiltonian is time-independent, while the system evolution is described by global unitary dynamics.

(A.4) Finite dimension: The Hilbert space associated with $\rho_{ColdHotMW}^0$ is finite dimensional but can be arbitrarily large. Moreover, the Hamiltonians $\hat{H}_{Cold}$, $\hat{H}_{Hot}$, $\hat{H}_M$ and $\hat{H}_W$ all have bounded pure point spectra, meaning that these Hamiltonians have eigenvalues which are bounded. This assumption comes from the resource theoretic approach of thermodynamics [15].

3. Quantifying work and efficiency

We have seen from equations (4) and (5) that a failure probability of work extraction is allowed. This probability injects a certain amount of entropy into the battery's final state, compromising the quality of extracted work. For an initially pure battery state, let ΔS denote the von Neumann entropy of the final battery state,

$$\Delta S := -\rho_W^1 \ln \rho_W^1 = -\varepsilon \ln \varepsilon - (1-\varepsilon)\ln(1-\varepsilon). \tag{6}$$

Since the distribution of the final battery state has its support on a two-dimensional subspace of the battery system, ΔS coincides with the binary entropy of ε, denoted by $h_2(\varepsilon)$.

The more entropy ΔS created in the battery, the more disordered is the energy one extracts, i.e. the larger are the heat contributions. Since work is ordered energy, therefore ideally, zero entropy is desirable; where the final state of the battery is simply another pure energy eigenstate $\rho_W^1 = |E_k\rangle\langle E_k|_W$. Not only then we obtain a net increase in energy, but also we have full knowledge of the state ρ_W^1, since it is also pure. This prompts the following characterization of work:

Definition 1 (**Perfect work** [14]). An amount of work extracted W_{ext} is referred to as perfect work when $\varepsilon = 0$.

Perfect work, although desirable in principle, is an extremely strict form of work where work extraction happens with zero failure probability, that is to say, $\Delta S = 0$. In fact, it has been proven in [14] that for any initial state of the cold bath which is of full rank, if we require perfect work, then $W_{ext} \leqslant 0$. Since thermal states are always of full rank, a positive amount of perfect work can never be extracted in a heat engine that operates only between two thermal heat baths. Such a phenomena is closely analogous to zero-error data compression: whenever a piece of information is represented by a random variable X over a probability distribution of full rank, then one cannot achieve zero-error in transmission if the data is compressed and transmitted in a message of shorter length [46].

Let us therefore proceed by considering another example: for a fixed amount of average energy increase from $\rho_W^0 \rightarrow \rho_W^1$, the entropy increase ΔS is maximized when the final state ρ_W^1 is thermal. However, another problem emerges: it is known that a thermal state by itself cannot be used to obtain work, if only energy-preserving unitaries are allowed. This is precisely why only multiple copies of thermal states (as long as they are of a fixed temperature) are allowed in the resource theory framework as free states [16]. For such a thermal state ρ_W^1 to be useful in work extraction, it has to be combined with other resources (for example another heat bath) in order to obtain ordered work. Therefore, while energy has increased, one cannot justify the full amount of average energy increase as work.

From the above example, we have seen the importance of constraining the amount of ε (or equivalently, the amount of ΔS), in order to properly justify that whatever energy stored in ρ_W^1 indeed corresponds to useful work. However, the absolute value of ΔS is not so important by itself. In particular, we could have cases where although ΔS is arbitrarily small, the amount of energy extracted could also be arbitrarily small, even comparable to ΔS. Indeed, many protocols for work extraction such as [9, 33] involve infinitesimal steps that extracts energy by small amounts in each step. In the light of such considerations, we may consider the following regimes:

Definition 2 (**Near perfect work** [14]). We say that a sequence of heat engine protocols leads to *near perfect work extraction* if

(1) For all protocols in the sequence, $0 < \varepsilon \leqslant l$, for some fixed $l < 1$ and

(2) For any $p > 0$, there exists a non-trivial subset of protocols where $\frac{\Delta S}{W_{ext}} < p$.

Definition 2 requires that if the amount of near perfect work W_{ext} for the whole sequence is bounded away from infinity, then there must always be a subset of protocols where the failure probability of work extraction governed by ε is arbitrarily small. However, it is also more stringent than just that: for near perfect work, whenever W_{ext} is finite, items (1) and (2) are both satisfied only in the limit $\varepsilon \to 0$, and if and only if $\lim_{\varepsilon \to 0^+} \frac{\Delta S}{W_{ext}} = 0$. If this limit is not satisfied, we say that the work extracted is *imperfect*.

Definition 3 (**Imperfect work**). We say that a sequence of heat engine protocols leads to *imperfect work extraction* if

(1) For all protocols in the sequence, $0 < \varepsilon_i \leqslant l$, for some fixed $l < 1$ and

(2) There exists some positive number $p > 0$, where for all protocols in the sequence, $\frac{\Delta S}{W_{ext}} \geqslant p$.

The reader might be concerned with using $\Delta S / W_{ext}$ as a parameter to characterize work quality, since $\Delta S / W_{ext}$ is not dimensionless. However, one can simply consider the rescaled and dimensionless quantity $k_B T \Delta S / W_{ext}$, for any value of T from the surrounding bath. Since $k_B T$ only comes into the characterization as a multiplicative factor which is positive but finite, one can therefore see that the regimes of perfect, near perfect and imperfect work would remain the same, had we use $k_B T \Delta S / W_{ext}$ instead of $\Delta S / W_{ext}$.

Next, we introduce the notion of a *quasi-static* heat engine. Traditionally in thermodynamics, the expression quasi-static refers to a process that happens slowly such that the system remains in thermal equilibrium at all times. In this manuscript, we use this term to denote a heat engine cycle that changes the state of the final cold bath only slightly, such that it remains a thermal state, however its temperature is slightly increased.

Definition 4 (**Quasi-static** [14]). Consider a sequence of heat engine protocols, where in each protocol, the final state of the cold bath is thermal with an inverse temperature of $\beta_f = \beta_c - g$. This heat engine (sequence) is called quasi-static, if for any positive number $G > 0$, there exists a non-trivial subset of protocols where $g \leqslant G$. The quasi-static limit refers to the subset of protocols in the limit where $G \to 0^+$.

In this manuscript, we constantly refer to g as the quasi-static parameter.

Having fully described the QHE in section 2, and expounding on different characterizations of extracted energy in definitions 1–3, one asks: for what values of W_{ext} can the transition $\rho^0_{ColdHotMW} \to \rho^1_{ColdHotMW}$ occur? The possibility of such a thermodynamic state transition depends on a set of conditions derived in [16], phrased in terms of quantities called generalized free energies (see appendix A). These conditions place upper bounds on the amount of work W_{ext} extractable, and since our initial states are block-diagonal in the energy eigenbasis, these second laws are necessary and sufficient to characterize a transition.

The efficiency of a particular heat engine is given by

$$\eta := \frac{W_{ext}}{\Delta H}, \tag{7}$$

where $\Delta H = \mathrm{tr}(\hat{H}_{Hot} \tau^0_{Hot}) - \mathrm{tr}(\hat{H}_{Hot} \rho^1_{Hot})$. This can be simplified by noting that the total Hamiltonian is simply the individual sum of each system's free Hamiltonian, and therefore for any state $\rho_{ColdHotMW}$, $\mathrm{tr}(\hat{H}_{ColdHotMW} \rho_{ColdHotMW}) = \mathrm{tr}(\hat{H}_{Hot} \rho_{Hot}) + \mathrm{tr}(\hat{H}_{Cold} \rho_{Cold}) + \mathrm{tr}(\hat{H}_W \rho_W) + \mathrm{tr}(\hat{H}_M \rho_M)$. If we define the terms $\Delta C = \mathrm{tr}(\hat{H}_{Cold} \rho^1_{Cold}) - \mathrm{tr}(\hat{H}_{Cold} \tau^0_{Cold})$, and $\Delta W = \mathrm{tr}(\hat{H}_W \rho^1_W) - \mathrm{tr}(\hat{H}_W \rho^0_W)$, then we see that since total energy is preserved in the process, by noting that $\rho^0_M = \rho^1_M$ and rearranging terms, we have $\Delta H = \Delta C + \Delta W$. Furthermore, note that because of equations (4) and (5), we have $\Delta W = (1 - \varepsilon) W_{ext}$. Hence, according to equation (7), we have

$$\eta^{-1} = 1 - \varepsilon + \frac{\Delta C}{W_{ext}}. \tag{8}$$

4. Results

We show that CE can be surpassed in a single-shot setting of work extraction, even without using non-thermal resources. We obtain this result through deriving an analytical expression for the efficiency of a QHE in the quasi-static limit, when extracting imperfect work.

Consider the probability ε where the final battery state is not in the state $|E_k\rangle \langle E_k|$, according to equation (5). This is also what we call the failure probability of extracting work. The limit $\varepsilon \to 0$ is the focus of our analysis for several reasons. Firstly, recall that when categorizing the quality of extracted work, one is interested not only in the absolute values of entropy change in the battery, which we denoted as ΔS. Rather, ΔS compared to the

Table 1. Different regimes of work corresponding to different limits of the ratio $\lim_{\varepsilon \to 0} \frac{\Delta S}{W_{\text{ext}}}$.

	Type	Maximum efficiency
Perfect work	$\varepsilon = 0$ [14]	Work extraction for any $W_{\text{ext}} > 0$ is not possible.
Near perfect work	$\lim_{\varepsilon \to 0} \frac{\Delta S}{W_{\text{ext}}} = 0$ [14]	η_C is the theoretical maximum, and can only be approached uniquely in the quasi-static limit. However, η_C can be approached only if certain conditions on the bath Hamiltonianare met. Otherwise, the maximum attainable efficiency is strictly upper bounded away from η_C.
Imperfect work (this paper)	$\lim_{\varepsilon \to 0} \frac{\Delta S}{W_{\text{ext}}} = p, \; p \in (0, \infty)$	
	∞	Unknown, however examples of exceeding CE can be found.

amount of extracted work W_{ext}, in other words the ratio $\frac{\Delta S}{W_{\text{ext}}}$ is the quantity of importance. For any given finite n number of cold bath qubits, the amount of work extractable is finite. Extracting near perfect work means that ΔS should be negligible compared with W_{ext}, as we have seen in definition 2. Since according to equation (6), $\Delta S = h_2(\varepsilon) \geqslant \varepsilon$, therefore we are concerned with the limit where ε is arbitrarily small. On the other hand, now consider imperfect work. The quasi-static limit, i.e. $g \to 0$ is the focus of our analysis that aims to provide examples of imperfect work extraction. In the quasi-static limit, since the cold bath changes only by an infinitesimal amount, therefore the amount of work extractable W_{ext} is also infinitesimally small. For most cases of imperfect work (when the ratio of $\frac{\Delta S}{W_{\text{ext}}}$ is finite) we know that ΔS is vanishingly small, and therefore so is ε.

In [14], it has been shown that perfect work is never achievable, while considering near perfect work allows us to sometimes achieve arbitrarily near to CE, but not always. Therefore, our results, when combining with [14] provide the full range of possible limits for $\frac{\Delta S}{W_{\text{ext}}}$, with the corresponding findings about maximum achievable efficiency, which we summarize in table 1. Theorem 1 formally states our main result. This theorem establishes a simplification of the efficiency of a quasi-static heat engine, given a cold bath consisting of n identical qubits, each with energy gap E. We consider a special case where the failure probability $\varepsilon \propto g$ is proportional to the quasi-static parameter g (see definition 4), and evaluate the efficiency in the limit $g \to 0$. We show that this corresponds to extracting imperfect work, in particular, $\lim_{\varepsilon \to 0} \frac{\Delta S}{W_{\text{ext}}} = \infty$. For such a case, we show that whenever $E < \frac{1}{2(\beta_c - \beta_h)}$, then for some parameter α^*, we can choose the proportionality constant $c(\alpha^*) = \frac{\varepsilon}{g}$ such that the corresponding efficiency of such a heat engine is given by a simple analytical expression. Therefore, by numerically evaluating such an expression for different parameters β_c, β_h, E, n, α^* etc, one can find examples of surpassing the CE.

Theorem 1 (**Main result**). *Consider a quasi-static heat engine with a cold bath consisting of n identical qubits with energy gap $E > 0$. Given the inverse temperatures of the hot and cold bath β_h, $\beta_c > 0$ respectively, and for $\alpha \in (1, \infty)$ define the functions*

$$B_\alpha = \frac{E}{1 + e^{\beta_c E}} \cdot \frac{e^{(\beta_h + \alpha \beta_c)E} - e^{(\beta_c + \alpha \beta_h)E}}{e^{\alpha \beta_h E} + e^{(\beta_h + \alpha \beta_c)E}} \tag{9}$$

and $B'_\alpha = \frac{\mathrm{d}B_\alpha}{\mathrm{d}\alpha}$ being the first derivative of B_α according to α. If the energy gap of the qubits satisfy

$$0 < E < \frac{1}{2(\beta_c - \beta_h)}, \tag{10}$$

then there exists $\alpha^* \in (1, 2)$ such that the failure probability $\varepsilon = g \cdot n[\alpha^*(\alpha^* - 1)B'_{\alpha^*} - B_{\alpha^*}] > 0$, and the inverse efficiency (equation (8)) of the described heat engine is given by

$$\eta^{-1} = 1 + \frac{\beta_h}{\beta_c - \beta_h} \frac{1}{\alpha^{*2}} \frac{B'_1}{B'_{\alpha^*}}. \tag{11}$$

We plot, in figures 2–4 the comparison between CE and the efficiency achievable according to theorem 1. In all these plots we observe that CE is always surpassed, therefore providing us with examples of heat engine cycles that surpass CE. However, this does not imply that the surpassing of CE when extracting imperfect work solely happens for quasi-static processes. The quasi-static limit is not a necessary restriction; it is simply a specific example we have chosen in order to demonstrate the consequences of considering imperfect work. The reason for such a choice, is because if we consider perfect or near perfect work instead, a quasi-static heat engine is most advantageous, i.e. whenever the CE is achieved, it is achieved only by a quasi-static process.
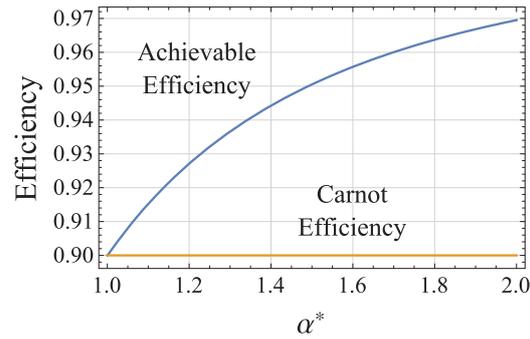
Figure 2. Achievable efficiency versus CE with respect to $\alpha^* \in (1, 2)$, $\beta_h = 1$, $\beta_c = 10$ and $E = \frac{0.4}{\beta_c - \beta_h}$.
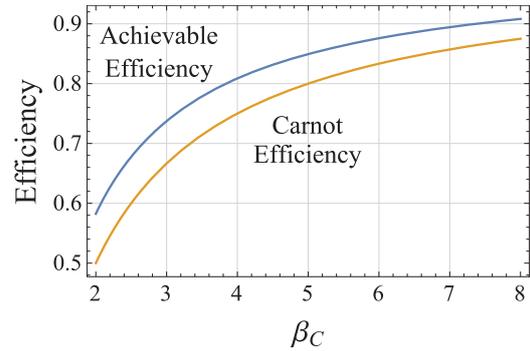

Figure 3. Achievable efficiency versus CE with respect to β_c, with $\alpha^* = 1.2$, $\beta_h = 1$, $E = \frac{0.4}{\beta_c - \beta_h}$.
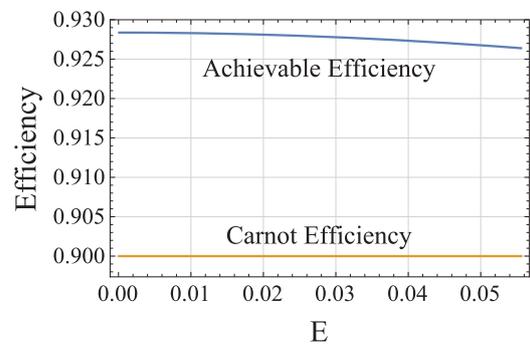

Figure 4. Achievable efficiency versus CE with respect to E, with $\alpha^* = 1.2$, $\beta_h = 1$, $\beta_c = 10$, $E = \frac{0.4}{\beta_c - \beta_h}$.

It is worth noting that equation (10) formulates a condition on the energy gap E of the cold bath qubits, as a function of β_c, β_h. This condition is also a sufficient condition for achieving the CE when extracting near perfect work [14]. Therefore, the blue curve never falls below the yellow line. The improvement in efficiency happens most when the parameter α^* is adjusted, since this is the parameter that determines how quickly the ratio $\frac{\Delta S}{W_{ext}} \to \infty$ in the quasi-static limit.

Given that in table 1, the case of $p \in (0, \infty)$ also corresponds to imperfect work, one might wonder if CE can also be surpassed in this regime. We show that this is not possible.

5. Methods

There are several steps taken in order to achieve the proof of theorem 1, which we outline in this section. For details, the reader is referred to corollary 1 and its proof in the appendix C.1, which directly implies theorem 1.

Theorem 1 is obtained by considering a cold bath of n-identical qubits, and calculating the ratio of extractable work W_{ext} against ΔC in the quasi-static limit, i.e. $g \to 0^+$. Then, by using equation (8), one can evaluate the efficiency. The main difficulty lies in evaluating W_{ext}, the amount of extractable work. This quantity represents the maximum value of the battery's energy gap, such that a transition $\tau_{\beta_c} \otimes \rho_W^0 \to \rho_{\text{Cold}}^1 \otimes \rho_W^1$ is possible according to the generalized second laws described in appendix A. Applying the generalized second laws, we can calculate W_{ext}, which is given by a minimization problem over the continuous range of a real-valued variable $\alpha > 0$,

$$W_{\text{ext}} = \inf_{\alpha > 0} W_\alpha, \tag{12}$$

where

$$W_\alpha = \frac{1}{\beta_h(\alpha - 1)}[\ln(A - \varepsilon^\alpha) - \alpha \ln(1 - \varepsilon)], \tag{13}$$

$$A = \left(\frac{\sum_i p_i^\alpha q_i^{1-\alpha}}{\sum_i p_i'^\alpha q_i^{1-\alpha}} \right), \tag{14}$$

where p_i are the eigenvalues of the state τ_{β_c}, p_i' are eigenvalues of ρ_{Cold}^1, q_i are eigenvalues of τ_{β_h} respectively. Therefore, the difficulty of evaluating the efficiency lies in performing the optimization of W_α over $\alpha \in (0, \infty)$, which is neither monotonic nor convex. However, by manipulating our freedom of choosing ε, we show that in certain parameter regimes of β_c, β_h, and E, one can evaluate a simple, analytical expression for W_{ext}. The steps taken are outlined as follows, while all the technical lemmas are proven in the appendix:

1. We start by choosing the failure probability to be $\varepsilon = \varepsilon_1 \cdot g$, where ε_1 is independent of the quasi-static parameter g.

2. Starting out from the expression for extractable work given in lemma 1, we prove that in the quasi-static limit, the regime $\alpha \in (0, 1)$ need not be considered in the optimization. This is proven in lemma 4.

3. We show that the function W_α which we desire to minimize has at most one unique local minima. To do so, we establish technical lemmas 7–9, in order to arrive at lemma 10.

4. We show that ε_1 can be chosen such that $\varepsilon > 0$ (lemma 11), and that we can choose it so that we know that a particular $\alpha^* \in (1, 2)$ corresponds to a local stationary point (lemma 12) and specifically a local minima (lemma 13). Since we have established item 3, this implies that we have identified a unique local minima.

5. We show that under certain conditions, $W_{\alpha^*} < W_\infty$. This implies that W_{α^*} corresponds to the global minima which we desire to evaluate[6].

6. The conditions for items 3–5 are summarized in corollary 1, where one can now, by choosing the parameter α^* directly evaluate W_{ext} analytically, and therefore use

$$\eta^{-1} = 1 - \varepsilon + \frac{\Delta C}{W_{\text{ext}}} \tag{15}$$

to calculate the efficiency. The calculation of ΔC is straightforward once ρ_{Cold}^0, ρ_{Cold}^1 are fixed, and for the quasi-static limit, we expand ΔC in terms of the quasi-static parameter g.

One can ask whether it is possible to always exceed CE when imperfect work is drawn. For example, observing in table 1 that the case of $p \in (0, \infty)$ also corresponds to imperfect work, one might wonder if a similar result of exceeding CE can be achieved in the regime where $\frac{\Delta S}{W_{\text{ext}}} \to p$ instead of $\frac{\Delta S}{W_{\text{ext}}} \to \infty$ (as in the case where $\varepsilon \propto g$). We show in appendix C.2 that this is not possible, i.e. CE remains the theoretical maximum when the ratio $\frac{\Delta S}{W_{\text{ext}}}$ remains finite in the quasi-static limit. It is interesting to note that, if only the standard free energy is responsible for determining state transitions, then CE again might be exceeded. In conclusion, in the regime where p is finite, the reason that one cannot exceed CE stems from the fact that there exists a continuous family of generalized free energies in the quantum microregime (see appendix A).

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[6] The reason why W_∞ is not the relevant quantity in our scenario, as in many other scenarios [15, 16, 33, 47], is noted by the fact that W_∞ usually provides the maximum possible amount of work extracted, which leads to the cold bath being in a final state that is thermalized with the surrounding hot bath. However, this process is not the one that maximizes efficiency, which is our goal in this calculation. The fact that we consider a process that is not completely thermalizing, gives rise to the importance of other W_α quantities.

6. Discussions and conclusion

Why is it important to distinguish between work and heat? Suppose we have two batteries A_1 and A_2, each containing the same amount of average energy. However, A_1 is in a pure, defined energy eigenstate; while A_2 is a thermal state corresponding to a particular temperature T_2. Note that there is an irreversibility via catalytic thermal operations for these two batteries: the transition $A_1 \rightarrow A_2$ might be possible, but certainly $A_2 \nrightarrow A_1$, since the free energy of A_1 is higher than of A_2. This makes A_1 a more valuable resource compared to A_2. Indeed, if we further consider the environment to be of temperature T_2, then having A_2 is completely useless: it is passive compared to the environment and cannot be used as a resource to enable more state transitions. Even more crucially, the full amount of energy contained in A_1 can be transferred out, because we have full knowledge of the quantum state.

Indeed, for the case of extracting imperfect work, and in particular for the choice of ε proportional to g, heat contributions are dominant. This is because in such an example, the average energy in the battery increases, its *free energy* actually decreases. This can be seen because by using equations (4)–(6), the free energy difference can be written as $\Delta F = (1 - \varepsilon) W_{\text{ext}} - \beta^{-1} \Delta S$, and when $\varepsilon \propto g$ in the quasi-static limit, ΔS is much larger than W_{ext}. This indicates that the free energy difference, instead of average energy difference in the battery would serve as a more accurate quantifier of work. Indeed, by adopting an operational approach towards this problem, [40] has also identified the free energy to be a potentially suitable quantifier.

Our result serves as a note of caution when it comes to analyzing the performance of heat engines, that quantifying microscopic work simply by the average energy increase in the battery does not adequately account for heat contribution in the work extraction process. Therefore, this might lead to the possibility of surpassing the CE, despite finite-size effects, even in the absence of non-thermal resources. For example, the work extraction protocol proposed in [9] indeed corresponds to $\frac{\Delta S}{W_{\text{ext}}} \rightarrow \infty$, when the intial battery state is a pure energy eigenstate. With each step in the protocol, an infinitesimal amount of energy is extracted, while a finite amount of entropy is injected into the battery. This reminds us that work and heat, although both may contribute to an energy gain, are distinctively different in quality (i.e. orderliness). Therefore, when considering small QHEs, it is not only important to propose schemes that extract energy on average, but also ensure that work is gained, rather than heat.

Acknowledgments

Appendix A. Second laws: the conditions for thermodynamical state transitions

Macroscopic thermodynamics says that for a system undergoing heat exchange with a background thermal bath (at inverse temperature β), the Helmholtz free energy

$$F(\rho) := \langle \hat{H} \rangle_\rho - \frac{1}{\beta} S(\rho), \tag{A1}$$

is necessarily non-increasing. For macroscopic systems, this also constitutes a sufficient condition: whenever the free energy does not increase, we know that a state transition is possible.

However, in the microscopic quantum regime, where only a few quantum particles are involved, it has been shown that macroscopic thermodynamics is not a complete description of thermodynamical transitions. More precisely, not only the Helmholtz free energy, but a whole other family of generalized free energies have to decrease during a state transition. This places further constraints on whether a particular transition is allowed. In particular, if the final target state ρ^1_{ColdW} is diagonal in the energy eigenbasis, these laws also give necessary and sufficient conditions for the possibility of a transition $\rho^0_{\text{ColdW}} \rightarrow \rho^1_{\text{ColdW}}$ via catalytic thermal operations.

We can apply these second laws to our scenario by associating the catalyst with ρ^0_{M}, and considering the heat engine state transition $\rho^0_{\text{W}} \otimes \tau^0_{\text{Cold}} \rightarrow \rho^1_{\text{ColdW}}$. In this scenario, the hot bath with inverse temperature β_h is treated as the background temperature used in the resource theory approach. Since we start with $\rho^0_{\text{W}} \otimes \tau^0_{\text{Cold}}$ which is diagonal in the energy eigenbasis, and since catalytic thermal operations do not create coherences between energy levels, the final state ρ^1_{ColdW} is also diagonal in the energy eigenbasis. Hence, the transition from $\rho^0_{\text{W}} \otimes \tau^0_{\text{Cold}} \rightarrow \rho^1_{\text{W}} \otimes \rho^1_{\text{Cold}}$ is possible via catalytic thermal operations iff $\forall \alpha \geqslant 0$ [16],

$$F_\alpha(\tau_{\mathrm{Cold}}^0 \otimes \rho_{\mathrm{W}}^0, \tau_{\mathrm{ColdW}}^h) \geqslant F_\alpha(\rho_{\mathrm{Cold}}^1 \otimes \rho_{\mathrm{W}}^1, \tau_{\mathrm{ColdW}}^h), \tag{A2}$$

where τ_{ColdW}^h is the thermal state of the system at temperature T_{Hot} of the surrounding bath. The quantity $F_\alpha(\rho, \sigma)$ for $\alpha \geqslant 0$ corresponds to a family of free energies defined in [16], which can be written in the form

$$F_\alpha(\rho, \tau_{\beta_h}) = \frac{1}{\beta_h}[D_\alpha(\rho\|\tau) - \ln Z_{\beta_h}], \tag{A3}$$

where $D_\alpha(\rho\|\tau)$ are known as α-Rényi divergences. Sometimes we will use the short hand $F_\infty := \lim_{\alpha\to\infty} F_\alpha$. On occasion, we will refer to a particular transition as being possible/impossible according to the F_α free energy constraint. By this, we mean that for that particular value of α and transition, equation (A2) is satisfied/not satisfied. The α-Rényi divergences can be defined for arbitrary quantum states, giving us necessary (but insufficient) second laws for state transitions [16, 48]. However, since we are analyzing states which are diagonal in the same eigenbasis (namely the energy eigenbasis), these laws are both neccesary and sufficient. Also, the Rényi divergences can be simplified to

$$D_\alpha(\rho\|\tau) = \frac{1}{\alpha - 1} \ln \sum_i p_i^\alpha q_i^{1-\alpha}, \tag{A4}$$

where p_i, q_i are the eigenvalues of ρ and the state τ. The cases $\alpha = 0$ and $\alpha \to 1$ are defined by continuity, namely

$$D_0(\rho\|\tau) = \lim_{\alpha\to 0^+} D_\alpha(\rho\|\tau) = -\ln \sum_{i:p_i\neq 0} q_i, \qquad D_1(\rho\|\tau) = \lim_{\alpha\to 1} D_\alpha(\rho\|\tau) = \sum_i p_i \ln \frac{p_i}{q_i}, \tag{A5}$$

and we also define D_∞ as

$$D_\infty(\rho\|\tau) = \lim_{\alpha\to\infty^+} D_\alpha(\rho\|\tau) = \ln \max_i \frac{p_i}{q_i}. \tag{A6}$$

The quantity $D_1(\rho\|\tau)$ is also known as the *relative entropy*, while it can be checked that $F_1(\rho, \tau)$ coincides with the Helmholtz free energy. We will often use the convention $D(\rho\|\tau)$, $F(\rho, \tau)$ in place of $D_1(\rho\|\tau)$ and $F_1(\rho, \tau)$.

Appendix B. Optimizing over W_α in the quasi-static limit

B.1. Basic technical tools
In this section, we write out the analytical expressions for the amount of extractable work in the case of a quasi-static heat engine, where the cold bath comprises of n identical systems. In particular, we use the expression of extractable work in lemma 1 in order to evaluate the efficiency of our heat engine.

Consider a state transition via catalytic thermal operations

$$\tau_{\beta_{\mathrm{Cold}}} \otimes \rho_{\mathrm{W}}^0 \to \rho_{\mathrm{Cold}}^1 \otimes \rho_{\mathrm{W}}^1, \tag{B1}$$

where $\tau_{\beta_{\mathrm{Cold}}}$ is the initial state of the cold bath (at inverse temperature β_{Cold}), ρ_{Cold}^1 is the final state of the cold bath, and the battery states are given by

$$\rho_{\mathrm{W}}^0 = |E_{\mathrm{W}}^j\rangle \langle E_{\mathrm{W}}^j|, \tag{B2}$$

$$\rho_{\mathrm{W}}^1 = \varepsilon |E_{\mathrm{W}}^j\rangle \langle E_{\mathrm{W}}^j| + (1 - \varepsilon)|E_{\mathrm{W}}^k\rangle \langle E_{\mathrm{W}}^k|. \tag{B3}$$

Lemma 1. *Consider the state transition described in equations (B1)–(B3), and assume that the cold bath Hamiltonian is taken to be of n identical systems,*

$$\hat{H}_{\mathrm{Cold}} = \sum_{i=1}^n \mathbb{1}^{\otimes(i-1)} \otimes \hat{H}_c \otimes \mathbb{1}^{\otimes(n-i)}. \tag{B4}$$

Then whenever the failure probability $0 < \varepsilon \ll 1$, the maximum extractable work is

$$W_{\mathrm{ext}} = \inf_{\alpha>0} W_\alpha, \tag{B5}$$

where

$$W_\alpha = \frac{1}{\beta_h(\alpha - 1)}[\ln(A - \varepsilon^\alpha) - \alpha \ln(1 - \varepsilon)], \tag{B6}$$

$$A = \left(\frac{\sum_i p_i^\alpha q_i^{1-\alpha}}{\sum_i p_i'^\alpha q_i^{1-\alpha}}\right)^n, \tag{B7}$$

where $p_i = \frac{e^{-\beta_c E_i}}{Z_{\beta_c}}$ are the eigenvalues of the thermal state for $\hat{H}_c$ at inverse temperature β_c, and $q_i = \frac{e^{-\beta_h E_i}}{Z_{\beta_h}}$ are the probabilities corresponding to the thermal state of the cold bath with respect to β_h. Furthermore, W_∞ denotes the shorthand notation for $\lim_{\alpha \to \infty} W_\alpha$.

Proof. The proof comes from directly applying the generalized second laws for block-diagonal states, i.e. noticing that equation (A2) is necessary and sufficient for the transition in equation (B1) to occur. Noting that Rényi divergences for all $\alpha \geqslant 0$ are additive, equation (A2) is equivalent to having

$$D_\alpha(\rho_W^0 \| \tau_W) + D_\alpha(\tau_{\beta_c} \| \tau_{\beta_h}) \geqslant D_\alpha(\rho_W^1 \| \tau_W) + D_\alpha(\rho_{\text{Cold}}^1 \| \tau_{\beta_h}), \tag{B8}$$

where τ_W is the thermal state with Hamiltonian $\hat{H}_W$ at inverse temperature β_h. We define W_α to be the value of $E_k^W - E_j^W$ that satisfies equation (B8) with equality. A straightforward manipulation of these equations will produce the expression for W_α in equation (B7). Then $W_{\text{ext}} = \inf_{\alpha \geqslant 0} W_\alpha$ is the maximum value that satisfies the inequalities equation (B8) for all $\alpha \geqslant 0$. $\qquad\square$

In the quasi-static limit, where recall that this implies $\rho_{\text{Cold}}^1 = \tau_{\beta_f}$, such that $\beta_f - \beta_c = g \ll 1$, one may rewrite equation (B6) into an approximation for small g, ε; this is done by expanding equation (B6) according to variables g and ε. More precisely, let us define the order terms as follows:

Definition 5 (**Big Θ notation** [49]). Consider two real-valued functions $P(x)$, $Q(x)$. We say that $P(x) = \Theta(Q(x))$ in the limit $x \to a$ iff there exists c_1, $c_2 > 0$ and $\delta > 0$ such that for all $|x - a| \leqslant \delta$, $c_1 \leqslant \left| \frac{P(x)}{Q(x)} \right| \leqslant c_2$.

Remark 1. In definition 5, if the limit of x is unspecified, by default we take $a = 0$. In [49], these order terms were originally defined for $x \to \infty$. However, choosing a general limit $x \to a$ can be done by simply defining the variable $x' = 1/(x - a)$, and $x \to a^+$ is the same as taking $x' \to \infty$.

By the use of the notation for such order functions, one can first simplify A in equation (B7) for small g by Taylor expanding A in g, i.e. for $g \ll 1$,

$$A = 1 + \left.\frac{dA}{dg}\right|_{g=0} \cdot g + \Theta(g^2). \tag{B9}$$

On the other hand, the function $\ln(1 - x)$ when $|x| \ll 1$ (in our case, x depends on both g and ε) can also be written as

$$\ln(1 + x) = 1 + x + \Theta(x^2). \tag{B10}$$

Therefore, the expansion of W_α for any $\alpha > 0$, in the regime where g, $\varepsilon \to 0$ can be written as the following:

$$W_\alpha = \begin{cases} \frac{1}{\beta_h(\alpha - 1)}[\alpha n g B_\alpha - \varepsilon^\alpha + \alpha \varepsilon] + \Theta(g^2) + \Theta(\varepsilon^{2\alpha}) + \Theta(g\varepsilon^\alpha) + \Theta(\varepsilon^2), & \text{if } \alpha \in (0, \infty) \backslash \{1\}, \\ \lim_{\alpha \to 1^+} \frac{1}{\beta_h(\alpha - 1)}[\alpha n g B_\alpha - \varepsilon^\alpha + \alpha \varepsilon] + \Theta(\varepsilon g) + \Theta(\varepsilon^2 \ln \varepsilon) + \Theta(\varepsilon^2) + \Theta(g^2), & \text{if } \alpha = 1, \end{cases} \tag{B11}$$

where the function B_α is given by

$$B_\alpha = \frac{1}{\sum_i p_i^\alpha q_i^{1-\alpha}} \sum_i p_i^\alpha q_i^{1-\alpha}(\langle \hat{H}_c \rangle_{\beta_c} - E_i). \tag{B12}$$

While multi-variable order terms can be defined in a much more general way, it is not necessary in our case. Here, when the order functions depend on both variables g, ε, we have simply adapted a shorthand notation that for any functions $P_1(g)$ and $P_2(\varepsilon)$, the order function $\Theta(P_1(g)P_2(\varepsilon)) = \Theta(P_1(g)) \cdot \Theta(P_2(\varepsilon))$. Furthermore, we also checked explicitly that by first taking the limit $\lim_{\alpha \to \infty} W_\alpha$, then expanding in small g, ε gives the same expression, i.e. equation (B11) holds also in the limit $\alpha \to \infty$.

As in this article, we are considering the limit where both g, $\varepsilon \to 0$. Throughout our proof, we have dealt directly with the general expression found in equation (B6). However, in the end we shall see that in this limit, only the largest order terms in equation (B11) matter. In other words, we will show that when g, $\varepsilon \to 0$, taking the infimum over the largest order term in equation (B11) for all $\alpha > 0$ yields the same solution α_1, which also achieves the infimum over equation (B6).

In the special case where the cold bath consists of n identical qubits, i.e. $\hat{H}_c = E|1\rangle\langle 1|$ with E being the energy gap of each qubit, the expression for B_α simplifies to

$$B_\alpha = \frac{E}{1 + e^{\beta_c E}} \cdot \frac{e^{(\beta_h + \alpha\beta_c)E} - e^{(\beta_c + \alpha\beta_h)E}}{e^{\alpha\beta_h E} + e^{(\beta_h + \alpha\beta_c)E}}. \tag{B13}$$

We also list several expressions that will be useful in deriving our results later. Taking the derivatives of B_α as defined in equation (B13) w.r.t. α, we have

$$B'_\alpha = \frac{dB_\alpha}{d\alpha} = \frac{1}{[e^{\alpha\beta_h E} + e^{(\beta_h + \alpha\beta_c)E}]^2} \cdot E^2(\beta_c - \beta_h) \cdot e^{[\beta_h + \alpha(\beta_c + \beta_h)]E} \tag{B14}$$

$$> 0 \text{ whenever } \beta_c > \beta_h, \forall \alpha > 0, \tag{B15}$$

$$B''_\alpha = \frac{d^2 B_\alpha}{d\alpha^2} = \frac{1}{[e^{\alpha\beta_h E} + e^{(\beta_h + \alpha\beta_c)E}]^3} \cdot E^3(\beta_c - \beta_h)^2 \cdot e^{[\beta_h + \alpha(\beta_c + \beta_h)]E} \cdot [e^{\alpha\beta_h E} - e^{(\alpha\beta_c + \beta_h)E}] \tag{B16}$$

$$< 0 \text{ whenever } \beta_c > \beta_h, \forall \alpha > 0. \tag{B17}$$

Next, an simple identity will be important for the evaluation of efficiency for a quasi-static heat engine as well. This we present as a lemma here.

Lemma 2. *Consider a quasi-static heat engine where the cold bath consists of n identical systems (with individual Hamiltonians $\hat{H}_c$) at inverse temperature β_c. Denote the inverse temperature of the hot bath as β_h, and the following function*

$$\Delta C := \text{tr}(\hat{H}_c \rho_C^1) - \text{tr}(\hat{H}_c \tau_{\beta_c}). \tag{B18}$$

Then in the quasi-static limit, where the cold bath final state is a thermal state of inverse temperature $\beta_f = \beta_c - g$, where $0 < g \ll 1$,

$$\Delta C = \frac{nB'_1}{\beta_c - \beta_h} \cdot g + \Theta(g^2), \tag{B19}$$

where $B'_\alpha = \frac{dB_\alpha}{d\alpha}$ and B_α is defined in equation (B12).

Proof. This lemma is directly obtained by Taylor expansion of equation (B18), noting two things: (1) that $\Delta C|_{g=0} = 0$, and that (2) when $\rho^1_{\text{Cold}} = \tau_{\beta_f}$,

$$\left.\frac{d\Delta C}{dg}\right|_{g=0} = \frac{nB'_1}{\beta_c - \beta_h}. \tag{B20}$$

$\square$

The third tool is an observation initially made in [14] for choices of $\varepsilon(g)$ as a function of the quasi-static parameter g. There, it is shown that one can characterize any choice of continuous function $\varepsilon(g)$ by the real parameters $\bar{\kappa}, \sigma \in \mathcal{R}_{\geqslant 0}$.

Lemma 3 (**Lemma 11, [14]**). *For every continuous function $\varepsilon(g) > 0$ satisfying $\lim_{g\to 0^+} \varepsilon(g) = 0, \exists \bar{\kappa} \in \mathbb{R}_{\geqslant 0}$ s.t.*

$$\delta(\kappa) = \lim_{g\to 0^+} \frac{\varepsilon^\kappa(g)}{g} = \begin{cases} 0, & \text{if } \kappa > \bar{\kappa}, \\ \sigma \geqslant 0, & \text{if } \kappa = \bar{\kappa}, \\ \infty, & \text{if } \kappa < \bar{\kappa}, \end{cases} \tag{B21}$$

where $\bar{\kappa} = +\infty$ is allowed (that is to say, $\lim_{g\to 0^+} \frac{\varepsilon^\kappa(g)}{g}$ diverges for every κ) and $\sigma = +\infty$ is also allowed.

Therefore, we summarize some results from [14] into the following table B1, for any continuous function $\varepsilon(g)$ such that $\lim_{g\to 0} \varepsilon(g) = 0$. The regime of near perfect work, i.e. $\lim_{g\to 0} \frac{\Delta S}{W_{\text{ext}}} = 0$ is thoroughly investigated in [14]. In this paper, we investigate the full regime of imperfect work by first analyzing in section C.1 an example where $\lim_{g\to 0} \frac{\Delta S}{W_{\text{ext}}} = \infty$, and in section C.2 for all cases where $\lim_{g\to 0} \frac{\Delta S}{W_{\text{ext}}} = p > 0$.

B.2. Technical lemmas
Building on the results adapted from [14] and summarized in section B.1, this section contains the technical lemmas and proofs used to develop the proof of theorem 1.

Lemma 4. *Given any heat engine, consider the state transition*

$$\tau_{\beta_{\text{Cold}}} \otimes \rho_W^0 \to \rho_{\text{Cold}}^1 \otimes \rho_W^1, \tag{B22}$$

where $\rho_W^0 = |E_j\rangle\langle E_j|_W$, $\rho_W^1 = (1-\varepsilon)|E_k\rangle\langle E_k|_W + \varepsilon|E_j\rangle\langle E_j|_W$ respectively, where $W_{\text{ext}} = E_k - E_j$. Let $\varepsilon = \varepsilon_1 \cdot g$, where $\varepsilon_1 > 0$ is independent of α and g. Then there exists $g' > 0$ such that for all $0 < g \leqslant g'$,

Table B1. Each choice of a continuous function ε such that $\lim_{g \to 0} \varepsilon = 0$, can lead to different regimes of $\frac{\Delta S}{W_{\text{ext}}}$ in the quasi-static limit, depending on the values of $\bar{\kappa}$, σ and $\lim_{g \to 0} \frac{-\varepsilon \ln \varepsilon}{g}$. Recall lemma 3 for the definitions of $\bar{\kappa}$ and σ.

	$\lim_{g \to 0} \frac{\Delta S}{W_{\text{ext}}}$	Characterization
Near perfect work	0	$\bar{\kappa} \in [0, 1)$
		$\bar{\kappa} = 1 \;\wedge\; \lim_{g \to 0} \frac{\varepsilon \ln \frac{1}{\varepsilon}}{g} = 0$
	$p > 0$	$\bar{\kappa} = 1 \;\wedge\; \lim_{g \to 0} \frac{\varepsilon \ln \frac{1}{\varepsilon}}{g} = p', \;\; 0 < p' < \infty$
Imperfect work		$\bar{\kappa} \in (1, \infty)$
	∞	$\bar{\kappa} = 1 \wedge \sigma = p'' > 0$
		$\left(\text{This implies that } \lim_{g \to 0} \frac{\varepsilon \ln \frac{1}{\varepsilon}}{g} = \infty \right)$

$$W_{\text{ext}} = \inf_{\alpha > 0} W_\alpha = \inf_{\alpha > 1} W_\alpha, \tag{B23}$$

where W_α is defined in equation (B6).

Proof. We start out from the most general expression of extractable work, given by equation (B6). Let us first note that for any $\alpha \in [0, \infty)$, W_α is a continuous function of g, and that $\lim_{g \to 0} W_\alpha = 0$. This can be seen by directly plugging in $g = 0$ into equation (B7), and since $\varepsilon = 0$, $|A|_{g=0} = 1$, therefore for all $\alpha > 0$, we have $W_\alpha = 0$ (the case of W_1 is automatically true as well, since W_1 is defined by taking the limit $\alpha \to 1$). Furthermore, for different values of $g > 0$, the value $W_{\text{ext}} = \inf_{\alpha > 0} W_\alpha$ can be obtained at different values of α such that the optimal α depends on g. However, in the quasi-static limit, there must exist a particular $\alpha_1 > 0$ that achieves the minimum value, i.e.

$$\lim_{g \to 0} \frac{W_{\text{ext}}}{W_{\alpha_1}} = 1, \tag{B24}$$

where this implies that for any $\alpha' \neq \alpha_1$, we have that

$$\lim_{g \to 0} \frac{W_{\alpha'}}{W_{\alpha_1}} \geqslant 1. \tag{B25}$$

However, since we know that both $\lim_{g \to 0} W_{\alpha'} = 0$ and $\lim_{g \to 0} W_{\alpha_1} = 0$. Therefore by L'Hospital rule, this implies that if we define the first derivative of W_α w.r.t. g:

$$I(\alpha) = \frac{\mathrm{d} W_\alpha}{\mathrm{d} g}, \tag{B26}$$

then for any α', we also have

$$\lim_{g \to 0} \frac{I(\alpha')}{I(\alpha_1)} \geqslant 1. \tag{B27}$$

This implies that the solution α_1 to the minimization problem of $\inf_{\alpha > 0} W_\alpha$, in the limit where $g \to 0$, is also the solution for the minimization problem $\inf_{\alpha > 0} I(\alpha)$. Substituting $\varepsilon = \varepsilon_1 \cdot g$ into equation (B26), we obtain

$$I(\alpha) = \frac{1}{\beta_h (\alpha - 1)} \left[\frac{1}{A - \varepsilon^\alpha} \left(\frac{\mathrm{d} A}{\mathrm{d} g} - \alpha \varepsilon_1^\alpha g^{\alpha - 1} \right) + \frac{\alpha \varepsilon_1}{1 - \varepsilon} \right]. \tag{B28}$$

We now see how equation (B28) behaves in the limit $g \to 0$. For any $\alpha < 1$, the terms involved are

$$A|_{g=0} = 1, \tag{B29}$$

$$\varepsilon|_{g=0} = 0, \tag{B30}$$

$$\frac{\mathrm{d} A}{\mathrm{d} g} \bigg|_{g=0} = \alpha n B_\alpha, \tag{B31}$$

$$g^{\alpha - 1} \to \infty. \tag{B32}$$

Equation (B32) in particular implies that in the limit of $g \to 0$, $I(\alpha)$ diverges to infinity in the interval $\alpha \in (0, 1)$. Furthermore, note that since this does not happen for the regime $\alpha > 1$, and all other terms do not diverge, therefore in the $\alpha > 1$ regime there must be some α such that $I(\alpha) < \infty$ is finite. This allows us to conclude that $\alpha_1 \notin (0, 1)$.

We will now exclude the point $\alpha = 1$ from the minimization. We make use of the small ε, g expansion of W_α in equation (B11) to see why this is so, by calculating the limit $\lim_{g \to 0} \frac{W_1}{W_\infty}$. Let us first substitute $\varepsilon = \varepsilon_1 \cdot g$, and write out the expression for W_∞:

$$W_\infty = \lim_{\alpha \to \infty} W_\alpha = \frac{ng}{\beta_h} \left[\lim_{\alpha \to \infty} \frac{\alpha B_\alpha}{\alpha - 1} + \frac{\varepsilon_1}{n} \right] + \Theta(g^2) + \Theta(\varepsilon^{2\alpha}) + \Theta(g\varepsilon^\alpha) + \Theta(\varepsilon^2) \tag{B33}$$

$$= \frac{ng}{\beta_h} \left[B_\infty + \frac{\varepsilon_1}{n} \right] + \Theta(g^2), \tag{B34}$$

where by definition of $\Theta(x)$ it is sufficient to keep the largest order term when several order functions are summed. One can check that the quantity $B_\infty = \lim_{\alpha \to \infty} B_\alpha$ is finite, for all finite dimensional $\hat{H}_c$. On the other hand, from equation (B11), by substituting our choice of ε we also have

$$W_1 = \frac{1}{\beta_h} \lim_{\alpha \to 1^+} \frac{\alpha n g B_\alpha - \varepsilon^\alpha + \alpha\varepsilon}{\alpha - 1} + \Theta(\varepsilon g) + \Theta(\varepsilon^2 \ln \varepsilon) + \Theta(\varepsilon^2) + \Theta(g^2) \tag{B35}$$

$$= \frac{1}{\beta_h} \lim_{\alpha \to 1^+} n g B_\alpha' + \alpha n g B_\alpha - \varepsilon^\alpha \ln \varepsilon + \varepsilon + \Theta(g^2 \ln g) \tag{B36}$$

$$= \frac{ng}{\beta_h} [B_1' - \varepsilon_1 \ln \varepsilon + \varepsilon_1] + \Theta(g^2 \ln g) \tag{B37}$$

$$> \frac{ng}{\beta_h} \cdot \varepsilon_1 \cdot \ln \frac{1}{\varepsilon} + \Theta(g^2 \ln g). \tag{B38}$$

The second equality comes by applying L'Hospital rule for differentiation limits, and the third equality comes by substituting $\alpha = 1$ into the equation, while noting that $B_1 = 0$, and using $\varepsilon = \varepsilon_1 \cdot g$. The last inequality sign comes from noting that B_1', $\varepsilon_1 > 0$. Comparing equation (B34) and (B38), we see that

$$\lim_{g \to 0} \frac{W_1}{W_\infty} > \lim_{g \to 0} \frac{\frac{ng}{\beta_h} \cdot \varepsilon_1 \cdot \ln \frac{1}{\varepsilon} + \Theta(g^2 \ln g)}{\frac{ng}{\beta_h} \left[B_\infty + \frac{\varepsilon_1}{n} \right] + \Theta(g^2)} = \lim_{g \to 0} \frac{\varepsilon_1 \cdot \ln \frac{1}{\varepsilon_1 \cdot g}}{B_\infty + \frac{\varepsilon_1}{n}} = \infty, \tag{B39}$$

and therefore in the quasi-static regime, $W_1 > W_\infty$.

We have thus proven that in the quasi-static limit, the global minima for $W_{\text{ext}} = \inf_{\alpha > 0} W_\alpha$ will not be obtained in the interval $\alpha \in (0, 1]$. This in turn implies that

$$\inf_{\alpha > 0} W_\alpha = \inf_{\alpha > 1} W_\alpha. \tag{B40}$$

$\square$

With lemma 4, one can dismiss the regime $0 < \alpha \leqslant 1$ when considering the infimum over W_α in equation (B11). In this lemma, we have also shown that in the quasi-static limit, the solution α_1 that corresponds to the infimum in W_{ext} coincides with the solution of the infimum for the function $I(\alpha) = \frac{\mathrm{d}W_\alpha}{\mathrm{d}g}$. By again making use of this function $I(\alpha)$, in the next step, we show that since we are interested in the quasi-static limit and the case where $\varepsilon = \varepsilon_1 \cdot g$, another useful simplification will help us to obtain the minimum for W_{ext}.

Lemma 5. *For $\varepsilon = \varepsilon_1 \cdot g$ where ε_1 is independent of α and g, consider the function*

$$I(\alpha) = \frac{\mathrm{d}W_\alpha}{\mathrm{d}g} = \frac{1}{\beta_h(\alpha - 1)} \left[\frac{1}{A - \varepsilon^\alpha} \left(\frac{\mathrm{d}A}{\mathrm{d}g} - \alpha\varepsilon_1^\alpha g^{\alpha-1} \right) + \frac{\alpha\varepsilon_1}{1 - \varepsilon} \right], \tag{B41}$$

where W_α is given by equations (B6) and (B7). Let α_1 be the solution that achieves the infimum in the quasi-static limit, such that for all other $\alpha' > 0$,

$$\lim_{g \to 0} \frac{I(\alpha')}{I(\alpha_1)} \geqslant 1. \tag{B42}$$

Then, α_1 is also the solution that achieves the infimum for $G(\alpha) = \frac{1}{\beta_h(\alpha - 1)}(\alpha n B_\alpha + \alpha\varepsilon_1)$ in the regime $\alpha \in (1, \infty)$, i.e.

$$G(\alpha_1) = \inf_{\alpha > 1} G(\alpha). \tag{B43}$$

Proof. To see this, note that in lemma 4 we have established that α_1 is not within the interval $(0, 1]$, since within this interval, $\lim_{g \to 0} I(\alpha) = \infty$. On the other hand, for any $\alpha \in (1, \infty)$,

$$\lim_{g \to 0} I(\alpha) = \lim_{g \to 0} \frac{1}{\beta_h(\alpha - 1)} \left[\frac{dA}{dg} - \alpha \varepsilon_1^\alpha g^{\alpha - 1} + \alpha \varepsilon_1 \right] = G(\alpha). \tag{B44}$$

This concludes the lemma. □

Lemma 5 implies that while analyzing $W_{\text{ext}} = \inf_{\alpha > 0} W_\alpha$ in the quasi-static limit, where we are interested in finding the solution α_1 that satisfies equation (B24), it suffices to analyze a much simpler function

$$G(\alpha) = \frac{1}{\beta(\alpha - 1)}(\alpha n B_\alpha + \alpha \varepsilon_1), \tag{B45}$$

since $G(\alpha_1) = \inf_{\alpha > 1} G(\alpha)$. Looking back to compare $G(\alpha)$ with the Taylor expansion of W_α evaluated in equation (B11), we see intuitively why this function provides us the same solution to α_1 as for W_{ext} in the quasi-static limit: $G(\alpha)$ is simply the largest order term (more precisely, it is the term associated with order g) of the Taylor expansion in the interval $\alpha \in (1, \infty)$.

To calculate the infimum of $G(\alpha)$ over the interval $\alpha > 1$, we compute

$$\frac{dG(\alpha)}{d\alpha} = \frac{n}{\beta_h} \frac{B'_\alpha}{(\alpha - 1)^2} \left\{ \alpha(\alpha - 1) - \frac{B_\alpha}{B'_\alpha} - \frac{\varepsilon_1}{n B'_\alpha} \right\}. \tag{B46}$$

Furthermore, we can already apply lemma 4 to understand how $\frac{\Delta S}{W_{\text{ext}}}$ behaves in the quasi-static limit, which we prove in lemma 6.

Lemma 6. *For any heat engine where $\varepsilon = \varepsilon_1 \cdot g$, with ε_1 independent of g, in the quasi-static limit $g \to 0^+$, we have*

$$\lim_{g \to 0^+} \frac{\Delta S}{W_{\text{ext}}} = \infty. \tag{B47}$$

Proof. From lemma 4, and by using equation (B11) we see that for some particular $\alpha_1 \in (1, \infty)$,

$$W_{\text{ext}} = \frac{1}{\beta_h(\alpha_1 - 1)} [\alpha n g B_{\alpha_1} - \varepsilon^\alpha + \alpha \varepsilon] + \Theta(g^2) + \Theta(\varepsilon^{2\alpha}) + \Theta(g\varepsilon^\alpha) + \Theta(\varepsilon^2) \tag{B48}$$

$$= \frac{g}{\beta_h(\alpha_1 - 1)} [\alpha_1 n B_{\alpha_1} + \alpha_1 \varepsilon_1] + \Theta(g^{\alpha_1}) + \Theta(g^2) + \Theta(g^{2\alpha_1}) + \Theta(g^{1 + \alpha_1}). \tag{B49}$$

This implies that the leading order term in W_{ext} is of first order in g. On the other hand,

$$\Delta S = -\varepsilon \ln \varepsilon - (1 - \varepsilon)\ln(1 - \varepsilon) \tag{B50}$$

$$= -\varepsilon_1 \cdot g \ln(\varepsilon_1 \cdot g) - (1 - \varepsilon)[-\varepsilon + \Theta(\varepsilon^2)] \tag{B51}$$

$$= -\varepsilon_1 \cdot g \ln g + \varepsilon_1 \ln \varepsilon_1 \cdot g + \varepsilon + \Theta(\varepsilon^2) + \Theta(\varepsilon^3) \tag{B52}$$

$$= -\varepsilon_1 \cdot g \ln g + \Theta(g) + \Theta(g^2) + \Theta(g^3). \tag{B53}$$

The second equality is obtained by substituting $\varepsilon = \varepsilon_1 \cdot g$ and writing $\ln(1 - \varepsilon) = -\varepsilon + \Theta(\varepsilon^2)$ in terms of Taylor expansion. The third equality is obtained by expanding out all the multiplied brackets, while the last equality is obtained by noting that $\Theta(\varepsilon) = \Theta(g)$, and therefore concluding that the leading order term (which has the slowest convergence rate as $g \to 0$) is of order $g \ln g$. With this, one can evaluate the limit

$$\lim_{g \to 0^+} \frac{\Delta S}{W_{\text{ext}}} = \lim_{g \to 0^+} \frac{-\varepsilon_1 \cdot g \ln g + \Theta(g) + \Theta(g^2) + \Theta(g^3)}{\frac{g}{\beta_h(\alpha_1 - 1)} [\alpha_1 n B_{\alpha_1} + \alpha_1 \varepsilon_1] + \Theta(g^{\alpha_1}) + \Theta(g^2) + \Theta(g^{2\alpha_1}) + \Theta(g^{1 + \alpha_1})} \tag{B54}$$

$$= \lim_{g \to 0^+} \frac{-\varepsilon_1 \cdot \ln g + \Theta(1) + \Theta(g) + \Theta(g^2)}{\frac{1}{\beta_h(\alpha_1 - 1)} [\alpha_1 n B_{\alpha_1} + \alpha_1 \varepsilon_1] + \Theta(g^{\alpha_1 - 1}) + \Theta(g) + \Theta(g^{2\alpha_1 - 1}) + \Theta(g^{\alpha_1})} \tag{B55}$$

$$= \infty. \tag{B56}$$

The second equality is obtained by dividing both numerator and denominator with g. Then we see that in the numerator, $-\varepsilon_1 \cdot \ln g$ goes to infinity, while the other terms remain finite. On the other hand, the denominator goes to a finite constant. Therefore, we conclude that $\lim_{g \to 0^+} \frac{\Delta S}{W_{\text{ext}}} = \infty$. □

Recall that we have previously estalished in lemmas 4 and 5 that the solution α_1 for the optimization of W_{ext} in equation (B5)–(B7), in the quasi-static limit, will be the same value that minimizes the function $G(\alpha)$ in equation (B45). From here onwards, we focus our analysis to the case where the cold bath consists of qubits. Therefore, B_α is given by equation (B13), and B'_α, B''_α in equations (B14) and (B16) respectively. Furthermore, it is also useful for us to evaluate

$$B_\infty = \lim_{\alpha \to \infty} B_\alpha = \frac{E}{1 + e^{\beta_c E}}, \tag{B57}$$

by applying equation (B13).

The next lemmas 7 and 8 would establish a useful property of $\frac{dG(\alpha)}{d\alpha}$, namely that this function has not more than 3 roots in the regime $\alpha \in (1, \infty)$, i.e. $G(\alpha)$ does not have more than 3 stationary points. Then in lemma 9 we show how the value of $\lim_{\alpha \to \infty} G(\alpha)$ is approached.

Lemma 7. *Consider the function $f(\alpha) := \alpha(\alpha - 1) - \frac{B_\alpha}{B_\alpha'} - \frac{\varepsilon_1}{nB_\alpha'}$, which is found in the rhs of equation (B46). Then its first derivative w.r.t α, $f'(\alpha) = \frac{df(\alpha)}{d\alpha}$ is strictly concave in the domain $\alpha \in (1, \infty)$. This also implies that $f(\alpha)$ has at most 3 roots in the regime $\alpha \in (1, \infty)$.*

Proof. Note that $f'(\alpha) = g'(\alpha) + \frac{\varepsilon_1}{n} \frac{B_\alpha''}{B_\alpha'^2}$, where $g'(\alpha) = \frac{d}{d\alpha}[\alpha(\alpha - 1) - \frac{B_\alpha}{B_\alpha'}]$. It has been shown in lemma 12, supplementary material of [14] that $g'(\alpha)$ is a strictly concave function. On the other hand, by using the definitions in equations (B14) and (B16), one can evaluate the second derivative of

$$\frac{d^2}{d\alpha^2} \frac{B_\alpha''}{B_\alpha'^2} = (\beta_c - \beta_h)^2 e^{-[\beta_h + \alpha(\beta_c + \beta_h)]E} E \cdot [e^{2\alpha\beta_h E} - e^{2(\alpha\beta_c + \beta_h)E}].$$

All the terms in the equation above are positive, except for the last term which is always negative when $\beta_h < \beta_c$. Therefore, the function $\frac{B_\alpha''}{B_\alpha'^2}$ is strictly concave as well. This implies that $f'(\alpha)$ is the addition of two strictly concave functions, and therefore is also strictly concave itself. $\qquad \square$

One can apply lemma 7 to analyze the function $G(\alpha)$ to show that it does not have more than 3 stationary points.

Lemma 8 ($G(\alpha)$ **has not more than 3 stationary points**). *Consider the continuous function $G(\alpha) = \frac{1}{\beta_h(\alpha - 1)}[\alpha nB_\alpha + \alpha\varepsilon_1]$ in the regime $\alpha \in (1, \infty)$. Then the equation $\frac{dG(\alpha)}{d\alpha} = 0$ has at most 3 roots, i.e. the function $G(\alpha)$ has not more than 3 stationary points.*

Proof. Let us begin by writing out the function

$$\frac{dG(\alpha)}{d\alpha} = \frac{n}{\beta_h} \frac{1}{(\alpha - 1)^2} B_\alpha' \left\{ \alpha(\alpha - 1) - n\frac{B_\alpha}{B_\alpha'} - \frac{\varepsilon_1}{nB_\alpha'} \right\}.$$

Since from the expression in equation (B14), we see that $B_\alpha' > 0$ whenever $\beta_h < \beta_c$, by lemma 7, we know that $\frac{dW_\alpha}{d\alpha}$ can have at most 3 roots. $\qquad \square$

Lemma 9 (W_∞ **is approached from above**). *Consider the continuous function $G(\alpha) = \frac{n}{\beta_h(\alpha - 1)}\left[\alpha B_\alpha + \frac{\alpha\varepsilon_1}{n}\right]$ in the regime $\alpha \in (1, \infty)$. Then the limit $\lim_{\alpha \to \infty} G(\alpha)$ exists and is approached from above.*

Proof. We have seen from equation (B34) that $\lim_{\alpha \to \infty} G(\alpha)$ exists and is some finite number. We then only need to prove that in the limit of large α, the quantity $\frac{dG(\alpha)}{d\alpha} < 0$. This can be seen from equation (B46), which we rewrite here again

$$\frac{dG(\alpha)}{d\alpha} = \frac{1}{\beta_h} \frac{1}{(\alpha - 1)^2} \{\alpha(\alpha - 1)nB_\alpha' - nB_\alpha - \varepsilon_1\}. \tag{B58}$$

Let us compare the terms in the large bracket of the rhs. The first term

$$\alpha(\alpha - 1)B_\alpha' = \alpha(\alpha - 1)E^2(\beta_c - \beta_h)e^{-\beta_h E}e^{-\alpha(\beta_c - \beta_h)E} \tag{B59}$$

has a quadratic term in α multiplied by a term which decreases exponentially in α, i.e. $\lim_{\alpha \to \infty} \alpha(\alpha - 1)B_\alpha' = 0$. On the other hand, the remaining terms can be expressed by using equation (B57):

$$\lim_{\alpha \to \infty} -B_\alpha - \frac{\varepsilon_1}{n} = -\left[\frac{E}{1 + e^{\beta_c E}} + \frac{\varepsilon_1}{n}\right] < 0. \tag{B60}$$

Since for large $\alpha \gg 1$, the multiplicative factor in equation (B58) is positive, we have that $\frac{dG(\alpha)}{d\alpha} < 0$. This implies that the function $G(\alpha)$ approaches the limit $\alpha \to \infty$ from above. $\qquad \square$

Lemma 10. *The function $G(\alpha)$ does not have more than one distinct local minimas in the regime $\alpha \in (1, \infty)$.*

Proof. By lemma 8, we know that the function $G(\alpha)$ has at most 3 stationary points in the regime $\alpha \in (1, \infty)$. Firstly, suppose that $G(\alpha)$ has only 1 or 2 stationary points. Then it is clear that there cannot exist two distinct local minimas, since for a continuous function with two local minimas, there has to be at least another local maxima in between, which is also a stationary point.

Now, suppose that $G(\alpha)$ has 3 stationary points, found at $1 < \alpha_1 < \alpha_2 < \alpha_3 < \infty$ respectively. Note that two neighboring stationary points cannot both correspond to local minimas, as reasoned out in the previous paragraph. Therefore, the only way for there to exist 2 local minimum points, is to have α_1, α_3 corresponding to local minimas. If there are no more stationary points in the regime $\alpha > \alpha_3$, then the function $G(\alpha)$ can only be non-decreasing, and the limit $\alpha \to \infty$ has to be approached from below. However, by lemma 9 we know that this cannot be true.

This establishes the fact that $G(\alpha)$ does not have two distinct local minimas. Therefore, it implies that whenever we find some α^* corresponding to a local minima, it will be the unique local minima of the entire function. This simplifies the minimization of $G(\alpha)$ in equation (B23) to comparing $G(\alpha^*)$ with $\lim_{\alpha \to \infty} G(\alpha)$.   □

In the next lemma, we then prove that by making use of our liberty to choose ε_1, we can design it such that $\inf_{\alpha > 1} G(\alpha)$ is obtained at any α^* we desire (albeit still within a certain range).

Lemma 11 (**Conditions for positive** ε_1). *Consider the function*

$$\varepsilon_1(a, n) := n[a(a-1)B_a' - B_a]. \tag{B61}$$

When the condition

$$E < \frac{2}{\beta_c - \beta_h} \frac{1 + e^{\beta_c E}}{e^{\beta_c E} - 1} \tag{B62}$$

holds, then there exists some $\alpha^* > 1$ such that $\varepsilon_1(\alpha^*, n) > 0$.

Proof. We begin by noting that $\varepsilon(1, n) = 0$ for any n. A Taylor's expansion around $a = 1$ would determine the positivity of $\varepsilon(a, n)$ for $a = 1 + \delta$ where $\delta \ll 1$. Therefore, we calculate the derivative $\frac{d\varepsilon_1(a, n)}{da}$

$$\frac{d\varepsilon_1(a, n)}{da} = n[(a-1)B_a' + aB_a' + a(a-1)B_a'' - B_a'] = n(a-1)[2B_a' + aB_a'']. \tag{B63}$$

It is easy to see from equation (B63) that $\frac{d\varepsilon_1(a, n)}{da}\Big|_{a=1} = 0$. Therefore, the term that determines positivity of $\varepsilon_1(a, n)$ around $a = 1$ is the second derivative,

$$\frac{d^2\varepsilon_1(a, n)}{da^2} = n[2B_a' + aB_a'' + (a-1)(2B_a'' + B_a'' + aB_a''')]. \tag{B64}$$

The quantity $\frac{d^2\varepsilon_1(a, n)}{da^2}\Big|_{a=1} = n[2B_1' + B_1'']$ we can expressed in a simplified form,

$$\frac{d^2\varepsilon_1(a, n)}{da^2}\Big|_{a=1} = \frac{n(\beta_c - \beta_h)e^{(\beta_c + 3\beta_h)E}E^2}{[e^{\beta_h E} + e^{(\beta_c + \beta_h)E}]^3}[2 + (\beta_c - \beta_h)E + e^{\beta_c E}(2 - \beta_c E + \beta_h E)]. \tag{B65}$$

For this to be positive, it implies that $2 + (\beta_c - \beta_h)E + e^{\beta_c E}(2 - \beta_c E + \beta_h E) > 0$. Rearranging terms, we find

$$(\beta_c - \beta_h)E(1 - e^{\beta_c E}) > -2(1 + e^{\beta_c E}) \tag{B66}$$

$$E < \frac{2}{\beta_c - \beta_h} \frac{e^{\beta_c E} + 1}{e^{\beta_c E} - 1}. \tag{B67}$$

□

Lemma 12. *Consider the function $G(\alpha)$ as described in equation (B45). When ε_1 is given by equation (B61) for some $a = \alpha^* > 1$, then $\frac{dG(\alpha)}{d\alpha}\Big|_{\alpha = \alpha^*} = 0$.*

Proof. To see this, let us write out the final form of the first derivative of $G(\alpha)$ in equation (B46),

$$\frac{dG(\alpha)}{d\alpha} = \frac{1}{\beta_h} \frac{1}{(\alpha - 1)^2}[\alpha(\alpha - 1)nB_\alpha' - nB_\alpha - \varepsilon_1], \tag{B68}$$

and observe that substituting equation (B61) into the equation above gives us $\frac{dG(\alpha)}{d\alpha}\Big|_{\alpha = \alpha^*} = 0$. This concludes the proof.   □

So far, for a specific design of ε_1, we have found conditions expressed in equation (B62) such that $\varepsilon_1 > 0$ and $G(\alpha^*)$ is a stationary point. Next, we can write down further conditions for when given α^* and $\varepsilon_1(\alpha^*, n)$ as defined in lemma 11, one can now find conditions on E such that $G(\alpha^*)$ not only is a stationary point, but also a local minima.

Lemma 13. *Consider the functions*

$$\frac{\mathrm{d}G(\alpha^*)}{\mathrm{d}\alpha} = \frac{1}{\beta_h}\frac{1}{(\alpha-1)^2}[\alpha(\alpha-1)nB_\alpha' - nB_\alpha - \varepsilon_1], \tag{B69}$$

and

$$B_\alpha = \frac{E}{1+\mathrm{e}^{\beta_c E}} \cdot \frac{\mathrm{e}^{(\beta_h+\alpha\beta_c)E} - \mathrm{e}^{(\beta_c+\alpha\beta_h)E}}{\mathrm{e}^{\alpha\beta_h E} + \mathrm{e}^{(\beta_h+\alpha\beta_c)E}}, \tag{B70}$$

If the following condition holds:

$$E < \frac{1}{\beta_c - \beta_h}, \tag{B71}$$

there one can find some $\alpha^* > 1$ in the vicinity of $\alpha = 1$ such that when we define $\varepsilon_1(\alpha^*, n) := n[\alpha^*(\alpha^*-1)B_{\alpha^*}' - B_{\alpha^*}]$, then $\varepsilon_1(\alpha^*, n) > 0$. Furthermore if $1 < \alpha^* < 2$ is chosen, then

$$\frac{\mathrm{d}^2 G(\alpha^*)}{\mathrm{d}\alpha^2}\bigg|_{\alpha=\alpha^*} > 0. \tag{B72}$$

Proof. We first note that if $E < \frac{1}{\beta_c - \beta_h}$, then equation (B62) holds and therefore by lemma 11, one can choose some $\alpha^* > 1$ and close to 1 such that $\varepsilon_1(\alpha^*, n) > 0$. Next, we calculate the analytical expression of $\frac{\mathrm{d}^2 G(\alpha)}{\mathrm{d}\alpha^2}$ in terms of first and second derivatives of B_α. Differentiating equation (B69),

$$\begin{aligned}\frac{\mathrm{d}^2 G(\alpha)}{\mathrm{d}\alpha^2} &= \frac{1}{\beta_h}\frac{1}{(\alpha-1)^4}\{(\alpha-1)^2[(\alpha-1)nB_\alpha' + \alpha nB_\alpha' + \alpha(\alpha-1)nB_\alpha'' - nB_\alpha'] \\ &\quad - 2(\alpha-1)[\alpha(\alpha-1)nB_\alpha' - nB_\alpha - \varepsilon_1]\} \\ &= \frac{1}{\beta_h}\frac{1}{(\alpha-1)^3}\{(\alpha-1)[2(\alpha-1)nB_\alpha' + \alpha(\alpha-1)nB_\alpha''] - 2[\alpha(\alpha-1)nB_\alpha' - nB_\alpha - \varepsilon_1]\} \\ &= \frac{1}{\beta_h}\frac{1}{(\alpha-1)^3}\{n(\alpha-1)^2[2B_\alpha' + \alpha B_\alpha''] - 2[\alpha(\alpha-1)nB_\alpha' - nB_\alpha - \varepsilon_1]\}. \end{aligned} \tag{B73}$$

Substituting $\alpha = \alpha^*$ into equation (B73), one sees that the last term vanishes, and therefore

$$\frac{\mathrm{d}^2 G(\alpha)}{\mathrm{d}\alpha^2}\bigg|_{\alpha=\alpha^*} = \frac{n}{\beta_h}\frac{1}{(\alpha^*-1)}[2B_{\alpha^*}' + \alpha B_{\alpha^*}'']. \tag{B74}$$

Since $\alpha^* > 1$, we see that to guarantee positivity of equation (B73) is equivalent to showing that the last term $2B_{\alpha^*}' + \alpha B_{\alpha^*}''$ is strictly positive. To do so, we evaluate the terms B_α' and B_α''. By both hand derivation and Mathematica, we obtain the expressions

$$B_\alpha' = \frac{1}{[\mathrm{e}^{\alpha\beta_h E} + \mathrm{e}^{(\beta_h+\alpha\beta_c)E}]^2} \cdot E^2(\beta_c - \beta_h) \cdot \mathrm{e}^{[\beta_h+\alpha(\beta_c+\beta_h)]E} \tag{B75}$$

and

$$B_\alpha'' = \frac{1}{[\mathrm{e}^{\alpha\beta_h E} + \mathrm{e}^{(\beta_h+\alpha\beta_c)E}]^3} \cdot E^3(\beta_c - \beta_h)^2 \cdot \mathrm{e}^{[\beta_h+\alpha(\beta_c+\beta_h)]E} \cdot [\mathrm{e}^{\alpha\beta_h E} - \mathrm{e}^{(\alpha\beta_c+\beta_h)E}]. \tag{B76}$$

One can then calculate the last term in equation (B73), which we again obtain a simplified expression via Mathematica,

$$2B_{\alpha^*}' + \alpha B_{\alpha^*}'' = \underbrace{\frac{(\beta_c - \beta_h)E^2}{[\mathrm{e}^{\alpha\beta_h E} + \mathrm{e}^{(\beta_h+\alpha\beta_c)E}]^3}}_{>0} \cdot \underbrace{\mathrm{e}^{[\beta_h+\alpha(\beta_c+\beta_h)]E}}_{>0} \cdot f(\alpha^*), \tag{B77}$$

where

$$f(\alpha^*) := \mathrm{e}^{\alpha^*\beta_h E}[2 + \alpha^*(\beta_c - \beta_h)E] + \mathrm{e}^{(\alpha^*\beta_c+\beta_h)E}[2 - \alpha^*(\beta_c - \beta_h)E] \tag{B78}$$

$$= \underbrace{2[\mathrm{e}^{\alpha^*\beta_h E} + \mathrm{e}^{(\alpha^*\beta_c+\beta_h)E}]}_{>0} + \underbrace{\alpha^*(\beta_c - \beta_h)E[\mathrm{e}^{\alpha^*\beta_h E} - \mathrm{e}^{(\alpha^*\beta_c+\beta_h)E}]}_{<0}. \tag{B79}$$

Note that the second term is always negative because $\alpha^* > 1$ and $\beta_c > \beta_h$. Therefore, to lower bound $f(\alpha^*)$ we want to upper bound the factor $\alpha^*(\beta_c - \beta_h)E$. By letting $1 < \alpha^* < 2$ and $E < \frac{1}{\beta_c - \beta_h}$, one can have

$\alpha^*(\beta_c - \beta_h)E < 2$, which gives

$$2B'_{\alpha^*} + \alpha B''_{\alpha^*} > 2[e^{\alpha^*\beta_h E} + e^{(\alpha^*\beta_c + \beta_h)E}] + 2[e^{\alpha^*\beta_h E} - e^{(\alpha^*\beta_c + \beta_h)E}] = 4e^{\alpha^*\beta_h E} > 0. \tag{B80}$$

Note that the constraints on α^* and E are not necessary, however sufficient and takes a relatively simple form. $\quad\square$

Finally, for $G(\alpha^*)$ to be the global minima, we need to compare it with the limits $\alpha \to 1, \infty$. Firstly, by using equation (B45),

$$G(1) = \lim_{\alpha \to 1} G(\alpha) = \frac{n}{\beta}\left[B_1 + B'_1 + \frac{\varepsilon_1}{n} \cdot \lim_{\alpha \to 1}\frac{\alpha}{\alpha - 1}\right] = \infty. \tag{B81}$$

We need one last condition: that $G(\alpha^*) < G(\infty)$. In the next lemma, we again develop conditions such that this is true.

Lemma 14. *Suppose $\alpha^*E < \frac{1}{\beta_c - \beta_h}$. Then for $\varepsilon_1(\alpha^*, n)$ defined as in equation (B61), we have that $G(\alpha^*) < G(\infty)$.*

Proof. To do so, we write out the expressions for $G(\alpha^*)$ and $G(\infty)$:

$$G(\alpha^*) = \frac{n}{\beta_h}\frac{\alpha^*}{\alpha^* - 1}[B_{\alpha^*} + \alpha^*(\alpha^* - 1)B'_{\alpha^*} - B_{\alpha^*}] = \frac{n}{\beta_h}\alpha^{*2}B'_{\alpha^*}, \tag{B82}$$

$$G(\infty) = \frac{n}{\beta_h}\left[B_\infty + \frac{\varepsilon_1}{n}\right] = \frac{n}{\beta_h}\left[\frac{E}{1 + E^{\beta_c E}} + \frac{\varepsilon_1}{n}\right]. \tag{B83}$$

For $G(\alpha^*) < G(\infty)$, this means

$$\alpha^{*2}B'_{\alpha^*} < \frac{E}{1 + E^{\beta_c E}} + \alpha^*(\alpha^* - 1)B'_{\alpha^*} - B_{\alpha^*} \tag{B84}$$

$$\frac{E}{1 + E^{\beta_c E}} - \alpha^*B'_{\alpha^*} - B_{\alpha^*} > 0. \tag{B85}$$

Expanding equation (B85), and using the shorthand $X := e^{\alpha^*\beta_h E} + e^{(\beta_h + \alpha^*\beta_c)E}$ we obtain

$$\frac{E}{1 + E^{\beta_c E}} - \frac{(\beta_c - \beta_h)\alpha^{*2}E^2}{X^2}e^{[\beta_h + \alpha^*(\beta_c + \beta_h)]E} - \frac{E}{1 + E^{\beta_c E}}\frac{e^{(\beta_h + \alpha^*\beta_c)E} + e^{(\beta_c + \alpha^*\beta_h)E}}{X} \tag{B86}$$

$$= \frac{E}{1 + E^{\beta_c E}}\frac{1}{X^2}\{X^2 - \alpha^*E(\beta_c + \beta_h)(1 + e^{\beta_c E})e^{[\beta_h + \alpha^*(\beta_c + \beta_h)]E} - X[e^{\alpha^*\beta_h E} + e^{(\beta_h + \alpha^*\beta_c)E}]\} \tag{B87}$$

$$= \frac{E}{1 + E^{\beta_c E}}\frac{1}{X^2}\{X[e^{\alpha^*\beta_h E} + E^{(\alpha^*\beta_h + \beta_c)E}] - \alpha^*E(\beta_c + \beta_h)(1 + e^{\beta_c E})e^{[\beta_h + \alpha^*E(\beta_c + \beta_h)]E}\} \tag{B88}$$

$$= \frac{E}{X^2} \cdot e^{\alpha^*\beta_h E}\{X - \alpha^*E(\beta_c + \beta_h)e^{(\beta_h + \alpha^*\beta_c)E}\} \tag{B89}$$

$$= \frac{E}{X^2} \cdot e^{\alpha^*\beta_h E}\{e^{\alpha^*\beta_h E} + e^{(\beta_h + \alpha^*\beta_c)E}[1 - \alpha^*E(\beta_c - \beta_h)]\}. \tag{B90}$$

The calculation above can be checked as follows: the first equality is obtained by taking out a common factor from all the three terms. The second equality focuses on the large bracket, and combines the first and third terms by expanding one of the X in the first term. In the third equality, one recognizes more common factors in equation (B88), and therefore pulls out $e^{\alpha^*\beta_h E} \cdot (1 + e^{\beta_c E})$. The fourth equality is obtained by expanding X, while regrouping terms. To demand that $W_{\alpha^*} < W_\infty$, implies that we demand

$$e^{\alpha^*\beta_h E} + e^{(\beta_h + \alpha^*\beta_c)E}[1 - \alpha^*E(\beta_c - \beta_h)] > 0. \tag{B91}$$

Rearranging equation (B91), we have

$$e^{\alpha^*\beta_h E} > e^{(\beta_h + \alpha^*\beta_c)E}[\alpha^*E(\beta_c - \beta_h) - 1]. \tag{B92}$$

One can continue to simplify the expression by bringing $e^{(\beta_h + \alpha^*\beta_c)E}$, and subsequently the -1 to the lhs, yielding

$$1 + e^{-\alpha^*(\beta_c - \beta_h)E}e^{-\beta_h E} > \alpha^*E(\beta_c - \beta_h). \tag{B93}$$

Then finally, one obtains an expression for α^*E:

$$\alpha^*E < \frac{1 + e^{-\alpha^*(\beta_c - \beta_h)E}e^{-\beta_h E}}{\beta_c - \beta_h}. \tag{B94}$$

Since $\beta_c - \beta_h > 0$, and we have that $e^{-\alpha^*(\beta_c - \beta_h)E}e^{-\beta_h E} > 0$, therefore as long as $\alpha^*E < \frac{1}{\beta_c - \beta_h}$, equation (B85) is satisfied and $G(\alpha^*) < G(\infty)$. This concludes our proof. $\quad\square$

Lemma 4, box 10, lemma 11–13 together presents a set of mathematical conditions such that α^* can be chosen such that $G(\alpha)$ has a *global* minima at $\alpha = \alpha^*$.

Appendix C. Results

In [14] it has been shown that for a heat engine to extract any positive amount of work at all, $\varepsilon > 0$ has to be true. Therefore, perfect work can never be drawn. Also, in [14] the regime of near perfect work was analyzed. There, it was found that the maximum efficiency can never exceed the CE.

In this paper, we develop an example of a heat engine which extracts imperfect work. In section C.1, we show (our main result) how to find examples where CE is surpassed. More specifically, this occurs in the quasi-static limit where $\frac{\Delta S}{W_{\text{ext}}} \to \infty$. In section C.2 we analyze the regime where $\frac{\Delta S}{W_{\text{ext}}} \to p$, with $0 < p < \infty$. We find that in this regime, according to the generalized second laws, CE cannot be surpassed.

C.1. Main result: an example of drawing imperfect work surpassing the CE

Our main result is stated in theorem 1 of the paper. Here, we present corollary 1, which is a direct consequence of combining all the technical lemmas derived in section B. This corollary is a more detailed version of theorem 1 in the main text.

Corollary 1. *Consider a quasi-static heat engine with a cold bath consisting of n identical qubits with energy gap E. Given the inverse temperatures of the hot and cold bath* β_h, $\beta_c > 0$ *respectively, and for* $\alpha \in (1, \infty)$ *define the functions*

$$B_\alpha = \frac{E}{1 + e^{\beta_c E}} \cdot \frac{e^{(\beta_h + \alpha \beta_c)E} - e^{(\beta_c + \alpha \beta_h)E}}{e^{\alpha \beta_h E} + e^{(\beta_h + \alpha \beta_c)E}}. \tag{C1}$$

If the energy gap of the qubits satisfies

$$E < \frac{1}{2(\beta_c - \beta_h)}, \tag{C2}$$

then there exists an $\alpha^* \in (1, 2)$ *such that the following holds:*

1. *The failure probability of the heat engine, can be chosen as* $\varepsilon = g \cdot n[\alpha^*(\alpha^* - 1)B'_{\alpha^*} - B_{\alpha^*}] > 0$, *where* $B'_\alpha = \frac{dB_\alpha}{d\alpha}$ *is the first derivative of* B_α *according to* α.

2. *In the quasi-static limit, the amount of extractable work* W_{ext} *is achieved by* W_{α^*}, *i.e.*

$$\lim_{g \to 0} \frac{W_{\text{ext}}}{W_{\alpha^*}} = 1. \tag{C3}$$

3. *The (inverse) efficiency of the described heat engine in the quasi-static limit is given by* $\eta^{-1} = 1 + \frac{\beta_h}{\beta_c - \beta_h} \frac{1}{\alpha^{*2}} \frac{B'_1}{B'_{\alpha^*}}$.

Proof. Since $\frac{1 + e^{\beta_c E}}{e^{\beta_c E} - 1} > 1$, if equation (C2) holds, then equation (B62) holds. Therefore item 1 is a direct result of lemma 11.

Item 2 concerns the quantity W_{ext}, given by equations (B5)–(B7). Suppose that α_1 is the solution such that equation (C3) holds. Since we have made a choice of ε according to item (1), then in the proof of lemma 4, we have shown that α_1 is also the solution that provides the infimum for

$$I(\alpha) = \frac{dW_\alpha}{dg} \tag{C4}$$

in the quasi-static limit. Furthermore, in lemma 5, we have also shown that α_1 is also the solution that achieves the minimum for $\inf_{\alpha > 1} G(\alpha)$, where $G(\alpha)$ is given by equation (B45) (it is the leading order term of $I(\alpha)$ with respect to g). Therefore, if we can show that α^* achieves the global minima for $G(\alpha)$ in the region $\alpha \in (1, \infty)$, then we know that W_{α^*} satisfies equation (C3).

Let us now see why the infimum $\inf_{\alpha > 1} G(\alpha) = G(\alpha^*)$. If one chooses $\alpha^* \in (1, 2)$ and that equation (C2) holds, then equation (B71) holds as well, and so lemmas 13 and 14. Therefore,

- By lemma 10 we know $G(\alpha)$ does not have more than one distinct local minima.

- By lemmas 12 and 13, $G(\alpha^*)$ is a unique local minima.

- By lemma 14, $G(\alpha^*) < G(\infty)$. Therefore, $G(\alpha^*)$ is the global minima.

Finally, for the fixed paramaters $n \in \mathbb{Z}^+$, $E \in \mathbb{R}$, $\alpha^* \in (1, 2)$, we can evaluate the efficiency of our quasi-static heat engine for a cold bath comprising of identical qubits. This can be done by evaluating the efficiency for our heat engine:

$$\eta^{-1} = \lim_{g \to 0^+} 1 - \varepsilon + \frac{\Delta C}{W_{\text{ext}}}. \tag{C5}$$

The term $\varepsilon = \varepsilon_1 \cdot g = \Theta(g)$, where $\varepsilon_1 = n[\alpha^*(\alpha^* - 1)B'_{\alpha^*} - B_{\alpha^*}]$ is a finite constant. Therefore we know $\lim_{g \to 0^+} \varepsilon = 0$. On the other hand, we have

$$\lim_{g \to 0} \frac{\Delta C}{W_{\text{ext}}} = \lim_{g \to 0} \frac{\Delta C}{W_{\text{ext}}} \cdot \frac{W_{\text{ext}}}{W_{\alpha^*}} = \lim_{g \to 0} \frac{\frac{\mathrm{d}\Delta C}{\mathrm{d}g}}{\frac{\mathrm{d}W_{\alpha^*}}{\mathrm{d}g}} \tag{C6}$$

$$= \frac{nB'_1}{\beta_c - \beta_h} \cdot \left[\lim_{g \to 0} I(\alpha^*) \right]^{-1} \tag{C7}$$

$$= \frac{nB'_1}{\beta_c - \beta_h} \cdot \left[\lim_{g \to 0} G(\alpha^*) \right]^{-1} \tag{C8}$$

$$= \frac{nB'_1}{\beta_c - \beta_h} \cdot \frac{\beta_h}{n\alpha^{*2}B'_{\alpha^*}}. \tag{C9}$$

The second equality holds by noting that both ΔC and W_α^* vanish in the limit $g \to 0$, and therefore apply the L'Hospital rule. In the third equality, we used the first derivative of ΔC as calculated in equation (B20) of lemma 2. Subsequently, we have used lemma 5 to calculate the value of $I(\alpha^*)$ in the quasi-static limit. where we have made use of therefore, substituting equation (C9) into the expression for efficiency in equation (C5), we have item 3, i.e.

$$\eta^{-1} = 1 + \lim_{g \to 0^+} \frac{\Delta C}{W_{\text{ext}}} \tag{C10}$$

$$= 1 + \frac{\beta_h}{\beta_c - \beta_h} \frac{1}{\alpha^{*2}} \frac{B'_1}{B'_{\alpha^*}}. \tag{C11}$$

$$\square$$

With this, we can numerically plot out the achievable efficiency as a function of β_c, β_h, n, E, α^*, in the limit where $g \to 0^+$. These expression are plotted out in several regimes in figures 2–4. It is worth noting that from the expression for inverse efficiency in corollary 1, we see that η^{-1} contains terms that originate from the expression of ε_1 chosen in item 1 of corollary 1. It is then, perhaps, unsurprising that we observe the surpassing of CE (for some values of $\alpha^* > 1$). Indeed, although the average energy change in the battery is positive, i.e. $\Delta W = (1 - \varepsilon)W_{\text{ext}} > 0$, the change in *free energy* of the battery,

$$\Delta F_{\text{W}} = F(\rho_{\text{W}}^1) - F(\rho_{\text{W}}^0) = \Delta W - \beta_h^{-1}\Delta S \tag{C12}$$

is actually negative. This can be seen when we compute the limit

$$\lim_{g \to 0^+} \frac{\Delta F_{\text{W}}}{\Delta W} = \lim_{g \to 0^+} \frac{\Delta W - \beta_h^{-1}\Delta S}{\Delta W} = 1 - \beta_h^{-1} \lim_{g \to 0^+} \frac{\Delta S}{(1 - \varepsilon)W_{\text{ext}}} = -\infty,$$

where the last limit comes from noting that $\lim_{g \to 0^+} \varepsilon = 0$, and applying lemma 6.

C.2. Drawing imperfect work with entropy comparable with W_{ext}

In this section we analyze the achievable efficiency when considering the quasi-static limit where

$$\frac{\Delta S}{W_{\text{ext}}} \to c \qquad \text{for some } c > 0. \tag{C13}$$

One can see that only certain choices of $\varepsilon(g)$ will lead to having such a limit, which we shall see later in detail on table B1. We prove that for all choices of ε such that equation (C13) is true, one cannot surpass the CE.

Theorem 2. *Consider a quasi-static heat engine where the failure probability of extracting work is $\varepsilon(g)$, g being the quasi-static parameter (see definition in main text), such that*

$$\lim_{g \to 0^+} \frac{\varepsilon^\kappa(g)}{g} = \begin{cases} 0, & \text{if } \kappa \geqslant 1, \\ \infty, & \text{if } \kappa < 1 \end{cases} \tag{C14}$$

and $\lim_{g \to 0} \frac{\varepsilon \ln \frac{1}{\varepsilon}}{g} = c > 0$. *Then the maximum achievable efficiency is upper bounded by the CE.*

Proof. Firstly, note that an example for such a choice of ε can be constructed, i.e. $\varepsilon \ln \frac{1}{\varepsilon} = c \cdot g$.

We make use of equation (C14) to analyze W_{ext}, which is given in appendix B. Rewriting equation (B11) by first drawing out a factor of g,

$$W_{\text{ext}} = g \cdot \inf_{\alpha > 0} \tilde{W}_\alpha, \tag{C15}$$

where

$$\tilde{W}_\alpha$$
$$= \begin{cases} \frac{1}{\beta_h(\alpha - 1)} \left[\alpha n B_\alpha - \frac{\varepsilon^\alpha}{g} + \frac{\alpha \varepsilon}{g} \right] + \Theta(g) + \Theta\left(\frac{\varepsilon^{2\alpha}}{g}\right) + \Theta(\varepsilon^\alpha) + \Theta\left(\frac{\varepsilon^2}{g}\right), & \text{if } \alpha \in (0, \infty) \setminus \{1\}, \\ \beta_h^{-1} \left[\lim_{\alpha \to 1^+} \frac{\alpha n B_\alpha}{\alpha - 1} + \frac{\varepsilon \ln \frac{1}{\varepsilon}}{g} \right] + \Theta(\varepsilon) + \Theta\left(\frac{\varepsilon^2 \ln \varepsilon}{g}\right) + \Theta\left(\frac{\varepsilon^2}{g}\right) + \Theta(g), & \alpha = 1. \end{cases} \tag{C16}$$

Note that the (inverse) efficiency in the quasi-static limit is given by

$$\eta^{-1} = \lim_{g \to 0^+} 1 - \varepsilon + \frac{\Delta C}{W_{\text{ext}}} = 1 + \lim_{g \to 0^+} \frac{\Delta C}{W_{\text{ext}}} \geqslant 1 + \lim_{g \to 0^+} \frac{\Delta C}{W_\alpha}, \tag{C17}$$

where any $\alpha > 0$ gives an upper bound. However, since ΔC and W_α are both vanishing in the quasi-static limit (for any $\alpha > 0$), we can also evaluate the limit by using equation (B19),

$$\lim_{g \to 0^+} \frac{\Delta C}{W_\alpha} = \frac{n B_1'}{\beta_c - \beta_h} \cdot \left(\lim_{g \to 0} \tilde{W}_\alpha \right)^{-1}. \tag{C18}$$

We are, then, interested in picking α that gives us the tightest bound, i.e. the smallest value for $\lim_{g \to 0} \tilde{W}_\alpha$. This leads us to scrutinize equation (C16) in the light of $\varepsilon(g)$ that satisfies the statement of the theorem. First of all, note that equation (C14) implies that for values of $\alpha \in (0, 1)$, the term $\frac{-\varepsilon^\alpha}{g(\alpha - 1)}$ goes to infinity as $g \to 0^+$, while other terms are finite. This implies that the minimization can be restricted to parameters $\alpha \geqslant 1$. Notice also all the order terms vanish when we take the limit $g \to 0$, therefore we need only to deal with the largest order terms in equation (C16).

Consider the case where $\alpha = 1$. We have that

$$\lim_{g \to 0} \tilde{W}_1 = \frac{1}{\beta_h} \left[\lim_{\alpha \to 1^+} \frac{\alpha n B_\alpha}{\alpha - 1} + c \right], \tag{C19}$$

where we have seen that $c > 0$, by choice of $\varepsilon(g)$. On the other hand, for $\alpha > 1$ the expression for $\tilde{W}_\alpha$ can be further simplified in the quasi-static limit,

$$\lim_{g \to 0} \tilde{W}_\alpha = \frac{\alpha n B_\alpha}{\beta_h(\alpha - 1)} \qquad \text{if } \alpha \in (1, \infty). \tag{C20}$$

This is because the terms $\frac{\varepsilon^\alpha}{g}$, $\frac{\alpha \varepsilon}{g}$ now vanish as $g \to 0^+$. From this we also see that since $\tilde{W}_1 > \beta_h^{-1} \lim_{\alpha \to 1^+} \frac{\alpha n B_\alpha}{\alpha - 1}$, and by continuity of the function $\frac{\alpha n B_\alpha}{\alpha - 1}$ for $\alpha \in (1, \infty)$, $\tilde{W}_1$ can also be disregarded in the minimization (see figure C1 for a pictorial understanding).

Upon scrutiny, one sees that in the quasi-static limit, the contribution from ε has dropped out of the expression for W_{ext}. Intuitively this tells us that having such a probability of failure ε does not help to boost W_{ext}, and in turn the efficiency. In particular, we can use the lower bound:

$$\eta^{-1} = 1 + \lim_{g \to 0^+} \frac{\Delta C}{W_{\text{ext}}} \geqslant \frac{n B_1'}{\beta_c - \beta_h} \left[\lim_{\alpha \to 1} \frac{\alpha n B_\alpha}{\beta_h(\alpha - 1)} \right]^{-1}, \tag{C21}$$

where we have substituted equations (C18) and (C20) into equation (C17), while picking $\alpha \to 1$ as our bound. This limit is evaluated as

$$\lim_{\alpha \to 1^+} \frac{\alpha n B_\alpha}{\alpha - 1} = \frac{n}{\beta_h}(B_1 + B_1') = \frac{n}{\beta_h} B_1'. \tag{C22}$$

The first equality in equation (C22) comes by noting that $B_1 = 0$, and therefore applying the L'Hospital rule. The second equality comes again from noting that $B_1 = 0$. Finally, substituting this into equation (C21), we have
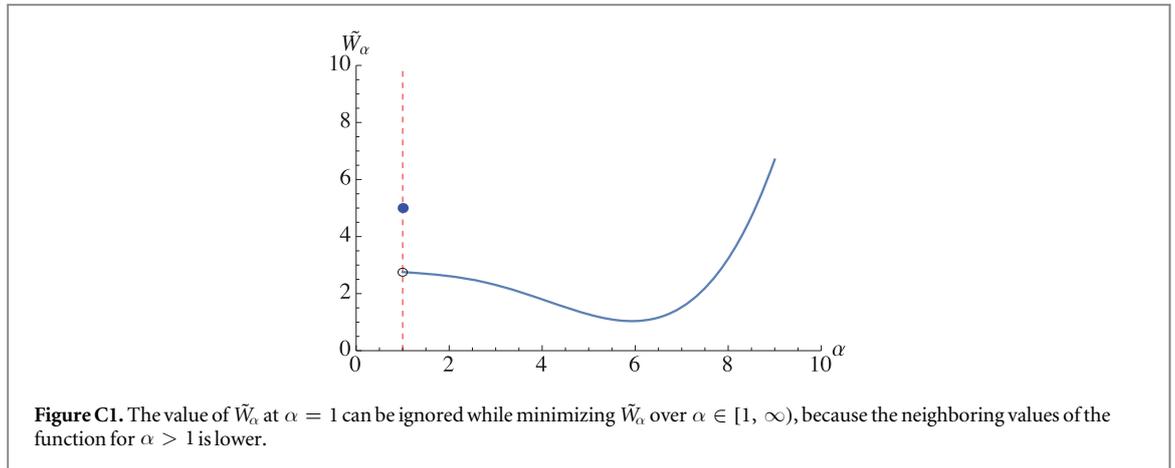
Figure C1. The value of $\tilde{W}_\alpha$ at $\alpha = 1$ can be ignored while minimizing $\tilde{W}_\alpha$ over $\alpha \in [1, \infty)$, because the neighboring values of the function for $\alpha > 1$ is lower.

$$\eta^{-1} \geqslant 1 + \frac{nB_1'}{\beta_c - \beta_h} \cdot \frac{\beta_h}{nB_1'} \tag{C23}$$

$$= 1 + \frac{\beta_h}{\beta_c - \beta_h} = \eta_C^{-1}. \tag{C24}$$

one finds that the upper bound on efficiency yields the Carnot expression, i.e. $\eta \leqslant \eta_C$. This means that for choices of $\varepsilon(g)$ according to the statement of the theorem, CE cannot be surpassed.   □

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