Clebsch-Gordan Coefficients

A Quantum Mechanical and Mathematical Perspective

by

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1 Introduction

Every physics undergraduate's first encounter with Clebsch-Gordan coefficients will be in their quantum mechanics course. These numbers are usually presented in dense tables that come out of nowhere. Where this table comes from and how these coefficients are calculated is deferred to more advanced quantum mechanics courses. This often means that the student is left puzzled as to what these numbers mean. As David J. Griffiths notes in his 'Introduction to Quantum Mechanics' book: "If you think this is starting to sound like mystical numerology, I don't blame you. [...] In a mathematical sense this is all applied group theory - what we are talking about is the decomposition of the direct product of two irreducible representations of the rotation group into a direct sum of irreducible representations."[1] Even though this explanation is correct, a physics undergraduate will not be any wiser for it, as representation theory is graduate mathematics. The aim of this project is to explain, in a way understandable for an undergraduate in physics and mathematics, from both a quantum mechanical perspective as well as a mathematical perspective what Clebsch-Gordan coefficients are, why they are useful, how they are used, and how they are calculated. The main focus lies with the quantum mechanical perspective of these coefficients. From this perspective, we will build from first principles the necessary knowledge to understand and calculate the Clebsch-Gordan coefficients. To do this we will discuss the necessary postulates of quantum mechanics, which we will need to discuss quantum angular momentum. Then combining quantum systems is discussed, which is followed by the definition, use, and calculation of the Clebsch-Gordan coefficients. From the mathematical perspective we will discuss the underpinnings of the coefficients, which will mostly entail a discussion about unitary representations, direct sums and tensor products of representations, and how they decompose into irreducible representations. Special care is given to the spherical harmonics and rotations.

2 Quantum Mechanics

2.1 A Brief Overview

In the 1930s, after more than 30 years of being on shaky grounds, quantum mechanics got a solid mathematical foundation. This is due to the work of, among others, Paul Dirac, Erwin Schrödinger, Werner Heisenberg, and John von Neumann. In this thesis we will use the Dirac-Von Neumann formulation of quantum mechanics in order to work from first principles towards the Clebsch-Gordan coefficients and their use in quantum mechanics. Afterwards we will take a brief look at uses of Clebsch-Gordan coefficients in mathematics. So without any further ado, let us get started by explaining the Dirac-Von Neumann formulation of quantum mechanics, as well as the notation used.

We use the Bra-Ket notation, introduced by Dirac. In this notation, vectors in a Hilbert space are written as $|\psi\rangle$, which is called a *ket*. An element of the dual space is called a *bra*, and written as $\langle\phi|$. In a Hilbert space,

$$|\psi\rangle = \langle\psi|^\dagger \qquad \text{and} \qquad \langle\psi| = |\psi\rangle^\dagger ,$$

where † denotes the hermitian conjugate. In finite dimensional Hilbert spaces, bras and kets become row and column vectors, so it is intuivitively easiest to think of $|\psi\rangle$ as a column vector, and $\langle\phi|$ as a row vector. The inner product between two kets $|\phi\rangle$ and $|\psi\rangle$ then becomes

$$(|\phi\rangle , |\psi\rangle) = |\phi\rangle^\dagger (|\psi\rangle) = \langle\phi| (|\psi\rangle) = \langle\phi|\psi\rangle .$$

An operator A working on a vector $|\psi\rangle$ can be written in two ways: $A|\psi\rangle$ or $|A\psi\rangle$. These mean the same thing. With $A^\dagger$ we mean the Hermitian adjoint of operator A, which is the operator such that

$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|A\psi\rangle .$$

The Dirac-Von Neumann formulation has a few important postulates, we will now discuss the postulates important to this thesis.

Postulate 2.1. For each quantum system, there is an associated (complex) Hilbert space with inner product $\langle\psi|\phi\rangle$. For each state that the quantum system can be in, there is an asociated one dimensionial subspace of said Hilbert space.

These one dimensional subspaces are usually denoted by the equivalence classes of vectors with unit length. This means that two unit length vectors correspond with the same state if they differ only by a phase factor. Often, the Hilbert space associated with a system is the space of square integrable (complex) functions on some set (such as $\mathbb{R}$). The inner product associated with such a Hilbert space is

$$\langle f|g\rangle = \int f^* g dr, \tag{1}$$

where f^* denotes the complex conjugate of f.

Since the states of a system can be identified by the vectors of length 1, we can get a normalization condition:

$$\int |\psi|^2 dr = 1. \tag{2}$$

Postulate 2.2. For each observable of a quantum system there is an associated self-adjoint operator on the Hilbert space associated with the quantum system.

Postulate 2.3. The eigenstates of an observable operator are complete, i.e. they span the Hilbert space.

Recall that an operator A is *self-adjoint*, or *Hermitian* if $A = A^\dagger$. With an observable we mean a physically observable quantity, such as position, momentum, or energy. When we measure such an observable A of a quantum system in state $|\Psi\rangle$, something strange happens. The state collapses to an eigenstate of the observable. The eigenvalue associated with that eigenstate is the measured value of the observable. Measuring complex values of real world observables is impossible, so we would like the measured values to be real. This is the case because eigenvalues of self-adjoint operators are real. More precisely: because of postulate 2.3, we can write $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_i c_i |\phi_i\rangle \tag{3}$$

where $|\phi_i\rangle$ are the eigenstates of observable A.[1] These eigenstates are not necessarily orthogonal, but can be made orthogonal via the Gram-Schmidt process. When measuring A, the state $|\Psi\rangle$ collapses to (i.e. becomes) the state $|\phi_k\rangle$ with probability $c_k^* c_k$.[2]This means we have a normalization condition on the c_i:

$$\sum_i |c_i|^2 = 1 \tag{4}$$

Even though, as postulate 2.3 states, eigenstates of observables are complete, it is important to note that the different eigenstates do not necessarily have different eigenvalues, i.e. the eigenspace of an eigenvalue need not be one dimensional. When the dimension of an eigenspace is higher than 1, we call the space *degenerate*. The *degeneracy* of an eigenspace is the same as its dimension. We would very much like the eigenspaces of eigenvalues to be one dimensional though, because then for each eigenvalue we know that there is exactly one state associated with it. This allows us to specify a basis for the Hilbert space of the system using the eigenvalues only. The theory of commutators gives us a very handy way to make this possible in the case of degeneracy.

[1]Equation (3) only holds when the eigenbasis of A is finite or countable. When it is uncountable, the sum becomes an integral. This is however not of importance in this project as all the eigenspaces will be finite or countable.

[2]This follows from the postulate about the wave equation which is of no further interest to us.

We denote the commutator of two observables A and B as $[A, B]$, with

$$[A, B] = AB - BA.$$

When $[A, B] = 0$, we say that A and B commute. Physically it means that the order of measuring A and B on a quantum state $|\psi\rangle$ is irrelevant, the result will be the same. We can see this, because when A and B commute, $[A, B] = 0$, so $AB = BA$. This means that when we measure state $|\psi\rangle$ we have

$$AB\,|\psi\rangle = BA\,|\psi\rangle\,.$$

The left hand side measures B first and then A, while the right hand side measures A first and then B. The following theorem shows why we can use commuting observables to specify a basis for the Hilbert space of the system. We prove this for the finite dimensional case. For the infinite dimensional case we need more assumptions, such as postulate 2.3 For more information, see [2].

Theorem 2.4. *Observables A and B working on a finite Hilbert space H are compatible, meaning that A and B share an eigenbasis, if and only if A and B commute.*

Proof. "⇒" Assume $\{|\phi_n\rangle\}$ is an eigenbasis for A and B with eigenvalues a_i and b_i for $|\phi_i\rangle$ respectively. Let $|\psi\rangle \in H$. Then we have

$$\begin{aligned}
AB\,|\psi\rangle &= AB \sum_n c_n\,|\phi_n\rangle \\
&= \sum_n c_n AB\,|\phi_n\rangle \\
&= \sum_n c_n a_n b_n\,|\phi_n\rangle \\
&= \sum_n c_n b_n a_n\,|\phi_n\rangle \\
&= \sum_n c_n Ba_n\,|\phi_n\rangle \\
&= \sum_n c_n BA\,|\phi_n\rangle = BA\,|\psi\rangle\,.
\end{aligned}$$

Since this holds for any $|\psi\rangle$, A and B commute.
"⇐" Assume A and B commute, and let $|\alpha\rangle$ be an eigenstate of A with eigenvalue a. We then have

$$A(B\,|\alpha\rangle) = B(A\,|\alpha\rangle) = B(a\,|\alpha\rangle) = a(B\,|\alpha\rangle)$$

This shows that $B\,|\alpha\rangle$ must lie in the eigenspace of a. This eigenspace is a subspace of H, and therefore also a Hilbert space. Since B is Hermitian, it is diagonalizable in this eigenspace, that is, there is an eigendecomposition of B of this space. We can therefore build an eigenbasis for both A and B by splitting the eigenspaces of degenerate eigenvalues of A into eigenstates of B. □

When two observables A and B commute, they share an eigenbasis $|\psi_n\rangle$, that, according to postulate 2.3 spans the Hilbert space. This means that

$$A|\psi_n\rangle = a_n|\psi_n\rangle \qquad \text{and} \qquad B|\psi_n\rangle = b_n|\psi_n\rangle,$$

with a_n and b_n eigenvalues corresponding to $|\psi_n\rangle$. As the second part of the proof of theorem 2.4 shows, if the commuting observables are chosen in a clever way, they can reduce the degeneracy of eigenspaces of eigenvalues, because the eigenspaces are split up into smaller eigenspaces by the addition of a commuting observable. Each eigenvector then has an extra eigenvalues corresponding to the added observable. In other words, if we specify that a state has eigenvalues a_j and b_k, we take the intersection of the eigenspaces of a_j and b_k. We can keep adding commuting operators until we get eigenspaces that are one dimensional. When this is the case, we call such a set of observables a *complete set of commuting observables* (*CSCO*). There can be multiple CSCOs for a given quantum system.

2.2 Angular Momentum

The Clebsch-Gordon (C-G) coefficients arise in quantum mechanics when we add the angular momentum of two individual particles and look at the total angular momentum. Let us therefore discuss the theory of quantum angular momentum in more detail.

In classical mechanics, the angular momentum is defined as

$$\mathbf{L_c} = \mathbf{r} \times \mathbf{p_c} = m(\mathbf{r} \times \nabla), \tag{5}$$

Where $\mathbf{L_c}$ is the angular momentum vector, $\mathbf{r}$ is the position vector, $\mathbf{p_c}$ is the momentum vector, and m is the mass. In other words, this says (in cartesian coordinates):

$$L_{x,c} = yp_{z,c} - zp_{y,c}, \qquad L_{y,c} = zp_{x,c} - xp_{z,c}, \qquad L_{z,c} = xp_{y,c} - yp_{x,c}. \tag{6}$$

When we define the quantum version of the angular momentum, we must replace the classical position and momentum operators with the quantum version of the position and momentum operators. The quantum version of the position operator is the same as the classical position operator. The only difference between the quantum version of the classical momentum $\mathbf{p_c}$ and the quantum momentum $\mathbf{p}$ is that the mass m is replaced by $-i\hbar$, where $\hbar$ is the reduced Planck constant. We therefore get

$$L_x = yp_z - zp_y, \qquad L_y = zp_x - xp_z, \qquad L_z = xp_y - yp_x, \tag{7}$$

or, more succinctly,

$$\mathbf{L} = -i\hbar(\mathbf{r} \times \nabla). \tag{8}$$

The Hilbert space associated with the quantum angular momentum operators is the square integrable functions on the unit sphere, $\boldsymbol{L}^2(S^2)$, where S^2 denotes the sphere and $\boldsymbol{L}^2(S^2)$ denotes the square integrable functions on S^2.

It should be intuitively clear that knowing only L_x, L_y or L_z individually does not fully specify a quantum system. After all, when for instance, L_z is known, we know nothing about L_y or L_z, and thus we cannot describe a system fully by only one of the three observables. Mathematically this means that the eigenspaces of the eigenvalues corresponding of the angular momentum operators have more than one dimension. In order to describe systems concretely, we shall search for a complete set of commuting operators. It is natural to try $\{L_x, L_y, L_z\}$ as a CSCO. We must first check whether L_x and L_y commute. First, we give the fundamental commutator relations for the position and momentum vectors:

$$[k, l] = 0, \qquad [p_k, p_l] = 0, \qquad [k, p_l] = i\hbar\delta_{kl}, \tag{9}$$

where $k, l \in \{x, y, z\}$ and δ_{kl} is the Kronecker delta. Now we need a short lemma to calculate the commutator of L_x and L_y.

Lemma 2.5. *For all linear operators A, B, and C that work on the same Hilbert space,*

$$[A, B + C] = [A, B] + [A, C].$$

Proof. Linear operators are distributive with respect to addition:

$$A(B + C) = AB + BC.$$

This means

$$\begin{aligned}
[A, B + C] &= A(B + C) - (B + C)A \\
&= AB + AC - BA - CA \\
&= (AB - BA) + (AC - CA) \\
&= [A, B] + [A, C].
\end{aligned}$$

□

Since all the operators used in quantum mechanics are linear, we can use this result, to calculate the commutator of L_x and L_y:

$$\begin{aligned}
[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\
&= [yp_z, zp_x] - [zp_y, zp_x] - [yp_z, xp_z] + [zp_y, xp_z] \\
&= [yp_z, zp_x] + [zp_y, xp_z] \\
&= yp_x[p_z, z] + xp_y[z, p_z] \\
&= i\hbar(xp_y - yp_x) \\
&= i\hbar L_z
\end{aligned}$$

By the cyclic permutation of (x, y, z) we find

$$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y. \tag{10}$$

Clearly, L_x, L_y, and L_z are incompatible, and thus do not have simultaneous eigenfunctions, which is what we are after. We must try something else.

It turns out that the square of the magnitude of the angular momentum

$$L^2 \coloneqq L_x^2 + L_y^2 + L_z^2 \tag{11}$$

does commute with L_x:

$$\begin{aligned}
[L^2, L_x] &= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\
&= L_x[L_x, L_x] + [L_x, L_x]L_x + L_y[L_y, L_x] + [L_y, L_x]L_y \\
&\qquad + L_z[L_z, L_x] + [L_z, L_x]L_z \\
&= L_y(-i\hbar L_Z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_y)L_z \\
&= 0.
\end{aligned}$$

Here we used the commutator relation $[AB, C] = A[B, C] + [A, C]B$ and the fact that every operator commutes with itself. We can repeat the same argument to show that L^2 also commutes with L_y and L_z. This means that we can try to find simultaneous eigenstates and corresponding eigenvalues of L^2 and L_z (we could also pick L_x or L_y instead of L_z, but convention tells us to pick L_z). In order to find the eigenvalues of these observables, we introduce the ladder operator $L_\pm$ as

$$L_\pm \coloneqq L_x \pm iL_y. \tag{12}$$

It clearly commutes with L^2 and the commutator with L_z is

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar(L_x \pm iL_y). \tag{13}$$

Thus, $[L_z, L_\pm] = \pm\hbar L_\pm$ and $[L^2, L_\pm] = 0$.

Now we are ready to find the eigenvalues of the simultaneous eigenstates of L^2 and L_z. Assume f is an eigenfunction of both L^2 and L_z:

$$L^2 f = \lambda f \qquad\qquad L_z f = \mu f. \tag{14}$$

The following theorem describes the importance of the ladder operator.

Theorem 2.6. *If f is a simultaneous eigenfuncction of L^2 and L_z, then $L_\pm f = 0$ or $L_\pm f$ an eigenfunction of L^2 and L_z as well.*

Proof. Let $L^2 f = \lambda f$. Since L^2 and $L_\pm$ commute, we find

$$L^2(L_\pm f) = L_\pm(L^2 f) = L_\pm \lambda f = \lambda L_\pm f.$$

For L_z we have $L_z f = \mu f$, so we get

$$\begin{aligned}
L_z(L_\pm f) &= [L_z, L_\pm]f + L_\pm L_z f \\
&= \pm\hbar L_\pm f + L_\pm \mu f \\
&= \pm\hbar L_\pm f + \mu L_\pm f \\
&= (\mu \pm \hbar)L_\pm f
\end{aligned}$$

This shows that, when $L_\pm f$ is normalizable, it is an eigenfunction of both L^2 and L_z. When $L_\pm f$ is not normalizable, its norm is either 0 or infinite, but because $L_\pm \in \mathcal{L}(\boldsymbol{L}^2(S^2))$ and $f \in \boldsymbol{L^2}(S^2)$, we must have $L_\pm f \in \boldsymbol{L}^2(S^2)$, and therefore its norm cannot be infinite. □

Note that the new eigenvalue for $L_\pm f$ on L_z is $\mu \pm \hbar$, whereas the eigenvalue on L^2 stays the same. We call L_+ the raising operator because it raises μ with $\hbar$. Likewise, we call L_- the lowering operator because it lowers μ with $\hbar$. So for a given eigenvalue λ for L^2, we get a 'ladder' of eigenvalues for L_z. It seems like we can do this indefinetely, but that cannot be, otherwise the angular momentum in the z-direction L_z becomes larger than the total angular momentum L^2! This implies that there must be a 'top rung' f_t such that $L_+ f_t$ is not normalizable, which means that its norm is zero or infinite. However, since our Hilbert space consists of square integrable functions, and L_+ works on this Hilbert space, it cannot send f_t outside the Hilbert space, so its norm cannot be infinite. This means that for our top rung we have

$$L_+ f_t = 0 \tag{15}$$

Let $\hbar l$ be the eigenvalue of L_z at this top rung f_t, and let λ be the eigenvalue of L^2:

$$L_z f_t = \hbar l f_t \qquad\qquad L^2 f_t = \lambda f_t \tag{16}$$

Note that l can be any real number.

In order to continue, we must first prove an important lemma.

Lemma 2.7. $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$

Proof. Look at $L_\pm L_\mp$:

$$\begin{aligned} L_\pm L_\mp &= (L_x \pm iL_y)(L_x \mp iL_y) \\ &= L_x^2 + L_y^2 \mp iL_xL_y \pm iL_yL_x \\ &= L_x^2 + L_y^2 \mp i(L_xL_y - L_yL_x) \\ &= L^2 - L_z^2 \mp i(i\hbar L_z) \\ &= L^2 - L_z^2 \pm \hbar L_z. \end{aligned}$$

Isolating L^2 we get $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$ □

Using this lemma, we shall apply L^2 to f_t:

$$\begin{aligned} L^2 f_t &= (L_- L_+ + L_z^2 + \hbar L_z) f_t \\ &= L_- L_+ f_t + L_z^2 f_t + \hbar L_z f_t \\ &= L_- 0 + \hbar^2 l^2 + \hbar^2 l \\ &= 0 + \hbar^2 l^2 + \hbar^2 l = \hbar^2 l(l+1), \end{aligned}$$

and therefore

$$\lambda = \hbar^2 l(l+1). \tag{17}$$

For the same reason that there is a top rung, there must also be a bottom rung f_b such that

$$L_- f_b = 0. \tag{18}$$

Let $\hbar\hat{l}$ be the eigenvalue of L_z at the bottom rung, with λ still the eigenvalue for L^2:

$$L_z f_b = \hbar\hat{l} f_b; \qquad L^2 f_b = \lambda f_b. \tag{19}$$

Using Lemma 2.7 again we get

$$\begin{aligned} L^2 f_b &= (L_+ L_- + L_z^2 - \hbar L_z) f_b \\ &= L_+ L_- f_b + L_z^2 f_b - \hbar L_z f_b \\ &= 0 + \hbar^2 \hat{l}^2 - \hbar^2 \hat{l} \\ &= \hbar^2 \hat{l}(\hat{l} - 1), \end{aligned}$$

and therefore

$$\lambda = \hbar^2 \hat{l}(\hat{l} - 1). \tag{20}$$

Combining (17) and (20) gives us $l(l+1) = \hat{l}(\hat{l} - 1)$. This means that $\hat{l} = l + 1$, or $\hat{l} = -l$. The former implies that the bottom rung is higher than the top rung, which is impossible. This means that the latter is true: $\hat{l} = -l$. For a certain λ, the eigenvalues for L_z range from $-\hbar l$ to $\hbar l$ in steps of $\hbar$. In other words, the eigenvalues are $\hbar m$, where m ranges from $-l$ to l in N integer steps. This implies that $l = -l + N$, or $l = N/2$, so l must be an integer or a half-integer. The numbers l and m characterize the eigenfunctions:

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \qquad L_z f_l^m = \hbar m f_l^m \tag{21}$$

Let us now take a look at what happens when we want to normalize the eigenfunctions acquired through using the ladder operators $L_\pm$. Say we have a normalized state $|f_l^m\rangle$, then we have $L_\pm |f_l^m\rangle = A_l^m f_l^{m\pm1}$, where A_l^m is the normalization constant. In order to find this constant, we prove a short lemma first.

Lemma 2.8. *The Hermitian conjugate of L_+ is L_- and the Hermitian conjugate of L_- is L_+, or more succinctly: $L_\pm^\dagger = L_\mp$*

Proof. For any $\phi, \psi \in \mathbf{L^2}(S^2)$ we have

$$\begin{aligned} \langle \phi \mid L_\pm \psi \rangle &= \langle \phi \mid (L_x \pm i L_y)\psi \rangle \\ &= \langle \phi \mid L_x \psi \rangle \pm i \langle \phi \mid L_y \psi \rangle \\ &= \langle L_x \phi \mid \psi \rangle \pm i \langle L_y \phi \mid \psi \rangle \\ &= \langle (L_x \mp i L_y)\phi \mid \psi \rangle \\ &= \langle L_\mp \phi \mid \psi \rangle, \end{aligned}$$

where we used the fact that L_x and L_y are physical observables, and therefore self-adjoint. We now see that $\langle \phi \mid L_\pm \psi \rangle = \langle L_\mp \phi \mid \psi \rangle$, which is to say that $L_\pm^\dagger = L_\mp$. □

Now we can find the normalization constant A_l^m. Notice that

$$\begin{aligned}
\langle f_l^m \mid L_\mp L_\pm f_l^m \rangle &= \langle f_l^m \mid L_\mp (L_\pm f_l^m) \rangle \\
&= \langle (L_\mp)^\dagger f_l^m \mid L_\pm f_l^m \rangle \\
&= \langle L_\pm f_l^m \mid L_\pm f_l^m \rangle \\
&= \langle A_l^m f_l^{m\pm 1} \mid A_l^m f_l^{m\pm 1} \rangle \\
&= |A_l^m|^2 \langle f_l^{m\pm 1} \mid f_l^{m\pm 1} \rangle \\
&= |A_l^m|^2.
\end{aligned}$$

Using lemma 2.7, we see that $L_\mp L_\pm = L^2 - L_z^2 \mp \hbar L_z$, so we also have

$$\begin{aligned}
\langle f_l^m \mid L_\mp L_\pm f_l^m \rangle &= \langle f_l^m \mid (L^2 - L_z^2 \mp \hbar L_z) f_l^m \rangle \\
&= \langle f_l^m \mid (\hbar^2 l(l+1) - (\hbar m)^2 \mp \hbar^2 m) f_l^m \rangle \\
&= (\hbar^2 l(l+1) - (\hbar m)^2 \mp \hbar^2 m) \langle f_l^m \mid f_l^m \rangle \\
&= \hbar^2 (l(l+1) - m^2 \mp m) \\
&= \hbar^2 (l(l+1) - m(m \pm 1)).
\end{aligned}$$

Combining these results we get $A_l^m = c\hbar\sqrt{l(l+1) - m(m \pm 1)}$, where c is a complex phase factor. This phase factor is not physically relevant, so by convention we set it to 1. Therefore, as a final result we have:

$$A_l^m = \hbar\sqrt{l(l+1) - m(m \pm 1)}. \tag{22}$$

It is important to realise that we have managed to find the eigenvalues of the angular momentum operators using solely algebraic means. The only information necessary is the commutation relations of equation (10).

These commutation relations will only take us so far however. If we want to find out whether L^2 and L_z form a CSCO we will need the eigenfunctions f_l^m of these operators as well. In order to do this, we must solve the simultaneous eigenvalue equation (21). Because the Hilbert space we are working in is $\boldsymbol{L}^2(S^2)$, it is natural to work in spherical coordinates. Using equation (8), we can find the expressions of L_x, L_y, and L_z in spherical coordinates. In turn, we can find an expression for $L_\pm$, and using lemma 2.7, we find an expression for L^2. We find

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \tag{23}$$

and

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right], \tag{24}$$

where θ and ϕ are the polar and azimuthal angles respectively. Plugging this into our eigenvalue equation (21) we get

$$-\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] f_l^m = \hbar^2 l(l+1) f_l^m \tag{25}$$

and

$$\frac{\hbar}{i} \frac{\partial}{\partial\phi} f_l^m = \hbar m f_l^m. \tag{26}$$

The solutions can be found using separation of variables, however we will not do that here. The solutions, as it turns out, are the spherical harmonics Y_l^m. These functions are defined as follows:

$$Y_l^m(\theta,\phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{l+|m|)!}} e^{im\phi} P_l^m(cos\theta), \tag{27}$$

where $\epsilon = (-1)^m$ for $m \geq 0$ and $\epsilon = 1$ for $m < 0$. The function P_l^m is the associated Legendre polynomial, defined as

$$P_l^m(x) = (1-x^2)^{|m|/2} \left(\frac{d}{dx} \right)^{|m|} P_l(x),$$

where $p_l(x)$ is the Legendre polynomial, defined by the Rodrigues formula:

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2-1)^l.$$

These functions become complicated very quickly, but they are explicit by nature. There are more ways to define the spherical harmonics, but this definition satisfies equations (25) and (26). It is known that the spherical harmonics span the space of the square integrable functions on the unit sphere. Also, each combination of m and l specify exactly one spherical harmonic function. This means that the operators L^2 and L_z form a complete set of commuting operators. By writing the unique eigenstate associated with Y_l^m as $|l\, m\rangle$, we can now easily write any quantum state as a weighted, normalized sum of the eigenstates $|l\, m\rangle$:

$$|\psi\rangle = \sum_i c_i \, |l\, m\rangle \tag{28}$$

Often the eigenstates are written as $|l\, m\rangle$, or $|s\, m\rangle$. Usually this means that respectively only the orbital angular momentum or the spin angular momentum are considered. The functions are the same spherical harmonics, the only difference is that l can only be integer and s can only be half integer. A third notation is $|j\, m\rangle$, where j can be both integer or half integer. From here on out we shall only look at $|l\, m\rangle$.

2.3 Combining Quantum Systems

It often happens that we have two quantum systems with their own Hilbert spaces, and we want to combine them into one system. Consider for instance putting two hydrogen atoms next to each other to calculate how they interact. In order to work with composite systems, there is another postulate.

Postulate 2.9. The Hilbert space of a composite system is the tensor product of the constituent Hilbert systems.

A tensor product is a very general and abstract concept. In this project we will only use it in the context of Hilbert spaces and operators. Let us define it as such.[3]

Definition 2.10. Let U and V be complex valued vector spaces.

- The *free vector space* $F(U \times V)$ over $U \times V$ is the vector space that has as basis the elements of the Cartesian product $U \times V$[4].
- The *tensor product* $U \otimes V$ of U and V is the quotient vector space

$$U \otimes V = F(U \times V)/\sim,$$

  where $\sim$ is the equivalence relation for which the following hold for

$$(u, v), (u', v'), (u'', v'') \in F(U \times V) \text{ and } c \in \mathbb{C}:$$

  1. *Identity*: $(u, v) \sim (u, v)$.
  2. *Symmetry*: $(u, v) \sim (u', v')$ implies $(u', v') \sim (u, v)$.
  3. *Transitivity*: $(u, v) \sim (u', v')$ and $(u', v') \sim (u'', v'')$ implies $(u, v) \sim (u'', v'')$.
  4. *Distributivity*: $(v, w) + (v', w) \sim (v + v', w)$ and $(v, w) + (v, w') \sim (v, w + w')$.
  5. *Scalar product*: $c(u, v) \sim (cu, v)$ and $c(u, v) \sim (u, cv)$.

- Given $u \in U$ and $v \in V$, the equivalence class that contains (u, v) is denoted by $u \otimes v$.
- An element in $U \otimes V$ that can be written as $u \otimes v$ is called a *simple tensor*.

[3] This definition works only for the tensor products of vector spaces. When the tensor product of other mathematical objects is investigated, a different, more general definition of tensor product is required.

[4] It is not clear that $F(U \times V)$ is a vector space itself. It is for instance not clear what a scalar times a vector in this free vector space would look like. We solve this by enforcing the rules and axioms for a vector space on the Cartesian product $U \times V$, and acting like it works that way. This is very similar to the way we handle imaginary numbers. We act as if $\sqrt{-1}$ exists and find that we can work with this without too much trouble. Seeing every element of $U \times V$ as a base vector creates a very unwieldy, massive vector space. The tensor product 'tames' this space and reduces it into a more easily to handle vector space.

Remark. Note the following:

- In general, not every element of $U \otimes V$ can be written as a simple tensor. To see this, let $u_1, u_2 \in U$ be linearly independent and let $v_1, v_2 \in V$ be linearly independent. Then $u_1 \otimes v_1 + u_2 \otimes v_2$ cannot be simplified. Every element of $U \otimes V$ can be written as a linear combination of simple tensors.
- The definition of a tensor product of vector spaces U and V is independent of bases for U and V. From here on out we shall always specify a basis for U and V. This lets us specify a basis for $U \otimes V$. For instance, let $\{u_1, u_2, ...\}$ be a basis for U, and let $\{v_1, v_2, ...\}$ be a basis for V. Then the set of all possible combinations of simple tensors of the bases, $\{u_i \otimes v_j | i, j \in \mathbb{N}\}$ forms a basis for $U \otimes V$.

This definition is fairly abstract, so let us look at some examples.

Example 2.11. Consider the following:

- Let V be a one dimensional vector space with basis $\{v\}$. Then $V \otimes V$ will be the span of $\{v \otimes v\}$, which is isomorphic to V. Therefore, $V \otimes V$ is also one dimensional
- Similarly, when U is one dimensional and V is two dimensional, $U \otimes V$ will be two dimensional and isomorphic to V.
- Let V be two dimensional with basis $\{v_1, v_2\}$. Then

$$V \otimes V = \text{span}\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}.$$

  Take a look at $v_1 \otimes v_2 + v_2 \otimes v_1 \in V \otimes V$. We shall prove that we cannot write this as an simple tensor. In order to do this, assume that for some $u, v \in V$, we have that $v_1 \otimes v_2 + v_2 \otimes v_1 = u \otimes v$. Write $u = \alpha_1 v_1 + \alpha_2 v_2$ and $v = \beta_1 v_1 + \beta_2 v_2$, then

$$\begin{aligned} v_1 \otimes v_2 + v_2 \otimes v_1 &= u \otimes v \\ &= (\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_1 v_1 + \beta_2 v_2) \\ &= \alpha_1\beta_1(v_1 \otimes v_1) + \alpha_1\beta_2(v_1 \otimes v_2) \\ &\quad + \alpha_2\beta_1(v_2 \otimes v_1) + \alpha_2\beta_2(v_2 \otimes v_2). \end{aligned}$$

  This implies that $\alpha_1\beta_1 = \alpha_2\beta_2 = 0$ and $\alpha_1\beta_2 = \alpha_2\beta_1 = 1$, which is a contradiction. The fact that some states can only be written as the sum of multiple simple tensors is crucial for the study of quantum entanglement.

- In general, let U and V be finite dimensional vector spaces with bases $\{u_1, ..., u_m\}$ and $\{v_1, ..., v_n\}$ respectively. Then

$$U \otimes V = \{u_i \otimes v_j | i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}\}.$$

  This implies that $U \otimes V$ is nm dimensional.

In quantum mechanics, we work with Hilbert spaces, which have an inner product and where the topology induced by the inner product is complete. Composite quantum systems must be a Hilbert space as well. We must therefore define an inner product on the tensor product of Hilbert spaces. We do this as follows:

Definition 2.12. Let H_1 and H_2 be Hilbert spaces with respective inner products $\langle .|. \rangle_1$ and $\langle .|. \rangle_2$. Then for all $\phi_1, \psi_1 \in H_1$ and $\phi_2, \psi_2 \in H_2$, the inner product of $H_1 \otimes H_2$ is

$$\langle \phi_1 \otimes \phi_2 \mid \psi_1 \otimes \psi_2 \rangle = \langle \phi_1 | \psi_1 \rangle_1 \langle \phi_2 | \psi_2 \rangle_2 . \tag{29}$$

The reader might notice that this definition is only for the simple tensors. We therefore extend the definition by linearity, meaning that the inner product of a sum of simple tensors equals the sum of the inner products of the individual simple tensors. For example:

$$\langle \phi_1 \otimes \phi_2 + \psi_1 \otimes \psi_2 \mid \psi_1 \otimes \psi_2 \rangle = \langle \phi_1 \mid \psi_1 \rangle_1 \langle \phi_2 \mid \psi_2 \rangle_2 + \langle \psi_1 \mid \psi_1 \rangle_1 \langle \psi_2 \mid \psi_2 \rangle_2 .$$

Now we still are in a bit of trouble, because there is no guarantee that the tensor product of the Hilbert spaces $H_1 \otimes H_2$ is a Hilbert space itself. In order to fix this, we take the completion in the topology defined by the inner product as defined above. This ensures that the tensor product is complete and thus a Hilbert space. We denote this completion by $H_1 \hat{\otimes} H_2$. Note that for simple tensors, the notation $\phi_1 \hat{\otimes} \phi_2$ is nonsensical, and even for vector spaces U and V that are not Hilbert spaces, $U \hat{\otimes} V$ makes little notational sense.

We also require to define the tensor products of operators.

Definition 2.13. Let H_1 and H_2 be Hilbert spaces and let A_1 and A_2 be operators working on H_1 and H_2 respectively. The operator $A_1 \otimes A_2$ working on a simple tensor $\psi_1 \otimes \psi_2 \in H_1 \otimes H_2$ is defined as

$$A_1 \otimes A_2(\psi_1 \otimes \psi_2) = A_1\psi_1 \otimes A_2\psi_2, \tag{30}$$

where we extend this definition by linearity.

We now have the mathematical basis to look at the angular momentum of composite systems. Say we have two quantum angular momentum systems. The first is in the state $|l_1\, m_1\rangle$ and the second is in the state $|l_2\, m_2\rangle$. The Hilbert space associated to each systems is $\mathbf{L^2}(S^2)$, so the Hilbert space associated to the composite system will be $\mathbf{L^2}(S^2) \hat{\otimes} \mathbf{L^2}(S^2)$ and the state that the composite system is in is $|l_1\, m_1\rangle \otimes |l_2\, m_2\rangle$, which we can abreviate as $|l_1\, m_1\rangle\, |l_2\, m_2\rangle$ or as $|l_1\, l_2\, m_1\, m_2\rangle$. Now we also need operators to work with in this new quantum system. Say that L^2 and L_z are the operators working on a single system. We can then define the operators on composite systems: $L^2 \otimes I$, $I \otimes L^2$, $L_z \otimes I$, and $I \otimes L_z$, where I is the identity operator. Let us consider an example to see how this works explicitly.

Example 2.14.

$$\begin{aligned}(L_z \otimes I)(I \otimes L^2)\,|6\,3\text{ -}4\,1\rangle &= (L_z \otimes I)(I \otimes L^2)\,|6\text{ -}4\rangle \otimes |3\,1\rangle \\ &= (L_z \otimes I)(I\,|6\text{ -}4\rangle \otimes L^2\,|3\,1\rangle) \\ &= (L_z \otimes I)(|6\text{ -}4\rangle \otimes 3(3+1)\hbar^2\,|3\,1\rangle) \\ &= 12\hbar^2(L_z\,|6\text{ -}4\rangle \otimes I\,|3\,1\rangle) \\ &= 12\hbar^2(-4\hbar\,|6\text{ -}4\rangle \otimes |3\,1\rangle) \\ &= -48\hbar^3\,|6\,3\text{ -}4\,1\rangle\end{aligned}$$

Note that since the identity operator does nothing to the system, and because the tensor product is bilinear, we can also write $(L_z \otimes I)(I \otimes L^2)$ as $L_z \otimes L^2$. We now have a perfectly fine way to describe composite systems of angular momentum. If we have two systems $|l_1\, m_1\rangle$ and $|l_2\, m_2\rangle$, then the composite system is $|l_1\, m_1\rangle \otimes |l_2\, m_2\rangle$. These composite states are a basis for the composite Hilbert space $\mathbf{L^2}(S^2)\hat{\otimes}\mathbf{L^2}(S^2)$, which implies that the set

$$\{L^2 \otimes I, I \otimes L^2, L_z \otimes I, I \otimes L_z\}$$

is a CSCO. However, we have not really gained any information on this composite system by doing this. We often want to see the composite system as one bigger system and calculate or measure the total angular momentum for instance. In order to gain this information, we need to change the basis for the Hilbert space in a specific way, which we shall now proceed to do.

First we define the total angular momentum operators.

Definition 2.15. Let $\mathbf{L^2}(S^2)\hat{\otimes}\mathbf{L^2}(S^2)$ be the Hilbert space of the composite system of two angular momentum systems. We define the total angular momentum in the z-direction, $L_{z,\text{tot}}$ as follows:

$$L_{z,\text{tot}} = L_z \otimes I + I \otimes L_z. \tag{31}$$

The total angular momentum in the x- and y-direction, $L_{x,\text{tot}}$ and $L_{y,\text{tot}}$ are defined in the same way. Now we define the total angular momentum L^2_{tot} in the same way as equation (11):

$$L^2_{\text{tot}} = (L_{x,\text{tot}})^2 + (L_{y,\text{tot}})^2 + (L_{z,\text{tot}})^2 \tag{32}$$

If a system is in the $|l_1\, l_2\, m_1\, m_2\rangle$ state and we measure $L_{z,\text{tot}}$, we get

$$\begin{aligned}L_{z,\text{tot}}\,|l_1\, l_2\, m_1\, m_2\rangle &= (L_z \otimes I + I \otimes L_z)\,|l_1\, m_1\rangle \otimes |l_2\, m_2\rangle \\ &= L_z\,|l_1\, m_1\rangle \otimes I\,|l_2\, m_2\rangle + I\,|l_1\, m_1\rangle \otimes L_z\,|l_2\, m_2\rangle \\ &= \hbar m_1\,|l_1\, m_1\rangle \otimes |l_2\, m_2\rangle + |l_1\, m_1\rangle \otimes \hbar m_2\,|l_2\, m_2\rangle \\ &= \hbar(m_1+m_2)\,|l_1\, m_1\rangle \otimes |l_2\, m_2\rangle \\ &= \hbar(m_1+m_2)\,|l_1\, l_2\, m_1\, m_2\rangle\end{aligned}$$

as desired. Measuring L^2_{tot} is a bit more complicated. First we look at what $(L_{z,\text{tot}})^2$ looks like:

$$\begin{aligned}(L_{z,\text{tot}})^2 &= (L_z \otimes I + I \otimes L_z)(L_z \otimes I + I \otimes L_z) \\ &= (L_z \otimes I)^2 + (I \otimes L_z)^2 + (L_z \otimes I)(I \otimes L_z) + (I \otimes L_z)(L_z \otimes I) \\ &= (L_z^2 \otimes I) + (I \otimes L_z^2) + 2(L_z \otimes L_z).\end{aligned}$$

Of course we can replace the z with x and y to get the same result for $(L_{x,\text{tot}})^2$ and $(L_{y,\text{tot}})^2$. Combining these results we get

$$L^2_{\text{tot}} = (L^2 \otimes I) + (I \otimes L^2) + 2(L_x \otimes L_x + L_y \otimes L_y + L_z \otimes L_z). \tag{33}$$

We shall now calculate the commutators for $L_{x,\text{tot}}$, $L_{y,\text{tot}}$ and $L_{z,\text{tot}}$:

$$\begin{aligned}[L_{x,\text{tot}}, L_{y,\text{tot}}] &= L_{x,\text{tot}}L_{y,\text{tot}} - L_{y,\text{tot}}L_{x,\text{tot}} \\ &= (L_x \otimes I + I \otimes L_x)(L_y \otimes I + I \otimes L_y) \\ &\qquad - (L_y \otimes I + I \otimes L_y)(L_x \otimes I + I \otimes L_x) \\ &= L_xL_y \otimes I + L_y \otimes L_x + L_x \otimes L_y + I \otimes L_xL_y \\ &\qquad - (L_yL_x \otimes I + L_x \otimes L_y + L_y \otimes L_x + I \otimes L_yL_x) \\ &= L_xL_y \otimes I - L_yL_x \otimes I + I \otimes L_xL_y - I \otimes L_yL_x \\ &= (L_xL_y - L_yL_x) \otimes I + I \otimes (L_xL_y - L_yL_x) \\ &= [L_x, L_y] \otimes I + I \otimes [L_x, L_y] \\ &= i\hbar(L_z \otimes I + I \otimes L_z) \\ &= i\hbar L_{z,\text{tot}}.\end{aligned}$$

Here we used equation (10) and the fact that the distributivity of the tensor product also holds for the tensor product of operators. As we have done before, we immediately know the commutators for all three observables by the cyclic permutation of (x, y, z). This gives us the identitical commutators for the composite system as for equation (10):

$$\begin{aligned}[L_{x,\text{tot}}, L_{y,\text{tot}}] &= i\hbar L_{z,\text{tot}} \\ [L_{y,\text{tot}}, L_{z,\text{tot}}] &= i\hbar L_{x,\text{tot}} \\ [L_{z,\text{tot}}, L_{x,\text{tot}}] &= i\hbar L_{y,\text{tot}}.\end{aligned} \tag{34}$$

It is important to realise that only the commutation relations of equation (10) were necessary in order to find the eigenvalues for the L_z and L^2 operators. Since the commutation relations in equation (34) are identitical, we can use the exact same derivation as we have done before in order to acquire the eigenvalues of the combined system observables L^2_{tot} and $L_{z,\text{tot}}$. Again, the numbers l and m characterize the common eigenfunction f_l^m:

$$L^2_{\text{tot}} f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_{z,\text{tot}} f_l^m = \hbar m f_l^m, \tag{35}$$

where $l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\}$ and $m \in \{-l, -l+1, ..., l-1, l\}$. We shall again only look at the case when l is integer. Note also that, using this derivation, we define $L_{\pm,\text{tot}}$:

$$L_{\pm,\text{tot}} = L_{x,\text{tot}} \pm iL_{y,\text{tot}}. \tag{36}$$

We also get the version of lemma 2.7 for the total angular momentum:

$$L^2_{\text{tot}} = L_{\pm,\text{tot}} L_{\mp,\text{tot}} + L^2_{z,\text{tot}} \mp \hbar L_{z,\text{tot}}. \tag{37}$$

These will come in handy later.

As stated before, $\{L^2 \otimes I, I \otimes L^2, L_z \otimes I, I \otimes L_z\}$ is a CSCO of the composite Hilbert space $\mathbf{L^2}(S^2)\hat{\otimes}\mathbf{L^2}(S^2)$. To check whether L^2_{tot} has eigenfunctions of the form $|l_1\, l_2\, m_1\, m_2\rangle$, we calculate the commutator of L^2_{tot} and $L_z \otimes I$:

$$\begin{aligned}
[L^2_{\text{tot}}, L_z \otimes I] &= L^2_{\text{tot}} L_z \otimes I - L_z \otimes I L^2_{\text{tot}} \\
&= 2((L_x L_z \otimes L_x - L_z L_x \otimes L_x) + (L_y L_z \otimes L_y - L_z L_y \otimes L_y)) \\
&= 2([L_x, L_z] \otimes L_x + [L_y, L_z] \otimes L_y) \\
&= 2i\hbar(L_x \otimes L_y - L_y \otimes L_x) \\
&\neq 0,
\end{aligned}$$

where we skipped a few steps that are similar to the calculation of $[L_{x,\text{tot}}, L_{y,\text{tot}}]$ given earlier. This means that L^2_{tot} and $L_z \otimes I$ do not commute and therefore we know that not all eigenfunctions $|l_1\, l_2\, m_1\, m_2\rangle$ of $L_z \otimes I$ can be eigenfunctions of L^2_{tot}. We must therefore find a different set of eigenfunctions for this observable. Ideally, we can find a CSCO of $\mathbf{L^2}(S^2)\hat{\otimes}\mathbf{L^2}(S^2)$ that includes L^2_{tot}.

In order to do this, we find in ways similar to what is shown above that the operators $I \otimes L^2$, $L^2 \otimes I$, L^2_{tot}, and $L_{z,\text{tot}}$ do commute, which implies that they share an eigenbasis. We shall write the eigenfunctions of this basis, just as with the basis we already have, with the associated eigenvalues. We do not know yet whether the eigenspaces of this set of commuting observables are one dimensional. We do not even know yet if they exist! Nonetheless, let $|l_1\, l_2\, l\, m\rangle$ be a state that is in the eigenspace with the given eigenvaluess. We shall call this a *coupled eigenvector*. A state $|l_1\, l_2\, m_1\, m_2\rangle = |l_1\, m_1\rangle \otimes |l_2\, m_2\rangle$ is called an *uncoupled eigenvector*. We will now prove that the eigenspaces are indeed one dimensional. This proof is adapted from [3]. It is long and has many arguments, so we shall prove it in steps. We start with a remark

Remark. Notice the following:

- Since the states $|l_1\, l_2\, m_1\, m_2\rangle$ and $|l_1\, l_2\, l\, m\rangle$ have the quantum numbers l_1 and l_2 in common, we can leave l_1 and l_2 out of the notation to simplify it to the states $|m_1\rangle\,|m_2\rangle$ and $|l\, m\rangle$ respectively. Note that we write $|m_1\rangle\,|m_2\rangle$ to keep it very clear that this concerns an uncoupled eigenvector, whereas $|l\, m\rangle$ is a coupled eigenvector.

- The states $|m_1\rangle\,|m_2\rangle$ are simple tensors and form the basis of our Hilbert space. We can write any state $|\psi\rangle$ in this Hilbert space as.

$$|\psi\rangle = \sum_{m_1}\sum_{m_2} C_{m_1\,m_2}\,|m_1\rangle\,|m_2\rangle\,.$$

- When l_1 and l_2 are fixed, the Hilbert space becomes finitely dimensional. The basis states are $|m_1\rangle\,|m_2\rangle$, where $-l_1 \leq m_1 \leq l_1$ and $-l_2 \leq m_2 \leq l_2$. This implies that the Hilbert space has dimension $(2l_1+1)(2l_2+1)$. We shall use this fact in our proof.

We shall now relate the quantum numbers m_1 and m_2 to the quantum number m.

Lemma 2.16. *Let l_1 and l_2 be fixed, and suppose*

$$|l\;m\rangle = \sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2} C_{m_1\,m_2}\,|m_1\rangle\,|m_2\rangle\,.$$

Then $C_{m_1\,m_2} = 0$ for $m_1 + m_2 \neq m$ and $C_{m_1\,m_2} = 0$ for $l > l_1 + l_2$.

Proof. When we apply $L_{z,\mathrm{tot}}$ to a basis state $|m_1\rangle\,|m_2\rangle$ we get

$$L_{z,\mathrm{tot}}\,|m_1\rangle\,|m_2\rangle = \hbar(m_1+m_2)\,|m_1\rangle\,|m_2\rangle\,. \tag{38}$$

This implies that, because by definition $L_{z,\mathrm{tot}}\,|l\;m\rangle = \hbar m\,|l\;m\rangle$, if we have a state $|l\;m\rangle$ and we want to write it as a linear combination of the basis states $|m_1\rangle\,|m_2\rangle$, the states in the sum must have $m_1 + m_2 = m$.
Now assume that $l > l_1 + l_2$. Consider the state $|l\;l\rangle$:

$$|l\;l\rangle = \sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2} C_{m_1\,m_2}\,|m_1\rangle\,|m_2\rangle\,.$$

For every $|m_1\rangle\,|m_2\rangle$, it holds that $m_1 + m_2 < m = l$. Therefore all $C_{m_1\,m_2} = 0$. Now for any $|\psi\rangle$ in the eigenspace with $L_{\mathrm{tot}} = l$ and $L_{z,tot} = m$ with $-l \leq m \leq l$, it holds that

$$|\psi\rangle = a(L_{-,\mathrm{tot}})^{l-m}\,|l\;l\rangle\,,$$

where a is a constant. Because $|l\;l\rangle = 0$, it follows that $|\psi\rangle = 0$. □

Lemma 2.17. *Let l_1 and l_2 be fixed. The eigenspaces where $|l_1 - l_2| \leq l \leq l_1 + l_2$ and where $-l \leq m \leq l$ exist, that is, they are at least one dimensional.*

Proof. We shall first take a look at the state where the angular momentum is maximally aligned in the z direction. This is (up to a phase factor) the state $|m_1\rangle\,|m_2\rangle$ with $m_1 = l_1$ and $m_2 = l_2$. There is only one state in the whole Hilbert space where this is the case. Lemma 2.16 implies that the only coupled eigenspace that is possible is the space $|l\;m\rangle$ with $l = l_1 + l_2$ and $m = l_1 + l_2$.

The problem is, we do not know if this space even exists. If a non-zero vector satisfies the eigenfunction equations (35) for L^2_{tot} and $L_{z,\text{tot}}$, the space is non-empty. Luckily, we already have one contender: $|m_1\rangle |m_2\rangle = |l_1\rangle |l_2\rangle$. We can plug this into the eigenvalue equations to see what we get. Equation (38) already shows that $|l_1\rangle |l_2\rangle$ satisfies the eigenvalue equation for $L_{z,\text{tot}}$. For L^2_{tot}, we shall first calculate a useful formula. Use equation (12), then apply the rules in definition 2.10 to simplify, in order to attain the following equality:

$$(L_+ \otimes L_-) + (L_- \otimes L_+) = 2(L_x \otimes L_x + L_y \otimes L_y). \tag{39}$$

Now, using equation (33) we can rewrite L^2_{tot}:

$$L^2_{\text{tot}} = (L^2 \otimes I) + (I \otimes L^2) + 2(L_z \otimes L_z) + (L_+ \otimes L_-) + (L_- \otimes L_+). \tag{40}$$

Now we can easily apply L^2_{tot} to the uncoupled state $|l_1\rangle |l_2\rangle$:

$$\begin{aligned} L^2_{\text{tot}} |l_1\rangle |l_2\rangle &= (L^2 \otimes I + I \otimes L^2 + 2L_z \otimes L_z + L_+ \otimes L_- + L_- \otimes L_+) |l_1\rangle |l_2\rangle \\ &= \hbar^2(l_1(l_1+1) + l_2(l_2+1) + m_1 m_2) |l_1\rangle |l_2\rangle , \end{aligned}$$

where the ladder operators vanish because both uncoupled states are at the maximum rung, so applying the raising operator to either one will result in a zero coupled vector. So we see that the eigenvector $|l_1\rangle |l_2\rangle$ is also an eigenvector of the operators L^2_{tot} and $L_{z,\text{tot}}$. Hence, we know that a $|l_1{+}l_2 \; l_1{+}l_2\rangle$ exists. Furthermore, the eigenspace that contains $|l_1\rangle |l_2\rangle$ is one dimensional, so the eigenspace that contains $|l_1{+}l_2 \; l_1{+}l_2\rangle$ cannot be larger than one dimensional, since the states in the former eigenspace build the states in the latter eigenspace by a linear combination. We conclude that

$$|l_1\rangle |l_2\rangle = |l_1{+}l_2 \; l_1{+}l_2\rangle$$

The next step is to apply the lowering operator $L_{-,\text{tot}}$ to this state $|l_1{+}l_2 \quad l_1{+}l_2\rangle$ to get $|l_1{+}l_2 \quad l_1{+}l_2{-}1\rangle$. We can repeatedly do this $2(l_1 + l_2)$ times to get $2(l_1 + l_2) + 1$ linearly independent eigenstates. They exist because theorem 2.6 holds for $L_{-,\text{tot}}$, $L_{z,\text{tot}}$, and L^2_{tot}, and, by extension, also formula (22) holds, which tells us that the normalization constant is nonzero. $L_{-,\text{tot}}$ does however change the state to a state from another eigenspace, and we know eigenspaces are orthogonal. Therefore the states are linearly independent.

Next we look at the eigenspace where m is fixed to be $m = l_1 + l_2 - 1$. There are two basis states that have this property: $|l_1{-}1\rangle |l_2\rangle$ and $|l_1\rangle |l_2{-}1\rangle$. This is one more than when $m = l_1 + l_2$. This means the eigenspace is two dimensional. We already have one coupled state in this eigenspace, $|l_1{+}l_2 \quad l_1{+}l_2{-}1\rangle$. We can write it as a linear combination of the two basis states:

$$|l_1{+}l_2 \quad l_1{+}l_2{-}1\rangle = a |l_1{-}1\rangle |l_2\rangle + b |l_1\rangle |l_2{-}1\rangle . \tag{41}$$

Next, we take the orthogonal state to this state. It will still be a linear combination of the basis states, so m will still equal $l_1 + l_2 - 1$ for this state.

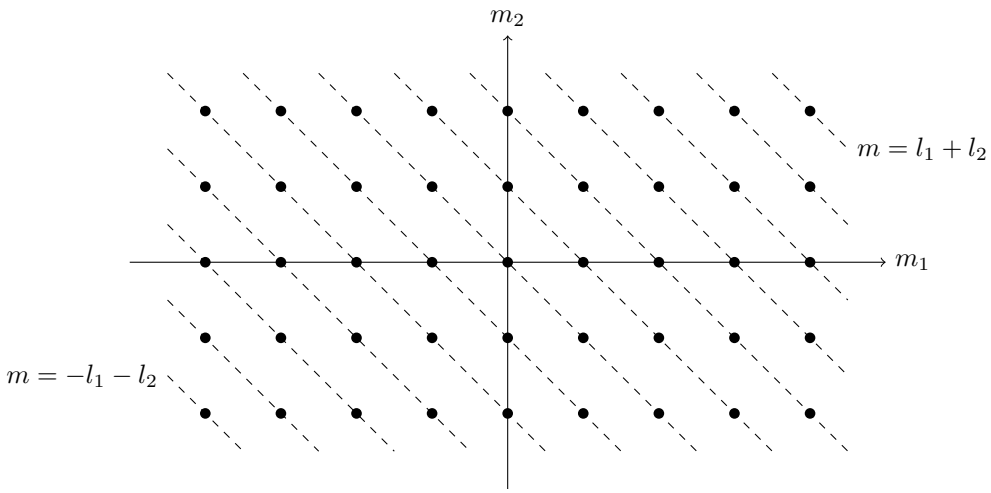

Figure 1: All posible combinations of m_1 and m_2 for the case $l_1 = 4$ and $l_2 = 2$. The dashed diagonal lines are the lines with constant $m = m_1 + m_2$.

Therefore it is in the eigenspace where $m = l_1 + l_2 - 1$, but linearly independent to $|l_1{+}l_2 \quad l_1{+}l_2{-}1\rangle$. Considering the rules for l and m, the only possible state this can be is $|l_1{+}l_2{-}1 \quad l_1{+}l_2{-}1\rangle$. Now we can again apply the lowering operator $L_{-,\text{tot}}$ repeatedly to this state to get $2(l_1 + l_2) - 1$ new, linearly independent states that have the same l value and are therefore in the eigenspace where $l = l_1 + l_2 - 1$.

Now we look at the eigenspace for $m = l_1 + l_2 - 2$ and repeat the argument. There are three uncoupled basis states that satisfy $m = l_1 + l_2 - 2$, namely $|l_1{-}2\rangle\,|l_2\rangle$, $|l_1\rangle\,|l_2{-}2\rangle$, and $|l_1{-}1\rangle\,|l_2{-}1\rangle$. This is one more basis state than when $m = l_1 + l_2 - 1$. We already have two coupled states in this eigenspace, $|l_1{+}l_2 \quad l_1{+}l_2{-}2\rangle$ and $|l_1{+}l_2{-}1 \quad l_1{+}l_2{-}2\rangle$. We can write these two states as linear combinations of the three basis states and build the state orthogonal to both of those. This state must be in the eigenspace, since it has the same eigenvalue $m = l_1{+}l_2{-}2$. The only possible state this can be is $|l_1{+}l_2{-}2 \quad l_1{+}l_2{-}2\rangle$. And we can repeatedly apply the $L_{-,\text{tot}}$ operator to this state to get $2(l_1{+}l_2){-}3$ linearly independent states.

It might seem like we can repeat this argument until $m = -(l_1{+}l_2)$, but this is not the case. There comes a point where lowering m by one will not give an extra basis state. When this happens, we cannot repeat the argument anymore and we stop. Figure 1 will show why this is.

The dashed diagonal lines indicate the states for which m is the same. Moving a diagonal line to the left means lowering m by one. If we start at the state for which m is the highest, which is at the top right, we see that lowering m by one increases the degeneracy by one, meaning the dimension of eigenspace for this m has become one larger than the eigenspace for $m + 1$. The degeneracy increases until a corner is reached. If we move left from the diagonal that goes

through the lower left corner, the degeneracy stays the same (if the grid is a square, which happens when $l_1 = l_2$, the degeneracy decreases immediately). At this point we cannot repeat the argument, and we cannot find new eigenstates. In the case of figure 2 the argument stops at $m = 2$. More generally, a corner is reached when

$$m = l_1 + l_2 - 2\min\{l_1, l_2\} = |l_1 - l_2|,$$

thus argument stops for $m = |l_1 - l_2|$. □

Theorem 2.18. *The simultaneous eigenspaces of the commuting observables $I \otimes L^2$, $L^2 \otimes I$, L^2_{tot}, and $L_{z,\text{tot}}$ are one dimensional, that is, these observables form a CSCO.*

Proof. This proof essentially boils down to different ways of counting the possible states, which are represented by dots on a rectangular part of a grid, such as in figure 2.

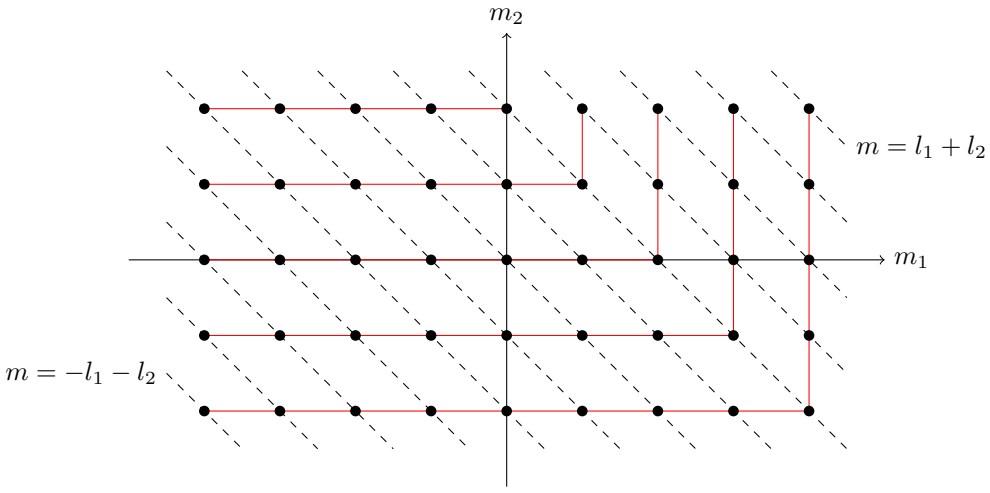

Figure 2: All possible combinations of m_1 and m_2 for the case $l_1 = 4$ and $l_2 = 2$. The dashed diagonal lines are the lines with constant $m = m_1 + m_2$. The red lines help us with counting.

We can count the states in the figure by taking the product of the number of rows and columns, or by counting the red "hooks" that can be seen in the figure. Let us make this more exact. These hooks show the eigenstates obtained by repeatedly applying $L_{-,\text{tot}}$ to the obtained states

$$|l_1{+}l_2\, l_1{+}l_2\rangle\,, |l_1{+}l_2{-}1\, l_1{+}l_2{-}1\rangle\,, \ldots, |l_1{-}l_2\, l_1{-}l_2\rangle\,.$$

This means the dots connected by the red line represent the states that have the same l. These are not actually the states with the same l. States with the same l are mostly linear combinations of the $|l_1\rangle\,|l_2\rangle$ states. There is however

the same amount of states in the hooks as the $2l+1$ states with the same l, and since we are only concerned with counting, this is enough.

First we fix l_1 and l_2 such that $l_1 \geq l_2$. Next we can count the amount of unique and linearly independent eigenstates states that are possible for $|l\, m\rangle$. By lemma 2.17 we know that the eigenspaces for $l_1 - l_2 \leq l \leq l_1 + l_2$ and $-l \leq m \leq l$ are at least one dimensional, meaning that for each $l_1 - l_2 \leq l \leq l_1 + l_2$ there are at least $2l+1$ unique eigenstates. Taking the sum over l we find

$$\begin{aligned}\sum_{l=l_1-l_2}^{l_1+l_2} 2l+1 &= 2\sum_{l=l_1-l_2}^{l_1+l_2} l + \sum_{l=l_1-l_2}^{l_1+l_2} 1 \\ &= (2l_2+1)(2l_1) + (2l_2+1) \\ &= (2l_2+1)(2l_1+1),\end{aligned}$$

unique coupled eigenstates, where we used the arithmetic series

$$\sum_{k=m}^{n} k = \frac{1}{2}(m-n+1)(m+n).$$

However, we know that the uncoupled eigenspace for fixed l_1 and l_2 has $(2l_2+1)(2l_1+1)$ dimensions. Therefore these eigenspaces in the coupled basis must be one dimensional, and hence the coupled basis is a CSCO. This means there are no other coupled eigenvectors than the ones we have found. In the case where $l_1 < l_2$, we can swap the indices 1 and 2, and the same argument holds. □

Remark. Lemma 2.16 and theorem 2.18 give us two constraints that hold in general. One for m_1, m_2, and m and one for l_1, l_2, and l:

- $$m_1 + m_2 = m \tag{42}$$
- $$|l_1 - l_2| \leq l \leq l_1 + l_2 \tag{43}$$

2.4 Clebsch-Gordan Coefficients

The proof of the last section is significant because it shows the states $|l_1\, l_2\, l\, m\rangle$ in their eigenspaces are unique (up to a phase factor), which implies that we can use them as a basis for the Hilbert space $\mathbf{L^2}(S^2) \otimes \mathbf{L^2}(S^2)$. We call this the *coupled basis*. We now have two othonormal bases for our Hilbert space, meaning that, when l_1 and l_2 are fixed, we can transform one base into the other

$$|l_1\, l_2\, l\, m\rangle = \sum_{m_1,m_2} C^{l\, m}_{l_1\, l_2\, m_1\, m_2} |l_1\, l_2\, m_1\, m_2\rangle, \tag{44}$$

and vice versa

$$|l_1\ l_2\ m_1\ m_2\rangle = \sum_{l,m} G^{l\,m}_{l_1\,l_2\,m_1\,m_2} |l_1\ l_2\ l\ m\rangle. \tag{45}$$

Note that, since both sets of eigenfunctions have the eigenvalues l_1 and l_2 in common, we do not need to sum over l_1 and l_2. The coefficients $C^{l\,m}_{l_1\,l_2\,m_1\,m_2}$ are called the *Clebsch-Gordan (C-G) coefficients* and $G^{l\,m}_{l_1\,l_2\,m_1\,m_2}$ are called the inverse Clebsch-Gordan coefficients. We will show that the C-G coefficients equal their inverse, i.e.

$$C^{l\,m}_{l_1\,l_2\,m_1\,m_2} = G^{l\,m}_{l_1\,l_2\,m_1\,m_2}. \tag{46}$$

To see why this is the case, we shall first find a different way to characterise the C-G coefficients.

We start with an important theorem:

Theorem 2.19. *Let $\{|\phi_1\rangle, |\phi_2\rangle, ..., |\phi_n\rangle\}$ be an orthonormal basis for the Hilbert space H of a system. Then for any $|\psi\rangle \in H$ we have*

$$|\psi\rangle = \sum_i (|\phi_i\rangle \langle\phi_i|)\, |\psi\rangle\,. \tag{47}$$

Proof. Let $|\psi\rangle \in H$. We can write this as $|\psi\rangle = \sum_j c_j |\phi_j\rangle$. Now we multiply both sides by $\langle\phi_i|$ for some $i \in \mathbb{N}$:

$$\begin{aligned} \langle\phi_i|\psi\rangle &= \sum_j c_j \langle\phi_i|\phi_j\rangle \\ &= c_i \langle\phi_i|\phi_i\rangle \\ &= c_i. \end{aligned}$$

So now we can write

$$\begin{aligned} |\psi\rangle &= \sum_i \langle\phi_i|\psi\rangle\, |\phi_i\rangle \\ &= \sum_i |\phi_i\rangle \langle\phi_i|\psi\rangle \\ &= \sum_i (|\phi_i\rangle \langle\phi_i|)\, |\psi\rangle \end{aligned}$$

as desired. The last step might seem like an abuse of notation, but it is allowed. It is intuitively more clear to see what we are doing when we see bras as row vectors and kets as column vectors. We are allowed to do this because H has a finite dimension. In this way we see that a ket multiplied with a bra is an outer product. □

Notice that the right hand part of equation 47 is written as an operator working on a certain vector. Since this vector is arbitrary, we can also write the relation as

$$\sum_n |\phi_n\rangle \langle \phi_n| = I. \tag{48}$$

We call equation (48) the *completion relation*. Using this completion relation, we can find a different way to characterize the C-G coefficients:

$$\begin{aligned} |l_1\ l_2\ l\ m\rangle &= I\,|l_1\ l_2\ l\ m\rangle \\ &= \sum_{m_1,m_2} |l_1\ l_2\ m_1\ m_2\rangle \langle l_1\ l_2\ m_1\ m_2|l_1\ l_2\ l\ m\rangle \\ &= \sum_{m_1,m_2} \langle l_1\ l_2\ m_1\ m_2|l_1\ l_2\ l\ m\rangle\, |l_1\ l_2\ m_1\ m_2\rangle \end{aligned}$$

Comparing this to equation (44) we find that

$$C^{l\,m}_{l_1\,l_2\,m_1\,m_2} = \langle l_1\ l_2\ m_1\ m_2|l_1\ l_2\ l\ m\rangle\,. \tag{49}$$

We can repeat this argument for $|l_2\ l_2\ m_1\ m_2\rangle$ and find that

$$G^{l\,m}_{l_1\,l_2\,m_1\,m_2} = \langle l_1\ l_2\ l\ m|l_1\ l_2\ m_1\ m_2\rangle\,. \tag{50}$$

Therefore, we conclude that

$$C^{l\,m}_{l_1\,l_2\,m_1\,m_2} = (G^{l\,m}_{l_1\,l_2\,m_1\,m_2})^*. \tag{51}$$

In the next section we will calculate the C-G coefficients and find out that we can choose them to be real. Combining this with (51), we can conclude (46). We can use this result to derive interesting and useful relationships. If (46) holds, then we can write

$$\begin{aligned} \sum_{l,m} C^{l\,m}_{l_1\,l_2\,m_1\,m_2} C^{l\,m}_{l_1\,l_2\,m_1'\,m_2'} &= \sum_{l,m} = \langle l_1\ l_2\ m_1\ m_2|l_1\ l_2\ l\ m\rangle \langle l_1\ l_2\ l\ m|l_1\ l_2\ m_1\ m_2\rangle \\ &= \sum_{l,m} \langle l_1\ l_2\ m_1\ m_2|\,(|l_1\ l_2\ l\ m\rangle \langle l_1\ l_2\ l\ m|)\,|l_1\ l_2\ m_1\ m_2\rangle \\ &= \langle l_1\ l_2\ m_1\ m_2|\sum_{l,m}(|l_1\ l_2\ l\ m\rangle \langle l_1\ l_2\ l\ m|)\,|l_1\ l_2\ m_1\ m_2\rangle \\ &= \langle l_1\ l_2\ m_1\ m_2|\,(|l_1\ l_2\ m_1\ m_2\rangle) \\ &= \langle l_1\ l_2\ m_1\ m_2|l_1\ l_2\ m_1\ m_2\rangle \\ &= \delta_{m_1 m_1'}\delta_{m_2 m_2'}, \end{aligned}$$

where we used associativity and linearity of the matrix product, as well as theorem 2.19 and the fact that the basisvectors are orthonormal. So we find

$$\sum_{l,m} C^{l\,m}_{l_1\,l_2\,m_1\,m_2} C^{l\,m}_{l_1\,l_2\,m_1'\,m_2'} = \delta_{m_1 m_1'}\delta_{m_2 m_2'} \tag{52}$$

and, with a similar argument, we can find

$$\sum_{m_1,m_2} C^{l\,m}_{l_1\,l_2\,m_1\,m_2} C^{l'\,m'}_{l_1\,l_2\,m_1\,m_2} = \delta_{ll'}\delta_{mm'}. \tag{53}$$

Equations (52) and (53) are called the *orthogonality relations* of the Clebsch-Gordan coefficients. They will come in handy later.

2.5 Calculating the Clebsch-Gordan Coefficients

In order to calculate the C-G Coefficients we shall again make use of the ladder operator. In the end we will acquire a recursion relation that we can use to explicitly calculate the C-G coefficients. The simplest way to do this is to apply the ladder operator to (44). However, the recursion relation itself is difficult to understand and not very efficient for calculation. The method we show here is adapted from [4] and essentially the same idea, except we apply L^2_{tot} to (44). The derivation is a bit more involved than that of the ladder operator, but it gives a much simpler and more efficient recursion relation.

Note that in equation (44) the sum is only over m_1 and m_2, therefore, we can see l_1, l_2, l, and m as fixed for the C-G coefficients $C^{l\,m}_{l_1\,l_2\,m_1\,m_2}$. Now, due to constraint (42), many of the C-G coefficients will become zero. The next lemma shows what values for m_1 will give (potentially) non-zero C-G coefficients.

Lemma 2.20. *The C-G coefficient $C^{l\,m}_{l_1\,l_2\,m_1\,m_2}$ is zero for*

$$m_1 < \frac{m - l_1 - l_2 + |l_1 - l_2 + m|}{2}, \tag{54}$$

and for

$$m_1 > \frac{m + l_1 + l_2 - |l_1 - l_2 - m|}{2}. \tag{55}$$

Proof. First, we check the lowest possible value for m_1, which is $-l_1$. We can choose m_1 to be $-l_1$ only when m_2 can be chosen high enough such that

$$m_1 + m_2 = m.$$

Since the highest possible value for m_2 is l_2, we can choose $m_1 = -l_1$ when $l_2 - l_1 \geq m$. Then m_2 will have its maximum value at $m + l_1$. If $l_2 - l_1 < m$, then m_2 can reach its maximum value l_2 and therefore the minimum value for m_1 is $m - l_2$.

Conversely, the highest possible value for m_1 is l_1. Since the lowest possible value for m_2 is $-l_2$, we can choose $m_1 = l_1$ only when $l_1 - l_2 \leq m$. Then m_2 will have minimum value $m - l_1$. If $l_1 - l_2 > m$, then m_2 can reach its minimum value $-l_2$, and therefore the maximum value for m_1 is $m + l_2$. Figure 3 should help the reader understand this proof.

Using this knowledge, it is now easy to verify the correctness of the formulas. □

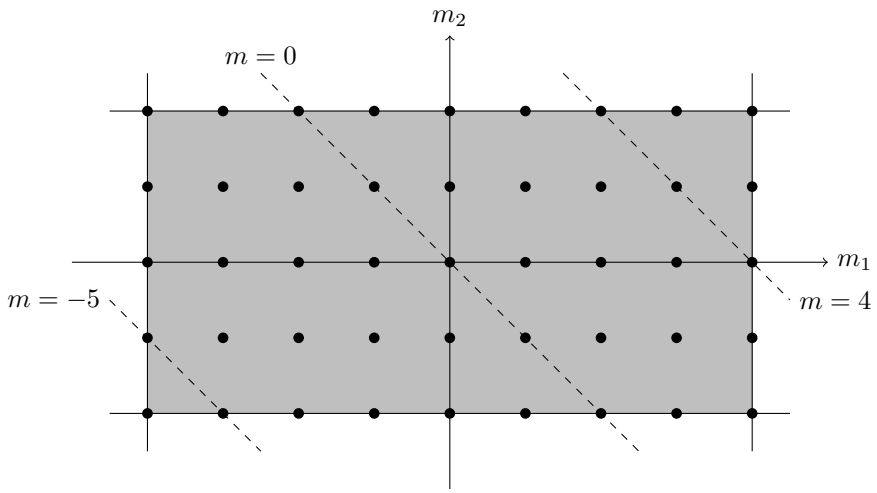

Figure 3: All posible combinations of m_1 and m_2 for the case $l_1 = 4$ and $l_2 = 2$. The grey area shows the possible states subject to the constraints $-l_1 \leq m_1 \leq l_1$ and $-l_2 \leq m_2 \leq l_2$. The diagonal lines show some cases of the proof.

Once we know m_1, we immediately know m_2, since $m_2 = m - m_1$. Now we can significantly ease up the notation for the Clebsch-Gordan coefficients. Let $m_{1,1}$ be the lowest possible value of m_1 for which the C-G coefficient is (potentially) non-zero, and let $m_{1,n}$ be the highest value of m_1 for which the C-G coefficient is (potentially) non-zero, i.e.

$$m_{1,1} = \frac{m - l_1 - l_2 + |l_1 - l_2 + m|}{2} \tag{56}$$

$$m_{1,n} = \frac{m + l_1 + l_2 - |l_1 - l_2 - m|}{2}. \tag{57}$$

Now we define $m_{1,k} := m_{1,1} + k - 1$, and $m_{2,k} := m - m_{1,k}$ for $1 \leq k \leq n$. We can now ease up the notation for equation (44):

$$|l_1\, l_2\, l\, m\rangle = \sum_{k=1}^{n} C_k\, |l_1\, l_2\, m_{1,k}\, m_{2,k}\rangle\,, \tag{58}$$

with

$$C_k = C^{l\,m}_{l_1\, l_2\, m_{1,k}\, m_{2,k}}.$$

In order to find the recursion relation, we shall apply L^2_{tot} to both sides of equation (58).

$$L_{\text{tot}}^2 \left| l_1\ l_2\ l\ m \right\rangle = L_{\text{tot}}^2 \sum_{k=1}^{n} C_k \left| l_1\ l_2\ m_{1,k}\ m_{2,k} \right\rangle$$

$$\hbar^2 l(l+1) \left| l_1\ l_2\ l\ m \right\rangle = \sum_{k=1}^{n} C_k L_{\text{tot}}^2 \left| l_1\ l_2\ m_{1,k}\ m_{2,k} \right\rangle$$

$$\hbar^2 l(l+1) \sum_{k=1}^{n} C_k \left| l_1\ l_2\ m_{1,k}\ m_{2,k} \right\rangle = \sum_{k=1}^{n} C_k L_{\text{tot}}^2 \left| l_1\ l_2\ m_{1,k}\ m_{2,k} \right\rangle .$$

On the right hand side, we apply L_{tot}^2 to uncoupled basis vectors, so we use equation (40), and by using the normalization constant for $L_\pm$ in equation (22), we obtain

$$\begin{aligned} 0 = \sum_{k=1}^{n} & C_k(l_1(l_1+1) + l_2(l_2+1) + 2m_{1,k}m_{2,k} - l(l+1)) \left| l_1\ l_2\ m_{1,k}\ m_{2,k} \right\rangle \\ & + C_k (A_{m_{1,k}}^{l_1})^+ (A_{m_{2,k}}^{l_2})^- \left| l_1\ l_2\ m_{1,k}{+}1\ m_{2,k}{-}1 \right\rangle \\ & + C_k (A_{m_{1,k}}^{l_1})^- (A_{m_{2,k}}^{l_2})^+ \left| l_1\ l_2\ m_{1,k}{-}1\ m_{2,k}{+}1 \right\rangle , \end{aligned}$$

where

$$(A_l^m)^+ = \hbar\sqrt{l(l+1) - m(m+1)}$$

and

$$(A_l^m)^- = \hbar\sqrt{l(l+1) - m(m-1)}.$$

Notice that for $k = 1$, the state $\left| l_1\ l_2\ m_{1,k}{-}1\ m_{2,k}{+}1 \right\rangle$ becomes zero, since we either apply the lowering operator to the state 1 where $m_{1,k} = -l_1$ or we apply the raising operator to state 2 where $m_{2,k} = l_2$. Similarly, for $k = n$, the state $\left| l_1\ l_2\ m_{1,k+1}\ m_{2,k-1} \right\rangle$ becomes zero, since we either apply the raising operator to the state 1 where $m_{1,k} = l_1$ or we apply the lowering operator to state 2 where $m_{2,k} = -l_2$.

Because of the way we have defined $m_{1,k}$ and $m_{2,k}$, it is true that

$$m_{1,k} + 1 = m_{1,k+1} \qquad \text{and} \qquad m_{2,k} - 1 = m_{2,k+1}.$$

Conversely,

$$m_{1,k} - 1 = m_{1,k-1} \qquad \text{and} \qquad m_{2,k} + 1 = m_{2,k-1}.$$

Using this, we find

$$\begin{aligned} 0 = \sum_{k=1}^{n} & C_k(l_1(l_1+1) + l_2(l_2+1) + 2m_{1,k}m_{2,k} - l(l+1)) \left| l_1\ l_2\ m_{1,k}\ m_{2,k} \right\rangle + \\ & C_k \sqrt{l_1(l_1+1) - m_{1,k}m_{1,k+1}} \sqrt{l_2(l_2+1) - m_{2,k}m_{2,k+1}} \left| l_1\ l_2\ m_{1,k+1}\ m_{2,k+1} \right\rangle + \\ & C_k \sqrt{l_1(l_1+1) - m_{1,k}m_{1,k-1}} \sqrt{l_2(l_2+1) - m_{2,k}m_{2,k-1}} \left| l_1\ l_2\ m_{1,k-1}\ m_{2,k-1} \right\rangle . \end{aligned}$$

Now we can gather terms, and knowing that every term $|l_1\, l_2\, m_{1,k}\, m_{2,k}\rangle$ must become zero, we find the following recursion relation:

$$\begin{aligned} &C_k[l_1(l_1+1)+l_2(l_2+1)+2m_{1,k}m_{2,k}-l(l+1)]+ \\ &C_{k+1}\sqrt{l_1(l_1+1)-m_{1,k}m_{1,k+1}}\sqrt{l_2(l_2+1)-m_{2,k}m_{2,k+1}}+ \\ &C_{k-1}\sqrt{l_1(l_1+1)-m_{1,k}m_{1,k-1}}\sqrt{l_2(l_2+1)-m_{2,k}m_{2,k-1}}=0 \end{aligned} \tag{59}$$

Notice that the recursion relation has only real coupling coefficients. This means we can choose the Clebsch-Gordan coefficients as real. Since the inverse C-G coefficients are the complex conjugate of the C-G coefficients, we conclude that the inverse coefficients equal the normal C-G coefficients.

Notice that in (59), the factor next to C_{k+1} is the same as the factor next to $C_{k'-1}$, in the case where $k'=k+1$. This means we can write the recursion relation as the matrix product $A\mathbf{C}=\mathbf{0}$, where A is symmetric:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\ 0 & a_{23} & a_{33} & a_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2\,n-1} & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1\,n} & a_{nn} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_{n-1} \\ C_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{60}$$

Here,

$$\begin{aligned} a_{kk} &= l_1(l_1+1)+l_2(l_2+1)+2m_{1,k}m_{2,k}-l(l+1) \\ a_{k\,k+1} &= \sqrt{l_1(l_1+1)-m_{1,k}m_{1,k+1}}\sqrt{l_2(l_2+1)-m_{2,k}m_{2,k+1}} \end{aligned} \tag{61}$$

It is known that a tridiagonal matrix has either rank n or $n-1$. If matrix A has rank n, then according to equation (60), A has only the one solution, which is the zero solution. However, A is a basis transformation matrix, so it cannot have the zero solution only. Therefore its rank is $n-1$. We reduce the amount of solutions to one, up to a minus sign, by requiring that $|\mathbf{C}|=1$. By convention we choose the C-G coefficient corresponding to the highest m_1 to be positive. In our case, this is C_n. We initiante the recursion relation by requiring that $C_n=1$. Equation (60) can now be solved by solving

$$\begin{pmatrix} a_{11} & a_{21} & 0 & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\ 0 & a_{23} & a_{33} & a_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2\,n-1} & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C'_1 \\ C'_2 \\ C'_3 \\ \vdots \\ C'_{n-1} \\ C'_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{62}$$

Then we obtain $\mathbf{C}$ by dividing $\mathbf{C}'$ by its norm. The main difference between (60) and (62) is that in the latter we required C'_n to equal 1 in order to initiate the recursion. This also changes the rank of A to n, so that there is only one solution. Equation (62) can be solved very efficiently by computer algorithms. In appendix A there is a sample code written in Julia that uses this algorithm to calculate the C-G coefficients.

Let us consider an example to see how the C-G coefficients are found.

Example 2.21. Consider the case where we have two particles, with $l_1 = 2$, and $l_2 = 4$, and they are in a combined state with $l = 5$ and $m = -3$. Then by equations (56) and (57) we find that $m_{1,1} = -2$ and $m_{1,n} = 1$, which implies $n = 4$. Now, we have $m_{1,k} = -3 + k$ and $m_{2,k} = -k$ for $k = 1, ..., 4$. Using equations (62) and (61), we find the following matrix equation:

$$\begin{pmatrix} 0 & 6\sqrt{2} & 0 & 0 \\ 6\sqrt{2} & 0 & 2\sqrt{21} & 0 \\ 0 & 2\sqrt{21} & -4 & 4\sqrt{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C'_1 \\ C'_2 \\ C'_3 \\ C'_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Solving this system we find

$$\begin{pmatrix} C'_1, & C'_2, & C'_3, & C'_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{14}, & 0, & \sqrt{3}, & 1 \end{pmatrix}$$

Now dividing C' by its norm we obtain

$$\begin{pmatrix} \langle 2\ 4\ -2\ -1 | 2\ 4\ 5\ -3 \rangle \\ \langle 2\ 4\ -1\ -2 | 2\ 4\ 5\ -3 \rangle \\ \langle 2\ 4\ 0\ -3 | 2\ 4\ 5\ -3 \rangle \\ \langle 2\ 4\ 1\ -4 | 2\ 4\ 5\ -3 \rangle \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{7}{15}} \\ 0 \\ \sqrt{\frac{6}{15}} \\ \sqrt{\frac{2}{15}} \end{pmatrix}$$

3 Mathematics

So far we have seen in detail what the Clebsch-Gordan coefficients mean for quantum mechanics, how physicists work with them, and how they are calculated. We will now use the C-G coefficients in a more mathematical setting. This setting is firmly grounded in representation theory. We will start with recalling the necessary group theory, then we will discuss unitary representations and how they relate to the spherical harmonics, and finally we will discuss the tensor product of irreducible representations and how they relate to the Clebsch-Gordon coefficients. But first, a remark:

Remark. On notation:

- S^{n-1} denotes the unit n-sphere, i.e.

$$S^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$$

- $\mathbf{L^2}(S^{n-1})$ denotes the set of square integrable functions $f : S^{n-1} \to \mathbb{R}$.
- Let H be a Hilbert space, then $U(H)$ denotes the space of linear bounded operators $T : H \to H$ that are surjective and unitary, i.e.

$$\langle Th_1 | Th_2 \rangle = \langle h_1 | h_2 \rangle$$

  for all $h_1, h_2 \in H$.

- $SO(n)$ denotes the set of n by n rotation matrices, i.e.

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : R^T R = I \wedge \det(R) = 1\}$$

We state without proof that $\mathbf{L^2}(S^{n-1})$ is a Hilbert space with inner product

$$\langle f | g \rangle = \int_{S^{n-1}} f^* g dx,$$

as in equation (1).

3.1 Group Theory

Definition 3.1. A *group* is a set G, combined with a binary operator '$\circ$', with $\circ : G \times G \to G$ that satisfies the following conditions (also called the *group axioms*):

1. *(associativity):* $\forall a, b, c \in G : a \circ (b \circ c) = (a \circ b) \circ c$
2. *(identity element):* $\exists e \in G : \forall a \in G : a \circ e = e \circ a = a$
3. *(inverse element):* $\forall a \in G : \exists b \in G : a \circ b = b \circ a = e$

Example 3.2. $SO(n)$ is a group with matrix multiplication as binary operation. For $R_1, R_2, R_3 \in SO(n)$ we have $R_1 \circ (R_2 \circ R_3) = R_1(R_2R_3) = (R_1R_2)R_3 = (R_1 \circ R_2) \circ R_3$. The identity element $e = I$ the identity matrix, and the inverse of R is R^T. Note that we do not necessarily have $R_1 \circ R_2 = R_2 \circ R_1$.

Example 3.3. Let H be a Hilbert space, then $U(H)$ is a group under composition of maps '$\circ$'. Let $A, B \in U(H)$.

Proof. We must first show that $A \circ B \in U(H)$. Let $h_1, h_2 \in H$, then we have

$$\begin{aligned}\langle A \circ B(h_1) | A \circ B(h_2)\rangle &= \langle A(B(h_1)) | A(B(h_2))\rangle \\ &= \langle B(h_1) | B(h_2)\rangle \\ &= \langle h_1 | h_2\rangle ,\end{aligned}$$

where we used the fact that A and B are unitary. Furthermore, we know that composition of maps is associative. We also know that the identity map I with $I(h) = h$ for every $h \in H$ is unitary and for every $A \in U(H)$ we have

$$A \circ I(h) = A(I(h)) = A(h) = I(A(h)) = I \circ A(h).$$

Finally, A is injective. To see this, let $h_1, h_2 \in H$, then

$$\begin{aligned}\|h_1 - h_2\| &= \sqrt{\langle h_1 - h_2 | h_1 - h_2\rangle} \\ &= \sqrt{\langle A(h_1 - h_2) | A(h_1 - h_2)\rangle} \\ &= \|A(h_1 - h_2)\| \\ &= \|A(h_1) - A(h_2)\| a.\end{aligned}$$

This shows A is an *isometry*, meaning that it is a distance preserving function. This implies that if $h_1 \neq h_2$, then $A(h_1) \neq A(h_2)$, since their distance is larger than 0. This means that A is injective. Since A is also surjective, we can define an inverse A^{-1}. We have now shown that $U(H)$ satisfies the group axioms. Therefore, it is a group. □

We can actually calculate the inverse A^{-1} of $A \in U(H)$ quite easily. In fact,

$$\langle h_1 | h_2\rangle = \langle Ah_1 | Ah_2\rangle = \langle A^\dagger A h_1 | h_2\rangle$$

implies that $A^\dagger A = I$, meaning that

$$A^\dagger = A^{-1}.$$

From here on out we will drop the '$\circ$' out of the notation as the binary operator of a group, so $a \circ b$ becomes ab for a and b in a group G.

Definition 3.4. Let G_1 and G_2 be groups. A function $\phi : G_1 \to G_2$ is called a *homomorphism* if

$$\phi(gg') = \phi(g)\phi(g')$$

for all $g, g' \in G_1$. We call ϕ a *group isomorphism* if it is bijective as well. We call G_1 and G_2 *isomorphic* if there exists an isomorphism between G_1 and G_2. This is denoted by $G_1 \equiv G_2$.

When $G_1 \equiv G_2$, the groups G_1 and G_2 have essentially the same group structure, meaning that the only difference between the groups is the notation of the elements in both groups. The behaviour of both groups is exactly the same. The idea of isomorphisms is not limited to groups only. In general, an isomorphism is a structure preserving bijection between two mathematical objects. For instance, two vector spaces U and V over the same field are isomorphic if there is a linear bijective map $T : U \to V$ that preserves addition and scalar multiplication, i.e.

$$T(u_1 + u_2) = T(u_1) + T(u_2) \qquad \text{and} \qquad cT(u) = T(cu).$$

Similarly, two Hilbert spaces H_1 and H_2 are isomorphic if there exists a linear bijective map $T : H_1 \to H_2$ that preserves the inner product, i.e.

$$\langle h|h'\rangle = \langle T(h)|T(h')\rangle$$

for $h, h' \in H_1$. We will use the notation '$\equiv$' to indicate an isomorphism for any type of object.

Definition 3.5. Let G be a group with identity element e and let X be a set. The function

$$\alpha : G \times X \to X \qquad \text{with} \qquad \alpha(g, x) = g \cdot x$$

is called a *left group action* if it satisfies the following axioms:

1. *(identity)*: $e \cdot x = x$ for all $x \in X$
2. *(compatability)*: $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and $x \in X$

It is often said that G *acts* on X by the left group action.

It is important to realise that the left hand side in the second axiom has two subsequent group actions on an element x, and the right hand side has a group operation along with one group action on x. The next examples will be very illustrative.

Example 3.6. The group $SO(n)$ acts on the set S^{n-1} in the following manner

$$\alpha : SO(n) \times S^{n-1} \to S^{n-1} \qquad \text{with} \qquad R \cdot x = Rx.$$

We simply rotate a point on the sphere by the given rotation. It follows the left action axioms, because $I \cdot x = Ix = x$ and

$$R_1 \cdot (R_2 \cdot x) = R_1(R_2 \cdot x) = R_1(R_2 x) = (R_1 R_2)x = (R_1 R_2) \cdot x$$

by associativity of the matrix product.

Example 3.7. The group $SO(n)$ also acts on the set $\mathbf{L^2}(S^{n-1})$. Define

$$\alpha : SO(n) \times \mathbf{L^2}(S^{n-1}) \to \mathbf{L^2}(S^{n-1}) \qquad \text{by} \qquad R \cdot f(x) = f(R^{-1}x).$$

For the first axiom we have

$$I \cdot f(x) = f(I^{-1}x) = f(x).$$

For the second axiom we must show that $R_1 \cdot (R_2 \cdot f(x)) = (R_1 R_2) \cdot f(x)$. Write $R_2 \cdot f(x)$ as $g(x)$. We then have

$$\begin{aligned} R_1 \cdot (R_2 \cdot f)(x) &= R_1 \cdot g(x) \\ &= g(R_1^{-1}x) \\ &= R_2 \cdot f(R_1^{-1}x) \\ &= f(R_2^{-1}R_1^{-1}x) \\ &= f((R_1R_2)^{-1}x) \\ &= (R_1R_2) \cdot f(x). \end{aligned}$$

There is an interesting interplay between certain kinds of group actions on Hilbert spaces and group homomorphisms on the unitary operators of the same Hilbert space:

Example 3.8. Let G act linearly on Hilbert space H in a unitary way, i.e.

$$\langle g \cdot h_1 | g \cdot h_2 \rangle = \langle h_1 | h_2 \rangle$$

for every $g \in G$ and $h_1, h_2 \in H$. Then there is a homomorphism $\phi : G \to U(H)$. This is precisely the map $\phi(g)(h) = g \cdot h$. Here h is the variable of a function in $U(H)$, not a variable of ϕ.

Conversely, when we have a homomorphism $\phi : G \to U(H)$, we can define a group action on H:

$$g \cdot h = \phi(g)(h).$$

Here h is a variable of the action.

As a concrete example, the action defined by example 3.7 yields a group homomorphism $\phi : SO(n) \to U(\mathbf{L^2}(S^{n-1}))$. since the action of rotation is linear and unitary.

Example 3.8 leads nicely to the definition of a unitary representation.

3.2 Unitary Representations and Irreducibility

Definition 3.9. A *unitary representation* π of a topological group G on a Hilbert space H is a homomorphism

$$\pi : G \to U(H)$$

with

$$g \mapsto \pi(g)h$$

such that π is a norm continuous function for all $h \in H$.

A topological group is a topological space that is also a group, where multiplication and inversion are continuous maps. The main point of a topological space is that there is a sense of how close two elements in the space are to each other. For instance a metric space is a topological space, because of the distance function. A Hilbert space is a topological space as well, since the inner product induces a metric, which induces a topology in its turn. The distance function of a metric gives an actual number to how close two elements are, but this is not necessarily the case for a topological space. The details of how this works do not matter too much to us however. With a norm continuous function we mean that a small perturbation in the input $g \in G$ will cause only a small perturbation in the output $h \in H$, where we use the inner product of the Hilbert space to define the topology. Again, the details do not matter much to us.

It might seem strange to define such a specific homomorphism in definition 3.9. An important reason we do this is because groups are often difficult to understand. By using a representation π, we can translate each element of the group G to an element of a group that we can represent by a matrix. We can understand this group better by using the tools of linear algebra. Often we can translate what we learn this way back to group G.

In our case however, we can also use irreducible representations to do the opposite. We are interested in $\mathbf{L^2}(S^2)$, the space that our angular momentum operators work on. We will use the group of rotations $SO(3)$, or more accurately, the representation of $SO(3)$ on $\mathbf{L^2}(S^2)$ to understand this Hilbert space, its subspaces, and tensor products of its subspaces better.

Example 3.10. From the action defined by example 3.7, we define the homomorphism

$$L : SO(n) \to U(L^2(S^{n-1})) \qquad by \qquad (L(R)f)(x) = f(R^{-1}x)$$

for $f \in \mathbf{L^2}(S^{n-1})$. Then L is a unitary representation of $SO(n)$. In order to see that this is true, we must first show that $L(R)$ is actually an element of $U(\mathbf{L^2}(S^{n-1}))$, meaning that $L(R)$ is a unitary operator. To see that this is true, let $g(x), f(x) \in \mathbf{L^2}(S^{n-1})$ and $R \in SO(n)$. Then we have

$$\begin{aligned}
\langle (L(R)f)(x) | (L(R)g)(x) \rangle &= \int_{S^{n-1}} (L(R)f^*)(x)(L(R)g)(x)dx \\
&= \int_{S^{n-1}} f^*(R^{-1}x)g(R^{-1}x)dx \\
&= \int_{S^{n-1}} f^*(x)g(x)dx \\
&= \langle f(x) | g(x)) \rangle
\end{aligned}$$

We must also show that L is a homomorphism, i.e. $(L(R_1)(L(R_2)f))(x) = (L(R_1R_2)f)(x)$. This argument is the same as that of example 3.8, but the

notation is a bit different. Write $(L(R_2)f)(x)$ as $g(x)$. We then have

$$\begin{aligned}(L(R_1)(L(R_2)f))(x) &= (L(R_1)g)(x)\\ &= g(R_1^{-1}x)\\ &= (L(R_2)f)(R_1^{-1}x)\\ &= f(R_2^{-1}R_1^{-1}x)\\ &= f((R_1R_2)^{-1}x)\\ &= (L(R_1R_2)f)(x).\end{aligned}$$

We often say that the Hilbert space H is the unitary representation of group G, instead of the homomorphism π. This is done when it is clear what the homomorphism is. This abuse of notation can be very confusing for someone new to representation theory. In example 3.10 for instance, we could have also said that $\mathbf{L^2}(S^{n-1})$ is a representation of $SO(n)$.

We will now work towards the definition of irreducible representations. For this, we need the definition of the direct sum of vector spaces and of the direct sum of representations.

Definition 3.11. Let U and V be vector spaces. Then the *outer direct sum* $U \oplus V$ is the vector space with underlying set $U \times V$ and operations defined by

1. $(u \oplus v) + (u' \oplus v') = (u + u') \oplus (v + v')$ for any $u, u' \in U$ and $v, v' \in V$
2. $c(u \oplus v) = (cu) \oplus (cv)$ for any $U \in U$, $v \in V$, and $c \in \mathbb{C}$.

Here we write an element of $U \times V$ as $u \oplus V$. When U and V are inner product spaces with inner products $\langle .|. \rangle_U$ and $\langle .|. \rangle_V$ respectively, we define the direct sum inner product of $U \oplus V$ as

$$\langle u \oplus v | u' \oplus v' \rangle_{U \oplus V} = \langle u|u' \rangle_U + \langle v|v' \rangle_V$$

for any $u, u' \in U$ and $v, v' \in V$.

Note that $\dim(U \oplus V) = \dim(U) + \dim(V)$. When U and W are orthogonal subspaces of V, we can write $u \oplus w \in U \oplus W$ as $u + w$. We can do this because $u, w \in V$, so addition is well defined and since the U and W are orthogonal, the sum of the dimensions of the subspaces still equals the dimension of the direct sum. In this case, $U \oplus W$ is called an *internal direct sum*.

Example 3.12. We have

$$\mathbb{R} \oplus \mathbb{R} = \text{span}\{1 \oplus 0, 0 \oplus 1\}.$$

This is only different in notation from

$$\text{span}\{(1, 0), (0, 1)\} = \mathbb{R}^2.$$

Therefore we have $\mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2$. In general we have

$$\mathbb{R}^m \oplus \mathbb{R}^n = \mathbb{R}^{m+n}.$$

Now that we know the direct sum of vector spaces, we can also define the direct sum of representations.

Definition 3.13. Let G be a group with $g \in G$, and let H_1 and H_2 be Hilbert spaces with $h_1 \in H_1$ and $h_2 \in H_2$. Let $\pi_1 : G \to U(H_1)$ and $\pi_2 : G \to U(H_2)$ be unitary representations of G on H_1 and H_2 respectively. Then the *direct sum representation*

$$\pi_1 \oplus \pi_2 : G \to U(H_1 \oplus H_2)$$

is defined by

$$\pi_1 \oplus \pi_2(g)(h_1 \oplus h_2) = \pi_1(g)h_1 \oplus \pi_2(g)h_2.$$

Theorem 3.14. *The direct sum representation $\pi_1 \oplus \pi_2$ is a unitary representation of $H_1 \oplus H_2$.*

Proof. Let $g_1, g_2 \in G$. We must show that

$$\pi_1 \oplus \pi_2(g_1 g_2) = (\pi_1 \oplus \pi_2(g_1))(\pi_1 \oplus \pi_2(g_2)).$$

Write $\pi_1(g_2)$ as u_1 and $\pi_2(g_2)$ as u_2. Then we have

$$\begin{aligned}
\pi_1 \oplus \pi_2(g_1 g_2)(v_1 \oplus v_2) &= \pi_1(g_1 g_2)v_1 \oplus \pi_2(g_1 g_2)v_2 \\
&= \pi_1(g_1)\pi_1(g_2)v_1 \oplus \pi_2(g_1)\pi_2(g_2)v_2 \\
&= \pi_1(g_1)u_1 \oplus \pi_2(g_1)u_2 \\
&= \pi_1 \oplus \pi_2(g_1)(u_1 \oplus u_2) \\
&= \pi_1 \oplus \pi_2(g_1)(\pi_1(g_2) \oplus \pi_2(g_2)) \\
&= \pi_1 \oplus \pi_2(g_1)(\pi_1 \oplus \pi_2(g_2)).
\end{aligned}$$

We will also show that the action of $\pi_1 \oplus \pi_2$ on $H_1 \oplus H_2$ is unitary. For $g \in G$, $h_1, h_1' \in H_1$, and $h_2, h_2' \in H_2$, we have

$$\begin{aligned}
&\langle \pi_1 \oplus \pi_2(g)(h_1 \oplus h_2) | \pi_1 \oplus \pi_2(g)(h_1' \oplus h_2) \rangle \\
&= \langle \pi_1(g)h_1 \oplus \pi_2(g)h_2 | \pi_1(g)h_1' \oplus \pi_2(g)h_2' \rangle \\
&= \langle \pi_1(g)h_1 | \pi_1(g)h_1' \rangle + \langle \pi_2(g)h_2 | \pi_2(g)h_2' \rangle \\
&= \langle h_1 | h_1' \rangle + \langle h_2 | h_2' \rangle \\
&= \langle h_1 \oplus h_2 | h_1' \oplus h_2' \rangle ,
\end{aligned}$$

where we used the fact that $\pi_1(g)$ and $\pi_2(g)$ are unitary actions on H_1 and H_2 respectively. □

Definition 3.15. Let V be a vector space and T be a linear mapping $T : V \to V$. A subspace W of V is called *invariant* of T, or *T-invariant* if $T(w) \in W$ for all $w \in W$.

Example 3.16. The subspaces $\{0\}$ and V are trivially invariant subspaces of V.

Example 3.17. Let $W = \text{span}\{(1,0,0),(0,1,0)\}$ be the x-y plane in $\mathbb{R}^3$. This is a subspace. Then W will be invariant of any rotation around the z-axis. This is because any vector in W will be rotated to another vector in W.

Say we have a unitary representation π of group G on Hilbert space $H = K_1 \oplus K_2$, with K_1 and K_2 *proper subspaces* of H, meaning that they are not equal to $\{0\}$ or H. Now assume K_1 and K_2 are invariant of $\pi(g)$ for every $g \in G$. Then the restriction of π to K_1 and K_2 will be a unitary representation of K_1 and K_2 respectively. These two unitary representations are called *subrepresentations* of H, and since $H = K_1 \oplus K_2$, they contain all the information contained in the original unitary representation on H. It is usually easier to understand the subrepresentations of the proper subspaces than the unitary representation of the whole space. Therefore, if we want to study a unitary representation on H, it is useful to ask whether it can be broken up into subrepresentations of proper subspaes that themselves cannot be broken up into proper subspaces, and study those representations instead. The next definition defines such representations.

Definition 3.18. A unitary representation π of group G on Hilbert space H is *irreducible* if the only invariant subspaces of $\pi(g)$ are $\{0\}$ and H for all $g \in G$.

Example 3.19. The Hilbert space $\mathbb{R}^3$ is an irreducible representation of $SO(3)$. A proper subspace of $\mathbb{R}^3$, which is a line or a plane through the origin, might be invariant of some rotations, but not to all rotations. Therefore we cannot break $\mathbb{R}^3$ up into smaller representations of $SO(3)$.

3.3 Spherical Harmonics

It turns out that the unitary representation $\mathbf{L^2}(S^2)$ of $SO(3)$ is not irreducible. In fact, they can be written as the direct sum of the spherical harmonics we have seen before. Recall that they are the eigenfunctions of the angular momentum operators. Let us define these functions more precisely.

Definition 3.20. The spherical harmonics of degree l (with $l \in \mathbb{N}$) are the (real) homogeneous polynomials p that satisfy the Laplace equation $\nabla^2 p = 0$. Let $\mathbb{R}[x,y,z]_k$ denote the vector space of real homogeneous polynomials of degree k. We can then define

$$H_l = \{p \in \mathbb{R}[x,y,z]_l : \nabla^2 p = 0\}.$$

Example 3.21. We have

- $H_0 = \text{span}\{1\}$
- $H_1 = \text{span}\{x, y, z\}$
- $H_2 = \text{span}\{xy, yz, zx, x^2 - y^2, 2z^2 - x^2 - y^2\}$

There are $2l + 1$ functions in H_l. In quantum mechanics we use a specific basis for the spherical harmonics in order to refer to a specific function in H_l. This basis is the functions Y_l^m as defined by equation (27).

Theorem 3.22. *H_l is an irreducible unitary representation of $SO(3)$.*

Proof. We take the same homomorphism L as example 3.10. The proof is exactly the same, except we must also show that H_l is $SO(3)$-invariant. To show why this is the case, we note first that the total angular momentum operator L^2 and $R \in SO(3)$ commute. This is intuitively clear, since a rotation can change the angular momentum in the x, y, or z direction, but it will not change the total angular momentum, which is conserved. With this knowledge, let $|\psi_l\rangle \in H_l$. Then we get

$$L^2 R\,|\psi_l\rangle = RL^2\,|\psi_l\rangle = Rl(l+1)\hbar^2\,|\psi_l\rangle = l(l+1)\hbar^2 R\,|\psi_l\rangle\,.$$

This means $R\,|\phi_l\rangle$ is an eigenfunction of L^2 with eigenvalue $l(l+1)\hbar^2$, which implies that $R\,|\phi_l\rangle \in H_l$. We omit the proof that H_l is irreducible. □

Theorem 3.23.

$$\mathbf{L^2}(S^2) = \bigoplus_{l=0}^{\infty} H_l,$$

Where we require that

$$\sum_{i=1}^{\infty} \|h_i\|_{H_i} < \infty$$

for every $h_i \in H_i$. Here, $\|.\|_{H_i}$ is the norm induced by the inner product of H_i.

For a proof, see [5]. Theorems 3.22 and 3.23 are significant. These theorems show us how we can break up $\mathbf{L^2}(S^2)$ into their irreducible parts. We can study these parts individually, and through doing this, discover a lot about the structure of $\mathbf{L^2}(S^2)$. Theorem 3.23 also proves postulate 2.3 in the case where the Hilbert space is $\mathbf{L^2}(S^2)$ and the observable is L^2. It is a bit strange that a postulate can be proven in some cases, but it cannot be proven in all situations, so we must keep it as a postulate.

We already now the direct sum of unitary representations, it is time to learn the tensor product of unitary representations.

Definition 3.24. Let π_1 and π_2 be representations of group G on Hilbert spaces H_1 and H_2 respectively. Then the *inner tensor product* of π_1 and π_2

$$\pi_1 \otimes \pi_2 : G \to U(H_1 \otimes H_2)$$

is defined by

$$\pi_1 \otimes \pi_2(g)(h_1 \otimes h_2) = \pi_1(g)h_1 \otimes \pi_2(g)h_2.$$

The proof that the inner tensor product of unitary representations is again a unitary representation is completely analogous to the proof that the direct sum of unitary representations is a unitary representation. Note that if π_1 and π_2 are irreducible representations, then $\pi_1 \oplus \pi_2$ is not irreducible. This is because π_1 and π_2 are irreducible subprepresentations. If π_1 and π_2 are irreducible representations, then, in general, $\pi_1 \otimes \pi_2$ is not irreducible. For

instance, $H_1 \otimes H_2$ is a unitary representation of $SO(3)$, but is not irreducible. In fact, $H_1 \otimes H_2 \equiv H_1 \oplus H_2 \oplus H_3$. In general,

$$H_{l_1} \otimes H_{l_2} \equiv \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} H_l. \tag{63}$$

It should be noted that this notation is ambiguous. This is because we can see the objects $H_{l_1} \otimes H_{l_2}$ and $\bigoplus_{l=|l_1-l_2|}^{l_1+l_2} H_l$ as Hilbert spaces, but also as unitary representations. That equation (63) holds for the objects as vector spaces is not hard to see.

This is because equation (63) looks very familiar to

$$|l_1\, m_1\rangle \otimes |l_2\, m_2\rangle = \sum_{l,m} C^{l\, m}_{l_1\, l_2\, m_1\, m_2} |l_1\, l_2\, l\, m\rangle, \tag{64}$$

which we have seen before in equation (45). The problem is that $|l_1\, l_2\, l\, m\rangle$ in equation (64) is still in the Hilbert space $H_{l_1} \otimes H_{l_2}$. If we change this equation to the linear transformation

$$C : H_{l_1} \otimes H_{l_2} \to \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} H_l$$

such that

$$C(Y^{m_1}_{l_1} \otimes Y^{m_2}_{l_2}) = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m=-l}^{l} C^{l\, m}_{l_1\, l_2\, m_1\, m_2} Y^m_l, \tag{65}$$

then the isomorphism of (63) as vector spaces is given by this transformation C. To prove this, we must show that

$$\langle Y^{m_1}_{l_1} \otimes Y^{m_2}_{l_2} | Y^{m_1'}_{l_1} \otimes Y^{m_2'}_{l_2} \rangle = \langle C(Y^{m_1}_{l_1} \otimes Y^{m_2}_{l_2}) | C(Y^{m_1'}_{l_1} \otimes Y^{m_2'}_{l_2}) \rangle .$$

The left hand side equates to

$$\langle Y^{m_1}_{l_1} \otimes Y^{m_2}_{l_2} | Y^{m_1'}_{l_1'} \otimes Y^{m_2'}_{l_2'} \rangle = \langle Y^{m_1}_{l_1} | Y^{m_1'}_{l_1'} \rangle \langle Y^{m_1}_{l_1} | Y^{m_1'}_{l_1'} \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'},$$

where the last step follows from orhonormality of the basis functions. The right hand side equates to

$$\begin{aligned}
&\langle C(Y^{m_1}_{l_1} \otimes Y^{m_2}_{l_2}) | C(Y^{m_1'}_{l_1'} \otimes Y^{m_2'}_{l_2'}) \rangle \\
&= \langle \Sigma_{l,m} C^{l\, m}_{l_1\, l_2\, m_1\, m_2} Y^m_l | \Sigma_{l',m'} C^{l'\, m'}_{l_1\, l_2\, m_1'\, m_2'} Y^{m'}_{l'} \rangle \\
&= \sum_{l,m} \sum_{l',m'} C^{l\, m}_{l_1\, l_2\, m_1\, m_2} C^{l'\, m'}_{l_1\, l_2\, m_1'\, m_2'} \langle Y^m_l | Y^{m'}_{l'} \rangle \\
&= \sum_{l,m} C^{l\, m}_{l_1\, l_2\, m_1\, m_2} C^{l\, m}_{l_1\, l_2\, m_1'\, m_2'} \\
&= \delta_{m_1 m_1'} \delta_{m_2 m_2'}.
\end{aligned}$$

The last step comes from the orthogonality relation of equation (52). So we see that the map C preserves the inner product for the basis functions. By linearity of the inner product, C must also preserve the inner product for any vector in $H_{l_1} \otimes H_{l_2}$, and therefore the vector spaces are isomorphic.

The fact that equation (63) holds for the objects as unitary representations is less clear. We must first define what we mean by isomorphic representations.

Definition 3.25. Let $\pi_1 : G \to U(H_1)$ and $\pi_2 : G \to U(H_2)$ be unitary representations of the same group G. Then π_1 and π_2 are *equivalent*, or *isomorphic as a representation* if there exists a unitary isomorphism $T : H_1 \to H_2$ such that

$$T\pi_1(g) = \pi_2(g)T \tag{66}$$

for every $g \in G$.

Theorem 3.26. *The representations*

$$H_{l_1} \otimes H_{l_2} \qquad \textit{and} \qquad \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} H_l$$

are equivalent.

This theorem is very useful. We know that $\mathbf{L^2}(S^2 \times S^2) \equiv \mathbf{L^2}(S^2)\hat{\otimes}\mathbf{L^2}(S^2)$, and combining this with theorem 3.26, we can understand the decomposition of $\mathbf{L^2}(S^2 \times S^2)$ as a direct sum of $H_j \otimes H_k$, which we can further decompose into a direct sum of H_l, and these individual H_l are irreducible. So just as we can use spherical harmonics to decompose $L^2(S^2)$ into irreducible subspaces, we can now use spherical harmonics and Clebsch-Gordan coefficients to decompose $\mathbf{L^2}(S^2 \times S^2)$ into irreducible subspaces. See [6] for an interesting application in optimization. Here, theorem 3.26 is used in energy minimization problems of interacting particle systems.

References

[1] David J. Griffiths. "Introduction to Quantum Mechanics". In: 2nd ed. Cambridge: Cambridge Unirversity Press, 2017. Chap. 3, 4.

[2] Paolo Glorioso. "On common eigenbases of commuting operators". In: *Massachusetts Inst. Technol., Cambridge, MA, USA, Tech. Rep* (2013).

[3] Knowino. *Angular momentum coupling — Knowino, an encyclopedia*. `http://knowino.org/w/index.php?title=Angular_momentum_coupling&oldid=6243`. Last accessed 23 June 2021. 2011.

[4] Williom O. Straub. "Efficient Computation of Clebsch-Gordan Coefficients". preprint. `https://vixra.org/abs/1403.0263`. 2014.

[5] S.A. Boere. *Lie Groups and Spherical Harmonics*. Bachelor thesis. Utrecht University. 2017.

[6] David de Laat. "Moment methods in energy minimization: New bounds for Riesz minimal energy problems". In: *Transactions of the American Mathematical Society* 373.2 (Oct. 2019), pp. 1407–1453. ISSN: 1088-6850. DOI: `10.1090/tran/7976`. URL: `http://dx.doi.org/10.1090/tran/7976`.

Appendix A Julia Code for the Calculation of C-G Coefficients

```
using LinearAlgebra

function clebschgordan(j1, j2, j, m)
    @assert j2>=j1

    m11 = div(m - j1 - j2 + abs(j1-j2+m), 2)
    m1n = div(m + j1 + j2 - abs(j1-j2-m), 2)
    n = m1n-m11+1

    A = zeros(n, n)

    for k = 1:n
        m1k = m11 + k - 1
        m1kp1 = m1k+1
        m2k = m - m1k
        m2kp1 = m2k-1
        A[k,k] = j1*(j1+1)+j2*(j2+1)+2m1k*m2k-j*(j+1)
        if k < n
            A[k,k+1] = A[k+1,k] = sqrt(j1*(j1+1) - m1k*m1kp1) *
            ↪  sqrt(j2*(j2+1) - m2k*m2kp1)
        end
    end

    A[n,n] = 1
    A[n,n-1] = 0
    b = zeros(n)
    b[n] = 1

    x = Tridiagonal(A) \ b

    x ./ norm(x)
end
```