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DOI
10.1137/24M1646686

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Publication date
2025

Document Version
Final published version

Published in
SIAM Journal on Applied Dynamical Systems

Citation (APA)
Gnann, M. V., Westdorp, R. W. S., & Winden, J. V. (2025). Solitary Waves in a Stochastic Parametrically Forced Nonlinear Schrödinger Equation. *SIAM Journal on Applied Dynamical Systems*, *24*(4), 3012-3044. https://doi.org/10.1137/24M1646686

SIAM J. Applied Dynamical Systems
Vol. 24, No. 4, pp. 3012–3044

Solitary Waves in a Stochastic Parametrically Forced Nonlinear Schrödinger Equation*

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Abstract. We study a parametrically forced nonlinear Schrödinger (PFNLS) equation, driven by multiplicative translation-invariant noise. We show that a solitary wave in the stochastic equation is orbitally stable on a timescale which is exponential in the inverse square of the noise strength. We give explicit expressions for the phase shift and fluctuations around the shifted wave which are accurate to second order in the noise strength. This is done by developing a new perspective on the phase-lag method introduced by Krüger and Stannat. Additionally, we show well-posedness of the equation in the fractional Bessel space H^s for any $s \in [0, \infty)$, demonstrating persistence of regularity.

Key words. stochastic partial differential equation, nonlinear Schrödinger equation, solitary wave

MSC codes. 37H30, 35C08, 35Q55, 35Q60, 35R60, 60H15

DOI. 10.1137/24M1646686

1. Introduction.

1.1. The parametrically forced nonlinear Schrödinger equation.

Optic fibers that act as waveguides for electromagnetic signals form the basis for systems of fiber-optic communications, enabling long-distance communication at high bandwidth [2]. The behavior of a pulse propagating through an optic fiber is governed by the nonlinear Schrödinger (NLS) equation [1], which is an archetypal example of a nonlinear dispersive equation that is known to support solitary waves. The NLS equation has many applications in physics, for instance in the description of Bose–Einstein condensates [11], deep-water waves [40], and plasma oscillations [36]. In these applications, the NLS equation describes the complex amplitude of a wave packet propagating through a nonlinear medium. We refer to [38] for a detailed treatment of the physical background.

*Received by the editors March 25, 2024; accepted for publication (in revised form) by J. Bronski July 21, 2025; published electronically October 24, 2025.
https://doi.org/10.1137/24M1646686
Funding: The work of the second author was supported by Dutch Research Council (NWO) grant 613.009.137. The work of the third author was supported by a DIAM fast-track scholarship. Early versions of the results contained in this paper can be found in the MSc theses of the second and third authors, both prepared under the supervision of the first author at Delft University of Technology.
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In optic fibers, the nonlinear behavior arises due to a response of the refractive index of the fiber to an applied electric field known as the Kerr effect, leading to a cubic nonlinear term in the equation. Effective signal transmission in optic communication systems may be obstructed by the presence of linear loss in the fiber, weakening the signal as it propagates. Kutz et al. [28] proposed a method of compensating loss using periodic phase-sensitive amplification [28], which has since become a popular approach for increasing feasible transmission lengths. The approach is modelled by the parametrically forced nonlinear Schrödinger (PFNLS) equation:

$$\mathrm{d}u = (i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}))\mathrm{d}t + i\kappa|u|^2 u\mathrm{d}t \quad \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+. \tag{1.1}$$

Here, the complex-valued function $u(x,t)$ denotes the envelope of the electric field in an optic fiber, t is the distance along the fiber, and x denotes time in a translating frame that moves with the group velocity of light. The constants $\gamma > 0$ and $\mu > 0$ model linear loss in the fiber and phase-sensitive amplification, respectively. The constant $\nu \in \mathbb{R}$ models a phase advance of the signal carrier, and the constant $\kappa > 0$ denotes the strength of the Kerr effect in the fiber. In this model, the local effect of the periodically spaced phase-sensitive amplifiers is averaged over the spacing length of the amplifiers. This description assumes that the amplifiers are closely spaced, which is valid for long propagation lengths [32]. In particular, the model applies well to a recirculating loop used for long-term storage of pulses in optical networks.

In case that $\mu > \gamma$, i.e., enough amplification is supplied, (1.1) admits stationary solutions $u(t,x) = u^*(x)$ called *solitary waves,* with the profile u^* given by

$$u^*(x) = \sqrt{\frac{2(\nu + \epsilon\mu\sin(2\theta))}{\kappa}}\operatorname{sech}(\sqrt{\nu + \epsilon\mu\sin(2\theta)}\,x)e^{i\theta}, \tag{1.2}$$

where θ is a solution to $\cos(2\theta) = \gamma/\mu$. This can be seen from [24, equation (1.8)] after scaling in κ by setting $\phi = \frac{1}{2}\sqrt{\kappa}u$. As (1.1) is translation invariant, shifting the solitary waves by an arbitrary constant $a \in \mathbb{R}$ produces a family of solutions. The solitary waves for which $\sin(2\theta) > 0$ were shown to be orbitally exponentially stable by Kapitula and Sandstede [24]: small perturbations of the solitary wave converge at an exponential rate to a suitable translate of the solitary wave. Solitary waves for which $\sin(2\theta) < 0$ are known to be unstable [29]. Here and throughout the rest of this work, the "orbit" in orbital stability refers to the orbit of u^* under the action of the group of spatial translations.

We briefly note that in the physical application of optic fiber loops, the term standing wave is misleading, as the equation describes the electric field in a moving frame. The standing waves (1.2) represent traveling pulses, and their stability is crucial for attaining long transmission lengths of signals and for the feasibility of long-time storage.

The stability analysis in [24] relies on computing the spectrum of the (real-)linear operator

$$\mathcal{L}v = i\Delta v - i\nu v - \epsilon(\gamma v - \mu\overline{v}) + i\kappa(2|u^*|^2 v + (u^*)^2\overline{v})$$

on $L^2(\mathbb{R};\mathbb{C})$ associated with the linearization of (1.1) around the solitary wave. It is known that the spectrum of the linearization is located at an $\mathcal{O}(\epsilon)$ distance to the left of the imaginary axis, except for a simple eigenvalue at zero [3, 24]. This eigenvalue arises due to the translation invariance of (1.1). For $\epsilon = \nu = 0$, the operator $\mathcal{L}$ corresponds to the linearization around the primary soliton in the NLS equation, and has continuous spectrum on the imaginary axis. The primary NLS soliton is also orbitally stable, but no exponential decay of perturbations can be expected [41, 33]. As such, parametric forcing entails stronger linear stability.

1.2. A stochastic equation. Mecozzi et al. [32] discusses two mechanisms that further inhibit signal transmission by introducing noise in the system, thereby transforming the description of pulse propagation into a stochastic partial differential equation. In this paper, we study the evolution of the solitary wave u^* (1.2) in the stochastic parametrically forced nonlinear Schrödinger (SPFNLS) equation:

$$\mathrm{d}u = (i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}))\,\mathrm{d}t + i\kappa|u|^2 u\,\mathrm{d}t - iu\circ(\phi * \mathrm{d}W) \quad \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+. \tag{1.3}$$

The symbol W denotes a cylindrical Wiener process in the Hilbert space $L^2(\mathbb{R},\mathbb{R})$, meaning that $\xi := \frac{\mathrm{d}W}{\mathrm{d}t}$ is a (formal) space-time white noise, and $\circ$ denotes the Stratonovich product. Here, ϕ is a real-valued function which serves to regularize the noise. Thus, u is multiplied by a noise process $\zeta := \phi * \xi$ which is white in time, and formally satisfies the covariance relation

$$\mathbb{E}[\zeta(t,x)\zeta(t,y)] = \int_{\mathbb{R}} \phi(z)\phi(z+x-y)\,\mathrm{d}z$$

in space. Because the covariance only depends on $x-y$, (1.3) preserves the physically relevant symmetry of translation invariance (in a statistical sense). This is highly relevant to our study of the motion of solitary waves.

The multiplicative noise term that we consider in (1.3) models phase noise induced by the coupling of light with the thermally excited acoustical modes of the fiber known as guided acoustic-wave Brillouin scattering (GAWBS) [32]. We use the Stratonovich product, as it is more realistic for physical applications. Indeed, in the absence of parametric forcing, it allows for conservation of the $L^2(\mathbb{R})$-norm [5, Proposition 4.1]. Because our variable x corresponds to physical time, our noise is correlated in time, which is a natural assumption in the context of GAWBS phase noise. The other noise effect proposed in [32] is due to quantum effects and results in an additive noise term. We focus in the present paper only on the multiplicative GAWBS phase noise.

1.3. Well-posedness. Our first result concerns well-posedness of the stochastic equation (1.3). We show that for any $s \geq 0$, ϕ in the fractional Bessel space $H^s(\mathbb{R};\mathbb{R})$ and $u(0)\in H^s_x$, (1.3) has a unique mild solution u taking values in the space $C([0,T];H^s_x)\cap L^r(0,T;L^p_x)$ for every $T>0$ and certain pairs (p,r) (see Theorem 3.1 and Definition 2.2).

The "standard" SNLS equation with linear multiplicative noise (corresponding to the case $\epsilon=\nu=\gamma=\mu=0$) was first shown to be well-posed in the spaces L^2_x (corresponding to $s=0$) [5] and H^1_x (corresponding to $s=1$) [6]. A proof of the L^2_x well-posedness using stochastic Strichartz estimates is given in [20]. Since the PFNLS equation differs from the NLS equation by linear terms, our proof of well-posedness is very similar. The main novelties are well-posedness in $H^s(\mathbb{R};\mathbb{R})$ for $s\in[0,\infty)\setminus\{0,1\}$ and the use of translation-invariant noise. The translation-invariant noise, aside from being motivated by physical symmetries, is relevant to our subsequent study of the solitary waves and is not directly covered by previous results. The well-posedness in H^s_x shows that, like its deterministic counterpart, the SPFNLS (and by extension, the one-dimensional cubic SNLS) equation has *persistence of regularity,* meaning that regularity of the solution is the same as the minimum of that of the noise and the initial data. Previous results on stochastic versions of these equations have mainly been concerned with the cases $s=0$ and $s=1$.

1.4. Orbital stability. With the well-posedness of (1.3) firmly established, we turn to the stability of the solitary wave u^* with $\sin(2\theta) > 0$ (see the discussion following (1.2)) in the stochastic equation. We establish that the solitary wave is orbitally stable under the multiplicative stochastic forcing in (1.3) on a timescale $T \sim \exp(\sigma^{-2})$, where σ denotes the strength of the noise. We describe the solution to (1.3) with initial condition close to u^* using the decomposition

$$u(x,t) = u^*(x + a(t)) + v(x,t),$$

where a is a real-valued stochastic process that tracks the wave position, and v an infinite-dimensional perturbation which is small when measured in the L^2_x-norm. In the parabolic setting, such problems are well-studied (see, e.g., [18, 23, 26, 31]). Rigorous results in a dispersive setting are more scarce but available [7, 8, 9, 10, 42]. However, these works are limited to the time scale $T \sim \sigma^{-2}$, and stability for $T \sim \exp(\sigma^{-2})$ has not been shown before in this setting as far as we are aware.

We give explicit expressions for $a(t)$ and $v(t)$ which are accurate to second order in σ. Second-order results in this setting are scarce, and mostly consist of formal computations [30]. By developing a new perspective on an established phase-tracking method (see section 1.5) we rigorously and efficiently prove accuracy of the second-order expressions for the first time.

To first order, the phase process $a(t)$ behaves like a Brownian motion with variance proportional to $t\sigma^2$, and the perturbation $v(t,x)$ behaves like an infinite-dimensional Ornstein–Uhlenbeck process. In particular, v satisfies an estimate of the form

$$\mathbb{E}[\|v(t)\|^2_{L^2_x}]^{1/2} \le C\sigma(e^{-at}\|v(0)\|_{L^2} + \min\{t^{\frac{1}{2}}, 1\}) + \mathcal{O}(\sigma^2) \tag{1.4}$$

(see Theorem 3.6). Using such bounds to control the development of a perturbation over short time-scales combined with a resetting procedure, we show that there exists a stochastic process $a(t)$ and constants $C, k, \varepsilon' > 0$ such that

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|u(\cdot,t) - u^*(\cdot + a(t))\|_{L^2_x} \ge \varepsilon\right] \le CTe^{-k\sigma^{-2}\varepsilon^2}$$

for all $T > 0$ and $0 < \sigma \le \varepsilon \le \varepsilon'$ (Proposition 3.7 and Theorem 3.8). This shows stability on a timescale $T \sim e^{k\sigma^{-2}\varepsilon^2}$. By a scaling argument, this is (up to better constants) the longest time for which the solitary wave can be expected to be stable, and matches the best results obtained in different settings, such as [19, 31].

1.5. Phase tracking. When showing stochastic orbital stability, there are several different ways of defining and tracking the phase process $a(t)$ (see, e.g., [26, 23, 18]). Our method is closely related to the one developed in [26, 27], which has also been applied by [16] to the FitzHugh–Nagumo equation [16]. Briefly, this method consists of defining an approximation process $a_m(t)$ using the random ODE

$$\frac{\mathrm{d}a_m(t)}{\mathrm{d}t} = -m\frac{\partial\|u(t,x) - u^*(x + a_m(t))\|^2_{L^2_x}}{\partial a_m},$$

and computing an SDE for $\frac{\mathrm{d}a_m(t)}{\mathrm{d}t}$. The idea is that $a_m(t)$ will dynamically move towards a minimizer of $a \mapsto \|u(t,x) - u^*(x+a)\|^2_{L^2_x}$, so that $u^*(x + a_m(t))$ is expected to serve as a good approximation to $u(t,x)$. By approximating the SDE to first order in σ and taking $m \to \infty$, orbital stability can be shown on timescales of the order $T \sim \sigma^{-2}$.

Our method obtains a similar phase process via a completely different route, which we briefly summarize. Before introducing our phase process, we first prove an asymptotic expansion of the form

$$u(t,x) = u^*(x) + \sigma v_1(t,x) + \sigma^2 v_2(t,x) + \mathcal{O}(\sigma^3) \tag{1.5}$$

(Proposition 3.2). This results in explicit representations of v_1 and v_2, as well as exact estimates relating to the validity of the expansion. On the linear level, the dynamics of v_1 and v_2 are governed by an operator $\mathcal{L}$ which is the linearization of (1.1) around u^* (see Theorem 2.6). Since the PFNLS equation is not parabolic, we rely on dispersive estimates to control the nonlinear terms. We also require Gaussian tail estimates on the remainder terms, for which we use a result by Seidler [35] to estimate L^p_Ω-norms of stochastic integrals with a constant which is $\mathcal{O}(\sqrt{p})$.

The next step is to introduce the following decomposition of v_1 and v_2:

$$v_1(t,x) = w_1(t,x) + a_1(t)u^*_x(x), \tag{1.6a}$$

$$v_2(t,x) = w_2(t,x) + a_2(t)u^*_x(x) + \frac{1}{2}a_1(t)^2 u^*_{xx}(x), \tag{1.6b}$$

where w_1 and w_2 should be regarded as being determined by (1.6) for a given choice of a_1 and a_2. The decomposition (1.6) is motivated by the fact that u^*_x is an eigenfunction of $\mathcal{L}$ with eigenvalue 0 (which is a consequence of the fact that (1.1) has a translational symmetry). By applying the spectral projection from Theorem 2.6 to (1.6) it is seen that there is a *unique* choice of a_1, a_2 such that w_1, w_2 both avoid the (secular) zero eigenmode. Since $\mathcal{L}$ has a spectral gap as shown in [24], the linear dynamics of w_1 and w_2 are then exponentially stable. We use this fact to control the nonlinear terms and prove Theorem 3.6, which captures the slower growth rates of w_1 and w_2 compared to v_1, v_2.

Directly combining the asymptotic expansion (1.5) with the decomposition (1.6) using a Taylor expansion finally results in

$$u(t,x) = u^*(x + \sigma a_1(t) + \sigma^2 a_2(t)) + \sigma w_1(t,x) + \sigma^2 w_2(t,x) + \mathcal{O}(\sigma^3),$$

which, combined with smallness of w_1 and w_2, shows orbital stability on a timescale for which the asymptotic expansion (1.5) is valid.

Asserting stability on longer timescales requires additional effort. The main issue is that (1.5) is a linearization around u^*, but after time t the solution is close to the translated wave $u^*(x + a(t))$. Thus, when $a(t)$ gets large enough (which happens on a timescale $T \sim \sigma^{-2}$), the linearization becomes completely inaccurate. We remedy this by resetting the linearization after a fixed time T, by linearizing around the shifted wave $u^*(x + a(T))$ instead. This makes it possible to combine the short-term estimates on each time interval $[NT, (N+1)T]$ to obtain long-term stability (Theorem 3.8). The cost of this procedure is that we incur a discontinuity in the phase process each time we reset, and our explicit representation is only valid in between resetting. We are not aware of any methods to obtain *explicit* descriptions of the phase which are accurate on long timescales. Surprisingly, the resetting procedure suggests that it is possible to show stability on long timescales without accurately tracking the phase on short timescales. This is something we aim to investigate in future work.

1.6. Outline. In section 2 we specify our notation and introduce the preliminaries necessary to state and prove the main results (Theorems 3.1, 3.6, and 3.8), which are contained in section 3. The proof of well-posedness of (1.3) is given in section 4, followed by the proof of the stability results in section 5. Appendices A and B contain some auxiliary results needed for the proofs.

2. Preliminaries.

We now give the preliminaries required to state and prove the main results, as well as some notational shorthands. We give a rigorous meaning to (1.3), and formulate the Strichartz estimates which are used to show well-posedness. Afterwards we state the deterministic stability of the solitary wave, along with additional Strichartz estimates related to the linearization around the solitary wave, which are needed for our stochastic stability results.

2.1. Notation and conventions. We denote the norm of general normed spaces X by $\|\cdot\|_X$, and the inner product of general inner product spaces H by $\langle\cdot,\cdot\rangle_H$. In the case where H is complex, we take the inner product to be conjugate-linear in the second variable. The space of bounded linear operators from a Banach space X to a Banach space Y is denoted by $\mathcal{L}(X;Y)$, and the space of Hilbert–Schmidt operators between separable Hilbert spaces H and $\tilde{H}$ as $\mathcal{L}_2(H;\tilde{H})$.

If X is a Banach space, we will write $C([0,T];X)$ for the space of continuous X-valued functions. For $p\in[1,\infty]$, we write $L^p(S;X)$ for the usual Bochner spaces defined on a measure space $(S,\mathcal{F},\mu)$ (which coincide with the Lebesgue spaces if $X=\mathbb{C}$ or $X=\mathbb{R}$). If $p=2$, and H is a Hilbert space, then $L^2(S;H)$ is a Hilbert space with the inner product given by

$$\langle f,g\rangle_{L^2(S;X)}=\int_S\langle f,g\rangle_H\,\mathrm{d}\mu.$$

For $z\in\mathbb{C}$, we write $\overline{z}$ for its complex conjugate. For $p\in[1,\infty]$, we write p' for its Hölder conjugate, which is the unique $p'\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{p'}=1$. Throughout this paper, all random variables will be defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a complete and right-continuous filtration $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,\infty)}$. We will make use of the following abbreviations:

$$\begin{aligned}L^p_x&:=L^p(\mathbb{R};\mathbb{C}),\\ L^p_\Omega(X)&:=L^p(\Omega;X),\\ L^p(T,T';X)&:=L^p([T,T'];X),\end{aligned}$$

where $\mathbb{R}$ and $[T,T']$ are equipped with the usual Lebesgue measure.

The weak derivative of a weakly differentiable function $f\in L^p_x$ is denoted by $\partial_x f$ and we write $\Delta=\partial_x^2$ for the Laplacian on the real line. We write u^*_x and u^*_{xx} for the first and second spatial derivatives of u^*. For $s\in[0,\infty)$ and $p\in(1,\infty)$, the Bessel space $H^{s,p}_x$ consists of the functions $f\in L^p_x$ for which the quantity

$$\|f\|_{H_x^{s,p}} = \|(1-\Delta)^{\frac{s}{2}} f\|_{L_x^p}$$

is finite. Here, the fractional power $(1-\Delta)^{\frac{s}{2}}$ is defined using the Fourier multiplier with symbol $\xi \mapsto (1+|\xi|^2)^{\frac{s}{2}}$. The space $H_x^{s,p}$ is a Banach space and we have continuous embeddings $H_x^{s_1,p} \hookrightarrow H_x^{s_2,p}$ if $s_1 \geq s_2$. When k is a nonnegative integer, the Bessel space $H_x^{k,p}$ is isomorphic to the classical Sobolev space $W_x^{k,p}$, which consists of the function in L_x^p for which all partial derivatives of order k or less are also in L_x^p. Proofs of these statements rely on the theory of singular integrals, and can, for example, be found in [37, Chapter 3]. We also note that $H_x^{s,2}$ is a Hilbert space with inner product $\langle f, g\rangle_{H_x^{s,2}} = \langle (1-\Delta)^{\frac{s}{2}} f, (1-\Delta)^{\frac{s}{2}} g\rangle_{L_x^2}$. In this case we will write $H_x^s := H_x^{s,2}$.

Lastly, we denote by $\{S(t)\}_{t\in\mathbb{R}}$ the C_0-group on L_x^2 generated by $i\Delta : L_x^2 \supset H_x^2 \to L_x^2$, which acts at $t \in \mathbb{R}$ as the Fourier multiplier with symbol $\xi \mapsto e^{-4\pi^2 i|\xi|^2 t}$. Using Plancherel's theorem, it can be seen that $S(t)$ is unitary on L_x^2. Since the Fourier multiplier of $S(t)$ commutes with that of $(1-\Delta)^{\frac{s}{2}}$, it is immediate that $S(t)$ is also a unitary group on H_x^s for any s.

2.2. Stochastic set-up. We let $W(t)$ be an $L^2(\mathbb{R};\mathbb{R})$-cylindrical Wiener process on Ω, which is adapted to $\mathbb{F}$. Then $W(t)$ has an interpretation as the time integral from 0 to t over a space-time white noise. To regularize the noise, fix some $\phi \in L^2(\mathbb{R};\mathbb{R})$ and define $\Phi\colon L^2(\mathbb{R};\mathbb{R}) \to L_x^\infty$ and $\beta \in \mathbb{R}$ as

$$\Phi f := \phi * f, \tag{2.1a}$$

$$\beta := \|\phi\|_{L_x^2}. \tag{2.1b}$$

We now convert (1.3) into an equivalent formulation in Itô form. Formally applying an Itô–Stratonovich correction to (1.3) results in

$$\mathrm{d}u = [i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}) + i\kappa|u|^2 u]\,\mathrm{d}t - \frac{1}{2}Fu\,\mathrm{d}t - iu\Phi\,\mathrm{d}W, \tag{2.2}$$

with F being defined as

$$F := \sum_{k\in\mathbb{N}} (\Phi e_k)^2, \tag{2.3}$$

where e_k is an orthonormal basis of $L^2(\mathbb{R};\mathbb{R})$. Let us collect some facts about Φ and F which will be used throughout this paper. The proof of Proposition 2.1 is contained in Appendix A.

Proposition 2.1. *Let $\phi \in L^2(\mathbb{R};\mathbb{R})$ and $u \in L_x^2$. Then the series in (2.3) is well-defined and we have the identities*

$$F = \beta^2, \tag{2.4a}$$

$$\|u\Phi\|_{\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});L_x^2)} = \beta\|u\|_{L_x^2}. \tag{2.4b}$$

If, additionally, $\phi \in H^s(\mathbb{R};\mathbb{R})$ and $u \in H_x^s$ for some $s \in [0,\infty)$, then we have the estimate

$$\|u\Phi\|_{\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});H_x^s)} \leq C_s\|\phi\|_{H_x^s}\|u\|_{H_x^s} \tag{2.4c}$$

for some $C_s > 0$ which depends only on s.

Substituting (2.4a) into (2.2), the stochastic PFNLS equation in Itô form reads

$$\mathrm{d}u = [i\Delta u - i\nu u - \epsilon(\gamma u - \mu\overline{u}) + i\kappa|u|^2 u]\,\mathrm{d}t - \frac{1}{2}\beta^2 u\,\mathrm{d}t - iu\Phi\,\mathrm{d}W. \tag{2.5}$$

From the definition of Φ (2.1a), it is clear that this operator commutes with translation. Furthermore, since $\xi = \frac{\mathrm{d}W}{\mathrm{d}t}$ formally represents a white noise, its statistics are also invariant under translation. Thus, the noise terms do not break the temporal- and spatial-translation symmetries inherent to (1.1) (in a statistical sense).

Before we proceed with the mathematical analysis, we give a meaningful interpretation to our noise. Since ξ formally has a covariance operator on $L^2(\mathbb{R};\mathbb{R})$ equal to the identity, it can be seen using (2.1a) that $\Phi\xi$ formally satisfies the covariance relation

$$\begin{aligned}\mathbb{E}\Big[(\Phi\xi)(t,x)\cdot(\Phi\xi)(t',x')\Big] &= \mathbb{E}\Big[\langle\Phi\xi(t),\delta_x\rangle_{L^2_x}\langle\Phi\xi(t'),\delta_{x'}\rangle_{L^2_x}\Big]\\ &= \delta_0(t-t')\langle\Phi^*\delta_x,\Phi^*\delta_{x'}\rangle_{L^2_x}\\ &= \delta_0(t-t')(\tilde{\phi}*\phi)(x-x'),\end{aligned}$$

where δ_a denotes a Dirac mass at the point $x=a$, and $\tilde{\phi}$ is defined via $\tilde{\phi}(x) := \phi(-x)$. Therefore, $g := \tilde{\phi}*\phi$ can be interpreted as the spatial correlation function of our noise. Note that g is an even function, so that the correlation only depends on $|x-x'|$. The variance at any point is given by $g(0)=\beta^2$, which means this quantity can be viewed as the strength of the noise.

2.3. Strichartz estimates. In the analysis of nonlinear Schrödinger equations, the dispersion displayed by the linear Schrödinger equation plays a major role. In our context, this dispersion manifests in the form of *Strichartz estimates.* These estimates give control over certain space-time mixed Lebesgue norms of solutions to the linear Schrödinger equation. In our one-dimensional setting, they take the following form.

Definition 2.2. *A pair (r,p) with $r\in[4,\infty]$, $p\in[2,\infty]$ is called* admissible *if it satisfies*

$$\frac{2}{r}+\frac{1}{p}=\frac{1}{2}. \tag{2.6}$$

Theorem 2.3 (Strichartz estimates). *Let $s\in[0,\infty)$, and let $(r,p)\neq(4,\infty)$ and (α,δ) be admissible. There exists a constant C, such that the estimates*

$$\|S(\cdot)f\|_{L^r(0,T;H^{s,p}_x)} \le C\|f\|_{H^s_x}, \tag{2.7a}$$

$$\left\|\int_0^{\cdot} S(\cdot-t')g(t')\,\mathrm{d}t'\right\|_{L^r(0,T;H^{s,p}_x)} \le C\|g\|_{L^{\alpha'}(0,T;H^{s,\delta'}_x)}, \tag{2.7b}$$

$$\left\|\int_0^{\cdot} S(\cdot-t')h(t')\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^r(0,T;H^{s,p}_x))} \le C\sqrt{q}\|\phi\|_{H^s_x}\|h\|_{L^q_\Omega(L^2(0,T;H^s_x))} \tag{2.7c}$$

hold for every $q\in[2,\infty)$, $T\in(0,\infty]$, $f\in H^s_x$, $g\in L^{\alpha'}(0,T;H^{s,\delta'}_x)$, $h\in L^q_\Omega(L^2(0,T;H^s_x))$, and $\phi\in L^2(\mathbb{R};\mathbb{R})\cap H^s_x$ (recall (2.1)).

Remark 2.4. In the case $(r,p)=(\infty,2)$, the relevant processes in Theorem 2.3 have continuous versions, and the L^∞-norm on the left-hand side of (2.7) can be replaced by $C([0,T])$. We will always use these continuous versions. This also applies to (2.14) further below.

Remark 2.5. Estimates (2.7a) and (2.7b) still hold in the case $(r,p)=(4,\infty)$. This also applies to (2.14a), (2.14b), (2.18a), and (2.18b) further below.

Estimates (2.7a), (2.7b), and (2.7c) are commonly referred to as the *homogeneous, convolution,* and *stochastic* Strichartz estimates, respectively. The homogeneous and convolution Strichartz estimates are well-known and can be found, for example, in [13, Theorem 2.3.3] or [25]. The stochastic Strichartz estimate is more recent, and was first shown in [12] for the case $r=q$. The proof of our formulation of (2.7c), which is contained in Appendix B, follows the same idea as [12], except that we use [35, Theorem 1.1] to obtain a constant which is $\mathcal{O}(\sqrt{q})$.

2.4. Solitary waves and linear stability. We now fix a set of parameters $\nu\in\mathbb{R}$, $\epsilon,\gamma,\mu>0$ which satisfy $\mu>\gamma$. We additionally let $\theta\in[0,\pi)$ be the unique solution to $\cos(2\theta)=\frac{\gamma}{\mu}$ which satisfies $\sin(2\theta)>0$. This ensures that the deterministic equation (1.1) has a stable solitary wave solution u^*, explicitly given by

$$u^*(x)=\sqrt{\frac{2(\nu+\epsilon\mu\sin(2\theta))}{\kappa}}\operatorname{sech}(\sqrt{\nu+\epsilon\mu\sin(2\theta)}x)e^{i\theta} \tag{2.8}$$

(see [24, equation (1.8)]). Since (1.1) is preserved under the transformation $u\mapsto -u$, it follows that $-u^*$ is also a stable solitary wave. Alternatively, this profile could be obtained by letting $\theta\in\mathbb{R}$. In that case the above conditions only select θ up to an additive multiple of π, giving rise to both u^* and $-u^*$ via the term $e^{i\theta}$ in (2.8).

We remark that u^* is infinitely often differentiable, and all of its derivatives are rapidly decaying.

We will frequently make use of expansions around the solitary wave u^*. Due to the cubic term in (1.3), this will require expansions of terms like $|a+b|^2(a+b)$. Here, the absolute value prevents the use of convenient multinomial expansion formulas. To remedy this, we introduce the following notation, which we call the *triple bracket*:

$$\begin{gathered}\{\cdot,\cdot,\cdot\}:\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}\\ \{a,b,c\}=ab\overline{c}+a\overline{b}c+\overline{a}bc.\end{gathered} \tag{2.9}$$

Observe that the triple bracket is symmetric, (real-)trilinear and that $|u|^2u=\frac{1}{3}\{u,u,u\}$. Therefore, we can compactly write binomial expansions like

$$|u+v|^2(u+v)=\frac{1}{3}\{u+v,u+v,u+v\}=\frac{1}{3}\{u,u,u\}+\{u,u,v\}+\{u,v,v\}+\frac{1}{3}\{v,v,v\}.$$

This notation is particularly useful when using multinomial expansions with more terms. For readability, we abbreviate

$$Lu:=-i\nu u-\epsilon(\gamma u-\mu\overline{u}). \tag{2.10}$$

Combining our new notation, we may compactly rewrite (1.1) as

$$\partial_t u=i\Delta u+Lu+\frac{1}{3}i\kappa\{u,u,u\}.$$

Using the additivity of the triple bracket, it is now straightforward to see that the operator

$$\mathcal{L}\colon v \mapsto i\Delta v + Lv + i\kappa\{u^*, u^*, v\} \tag{2.11}$$

corresponds to the linearization of (1.1) around the solitary wave u^*. The linear stability of the solitary wave (2.8) is captured in the following theorem, which has been shown in [24].

Theorem 2.6. *The operator $\mathcal{L}$ has the following properties:*

1. *$\mathcal{L}$ is the generator of a strongly continuous semigroup on L^2_x, denoted by $P(t)$.*
2. *u^*_x is an eigenfunction of $\mathcal{L}$ with eigenvalue 0, which has algebraic multiplicity one.*
3. *The spectrum of $\mathcal{L}$ on L^2_x is contained in $\{z \in \mathbb{C} : \operatorname{Re} z \le -b\} \cup \{0\}$ for some $b > 0$. Thus, the Riesz spectral projection*
$$\Pi^0 := \frac{1}{2\pi i}\oint_C (\lambda I - \mathcal{L})^{-1} d\lambda$$
is well-defined if C is a sufficiently small contour encircling 0 counterclockwise.
4. *If we additionally define $\Pi := I - \Pi^0$, then there exist constants M and $a > 0$ such that the inequalities*
$$\|P(t)\Pi\|_{\mathcal{L}(L^2_x)} \le Me^{-at}, \qquad \|P(t)\|_{\mathcal{L}(L^2_x)} \le M, \tag{2.12}$$
hold for all $t \in [0,\infty)$.

Remark 2.7. The operator $\mathcal{L}$ is not complex-linear, and the same applies to $P(t)$, Π^0, and Π. Additionally, Π^0 projects onto the real span of u^*_x as opposed to the complex span. Thus, in the context of the linearization we should regard $L^2_x \simeq L^2(\mathbb{R};\mathbb{R}^2)$ as a real vector space.

Using Π, we also define the linear operator $\mathcal{P}$ as follows:

$$\mathcal{P}\colon f \mapsto \frac{\langle f - \Pi f, u^*_x\rangle_{L^2_x}}{\|u^*_x\|^2_{L^2_x}}. \tag{2.13}$$

Proposition 2.8. *The operator $\mathcal{P}$ is bounded from L^2_x to $\mathbb{R}$, and for every $f \in L^2_x$ we have the decomposition*

$$f = \Pi f + \mathcal{P}(f) u^*_x.$$

Proof. The boundedness of $\mathcal{P}$ follows from the boundedness of Π and the Cauchy–Schwarz inequality. Now fix $f \in L^2_x$. Since $I = \Pi + \Pi^0$ and Π^0 projects onto the span of u^*_x, there exists a unique $a \in \mathbb{R}$ such that

$$f = \Pi f + \Pi^0 f = \Pi f + a u^*_x.$$

Rearranging this equation, taking inner products with u^*_x and dividing by $\|u^*_x\|^2_{L^2_x}$ shows that $a = \mathcal{P}(f)$. ∎

We now formulate appropriate Strichartz estimates for the semigroups $P(\cdot)\Pi$ and $P(\cdot)\Pi^0$ separately. Using the decomposition $P(t) = P(t)\Pi + P(t)\Pi^0$, we also obtain Strichartz estimates for $P(t)$.

Proposition 2.9 (Strichartz estimates for $P(\cdot)\Pi$). *Let $(r,p)\neq(4,\infty)$ be admissible. There exists a constant C, such that the estimates*

$$\|P(\cdot)\Pi f\|_{L^r(0,T;L^p_x)}\le C\|f\|_{L^2_x}, \tag{2.14a}$$

$$\Big\|\int_0^{\cdot} P(\cdot-t')\Pi g(t')\,\mathrm{d}t'\Big\|_{L^r(0,T;L^p_x)}\le C\|g\|_{L^1(0,T;L^2_x)}, \tag{2.14b}$$

$$\Big\|\int_0^{\cdot} P(\cdot-t')\Pi h(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(L^r(0,T;L^p_x))}\le C\sqrt{q}T^{\frac12-\frac{1}{q+\varepsilon}}\beta\|h\|_{L^q_\Omega(L^{q+\varepsilon}(0,T;L^2_x))} \tag{2.14c}$$

hold for all $q\in[2,\infty)$, $\varepsilon\in(0,\infty)$, $T\in(0,\infty)$, $f\in L^2_x$, $g\in L^1(0,T;L^2_x)$, $h\in L^q_\Omega(L^{q+\varepsilon}(0,T;L^2_x))$, and $\phi\in L^2(\mathbb{R};\mathbb{R})$ (recall (2.1)).

Proof. We first show (2.14a). Consider for some $f\in L^2_x$ the evolution equation

$$\begin{aligned} \mathrm{d}u&=[i\Delta u-i\nu u-\epsilon(\gamma u-\mu\overline{u})+i\kappa\{u^*,u^*,u\}]\,\mathrm{d}t,\\ u(0)&=\Pi f. \end{aligned} \tag{2.15}$$

By standard semigroup theory, it can be shown that (2.15) has a unique solution $u\in C([0,t];L^2_x)$, which satisfies the following equalities:

$$u(t)=P(t)\Pi f, \tag{2.16a}$$

$$u(t)=S(t)\Pi f+\int_0^t S(t-t')(-i\nu u-\epsilon(\gamma u-\mu\overline{u})+i\kappa\{u^*,u^*,u\})\,\mathrm{d}t'. \tag{2.16b}$$

Using the decay estimate (2.12) from Theorem 2.6, we first observe that

$$\|u\|_{L^1(0,T;L^2_x)}\overset{(2.16a)}{=}\|P(\cdot)\Pi f\|_{L^1(0,T;L^2_x)}\overset{(2.12)}{\le}\|f\|_{L^2_x}\int_0^T Me^{-at}\,\mathrm{d}t\le Ma^{-1}\|f\|_{L^2_x}. \tag{2.17}$$

From (2.16b) and Theorem 2.3, it now follows that

$$\begin{aligned} \|u\|_{L^r(0,T;L^p_x)}&\overset{(2.7a),(2.7b)}{\le}C(\|\Pi f\|_{L^2_x}+\|-i\nu u-\epsilon(\gamma u-\mu\overline{u})+i\kappa\{u^*,u^*,u\}\|_{L^1(0,T;L^2_x)})\\ &\le C\|\Pi f\|_{L^2_x}+C'\|u\|_{L^1(0,T;L^2_x)}\overset{(2.17)}{\le}C''\|f\|_{L^2_x}, \end{aligned}$$

which shows (2.14a). To show (2.14b), we use Minkowski's integral inequality and (2.14a):

$$\begin{aligned} \Big\|\int_0^{\cdot}P(\cdot-t')\Pi g(t')\,\mathrm{d}t'\Big\|_{L^r(0,T;L^p_x)}&=\Big\|\int_0^T \mathbb{1}_{[t',T]}(\cdot)P(\cdot-t')\Pi g(t')\,\mathrm{d}t'\Big\|_{L^r(0,T;L^p_x)}\\ &\le\int_0^T\|\mathbb{1}_{[t',T]}(\cdot)P(\cdot-t')\Pi g(t')\|_{L^r(0,T;L^p_x)}\,\mathrm{d}t'=\int_0^T\|P(\cdot)\Pi g(t')\|_{L^r(0,T-t';L^p_x)}\,\mathrm{d}t'\\ &\overset{(2.14a)}{\le}C\int_0^T\|g(t')\|_{L^2_x}\,\mathrm{d}t'. \end{aligned}$$

To obtain the stochastic estimate (2.14c) for $(r,p)\neq(\infty,2)$, we simply repeat the first part of the proof of (2.7c) from Appendix B, replacing all occurences of $S(t)$ with $P(t)\Pi$ and using (2.14a) instead of (2.7a) in the intermediate steps. Using Hölder's inequality at the end then gives (2.14c).

For the case $(r,p)=(\infty,2)$, the proof strategy in Appendix B is no longer applicable, since $P(t)\Pi$ is not unitary. Instead, we estimate the stochastic convolution using the well-known factorization method (see, for instance, [39, Theorem 4.5] for a version applicable to our setting), which gives the result after applying (2.4b). ∎

For $P(t)\Pi^0$, there is significantly more freedom in choosing the exponents, and the requirement of admissibility can be dropped. In this case, the estimates follow not from any dispersive phenomena, but rather from the fact that the range of Π^0 is one-dimensional, being spanned by u_x^*.

Proposition 2.10. *Let $p\in[1,\infty]$. There exists a constant C, such that the estimates*

$$\|P(\cdot)\Pi^0 u_0\|_{C([0,T];L_x^p)}\le C\|u_0\|_{L_x^2}, \tag{2.18a}$$

$$\left\|\int_0^{\cdot} P(\cdot-t')\Pi^0 f(t')\,\mathrm{d}t'\right\|_{C([0,T];L_x^p)}\le C\|f\|_{L^1(0,T;L_x^2)}, \tag{2.18b}$$

$$\left\|\int_0^{\cdot} P(\cdot-t')\Pi^0 g(t')\Phi\,\mathrm{d}W(t')\right\|_{L_\Omega^q(C([0,T];L_x^p))}\le C\sqrt{q}\beta\|g\|_{L_\Omega^q(L^2(0,T;L_x^2))}, \tag{2.18c}$$

hold for all $q\in[2,\infty)$, $T\in(0,\infty]$, $u_0\in L_x^2$, $f\in L^1(0,T;L_x^2)$, $g\in L_\Omega^q(L^2(0,T;L_x^2))$, and $\phi\in L^2(\mathbb{R};\mathbb{R})$ (recall (2.1)).

Proof. Since $\mathcal{L}u_x^*=0$ by Theorem 2.6, it holds that $P(t)u_x^*=u_x^*$. After observing that the range of Π^0 is spanned by u_x^*, it follows that $P(t)\Pi^0=\Pi^0$ for every t. Thus, we get

$$\|P(t)\Pi^0 u_0\|_{L_x^p}=\|\Pi^0 u_0\|_{L_x^p}=\frac{\|u_x^*\|_{L_x^p}}{\|u_x^*\|_{L_x^2}}\|\Pi^0 u_0\|_{L_x^2}\le C\|u_0\|_{L_x^2}, \tag{2.19}$$

where $\|u_x^*\|_{L_x^p}<\infty$ because u_x^* decays rapidly. Using Minkowski's inequality, we can additionally estimate

$$\left\|\int_0^t P(t-t')\Pi^0 f(t')\,\mathrm{d}t'\right\|_{L_x^p}\le\int_0^t\|P(t-t')\Pi^0 f(t')\|_{L_x^p}\,\mathrm{d}t'\overset{(2.19)}{\le}C\int_0^t\|f(t')\|_{L_x^2}\,\mathrm{d}t',$$

at which point (2.18b) follows by taking the supremum over $t\in[0,T]$. Finally, we estimate

$$\begin{aligned}\left\|\int_0^{\cdot} P(\cdot-t')\Pi^0 g(t')\Phi\,\mathrm{d}W(t')\right\|_{L_\Omega^q(C([0,T];L_x^p))}&=\left\|\int_0^{\cdot}\Pi^0 g(t')\Phi\,\mathrm{d}W(t')\right\|_{L_\Omega^q(C([0,T];L_x^p))}\\&\le C\left\|\int_0^{\cdot} g(t')\Phi\,\mathrm{d}W(t')\right\|_{L_\Omega^q(C([0,T];L_x^2))}\\&\le C'\sqrt{q}\|g\Phi\|_{L_\Omega^q(L^2(0,T;\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});L_x^2)))}\\&\overset{(2.4b)}{=}C'\sqrt{q}\beta\|g\|_{L_\Omega^q(L^2(0,T;L_x^2))},\end{aligned}$$

where we have used the Burkholder–Davis–Gundy inequality for the penultimate step. ∎

To get appropriate Gaussian tail bounds, we need the following elementary lemma.

Lemma 2.11. *Let ξ be a nonnegative real-valued random variable which satisfies*

$$\|\xi\|_{L_\Omega^p}\le C\sqrt{p}$$

for all sufficiently large $p < \infty$, where C is independent of p. Then ξ satisfies the Gaussian tail bound

$$\mathbb{P}[\xi \geq \lambda] \leq \exp(-e^{-2}C^{-2}\lambda^2)$$

for all sufficiently large λ.

Proof. By Markov's inequality and the assumption on ξ, we have

$$\mathbb{P}[\xi \geq \lambda] = \mathbb{P}[\xi^p \geq \lambda^p] \leq \lambda^{-p} C^p \sqrt{p}^p = (\lambda^{-1} C \sqrt{p})^p$$

for p sufficiently large. Choosing $p = e^{-2}C^{-2}\lambda^2$ (which can be made sufficiently large by increasing λ) gives the result. ■

3. Main results.

We now state the results of this paper. We begin by showing that (2.5) is well-posed in Theorem 3.1. Afterwards, we derive an asymptotic expansion (in terms of σ) of solutions to (2.5) around a solitary wave centered at the origin. This expansion is needed for the coming orbital stability results, and its validity is shown in Proposition 3.2.

Using the asymptotic expansion we then construct the phase processes a_1 and a_2 which describe the motion of the solitary wave to second order in σ. Our first main result in the direction of stochastic stability of the solitary wave is Theorem 3.6. Roughly speaking, it states that our choice of a_1 and a_2 eliminate the leading-order growth terms of the perturbations around the shifted wave. We then build on this result to prove Theorem 3.8, where we show an exponential estimate on the exit time from the shifted solitary wave and hence establish orbital stability on an exponential timescale.

3.1. Well-posedness.

Our first main result is the well-posedness of a mild formulation of (2.5). The proof is contained in section 4.

Theorem 3.1. *Let $\nu, \epsilon, \gamma, \mu, \kappa > 0$, let u_0 be an L^2_x-valued $\mathcal{F}_0$-measurable random variable, and let $T \in (0,\infty)$ and $\phi \in L^2(\mathbb{R};\mathbb{R})$. There exists a unique $\mathbb{F}$-adapted process $u \in C([0,T]; L^2_x) \cap L^6(0,T; L^6_x)$ satisfying the mild-solution equation*

$$\begin{aligned} u(t) = S(t)u_0 &+ \int_0^t S(t-t')(-i\nu u(t') - \epsilon(\gamma u(t') - \mu \overline{u}(t')) - \frac{1}{2}\beta^2 u(t'))\,\mathrm{d}t' \\ &+ i\kappa \int_0^t S(t-t')|u(t')|^2 u(t')\,\mathrm{d}t - i\int_0^t S(t-t')u(t')\Phi\,\mathrm{d}W(t'), \end{aligned} \tag{3.1}$$

for every $t \in [0,T]$, $\mathbb{P}$-a.s. Furthermore, $u \in L^r(0,T; L^p_x)$ for any $(r,p) \neq (4,\infty)$ which satisfies (2.6), and we have the a priori estimate

$$\|u(t)\|_{L^2_x} \leq e^{\epsilon(\mu-\gamma)t}\|u_0\|_{L^2_x}, \tag{3.2}$$

for every $t \in [0,T]$, $\mathbb{P}$-a.s.

If we additionally assume that $\phi \in H^s_x$ and u_0 takes values in H^s_x for some $s \in [0,\infty)$, then also $u \in C([0,T]; H^s_x) \cap L^r(0,T; H^{s,p}_x)$ for any $(r,p) \neq (4,\infty)$ which satisfies (2.6).

3.2. Asymptotic expansion. We now derive the asymptotic expansions which will be used in section 3.3 to show orbital stability. From now on, let $\nu, \epsilon, \gamma, \mu, \kappa$, and u^* be as described in section 2.4. Consider the SPFNLS equation (2.5), now written using our notational shorthands (cf. (2.1), (2.9), (2.11)), and including an additional parameter $\sigma > 0$ which controls the strength of the noise:

$$\mathrm{d}u = \left[i\Delta u + Lu + \frac{1}{3} i\kappa\{u,u,u\} - \frac{1}{2}\beta^2\sigma^2 u \right] \mathrm{d}t - i\sigma u \Phi \,\mathrm{d}W. \tag{3.3}$$

The first step towards showing orbital stability of the solitary wave is to construct an asymptotic expansion to second order in σ. For this we use the following ansatz:

$$u = u^* + \sigma v_1 + \sigma^2 v_2 + z, \tag{3.4}$$

where z should be regarded as being $\mathcal{O}(\sigma^3)$. To match our ansatz, we supply (3.3) with the initial condition

$$u(0) = u^* + \sigma v_{1,0} + \sigma^2 v_{2,0}. \tag{3.5}$$

By using the additivity of the triple bracket, we see that (3.3) can be rewritten as

$$\begin{aligned}
\mathrm{d}u = &\left[(i\Delta + L)u^* + \frac{1}{3} i\kappa\{u^*,u^*,u^*\} \right] \mathrm{d}t \\
&+ \sigma([(i\Delta+L)v_1 + i\kappa\{u^*,u^*,v_1\}]\,\mathrm{d}t - iu^*\Phi\,\mathrm{d}W) \\
&+ \sigma^2 \left(\left[(i\Delta+L)v_2 + i\kappa\{u^*,u^*,v_2\} + i\kappa\{u^*,v_1,v_1\} - \frac{1}{2}\beta^2 u^* \right] \mathrm{d}t - iv_1\Phi\,\mathrm{d}W \right) \\
&+ [(i\Delta+L)z + i\kappa\{u^*,u^*,z\} + i\kappa R - \frac{1}{2}\beta^2(\sigma^3 v_1 + \sigma^4 v_2 + \sigma^2 z)]\,\mathrm{d}t \\
&- i(\sigma^3 v_2 + \sigma z)\Phi\,\mathrm{d}W,
\end{aligned} \tag{3.6}$$

where we have abbreviated

$$\begin{aligned}
R := {} & 2\{u^*,\sigma v_1,\sigma^2 v_2\} + \frac{1}{3}\{\sigma v_1,\sigma v_1,\sigma v_1\} \\
&+ 2\{u^*,\sigma v_1,z\} + \{u^*,\sigma^2 v_2,\sigma^2 v_2\} + \{\sigma v_1,\sigma v_1,\sigma^2 v_2\} \\
&+ 2\{u^*,\sigma^2 v_2,z\} + \{\sigma v_1,\sigma v_1,z\} + \{\sigma v_1,\sigma^2 v_2,\sigma^2 v_2\} \\
&+ \{u^*,z,z\} + 2\{\sigma v_1,\sigma^2 v_2,z\} + \frac{1}{3}\{\sigma^2 v_2,\sigma^2 v_2,\sigma^2 v_2\} \\
&+ \{\sigma v_1,z,z\} + \{\sigma^2 v_2,\sigma^2 v_2,z\} \\
&+ \{\sigma^2 v_2,z,z\} \\
&+ \frac{1}{3}\{z,z,z\}.
\end{aligned} \tag{3.7}$$

Note that the terms in (3.7) are organized according to their order in σ, and all terms are $\mathcal{O}(\sigma^3)$. Taking the differential of (3.4) and using (2.11) and (3.6), we see that if v_1 and v_2 satisfy

$$\mathrm{d}v_1 = \mathcal{L}v_1\,\mathrm{d}t - iu^*\Phi\,\mathrm{d}W, \tag{3.8a}$$

$$\mathrm{d}v_2 = \left[\mathcal{L}v_2 + i\kappa\{u^*, v_1, v_1\} - \frac{1}{2}\beta^2 u^*\right]\mathrm{d}t - iv_1\Phi\,\mathrm{d}W, \tag{3.8b}$$

$$v_1(0) = v_{1,0}, \tag{3.8c}$$

$$v_2(0) = v_{2,0}, \tag{3.8d}$$

then z satisfies

$$\mathrm{d}z = \left[\mathcal{L}z + i\kappa R - \frac{1}{2}\beta^2(\sigma^3 v_1 + \sigma^4 v_2 + \sigma^2 z)\right]\mathrm{d}t - i(\sigma^3 v_2 + \sigma z)\Phi\,\mathrm{d}W, \tag{3.9a}$$

$$z(0) = 0 \tag{3.9b}$$

(note that $\mathrm{d}u^* = [(i\Delta + L)u^* + \frac{1}{3}\{u^*, u^*, u^*\}]\,\mathrm{d}t$ always holds, since both sides vanish). We now formulate a proposition which states that on any fixed time interval $[0, T]$, the approximation $u \approx u^* + \sigma v_1 + \sigma^2 v_2$ is accurate to second order in σ with high probability, as long as v_1 and v_2 are not too large. The proof is contained in section 5.1.

Proposition 3.2 (asymptotic expansion, second order). *Let $v_{1,0}$ and $v_{2,0}$ be $\mathcal{F}_0$-measurable and L^2_x-valued random variables, and let u be the solution to (3.3) with initial condition (3.5). The system (3.8) has a unique mild solution given by*

$$v_1(t) = P(t)v_{1,0} - \int_0^t P(t-t')iu^*\Phi\,\mathrm{d}W(t'), \tag{3.10a}$$

$$v_2(t) = P(t)v_{2,0} + \int_0^t P(t-t')\left(i\kappa\{u^*, v_1, v_1\} - \frac{1}{2}\beta^2 u^*\right)\mathrm{d}t' - \int_0^t P(t-t')iv_1\Phi\,\mathrm{d}W(t'). \tag{3.10b}$$

We have $v_1, v_2 \in C([0,T]; L^2_x) \cap L^r(0,T; L^p_x)$ for every $T \in (0,\infty)$ and every admissible pair $(r,p) \neq (4,\infty)$, $\mathbb{P}$-a.s. With these v_1 and v_2, we have the asymptotic expansion

$$u(t) = u^* + \sigma v_1(t) + \sigma^2 v_2(t) + z(t), \tag{3.11}$$

where z satisfies (3.9). Furthermore, for every $T \in (0,\infty)$ and every admissible pair $(r,p) \neq (4,\infty)$, there exist strictly positive constants c_1, c_2, ε', independent of $v_{1,0}$, $v_{2,0}$, such that for the following stopping times

$$\tau_{v_1} := \sup\{t \in [0,T] : \|v_1\|_{L^\infty(0,t;L^2_x)\cap L^6(0,t;L^6_x)} \le \sigma^{-1}\varepsilon\}, \tag{3.12a}$$

$$\tau_{v_2} := \sup\{t \in [0,T] : \|v_2\|_{L^\infty(0,t;L^2_x)\cap L^6(0,t;L^6_x)} \le \sigma^{-2}\varepsilon^2\}, \tag{3.12b}$$

$$\tau_z := \sup\{t \in [0,T] : \|z\|_{L^\infty(0,t;L^2_x)\cap L^r(0,t;L^p_x)} \le c_1\varepsilon^3\}, \tag{3.12c}$$

we have the inequality

$$\mathbb{P}[\tau_z < \min\{\tau_{v_1}, \tau_{v_2}\}] \le \exp(-c_2\varepsilon^2\sigma^{-2}) \tag{3.13}$$

for all σ, ε which satisfy $0 < \sigma \le \varepsilon \le \varepsilon'$.

Remark 3.3. It would be sufficient in (3.12) to control v_1 and v_2 in a slightly weaker norm. However, the choice of $L^\infty(0,t;L_x^2)\cap L^6(0,t;L_x^6)$ permits a more convenient proof, and we will be able to control v_1 and v_2 in this norm due to the Strichartz estimates.

Remark 3.4. The main purpose of Proposition 3.2 is to characterize the short-term dynamics of (2.5), and to serve as a building block towards the stability results in section 3.3.

The following proposition is a first-order variant of Proposition 3.2, and will be used to show the long-term stability result (Theorem 3.8). The proof is a strictly simpler version of that of Proposition 3.2, so we choose to omit it.

Proposition 3.5. *Consider the setting of Proposition 3.2 with $v_{2,0}=0$ and define z' via*

$$u(t) =: u^* + \sigma v_1(t) + z'(t). \tag{3.14}$$

For every $T\in(0,\infty)$ and every admissible pair $(r,p)\neq(4,\infty)$ there exist strictly positive constants c_1, c_2, and ε', independent of $v_{1,0}$, such that if we introduce the additional stopping time

$$\tau_{z'} := \sup\{t\in[0,T] : \|z'\|_{L^\infty(0,t;L_x^2)\cap L^r(0,t;L_x^p)} \le c_1\varepsilon^2\}, \tag{3.15}$$

we have the inequality

$$\mathbb{P}[\tau_{z'} < \tau_{v_1}] \le \exp(-c_2\varepsilon^2\sigma^{-2}) \tag{3.16}$$

for all σ,ε which satisfy $0<\sigma\le\varepsilon\le\varepsilon'$.

3.3. Orbital stability. Proposition 3.2 implies that on any fixed timescale, we have the expansion $u=u^*+\sigma v_1+\sigma^2 v_2+\mathcal{O}(\sigma^3)$. However, from (3.10) it can be seen that, in general, the processes v_1 and v_2 grow with time. To show orbital stability of the solitary wave on long timescales, we need to control this growth. Therefore, we first decompose v_1 and v_2 in the following way:

$$v_1 = a_1 u_x^* + w_1, \tag{3.17a}$$

$$v_2 = a_2 u_x^* + \frac{1}{2}a_1^2 u_{xx}^* + w_2, \tag{3.17b}$$

where a_1 and a_2 are (real-valued) stochastic processes which we will specify later, at which point (3.17) determines w_1 and w_2. Substituting (3.17) into (3.4) and using Proposition 3.2, we get

$$u = u^* + \sigma a_1 u_x^* + \sigma^2 a_2 u_x^* + \frac{1}{2}\sigma^2 a_1^2 u_{xx}^* + \sigma w_1 + \sigma^2 w_2 + \mathcal{O}(\sigma^3).$$

The first four terms on the right-hand side exactly constitute a Taylor expansion of $u^*(x+\sigma a_1+\sigma^2 a_2)$ to second order in σ, and thus we have

$$u = u^*(x+\sigma a_1+\sigma^2 a_2) + \sigma w_1 + \sigma^2 w_2 + \mathcal{O}(\sigma^3), \tag{3.18}$$

still on the same fixed timescale. We will see that for some particular choice of a_1 and a_2, the processes w_1 and w_2 exhibit growth behavior which is much more favorable than that of their counterparts v_1 and v_2. This is the statement of Theorem 3.6, which gives explicit expressions for a_1 and a_2, and characterizes the growth behavior of w_1 and w_2. This is made possible by the exponential decay of $P(t)\Pi$ (2.12), which is essentially the content of the deterministic stability result.

As an example, from (3.10a) it is clear that v_1 is expected to grow like $\sqrt{t}$ (this can be made rigourous by combining (3.17a), (3.20a), and (3.22a)). On the other hand, from (3.22a) we see that the moments of w_1 remain bounded in time. Thus, the term $a_1 u_x^*$ in (3.17a) fully captures the growth of v_1. Similarly, v_2 is expected to grow at a rate of t^2, whereas (3.22b) shows that w_2 only grows like t.

From (3.18) it is then clear that a_1 and a_2 have an interpretation as the first- and second-order corrections to the phase of the solitary wave. Additionally, since Φ and u^* do not depend on t and ω, it can be seen from (3.20a) that a_1 is a Brownian motion rescaled by $\|\mathcal{P}iu^*\Phi\|_{\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});\mathbb{R})}$ and offset by $\mathcal{P}(v_{1,0})$. The proofs of Theorem 3.6, Propostion 3.7, and Theorem 3.8 are contained in section 5.2.

Theorem 3.6. *There exist predictable processes a_1, a_2, w_1, w_2, such that* (3.17) *and the condition*

$$\Pi^0 w_k = 0, \qquad k \in \{1,2\}, \tag{3.19}$$

both hold. The processes a_1 and a_2 are given by

$$a_1(t) = \mathcal{P}\left[v_{1,0} - \int_0^t iu^*\Phi \,\mathrm{d}W(t')\right], \tag{3.20a}$$

$$a_2(t) = \mathcal{P}\left[v_{2,0} + \int_0^t i\kappa\{u^*, v_1, v_1\} - \frac{1}{2}\beta^2 u^* \,\mathrm{d}t' - \int_0^t iv_1\Phi \,\mathrm{d}W(t') - \frac{1}{2}a_1(t)^2 u^*_{xx}\right], \tag{3.20b}$$

and the corresponding w_1 and w_2 are given by

$$w_1 = P(t)\Pi v_{1,0} - \int_0^t P(t-t')\Pi iu^*\Phi \,\mathrm{d}W(t'), \tag{3.21a}$$

$$\begin{aligned} w_2 = P(t)\Pi v_{2,0} &+ \int_0^t P(t-t')\Pi \left(i\kappa\{u^*, v_1, v_1\} - \frac{1}{2}\beta^2 u^*\right) \mathrm{d}t' \\ &- \int_0^t P(t-t')\Pi iv_1\Phi \,\mathrm{d}W(t') - \frac{1}{2}a_1(t)^2 \Pi u^*_{xx}. \end{aligned} \tag{3.21b}$$

Finally, there exists a constant C, such that the estimates

$$\|w_1(t)\|_{L^q_\Omega(L^2_x)} \le C(e^{-at}\|v_{1,0}\|_{L^q_\Omega(L^2_x)} + \sqrt{q}\beta \min\{t^{\frac{1}{2}}, 1\}), \tag{3.22a}$$

$$\|w_2(t)\|_{L^q_\Omega(L^2_x)} \le C(e^{-at}\|v_{2,0}\|_{L^q_\Omega(L^2_x)} + \|v_{1,0}\|^2_{L^{2q}_\Omega(L^2_x)} + q\beta^2 t) \tag{3.22b}$$

hold for every $q \in [2,\infty)$, $v_{1,0} \in L^{2q}_\Omega(L^2_x)$, $v_{2,0} \in L^q_\Omega(L^2_x)$, $t \in [0,\infty)$, and $\phi \in L^2(\mathbb{R};\mathbb{R})$ (recall (2.1)*).*

Proposition 3.5 and Theorem 3.6 then allow us to show the following proposition.

Proposition 3.7. *Consider* (3.3) *with initial data* $u(0) = u^* + v_0$, *where* v_0 *is an* L^2_x*-valued* $\mathcal{F}_0$*-measurable random variable. There exist strictly positive constants* T, $\tilde{c}_1$, $\tilde{c}_2$, λ, ε' *such that the estimates*

$$\mathbb{P}[\|u(T) - u^*(x + \sigma a_1(T))\|_{L^2_x} \geq \tilde{c}_1\varepsilon] \leq 4\exp(-\tilde{c}_2\sigma^{-2}\varepsilon^2), \tag{3.23a}$$

$$\mathbb{P}[\|u(t) - u^*(x + \sigma a_1)\|_{L^\infty(0,T;L^2_x)} \geq \varepsilon] \leq 4\exp(-\tilde{c}_2\sigma^{-2}\varepsilon^2) \tag{3.23b}$$

hold for every $0 < \lambda\sigma \leq \varepsilon \leq \varepsilon'$, *and every* v_0 *which satisfies* $\|v_0\|_{L^2_x} \leq \tilde{c}_1\varepsilon$, $\mathbb{P}$*-a.s.*

From the translation invariance of the equation, it is immediate that the previous proposition also holds if we consider an initial condition of the form $u(0) = u^*(x + a) + v_0$ for any $a \in \mathbb{R}$. Thus, by (3.23a) we are at time T in essentially the same situation as at time 0 (with high probability). In this way, we can "chain" Proposition 3.7 to finally obtain the long-term stability result.

Theorem 3.8. *Let* v_0 *be as in Proposition* 3.7. *There exist constants* $c, \lambda, \varepsilon' > 0$, *such that the estimate*

$$\mathbb{P}\left[\sup_{t\in[0,T]} \inf_a \|u(t) - u^*(x + a)\|_{L^2_x} \geq \varepsilon\right] \leq (T + 1)\exp(-c\sigma^{-2}\varepsilon^2) \tag{3.24}$$

holds for every $T > 0$, $0 < \lambda\sigma \leq \varepsilon \leq \varepsilon'$, *and every* v_0 *which satisfies* $\|v_0\|_{L^2_x} \leq c\varepsilon$, $\mathbb{P}$*-a.s.*

Note that in the presence of noise, there is (generically) a nonzero probability for u to wander far from a shifted solitary wave in finite time. Hence, an estimate on the exit time is the best one could hope for.

4. Proof of well-posedness.

4.1. Local well-posedness.

Following the approach of de Bouard and Debussche in [5, 6] and Hornung in [20], we first establish well-posedness of a modified version of (3.1) in which the nonlinear term $|u|^2u$ is truncated. The truncation allows us to control the nonlinearity, which is otherwise not Lipschitz continuous.

We now fix $T_0 \in (0, \infty)$, $s \in [0, \infty)$, $\phi \in L^2(\mathbb{R}; \mathbb{R}) \cap H^s_x$, and $(r, p) \neq (4, \infty)$ which satisfies (2.6). All of these will remain fixed throughout the proof. For $T \in (0, \infty)$, we also introduce the following spaces:

$$X^s_T := C([0, T]; H^s_x) \cap L^6(0, T; H^{s,6}_x) \cap L^r(0, T; H^{s,p}_x), \tag{4.1a}$$

$$X_T := C([0, T]; L^2_x) \cap L^6(0, T; L^6_x). \tag{4.1b}$$

Since the pairs (r, p) and $(\infty, 2)$ both satisfy (2.6), we can freely replace the norms on the left-hand side of (2.7) by the X^s_T-norm, and will do so throughout.

For $R \geq 1$, let θ_R be the function which takes the value 1 on $[0, R]$, interpolates linearly between 1 and 0 on $[R, 2R]$, and is identically zero on $[2R, \infty)$. Also define

$$(\Theta_R(u))(t) := \theta_R(\|u\|_{L^6(0,t;L^6_x)})u(t).$$

The function Θ_R will serve to truncate the nonlinearity in (3.1). Notice that Θ_R preserves adaptedness of u. The truncated mild equation now takes the form

$$u(t) = S(t)u_0 - \int_0^t S(t-t')\left(i\nu u(t') + \epsilon(\gamma u(t') - \mu\overline{u}(t')) + \frac{1}{2}\beta^2 u(t')\right) \mathrm{d}t' \\ + i\kappa \int_0^t S(t-t')(|\Theta_R(u)(t')|^2 \Theta_R(u)(t'))\,\mathrm{d}t' - i\int_0^t S(t-t')u(t')\Phi\,\mathrm{d}W(t'). \tag{4.2}$$

Proposition 4.1 (global well-posedness of truncated equation). *For every $R \geq 1$ and every $\mathcal{F}_0$-measurable $u_0 \in L^2_\Omega(L^2_x)$, there is a unique $u \in L^2_\Omega(X_{T_0})$ which satisfies (4.2) for every $t \in [0,T_0]$, $\mathbb{P}$-a.s.*

Proof. Since the PFNLS equation differs from the nonlinear Schrödinger equation only by linear terms, the existence and uniqueness of a solution $u \in L^2_\Omega(X_{T_0})$ to (4.2) follows from the same arguments as in [20, 5, 6]. Broadly speaking, the proof consists of a fixed-point argument in $L^2_\Omega(X_T)$ and uses the fact that the truncated nonlinearity satisfies a global Lipschitz estimate in the space X_T. For detailed expositions we refer the reader to [20, Proposition 3] and [5, Proposition 3.1]. ■

Let us denote by u_R the unique solution to the truncated equation (4.2) with radius R given by Proposition 4.1. We define for $R \geq 1$ the stopping time

$$\tau_R := \sup\{t \in [0,T_0] : \|u_R\|_{L^6(0,t;L^6_x)} \leq R\}, \tag{4.3}$$

which corresponds to the first time the norm $\|u_R\|_{L^6(0,t;L^6_x)}$ reaches size R, and before this time no truncation takes place.

We now prove additional regularity of u_R. Since no similar statement is shown in [5, 6, 20], we give the complete proof. We begin by stating a lemma relating to the regularity of the nonlinearity.

Lemma 4.2. *There exists a constant C, such that the estimate*

$$\||u|^2 u\|_{L^1(0,T;H^s_x)} \leq CT^{\frac{1}{2}}\|u\|_{L^6(0,T;H^{s,6}_x)}\|u\|^2_{L^6(0,T;L^6_x)} \tag{4.4}$$

holds for all $T \in (0,\infty)$ and $u \in L^6(0,T;H^{s,6}_x)$. In the case $s=0$, we can take $C=1$.

Proof. Since $|u|^2u$ can be written as $u^2\bar{u}$, the estimate (4.2) follows by repeated application of Hölder's inequality and the Kato–Ponce inequality (see [17, Theorem 1.4]). ■

Proposition 4.3. *Let $u_0 \in L^2_\Omega(H^s_x)$ be $\mathcal{F}_0$-measurable, and let $R \geq 1$. Then there exist $T > 0$ and C, which do not depend on u_0, such that*

$$\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(X^s_T)} \leq C\|u_0\|_{L^2_\Omega(H^s_x)}. \tag{4.5}$$

Proof. For $t \leq \tau_R$, u_R satisfies (4.2):

$$u_R(t) = S(t)u_0 - \int_0^t S(t-t')\left(i\nu u_R(t') + \epsilon(\gamma u_R(t') - \mu\overline{u}_R(t')) + \frac{1}{2}\beta^2 u_R(t')\right) \mathrm{d}t' \\ + i\kappa\int_0^t S(t-t')(|u_R(t')|^2 u_R(t'))\,\mathrm{d}t' - i\int_0^t S(t-t')u_R(t')\Phi\,\mathrm{d}W(t') \\ =: I_1 + I_2 + I_3 + I_4.$$

Using the deterministic Strichartz estimates (2.7a)–(2.7b) together with (4.3) and (4.4) we find a constant C such that

$$\begin{aligned}
&\|\mathbb{1}_{[0,\tau_R]}(I_1+I_2+I_3)\|_{X^s_T}\\
&\quad\le C\|u_0\|_{H^s_x}+CT\|\mathbb{1}_{[0,\tau_R]}u_R\|_{C([0,T];H^s_x)}+CR^2T^{1/2}\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^6(0,T;H^{s,6}_x)}\\
&\quad\overset{(4.1)}{\le} C\|u_0\|_{H^s_x}+C(T+R^2T^{1/2})\|\mathbb{1}_{[0,\tau_R]}u_R\|_{X^s_T}.
\end{aligned}$$

From the stochastic Strichartz estimate (2.7c) there exists a constant C' such that

$$\|\mathbb{1}_{[0,\tau_R]}I_4\|_{L^2_\Omega(X^s_T)}\le C'\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(L^2(0,T;H^s_x))}\overset{(4.1)}{\le} C'T^{1/2}\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(X^s_T)}.$$

Combining the two above estimates, we find

$$\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(X^s_T)}\le C\|u_0\|_{L^2_\Omega(H^s_x)}+(CT+CR^2T^{1/2}+C'T^{1/2})\|\mathbb{1}_{[0,\tau_R]}u_R\|_{L^2_\Omega(X^s_T)}.$$

Choosing T sufficiently small based on R, C, C' we can absorb the rightmost term into the left-hand side, at which point the desired estimate follows. ∎

By iterating the above proposition on the time intervals $[T,2T]$ and so on, we find that $\mathbb{1}_{[0,\tau_R]}u(\cdot)\in L^2_\Omega(X^s_{T_0})$.

Now notice that for $t<\tau_R$, we have $\Theta_R(u_R(t))=u_R(t)$. Hence, u_R is a genuine solution to (3.1) up until time τ_R. Moreover, solutions to (4.2) with different values of R will be the same until the truncation time. Hence, we may patch together these solutions to obtain a maximal solution u which solves (3.1) up until a maximal stopping time τ^*, defined as

$$\tau^*:=\sup_{R\ge1}\,\tau_R. \tag{4.6}$$

The following proposition summarizes the results of this procedure (see [20, Proposition 4] for more details).

Proposition 4.4 (local well-posedness of SPFNLS). *There exists a unique adapted process u which satisfies the following properties $\mathbb{P}$-a.s.:*

1. *$u\in X^s_t$ for every $t\in[0,\tau^*)$,*
2. *u satisfies (3.1) for all $t\in[0,\tau^*)$,*
3. *$\tau^*<T_0$ implies $\lim_{t\nearrow\tau^*}\|u(t)\|_{L^6(0,t;L^6_x)}=\infty$.*

4.2. Blow-up. We now show that the constructed solution can only fail to exist globally if its L^2_x-norm blows up.

Proposition 4.5 (blow-up criterion). *The implication*

$$\sup_{t\in[0,\tau^*)}\|u\|_{C([0,t];L^2_x)}<\infty\Longrightarrow\sup_{t\in[0,\tau^*)}\|u\|_{L^6(0,t;L^6_x)}<\infty$$

holds, $\mathbb{P}$-a.s.

Proof. Fix some $M \geq 1$, and define the stopping time

$$\tau := \sup\{\, t \in [0,\tau^*) : \|u\|_{C([0,t];L^2_x)} \leq M \,\}, \tag{4.7a}$$

as well as a recursive sequence of stopping times according to $\tau_0 = 0$ and

$$\tau_{N+1} := \sup\{\, t \in [\tau_N,\tau] : \|u\|_{L^6(\tau_N,t;L^6_x)} \leq 3KM \,\}, \quad N \in \mathbb{N}_0, \tag{4.7b}$$

where K is the constant C from the right-hand side of (2.7a). Additionally, we define the event

$$A := \{\omega \in \Omega : \tau_N < \tau \,\forall N \in \mathbb{N}_0\},$$

and claim that $\mathbb{P}(A) = 0$. To see this, we start the solution from time τ_N and get the $\mathbb{P}$-a.s. equality

$$\begin{aligned} u(t) &= S(t-\tau_N)u(\tau_N) - \int_{\tau_N}^t S(t-t')\left(i\nu u(t') + \epsilon(\gamma u(t') - \mu\overline{u}(t')) + \frac{1}{2}\beta^2 u(t')\right) \mathrm{d}t' \\ &\quad + i\kappa \int_{\tau_N}^t S(t-t')(|u(t')|^2 z(t'))\,\mathrm{d}t' - i\int_{\tau_N}^t S(t-t')u(t')\Phi\,\mathrm{d}W(t') \\ &=: I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{4.8}$$

for every $t \in [\tau_N,\tau^*)$. Since the Strichartz estimates from Theorem 2.3 are invariant under time translation and the pair $(6,6)$ is admissible (cf. (2.6)), we see that

$$\|I_1\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} \overset{(2.7a)}{\leq} K\|u(\tau_N)\|_{L^2_x} \overset{(4.7a)}{\leq} KM, \tag{4.9a}$$

$$\|I_2\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} \overset{(2.7b)}{\leq} C(\tau_{N+1}-\tau_N)\|u\|_{C([\tau_N,\tau_{N+1}];L^2_x)} \overset{(4.7a)}{\leq} C(\tau_{N+1}-\tau_N)M. \tag{4.9b}$$

To estimate I_3 we use Theorem 2.3 and Hölder's inequality:

$$\begin{aligned} \|I_3\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} &\overset{(2.7b)}{\leq} C\||u|^2u\|_{L^1(\tau_N,\tau_{N+1};L^2_x)} \leq C(\tau_{N+1}-\tau_N)^{\frac{1}{2}}\|u\|^3_{L^6(\tau_N,\tau_{N+1};L^6_x)} \\ &\overset{(4.7b)}{\leq} 27CK^3M^3(\tau_{N+1}-\tau_N)^{\frac{1}{2}}. \end{aligned} \tag{4.9c}$$

Taking the $L^6(\tau_N,\tau_{N+1};L^6_x)$-norm of (4.8) and using the triangle inequality along with (4.9a)–(4.9c) gives

$$\begin{aligned} \|u\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} &\leq KM + CM(\tau_{N+1}-\tau_N) + 27CK^3M^3(\tau_{N+1}-\tau_N)^{\frac{1}{2}} \\ &\quad + \left\|\int_{\tau_N}^{\cdot} S(\cdot-t')u(t')\Phi\,\mathrm{d}W(t')\right\|_{L^6(\tau_N,\tau_{N+1};L^6_x)}. \end{aligned} \tag{4.10}$$

From (4.7b) it is clear that we must have the equality $\|u\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} = 3KM$ for every N if $\omega \in A$. On the other hand, since τ_N is nondecreasing with N and bounded by T_0, the second and third term on the right-hand side of (4.10) converge to zero as $N \to \infty$. Combining these facts, we see that $\mathbb{P}(A)$ is bounded by the probability that the events

$$A_N := \left\{\omega \in \Omega : \left\|\int_{\tau_N}^{\cdot} S(\cdot-t')u(t')\Phi\,\mathrm{d}W(t')\right\|_{L^6(\tau_N,\tau_{N+1};L^6_x)} \geq KM\right\}$$

occur for infinitely many N. However, using Markov's inequality and Theorem 2.3, we can estimate

$$\begin{aligned} K^2M^2\mathbb{P}(A_N) &\leq \mathbb{E}\left[\left\|\int_{\tau_N}^{\cdot} S(\cdot - t')u(t')\Phi\,\mathrm{d}W(t')\right\|^2_{L^6(\tau_N,\tau_{N+1};L^6_x)}\right] \\ &\leq \mathbb{E}\left[\left\|\int_0^{\cdot} S(\cdot - t')\mathbb{1}_{[\tau_N,\tau_{N+1}]}(t')u(t')\Phi\,\mathrm{d}W(t')\right\|^2_{L^6(0,T_0;L^6_x)}\right] \\ &\overset{(2.7c)}{\leq} C^2\mathbb{E}\left[\|u\|^2_{L^2(\tau_N,\tau_{N+1};L^2_x)}\right]. \end{aligned}$$

Since

$$\sum_{N=0}^{\infty}\mathbb{E}\Big[\|u\|^2_{L^2(\tau_N,\tau_{N+1};L^2_x)}\Big] \leq \mathbb{E}\Big[\|u\|^2_{L^2(0,\tau;L^2_x)}\Big] \overset{(4.7a)}{\leq} M^2T_0 < \infty$$

by Fubini's theorem, we see that the probabilities $\mathbb{P}(A_N)$ are summable. Thus, $\mathbb{P}(A)=0$ by the Borel–Cantelli lemma. By definition of A, this implies $\sup_{t\in[0,\tau)}\|u\|_{L^6(0,t;L^6_x)}<\infty$, $\mathbb{P}$-a.s. Recalling that M was arbitrary, we finish the proof by choosing M larger than $\sup_{t\in[0,\tau^*)}\|u\|_{C([0,t];L^2_x)}$ (if this quantity is finite) so that $\tau=\tau^*$ by (4.7a). ∎

4.3. Conservation.

Having formulated a blow-up criterion in terms of the L^2_x-norm, we now show that this norm can be controlled pathwise. This will yield global well-posedness of (3.1) in combination with Proposition 4.5.

Proposition 4.6. *The inequality*

$$\|u(t)\|_{L^2_x} \leq e^{\epsilon(\mu-\gamma)t}\|u(0)\|_{L^2_x} \tag{4.11}$$

holds, $\mathbb{P}$-a.s., for every $t\in[0,\tau^*)$.

Recall that we need $\epsilon>0$ and $\mu>\gamma$ in section 2.4 in order for the solitary wave to exist. In that case, the bound (4.11) is exponentially growing in time.

Proof. Formally applying the Itô formula to $\|u\|^2_{L^2_x}$ and omitting terms which are identically zero gives the identity:

$$\|u(t)\|^2_{L^2_x} = \|u(0)\|^2_{L^2_x} + 2\epsilon\int_0^t \mu\,\mathrm{Re}\,\langle\overline{u}(t'),u(t')\rangle_{L^2_x} - \gamma\|u(t')\|^2_{L^2_x}\,\mathrm{d}t'$$

for all $t\in[0,\tau^*)$. The above can be justified by applying the mild Itô formula proved by Da Prato, Jentzen, and Röckner [14, Theorem 1], which in our case reduces to the standard Itô formula since $S(t)$ is unitary on L^2_x. Applying the Cauchy–Schwarz inequality allows us to deduce

$$\|u(t)\|^2_{L^2_x} \leq \|u(0)\|^2_{L^2_x} + 2\epsilon\int_0^t(\mu-\gamma)\|u(t')\|^2_{L^2_x}\,\mathrm{d}t',$$

which implies (4.11) after using Grönwall's lemma and taking square roots. ∎

Proof of Theorem 3.1. From (4.11) it is immediate that $\mathbb{P}[\sup_{t\in[0,\tau^*)}\|u(t)\|_{L^2}=\infty]=0$. Thus, by Proposition 4.5 the solutions constructed in Proposition 4.4 exist on the entire interval $[0,T_0]$, $\mathbb{P}$-a.s. It only remains to lift the assumption that $u_0\in L^2_\Omega$. This can be done by considering the initial conditions $u_0^M=\mathbb{1}_{\|u_0\|_{L^2_x}\leq M}u_0$ and taking M to infinity, using pathwise uniqueness to patch together the solutions. Since this is a well-known standard procedure, we will not elaborate. ∎

5. Proof of stability.

5.1. Asymptotic expansion.

Proof of Proposition 3.2. Throughout the proof, we will use the notation $A\lesssim B$ to denote that there exists a constant C, independent of v_1, v_2, ε, σ, and c_1, such that $A\leq CB$.

Fix $T\in(0,\infty)$ and an admissible pair (r,p) with $p\in[6,\infty)$. If we prove the theorem for such p, it follows from an iterated application of Hölder's inequality that the theorem also holds for admissible pairs with $p\in[2,6)$, so the restriction on p does not entail any loss of generality.

The existence and uniqueness of the mild solution $v_1\in C([0,T];L^2_x)$ to (3.8a) follows from standard theory (see, for example, [15, Theorem 5.4]). Using (2.14a), (2.14c) and (2.18a), (2.18c) of Propositions 2.9 and 2.10, we obtain from (3.10a) that $v_1\in L^r(0,T;L^p_x)$, so that also $v_1\in L^6(0,T;L^6_x)$, $\mathbb{P}$-a.s. Combining this with Hölder's inequality shows

$$\|\{u^*,v_1,v_1\}\|_{L^1(0,T;L^2_x)}\leq 3T^{\frac{1}{2}}\|u^*\|_{L^6(0,T;L^6_x)}\|v_1\|^2_{L^6(0,T;L^6_x)}.$$

By a standard localization procedure we can also get integrability in ω, so that the terms on the right-hand side of (3.10b) are well-defined and this is indeed the unique solution for v_2. Again, $v_2\in L^r(0,T;L^p_x)$ by Propositions 2.9 and 2.10.

From the definition $z(t):=u(t)-u^*-\sigma v_1(t)-\sigma^2v_2(t)$, it follows that z satisfies (3.9) in the mild sense, meaning for every $t\in[0,T]$ we have the $\mathbb{P}$-a.s. equality

$$\begin{aligned}z(t)=&\int_0^t P(t-t')i\kappa R(t')\,\mathrm{d}t'-\frac{1}{2}\beta^2\int_0^t P(t-t')(\sigma^3v_1+\sigma^4v_2+\sigma^2z)\,\mathrm{d}t'\\&-\int_0^t P(t-t')i(\sigma^3v_2+\sigma z)\Phi\,\mathrm{d}W(t')=:I_1+I_2+I_3.\end{aligned}\tag{5.1}$$

To show (3.13) we define the stopping time $\tau:=\min\{\tau_{v_1},\tau_{v_2},\tau_z\}$, and notice that

$$\mathbb{P}[\tau_z<\min\{\tau_{v_1},\tau_{v_2}\}]=\mathbb{P}[\tau_z<T,\tau_z\leq\tau].$$

To estimate the latter probability, we first estimate I_1 and I_2 on the interval $[0,\tau]$. We assume $\varepsilon'\leq 1$ and $c_1\geq 1$, so that we can estimate $\sigma^m\leq\varepsilon^m\leq 1$ for any $m\geq 0$. We will use this frequently and without further mention.

To estimate I_2, note that by Propositions 2.9 and 2.10 we have

$$\|I_2\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)}\overset{(2.14b),(2.18b)}{\lesssim}\|\sigma^3v_1+\sigma^4v_2+\sigma^2z\|_{L^1(0,\tau;L^2_x)}\overset{(3.12)}{\lesssim}\varepsilon^3.\tag{5.2}$$

Using Propositions 2.9 and 2.10 again, carefully inspecting every term in (3.7) and using Hölder's inequality on the triple bracket, we see that we can also estimate

$$\|I_1\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \overset{(2.14b),(2.18b)}{\lesssim} \|R\|_{L^1(0,\tau;L^2_x)} \overset{(3.12)}{\lesssim} \varepsilon^3 + c_1^3\varepsilon^4. \tag{5.3}$$

Combining (5.1), (5.2), and (5.3) with the triangle inequality we get the estimate

$$\|z\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \le C(\varepsilon^3 + c_1^3\varepsilon^4) + \|I_3\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \tag{5.4}$$

for some constant C which is independent of v_1, v_2, ε, σ, and c_1. This allows us to set $c_1 = 4C$ and $\varepsilon' = c_1^{-3}$. Suppose now that $\tau_z < T$ and $\tau_z \le \tau$. Then since $z \in C([0,T];L^2_x) \cap L^r(0,T;L^p_x)$, we have by continuity:

$$c_1\varepsilon^3 \overset{(3.12c)}{=} \|z\|_{L^\infty(0,\tau_z;L^2_x)\cap L^r(0,\tau_z;L^p_x)} \overset{(5.4)}{\le} \frac{1}{2}c_1\varepsilon^3 + \|I_3\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)}.$$

Since this can only happen if I_3 is sufficiently large, we can now estimate

$$\begin{aligned}\mathbb{P}[\tau_z < T, \tau_z \le \tau] &\le \mathbb{P}\left[\|\varepsilon^{-3}I_3\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \ge \frac{1}{2}c_1\right] \\ &= \mathbb{P}\left[\|\sigma^{-1}\varepsilon^{-2}I_3\|_{L^\infty(0,\tau;L^2_x)\cap L^r(0,\tau;L^p_x)} \ge \frac{1}{2}c_1\sigma^{-1}\varepsilon\right].\end{aligned}$$

It only remains to estimate the latter probability. We note that for $t \le \tau$ we have the equality

$$\sigma^{-1}\varepsilon^{-2}I_3(t) \overset{(5.1)}{=} -\int_0^t P(t-t')(\mathbb{1}_{[0,\tau]}(t')i(\sigma^2\varepsilon^{-2}v_2(t') + \varepsilon^{-2}z(t')))\Phi\,\mathrm{d}W(t').$$

After estimating the integrand as

$$\|\mathbb{1}_{[0,\tau]}(t')(\sigma^2\varepsilon^{-2}v_2(t') + \varepsilon^{-2}z(t'))\|_{L^\infty_\Omega(L^\infty(0,T;L^2_x))} \overset{(3.12)}{\le} (1+c_1)\varepsilon \le 2,$$

it follows from (2.14c), (2.18c), and Lemma 2.11 that the Gaussian tail estimate

$$\mathbb{P}[\tau_z < T, \tau_z \le \tau] \le \exp(-c_2c_1^2\sigma^{-2}\varepsilon^2)$$

holds for some $c_2 > 0$ which is independent of ε, σ, c_1, as long as $c_1\sigma^{-1}\varepsilon$ is sufficiently large. But since $\varepsilon\sigma^{-1} \ge 1$, this can be accomplished by rechoosing c_1 to be larger than before if necessary (and also rechoosing $\varepsilon' = c_1^{-3}$). ■

5.2. Orbital stability. Before we prove Theorem 3.6, we isolate some convolution estimates which are used multiple times in the proof. These estimates essentially follow from Young's convolution inequality and the exponential decay of $P(t)\Pi$ (which we have not used before this point).

Lemma 5.1. *Let $r \in [1,\infty]$. There exists a constant C, such that the estimates*

$$\left\| \int_0^\cdot P(\cdot - t')\Pi f(t')\, \mathrm{d}t' \right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \le C \min\{T^{\frac{1}{r'}}, 1\} \|f\|_{L^r(0,T;L^q_\Omega(L^2_x))}, \tag{5.5a}$$

$$\left\| \int_0^\cdot P(\cdot - t')\Pi g(t')\, \mathrm{d}t' \right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \le C \min\{T^{\frac{1}{r'}}, 1\} \|g\|_{L^q_\Omega(L^r(0,T;L^2_x))}, \tag{5.5b}$$

$$\left\| \int_0^\cdot P(\cdot - t')\Pi h(t')\Phi\, \mathrm{d}W(t') \right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \le C\sqrt{q}\beta \min\{T^{\frac{1}{2}}, 1\} \|h\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \tag{5.5c}$$

hold for any $q \in [2,\infty)$, $T \in (0,\infty)$, $f \in L^r(0,T;L^q_\Omega(L^2_x))$, $g \in L^q_\Omega(L^r(0,T;L^2_x))$, $h \in L^\infty(0,T;L^q_\Omega(L^2_x))$, and $\phi \in L^2(\mathbb{R};\mathbb{R})$ (recall (2.1)).

Proof. First, we compute

$$\alpha_r(T) := \|P(\cdot)\Pi\|_{L^r(0,T;\mathcal{L}(L^2_x))} \overset{(2.12)}{\le} \|M\exp(-a\cdot)\|_{L^r(0,T)} \le C\min\{T^{\frac{1}{r}}, 1\}, \tag{5.6}$$

for some C which does not depend on T. It then follows from Young's convolution inequality that

$$\left\| \int_0^\cdot P(\cdot - t')\Pi f(t')\, \mathrm{d}t' \right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \le \alpha_{r'}(T)\|f\|_{L^r(0,T;L^q_\Omega(L^2_x))},$$

and also

$$\begin{aligned}\left\| \int_0^\cdot P(\cdot - t')\Pi g(t')\, \mathrm{d}t' \right\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} &\le \left\| \int_0^\cdot P(\cdot - t')\Pi g(t')\, \mathrm{d}t' \right\|_{L^q_\Omega(L^\infty(0,T;L^2_x))} \\ &\le \alpha_{r'}(T)\|g\|_{L^q_\Omega(L^r(0,T;L^2_x))},\end{aligned}$$

which, in combination with (5.6), shows (5.5a) and (5.5b). Finally, for $t \in [0,T]$ we estimate

$$\begin{aligned}\left\| \int_0^t P(t-t')\Pi h\Phi\, \mathrm{d}W(t') \right\|_{L^q_\Omega(L^2_x)} &\le C\sqrt{q}\|P(t-\cdot)\Pi h(\cdot)\Phi\|_{L^q_\Omega(L^2(0,t;\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});L^2_x)))} \\ &\overset{(2.4b)}{=} C\sqrt{q}\beta\|P(t-\cdot)\Pi h(\cdot)\|_{L^q_\Omega(L^2(0,t;L^2_x))} \\ &\le C\sqrt{q}\beta\|P(t-\cdot)\Pi h(\cdot)\|_{L^2(0,t;L^q_\Omega(L^2_x))} \\ &\le C\sqrt{q}\beta\|P(t-\cdot)\Pi\|_{L^2(0,t;\mathcal{L}(L^2_x))}\|h\|_{L^\infty(0,T;L^q_\Omega(L^2_x))} \\ &= C\sqrt{q}\beta\alpha_2(T)\|h\|_{L^\infty(0,T;L^q_\Omega(L^2_x))},\end{aligned}$$

where we have used [35, Theorem 1.1] for the first inequality, and the fact that $q \ge 2$ for the third inequality. Taking the supremum over $t \in [0,T]$ and using (5.6) gives (5.5c). ∎

Proof of Theorem 3.6. From Proposition 2.8 we obtain

$$\begin{aligned} v_1 &= \mathcal{P}[v_1]u^*_x + \Pi v_1, \\ v_2 &= \mathcal{P}\left[v_2 - \frac{1}{2}\mathcal{P}[v_1]^2 u^*_{xx}\right]u^*_x + \frac{1}{2}\mathcal{P}[v_1]^2 u^*_{xx} + \Pi\left(v_2 - \frac{1}{2}\mathcal{P}[v_1]^2 u^*_{xx}\right).\end{aligned}$$

If we define

$$a_1 := \mathcal{P}[v_1], \qquad w_1 := \Pi v_1,$$
$$a_2 := \mathcal{P}\left[v_2 - \frac{1}{2}\mathcal{P}[v_1]^2 u^*_{xx}\right], \qquad w_2 := \Pi\left(v_2 - \frac{1}{2}\mathcal{P}[v_1]^2 u^*_{xx}\right),$$

then (3.17) and (3.19) hold. Equations (3.20) and (3.21) follow by substitution using (3.10) and noting that Π commutes with $P(t)$.

We will now show (3.22). Throughout the proof, $A \lesssim B$ means that there exists a constant C, independent of $v_{1,0}$, $v_{2,0}$, t, q, and ϕ (recall (2.1)) such that $A \le CB$. We first estimate w_1 as follows:

$$\begin{aligned}\|w_1(t)\|_{L^q_\Omega(L^2_x)} &\overset{(3.21a)}{\le} \|P(t)\Pi v_{1,0}\|_{L^q_\Omega(L^2_x)} + \left\|\int_0^t P(t-t')\Pi i u^*\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^2_x)} \\ &\overset{(2.12),(5.5c)}{\lesssim} e^{-at}\|v_{1,0}\|_{L^q_\Omega(L^2_x)} + \sqrt{q}\beta\min\{t^{\frac{1}{2}},1\},\end{aligned}$$

which is (3.22a). In order to show (3.22b), we will need two intermediate estimates. First, by Proposition 2.9 we have

$$(5.7)\quad \begin{aligned}\|w_1\|_{L^q_\Omega(L^6(0,t;L^6_x))} &\le \|P(\cdot)\Pi v_{1,0}\|_{L^q_\Omega(L^6(0,t;L^6_x))} + \left\|\int_0^\cdot P(\cdot - t')\Pi i u^*\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^6(0,t;L^6_x))} \\ &\overset{(2.14a),(2.14c)}{\lesssim} \|v_{1,0}\|_{L^q_\Omega(L^2_x)} + \sqrt{q}\beta t^{\frac{1}{2}}.\end{aligned}$$

It also follows from [35, Theorem 1.1] that

$$(5.8)\quad \|a_1(t)\|_{L^q_\Omega} \overset{(3.20a)}{\lesssim} \|v_{1,0}\|_{L^q_\Omega(L^2_x)} + \sqrt{q}\beta t^{\frac{1}{2}}.$$

Now we have all the ingredients needed to estimate w_2. We first replace the occurrences of v_1 in (3.21b) by $w_1 + a_1 u^*_x$, in accordance with (3.17). This results in the equality

$$(5.9)\quad \begin{aligned} w_2(t) = {} & P(t)\Pi v_{2,0} \\ & + \int_0^t P(t-t')\Pi i\kappa\{u^*, w_1, w_1\}\,\mathrm{d}t' \\ & + 2\int_0^t P(t-t')\Pi i\kappa a_1\{u^*, u^*_x, w_1\}\,\mathrm{d}t' \\ & + \int_0^t P(t-t')\Pi i\kappa a_1^2\{u^*, u^*_x, u^*_x\}\,\mathrm{d}t' \\ & - \frac{1}{2}\int_0^t P(t-t')\Pi\beta^2 u^*\,\mathrm{d}t' \\ & - \int_0^t P(t-t')\Pi i w_1\Phi\,\mathrm{d}W(t') \\ & - \int_0^t P(t-t')\Pi i a_1 u^*_x\Phi\,\mathrm{d}W(t') \\ & - \frac{1}{2}a_1^2\Pi u^*_{xx}.\end{aligned}$$

We estimate the $L^q_\Omega(L^2_x)$-norm of each term separately, which will show (3.22b). First, we have

$$\|P(t)\Pi v_{2,0}\|_{L^q_\Omega(L^2_x)} \overset{(2.12)}{\lesssim} e^{-at}\|v_{2,0}\|_{L^q_\Omega(L^2_x)},$$

$$\|a_1(t)^2\Pi u^*_{xx}\|_{L^q_\Omega(L^2_x)} \lesssim \|a_1(t)^2\|_{L^q_\Omega} = \|a_1(t)\|^2_{L^{2q}_\Omega} \overset{(5.8)}{\lesssim} \|v_{1,0}\|^2_{L^{2q}_\Omega(L^2_x)} + q\beta^2 t.$$

Next, we use our first intermediate estimate on the term which is quadratic in w_1.

$$\left\|\int_0^t P(t-t')\Pi i\kappa\{u^*, w_1, w_1\}\,\mathrm{d}t'\right\|_{L^q_\Omega(L^2_x)} \overset{(5.5b)}{\lesssim} \|\{u^*, w_1, w_1\}\|_{L^q_\Omega(L^3(0,t;L^2_x))}$$

$$\lesssim \|u^*\|_{L^\infty(0,t;L^6_x)}\|w_1\|^2_{L^{2q}_\Omega(L^6(0,t;L^6_x))} \overset{(5.7)}{\lesssim} \|v_{1,0}\|^2_{L^{2q}_\Omega(L^2_x)} + q\beta^2 t,$$

where we have used Hölder's inequality for the second step. We also estimate

$$\left\|\int_0^t P(t-t')\Pi i\kappa a_1\{u^*, u^*_x, w_1\}\,\mathrm{d}t'\right\|_{L^q_\Omega(L^2_x)} \overset{(5.5a)}{\lesssim} \|a_1\{u^*, u^*_x, w_1\}\|_{L^\infty(0,t;L^q_\Omega(L^2_x))}$$

$$\lesssim \|a_1\|_{L^\infty(0,t;L^{2q}_\Omega)}\|w_1\|_{L^\infty(0,t;L^{2q}_\Omega(L^2_x))} \overset{(3.22a),(5.8)}{\lesssim} \|v_{1,0}\|^2_{L^{2q}_\Omega(L^2_x)} + q\beta^2 t,$$

as well as

$$\left\|\int_0^t P(t-t')\Pi i\kappa a_1^2\{u^*, u^*_x, u^*_x\}\,\mathrm{d}t'\right\|_{L^q_\Omega(L^2_x)} \overset{(5.5a)}{\lesssim} \|a_1^2\{u^*, u^*_x, u^*_x\}\|_{L^\infty(0,t;L^q_\Omega(L^2_x))}$$

$$\lesssim \|a_1\|^2_{L^\infty(0,t;L^{2q}_\Omega)} \overset{(5.8)}{\lesssim} \|v_{1,0}\|^2_{L^{2q}_\Omega} + q\beta^2 t,$$

and

$$\left\|\int_0^t P(t-t')\Pi\beta^2 u^*\,\mathrm{d}t'\right\|_{L^q_\Omega(L^2_x)} \overset{(2.12)}{\lesssim} \beta^2 t.$$

It only remains to estimate the stochastic integrals in (5.9). For the first we have

$$\left\|\int_0^t P(t-t')\Pi i w_1\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^2_x)} \overset{(5.5c)}{\lesssim} \sqrt{q}\beta t^{\frac{1}{2}}\|w_1\|_{L^\infty(0,t;L^q_\Omega(L^2_x))}$$

$$\leq \frac{1}{2}\|w_1\|^2_{L^\infty(0,t;L^q_\Omega(L^2_x))} + \frac{1}{2}q\beta^2 t \overset{(3.22a)}{\lesssim} \|v_{1,0}\|^2_{L^q_\Omega(L^2_x)} + q\beta^2 t,$$

and for the second

$$\left\|\int_0^t P(t-t')\Pi i a_1 u^*_x\Phi\,\mathrm{d}W(t')\right\|_{L^q_\Omega(L^2_x)} \overset{(5.5c)}{\lesssim} \sqrt{q}\beta t^{\frac{1}{2}}\|a_1\|_{L^\infty(0,t;L^q_\Omega)}$$

$$\leq \frac{1}{2}\|a_1\|^2_{L^\infty(0,t;L^q_\Omega)} + \frac{1}{2}q\beta^2 t \overset{(5.8)}{\lesssim} \|v_{1,0}\|^2_{L^q_\Omega(L^2_x)} + q\beta^2 t. \quad \blacksquare$$

Proof of Proposition 3.7. From our previous ansatz for u and v_1 we have the equalities

$$u(t) - u^*(x+\sigma a_1(t)) \overset{(3.14)}{=} u^* - u^*(x+\sigma a_1(t)) + \sigma v_1(t) + z'(t) \tag{5.10a}$$

$$\overset{(3.17a)}{=} u^* + \sigma a_1(t) u^*_x - u^*(x+\sigma a_1(t)) + \sigma w_1(t) + z'(t). \tag{5.10b}$$

From (5.10a) and a zeroth-order Taylor expansion we may obtain

$$\|u(t) - u^*(x+\sigma a_1(t))\|_{L^2_x} \le C_1 \sigma |a_1(t)| + \sigma \|v_1(t)\|_{L^2_x} + \|z'(t)\|_{L^2_x}, \tag{5.11a}$$

for some constant C_1 derived from u^*. From (5.10b) and a first-order Taylor expansion we also get

$$\|u(t) - u^*(x+\sigma a_1(t))\|_{L^2_x} \le C_2 \sigma^2 |a_1(t)|^2 + \sigma \|w_1(t)\|_{L^2_x} + \|z'(t)\|_{L^2_x}, \tag{5.11b}$$

for some constant C_2 also derived from u^*. Now set $T = a^{-1}\log(6M)$, where a and M are the constants from (2.12), and fix some c_1, c_2, ε' such that Proposition 3.5 holds with this choice of T (note that our initial condition corresponds to setting $v_{1,0} = \sigma^{-1} v_0$). Additionally, set $\tilde{c}_1 = \frac{1}{6}\min\{M^{-1}, C_1^{-1}\|\mathcal{P}\|_{\mathcal{L}(L^2_x;\mathbb{R})}\}$. From the assumption that $\|v_0\| \le \tilde{c}_1 \varepsilon$ we obtain

$$\begin{aligned}
\sigma|a_1(t)| &\overset{(3.20a)}{\le} C_1^{-1}\frac{\varepsilon}{6} + \sigma \|\mathcal{P}\|_{\mathcal{L}(L^2_x;\mathbb{R})} \left\| \int_0^t u^* \Phi \, \mathrm{d}W(t') \right\|_{L^2_x}, \\
\sigma\|v_1(t)\|_{L^2_x} &\overset{(3.10a)}{\le} \frac{\varepsilon}{6} + \sigma \left\| \int_0^t P(t-t') u^* \Phi \, \mathrm{d}W(t') \right\|_{L^2_x}, \\
\sigma\|w_1(T)\|_{L^2_x} &\overset{(3.21a)}{\le} \tilde{c}_1 \frac{\varepsilon}{6} + \sigma \left\| \int_0^T P(T-t') \Pi u^* \Phi \, \mathrm{d}W(t') \right\|_{L^2_x},
\end{aligned}$$

where the third inequality follows from (2.12) since $Me^{-aT} = \frac{1}{6}$ by our choice of T. Using (2.14c), (2.18c), and Lemma 2.11, we can find constants λ, $c_2' > 0$, such that

$$\mathbb{P}\left[C_1 \sigma |a_1|_{L^\infty(0,T)} \ge \frac{\varepsilon}{3} \right] \le \exp(-c_2' \sigma^{-2} \varepsilon^2), \tag{5.12a}$$

$$\mathbb{P}\left[\sigma \|v_1\|_{L^\infty(0,T;L^2_x)} \ge \frac{\varepsilon}{3} \right] \le \exp(-c_2' \sigma^{-2} \varepsilon^2), \tag{5.12b}$$

$$\mathbb{P}\left[\sigma \|w_1(T)\|_{L^2_x} \ge \tilde{c}_1 \frac{\varepsilon}{3} \right] \le \exp(-c_2' \sigma^{-2} \varepsilon^2), \tag{5.12c}$$

whenever $\sigma^{-1}\varepsilon \ge \lambda$. If we take ε' small enough such that $\tilde{c}_1 \frac{\varepsilon'}{3} \ge c_1 \varepsilon'^2$ (if necessary), then by Proposition 3.5, this also results in

$$\begin{aligned}
\mathbb{P}\left[\|z'\|_{L^\infty(0,T;L^2_x)} \ge \tilde{c}_1 \frac{\varepsilon}{3} \right] &\le \mathbb{P}[\|z'\|_{L^\infty(0,T;L^2_x)} \ge c_1 \varepsilon^2] \\
&= \mathbb{P}[\tau_{z'} < T] \\
&\le \mathbb{P}[\tau_{z'} < \tau_{v_1}] + \mathbb{P}[\tau_{v_1} < T] \\
&\overset{(3.16),(5.12b)}{\le} \exp(-c_2 \sigma^{-2} \varepsilon^2) + \exp(-c_2' \sigma^{-2} \varepsilon^2)
\end{aligned} \tag{5.12d}$$

for all $\varepsilon \le \varepsilon'$. If we additionally take ε' smaller (if necessary) such that $\frac{C_1\sqrt{3\tilde{c}_1}}{\sqrt{C_2\varepsilon'}} \ge 1$, then we also get

$$\mathbb{P}\left[C_2\sigma^2|a_1|^2_{L^\infty(0,T)} \ge \tilde{c}_1\frac{\varepsilon}{3}\right] = \mathbb{P}\left[C_1\sigma|a_1|_{L^\infty(0,T)} \ge \frac{C_1\sqrt{3\tilde{c}_1}}{\sqrt{C_2\varepsilon}}\frac{\varepsilon}{3}\right] \overset{(5.12a)}{\le} \exp(-c_2'\sigma^{-2}\varepsilon^2), \tag{5.12e}$$

for all $\varepsilon \le \varepsilon'$. Equation (5.11a), a simple union bound, and the fact that $\tilde{c}_1 \le 1$ now gives

$$\begin{aligned}\mathbb{P}[\|u(\cdot) - u^*(x+\sigma a_1(\cdot))\|_{L^\infty(0,T;L^2_x)} \ge \varepsilon] &\le \mathbb{P}\left[C_1\sigma|a_1|_{L^\infty(0,T)} \ge \frac{\varepsilon}{3}\right]\\ &\quad + \mathbb{P}\left[\sigma\|v_1\|_{L^\infty(0,T;L^2_x)} \ge \frac{\varepsilon}{3}\right]\\ &\quad + \mathbb{P}\left[\|z'\|_{L^\infty(0,T;L^2_x)} \ge \tilde{c}_1\frac{\varepsilon}{3}\right]\\ &\overset{(5.12)}{\le} 3\exp(-c_2'\sigma^{-2}\varepsilon^2) + \exp(-c_2\sigma^{-2}\varepsilon^2).\end{aligned}$$

Similarly, from (5.11b) we get

$$\begin{aligned}\mathbb{P}[\|u(T) - u^*(x+\sigma a_1(T))\|_{L^2_x} \ge \tilde{c}_1\varepsilon] &\le \mathbb{P}\left[C_2\sigma^2|a_1(T)|^2 \ge \tilde{c}_1\frac{\varepsilon}{3}\right]\\ &\quad + \mathbb{P}[\sigma\|w_1(T)\|_{L^2_x} \ge \tilde{c}_1\frac{\varepsilon}{3}]\\ &\quad + \mathbb{P}\left[\|z'(T)\|_{L^2_x} \ge \tilde{c}_1\frac{\varepsilon}{3}\right]\\ &\overset{(5.12)}{\le} 3\exp(-c_2'\sigma^{-2}\varepsilon^2) + \exp(-c_2\sigma^{-2}\varepsilon^2)\end{aligned}$$

(note that although we wrote $L^\infty(0,T)$ in (5.12), we could have also written $C([0,T])$ so the estimate is valid). The result follows by choosing $\tilde{c}_2 = \min\{c_2, c_2'\}$. ■

Appendix A. Hilbert–Schmidt operators.

Proof of Proposition 2.1. Fix some $\phi \in L^2(\mathbb{R};\mathbb{R})$, and define for any $\psi \in L^2_x$ the following map:

$$\Phi_\psi \colon f \mapsto \psi * f.$$

Recall that with this notation $\Phi = \Phi_\phi$ (see (2.1a)). Now let e_k, $k \in \mathbb{N}$ be any orthonormal basis of $L^2(\mathbb{R};\mathbb{R})$. We see using Parseval's identity that

$$\sum_{k\in\mathbb{N}}(\Phi e_k(x))^2 = \sum_{k\in\mathbb{N}}\langle\phi(\cdot - x), e_k\rangle^2_{L^2_x} = \|\phi(\cdot - x)\|^2_{L^2_x} \overset{(2.1b)}{=} \beta^2,$$

which shows (2.4a). Using Fubini's theorem and Parseval's identity, we can also compute

$$\begin{aligned}\|u\Phi\|^2_{\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});L^2_x)} &= \sum_{k\in\mathbb{N}}\|u\Phi e_k\|^2_{L^2_x} = \sum_{k\in\mathbb{N}}\int_{\mathbb{R}}|u(x)|^2\langle\phi(\cdot - x), e_k\rangle^2_{L^2_x}\,\mathrm{d}x\\ &= \int_{\mathbb{R}}|u(x)|^2\sum_{k\in\mathbb{N}}\langle\phi(\cdot - x), e_k\rangle^2_{L^2_x}\,\mathrm{d}x = \int_{\mathbb{R}}|u(x)|^2\|\phi(\cdot - x)\|^2_{L^2_x}\,\mathrm{d}x\\ &= \|u\|^2_{L^2_x}\|\phi\|^2_{L^2_x},\end{aligned}$$

which shows (2.4b).

To show (2.4c) we will make use of complex interpolation. Thus, we will now break convention and regard H^s_x and L^2_x as complex spaces for the rest of this section. We will show the complexified estimate

$$\|u\Phi\|_{\mathcal{L}_2(L^2_x;H^s_x)} \le C_s\|\phi\|_{H^s_x}\|u\|_{H^s_x}. \tag{A.1}$$

The result then follows after noting that an orthonormal basis of the real Hilbert space $L^2(\mathbb{R};\mathbb{R})$ is also an orthonormal basis of L^2_x when the latter is regarded as a complex Hilbert space. We first show by induction that (A.1) holds when $s=2n$ for some nonnegative integer n. By repeating the previous calculation, we find again that

$$\|u\Phi\|_{\mathcal{L}_2(L^2_x;L^2_x)} = \|u\|_{L^2_x}\|\phi\|_{L^2_x},$$

which implies the base case. Therefore, we now assume that the statement holds for some n. By elementary computations, we find

$$\begin{aligned}(1-\Delta)(u\Phi f) &= (1-\Delta)(u(\phi * f))\\ &= u(\phi*f) - \Delta u(\phi*f) - 2\,\partial_x u(\partial_x\phi * f) - u(\Delta\phi * f)\\ &= u\Phi f - \Delta u\Phi f - 2\,\partial_x u(\Phi_{\partial_x\phi} f) - u(\Phi_{\Delta\phi} f),\end{aligned}$$

so that

$$(1-\Delta)(u\Phi) = u\Phi - \Delta u\Phi - 2\,\partial_x u\Phi_{\partial_x\phi} - u\Phi_{\Delta\phi}.$$

Combining this with the triangle inequality and the induction hypothesis gives

$$\begin{aligned}\|u\Phi\|_{\mathcal{L}_2(L^2_x;H^{n+2}_x)} &= \|(1-\Delta)(u\Phi)\|_{\mathcal{L}_2(L^2_x;H^n_x)}\\ &\le C(\|u\|_{H^n_x}\|\phi\|_{H^n_x} + \|\Delta u\|_{H^n_x}\|\phi\|_{H^n_x} + 2\|\partial_x u\|_{H^n_x}\|\partial_x\phi\|_{H^n_x} + \|u\|_{H^n_x}\|\Delta\phi\|_{H^n_x})\\ &\le C'\|u\|_{H^{n+2}_x}\|\phi\|_{H^{n+2}_x}.\end{aligned}$$

Now let $s\in[0,\infty)$ be arbitrary, let n be an integer such that $2n\ge s$, let $\theta\in[0,1]$ be such that $s=2n\theta$, and consider the bilinear map

$$B\colon (u,\phi)\mapsto u\cdot\Phi_\phi.$$

We have already shown that B is bounded from $L^2_x\times L^2_x$ to $\mathcal{L}_2(L^2_x;L^2_x)$ and from $H^{2n}_x\times H^{2n}_x$ to $\mathcal{L}_2(L^2_x,H^{2n}_x)$. Thus, by complex interpolation (using the notation $[\cdot,\cdot]_\theta$ for the intermediate space) it follows that B is also bounded from

$$[L^2_x,H^{2n}_x]_\theta\times[L^2_x,H^{2n}_x]_\theta = H^s_x\times H^s_x$$

to

$$[\mathcal{L}_2(L^2_x,L^2_x),\mathcal{L}_2(L^2_x,H^{2n}_x)]_\theta = \mathcal{L}_2(L^2_x,H^s_x). \tag{A.2}$$

For the interpolation of bilinear operators we have used [4, Theorem 4.4.1], and the isomorphism (A.2) is shown for γ-radonifying operators (which generalize Hilbert–Schmidt operators) in [21, Theorem 9.1.25]. ■

Appendix B. Stochastic Strichartz estimates. To prove (2.7c) we distinguish between the cases $p=2$ and $p>2$.

Case $p>2$. For every $t'\in[0,T]$, define the operator

$$\begin{aligned}\Psi(t'): H^s_x &\to L^r(0,T;H^{s,p}_x)\\ \psi &\mapsto 1_{[t',T]}(\cdot)S(\cdot-t')\psi,\end{aligned}$$

and observe that $\|\Psi(t')\|_{\mathcal{L}(H^s_x;L^r(0,T;H^{s,p}_x))}\le\|\Psi(0)\|_{\mathcal{L}(H^s_x;L^r(0,T;H^{s,p}_x))}\le L$ for some $L<\infty$ which is independent of T by (2.7a).

Since $p\in(2,\infty)$, the space L^p_x is 2-smooth [22, Proposition 3.5.30]. Using the lifting operator $(1-\Delta)^{\frac{s}{2}}$, this property immediately extends to $H^{s,p}_x$. Since $r\in(4,\infty)$, the space $L^r(0,T;H^{s,p}_x)$ has this property as well (see, for instance, [34, Proposition 2.2]). Thus, using our definition of Ψ we can rewrite and estimate

$$\begin{aligned}\Big\|\int_0^{\cdot} S(\cdot-t')h(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(L^r(0,T;H^{s,p}_x))} &= \Big\|\int_0^T \Psi(t')h(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(L^r(0,T;H^{s,p}_x))}\\ &\le C\sqrt{q}\|\Psi h\Phi\|_{L^q_\Omega(L^2(0,T;\gamma(L^2(\mathbb{R};\mathbb{R});L^r(0,T;H^{s,p}_x))))}\\ &\le CL\sqrt{q}\|h\Phi\|_{L^q_\Omega(L^2(0,T;\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});H^s_x)))}\\ &\overset{(2.4c)}{\le} C'L\sqrt{q}\|\phi\|_{H^s_x}\|h\|_{L^q_\Omega(L^2(0,T;H^s_x))}.\end{aligned}$$

The first inequality follows from [35, Theorem 1.1], and the second follows from the left-ideal property of γ-radonifying operators (which can easily be seen from the definition) and the boundedness of Ψ. ■

Case $p=2$. Since (r,p) satisfies (2.6) we have $r=\infty$. Using the fact that $S(t)$ is unitary on H^s_x and using [35, Theorem 1.1] again we find

$$\begin{aligned}\Big\|\int_0^{\cdot} S(\cdot-t')h(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(L^\infty(0,T;H^s_x))} &= \Big\|\int_0^{\cdot} S(-t')h(t')\Phi\,\mathrm{d}W(t')\Big\|_{L^q_\Omega(L^\infty(0,T;H^s_x))}\\ &\le C\sqrt{q}\|S(-\cdot)h(\cdot)\Phi\|_{L^q_\Omega(L^2(0,T;\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});H^s_x)))}\\ &= C\sqrt{q}\|h\Phi\|_{L^q_\Omega(L^2(0,T;\mathcal{L}_2(L^2(\mathbb{R};\mathbb{R});H^s_x)))}\\ &\overset{(2.4c)}{\le} C'\sqrt{q}\|\phi\|_{H^s_x}\|h\|_{L^q_\Omega(L^2(0,T;H^s_x))}.\end{aligned}$$

The continuity in H^s_x follows by a routine approximation argument. ■

Acknowledgments. The authors thank Mark Veraar for discussions and valuable suggestions on this manuscript. The authors thank the anonymous reviewers for helpful suggestions.

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