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Minimal Input Structural Modifications for Strongly Structural Controllability

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Abstract— **This paper studies the problem of modifying the input matrix of a structured system to make the system strongly structurally controllable. We focus on the generalized structured systems that rely on zero/nonzero/arbitrary structure, i.e., some entries of system matrices are zeros, some are nonzero, and the remaining entries can be zero or nonzero (arbitrary). We derive the feasibility conditions of the problem, and if it is feasible, we reformulate it into another equivalent problem. This new formulation leads to a greedy heuristic algorithm. However, we also show that the greedy algorithm can give arbitrarily poor solutions for some special systems. Our alternative approach is a randomized Markov chain Monte Carlo-based algorithm. Unlike the greedy algorithm, this algorithm is guaranteed to converge to an optimal solution with high probability. Finally, we numerically evaluate the algorithms on random graphs to show that the algorithms perform well.**

Index Terms— **Network controllability, pattern matrices, structured system, zero forcing, Markov chain Monte Carlo**

I. Introduction

Network controllability is a fundamental property used to analyze the behavior of dynamical systems like social networks [1], power systems [2], and biological systems [3]. However, complex system dynamics, like those in vast social networks, may not be completely known due to unmeasurable influences and unknown interconnection structures. Such uncertainties in large systems like social, power, and biological networks can be modeled using *structured systems* [2], [3]. It comprises the linear dynamical systems whose system matrices follow a *zero/nonzero/arbitrary structure* (arbitrary values can be zero or nonzero). Further, structural changes occur over time due to external disturbances, component failures, or malicious attacks. Some examples are failed or new ties in social networks [4], [5], loss of interconnections in power and transportation networks [6]–[8] and alteration of cellular biochemical pathways in biological networks [9], [10]. Therefore, developing methods to sparingly modify the interconnections between the external inputs and state variables while preserving control properties is highly beneficial. Given a strongly structurally uncontrollable system, we aim to enforce strong structural controllability by minimally modifying the input matrix.

Several studies have investigated the strong structural controllability of structured systems with various zero/nonzero patterns. The related prior work addresses the problems such as graph-theoretic controllability tests [11]–[13], minimum input selection [14]–[17], targeted controllability [18], [19], robustness of controllability [10], and edge augmentation [20]. To the best of our knowledge, the input structural modification problem has not been studied in the context of strong structural controllability for either zero/nonzero or zero/nonzero/arbitrary systems. Nonetheless, structural modification has been investigated for (not strong) structural controllability of systems defined by zero/nonzero pattern matrices [21]–[23]. This is a relaxed version of strong structural controllability, and naturally, this formulation is not directly extendable to our strong structural controllability setting. Therefore, we look at the problem of making minimal changes to the input matrix of a structured system to guarantee its strong structural controllability.

Our specific contributions are as follows:

- In Section III, we formulate the minimal input structural modification problem and Proposition 1 discusses its feasibility conditions. Under the feasibility conditions, Theorem 3 presents a more intuitive problem reformulation.
- In Section IV-A, we present and analyze a novel greedy algorithm with a column-by-column update rule for the input matrix. Although the greedy algorithm performs well in general, in certain cases, the greedy solution can be arbitrarily larger than the optimal solution (Proposition 2).
- In Section IV-B, we devise a Markov chain Monte Carlo (MCMC)-based solution for the structural modification problem. This approach has guarantees on solution optimality (Theorem 4) and error probability (Proposition 3) if it is run sufficiently long.

Overall, our results provide interesting insights and devise design algorithms to achieve strong structural controllability. We note that while the MCMC algorithms have been used in the strong structural controllability literature [15], our approach is more rigorous. The proof techniques of our new results Theorem 3 and Proposition 3 can be used to derive similar results for [15]. Our design of the MCMC transition probability matrix that avoids bias towards certain solutions can also be adapted to improve the algorithm in [15].

II. Notation and Preliminaries

A structured system $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ is a class of linear dynamical systems whose state matrix $\boldsymbol{A} \in \mathbb{P}(\bar{\mathcal{A}})$ and input matrix $\boldsymbol{B} \in \mathbb{P}(\bar{\mathcal{B}})$. Here, for any given pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p\times q}$,

$$\mathbb{P}(\mathcal{M}) = \left\{ \boldsymbol{M} \in \mathbb{R}^{p\times q} : \boldsymbol{M}_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0 \text{ and } \boldsymbol{M}_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = * \right\}. \quad (1)$$

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We note that the entries in $\boldsymbol{M} \in \mathbb{P}(\mathcal{M})$ corresponding to ? entries in $\mathcal{M}$ can either be zero or nonzero (arbitrary). The system $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ is called strongly structurally controllable if the linear dynamical system $(\boldsymbol{A}, \boldsymbol{B})$ is controllable for any $\boldsymbol{A} \in \mathbb{P}(\bar{\mathcal{A}})$ and $\boldsymbol{B} \in \mathbb{P}(\bar{\mathcal{B}})$. It can be tested using the rank of pattern matrices, $\operatorname{rank}(\mathcal{M}) = \min_{\boldsymbol{M}\in\mathbb{P}(\mathcal{M})} \operatorname{rank}(\boldsymbol{M})$.

Theorem 1 ([24]). *The system $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ is strong structural controllable if and only if the following two conditions hold:*

1) *The pattern matrix $\begin{bmatrix}\bar{\mathcal{A}} & \bar{\mathcal{B}}\end{bmatrix}$ has full row rank.*
2) *The pattern matrix $\begin{bmatrix}\mathcal{Q}(\bar{\mathcal{A}}) & \bar{\mathcal{B}}\end{bmatrix}$ has full row rank, where the pattern matrix $\mathcal{Q}(\bar{\mathcal{A}})$ is*

$$[\mathcal{Q}(\bar{\mathcal{A}})]_{ij} = \begin{cases} \bar{\mathcal{A}}_{ij} & \text{if } i \neq j \\ * & \text{if } i = j \text{ and } \bar{\mathcal{A}}_{ii} = 0 \\ ? & \text{otherwise.} \end{cases} \tag{2}$$

Here, a graph-theoretic algorithm, summarized in Algorithm 1, can be used to test the rank condition [24].

Algorithm 1 Color change rule

Input: Pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p\times q}$
1: Set $\mathbb{E}_* = \{(j,i) : \mathcal{M}_{ij} = *\}$; $\mathbb{E}_? = \{(j,i) : \mathcal{M}_{ij} = ?\}$
2: Initialize the set of white vertices $W \leftarrow \{1, 2, \ldots, p\}$
3: **repeat**
4: Set $W_{\text{del}} \leftarrow \{i \in W : \exists j$ such that $(j,i) \in E_*$ and $(j,\tilde{i}) \notin \mathbb{E}_* \cup \mathbb{E}_?, \forall \tilde{i} \in W \setminus \{i\}\}$
5: Color vertex set W_{del} black, $W \leftarrow W \setminus W_{\text{del}}$
6: **until** $W_{\text{del}} = \emptyset$
7: Set $W(\mathcal{M}) \leftarrow W$
Output: Set of white vertices $W(\mathcal{M})$

Theorem 2 ([24]). *A given pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p\times q}$ is full row rank if and only if the set of white vertices $W(\mathcal{M})$ outputted by the color change rule in Algorithm 1 is empty.*

Further, we define a zero forcing set of a pattern matrix.

Definition 1. *For a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p\times q}$, the set $V^0 \subseteq \{1, 2, \ldots, p\}$ is called its zero forcing set if Algorithm 1 returns an empty set $W(\mathcal{M})$ when initialization in Step 2 is changed to $W \leftarrow \{1, 2, \ldots, p\} \setminus V^0$.*

Relying on the above definitions and results, the next section presents our problem formulation.

III. Minimal Structural Modification Problem

Consider a strongly structurally uncontrollable system $(\bar{\mathcal{A}} \in \{0, *, ?\}^{n\times n}, \bar{\mathcal{B}} \in \{0, *, ?\}^{n\times m})$ as given in Section II. We address the problem of making minimum changes to the pattern matrix $\bar{\mathcal{B}}$ to obtain $\mathcal{B} \in \{0, *, ?\}^{n\times m}$ such that the new system $(\bar{\mathcal{A}}, \mathcal{B})$ is strongly structurally controllable. The change in the pattern matrix is quantified using the distance metric, $\operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}) = \left|\{(i,j) : \mathcal{B}_{ij} \neq \bar{\mathcal{B}}_{ij}\}\right|$. Further, using Theorems 1 and 2, the resulting optimization problem is

$$\begin{aligned} \underset{\mathcal{B}\in\{0,*,?\}^{n\times m}}{\arg\min} \quad & \operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}) \\ \text{s. t. } & W([\bar{\mathcal{A}} \ \ \mathcal{B}]) \cup W([\mathcal{Q}(\bar{\mathcal{A}}) \ \ \mathcal{B}]) = \emptyset, \end{aligned} \tag{3}$$

where $W(\cdot)$ is Algorithm 1's output and $\mathcal{Q}(\cdot)$ is as in (2).

We next discuss the feasibility conditions of the problem, using the notion of zero forcing set in Definition 1.

Proposition 1. *Consider a given structured system $(\bar{\mathcal{A}} \in \{0, *, ?\}^{n\times n}, \bar{\mathcal{B}} \in \{0, *, ?\}^{n\times m})$ and $\mathcal{Q}(\bar{\mathcal{A}})$ as defined in (2). The structural modification problem (3) is feasible only if*

$$m \geq \max\{Z(\bar{\mathcal{A}}), Z(\mathcal{Q}(\bar{\mathcal{A}}))\}, \tag{4}$$

where $Z(\cdot)$ is the zero forcing number i.e., the minimum cardinality $|V^0|$ over all the zero forcing sets V^0. Also, the problem (3) is feasible if

$$\begin{aligned} m \geq Z^{\text{joint}} = & \min_{V\subset\{1,2,\ldots,n\}} |V| \\ & \text{s. t. } V \text{ is a zero forcing set of both } \bar{\mathcal{A}} \text{ and } \mathcal{Q}(\bar{\mathcal{A}}). \end{aligned} \tag{5}$$

Proof. See Appendix I. □

The necessary condition is not always sufficient, and vice versa. For example, consider the following pattern matrices,

$$\bar{\mathcal{A}}^{(1)} = \begin{bmatrix} 0 & ? & ? \\ 0 & * & ? \\ 0 & ? & ? \end{bmatrix}, \bar{\mathcal{A}}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \text{ and } \mathcal{B}^* = \begin{bmatrix} * & 0 \\ * & 0 \\ 0 & * \end{bmatrix}. \tag{6}$$

For $(\bar{\mathcal{A}}^{(1)}, \bar{\mathcal{B}})$ with $m = 2 = \max\{Z(\bar{\mathcal{A}}), Z(\mathcal{Q}(\bar{\mathcal{A}}))\}$, the problem (3) is infeasible, despite satisfying the necessary condition. For $(\bar{\mathcal{A}}^{(2)}, \bar{\mathcal{B}})$ with $m < 3 = Z^{\text{joint}}$, the problem is feasible even though the sufficient condition fails. Thus, determining the feasibility set is difficult. The following theorem reformulates (3) into an equivalent feasible one.

Theorem 3. *For a structured system $(\bar{\mathcal{A}} \in \{0, *, ?\}^{n\times n}, \bar{\mathcal{B}} \in \{0, *, ?\}^{n\times m})$, the the structural modification problem in (3), if feasible, is equivalent to following optimization problem,*

$$\underset{\mathcal{B}\in\mathbb{B}}{\arg\min}\ c(\mathcal{B}). \tag{7}$$

Here, the feasible set $\mathbb{B}$ and the cost function $c(\cdot)$ are

$$\mathbb{B} = \left\{\mathcal{B} \in \{0, *, ?\}^{n\times m} : \mathcal{B}_{ij} \neq ?\ \forall (i,j) \text{ with } \bar{\mathcal{B}}_{ij} \neq ?\right\} \tag{8}$$

$$c(\mathcal{B}) = \operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}) + \epsilon\left(\left|W([\bar{\mathcal{A}} \ \ \mathcal{B}])\right| + \left|W([\mathcal{Q}(\bar{\mathcal{A}}) \ \ \mathcal{B}])\right|\right), \tag{9}$$

where the set $W(\cdot)$ is the output of Algorithm 1, $\mathcal{Q}(\cdot)$ is defined in (2), and the constant $\epsilon > nm$.

Proof. See Appendix II. □

The feasibility assumption in Theorem 3 holds only if the number of columns m of the input matrix is large enough as given in Proposition 1. Therefore, one can add all additional columns $n - m$ with all entries as ? to $\bar{\mathcal{B}}$ to ensure feasibility, and consequently, equivalence between (3) and (7). So, feasibility is not a strong assumption. Further, the cost function ensures that all feasible solutions satisfy $c(\mathcal{B}) < \epsilon$ (see details in Appendix I), i.e., the solution to (7) ensures structural controllability only if the corresponding cost $c(\mathcal{B}) < \epsilon$. Also, from Proposition 1, the optimal solution changes at most Z^{joint} columns, implying $c^* \leq nZ^{\text{joint}} < \epsilon$.

Moreover, the feasibility set $\mathbb{B}$ in Theorem 3 offers an interesting insight. If $\bar{\mathcal{B}}$ does not have any arbitrary entries (i.e.,? entries), then $\mathbb{B} = \{0, *\}^{n\times m}$ and the optimal solution also does not have any ? entries. Hence, our formulation and algorithms directly apply to zero/nonzero structural systems.

IV. Structural Modification Algorithms

We present two algorithms to solve the minimal structural modification, starting with a greedy approach.

A. Greedy Algorithm

To design the greedy algorithm, we note that every iteration of Algorithm 1 considers the sub-pattern matrix $\mathcal{M}^W$ of the input pattern matrix $\mathcal{M}$ restricted to the rows indexed by the set of white nodes W. The algorithm removes an element i from the set W only if the sub-pattern matrix $\mathcal{M}^W$ has a column j with one $*$ entry and zeros elsewhere. Then, the ith row of $\mathcal{B}$ corresponding to the $*$ entry in the jth column of $\mathcal{M}^W$ is removed from W. Therefore, in the next iteration, the jth column of $\mathcal{M}^W$ has all zeros and can not induce more color changes. Consequently, once a column of the input pattern matrix induces a color change, it can not induce any other color change in the subsequent iterations. Based on these observations, our greedy algorithm iteratively changes one column of the current iterate $\mathcal{B}$ that is locally optimal in each iteration. Also, once a column of $\mathcal{B}$ is changed, it is kept fixed in the subsequent iterations. So, in every iteration of the greedy algorithm, we first compute the set of the white nodes in $[\bar{\mathcal{A}}\ \ \mathcal{B}]$ and $[\mathcal{Q}(\bar{\mathcal{A}})\ \ \mathcal{B}]$ using Algorithm 1 and the previous iterate $\mathcal{B}$, i.e.,

$$I = W([\bar{\mathcal{A}}\ \ \mathcal{B}]) \cup W([\mathcal{Q}(\bar{\mathcal{A}})\ \ \mathcal{B}]). \tag{10}$$

Then, the greedy algorithm solves for the optimal column of $\mathcal{B}$ restricted to rows indexed by I,

$$(i^*, j^*) = \underset{i\in I, j\in J}{\arg\min}\ c(\tilde{\mathcal{B}}) \text{ s. t. } \tilde{\mathcal{B}}_{\tilde{i}\tilde{j}} = \begin{cases} * & \text{if} (\tilde{i},\tilde{j}) = (i,j) \\ 0 & \text{if} \tilde{j} = j,\ \tilde{i} \in I\backslash\{i\} \\ \mathcal{B}_{\tilde{i}\tilde{j}} & \text{otherwise,} \end{cases} \tag{11}$$

where J is the index of columns of $\mathcal{B}$ that are identical to the corresponding columns of $\bar{\mathcal{B}}$, i.e., unchanged in the previous iterations. Here, j^* is the column changed in the current iteration, and i^* denotes the location of $*$ entry in the jth column of $\mathcal{B}$. The greedy algorithm stops when the feasibility set of (11) is empty, i.e., $I = \emptyset$ or $J = \emptyset$. The overall greedy algorithm is summarized in Algorithm 2.

Algorithm 2 Greedy structural modification

Input: System $(\bar{\mathcal{A}} \in \{0, *, ?\}^{n\times n}, \bar{\mathcal{B}} \in \{0, *, ?\}^{n\times m})$
1: Initialize $\mathcal{B} = \bar{\mathcal{B}}$ and $J = \{1, 2, \ldots, m\}$
2: Compute I using (10)
3: **while** $I \neq \emptyset$ and $J \neq \emptyset$ **do**
4: $\quad\mathcal{B}_{i^*j^*} \leftarrow *$ and $\mathcal{B}_{\tilde{i}j^*} \leftarrow 0$ for $\tilde{i} \neq I \setminus \{i^*\}$ using (11)
5: $\quad$Update I using (10) and $J \leftarrow J \setminus \{j^*\}$
6: **end while**
Output: Modified input pattern matrix $\mathcal{B} \in \{0, *, ?\}^{n\times m}$

The greedy algorithm is simple to implement, but it does not guarantee the solution's optimality. The following proposition presents a case where the cost returned by the greedy solution can be arbitrarily larger than the optimal cost.

Proposition 2. *For any given $\gamma > 0$, there exists integers $n, m > 0$ and a structured system $(\bar{\mathcal{A}} \in \{0, *, ?\}^{n\times n}, \bar{\mathcal{B}} \in \{0, *, ?\}^{n\times m})$ such that the solution $\mathcal{B}_{\text{greedy}}$ returned by the greedy algorithm in Algorithm 2 satisfies*

$$c(\mathcal{B}_{\text{greedy}}) > \gamma \left[\min_{\mathcal{B}\in\{0,*,?\}^{n\times m}} c(\mathcal{B}) \right]. \tag{12}$$

Proof. See Appendix III. □

Since the worst-case performance of the greedy algorithm is not bounded, we propose another algorithm for structural modification that relies on MCMC.

B. Monte Carlo Markov Chain Algorithm

MCMC is a powerful stochastic optimization technique used to solve discrete optimization problems. The underlying principle of this approach is to randomly generate pattern matrices from $\mathbb{B}$ using a probability distribution and return the sample with the lowest cost. A common technique to define the probability distribution is to use the softmax function to favor the pattern matrices with smaller $c(\mathcal{B})$,

$$p_T(\mathcal{B}) = e^{-c(\mathcal{B})/T}/G, \tag{13}$$

where $G = \sum_{\mathcal{B}'\in\mathbb{B}} e^{-c(\mathcal{B}')/T}$ is the normalization constant of the distribution. As T gets smaller, for $\mathcal{B} \notin \underset{\mathcal{B}\in\mathbb{B}}{\arg\min}\ c(\mathcal{B})$,

$$\lim_{T\to 0} p_T(\mathcal{B}) = \lim_{T\to 0} \frac{e^{-c(\mathcal{B})/T}}{\sum_{\mathcal{B}'\in\mathbb{B}} e^{-c(\mathcal{B}')/T}} = 0. \tag{14}$$

Therefore, we arrive at the optimal solution when T is small. Nonetheless, computing the distribution is cumbersome because the number of candidate solutions increases exponentially with nm. To solve this problem, we build a discrete-time Markov chain (DTMC), which converges to its stationary distribution equal to the desired distribution p_T in (13). If we simulate the DTMC for a sufficiently long period for a small value of T, its state arrives at an optimal solution.

In the following, we construct a DTMC $\{\mathcal{B}(t) \in \mathbb{B}\}_{t>0}$ whose stationary distribution is $p_T(\mathcal{B})$. Since the state space is of size $|\mathbb{B}|$ from Theorem 3, the DTMC is defined by the one-step probability transition matrix $\boldsymbol{P}_T \in [0,1]^{|\mathbb{B}|\times|\mathbb{B}|}$. Let $\mathcal{B}$ be the current state. We allow only transitions to the neighboring states that differ from the current state $\mathcal{B}$ by at most one entry, i.e., the set of next states is $\mathbb{S}(\mathcal{B}) \cup \{\mathcal{B}\}$ where

$$\mathbb{S}(\mathcal{B}) = \{\mathcal{B}' \in \mathbb{B} : \text{dist}(\mathcal{B}, \mathcal{B}') = 1\}. \tag{15}$$

Therefore, we arrive at

$$\boldsymbol{P}_T(\mathcal{B}, \mathcal{B}') = 0,\ \forall\ \mathcal{B}' \notin \mathbb{S}(\mathcal{B}) \cup \{\mathcal{B}\}. \tag{16}$$

We define the matrix $\boldsymbol{P}_T$ such that any neighboring state in $\mathbb{S}(\mathcal{B})$ is equally probable. First, we choose (i, j) from $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ obeying the distribution $d(i, j)$,

$$d(i,j) = \begin{cases} 1/|\mathbb{S}(\mathcal{B})| & \text{if } \bar{\mathcal{B}}_{ij} \neq ? \\ 2/|\mathbb{S}(\mathcal{B})| & \text{if } \bar{\mathcal{B}}_{ij} = ?, \end{cases} \tag{17}$$

where $\mathbb{S}(\mathcal{B})$ is defined in (15) and its size is $|\mathbb{S}(\mathcal{B})| = nm + |\{(i,j) : \bar{\mathcal{B}}_{ij} = ?\}| < 2nm$. Then, we replace the (i,j)th entry $\mathcal{B}_{ij}$ of $\mathcal{B}$ with a sample uniformly randomly chosen from $\{0,*\} \cup \{\bar{\mathcal{B}}_{ij}\} \setminus \{\mathcal{B}_{ij}\}$, to get the next state $\mathcal{B}'$. We note that if $\bar{\mathcal{B}}_{ij} = ?$, the entry $\mathcal{B}'_{ij}$ has two choices, and when $\bar{\mathcal{B}}_{ij} \neq ?$, the entry $\mathcal{B}'_{ij}$ has only one choice. So the above process chooses $\mathcal{B}'$ uniformly at random from $\mathbb{S}(\mathcal{B}(t)) \subset \mathbb{B}$.

Further, we also assign a non-zero probability to continue in the current state. If $c(\mathcal{B}) > c(\mathcal{B}')$, the DTMC jumps to the neighboring state $\mathcal{B}'$ with a lower cost with probability one. Also, if $c(\mathcal{B}) < c(\mathcal{B}')$, the DTMC jumps to the neighboring state $\mathcal{B}'$ with a higher cost with probability $e^{[c(\mathcal{B})-c(\mathcal{B}')]/T}$. The resulting DTMC transition probabilities are as follows.

$$\boldsymbol{P}_T(\mathcal{B},\mathcal{B}') = \frac{1}{|\mathbb{S}(\mathcal{B})|}\min\{1, e^{[c(\mathcal{B})-c(\mathcal{B}')]/T}\} \;\forall\, \mathcal{B}' \in \mathbb{S}(\mathcal{B}). \tag{18}$$

Since $\sum_{\mathcal{B}'\in\mathbb{B}} \boldsymbol{P}_T(\mathcal{B},\mathcal{B}') = 1$, we deduce

$$\boldsymbol{P}_T(\mathcal{B},\mathcal{B}) = 1 - \frac{1}{|\mathbb{S}(\mathcal{B})|}\sum_{\mathcal{B}'\in\mathbb{S}(\mathcal{B})}\min\{1, e^{[c(\mathcal{B})-c(\mathcal{B}')]/T}\}, \tag{19}$$

from (16) and (18). The resulting algorithm is summarized in Algorithm 3. We note that the MCMC algorithm uses an entry-by-entry update rule, unlike the column-by-column update of the greedy algorithm. Here, we decrease T in every $R_{\max}$ iteration. The initial large T value allows the MCMC to jump between the states quickly for a flexible random search. Later, we use small T values to converge to the unique steady state distribution, as guaranteed by Theorem 4.

Algorithm 3 MCMC-based structural modification

Input: System $(\bar{\mathcal{A}} \in \{0,*,?\}^{n\times n}, \bar{\mathcal{B}} \in \{0,*,?\}^{n\times m})$
Initialize the parameters $R_{\max}, T_{\text{stop}}, \alpha < 1$
Set $\epsilon = nm+1$, $T > T_{\text{stop}}$, and $\mathcal{B} = \bar{\mathcal{B}}$
while $T_{\text{stop}} \leq T$ **do**
  for $r = 1,2,\ldots,R_{\max}$ **do**
    Set $\mathcal{B}' \leftarrow \mathcal{B}$
    Generate (i,j) from the distribution d in (17)
    Choose $\mathcal{B}'_{ij}$ from $\{0,*\}\cup\{\bar{\mathcal{B}}_{ij}\}\setminus\{\mathcal{B}_{ij}\}$ uniformly at random
    $\mathcal{B} \leftarrow \mathcal{B}'$ with probability $\min\{1, e^{[c(\mathcal{B})-c(\mathcal{B}')]/T}\}$
  end for
  $T \leftarrow T\alpha$
end while
Output: Input pattern matrix $\mathcal{B}$

Theorem 4. *For any fixed $T > 0$, the DTMC with states from $\mathbb{B}$ in* (8) *and transition probabilities given by* (16), (18), *and* (19) *admits a unique steady state distribution given by* (13).

Proof. The proof is similar to the proof of [15, Lemma 2 and Theorem 3], and hence, omitted. □

The above theorem establishes that when T approaches 0 and $R_{\max}$ goes to ∞, the pattern matrix returned by Algorithm 3 is a solution to (3) almost surely due to (14). However, in practice, $T \neq 0$, leading to a suboptimal solution, as characterized by the following result.

Proposition 3. *For any $T < \infty$, the DTMC with states from $\mathbb{B}$ in* (8) *and transition probabilities given by* (16), (18), *and* (19) *converges to an optimal solution of* (7) *with probability exceeding $\frac{B^*}{|\mathbb{B}|}$, where we define*

$$B^* = \left|\underset{\mathcal{B}\in\mathbb{B}}{\arg\min}\; c(\mathcal{B})\right| < |\mathbb{B}|\,. \tag{20}$$

Further, for any $\frac{B^*}{|\mathbb{B}|} < \delta < 1$, the DTMC arrives at an optimal solution with probability δ if

$$T < \frac{1}{\log(\delta/(1-\delta)) + \log(|\mathbb{B}|/B^* - 1)}. \tag{21}$$

Proof. See Appendix IV. □

Proposition 3 depends on B^*, the number of optimal solutions of (7), which is hard to compute. So, we choose the lower bound of $B^* \geq 1$ to compute T_{stop} for a desired error probability δ, as follows:

$$T_{\text{stop}} \leq \frac{1}{\log(1/\delta - 1) + \log(|\mathbb{B}| - 1)}. \tag{22}$$

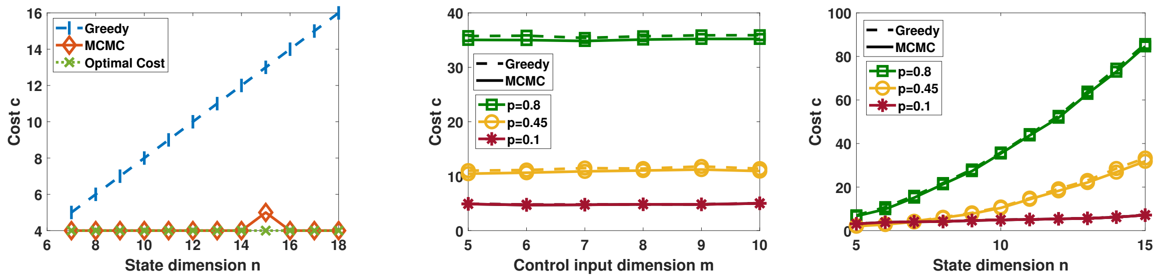
V. Numerical Results

For numerical evaluation, we choose the MCMC parameters $R_{\max} = 50000$, $T_{\text{Stop}} = 10^{-10}$, $\alpha = 10^{-1}$, and $T = 1$ as the starting value. In Figure 1, the cost is always less than $nm < \epsilon$, indicating that the resulting system is strongly structurally controllable. So, the cost is the number of changes required to make the system controllable.

For the example discussed in the proof of Proposition 2, Figure 1a shows that the greedy solution's cost is $n-2$, whereas the MCMC algorithm returns the optimal solution in most cases. Next, we look at the average performance of the algorithms (over 100 trials) when the adjacency matrix of Erdős Rényi graphs is used to generate both $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$. Figure 1b shows the algorithms' performances with $p = 0.1, 0.45$, and 0.8, where p is the probability of having an edge between any two graph nodes ($*$ entry). Also, the existence of an edge in the graph is unknown (? entry) with probability 0.1. Figure 1b indicates that the greedy and MCMC algorithms return similar solutions (mostly, the difference is less than 1), illustrating that the greedy algorithm performs well generally.

From Figure 1, the number of changes increases with the state dimension n, as expected. Further, if the system $(\bar{\mathcal{A}}, \mathcal{B}')$ is controllable for a submatrix $\mathcal{B}'$ of a matrix $\mathcal{B}$, the system $(\bar{\mathcal{A}}, \mathcal{B})$ is also controllable. So, the algorithm needs to change only a submatrix of $\bar{\mathcal{B}}$, making the required number of changes insensitive to the control dimension m (Figure 1b). We also infer that the number of changes increases with p, the probability of an $*$ entry. It is intuitive as a triangular structure of $\boldsymbol{B}$ favors strong structural controllability. So we need to make fewer changes if $\bar{\mathcal{B}}$ has a lot of zeros.

Also, the greedy algorithm runs much faster than the MCMC algorithm (details omitted due to lack of space). Nonetheless, we highlight that simplistic greedy approaches can yield arbitrarily poor solutions, as shown in Figure 1a.



(a) Example discussed in the proof of Proposition 2 where $n = m$

(b) Erdős Rényi graph with p as the probability of a $*$ entry

Fig. 1: Variation of the number of changes required by the greedy and MCMC algorithms with the state and control input dimensions. For the special case discussed in the proof of Proposition 2, the greedy algorithm performed poorly, while the MCMC algorithm returned the optimal cost in most cases. However, for Erdős Rényi graphs, the greedy algorithm performs comparably to that of the MCMC algorithm.

VI. Conclusion

We addressed the problem of making minimal changes to the input pattern matrix of a structured system to ensure strong structural controllability. We offered a greedy algorithm and an MCMC-based solution with provable guarantees. Our results open new interesting directions for future work, such as strong structural controllability under restricted structural modifications, analysis of the computational complexity, and extensions to time-varying systems.

Appendix I
Proof of Proposition 1

To prove the necessary conditions, consider Algorithm 1 with $\mathcal{M} = [\bar{\mathcal{A}} \;\; \mathcal{B}]$ as the input, for some $\mathcal{B} \in \{0, *, ?\}^{n \times m}$. In every iteration, Algorithm 1 removes an entry i from the set W if there exists a column j such that $\mathcal{M}_{ij} = *$ and $\mathcal{M}_{\tilde{i}j} = 0$ for all $\tilde{i} \in W \setminus \{i\}$. Therefore, in the next iteration, the jth column of $\mathcal{M}$ has all zeros corresponding to rows indexed by W and can not induce any more color changes. Thus, a column of the input matrix induces at most one color change. Further, by the definition of zero forcing number, Algorithm 1 does not return an empty set unless $Z(\bar{\mathcal{A}})$ columns of $\mathcal{B}$ induces a color change. Hence, if there is a feasible solution, the number of columns m of $\mathcal{B}$ satisfy $m \geq Z(\bar{\mathcal{A}})$. Using similar arguments with $\mathcal{M} = [\mathcal{Q}(\bar{\mathcal{A}}) \;\; \mathcal{B}]$, we deduce that $m \geq Z(\mathcal{Q}(\bar{\mathcal{A}})$. Hence, the minimal structural modification problem (3) is feasible only if (4) holds.

Next, to establish the sufficient condition in (5), it is enough to construct a matrix $\mathcal{B}^* \in \{0, *, ?\}^{n \times Z^{\text{joint}}}$ such that $W([\bar{\mathcal{A}} \;\; \mathcal{B}^*]) \cup W([\mathcal{Q}(\bar{\mathcal{A}}) \;\; \mathcal{B}^*]) = \emptyset$. Let V be a minimal zero forcing set of both $\bar{\mathcal{A}}$ and $\mathcal{Q}(\bar{\mathcal{A}})$ and $\mathcal{B}^*$ be a submatrix of the $n \times n$ diagonal matrix with $*$ along the diagonal, formed by Z^{joint} columns indexed by V. Then, the first iteration of Algorithm 1 with $[\bar{\mathcal{A}} \;\; \mathcal{B}^*]$ as input removes V from W. So we deduce that $W([\bar{\mathcal{A}} \;\; \mathcal{B}^*])$ is the same as $W(\bar{\mathcal{A}})$ when initialization in Step 2 of Algorithm 1 is changed to $W \leftarrow \{1, 2, \ldots, p\} \setminus V$. Further, from Definition 1, we conclude that $W([\bar{\mathcal{A}} \;\; \mathcal{B}^*]) = \emptyset$. Similarly, we can show that $W([\mathcal{Q}(\bar{\mathcal{A}}) \;\; \mathcal{B}^*]) = \emptyset$. Therefore, if $m \geq Z^{\text{joint}}$, (3) is feasible for any system $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$.

Appendix II
Proof of Theorem 3

We prove the equivalence of (3) and (7) in two steps under the assumption that (3) is feasible. Using (9), we first show that the following optimization is equivalent to (3),

$$\underset{\mathcal{B} \in \{0,*,?\}^{n \times m}}{\arg\min} \; c(\mathcal{B}). \tag{23}$$

In the second step, we show that, if $\mathcal{B} \notin \mathbb{B}$, then $\mathcal{B}$ is not an optimal solution of (23). This step establishes that (23) is equivalent to (7), and the proof is complete.

We start with the first step. Let $w(\mathcal{B}) = |W([\bar{\mathcal{A}} \;\; \mathcal{B}])| + |W([\mathcal{Q}(\bar{\mathcal{A}}) \;\; \mathcal{B}])|$, for any $\mathcal{B} \in \{0, *, ?\}^{n \times m}$. Then, $c(\mathcal{B}) = \operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}) + \epsilon w(\mathcal{B})$. Since $w(\mathcal{B}) \geq 1$ when $w(\mathcal{B}) \neq 0$ and $\epsilon > nm$, we have

$$\min_{\mathcal{B}: w(\mathcal{B}) \neq 0} c(\mathcal{B}) > \min_{\mathcal{B}: w(\mathcal{B}) \neq 0} \operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}) + nm \tag{24}$$

$$\geq nm \geq \min_{\mathcal{B}: w(\mathcal{B}) = 0} \operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}) \tag{25}$$

$$\geq \min_{\mathcal{B}: w(\mathcal{B}) = 0} c(\mathcal{B}), \tag{26}$$

where (25) follows because $0 \leq \operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}) \leq nm$, for any $\mathcal{B}, \bar{\mathcal{B}} \in \{0, *, ?\}$. Therefore, we obtain

$$\underset{\mathcal{B} \in \{0,*,?\}^{n \times m}}{\arg\min} \; c(\mathcal{B}) = \underset{\mathcal{B}: w(\mathcal{B}) = 0}{\arg\min} \; c(\mathcal{B}) = \underset{\mathcal{B}: w(\mathcal{B}) = 0}{\arg\min} \; \operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}). \tag{27}$$

Here, $w(\mathcal{B}) = 0$ if and only if $W([\bar{\mathcal{A}} \;\; \mathcal{B}]) \cup W([\mathcal{Q}(\bar{\mathcal{A}}) \;\; \mathcal{B}])$ is an empty set. Hence, problems (3) and (23) are equivalent.

Now, we present the second step using a proof by negation. Suppose there exists $\tilde{\mathcal{B}} \notin \mathbb{B}$ such that $\tilde{\mathcal{B}}$ is a solution to (23), i.e., there exists (i, j) for which $\tilde{\mathcal{B}}_{ij} = ?$ and $\bar{\mathcal{B}}_{ij} \neq ?$. Due to the equivalence of (3) and (23), $\tilde{\mathcal{B}}$ also belongs to the solution set of (3). Therefore, $(\bar{\mathcal{A}}, \tilde{\mathcal{B}})$ is strongly structurally controllable. Further, let pattern matrices $\tilde{\mathcal{B}}^{(0)}, \tilde{\mathcal{B}}^{(*)} \in \{0, *, ?\}^{n \times m}$ be such that they are identical to $\tilde{\mathcal{B}}$ except that $\tilde{\mathcal{B}}_{ij}^{(0)} = 0$ and $\tilde{\mathcal{B}}_{ij}^{(*)} = *$. However, from (1) we note that

$$\mathbb{P}(\tilde{\mathcal{B}}) = \mathbb{P}(\tilde{\mathcal{B}}^{(0)}) \cup \mathbb{P}(\tilde{\mathcal{B}}^{(*)}). \tag{28}$$

Hence, we see that $(\bar{\mathcal{A}}, \tilde{\mathcal{B}}^{(0)})$ and $(\bar{\mathcal{A}}, \tilde{\mathcal{B}}^{(*)})$ are strongly structurally controllable. However, since either $\tilde{\mathcal{B}}_{ij}^{(0)}$ or $\tilde{\mathcal{B}}_{ij}^{(*)}$ is the same as $\bar{\mathcal{B}}_{ij} \neq ?$ and $\tilde{\mathcal{B}}_{ij} \neq \bar{\mathcal{B}}_{ij}$, we arrive at

$$c(\tilde{\mathcal{B}}) = \operatorname{dist}(\tilde{\mathcal{B}}, \bar{\mathcal{B}}) > \min\left\{c(\tilde{\mathcal{B}}^{(0)}), c(\tilde{\mathcal{B}}^{(*)})\right\}. \quad (29)$$

Thus, the assumption that $\tilde{\mathcal{B}}$ minimizes the cost c does not hold, and the proof is complete.

Appendix III
Proof of Proposition 2

Consider a system $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ where $\bar{\mathcal{B}} = \mathbf{0} \in \{0, *, ?\}^{n \times n}$ with $n \geq 6$ and $\bar{\mathcal{A}} \in \{0, *, ?\}^{n \times n}$ is defined as follows:

$$\bar{\mathcal{A}}_{ij} = \begin{cases} ? & \text{if } i = j \\ 0 & \text{if } 1 \leq i, j \leq n-3 \text{ and } |i-j| > 1 \\ 0 & \text{if } n-3 \leq i, j \leq n \text{ and } |i-j| > 1 \\ * & \text{otherwise.} \end{cases} \quad (30)$$

Here, $\mathcal{Q}(\bar{\mathcal{A}}) = \bar{\mathcal{A}}$, and $c(\mathcal{B}) = \operatorname{dist}(\mathcal{B}, \bar{\mathcal{B}}) + 2\epsilon \left|W([\bar{\mathcal{A}} \;\; \mathcal{B}])\right|$. Let $\mathcal{B}^*$ be a submatrix of the diagonal matrix with $*$ along the diagonal, formed by columns indexed by $\{1, n-2, n-1, n\}$. Then, $\left|W([\bar{\mathcal{A}} \;\; \mathcal{B}^*])\right| = 0$, making $\mathcal{B}^*$ feasible, and we get

$$\min_{\mathcal{B} \in \{0,*,?\}^{n \times m}} c(\mathcal{B}) \leq c(\mathcal{B}^*) = 4. \quad (31)$$

Further, if we apply a greedy algorithm, in the first iteration, changing any entry of $\mathcal{B}$ to $*$ returns the cost $1 + 2\epsilon(n-1)$. Let the algorithm changes $\mathcal{B}_{11}$ to $*$. In the next iteration, changing any entry of $\mathcal{B}$, except from the first row and first column to $*$, reduces the cost to $c(\mathcal{B}) = 2 + 2\epsilon(n-2)$. Assume that the algorithm changes $\mathcal{B}_{22}$ to $*$. Similarly, after $n-2$ iterations, the algorithm terminates with $\mathcal{B}_{\text{greedy}}$ such that $[\mathcal{B}_{\text{greedy}}]_{ii} = *$ for $i = 1, 2, \ldots, n-2$ and zeros elsewhere. Thus, the greedy solution's cost is

$$c(\mathcal{B}_{\text{greedy}}) = n - 2 \geq \frac{n-2}{4}\left[\min_{\mathcal{B} \in \{0,*,?\}^{n \times m}} c(\mathcal{B})\right], \quad (32)$$

where we use (31). So, for any $\gamma > 0$, if we choose $n > 4\gamma + 2$, the lower bound (12) holds.

Appendix IV
Proof of Proposition 3

Let the optimal cost of the minimal structural modification problem in (7) be $c^* = \min_{\mathcal{B} \in \mathbb{B}} c(\mathcal{B})$. Then, for any $T > 0$, from the DTMC distribution in (13), the probability of the DTMC converging to an optimal solution is $B^* e^{-c^*/T}/G$. Since $c(\mathcal{B}') \geq c^* + 1$ for all $\mathcal{B}' \notin \arg\min_{\mathcal{B} \in \mathbb{B}} c(\mathcal{B})$, we have

$$B^* e^{-c^*/T}/G \geq \frac{B^* e^{-c^*/T}}{B^* e^{-c^*/T} + (|\mathbb{B}| - B^*) e^{-(c^*+1)/T}} \quad (33)$$

$$= \frac{B^*}{B^* + (|\mathbb{B}| - B^*) e^{-1/T}} > \frac{B^*}{|\mathbb{B}|}. \quad (34)$$

Therefore, we prove the first part of the result. Furthermore, the error probability exceeds $1 - \delta$ if $\frac{B^*}{B^* + (|\mathbb{B}| - B^*) e^{-1/T}} > 1 - \delta$. Rearranging this relation, we arrive at (21).

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