First - order Reliability Method: Concepts and Application

ADDITIONAL GRADUATION THESIS

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Abstract

First/second-order reliability method (FORM/SORM) is considered to be one of the most reliable computational methods for structural reliability. A relative advantage of such analytical methods is that they provide physical interpretations and do not require much computation time. Designs based on FORM/SORM are usually performed using commercial software packages in which the underlying concept of the Reliability method is hidden. Also, the available literature is not easy to read and the basic concept is buried in complex mathematical equations. This document aims to give a comprehensive understanding of First Order Reliability Methods.

In this document, practical application of FORM is demonstrated with a retaining wall and slope stability problem, both analysed using a spreadsheet model developed by Low (2003). Both applications presented are existing examples by Low (2003, 2005). These are briefly explained, and later modified to understand the efficiency of the model, and to investigate the effect of geometrical uncertainties in a slope's stability.

The efficiency of spreadsheet model is investigated by considering uncertainty of geometrical parameters. Taking advantage of FORM's ability to reflect sensitivity of the parameters, a sensitivity interpretation of the parameters involved in the slope stability problem is made. The influence of uncertainty of soil layering on the stability of the slope is analysed. Additional investigation on the effect of one dimensional spatial variation on the outcome of slope reliability is made.

The spreadsheet model uses intuitive First Order Reliability approach and MS Excels's inbuilt solver with constrained optimisation to compute Reliability index and probability of failure. It was found to be relatively less user friendly when compared to the existing commercial software packages but it serves as a very efficient tool to understand the concepts of FORM better.

The major disadvantage of Monte Carlo regarding its high computational cost has triggered the need to find better alternatives. In most applications, FORM only needs a small number of iterations for convergence, making it more computationally efficient than MCS. This is particularly so when the failure probabilities are low. With the limited research here, it is safe to say that FORM could serve as a first step in Reliability based design to study the relative importance of parameters.

Contents

Chapter 1

First order reliability methods

1.1 Introduction

In this chapter, the underlying concepts of first order reliability method are described. The advantages of FORM are discussed to understand its potential to be used as an alternative to the cumbersome Monte Carlo Process. It is a popular reliability method among academicians but remains less used in industry owing to its mathematical complexity to understand the concepts. This chapter gives an introduction to the concepts involved in FORM. The intent of this chapter is to break down the complex equations and explain it in a practical context. There are different approaches of FORM. These different approaches are explored.

In a nutshell, this chapter gives

1. The concept of risk based design
2. The basic concepts of reliability analysis
3. Theory behind FORM

1.2 Risk and Safety Factors Concept

The most basic criteria in the design of a structure is to make sure that the strength of the structure is greater than the impact of the loads applied. It is well known that geotechnical engineering is associated with many uncertain parameters mainly owing to the soil variability. The purpose of risk and reliability based design is to incorporate the information on uncertainty into actual design problems.

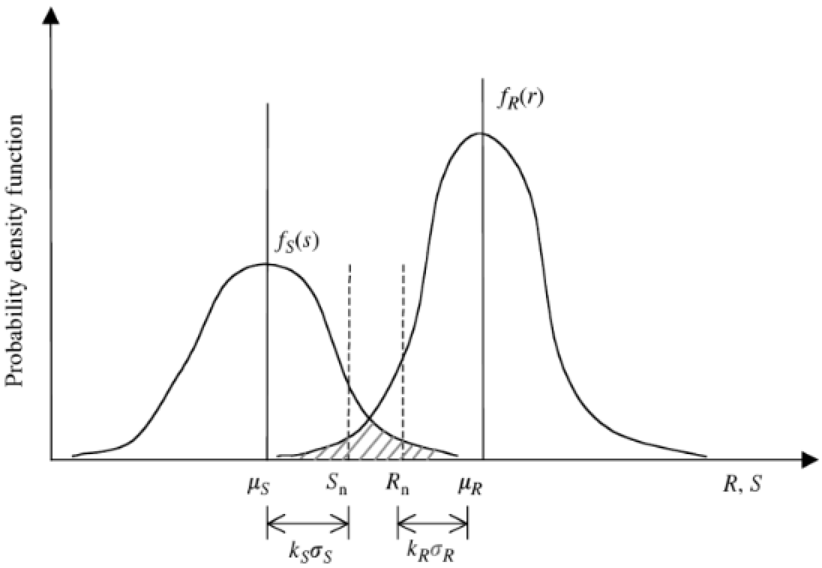


Figure 1 Fundamentals of Risk Evaluation

Figure 1 is a simple case considering two variables R, Resistance of the structure and S, Load on the structure. S and R are stochastic variables, meaning they are random in nature. These stochastic variables are defined by their probability density functions and statistical parameters are used to characterise their randomness. μ_R and μ_S are the respective means, σ_R and σ_S, their standard deviations and $f_S(s)$ and $f_R(s)$ their corresponding probability density functions. Referring to Figure 1, design safety is ensured in a deterministic approach by requiring that R_N be greater than S_N with a specified margin of safety.

$$\boldsymbol{Nominal\ S_F = R_N/S_N} \qquad \textbf{Equation 1}$$

S_F is the safety factor. In a deterministic design, the uncertainty of all the parameters is taken into account by a single number i.e the safety factor. There are different approaches or methods based on how he safety factor is applied, i.e. it can be applied to the load, resistance or both.

- Working stress method: Safety factor applied to resistance alone
- Ultimate Strength method: Safety factor applied to loads alone
- Concrete or steel (LRDF): Safety factors applied to both resistance and loads

Haldar and Mahadevan explains the intent of these conventional approaches by considering the area of overlap between the two curves. This area of overlap provides a qualitative measure of the probability of failure. The area of overlap depends on the following factors

1. The mean of the input parameters are a measure of the relative positions of the curves. More distance between the curves implies less overlap area which reduces the probability of failure.
2. Standard deviation is a measure of the dispersion of the curves. Narrow curves lead to small overlap area reducing the probability of failure and vice versa
3. The probability density function is a measure of the shapes of the curves. The shape of the curve plays a role in the area of overlap.

Ensuring safety in a deterministic design is achieved by selecting the design variables with the least area of overlap. Safety factors are employed to shift the positions of the curves. But such a design does not take into account all the overlap factors. Risk based design is a more rational approach as it minimises the overlap area by considering all the design variables to achieve an acceptable level of risk.

In terms of probability of failure, risk can be defined as:

$$p_f = P(failure) = P(R < S)$$

$$= \int_0^\infty \left[\int_0^s \boldsymbol{f_R(r)dr}\right] \boldsymbol{f_S(s)\,ds} \qquad \textbf{Equation 2}$$

$$= \int_0^\infty \boldsymbol{F_R(s) f_s(s) ds}$$

1.3 Basic Concepts of Reliability Analysis

The reliability of an engineering design is the probability that it meets certain demands under certain conditions. In geotechnics, an example of shallow foundation is often used to illustrate this. For the stability of a foundation, it should be designed such that it satisfies certain demands towards vertical loads. The bearing capacity (R) of the soil should exceed the total vertical load (S) acting on it, for the foundation to be stable. Mathematically, this can be represented as R>S.

This is mathematically expressed as

$$\boldsymbol{Z = R - S} \qquad \textbf{Equation 3}$$

Here Z is the performance function or limit state function of the foundation. This function differentiates the unsafe and safe zones with respect to R and S. This example has two stochastic variables R and S. This equation can be generalised as:

$$\boldsymbol{Z = g(x)} \qquad \textbf{Equation 4}$$

where g(x) constitutes the n basic variables x_1, x_2,.. x_n of the performance function. The performance function owes it’s name to the fact that it is a measure of the performance of any structure. Like any mathematical equation, the performance function could have three outcomes as follows:

- $(x) > 0$: Safe region
- $g(x) = 0$: Limit state
- $g(x) < 0$: Failure region

In figure, the curve is the performance function. The region to the right of the curve is unsafe where g(x)<0 while the region to the left of the curve is the safe region (g(X)>0). The boundary or the curve represents the combination of the variables that are on the verge of failure i.e. at limit state.

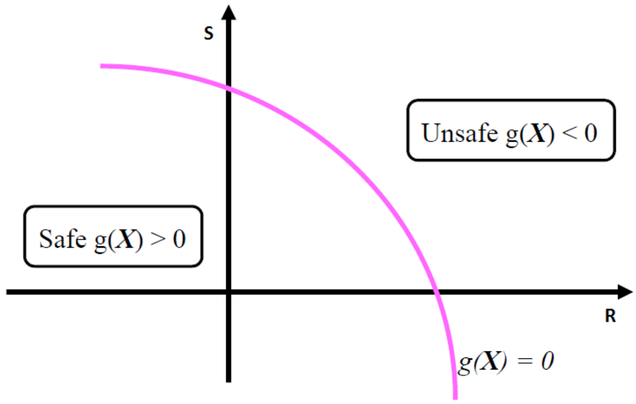


Figure 2 Limit State Concept

Figure 3 shows the joint probability density function and the corresponding contours. The contours are projections of the surface of f_x (x_1, x_2) on $x1$ - $x2$ plane. All the points on the contours have the same values of $f(x)$ or the same probability density.

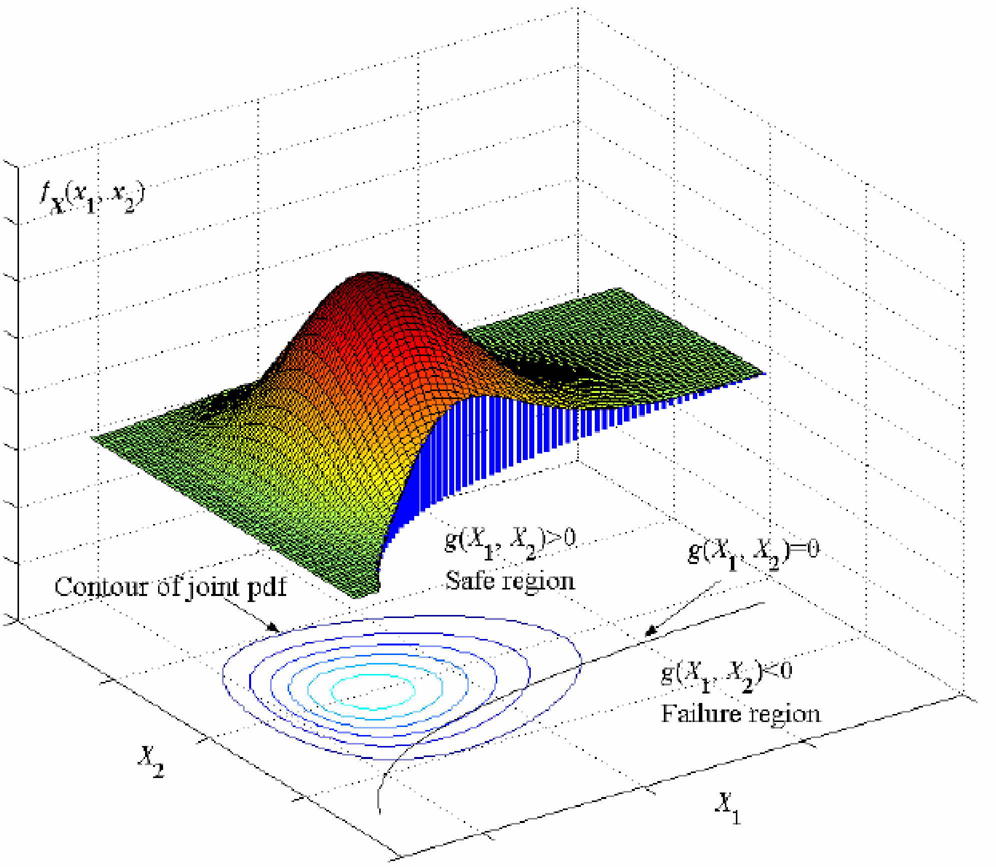


Figure 3 Safe and Unsafe Regions (Du, 2005)

1.4 First Order Reliability Methods (FORM)

This section gives some insight into the theory behind FORM; the underlying mathematical equations involved in it and identifies the advantages of FORM. FORM is considered as a good alternative to the cumbersome Monte Carlo Analysis. It's accuracy with lesser

computations makes up for its mathematical complexity. FORM was initially proposed by Hasofer et al. (1974). It is capable of handling non linear performance functions, and correlated non-normal variables.
FORM is also referred to as Mean Value First order second moment method (MVFOSM)
FORM linearizes the performance function using Taylor series approximation. Hence it's a first order approximation. FORM uses only mean and standard deviation of the variables.

The performance / limit state function is given by

$$\boldsymbol{Z} = \boldsymbol{R} - \boldsymbol{S}$$ **Equation 5**

As both R and S are assumed as normal random variables, Z can also be inferred as a random variable, that is $N(\mu_R - \mu_s, \sqrt{\sigma_R^2 + \sigma_s^2})$. Then probability of failure can be defined as

$$p_f = P(Z < 0)$$ **Equation 6**

$$p_f = \phi\left[\frac{0-(\mu_R-\mu_S)}{\sqrt{\sigma_R^2+\sigma_S^2}}\right]$$ **Equation 7**

$$p_f = 1 - \phi\left[\frac{\mu_R-\mu_S}{\sqrt{\sigma_R^2+\sigma_S^2}}\right]$$ **Equation 8**

ϕ is the CDF of the standard normal variate

Thus, the probability of failure is a function of the mean value of Z to its standard deviation.

$$\beta = \frac{\mu_z}{\sigma_z} = \frac{\mu_R-\mu_S}{\sqrt{\sigma_R^2+\sigma_S^2}}$$ **Equation 9**

The probability of failure can be expressed in terms of the safety index as follows.

$$p_f = \varphi(-\beta) = 1 - \varphi(\beta)$$ **Equation 10**

These variables are restricted to positive values. Hence log normally distribution is assumed. The generalized formulation of the performance function can be written as:

$$Z = g(X) = g(X_1, X_2 \dots, X_n)$$ **Equation 11**

X_1, X_2..X_n represents the random variables in the limit state function as mentioned before.

A Taylor series expansion of the limit state function about the mean gives

$$G = g(\mu_X) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}\left(X_i - \mu_{X_i}\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g}{\partial X_i \partial X_j}(X_i - \mu_{X_i})\left(X_j - \mu_{X_j}\right) + \cdots$$

Equation 12

The gradient is evaluated at the mean values. These calculations make FORM less attractive for practical use, although it is much less complicated than it is assumed to be. To lessen the assumed mathematical complexity, new FORM methods use iterative constrained optimisation algorithms that do not require evaluation of the gradient.

The Taylor series expansion is truncated for linear terms to obtain a first order approximate The mean and variance obtained from the truncated expansion is given by:

$\mu_Z \approx g(\mu_{X_1}, \mu_{X_2}, \ldots.. \mu_{X_n})$ ***and*** **Equation 13**

$\sigma_Z^2 \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial g}{\partial X_i} \frac{\partial g}{\partial X_j} Cov(X_i, X_j)$ **Equation 14**

$Cov(X_i, X_j)$ is the covariance of X_i and X_j.

This explains the concept behind MVFOSM, which basically a Taylor Series Approximation or rather, linearization of the performance function at the mean values of the random variables. But this linearization was later identified as a limitation. Later, a second order approximation of the Taylor Series Approximation was defined, and is referred to as SORM – Second order reliability method. The limitations of MVFOSM is summarized here:

Limitations of MVFOSM

1. Information regarding distribution of the variables is completely ignored
2. Truncation errors due to linearization at mean point for non-linear limit state function.
3. Different though mechanically equivalent equations did not give the same safety index. In other words, the safety indices depend on the how the limit state equation is formulated. This was commonly called the invariance problem.

To overcome the invariance problem, Hasofer and Lind proposed an advanced First Order Reliability method. This is discussed in the next section.

1.5 Advanced First Order Reliability Method or the Hasofer Lind Method

As the name says, this method is an advanced version of FORM which compensates for the non-invariance of the reliability index in FORM. This method transforms the variables to a standardized space of Normal variables. As it is known, standard normal variables have zero mean and standard deviation of 1. This transformation of the coordinate space is performed to aid in the computation of reliability Index. A random variable X_i is reduced as:

$X'_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad (i = 1,2,..,n)$ **Equation 15**

X_i' is a random variable characterized by a probability density function having zero mean and unit standard deviation. Equation 15 is implemented in the limit state equation to obtain the limit state in the new space – the reduced coordinate space. Each variable in the limit state equation is substituted by the respective reduced equation and the corresponding limit state equation is obtained. This is referred to as transformation of the coordinate space from the original coordinates to the reduced coordinates as shown in Figure 4. The limit state surface in the reduced coordinate system is referred as g(X') = 0.

Now the reliability index is defined in this new reduced space. The Hasofer-Lind reliability index β_{HL} is defined as the minimum possible distance between the origin and the limit state surface. Thus the determination of this point has two important aspects – Optimisation of the distance to find the right minimum distance point, with the Constraint that the point lies on the limit state surface. This minimum distance point on the limit state surface is called the 'design point (x*)'. Hasofer – Lind index can be mathematically written as

$$\beta_{HL} = \sqrt{(x'^*)^T(x'^*)}$$ **Equation 16**

The design point represents the most probable point of failure - MPP

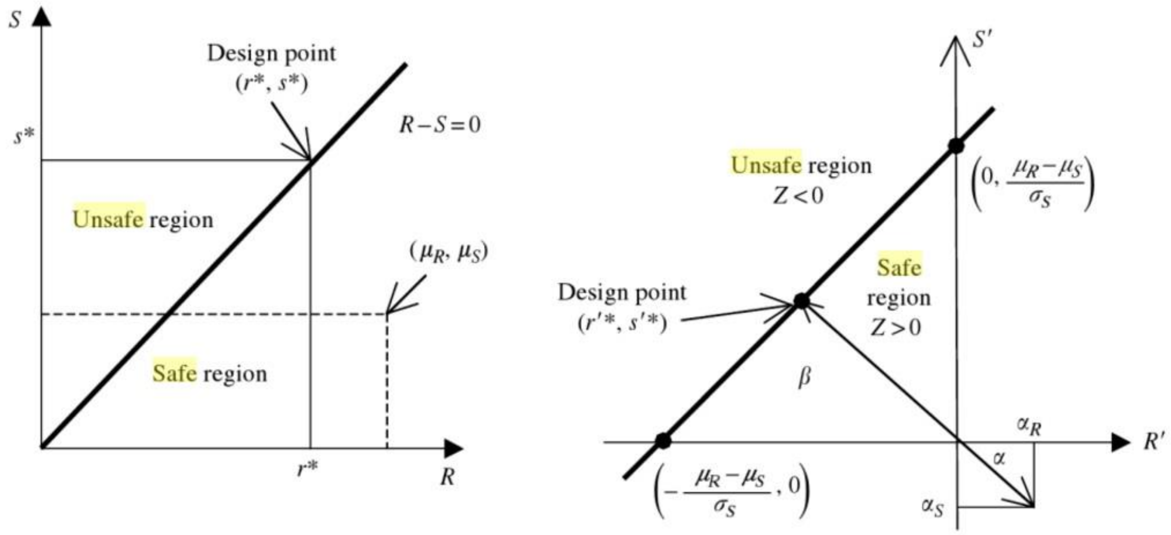


Figure 4 Original Coordinates, Reduced Coordinates (Haldar and Mahadevan)

Figure 4 shows the transformation of the random variables of a limit state function from the original coordinate system to the reduced coordinate system. Here, it is explained why the minimum distance as defined earlier is the reliability index.

The transformation of the variables, R and S is as follows:

$$R' = \frac{R_i - \mu_R}{\sigma_R}; \quad S' = \frac{S_i - \mu_S}{\sigma_S}$$ **Equation 17**

Now the reduced equations of R and S are substituted into the limit state equation to obtain the new limit state surface in the standard space of coordinates.

$$Z = \sigma_R R' - \sigma_S S' + \mu_R - \mu_S$$ **Equation 18**

The reliability index is calculated using Equation 16. Reliability index can be estimated as:

$$\beta = \frac{\mu_z}{\sigma_z} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$ **Equation 19**

Probability of failure can be calculated from the reliability index.

$$P_f = \Phi(-\beta_{HL})$$ **Equation 20**

Φ [ ] is the standard normal distribution function.

The physical meaning of reliability index in this definition is the minimum distance between the origin to the limit state surface in the reduced space of random variables. This point on the limit state surface is the most probable point of failure or the design point. It is aptly named because the design point represents the combination of stochastic variables that has the highest probability of failure. In other words, it is the worst possible combination of the input parameters.

Unlike FOSM reliability index, Hasofer-Lind reliability index is invariant. This is because the reliability index does not vary for mechanically equivalent limit states.

The actual problem here is to determine the design point that leads to the least distance between the origin and the limit state surface. This becomes a constrained optimisation problem where the distance between the origin and the limit state surface is optimised / minimised by constraining the design point to lie on the limit state.

- Minimise $D = \sqrt{(x'^*)^t (x'^*)}$
- Subjecting to constraint $g(X') = 0$.

Lagrange's multipliers is used to estimate the minimum distance as:

$$\beta_{HL} = -\frac{\sum_{i=1}^{n} x_i'^* \left(\frac{\partial g}{\partial X_i'}\right)^*}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i'}\right)^{2*}}}$$ **Equation 21**

$(\partial g/\partial X'_i)^*$ is the i^{th} partial derivative at the design point ($x_1'^*$, $x_2'^*$,…, $x_n'^*$).

The design point in the reduced coordinates is:

$$x_i'^* = -\alpha_i \beta_{HL}$$ **Equation 22**

α_i are the direction cosines along the coordinate axes X'_i.

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial X_i'}\right)}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i'}\right)^{2*}}}$$ **Equation 23**

Direction cosines give an estimate of the sensitivity of each variable. It represents the contribution of the parameter in the probability of failure. The direction cosines of all the variables together should add up to 1.

Substituting α in the design point of original equation gives:

$$x_i^* = \mu_{X_i} - \alpha_i \sigma_{X_i} \beta_{HL}$$ **Equation 24**

Algorithm to compute the Hasofer – Lind Reliability Index (Rackwitz, 1976)

For nonlinear performance functions, an iterative algorithm proposed by Rackwitz (1976) is utilized. This is shown in Figure 5.

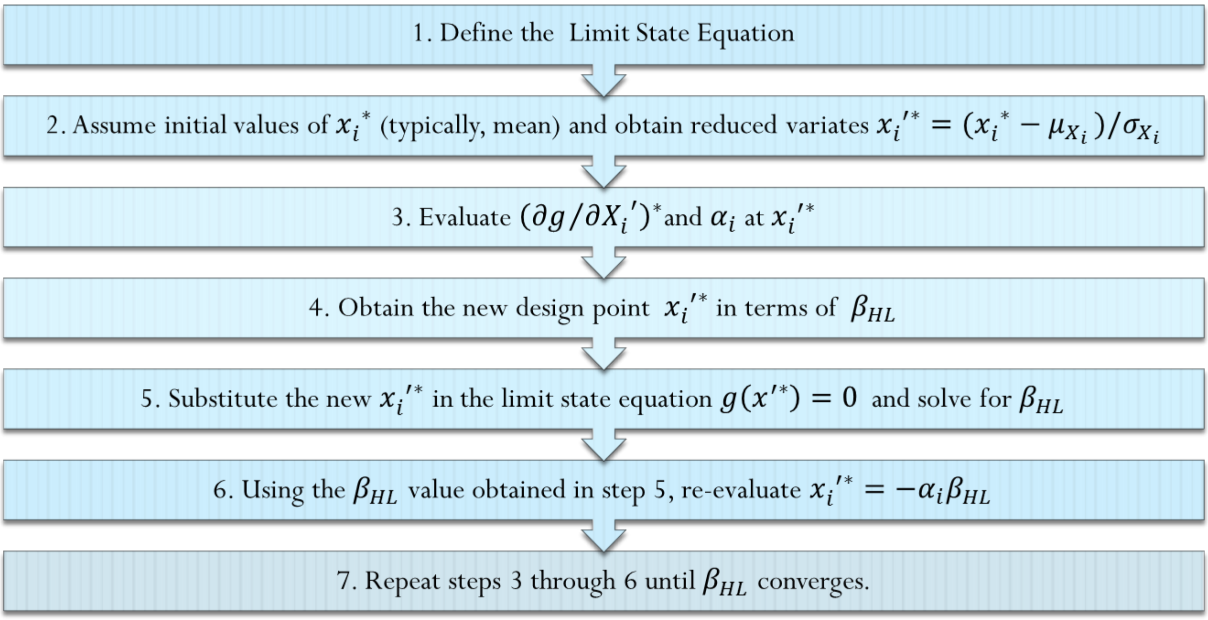


Figure 5 Algorithm to compute β_{HL}

Finally, the reliability index is used t compute the probability of failure as:
$p_f = \varphi(-\beta_{HL})$.

1.5.1 A different perspective of Hasofer – Lind Reliability Index

Hasofer - Lind rewrote the reliability index in a matrix formulation, as

$$\beta = min\sqrt{(x-m)^T C^{-1}(x-m)}$$ **Equation 25**

Or

$$\beta_{HL} = min\sqrt{\left(\frac{x_i - m_i}{\sigma_i}\right)^T R^{-1} \left(\frac{x_i - m_i}{\sigma_i}\right)}$$ **Equation 26**

x represents the input stochastic variables, m is the mean values of the variables, C is the covariance matrix that considers the negative or positive correlation between different input parameters and R is the correlation matrix.

In general, reliability index is the distance between the performance function and the the mean value point of the variables in units of standard deviation. There are computational barriers in reliability analysis by the classical methods. This is because the classical approaches require rotation of frame of reference and co-ordinate transformation.

To overcome these disadvantages, Low and Tang (2005) proposed a different interpretation with the perspective of an expanding ellipsoid. This does not require the transformation of the original space to the reduced or standardised space of variables.

The ellipsoidal approach is based on the fact that the quadratic form in Equation 25 is similar to the negative exponent of the multivariate normal probability density function. The iso density locus of a multivariate normal probability density function is an ellipse, which is the reason why Equation 25 can be represented by an ellipse. Minimising beta is equivalent to maximising the value of the multivariate normal pdf. Thus the design point or the most probable point of failure can be found as the smallest ellipsoid tangent to the limit state surface. For non-normal variables, Rackwitz Fiessler transformation is used.

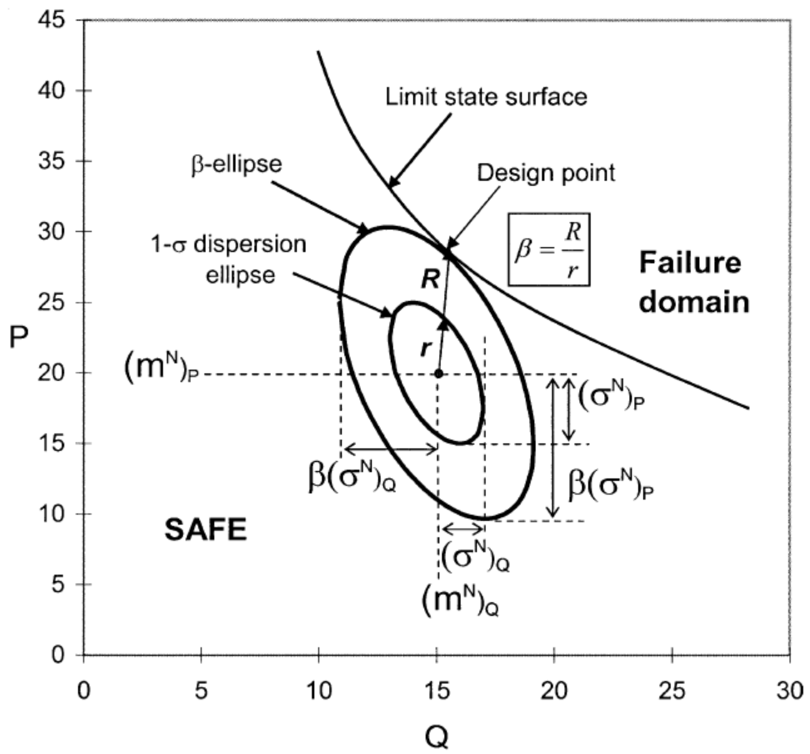


Figure 6 Ellipsoid approach for computing Hasofer Lind Reliability Index

Chapter 2

Reliability-based Retaining wall design

(B. K. Low, 2005)

2.1 Introduction

In this chapter, Reliability-based design of a retaining wall using constrained optimization approach in spreadsheet (Low, 2005) is discussed. The spreadsheet model is based on intuitive expanding dispersion ellipsoid perspective, as described in the previous chapter. Using this approach simplifies the computations and interpretations. Sensitivity information conveyed in a reliability analysis is discussed.

2.2 Reliability design

This design explicitly considers uncertainty in the design and gives the reliability index and probability of failure. Here, the retaining wall is designed using the ellipsoidal approach of FORM. The reliability index, probability of failure and the design points are obtained. The probability of failure (P_f) can be estimated from the reliability index, β using

$$\boldsymbol{P_f = 1 - \Phi(\beta) = \Phi(-\beta)}$$ **Equation 27**

2.2.1 Reliability-based approach and factor-of-safety approach

In figure 8a, A and B has the same of factor of safety, but clearly, A is safer than B. On the other hand, figure 8b shows a slope and foundation with different factor of safety but a reliability analysis shows that both structures had similar levels of reliability. This shows that a reliability based design gives a better measure of safety that lumped factor of safety approach.

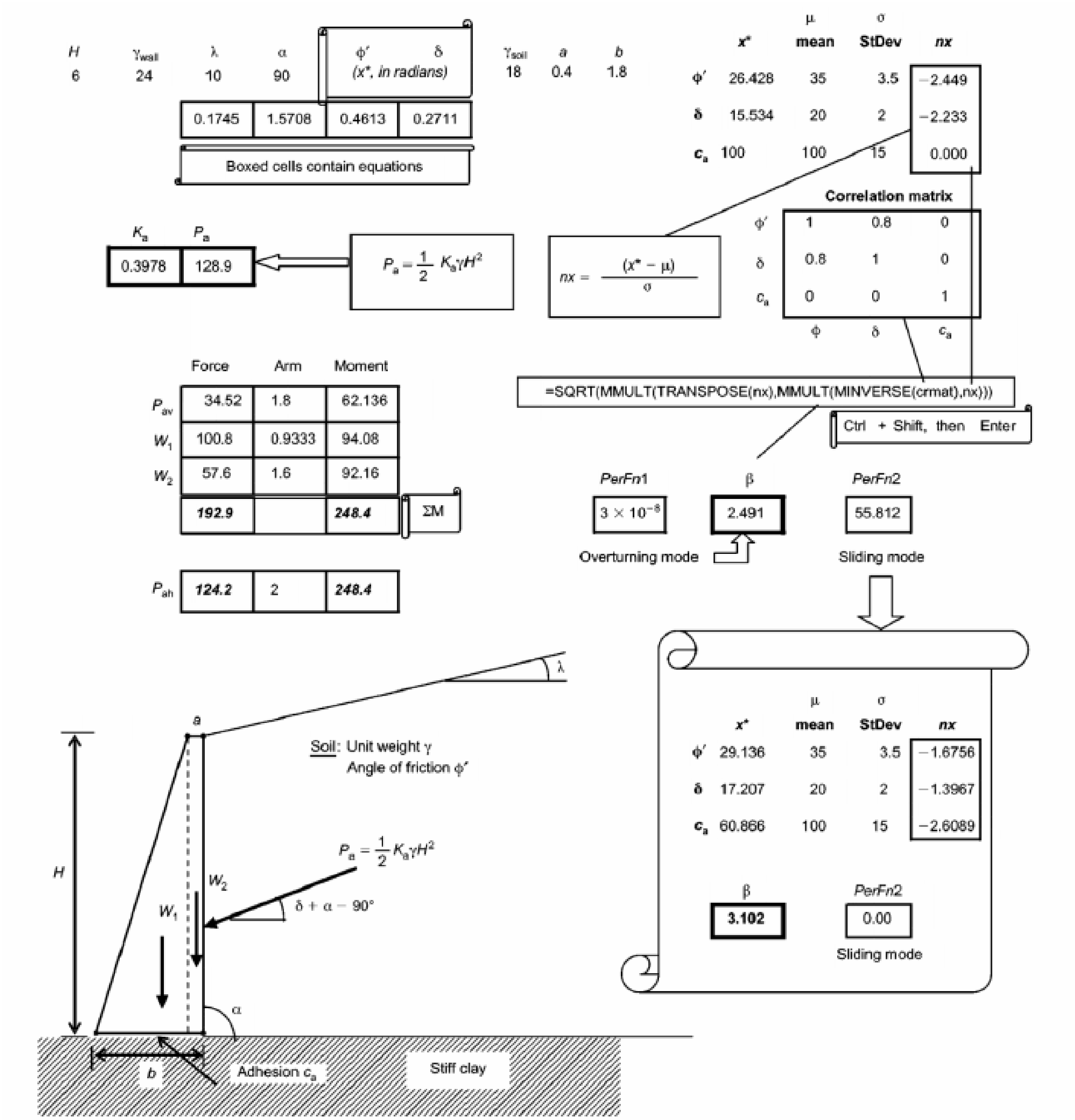


Figure 7 Reliability analysis of overturning failure mode and sliding mode, for correlated normal random variables using spreadsheet model (B K Low, 2005)

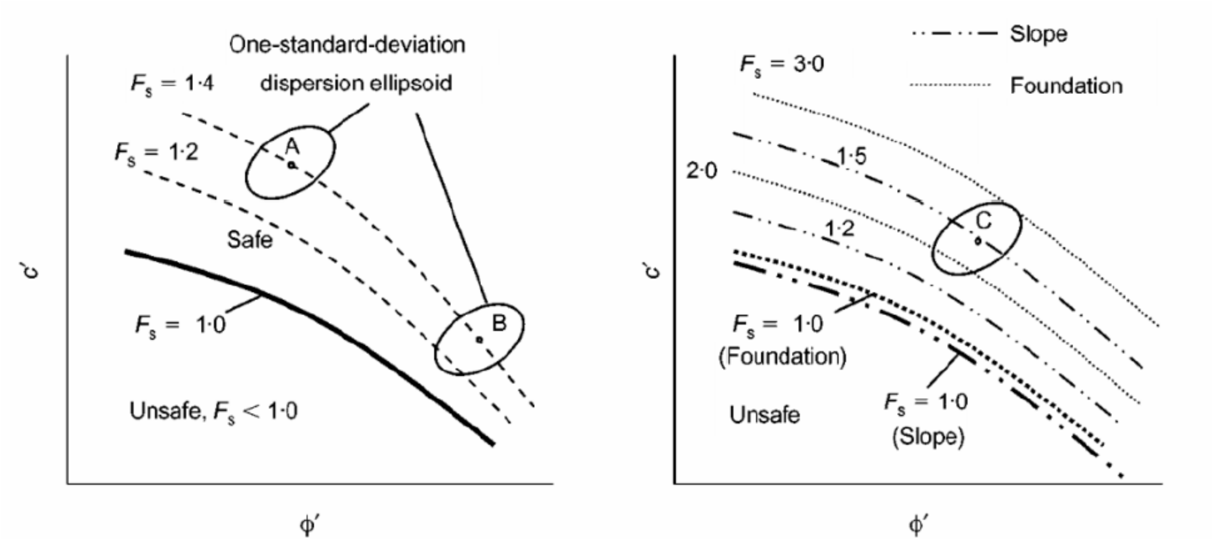


Figure 8a, b Limitations of lumped FoS (B K Low, 2005)

2.3 Limit State function

Performance Function or the limit state function is defined based on the input variables and failure limits. The performance functions (*PerFn1 and PerFn2*) are written for the rotational mode of failure and sliding failure mode:

$PerFn1 := W_1\ Arm_1 + W_2 Arm_2 + P_{av} Arm_{av} - P_{ah} Arm_{ah}$

$PerFn2 := b \times c_a - P_{ah}$

$P_{av} = P_a \sin\delta,\ Arm_{av} = b, W_1 = 0.5\gamma_{wall}(b - a)\,H, Arm_1 = \frac{2}{3}(b - a)H$

$W_2 = \gamma_{wall} aH\ , Arm_2 = b - \frac{a}{2},\ P_{ah} = P_a \cos\delta, Arm_{ah} = \frac{H}{3}$

2.4 Determination of Reliability Index

Equation 26 is used to compute the reliability index. The soil properties that are randomized are soil friction angle φ', the interface friction angle δ, and the base adhesion c_a. The statistical inputs of the variables are defined. The correlation matrix is set up based on the expected correlations between the parameters.

The design values of the parameters are initialized with the mean values and the solver is invoked. The solver is set to minimize the cell containing the reliability index by changing the design values and the values dependent on the design values by subjecting to the constraint that the respective performance function *PerFn* < = 0 . This is to make sure that the design point lies on the limit state surface. This optimizes the reliability index and searches for the most probable point of failure.

2.5 Results and Interpretation

- **Design point**

Design point lies on the failure surface and determines the value of the reliability index. As the design point lies on the limit state surface given by *PerFn,* it satisfies the corresponding limit state equation. The mean value points of (φ', δ, c_a) indicates points that can be called 'safe', against sliding /overturning but failure occurs when the mean values (φ', δ, c_a) are decreased to the design values (φ', δ, c_a).

- **Parametric Sensitivity:**

The column nx shows how much the design point x* deviates from the mean. In other words, nx reflects the sensitivity of each parameter to sliding / overturning failure. For example, in Figure 7, nx corresponding to c_a is 0 for *Perfn1*. This implies that c_a is insensitive to overturning failure whereas for sliding limit stat the values of nx show that c_a is the most sensitive. This ability to reflect parametric sensitivity is unique to FORM

- **Partial factors**

  Low (2005) has shown that the ratio of the mean values to the design point is similar to the partial factors in the limit state approach in the Eurocode7, although partial factors have not been used in this Reliability based design here.

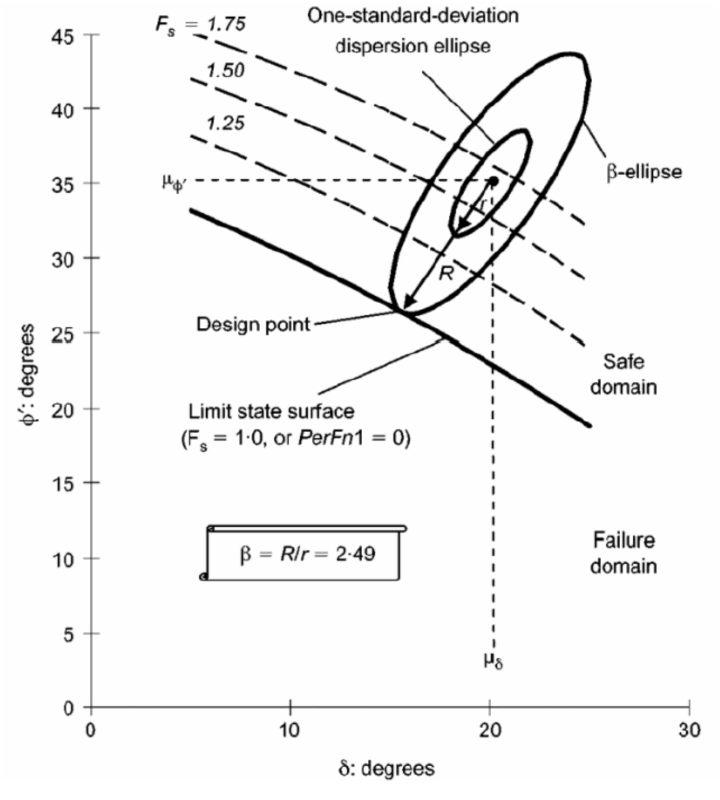


Figure 9 Design point and normal dispersion ellipsoids in φ' – δ space. Correlation coefficient is non zero. (B K Low, 2005)

- **Significance of correlation**

To investigate the significance of correlation, Low(2005) assumed the parameters as uncorrelated by giving a null correlation matrix. As expected, the reliability was a higher value, showing that a ignoring positive correlations leads to an unconservative estimate of reliability. Also it was observed that the two ellipses in Figure 10 are non-tilted when compared to the tilted ellipses in Figure 9.

Table 1 Results of Reliability Analysis performed by Low, 2005

	Overturning mode(Perfn1)		Sliding mode (Perfn2)	
Reliability Index	2.49		3.102	
Parametric Sensitivity	**nx**	**Sensitivity Scale**	**nx**	**Sensitivity Scale**
	-2.449	Highest	-1.67	Sensitive
	-2.23	High	-1.39	Sensitive
	0	Insensitive	-2.60	Highest

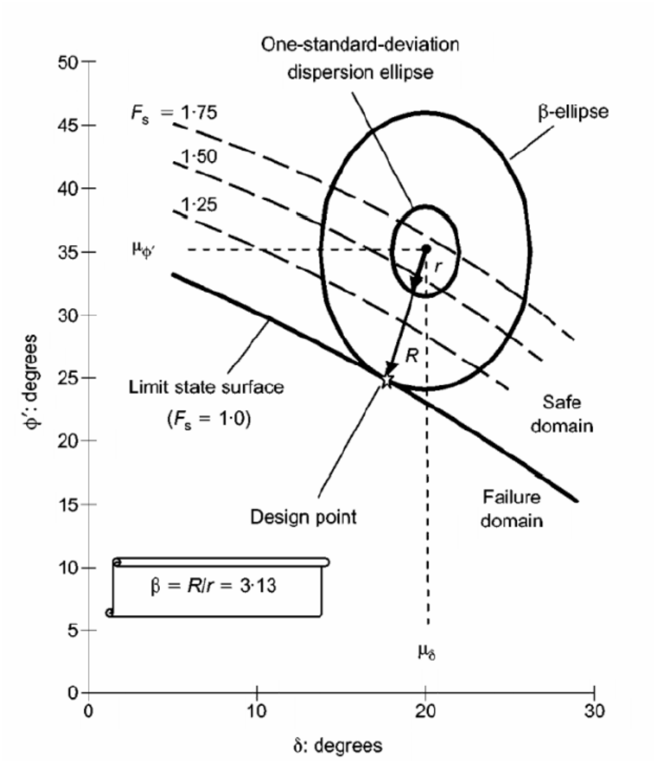


Figure 10 Design point and normal dispersion ellipsoids in ϕ’ – δ space. Correlation coefficient is 0 (Low, 2005)

Table 1 gives the results of the reliability analysis of the retaining wall. The retaining wall has a higher reliability index for the sliding mode when compared to the overturning mode. This implies that the probability of failure of the retaining wall due to overturning is greater than its probability of failure by sliding. The reason could be due to the insensitivity of one of the parameters, the base adhesion to overturning. The results also show the sensitivity of each parameter for the respective failure modes. Overturning failure mode is insensitive to base adhesion, as expected, whereas it is highly sensitive to the sliding failure mode.

2.6 Conclusions

This chapter gives an insight into Reliability based design of a retaining wall by Low (2005). The usefulness of the ellipsoidal approach of the Hasofer - Lind reliability index is shown. The design values are computed automatically using iterative constrained optimization. This approach considers correlation between variables and also gives an estimate of the importance of each parameter. Low (2005) proves that this approach could play a supplementary verification and comparison role to a design based on Eurocode7.

A similar example by Low (2005) shows the design of a retaining wall by back calculating the parameter design values by assuming a required level of reliability index.

Chapter 3

Probabilistic slope stability analysis

3.1 Introduction

One of the most common geotechnical problems is the slope stability problems. This chapter gives an insight into Reliability-based analysis of a slope stability problem using an intuitive First Order Method. For this purpose, a spreadsheet model developed by Low (2003) is used. Low (2003) analysed a clay slope using the intuitive first order reliability method. In this chapter, the example of the clay slope analysed by Low (2003) is briefly explained. Further investigation on the uncertainty of the depth levels that define the soil layers is performed. The effect of one dimensional spatial variation on the outcome of slope reliability is analyzed by altering the autocorrelation distance. Sensitivity analysis in FORM is discussed and an interpretation of the sensitivity of each parameter is made. The advantages and shortcomings of using the spreadsheet model to implement FORM are studied.

3.2 Methodology

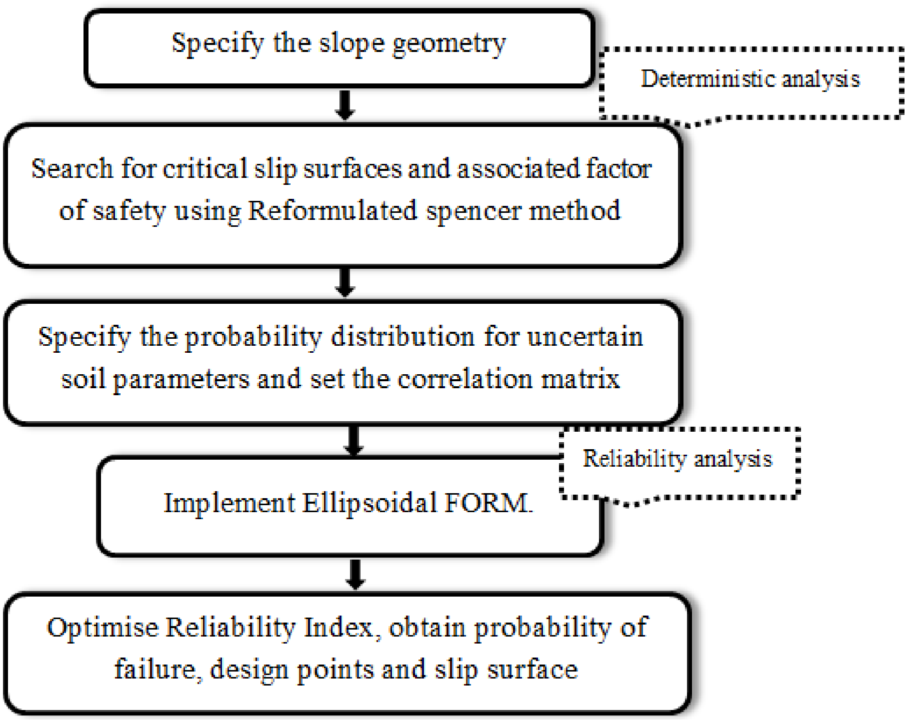
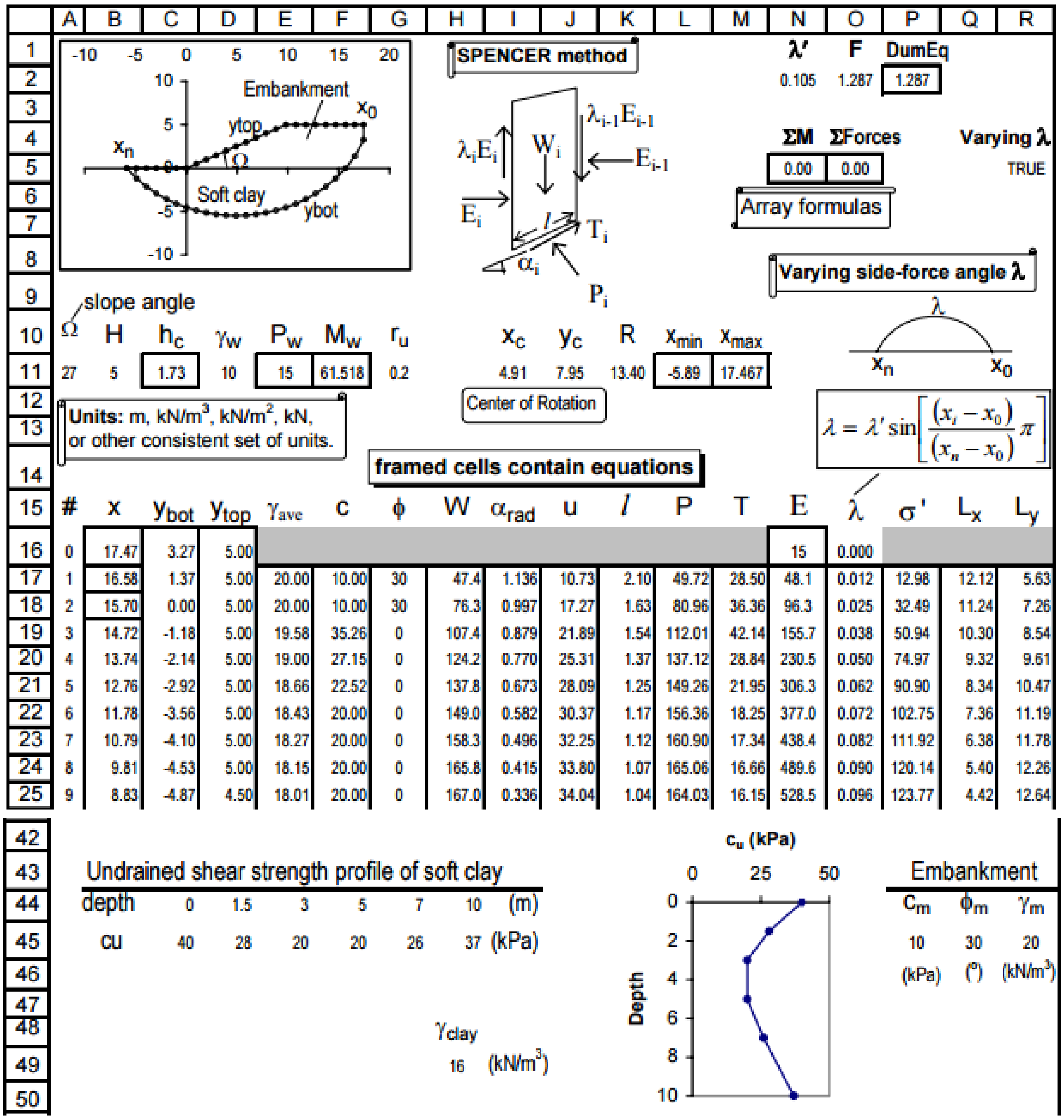


Figure 11 Schematic representation of Low's spreadsheet set up.

The flowchart is a basic representation of the scheme used by Low (2003) to set up the spreadsheet model.

3.3 Slope Stability Spreadsheet Model (Low, 2003)

Low (2003) uses reformulated spencer method that is compatible for being implemented in a spreadsheet. A deterministic analysis is first made which is then extended probabilistically to include parameter uncertainties. The principles of the reformulated spencer method are not discussed here, but it can be found in Low (2003). Different methods can be explored in the by varying the constraints of optimization.



#	x	y_{bot}	y_{top}	γ_{ave}	c	φ	W	α_{rad}	u	l	P	T	E	λ	σ'	L_x	L_y
0	17.47	3.27	5.00										15	0.000			
1	16.58	1.37	5.00	20.00	10.00	30	47.4	1.136	10.73	2.10	49.72	28.50	48.1	0.012	12.98	12.12	5.63
2	15.70	0.00	5.00	20.00	10.00	30	76.3	0.997	17.27	1.63	80.96	36.36	96.3	0.025	32.49	11.24	7.26
3	14.72	-1.18	5.00	19.58	35.26	0	107.4	0.879	21.89	1.54	112.01	42.14	155.7	0.038	50.94	10.30	8.54
4	13.74	-2.14	5.00	19.00	27.15	0	124.2	0.770	25.31	1.37	137.12	28.84	230.5	0.050	74.97	9.32	9.61
5	12.76	-2.92	5.00	18.66	22.52	0	137.8	0.673	28.09	1.25	149.26	21.95	306.3	0.062	90.90	8.34	10.47
6	11.78	-3.56	5.00	18.43	20.00	0	149.0	0.582	30.37	1.17	156.36	18.25	377.0	0.072	102.75	7.36	11.19
7	10.79	-4.10	5.00	18.27	20.00	0	158.3	0.496	32.25	1.12	160.90	17.34	438.4	0.082	111.92	6.38	11.78
8	9.81	-4.53	5.00	18.15	20.00	0	165.8	0.415	33.80	1.07	165.06	16.66	489.6	0.090	120.14	5.40	12.26
9	8.83	-4.87	4.50	18.01	20.00	0	167.0	0.336	34.04	1.04	164.03	16.15	528.5	0.096	123.77	4.42	12.64

Figure 12 Deterministic analysis of a 5 m high embankment on soft ground with depth-dependent undrained shear strength (Low, 2003)

Figure 12 shows the spreadsheet set up for the deterministic analysis of 5m embankment geometry. The undrained shear strength is depth dependent and distinguishes 5 soil layers. The user defined VBA function computes the co-ordinate values of the embankment geometry. The slip surface co-ordinates defined by the center of rotation (x_c and y_c) and its

radius is initialized as seen in the Figure 12. The factor of safety is set to 1 and Microsoft Excel's built in Solver is invoked. The solver asks for targets, variables and constraints. The target here is to minimize the cell having the factor of safety. The cells that are allowed to change values every iteration are the ones containing the factor of safety and the slip surface geometry. The changing cells are giving constraints to make sure that they are within their permissible limits. On invoking the solver, the model gives a minimized factor of safety and the slip surface of the slope (circular or non-circular depending on the constraints).
A reliability analysis of the same slope is performed by extending the deterministic analysis probabilistically. This considers uncertainties in the undrained shear strength of the soft clay layers, cohesion, friction angle and unit weight of the embankment. Only normal or log normal variables are considered. A VBA function converts the mean and standard deviation to the respective normal values by Rackwitz-Fiessler equivalent normal transformation. A correlation matrix models the spatial variation in the soft ground. An autocorrelation distance (δ) of 3 m is assumed in the following negative exponential model:

$$\boldsymbol{\rho_{ij} = e^{-\frac{Depth(i)-Depth(j)}{\delta}}} \quad \textbf{Equation 28}$$

The design values are initialized with mean values, and the solver is set. The target cell is the cell having the Reliability index which is a quadratic form (9 dimensional ellipsoid in original space). As the solver is invoked, the reliability index is minimized as the design values are updated. This process of optimizing the reliability index by subjecting the model to a certain set of constraints is referred as constrained optimization. During each iteration, the equivalent normal mean (m_N) and standard deviation (σ_N) are computed automatically for each trial design point. As mentioned in previous sections, the design point represents the worst possible combination of the random variables that can potentially lead to failure. The design values of the parameters are linked to the deterministic computations, which in turn compute the co-ordinates of the slip circle. The analysis was performed with both normal and log-normal variates. The critical slip surface of both cases with normal and lognormal variates was the same. However it is different from that of the deterministic slip surface.

3.4 Uncertainty in soil layering and the height of the embankment

In this section, the spreadsheet model discussed in the previous section is modified to include uncertainties in slope geometry. This section further investigates the uncertainty in the depth levels and slope geometry (embankment height). In this chapter, depth levels refer to the depth at which soil layers are distinguished with their shear strength parameter, as indicated in Figure 13. The depth level is highly uncertain as it is often hard to distinctly define the different soil layers. The uncertainty in slope geometry is often ignored as it is considered negligible. Here, the parameter defining slope height is randomized and the model response is studied.

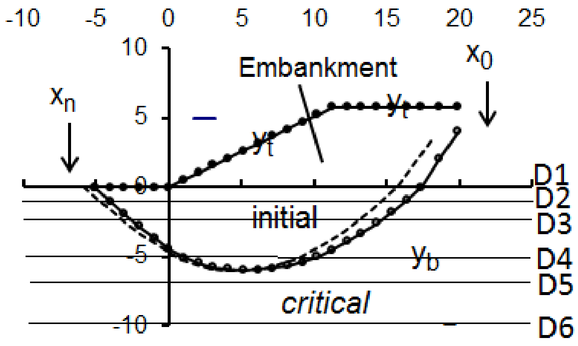


Figure 13 Slope geometry and Depth Levels

The upper most and lowermost depth levels were kept constant, and the depth levels in between were randomized. The depth levels were called D2, D3, D4, and D5 and the height of the embankment is referred as H. They were given a log normal distribution with mean values of 1.5, 3, 5, 7 and 5m respectively. The standard deviations are computed and the correlation matrix is updated. The embankment height is assumed to be uncorrelated. Adjacent depth levels are assumed to have a correlation coefficient of 0.5, and a correlation coefficient of 0.3 and 0.2 are assumed for the next layers. The design values of the variables are initialized with the mean values and the reliability index is computed using the Hasofer-Lind Matrix equation (Equation 25). The solver is then invoked to minimize the reliability index and compute the design value of the variables. These design values are linked to the deterministic computations to obtain the slip circle. Figure 14 shows the spreadsheet set up to compute the reliability index and the probability of failure.

Table 2 shows the reliability indices and the probability of failure as the number of uncertain parameters is increased. It is only logical to expect the reliability index decreasing and the corresponding probability of failure increasing with more uncertainty being considered. It can be seen that the inclusion of the embankment height does show a significant decrease in reliability. But the inclusion of depth levels does not influence the reliability index. The design values of the depth levels are very close to their mean values. This could be due to the respective shear strengths in the soil layers also being randomized.

DistName		v_i	mean m_i	StDev s	m^N	s^N	vn $(v-m^N)/s^N$
lognormal	c_m	10.158074	10	1.50	9.886	1.515	0.180
lognormal	ϕ_m	30.343898	30	3.00	29.847	3.027	0.164
lognormal	γ_m	20.336246	20	1.00	19.972	1.016	0.359
lognormal	c_{u1}	37.000355	40	6.00	39.473	5.519	-0.448
lognormal	c_{u2}	25.316775	28	4.20	27.585	3.776	-0.601
lognormal	c_{u3}	17.912028	20	3.00	19.688	2.672	-0.665
lognormal	c_{u4}	17.830734	20	3.00	19.679	2.660	-0.695
lognormal	c_{u5}	24.00851	26	3.90	25.655	3.581	-0.460
lognormal	c_{u6}	35.646365	37	5.55	36.578	5.317	-0.175
lognormal	H	5.72168	5	0.75	4.887	0.853	0.978
lognormal	D2	1.47465	1.5	0.23	1.483	0.220	-0.040
lognormal	D3	2.97666	3	0.45	2.967	0.444	0.022
lognormal	D4	5.09799	5	0.75	4.942	0.760	0.205
lognormal	D5	7.10943	7	1.05	6.920	1.060	0.179
						Reliability Index	1.363
						Probability of failure	0.0864

Correlation matrix, "**crmat**"

c_m	ϕ_m	γ_m	c_{u1}	c_{u2}	c_{u3}	c_{u4}	c_{u5}	c_{u6}	H	D2	D3	D4	D5
1	-0.3	0.5	0	0	0	0	0	0	0	0	0	0	0
-0.3	1	0.5	0	0	0	0	0	0	0	0	0	0	0
0.5	0.5	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0.6117	0.3708	0.1828	0.0935	0.0357	0	0	0	0	0
0	0	0	0.6117	1	0.6061	0.2989	0.1529	0.0583	0	0	0	0	0
0	0	0	0.3708	0.6061	1	0.4931	0.2522	0.0962	0	0	0	0	0
0	0	0	0.1828	0.2989	0.4931	1	0.5115	0.1951	0	0	0	0	0
0	0	0	0.0935	0.1529	0.2522	0.5115	1	0.3815	0	0	0	0	0
0	0	0	0.0357	0.0583	0.0962	0.1951	0.3815	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0.5	0.3	0.2
0	0	0	0	0	0	0	0	0	0	0.5	1	0.5	0.3
0	0	0	0	0	0	0	0	0	0	0.3	0.5	1	0.5
0	0	0	0	0	0	0	0	0	0	0.2	0.3	0.5	1

Figure 14 Spreadsheet set up - Reliability computation

Table 2 Reliability computation for different cases

	Random Variables	β	pf
Case 1	$c_m, \varphi_m, \gamma_m, c_{u1}, c_{u2}, c_{u3}, c_{u4}, c_{u5}, c_{u6}$	1.961	0.0249
Case 2	$c_m, \varphi_m, \gamma_m, c_{u1}, c_{u2}, c_{u3}, c_{u4}, c_{u5}, c_{u6}, H$	1.387	0.0827
Case 3	$c_m, \varphi_m, \gamma_m, c_{u1}, c_{u2}, c_{u3}, c_{u4}, c_{u5}, c_{u6}, H, D2, D3, D4, D5$	1.363	0.0864

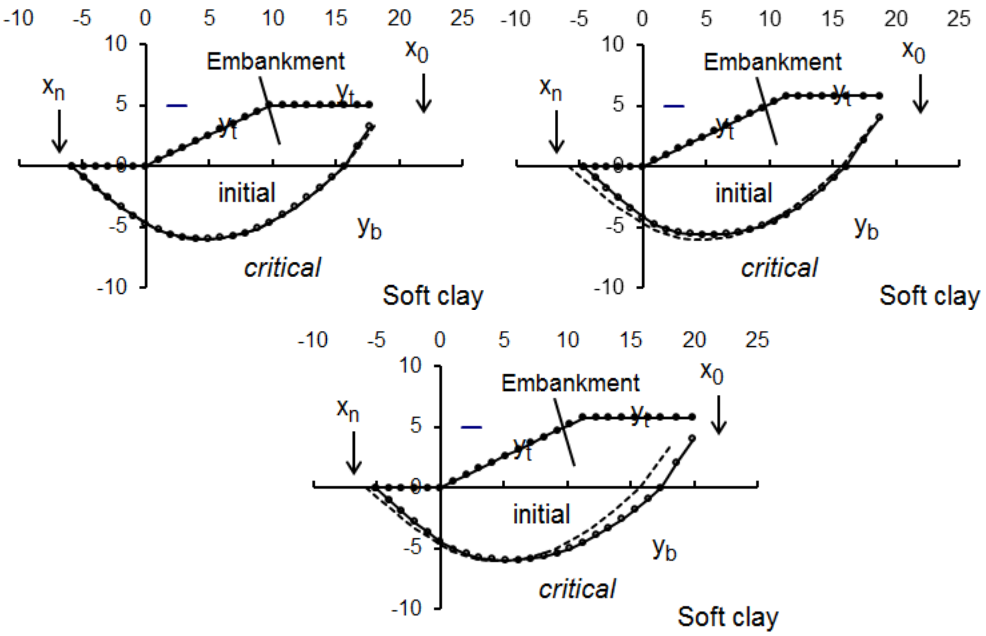


Figure 15 Comparison of reliability-based critical noncircular slip surface with deterministic critical noncircular slip surface (the lower dotted line) for Case1, Case2, and Case3 (clockwise)

Figure 15 shows that the reliability based non circular slip surface and deterministic critical non circular slip surfaces are indistinguishable for case 1, while case 2 shows that inclusion of embankment height affects the lower end of the slip surface. Case 3 shows that as the depth levels are included, upper end of the reliability based critical slip surface is different from the deterministic slip surface.

3.5 Sensitivity Interpretation

A very important advantage of using the ellipsoidal approach of FORM is its ability to reflect sensitivities. The design values reflect the sensitivities of the parameters. In Figure 16, v_i refers to the design values of the parameters. The embankment height, H seems to be the most sensitive parameter with a vn value of 0.987 (vn shows how much the design value, v_i deviates from the mean). Among the soil layers, it is to be noted that the fourth depth level D4 with shear strength of cu_4 is the most sensitive, as the vn values of D4 and cu_4 are comparatively on the higher side. The mid layer of the soft clay seems to be the most sensitive to the sliding of the slope.

DistName		v_i	mean m_i	StDev s	m^N	s^N	vn $(v-m^N)/s^N$
lognormal	c_m	10.155782	10	1.50	9.886	1.515	0.178
lognormal	ϕ_m	30.338167	30	3.00	29.847	3.026	0.162
lognormal	γ_m	20.332924	20	1.00	19.972	1.016	0.355
lognormal	c_{u1}	37.025732	40	6.00	39.475	5.523	-0.443
lognormal	c_{u2}	25.341992	28	4.20	27.588	3.780	-0.594
lognormal	c_{u3}	17.931483	20	3.00	19.690	2.675	-0.657
lognormal	c_{u4}	17.847938	20	3.00	19.681	2.662	-0.689
lognormal	c_{u5}	24.019102	26	3.90	25.655	3.583	-0.457
lognormal	c_{u6}	35.651585	37	5.55	36.578	5.318	-0.174
lognormal	H	5.72932	5	0.75	4.885	0.855	0.987
lognormal	D2	1.47485	1.5	0.23	1.483	0.220	-0.039
lognormal	D3	2.97666	3	0.45	2.967	0.444	0.022
lognormal	D4	5.09598	5	0.75	4.942	0.760	0.202
lognormal	D5	7.1083	7	1.05	6.920	1.060	0.178
					Reliability Index		1.363
					Probability of failure		0.0864

Figure 16 Sensitivity interpretation using FORM

3.6 Vertical autocorrelation distance

To analyze the influence of the auto correlation distance on the slope reliability, a series of slope reliability analysis is carried out by varying the auto correlation distance (δ) from 0.5 to 5m.

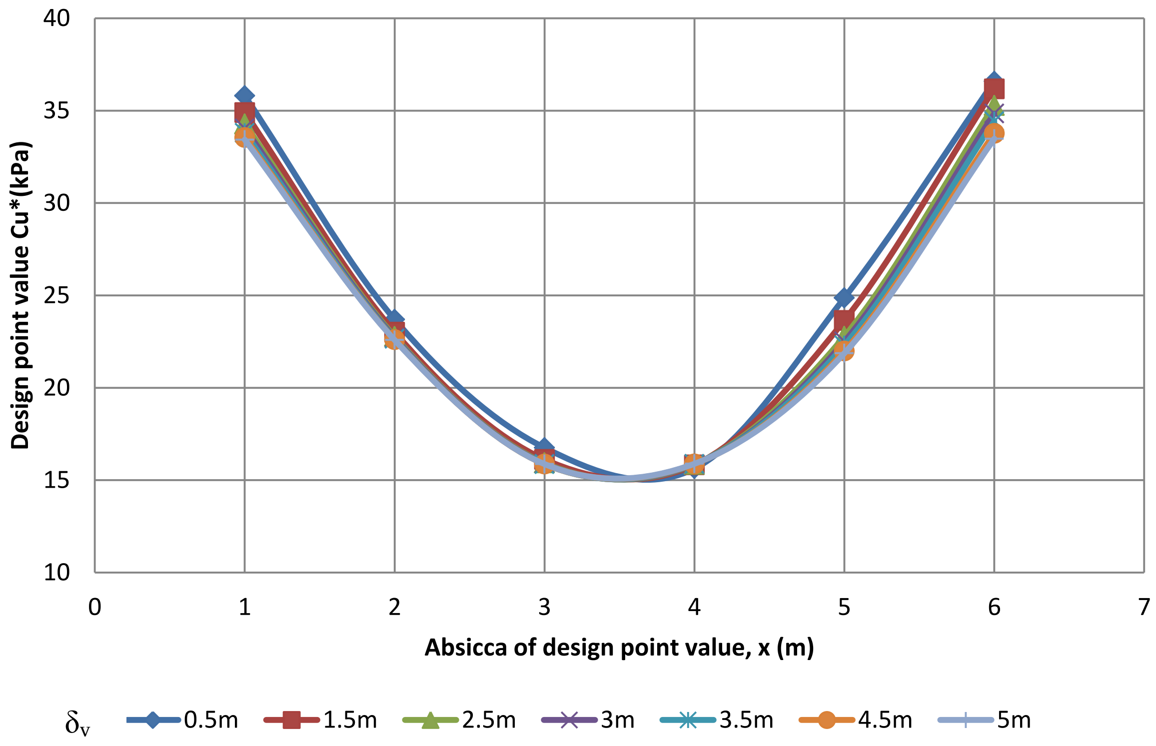


Figure 17 Slope Reliability Analysis for different correlation values

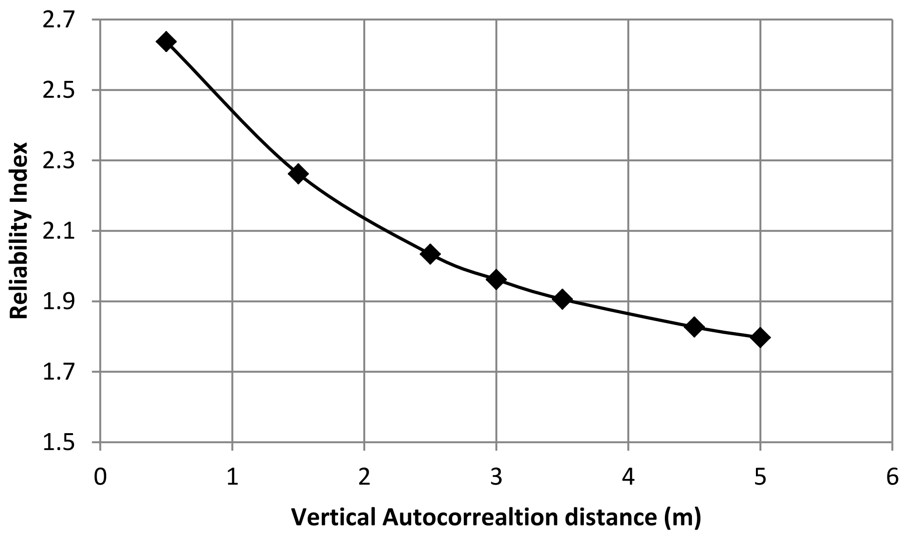


Figure 18 Reliability trend with Vertical Auto correlation distance

The design point is significant in the design as it gives the combination of parameters with the most failure probability. Thus the behavior of random variables at this design point was investigated. Figure 17 shows that the profiles of design point does not vary significantly at with slight increases in δ. Figure 18 shows that reliability index decreases with increase in auto correlation distance as expected.

3.7 Conclusions

This model allows any number of variables to be randomized and their effect on the reliability index could be studied. A spreadsheet model for a slope stability problem developed by Low (2003) was used to investigate the uncertainties in slope geometry and depth levels of the embankment. One of the most important advantages of using the spreadsheet model is the ease with which any number of parameters could be randomized. The slope geometry (embankment height) was found to have a major influence on the Reliability index while the inclusion of depth levels did not have a big impact on the reliability index. This lesser influence of depth levels on reliability index is probably due to their respective shear strengths also being randomized.

The spreadsheet model is not as user friendly as the commercial softwares but it provides a better understanding of the concept unlike commercial softwares where the underlying concept is hidden.

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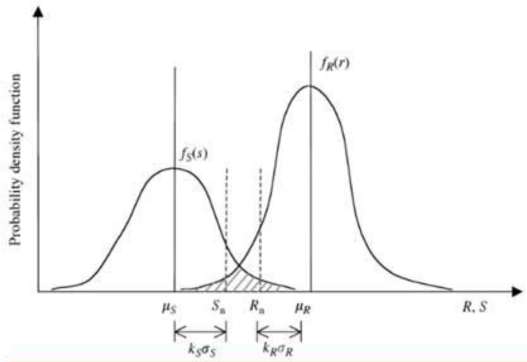
APPENDIX

FIRST ORDER RELIABILITY METHODS CONCEPTS AND APPLICATION

RELIABILITY BASED DESIGN CONCEPT

- Overlapped area gives the measure of the Probability of failure.
- Area of overlap factors: relative positions (μ), dispersion (σ) and shapes of the curves (pdfs).
- Objective of safe designs: To control the size of overlapped area.
- Conventional design approach achieve this by shifting position of the curves using S_F
- Risk based design accounts for all overlap factors.



S – Load on the structure
R – Resistance of the structure
R_N and **S_N** – Deterministic / Nominal values
Nominal Safety factor , $S_F = R_N/S_N$

PROBABILITY OF FAILURE

Basic equation of the risk based design concept:

$$p_f = P(R < S)$$
$$= \int_0^{\infty}\left[\int_0^{s} f_R(r)dr\right]f_S(s)\, ds$$
$$= \int_0^{\infty} F_R(s)f_s(s)ds$$

$F_R(s)$ - CDF of R evaluated at s

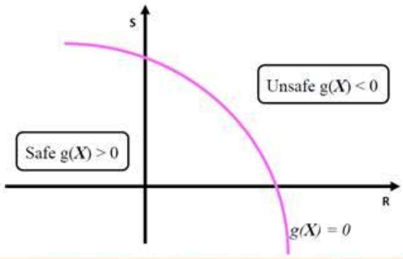
LIMIT STATE CONCEPT

$$p_f = \iint f_x(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_n$$

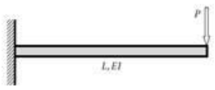
- *fx is the joint probability density function*
- *Integration performed over the failure region g(.)<0*
- *"Full distributional approach"*

Analytical approximations of the integral

FORM	SORM
• Limit state function linearized at mean values • First order Taylor series approximation • Based on second moment statistics.	• Non linear limit state function • Second order approximation



- Performance function : $\mathbf{Z = g(X_1, X_2, \ldots, X_n)}$
- Limit state or failure surface: $\boldsymbol{Z = g(X) = 0}$
- Failure event: $\mathbf{g(X)<0}$



$$LS = \frac{L}{325} - \frac{PL^3}{3EI}$$

$$G = R - S$$

First Order Second - Moment Method (FOSM) or MVFOSM method

Case 1: R & S are Independent Normal

Probability of failure is

$$p_f = P(G < 0)$$

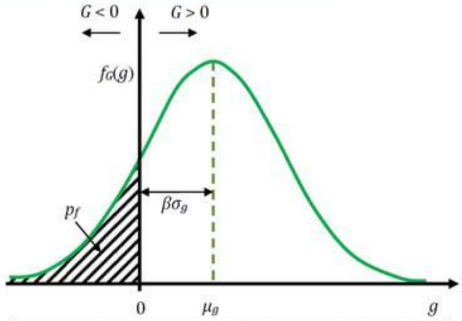
Or

$$p_f = \varphi\left(\frac{0-\mu_g}{\sigma_g}\right) = \varphi(-\beta) = 1 - \varphi(\beta)$$

$$\beta = \frac{\mu_g}{\sigma_g} = \left(\frac{\mu_R - \mu_S}{\sqrt{{\sigma_R}^2 + {\sigma_S}^2}}\right)$$

β is the reliability index

φ is the CDF of the standard normal variate



- $g(X) = G = R - S$
- $G \sim N\left(\mu_R - \mu_S, \sqrt{{\sigma_R}^2 - {\sigma_S}^2}\right)$

First Order Second - Moment Method (FOSM) or MVFOSM method

Case 2: R & S are Independent Log Normal

Probability of failure is

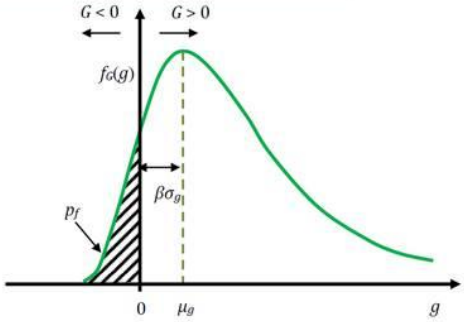
$$p_f = P\left[\left(\frac{R}{S}\right) < 1\right]$$

Or

$$p_f = 1 - \varphi\left(\frac{\ln {}^{\mu_R}/_{\mu_S}}{\sqrt{{\delta_R}^2 + {\delta_S}^2}}\right) = 1 - \varphi(\beta)$$

β is the reliability index

φ is the CDF of the standard normal variate



- Limit state function : $Y = {}^{R}/_{S}$ (or)
- $lnY = G = lnR - lnS$
- $G \sim N\left(\lambda_R - \lambda_S, \sqrt{{\zeta_R}^2 - {\zeta_S}^2}\right)$

First Order Second - Moment Method (FOSM) or MVFOSM method

Limit State function: $\boldsymbol{Z} = \boldsymbol{g}(\boldsymbol{X_1}, \boldsymbol{X_2} \dots, \boldsymbol{X_n})$

Taylor series expansion of the limit state function about the mean gives

$$G = g(\mu_X) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(X_i - \mu_{X_i}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g}{\partial X_i \partial X_j}(X_i - \mu_{X_i})\left(X_j - \mu_{X_j}\right) + \cdots$$

Truncating the series at linear terms, first order approximate mean and variance of G is

$$\mu_Z \approx g(\mu_{X_1}, \mu_{X_2}, \dots \dots \mu_{X_n}) \quad and \quad {\sigma_Z}^2 \approx \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial g}{\partial X_i}\frac{\partial g}{\partial X_j} Cov(X_i, X_j)$$

Reliability index: $\beta = {}^{\mu_Z}/_{\sigma_Z}$

Probability of failure: $p_f = \varphi(-\beta)$

Limit state function is linearized at the mean values of the random variables

Limitations of FOSM

Highly unlikely to have statistically independent normal or lognormal variables or limit state functions that are simple additive or multiplicative function of the variables.

1. Distribution information of variables ignored
2. Truncation errors due to linearization at mean point for non linear limit state function.
3. Safety index is not constant for different but mechanically equivalent formulations of the same limit state.

Eg. Safety margins defined as R-S<0 and R/S<1 are mechanically equivalent yet gives different probabilities of failure.

THIS LACK OF INVARINCE PROBLEM OVERCOME BY ADVANCED FOSM BY HASOFER AND LIND.

AFOSM Method for Normal Variables (Hasofer – Lind Method)

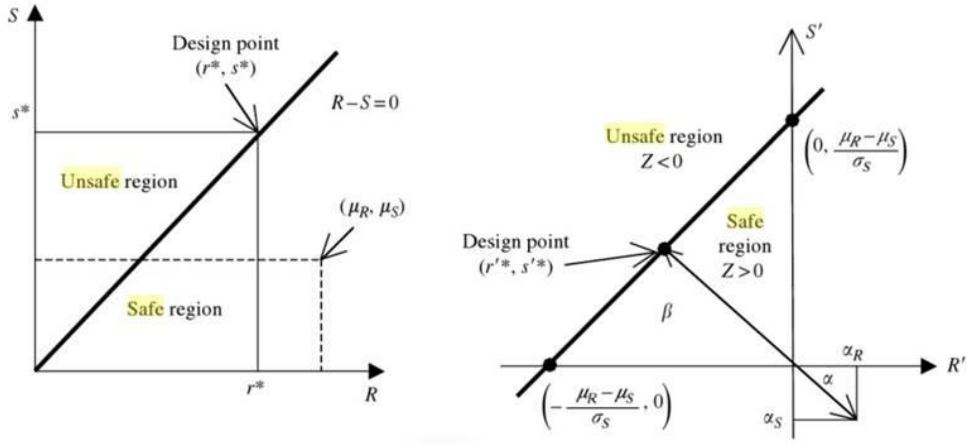
Transform original limit state $g(X) = 0$ to reduced limit state $g(X') = 0$

$$X_i' = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, \qquad X_i \sim N(\mu, \sigma) \text{ and } X_i' \sim N(0, 1)$$

	Original coordinate	Reduced coordinate
Random variables	R, S	$R' = \frac{R_i - \mu_R}{\sigma_R}$; $S' = \frac{S_i - \mu_S}{\sigma_S}$
Limit State function	$G = R - S = 0$	$g(\,) = \sigma_R R' - \sigma_S S' + \mu_R - \mu_S$
Reliability Index	$\beta = \left(\frac{\mu_R - \mu_S}{\sqrt{{\sigma_R}^2 + {\sigma_S}^2}}\right)$	$\beta = \left(\frac{\mu_R - \mu_S}{\sqrt{{\sigma_R}^2 + {\sigma_S}^2}}\right)$

Reliability index is the shortest distance of the linearised limit state from the origin in the reduced coordinate system.

HASOFER- LIND RELIABILITY INDEX: LINEAR LIMIT STATE FUNCTION



(a) Original coordinates → (b) Reduced coordinates

Hasofer - Lind Reliability Index : Non Linear Limit State function

$\beta_{HL} = \sqrt{(x'^*)^t(x'^*)}$ & $p_f = \varphi(-\beta_{HL})$
(x' - coordinates of the design point)

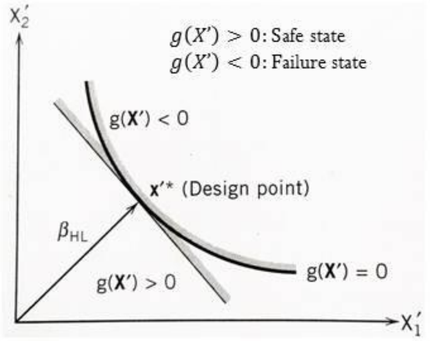
OPTMISATION OF X'^*

- Minimise $D = \sqrt{(x'^*)^t(x'^*)}$ by subjecting to constraint $g(X) = g(X') = 0$.
- Using Lagrange multiplier method, minimum distance is

$$\beta_{HL} = -\frac{\sum_{i=1}^{n} x_i'^* \left(\frac{\partial g}{\partial x_i'}\right)^*}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i'}\right)^{2*}}}$$

Evaluated at design points $x_1'^*, x_2'^* \ldots$

- The design point is $x_i'^* = -\alpha_i \beta_{HL}$
- The direction cosines, $\alpha_i = \frac{\left(\frac{\partial g}{\partial x_i'}\right)}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i'}\right)^{2*}}}$



x'^* - Minimum distance point on limit state surface

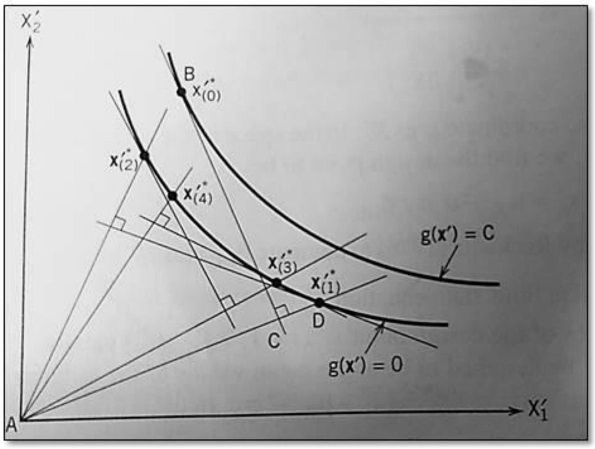
Smaller x'^* ⟷ Larger failure probability

x'^* — **M**ost **P**robable **P**oint of failure

ALGORITHM TO COMPUTE β_{HL} & $x_i'^*$

1. Define the Limit State Equation
2. Assume initial values of x_i^* (typically, mean) and obtain reduced variates $x_i'^* = (x_i^* - \mu_{X_i})/\sigma_{X_i}$
3. Evaluate $(\partial g/\partial X_i')^*$ and α_i at $x_i'^*$
4. Obtain the new design point $x_i'^*$ in terms of β_{HL}
5. Substitute the new $x_i'^*$ in the limit state equation $g(x'^*) = 0$ and solve for β_{HL}
6. Using the β_{HL} value obtained in step 5, re-evaluate $x_i'^* = -\alpha_i \beta_{HL}$
7. Repeat steps 3 through 6 until β_{HL} converges.

GEOMETRICAL INTEPREPTATION OF ALGORITHM TO COMPUTE β_{HL}



At every search point,

- A linear approximation to the limit state is constructed
- The distance from origin to the limit state is found.

- ❑ Point B: Initial design point.
- ❑ Tangent at B, BC is drawn.
- ❑ AD gives the β_{HL} in the first iteration

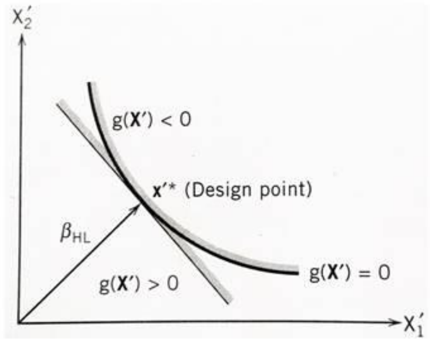
GENERALIZED RELIABILITY INDEX

- **Inconsistency** due to identical β_{HL} but different reliabilities
- Generalized reliability index, β_g to overcome inconsistency *(Ditlevsen 1979)*

$$\beta_g = \varphi^{-1}\left[\iint \phi(x_1')\phi(x_2') \ldots . \phi(x_n')dx_1'dx_2' \ldots . dx_n'\right]$$

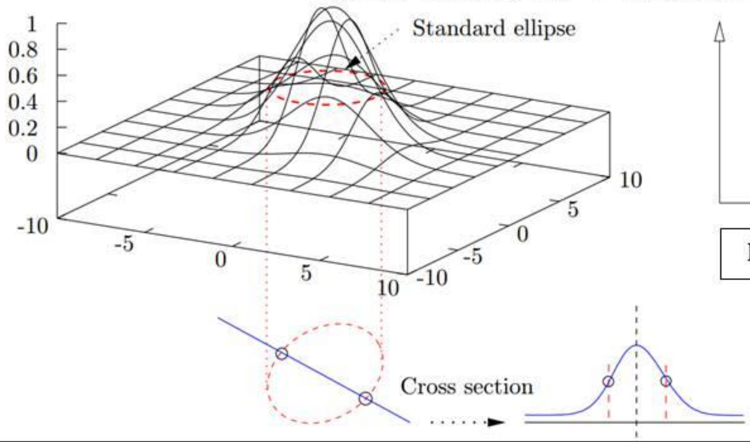
(integration over $g(x') = 0$)

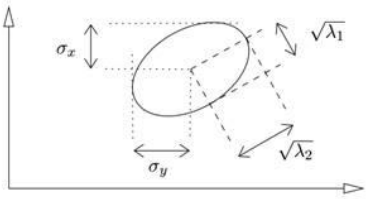
Polyhedral surface with tangent hyperplanes to approximate the non linear limit state.



- Two limit state surfaces: Flat and Curved
- Shaded portion indicates failure region

ERROR ELLIPSE : AN INTRODUCTION TO THE ELLIPSOID APPRAOCH





Parameters of error ellipse

A 2D Gaussian distribution function . A cross section gives the uncertainty in a certain direction, its standard deviation corresponding to the intersections with the (one-sigma) error ellipse.

ELLIPSOID APPROACH TO COMPUTE RELIABILITY INDEX

Matrix formulation of the Hasofer – Lind index β *is*

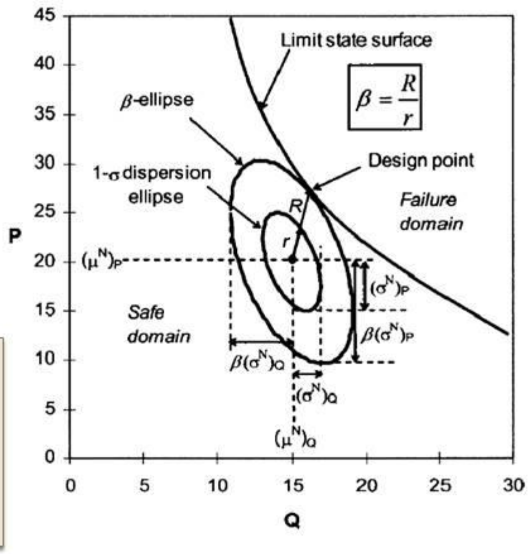
$$\beta_{HL} = min\sqrt{(x-m)^T C^{-1}(x-m)}$$

Or equivalently,

$$\beta_{HL} = min\sqrt{\left(\frac{x_i - m_i}{\sigma_i}\right)^T R^{-1} \left(\frac{x_i - m_i}{\sigma_i}\right)}$$

$\mathbf{x}$ – *Random variables vector,* $\mathbf{C}$ – *covariance matrix*
$\mathbf{R}$ – *correlation matrix ,* $\boldsymbol{m_i}$ – *mean values*
$\boldsymbol{\sigma_i}$ - *Standard deviation*

- 1σ dispersion ellipse is centred at mean and corresponds to $\beta_{HL} = 1$
- Expand or contract the ellipse to determine the smallest ellipse tangent to the limit state surface, which gives the most probable failure point/ design point.

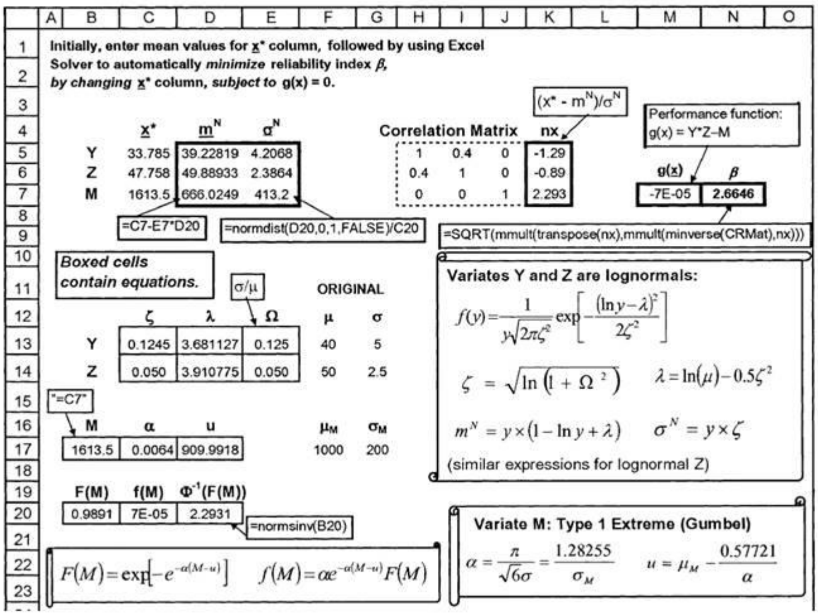


Reliability analysis of correlated non normals using constrained optimisation.

SPREADSHEET METHOD 1

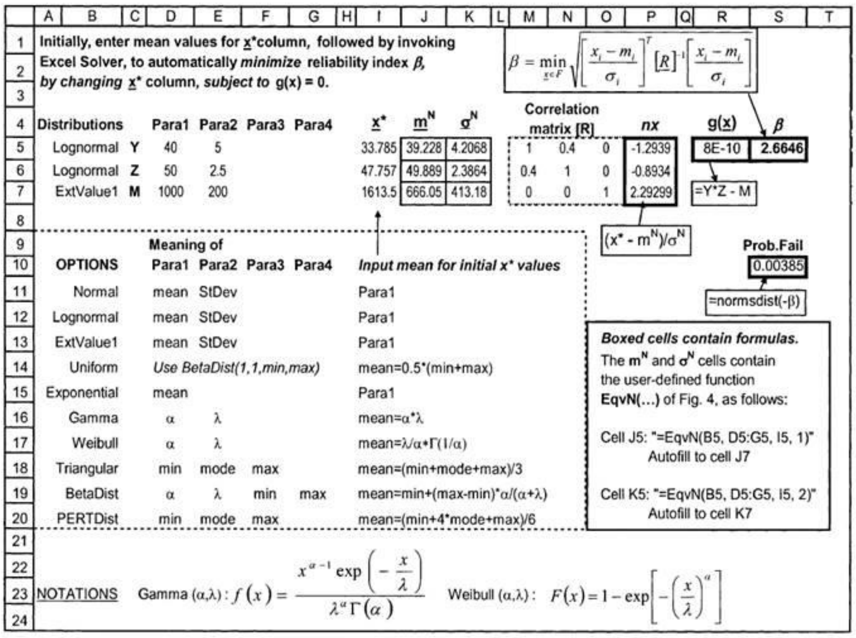
(Low and Tang 2004)

- Initialize x^* with values of mean μ of original random variables.
- Rackwitz – Fiessler equivalent normal transformation
- ✓ $\sigma^N = \frac{\phi\{\varphi^{-1}[F(x)]\}}{f(x)}$
- ✓ $m^N = x - \sigma^N \times \varphi^{-1}[F(x)]$
- Minimise β cell by changing x^* values, subject to constraint that $g(x) = 0$



Reliability analysis using object oriented constrained optimisation

(Low and Tang 2004)



SPREADSHEET METHOD 2: NEW EFFICIENT FORM ALGORITHM

(Low and Tang 2007)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1										Correlation				
2	Distribution		Para1	Para2	Para3	Para4	x_i^*			matrix [R]		n_i	g(**x***)	β
3	Lognormal	Y	40	5			33.78		1	0.4	0	-1.294	-7E-08	2.6646
4	Lognormal	Z	50	2.5			47.76		0.4	1	0	-0.893	=YZ - M	
5	ExtValue1	M	1000	200			1613		0	0	1	2.293	(x*values)	

More efficient reliability analyses *(Low and Tang 2007)*

- $\beta = min\sqrt{[n]^T[R]^{-1}[n]}$
- $n_i = \frac{x_i - \mu_i^N}{\sigma_i^N} = \varphi^{-1}[F(x_i)]$
- $x_i = F^{-1}[\varphi(n_i)]$

n – Column vector of ni

x_i - original non normal variate

$\varphi^{-1}[\,]$ - Inverse CDF of a standard normal distribution

$F(x_i)$ - Original non normal CDF

Differences from 2004 approach:

- Equivalent μ_N and σ_N not computed.
- During constrained optimisation,
  - Dimensionless numbers n_i is varied.
  - Each x_i is computed as a function of n_i.
- The objective is to find the value x_i such that the non normal cdf $F(x_i)$ at x_i is equal to the standard normal cdf $[\varphi(n_i)]$.

  $F(x_i) = [\varphi(n_i)]$
- $x \in F$ imposed as a constraint $g(x^*) = 0$ in Excel Solver.
- When $g(x^*) = 0$ and β is minimum, x^* becomes design values.

COMPARISON OF 2 METHODS

Method 1: minimize β by varying x_i

$$\beta = \min_{\underline{x} \in F} \sqrt{\left[\frac{x_i - \mu_i^N}{\sigma_i^N}\right]^T [R]^{-1} \left[\frac{x_i - \mu_i^N}{\sigma_i^N}\right]}$$

by changing x_i (via Excel's *Solver*).

$$\sigma^N = \frac{\phi\{\Phi^{-1}[F(x)]\}}{f(x)}$$

$$\mu^N = x - \sigma^N \times \Phi^{-1}[F(x)]$$

Subject to g($\underline{x}$) = 0

Method 2: minimize β by varying n_i

$$\beta = \min_{\underline{x} \in F} \sqrt{[\underline{n}]^T [\underline{R}]^{-1} [\underline{n}]}$$

by changing n_i (via Excel's *Solver*).

For a trial n_i, get $x_i = h(n_i)$

Functions *h(n_i)* as in Figure 5.

Subject to g($\underline{x}$) = 0

(More efficient and robust than method 1)

Method 1 requires computation of equivalent normal means and equivalent normal standard deviations. Method 2 does not.

RETAINING WALL– DIFFERENT SAFETY APPROACHES

Lumped Factor of safety approach

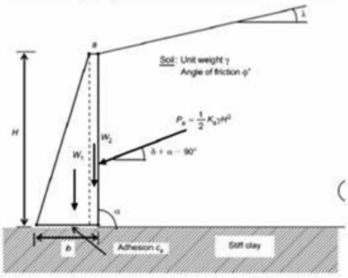
$$F_s = \frac{W_1 \times Arm_1 + W_2 \times Arm_2}{P_{ah} \times Arm_{ah} - P_{av} \times Arm_{av}} = f(\varphi', \delta, \ldots)$$

Limit state approach using partial factors

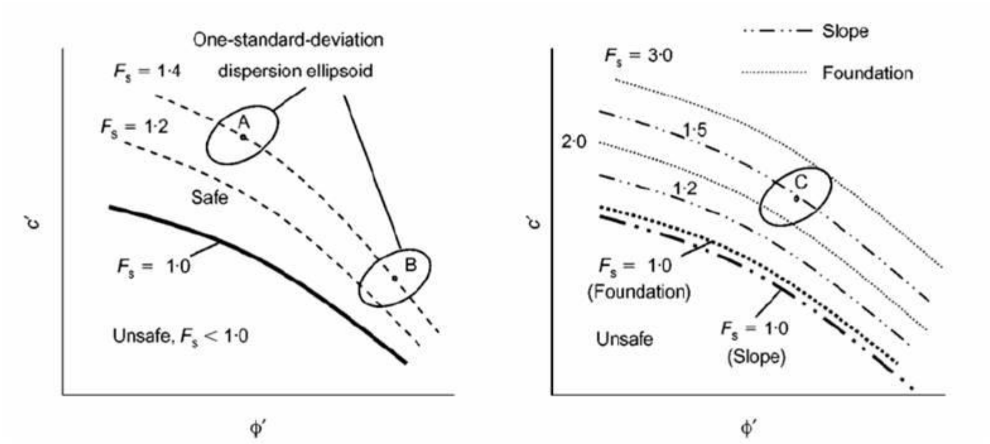
$$\sum(Resisting\ moments, factored) \geq \sum(Overturning\ moments, factored)$$

Reliability - based design

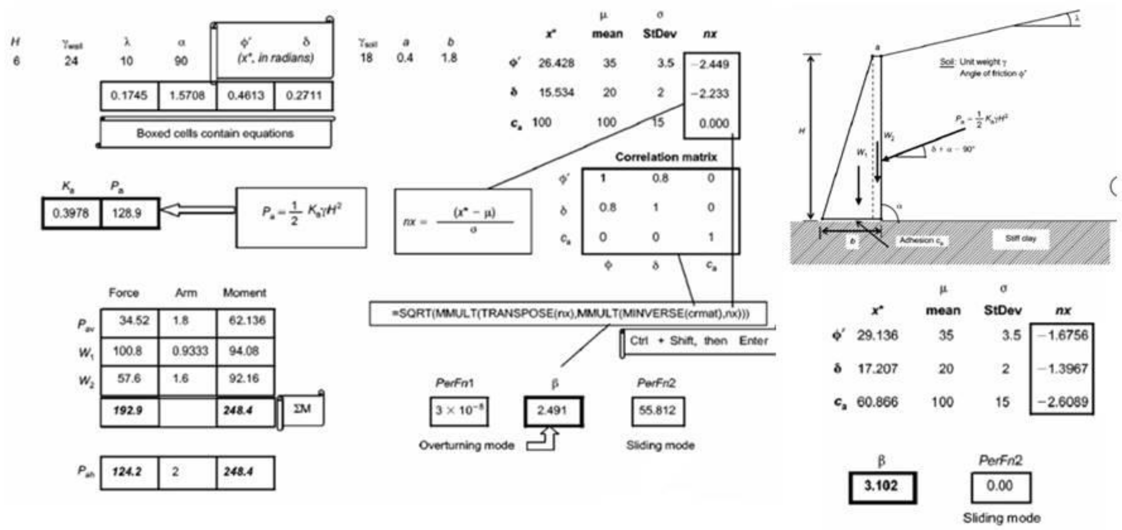
$$P_f = 1 - \Phi(\beta) = \Phi(-\beta)$$



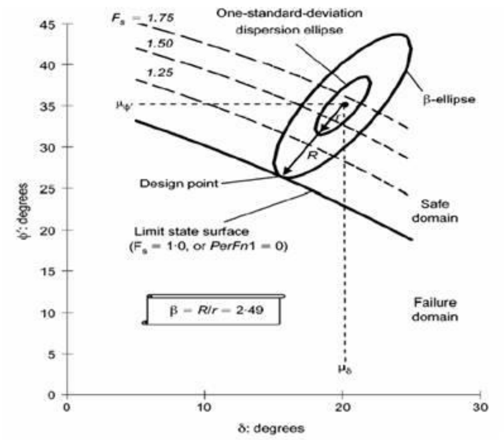
Merits of reliability-based approach over the lumped factor-of-safety approach



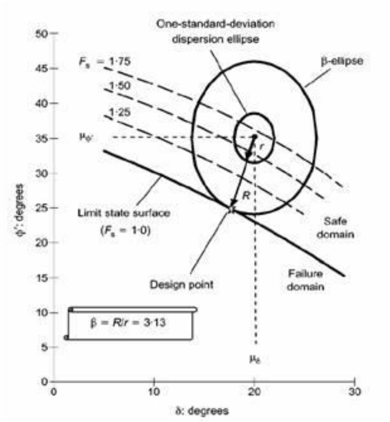
RELIABILITY ANALYSIS OF A SEMI-GRAVITY RETAINING WALL USING SPREADSHEET MODEL



Design point and normal dispersion ellipsoids in φ' – δ space



Correlation coefficient = 0.8



Correlation coefficient = 0

Probabilistic Slope stability analysis

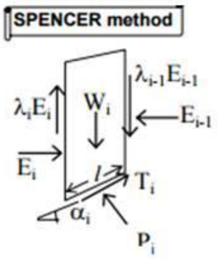
- Deterministic analysis, extended probabilistically in the spreadsheet.
- Robustness of non circular critical slip surface.
- Effects of auto correlation distance on slope reliability results

Deterministic Spencer Method

$$T_i = [c_i' l_i + (P_i - u_i l_i) tan\phi_i']/F$$

$$E_i = E_{i-1} + P_i sin\alpha_i - T_i cos\alpha_i$$

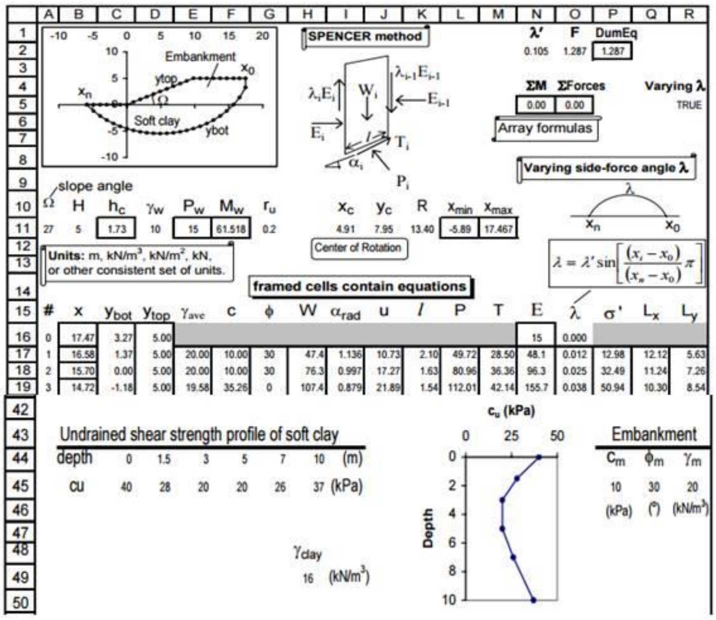
$$P_i = \frac{\left[W_i - (\lambda_i - \lambda_{i-1})E_{i-1} - \frac{1}{F}(c_i' l_i - u_i l_i tan\phi_i')(sin\alpha_i - \lambda_i cos\alpha_i)\right]}{\left[\lambda_i sin\alpha_i + cos\alpha_i + \frac{1}{F} tan\varphi_i'(sin\alpha_i - \lambda_i cos\alpha_i\right]}$$



$$\sum[T_i cos\alpha_i - P_i sin\alpha_i] - P_w = 0$$

$$\sum\left[(T_i sin\alpha_i + P_i cos\alpha_i - W_i) \times L_{xi} + (T_i cos\alpha_i - P_i sin\alpha_i) \times L_{yi}\right] - M_w = 0$$

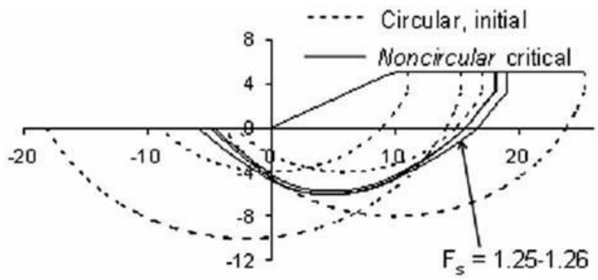
$$L_{xi} = 0.5(x_i + x_{i-1}) - x_c$$
$$L_{yi} = y_c - 0.5(y_i + y_{i-1})$$



<u>Deterministic stability analysis of 5m high embankment</u>

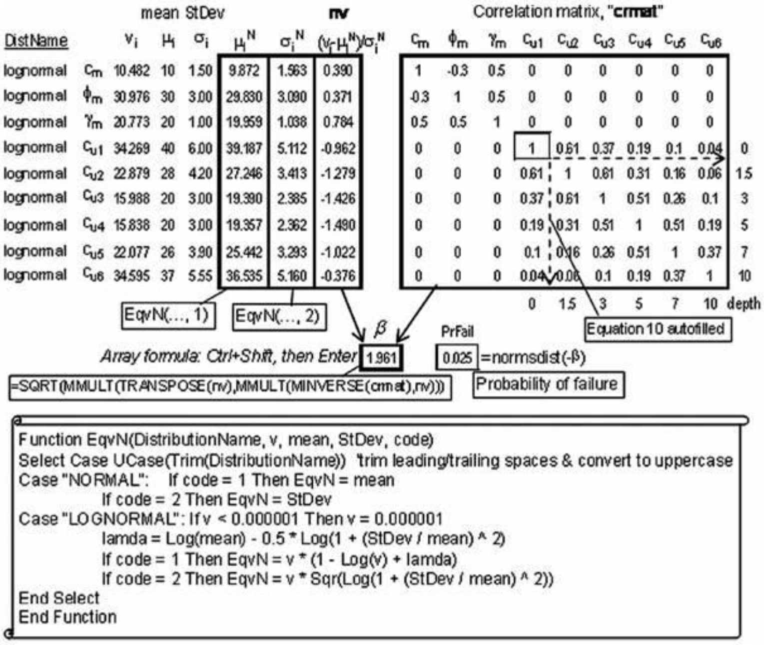
- Depth dependent undrained shear strength
- Initialize the centre, radius of critical slip surface and F = 1
- Invoke solver to obtain the critical slip surface and FoS

Testing the robustness of search for critical non circular slip surface



Different optimization settings for implementing different limit equilibrium methods

Method	Assumption for λ or λ'	By changing cells	Solver constraints regarding equilibrium
Spencer with varying side-force inclination	Varying λ	λ', F, ...	ΣForces = 0; ΣMoments = 0
Spencer with constant side-force inclination	Constant λ	λ', F, ...	ΣForces = 0; ΣMoments = 0
Bishop simplified method (circular slip surface)	Set $\lambda' = 0$	F, ...	ΣMoments = 0
Wedge method	Example: $\lambda = (\tan\phi)/F$	F, ...	ΣForces = 0



mean StDev | rrv | Correlation matrix, "**crmat**"

DistName		v_i	μ_i	σ_i	μ_i^N	σ_i^N	$(v_i-\mu_i^N)/\sigma_i^N$	c_m	ϕ_m	γ_m	c_{u1}	c_{u2}	c_{u3}	c_{u4}	c_{u5}	c_{u6}	
lognormal	c_m	10.482	10	1.50	9.872	1.563	0.390	1	-0.3	0.5	0	0	0	0	0	0	
lognormal	ϕ_m	30.976	30	3.00	29.830	3.090	0.371	-0.3	1	0.5	0	0	0	0	0	0	
lognormal	γ_m	20.773	20	1.00	19.959	1.038	0.784	0.5	0.5	1	0	0	0	0	0	0	
lognormal	c_{u1}	34.269	40	6.00	39.187	5.112	-0.962	0	0	0	1	0.61	0.37	0.19	0.1	0.04	0
lognormal	c_{u2}	22.879	28	4.20	27.246	3.413	-1.279	0	0	0	0.61	1	0.61	0.31	0.16	0.06	1.5
lognormal	c_{u3}	15.988	20	3.00	19.390	2.385	-1.426	0	0	0	0.37	0.61	1	0.51	0.26	0.1	3
lognormal	c_{u4}	15.838	20	3.00	19.357	2.362	-1.490	0	0	0	0.19	0.31	0.51	1	0.51	0.19	5
lognormal	c_{u5}	22.077	26	3.90	25.442	3.293	-1.022	0	0	0	0.1	0.16	0.26	0.51	1	0.37	7
lognormal	c_{u6}	34.595	37	5.55	36.535	5.160	-0.376	0	0	0	0.04	0.06	0.1	0.19	0.37	1	10
											0	1.5	3	5	7	10	depth

EqvN(..., 1) EqvN(..., 2)

β

Array formula: Ctrl+Shift, then Enter 1.961

=SQRT(MMULT(TRANSPOSE(rrv),MMULT(MINVERSE(crmat),rrv)))

PrFail 0.025 =normsdist(-β)

Probability of failure

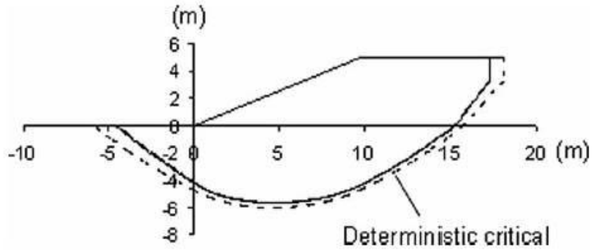
Equation 10 autofilled

```
Function EqvN(DistributionName, v, mean, StDev, code)
Select Case UCase(Trim(DistributionName))  'trim leading/trailing spaces & convert to uppercase
Case "NORMAL":    If code = 1 Then EqvN = mean
            If code = 2 Then EqvN = StDev
Case "LOGNORMAL": If v < 0.000001 Then v = 0.000001
            lamda = Log(mean) - 0.5 * Log(1 + (StDev / mean) ^ 2)
            If code = 1 Then EqvN = v * (1 - Log(v) + lamda)
            If code = 2 Then EqvN = v * Sqr(Log(1 + (StDev / mean) ^ 2))
End Select
End Function
```

- *Spatial Correlation*

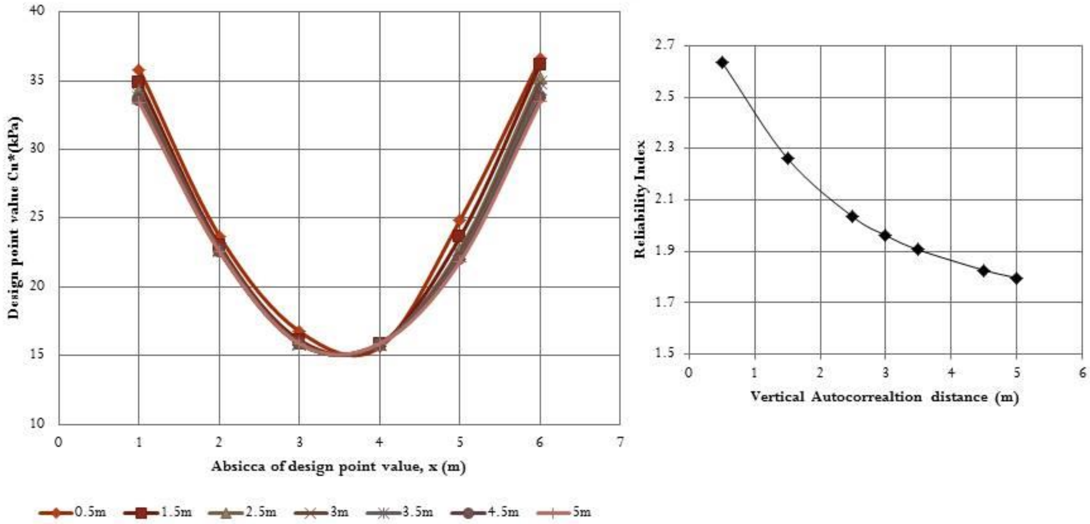
  $\rho_{ij} = e^{-\frac{Depth(i)-Depth(j)}{\delta}}$
- Minimize the quadratic form (, i.e. cell "β"
- By changing the nine random variables v_i, the λ' and the 25 coordinate values x_0, x_2, x_{24}, y_{b1}, y_{b3}:yb_{23} of the slip surface.
- F remains at 1, i.e. at failure,
- Subject to the constraints
  - $-1 \leq \lambda' \leq 1$,
  - $x_0 \geq H/\tan(\text{radians}(\Omega))$,
  - $x_0 \geq x_2$
  - $x_{24} \leq 0$
  - $yb_3 : yb_{23} \leq yt_3 : yt_{23}$
  - $\sum M = 0$, and
  - $\sigma_1' : \sigma_{24}' \geq 0$.

Comparison of reliability based critical non circular slip surface with the deterministic critical non circular slip surface



	Lognormal variates	Normal variates
Reliability index	1.96 (1.97)*	1.86 (1.87)*
Probability of failure	2.5% (2.4%)*	3.2% (3.1%)*

EFFECT OF AUTOCORELATION DISTANCE ON SLOPE RELIABILITY



Pros of the Spreadsheet Model

- That the enhanced participation will give rise to improved understanding and appreciation of the principles,
- Versatile because unique features (e.g. anisotropy) can be modeled by simple programming in the VBA programming environment of the spreadsheet software
- The ease with which the deterministic analysis can be extended into a first-order reliability analysis.

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THANK YOU!

SORRY FOR THE CANCELLED MEETINGS.