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# On Pointwise $\ell^r$ -Sparse Domination in a Space of Homogeneous Type

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## Abstract

We prove a general sparse domination theorem in a space of homogeneous type, in which a vector-valued operator is controlled pointwise by a positive, local expression called a sparse operator. We use the structure of the operator to get sparse domination in which the usual  $\ell^1$ -sum in the sparse operator is replaced by an  $\ell^r$ -sum. This sparse domination theorem is applicable to various operators from both harmonic analysis and (S)PDE. Using our main theorem, we prove the  $A_2$ -theorem for vector-valued Calderón–Zygmund operators in a space of homogeneous type, from which we deduce an anisotropic, mixed-norm Mihlin multiplier theorem. Furthermore, we show quantitative weighted norm inequalities for the Rademacher maximal operator, for which Banach space geometry plays a major role.

**Keywords** Sparse domination · Space of homogeneous type · Muckenhoupt weight · Singular integral operator · Mihlin multiplier theorem · Rademacher maximal operator

**Mathematics Subject Classification** Primary: 42B20 · Secondary: 42B15, 42B25, 46E40

## 1 Introduction

The technique of controlling various operators by so-called sparse operators has proven to be a very useful tool to obtain (sharp) weighted norm inequalities in the past decade.

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The key feature in this approach is that a typically signed and non-local operator is dominated, either in norm, pointwise or in dual form, by a positive and local expression.

The sparse domination technique comes from Lerner's work towards an alternative proof of the  $A_2$ -theorem, which was first proven by Hytönen in [38]. In [54] Lerner applied his local mean oscillation decomposition approach to the  $A_2$ -theorem, estimating the norm of a Calderón–Zygmund operator by the norm of a sparse operator. This was later improved to a pointwise estimate independently by Conde-Alonso and Rey [15] and by Lerner and Nazarov [57]. Afterwards, Lacey [51] obtained the same result for a slightly larger class of Calderón–Zygmund operators by a stopping cube argument instead of the local mean oscillation decomposition approach. This argument was further refined by Hytönen, Roncal, and Tapiola [35] and afterwards made strikingly clear by Lerner [55], where the following abstract sparse domination principle was shown:

If  $T$  is a bounded sublinear operator from  $L^{p_1}(\mathbb{R}^n)$  to  $L^{p_1, \infty}(\mathbb{R}^n)$  and the *grand maximal truncation operator*

$$\mathcal{M}_T f(s) := \sup_{Q \ni s} \operatorname{ess\,sup}_{s' \in Q} |T(f \mathbf{1}_{\mathbb{R}^n \setminus 3Q})(s')|, \quad s \in \mathbb{R}^n,$$

is bounded from  $L^{p_2}(\mathbb{R}^n)$  to  $L^{p_2, \infty}(\mathbb{R}^n)$  for some  $1 \leq p_1, p_2 < \infty$ , then there is an  $\eta \in (0, 1)$  such that for every compactly supported  $f \in L^p(\mathbb{R}^n)$  with  $p_0 := \max\{p_1, p_2\}$  there exists an  $\eta$ -sparse family of cubes  $\mathcal{S}$  such that

$$|Tf(s)| \lesssim \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{p_0, Q} \mathbf{1}_Q(s), \quad s \in \mathbb{R}^n. \quad (1.1)$$

Here  $\langle f \rangle_{p, Q}^p := \int_Q f^p := \frac{1}{|Q|} \int_Q f^p$  for  $p \in (0, \infty)$  and positive  $f \in L_{\text{loc}}^p(\mathbb{R}^n)$  and we call a family of cubes  $\mathcal{S}$   $\eta$ -sparse if for every  $Q \in \mathcal{S}$  there exists a measurable set  $E_Q \subseteq Q$  such that  $|E_Q| \geq \eta|Q|$  and such that the  $E_Q$ 's are pairwise disjoint.

This sparse domination principle was further generalized in the recent paper [58] by Lerner and Ombrosi, in which the authors showed that the weak  $L^{p_2}$ -boundedness of the more flexible operator

$$\mathcal{M}_{T, \alpha}^\# f(s) := \sup_{Q \ni s} \operatorname{ess\,sup}_{s', s'' \in Q} |T(f \mathbf{1}_{\mathbb{R}^n \setminus \alpha Q})(s') - T(f \mathbf{1}_{\mathbb{R}^n \setminus \alpha Q})(s'')|, \quad s \in \mathbb{R}^n,$$

for some  $\alpha \geq 3$  is already enough to deduce the pointwise sparse domination as in (1.1). Furthermore, they relaxed the weak  $L^{p_1}$ -boundedness condition on  $T$  to a condition in the spirit of the  $T(1)$ -theorem.

## 1.1 Main Result

Our main result is a generalization of the main result in [58] in the following four directions:

- (i) We replace  $\mathbb{R}^n$  by a space of homogeneous type  $(S, d, \mu)$ .

- (ii) We let  $T$  be an operator from  $L^{p_1}(S; X)$  to  $L^{p_1, \infty}(S; Y)$ , where  $X$  and  $Y$  are Banach spaces.
- (iii) We use structure of the operator  $T$  and geometry of the Banach space  $Y$  to replace the  $\ell^1$ -sum in the sparse operator by an  $\ell^r$ -sum for  $r \geq 1$ .
- (iv) We replace the truncation  $T(f \mathbf{1}_{\mathbb{R}^n \setminus \alpha Q})$  in the grand maximal truncation operator by an abstract localization principle.

The extensions (i) and (ii) are relatively straightforward. The main novelty of this paper is (iii), which controls the weight characteristic dependence that can be deduced from the sparse domination. Generalization (iv) will only make its appearance in Theorem 3.2 and can be used to make the associated grand maximal truncation operator easier to estimate in specific situations.

Let  $(S, d, \mu)$  be a space of homogeneous type and let  $X$  and  $Y$  be Banach spaces. For a bounded linear operator  $T$  from  $L^{p_1}(S; X)$  to  $L^{p_1, \infty}(S; Y)$  and  $\alpha \geq 1$  we define the following *sharp grand maximal truncation operator*

$$\mathcal{M}_{T, \alpha}^\# f(s) := \sup_{B \ni s} \operatorname{ess\,sup}_{s', s'' \in B} \|T(f \mathbf{1}_{S \setminus \alpha B})(s') - T(f \mathbf{1}_{S \setminus \alpha B})(s'')\|_Y, \quad s \in S,$$

where the supremum is taken over all balls  $B \subseteq S$  containing  $s \in S$ . Our main theorem reads as follows.

**Theorem 1.1** *Let  $(S, d, \mu)$  be a space of homogeneous type and let  $X$  and  $Y$  be Banach spaces. Take  $p_1, p_2, r \in [1, \infty)$  and set  $p_0 := \max\{p_1, p_2\}$ . Take  $\alpha \geq 3c_d^2/\delta$ , where  $c_d$  is the quasi-metric constant and  $\delta$  is as in Proposition 2.1. Assume the following conditions:*

- $T$  is a bounded linear operator from  $L^{p_1}(S; X)$  to  $L^{p_1, \infty}(S; Y)$ .
- $\mathcal{M}_{T, \alpha}^\#$  is a bounded operator from  $L^{p_2}(S; X)$  to  $L^{p_2, \infty}(S)$ .
- There is a  $C_r > 0$  such that for disjointly and boundedly supported  $f_1, \dots, f_n \in L^{p_0}(S; X)$

$$\left\| T\left(\sum_{k=1}^n f_k\right)(s) \right\|_Y \leq C_r \left( \sum_{k=1}^n \|Tf_k(s)\|_Y^r \right)^{1/r}, \quad s \in S.$$

Then there is an  $\eta \in (0, 1)$  such that for any boundedly supported  $f \in L^{p_0}(S; X)$  there is an  $\eta$ -sparse collection of cubes  $\mathcal{S}$  such that

$$\|Tf(s)\|_Y \lesssim_{S, \alpha} C_T C_r \left( \sum_{Q \in \mathcal{S}} (\|f\|_X)_{p_0, Q}^r \mathbf{1}_Q(s) \right)^{1/r}, \quad s \in S,$$

where  $C_T = \|T\|_{L^{p_1} \rightarrow L^{p_1, \infty}} + \|\mathcal{M}_{T, \alpha}^\#\|_{L^{p_2} \rightarrow L^{p_2, \infty}}$ .

As the assumption in the third bullet of Theorem 1.1 expresses a form of sublinearity of the operator  $T$  when  $r = 1$ , we will call this assumption *r-sublinearity*. Note that it is crucial that the constant  $C_r$  is independent of  $n \in \mathbb{N}$ . If  $C_r = 1$  it suffices to consider  $n = 2$ .

## 1.2 Sharp Weighted Norm Inequalities

One of the main reasons to study sparse domination of an operator is the fact that sparse bounds yield weighted norm inequalities and these weighted norm inequalities are sharp for many operators. Here sharpness is meant in the sense that for  $p \in (p_0, \infty)$  we have a  $\beta \geq 0$  such that

$$\|T\|_{L^p(S,w;X) \rightarrow L^p(S,w;Y)} \lesssim [w]_{A_{p/p_0}}^\beta, \quad w \in A_{p/p_0}, \quad (1.2)$$

and (1.2) is false for any  $\beta' < \beta$ .

The first result of this type was obtained by Buckley [9], who showed that  $\beta = \frac{1}{p-1}$  for the Hardy–Littlewood maximal operator. A decade later, the quest to find sharp weighted bounds attracted renewed attention because of the work of Astala, Iwaniec, and Saksman [4]. They proved sharp regularity results for the solution to the Beltrami equation under the assumption that  $\beta = 1$  for the Beurling–Ahlfors transform for  $p \geq 2$ . This linear dependence on the  $A_p$  characteristic for the Beurling–Ahlfors transform was shown by Petermichl and Volberg in [72]. Another decade later, after many partial results, sharp weighted norm inequalities were obtained for general Calderón–Zygmund operators by Hytönen in [38] as discussed before.

In Sect. 4, we will prove weighted  $L^p$ -boundedness for the sparse operators appearing in Theorem 1.1. As a direct corollary from Theorem 1.1 and Proposition 4.1 we have:

**Corollary 1.2** *Under the assumptions of Theorem 1.1 we have for all  $p \in (p_0, \infty)$  and  $w \in A_{p/p_0}$*

$$\|T\|_{L^p(S,w;X) \rightarrow L^p(S,w;Y)} \lesssim C_T C_r [w]_{A_{p/p_0}}^{\max\{\frac{1}{p-p_0}, \frac{1}{r}\}},$$

where the implicit constant depends on  $S$ ,  $p_0$ ,  $p$ ,  $r$ , and  $\alpha$ .

As noted before, the main novelty in Theorem 1.1 is the introduction of the parameter  $r \in [1, \infty)$ . The  $r$ -sublinearity assumption in Theorem 1.1 becomes more restrictive as  $r$  increases and the conclusions of Theorem 1.1 and Corollary 1.2 consequently become stronger. In order to check whether the dependence on the weight characteristic is sharp, one can employ, e.g., [65, Theorem 1.2], which provides a lower bound for the best possible weight characteristic dependence in terms of the operator norm of  $T$  from  $L^p(S; X)$  to  $L^p(S; Y)$ . For some operators, like Littlewood–Paley or maximal operators, sharpness in the estimate in Corollary 1.2 is attained for  $r > 1$  and thus Theorem 1.1 can be used to show sharp weighted bounds for more operators than precursors like [58, Theorem 1.1].

## 1.3 How to Apply Our Main Result

Let us outline the typical way how one applies Theorem 1.1 (or the local and more general version in Theorem 3.2) to obtain (sharp) weighted  $L^p$ -boundedness for an operator  $T$ :

- (i) If  $T$  is not linear it is often *linearizable*, which means that we can linearize it by putting part of the operator in the norm of the Banach space  $Y$ . For example, if  $T$  is a Littlewood–Paley square function, we take  $Y = L^2$ , and if  $T$  is a maximal operator, we take  $Y = \ell^\infty$ . Alternatively one can apply Theorem 3.2, which is a local and more abstract version of Theorem 1.1 that does not assume  $T$  to be linear.
- (ii) The weak  $L^{p_1}$ -boundedness of  $T$  needs to be studied separately and is often already available in the literature.
- (iii) The operator  $\mathcal{M}_{T,\alpha}^\#$  reflects the non-localities of the operator  $T$ . The weak  $L^{p_2}$ -boundedness of  $\mathcal{M}_{T,\alpha}^\#$  requires an intricate study of the structure of the operator. In many examples  $\mathcal{M}_{T,\alpha}^\#$  can be pointwise dominated by the Hardy–Littlewood maximal operator  $M_{p_2}$ , which is weak  $L^{p_2}$ -bounded. This is exemplified for Calderón–Zygmund operators in the proof of Theorem 6.1. Sometimes, one can choose a suitable localization in Theorem 3.2 such that the sharp maximal truncation operator is either zero (see Sect. 8 on the Rademacher maximal operator), or pointwise dominated by  $T$ .
- (iv) The  $r$ -sublinearity assumption on  $T$  is trivial for  $r = 1$ , which suffices if one is not interested in quantitative weighted bounds. To check the  $r$ -sublinearity for some  $r > 1$ , one needs to use the structure of the operator and often also the geometric properties of the Banach space  $Y$  like type  $r$ . See, for example, the proofs of Theorems 8.1 and [64, Theorem 6.4] how to check  $r$ -sublinearity in concrete cases.

## 1.4 Applications

The main motivation to generalize the results in [58] comes from the application in the recent work [64] by Veraar and the author, in which Calderón–Zygmund theory is developed for stochastic singular integral operators. In particular, in [64, Theorem 6.4] Theorem 1.1 is applied with  $p_1 = p_2 = r = 2$  to prove a stochastic version of the vector-valued  $A_2$ -theorem for Calderón–Zygmund operators, which yields new results in the theory of maximal regularity for stochastic partial differential equations. The fact that  $r = 2$  in [64, Theorem 6.4] was needed to obtain a sharp result motivated the introduction of the parameter  $r$  in this paper. In future work, further applications of Theorem 1.1 to both deterministic and stochastic partial differential equations will be given, for which it is crucial that we allow spaces of homogeneous type instead of just  $\mathbb{R}^n$ , as, in these applications,  $S$  is typically  $\mathbb{R}_+ \times \mathbb{R}^n$  with the parabolic metric.

In this paper, we will focus on applications in harmonic analysis. We will provide a few examples that illustrate the sparse domination principle nicely and comment on further potential applications in Sect. 9.

- As a first application of Theorem 1.1, we prove an  $A_2$ -theorem for vector-valued Calderón–Zygmund operators with operator-valued kernel in a space of homogeneous type. The  $A_2$ -theorem for vector-valued Calderón–Zygmund operators with operator-valued kernel in Euclidean space has previously been proven in [32] and the  $A_2$ -theorem for scalar-valued Calderón–Zygmund operators in spaces of homogeneous type in [3,69]. Our theorem unifies these two results.

- Using the  $A_2$ -theorem, we prove a weighted, anisotropic, mixed-norm Mihlin multiplier theorem, which is a natural supplement to the recent results in [24] and is particularly useful in the study of spaces of smooth, vector-valued functions.
- In our second application of Theorem 1.1, we study sparse domination and quantitative weighted norm inequalities for the Rademacher maximal operator, extending the qualitative bounds in Euclidean space in [50]. The proof demonstrates how one can use the geometry of the Banach space to deduce  $r$ -sublinearity for an operator. As a corollary, we deduce that the lattice Hardy–Littlewood and the Rademacher maximal operator are not comparable.

## 1.5 Outline

This paper is organized as follows: After introducing spaces of homogeneous type and dyadic cubes in such spaces in Sect. 2, we will set up our abstract sparse domination framework and deduce Theorem 1.1 in Sect. 3. We also give some further generalizations of our main results. In Sect. 4 we introduce weights and state weighted bounds for the sparse operators in the conclusions of Theorem 1.1, from which Corollary 1.2 follows. To prepare for our application sections, we will discuss some preliminaries on, e.g., Banach space geometry in Sect. 5. Afterwards we will use our main result to prove the previously discussed applications in Sects. 6, 7 and 8. Finally, in Sect. 9 we discuss some potential further applications of our main result.

## 2 Spaces of Homogeneous Type

A space of homogeneous type  $(S, d, \mu)$ , originally introduced by Coifman and Weiss in [14], is a set  $S$  equipped with a quasi-metric  $d$  and a doubling Borel measure  $\mu$ . That is, a metric  $d$  which instead of the triangle inequality satisfies

$$d(s, t) \leq c_d (d(s, u) + d(u, t)), \quad s, t, u \in S,$$

for some  $c_d \geq 1$ , and a Borel measure  $\mu$  that satisfies the doubling property

$$\mu(B(s, 2\rho)) \leq c_\mu \mu(B(s, \rho)), \quad s \in S, \quad \rho > 0,$$

for some  $c_\mu \geq 1$ , where  $B(s, \rho) := \{t \in S : d(s, t) < \rho\}$  is the ball around  $s$  with radius  $\rho$ . Throughout this paper, we will assume additionally that all balls  $B \subseteq S$  are Borel sets and that we have  $0 < \mu(B) < \infty$ .

It was shown in [78, Example 1.1] that it can indeed happen that balls are not Borel sets in a quasi-metric space. This can be circumvented by taking topological closures and adjusting the constants  $c_d$  and  $c_\mu$  accordingly. However, to simplify matters we just assume all balls to be Borel sets and leave the necessary modifications if this is not the case to the reader. The size condition on the measure of a ball ensures that taking the average  $\langle f \rangle_{p, B}$  of a positive function  $f \in L^p_{\text{loc}}(S)$  over a ball  $B \subseteq S$  is always well defined.

As  $\mu$  is a Borel measure, i.e., a measure defined on the Borel  $\sigma$ -algebra of the quasi-metric space  $(S, d)$ , the Lebesgue differentiation theorem holds and as a consequence the continuous functions with bounded support are dense in  $L^p(S)$  for all  $p \in [1, \infty)$ . The Lebesgue differentiation theorem and consequently our results remain valid if  $\mu$  is a measure defined on a  $\sigma$ -algebra  $\Sigma$  that contains the Borel  $\sigma$ -algebra as long as the measure space  $(S, \Sigma, \mu)$  is Borel semi-regular. See [1, Theorem 3.14] for the details.

Throughout we will write that an estimate depends on  $S$  if it depends on  $c_d$  and  $c_\mu$ . For a thorough introduction to and a list of examples of spaces of homogeneous type we refer to the monographs of Christ [12] and Alvarado and Mitréa [1].

### 2.1 Dyadic Cubes

Let  $0 < c_0 \leq C_0 < \infty$  and  $0 < \delta < 1$ . Suppose that for  $k \in \mathbb{Z}$  we have an index set  $J_k$ , pairwise disjoint collection  $\mathcal{D}_k = \{Q_j^k\}_{j \in J_k}$  of measurable sets and a collection of points  $\{z_j^k\}_{j \in J_k}$ . We call  $\mathcal{D} := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$  a *dyadic system* with parameters  $c_0, C_0$  and  $\delta$  if it satisfies the following properties:

- (i) For all  $k \in \mathbb{Z}$  we have

$$S = \bigcup_{j \in J_k} Q_j^k;$$

- (ii) For  $k \geq l, Q \in \mathcal{D}_k$  and  $Q' \in \mathcal{D}_l$  we either have  $Q \cap Q' = \emptyset$  or  $Q \subseteq Q'$ ;
- (iii) For each  $k \in \mathbb{Z}$  and  $j \in J_k$  we have

$$B(z_j^k, c_0 \delta^k) \subseteq Q_j^k \subseteq B(z_j^k, C_0 \delta^k);$$

We will call the elements of a dyadic system  $\mathcal{D}$  cubes and for a cube  $Q \in \mathcal{D}$  we define the *restricted dyadic system*  $\mathcal{D}(Q) := \{P \in \mathcal{D} : P \subseteq Q\}$ . We will say that an estimate depends on  $\mathcal{D}$  if it depends on the parameters  $c_0, C_0$  and  $\delta$ .

One can view  $z_j^k$  and  $\delta^k$  as the center and side length of a cube  $Q_j^k \in \mathcal{D}_k$ . These have to be with respect to a specific  $k \in \mathbb{Z}$ , as this  $k$  may not be unique. We therefore think of a cube  $Q \in \mathcal{D}$  to also encode the information of its center  $z$  and generation  $k$ . The structure of individual dyadic cubes  $Q \in \mathcal{D}$  in a space of homogeneous type can be very messy and consequently the dilations of such cubes do not have a canonical definition. Therefore for a cube  $Q \in \mathcal{D}$  with center  $z$  and of generation  $k$  we define the *dilations*  $\alpha Q$  for  $\alpha \geq 1$  as

$$\alpha Q := B(z, \alpha \cdot C_0 \delta^k),$$

which are actually dilations of the ball that contains  $Q$  by property (iii) of a dyadic system.

When  $S = \mathbb{R}^n$  and  $d$  is the Euclidean distance, the standard dyadic cubes form a dyadic system and, combined with its translates over  $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$ , it holds that any ball in  $\mathbb{R}^n$  is contained in a cube of comparable size from one of these dyadic systems (see, e.g., [43, Lemma 3.2.26]). We will rely on the following proposition for



the existence of dyadic systems with this property in a general space of homogeneous type. For the proof and a more detailed discussion, we refer to [40].

**Proposition 2.1** *Let  $(S, d, \mu)$  be a space of homogeneous type. There exist  $0 < c_0 \leq C_0 < \infty$ ,  $\gamma \geq 1$ ,  $0 < \delta < 1$  and  $m \in \mathbb{N}$  such that there are dyadic systems  $\mathcal{D}^1, \dots, \mathcal{D}^m$  with parameters  $c_0, C_0$  and  $\delta$ , and with the property that for each  $s \in S$  and  $\rho > 0$  there is a  $j \in \{1, \dots, m\}$  and a  $Q \in \mathcal{D}^j$  such that*

$$B(s, \rho) \subseteq Q, \quad \text{and} \quad \text{diam}(Q) \leq \gamma\rho.$$

The following covering lemma will be used in the proof of our main theorem:

**Lemma 2.2** *Let  $(S, d, \mu)$  be a space of homogeneous type and  $\mathcal{D}$  a dyadic system with parameters  $c_0, C_0$ , and  $\delta$ . Suppose that  $\text{diam}(S) = \infty$ , take  $\alpha \geq 3c_d^2/\delta$  and let  $E \subseteq S$  satisfy  $0 < \text{diam}(E) < \infty$ . Then there exists a partition  $\mathcal{D} \subseteq \mathcal{D}$  of  $S$  such that  $E \subseteq \alpha Q$  for all  $Q \in \mathcal{D}$ .*

**Proof** For  $s \in S$  and  $k \in \mathbb{Z}$  let  $Q_s^k \in \mathcal{D}_k$  be the unique cube such that  $s \in Q_s^k$  and denote its center by  $z_s^k$ . Define

$$K_s := \{k \in \mathbb{Z} : E \not\subseteq 2c_d Q_s^k\},$$

where  $c_d$  is the quasi-metric constant. If  $k \in \mathbb{Z}$  is such that

$$\text{diam}(2c_d Q_s^k) \leq 4c_d^2 C_0 \delta^k < \text{diam}(E),$$

then  $E \not\subseteq 2c_d Q_s^k$ , i.e.,  $k \in K_s$  so is  $K_s$  non-empty. On the other hand, if  $k \in \mathbb{Z}$  is such that  $C_0 \delta^k > \sup_{s' \in E} d(s, s')$ , then

$$\sup_{s' \in E} d(s', z_s^k) \leq c_d \left( \sup_{s' \in E} d(s, s') + d(s, z_s^k) \right) \leq 2c_d C_0 \delta^k$$

so  $E \subseteq 2c_d Q_s^k$  and thus  $k \notin K_s$ . Therefore  $K_s$  is bounded from below.

Define  $k_s := \min K_s$  and set  $\mathcal{D} := \{Q_s^{k_s} : s \in S\}$ . Then  $\mathcal{D}$  is a partition of  $S$ . Indeed, suppose that for  $s, s' \in S$  we have  $Q_s^{k_s} \cap Q_{s'}^{k_{s'}} \neq \emptyset$ . Then using property (ii) of a dyadic system we may assume without loss of generality that  $Q_s^{k_s} \subseteq Q_{s'}^{k_{s'}}$ . Property (ii) of a dyadic system then implies that  $k_s \geq k_{s'}$ . In particular  $s \in Q_{s'}^{k_{s'}}$ , so by the minimality of  $k_s$  we must have  $k_s = k_{s'}$ . Therefore, since the elements of  $\mathcal{D}_{k_s}$  are pairwise disjoint, we can conclude  $Q_s^{k_s} = Q_{s'}^{k_{s'}}$ .

To conclude note that  $z_s^{k_s} \in Q_s^{k_s} \subseteq Q_s^{k_s-1}$  by property (ii) of a dyadic system, so  $d(z_s^{k_s-1}, z_s^{k_s}) \leq C_0 \delta^{k_s-1}$ . Therefore using the minimality of  $k_s$  we obtain

$$E \subseteq 2c_d Q_s^{k_s-1} = B(z_s^{k_s-1}, 2c_d C_0 \delta^{k_s-1}) \subseteq B\left(z_s^{k_s}, \frac{3c_d^2}{\delta} \cdot C_0 \delta^{k_s}\right) \subseteq \alpha Q_s^{k_s},$$

which finishes the proof.  $\square$

### 2.2 The Hardy–Littlewood Maximal Operator

On a space of homogeneous type  $(S, d, \mu)$  with a dyadic system  $\mathcal{D}$ , we define the *dyadic Hardy–Littlewood maximal operator* for  $f \in L^1_{\text{loc}}(S)$  by

$$M^{\mathcal{D}} f(s) := \sup_{Q \in \mathcal{D}: s \in Q} \langle |f| \rangle_{1, Q}, \quad s \in S.$$

By Doob’s maximal inequality (see, e.g., [43, Theorem 3.2.2])  $M^{\mathcal{D}}$  is strong  $L^p$ -bounded for all  $p \in (1, \infty)$  and weak  $L^1$ -bounded. We define the (non-dyadic) *Hardy–Littlewood maximal operator* for  $f \in L^1_{\text{loc}}(S)$  by

$$Mf(s) := \sup_{B \ni s} \langle |f| \rangle_{1, B}, \quad s \in S,$$

where the supremum is taken over all balls  $B \subseteq S$  containing  $s$ . By Proposition 2.1 there are dyadic systems  $\mathcal{D}^1, \dots, \mathcal{D}^m$  such that

$$Mf(s) \lesssim_S \sum_{j=1}^m M^{\mathcal{D}^j} f(s), \quad s \in S,$$

so  $M$  is also strong  $L^p$ -bounded for  $p \in (1, \infty)$  and weak  $L^1$ -bounded. For  $p_0 \in [1, \infty)$  and  $f \in L^{p_0}_{\text{loc}}(S)$ , we define

$$M_{p_0} f(s) := \sup_{B \ni s} \langle |f| \rangle_{p_0, B} = M(|f|^{p_0})(s)^{1/p_0}, \quad s \in S,$$

which is strong  $L^p$ -bounded for  $p \in (p_0, \infty)$  and weak  $L^{p_0}$ -bounded. This follows from the boundedness of  $M$  by rescaling.

### 3 Pointwise $\ell^r$ -Sparse Domination

In this section, we will prove a local version of the sparse domination result in Theorem 1.1, from which we will deduce Theorem 1.1 by a covering argument using Lemma 2.2. This local version will use an abstract localization of the operator  $T$ , since it depends upon the operator at hand as to the most effective localization. For example, in the study of a Calderón–Zygmund operator it is convenient to localize the function inserted into  $T$ , for a maximal operator it is convenient to localize the supremum in the definition of the maximal operator and for a Littlewood–Paley operator it is most suitable to localize the defining integral.

**Definition 3.1** Let  $(S, d, \mu)$  be a space of homogeneous type with a dyadic system  $\mathcal{D}$ , let  $X$  and  $Y$  be Banach spaces,  $p \in [1, \infty)$  and  $\alpha \geq 1$ . For a bounded operator

$$T : L^p(S; X) \rightarrow L^{p, \infty}(S; Y),$$

we say that a family of operators  $\{T_Q\}_{Q \in \mathcal{D}}$  from  $L^p(S; X)$  to  $L^{p,\infty}(Q; Y)$  is an  $\alpha$ -localization family of  $T$  if for all  $Q \in \mathcal{D}$  and  $f \in L^p(S; X)$  we have

$$T_Q(f \mathbf{1}_{\alpha Q})(s) = T_Q f(s), \quad s \in Q, \quad (\text{Localization})$$

$$\|T_Q(f \mathbf{1}_{\alpha Q})(s)\|_Y \leq \|T(f \mathbf{1}_{\alpha Q})(s)\|_Y, \quad s \in Q, \quad (\text{Domination})$$

For  $Q, Q' \in \mathcal{D}$  with  $Q' \subseteq Q$  we define the difference operator

$$T_{Q \setminus Q'} f(s) := T_Q f(s) - T_{Q'} f(s), \quad s \in Q'.$$

and for  $Q \in \mathcal{D}$  the localized sharp grand maximal truncation operator

$$\mathcal{M}_{T,Q}^\# f(s) := \sup_{\substack{Q' \in \mathcal{D}(Q): \\ s \in Q'}} \operatorname{ess\,sup}_{s' \in Q'} \|(T_{Q \setminus Q'} f)(s') - (T_{Q \setminus Q'} f)(s'')\|_Y, \quad s \in S.$$

In order to obtain interesting results, one needs to be able to recover the boundedness of  $T$  from the boundedness of  $T_Q$  uniformly in  $Q \in \mathcal{D}$ . The canonical example of an  $\alpha$ -localization family is

$$T_Q f(s) := T(f \mathbf{1}_{\alpha Q})(s), \quad s \in Q.$$

for all  $Q \in \mathcal{D}$  and it is exactly this choice that will lead to Theorem 1.1. We are now ready to prove our main result, which is a local, more general version of Theorem 1.1.

**Theorem 3.2** *Let  $(S, d, \mu)$  be a space of homogeneous type with dyadic system  $\mathcal{D}$  and let  $X$  and  $Y$  be Banach spaces. Take  $p_1, p_2, r \in [1, \infty)$ , set  $p_0 := \max\{p_1, p_2\}$  and take  $\alpha \geq 1$ . Suppose that*

- $T$  is a bounded operator from  $L^{p_1}(S; X)$  to  $L^{p_1,\infty}(S; Y)$  with  $\alpha$ -localization family  $\{T_Q\}_{Q \in \mathcal{D}}$ .
- $\mathcal{M}_{T,Q}^\#$  is bounded from  $L^{p_2}(S; X)$  to  $L^{p_2,\infty}(S)$  uniformly in  $Q \in \mathcal{D}$ .
- For all  $Q_1, \dots, Q_n \in \mathcal{D}$  with  $Q_n \subseteq \dots \subseteq Q_1$  and any  $f \in L^p(S; X)$

$$\|T_{Q_1} f(s)\|_Y \leq C_r \left( \|T_{Q_n} f(s)\|_Y^r + \sum_{k=1}^{n-1} \|T_{Q_k \setminus Q_{k+1}} f(s)\|_Y^r \right)^{1/r}, \quad s \in Q_n.$$

Then for any  $f \in L^{p_0}(S; X)$  and  $Q \in \mathcal{D}$  there exists a  $\frac{1}{2}$ -sparse collection of dyadic cubes  $\mathcal{S} \subseteq \mathcal{D}(Q)$  such that

$$\|T_Q f(s)\|_Y \lesssim_{S, \mathcal{D}, \alpha} C_T C_r \left( \sum_{P \in \mathcal{S}} (\|f\|_X)_{p_0, \alpha P}^r \mathbf{1}_P(s) \right)^{1/r}, \quad s \in Q,$$

with  $C_T := \|T\|_{L^{p_1} \rightarrow L^{p_1,\infty}} + \sup_{P \in \mathcal{D}} \|\mathcal{M}_{T,P}^\#\|_{L^{p_2} \rightarrow L^{p_2,\infty}}$ .

The assumption in the third bullet in Theorem 3.2 replaces the  $r$ -sublinearity assumption in Theorem 1.1. We will call this assumption a localized  $\ell^r$ -estimate.

**Proof** Fix  $f \in L^p(S, X)$  and  $Q \in \mathcal{D}$ . We will prove the theorem in two steps: we will first construct the  $\frac{1}{2}$ -sparse family of cubes  $\mathcal{S}$  and then show that the sparse expression associated to  $\mathcal{S}$  dominates  $T_Q f$  pointwise.

**Step 1:** We will construct the  $\frac{1}{2}$ -sparse family of cubes  $\mathcal{S}$  iteratively. Given a collection of pairwise disjoint cubes  $\mathcal{S}^k$  for some  $k \in \mathbb{N}$  we will first describe how to construct  $\mathcal{S}^{k+1}$ . Afterwards we can inductively define  $\mathcal{S}^k$  for all  $k \in \mathbb{N}$  starting from  $\mathcal{S}^1 = \{Q\}$  and set  $\mathcal{S} := \bigcup_{k \in \mathbb{N}} \mathcal{S}^k$ .

Fix a  $P \in \mathcal{S}^k$  and for  $\lambda \geq 1$  to be chosen later define

$$\begin{aligned} \Omega_P^1 &:= \left\{ s \in P : \|T_P f(s)\|_Y > \lambda C_T \langle \|f\|_X \rangle_{p_0, \alpha P} \right\} \\ \Omega_P^2 &:= \left\{ s \in P : \mathcal{M}_{T,P}^\#(f)(s) > \lambda C_T \langle \|f\|_X \rangle_{p_0, \alpha P} \right\}, \end{aligned}$$

and  $\Omega_P := \Omega_P^1 \cup \Omega_P^2$ . Let  $c_1 \geq 1$ , depending on  $S, \mathcal{D}$  and  $\alpha$ , be such that  $\mu(\alpha P) \leq c_1 \mu(P)$ . By the domination property of the  $\alpha$ -localization family we have

$$\|T_P f(s)\|_Y \leq \|T(f \mathbf{1}_{\alpha P})(s)\|_Y, \quad s \in P,$$

and by the localization property

$$\mathcal{M}_{T,P}^\#(f)(s) = \mathcal{M}_{T,P}^\#(f \mathbf{1}_{\alpha P})(s), \quad s \in P.$$

Thus by the weak boundedness assumptions on  $T$  and  $\mathcal{M}_{T,P}^\#$  and Hölder’s inequality we have for  $i = 1, 2$

$$\mu(\Omega_P^i) \leq \left( \frac{\|f \mathbf{1}_{\alpha P}\|_{L^{p_i}(S; X)}}{\lambda \langle \|f\|_X \rangle_{p_0, \alpha P}} \right)^{p_i} = \frac{\langle \|f\|_X \rangle_{p_i, \alpha P}^{p_i}}{\lambda^{p_i} \langle \|f\|_X \rangle_{p_0, \alpha P}^{p_i}} \mu(\alpha P) \leq \frac{c_1}{\lambda} \mu(P). \quad (3.1)$$

Therefore it follows that

$$\mu(\Omega_P) \leq \frac{2c_1}{\lambda} \mu(P). \quad (3.2)$$

To construct the cubes in  $\mathcal{S}^{k+1}$  we will use a local Calderón–Zygmund decomposition (see, e.g., [26, Lemma 4.5]) on

$$\Omega_{P,\rho} := \left\{ s \in P : M^{\mathcal{D}(P)}(\mathbf{1}_{\Omega_P}) > \frac{1}{\rho} \right\}, \quad \rho > 0,$$

which will be a proper subset of  $P$  for our choice of  $\lambda$  and  $\rho$ . Here  $M^{\mathcal{D}(P)}$  is the dyadic Hardy–Littlewood maximal operator with respect to the restricted dyadic system  $\mathcal{D}(P)$ . The local Calderón–Zygmund decomposition yields a pairwise disjoint collection of cubes  $\mathcal{S}_P \subseteq \mathcal{D}(P)$  and a constant  $c_2 \geq 2$ , depending on  $S$  and  $\mathcal{D}$ , such that  $\Omega_{P,c_2} = \bigcup_{P' \in \mathcal{S}_P} P'$  and

$$\frac{1}{c_2} \mu(P') \leq \mu(P' \cap \Omega_P) \leq \frac{1}{2} \mu(P'), \quad P' \in \mathcal{S}_P. \quad (3.3)$$

Then by (3.2), (3.3) and the disjointness of the cubes in  $\mathcal{S}_P$  we have

$$\sum_{P' \in \mathcal{S}_P} \mu(P') \leq c_2 \sum_{P' \in \mathcal{S}_P} \mu(P' \cap \Omega_P) \leq c_2 \mu(\Omega_P) \leq \frac{2c_1 c_2}{\lambda} \mu(P).$$

Therefore, by choosing  $\lambda = 4c_1 c_2$ , we have  $\sum_{P' \in \mathcal{S}_P} \mu(P') \leq \frac{1}{2} \mu(P)$ . This choice of  $\lambda$  also ensures that  $\Omega_{P, c_2}$  is a proper subset of  $P$  by as claimed before. We define  $\mathcal{S}^{k+1} := \bigcup_{P \in \mathcal{S}^k} \mathcal{S}_P$ .

Now take  $\mathcal{S}^1 = \{Q\}$ , iteratively define  $\mathcal{S}^k$  for all  $k \in \mathbb{N}$  as described above and set  $\mathcal{S} := \bigcup_{k \in \mathbb{N}} \mathcal{S}^k$ . Then  $\mathcal{S}$  is  $\frac{1}{2}$ -sparse family of cubes, since for any  $P \in \mathcal{S}$  we can set

$$E_P := P \setminus \bigcup_{P' \in \mathcal{S}_P} P',$$

which are pairwise disjoint by the fact that  $\bigcup_{P' \in \mathcal{S}^{k+1}} P' \subseteq \bigcup_{P \in \mathcal{S}^k} P$  for all  $k \in \mathbb{N}$  and we have

$$\mu(E_P) = \mu(P) - \sum_{P' \in \mathcal{S}_P} \mu(P') \geq \frac{1}{2} \mu(P).$$

**Step 2:** We will now check that the sparse expression corresponding to  $\mathcal{S}$  constructed in Step 1 dominates  $T_Q f$  pointwise. Since

$$\lim_{k \rightarrow \infty} \mu\left(\bigcup_{P \in \mathcal{S}^k} P\right) \leq \lim_{k \rightarrow \infty} \frac{1}{2^k} \mu(Q) = 0,$$

we know that there is a set  $N_0$  of measure zero such that for all  $s \in Q \setminus N_0$  there are only finitely many  $k \in \mathbb{N}$  with  $s \in \bigcup_{P \in \mathcal{S}^k} P$ . Moreover by the Lebesgue differentiation theorem we have for any  $P \in \mathcal{S}$  that  $\mathbf{1}_{\Omega_P}(s) \leq M^{\mathcal{D}(P)}(\mathbf{1}_{\Omega_P})(s)$  for a.e.  $s \in P$ . Thus

$$\Omega_P \setminus N_P \subseteq \Omega_{P,1} \subseteq \Omega_{P,c_2} = \bigcup_{P' \in \mathcal{S}_P} P' \quad (3.4)$$

for some set  $N_P$  of measure zero. We define  $N := N_0 \cup \bigcup_{P \in \mathcal{S}} N_P$ , which is a set of measure zero.

Fix  $s \in Q \setminus N$  and take the largest  $n \in \mathbb{N}$  such that  $s \in \bigcup_{P \in \mathcal{S}^n} P$ , which exists since  $s \notin N_0$ . For  $k = 1, \dots, n$  let  $P_k \in \mathcal{S}^k$  be the unique cube such that  $s \in P_k$  and note that by construction we have  $P_n \subseteq \dots \subseteq P_1 = Q$ . Using the localized  $\ell^r$ -estimate of  $T$  we split  $\|T_Q f(s)\|_Y^r$  into two parts

$$\begin{aligned} \|T_Q f(s)\|_Y^r &\leq C_r^r \left( \|T_{P_n} f(s)\|_Y^r + \sum_{k=1}^{n-1} \|T_{P_k \setminus P_{k+1}} f(s)\|_Y^r \right) \\ &=: C_r^r \left( \boxed{\text{A}} + \boxed{\text{B}} \right). \end{aligned}$$

For  $\square A$  note that  $s \notin N_{P_n}$  and  $s \notin \bigcup_{P' \in \mathcal{S}^{n+1}} P'$  and therefore by (3.4) we know that  $s \in P_n \setminus \Omega_{P_n}$ . So by the definition of  $\Omega_{P_n}^1$

$$\square A \leq \lambda^r C_T^r \langle \|f\|_X \rangle_{p_0, \alpha P_n}^r.$$

For  $1 \leq k \leq n - 1$  we have by (3.2) and (3.3) that

$$\begin{aligned} \mu(P_{k+1} \setminus (\Omega_{P_{k+1}} \cup \Omega_{P_k})) &\geq \mu(P_{k+1}) - \mu(\Omega_{P_{k+1}}) - \mu(P_{k+1} \cap \Omega_{P_k}) \\ &\geq \mu(P_{k+1}) - \frac{1}{2c_2} \mu(P_{k+1}) - \frac{1}{2} \mu(P_{k+1}) > 0, \end{aligned} \tag{3.5}$$

so  $P_{k+1} \setminus (\Omega_{P_{k+1}} \cup \Omega_{P_k})$  is non-empty. Take  $s' \in P_{k+1} \setminus (\Omega_{P_{k+1}} \cup \Omega_{P_k})$ , then we have

$$\begin{aligned} \|T_{P_k \setminus P_{k+1}} f(s)\|_Y &\leq \|T_{P_k \setminus P_{k+1}} f(s) - T_{P_k \setminus P_{k+1}} f(s')\|_Y + \|T_{P_k \setminus P_{k+1}} f(s')\|_Y \\ &\leq \mathcal{M}_{T, P_k}^\# f(s') + \|T_{P_k}(s')\|_Y + \|T_{P_{k+1}}(s')\|_Y \\ &\leq 2\lambda C_T (\langle \|f\|_X \rangle_{p_0, \alpha P_k} + \langle \|f\|_X \rangle_{p_0, \alpha P_{k+1}}), \end{aligned}$$

where we used the definition of  $\mathcal{M}_{T, P_k}^\#$  and  $T_{P_{k+1} \setminus P_k}$  in the second inequality and  $s' \notin \Omega_{P_{k+1}} \cup \Omega_{P_k}$  in the third inequality. Using  $(a + b)^r \leq 2^{r-1}(a^r + b^r)$  for any  $a, b > 0$  this implies that

$$\begin{aligned} \square B &\leq \sum_{k=1}^{n-1} 2^r 2^{r-1} \lambda^r C_T^r (\langle \|f\|_X \rangle_{p_0, \alpha P_k}^r + \langle \|f\|_X \rangle_{p_0, \alpha P_{k+1}}^r) \\ &\leq \sum_{k=1}^n 4^r \lambda^r C_T^r \langle \|f\|_X \rangle_{p_0, \alpha P_k}^r. \end{aligned}$$

Combining the estimates for  $\square A$  and  $\square B$  we obtain

$$\begin{aligned} \|T_Q f(s)\|_Y &\leq 5\lambda C_T C_r \left( \sum_{k=1}^n \langle \|f\|_X \rangle_{p_0, \alpha P_k}^r \right)^{1/r} \\ &= 5\lambda C_T C_r \left( \sum_{P \in \mathcal{S}} \langle \|f\|_X \rangle_{p_0, \alpha P}^r \mathbf{1}_P(s) \right)^{1/r}. \end{aligned}$$

Since  $s \in Q \setminus N$  was arbitrary and  $N$  has measure zero, this inequality holds for a.e.  $s \in Q$ . Noting that  $\lambda = 4c_1c_2$  and  $c_1$  and  $c_2$  only depend on  $S, \alpha$  and  $\mathcal{D}$  finishes the proof of the theorem.  $\square$

As announced Theorem 1.1 now follows directly from Theorem 3.2 and a covering argument with Lemma 2.2.

**Proof of Theorem 1.1** We will prove Theorem 1.1 in three steps: we will first show that the assumptions of Theorem 1.1 imply the assumptions of Theorem 3.2, then we will improve the local conclusion of Theorem 3.2 to a global one and finally we will replace the averages over the dilation  $\alpha P$  in the conclusion of Theorem 3.2 by the average over larger cubes  $P'$ .

To start let  $\mathcal{D}^1, \dots, \mathcal{D}^m$  be as in Proposition 2.1 with parameters  $c_0, C_0, \delta$ , and  $\gamma$ , which only depend on  $S$ .

**Step 1:** For any  $Q \in \mathcal{D}^1$  define  $T_Q$  by  $T_Q f(s) := T(f \mathbf{1}_{\alpha Q})(s)$  for  $s \in Q$ . Then:

- $\{T_Q\}_{Q \in \mathcal{D}^1}$  is an  $\alpha$ -localization family of  $T$ .
- For any  $Q \in \mathcal{D}^1$  and  $f \in L^{p_1}(S; X)$  we have

$$\mathcal{M}_{T, Q}^\# f(s) \leq \mathcal{M}_{T, \alpha}^\#(f \mathbf{1}_{\alpha Q})(s), \quad s \in Q.$$

So by the weak  $L^{p_2}$ -boundedness of  $\mathcal{M}_{T, \alpha}^\#$  it follows that  $\mathcal{M}_{T, Q}^\# f$  is weak  $L^{p_2}$ -bounded uniformly in  $Q \in \mathcal{D}^1$ .

- For any  $f \in L^p(S; X)$  and  $Q_1, \dots, Q_n \in \mathcal{D}^1$  with  $Q_n \subseteq \dots \subseteq Q_1$  the functions  $f_k := f \mathbf{1}_{\alpha Q_k \setminus \alpha Q_{k+1}}$  for  $k = 1, \dots, n-1$  and  $f_n := f \mathbf{1}_{\alpha Q_n}$  are disjointly supported. Thus by the  $r$ -sublinearity of  $T$

$$\|T_{Q_1} f(s)\|_Y \leq C_r \left( \|T_{Q_n} f(s)\|_Y^r + \sum_{k=1}^{n-1} \|T_{Q_k \setminus Q_{k+1}} f(s)\|_Y^r \right)^{1/r}, \quad s \in Q_n.$$

So the assumptions of Theorem 3.2 follow from the assumptions of Theorem 1.1.

**Step 2:** Let  $f \in L^p(S; X)$  be boundedly supported. First suppose that  $\text{diam}(S) = \infty$  and let  $E$  be a ball containing the support of  $f$ . By Lemma 2.2 there is a partition  $\mathcal{D} \subseteq \mathcal{D}^1$  such that  $E \subseteq \alpha Q$  for all  $Q \in \mathcal{D}$ . Thus by Theorem 3.2 we can find a  $\frac{1}{2}$ -sparse collection of cubes  $S_Q \subseteq \mathcal{D}^1(Q)$  for every  $Q \in \mathcal{D}$  with

$$\|Tf(s)\|_Y \lesssim_{S, \alpha} C_T C_r \left( \sum_{P \in S_Q} \langle \|f\|_X \rangle_{p_0, \alpha P}^r \mathbf{1}_P(s) \right)^{1/r}, \quad s \in Q,$$

where we used that  $T_Q f = T(f \mathbf{1}_{\alpha Q}) = Tf$  as  $\text{supp } f \subseteq \alpha Q$ . Since  $\mathcal{D}$  is a partition,  $S := \bigcup_{Q \in \mathcal{D}} S_Q$  is also a  $\frac{1}{2}$ -sparse collection of cubes with

$$\|Tf(s)\|_Y \lesssim_{S, \alpha} C_T C_r \left( \sum_{P \in S} \langle \|f\|_X \rangle_{p_0, \alpha P}^r \mathbf{1}_P(s) \right)^{1/r}, \quad s \in S, \quad (3.6)$$

If  $\text{diam}(S) < \infty$ , then (3.6) follows directly from Theorem 3.2 since  $S \in \mathcal{D}$  in that case.

**Step 3:** For any  $P \in S$  with center  $z$  and sidelength  $\delta^k$  we can find a  $P' \in \mathcal{D}^j$  for some  $1 \leq j \leq m$  such that

$$\alpha P = B(z, \alpha C_0 \cdot \delta^k) \subseteq P', \quad \text{diam}(P') \leq \gamma \alpha C_0 \cdot \delta^k.$$

Therefore there is a  $c_1 > 0$  depending on  $S$  and  $\alpha$  such that

$$\mu(P') \leq \mu(B(z, \gamma\alpha C_0 \cdot \delta^k)) \leq c_1 \mu(B(z, c_0 \cdot \delta^k)) \leq c_1 \mu(P).$$

So by defining  $E_{P'} := E_P$  we can conclude that the collection of cubes  $S' := \{P' : P \in S\}$  is  $\frac{1}{2c_1}$ -sparse. Moreover since  $\alpha P \subseteq P'$  and  $\mu(P') \leq c_1 \mu(P) \leq c_1 \mu(\alpha P)$  for any  $P \in S$ , we have

$$\langle \|f\|_X \rangle_{p_0, \alpha P} \leq c_1 \langle \|f\|_X \rangle_{p_0, P'}.$$

Combined with (3.6), this proves the sparse domination in the conclusion of Theorem 1.1. □

**Remark 3.3** The assumption  $\alpha \geq 3c_d^2/\delta$  in Theorem 1.1 arises from the use of Lemma 2.2, which transfers the local sparse domination estimate of Theorem 3.2 to the global statement of Theorem 1.1. To deduce weighted estimates the local sparse domination estimate of Theorem 3.2 suffices by testing against boundedly supported functions. However the operator norm of  $\mathcal{M}_{T, \alpha}^\#$  usually becomes easier to estimate for larger  $\alpha$ , so the lower bound on  $\alpha$  is not restrictive.

### Further Generalizations

Our main theorems, Theorems 1.1 and 3.2, allow for various further generalizations. One can for instance change the boundedness assumptions on  $T$  and  $\mathcal{M}_{T, \alpha}^\#$ , treat multilinear operators, or deduce domination by sparse forms for operators that do not admit a pointwise sparse estimate. We end this section by sketching some of these possible generalizations.

In [58, Section 3] various variations and extensions of the main result in [58] are outlined. In particular they show:

- The sparse domination for an individual function follows from assumptions on the same function. This can be exploited to prove a sparse  $T(1)$ -type theorem, see [58, Section 4].
- One can use certain Orlicz estimates to deduce sparse domination with Orlicz averages.
- The method of proof extends to the multilinear setting (see also [60]).

Our results can also be extended in these directions, which we leave to the interested reader. In the remainder of this section, we will explore some further directions in which our results can be extended.

Sparse domination techniques have been successfully applied to *fractional integral operators*, see, e.g., [16–18, 47]. In these works, sparse domination and sharp weighted estimates are deduced for, e.g., the Riesz potentials, which for  $0 < \alpha < d$  and a Schwartz function  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  are given by

$$I_\alpha f(s) := \int_{\mathbb{R}^d} \frac{f(t)}{|s-t|^{d-\alpha}} dt, \quad s \in \mathbb{R}^d,$$



A key feature of such operators is that they are not (weakly)  $L^p$ -bounded, but bounded from  $L^p(\mathbb{R}^d)$  to  $L^q(\mathbb{R}^d)$ , where  $p, q \in (1, \infty)$  are such that  $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{d}$ . The sparse domination that one obtains in this case involves fractional sparse operators, in which the usual averages  $\langle |f| \rangle_{p, Q}$  are replaced by fractional averages.

These operators fit in our framework with minimal effort. Indeed, upon inspection of the proof of Theorem 3.2 it becomes clear that the only place where we use the boundedness of  $T$  and  $\mathcal{M}_{T, \alpha}^\#$  is in (3.1). Replacing the bounds with the off-diagonal bounds arising from fractional integral operators, we obtain the following variant of Theorem 1.1.

**Theorem 3.4** *Let  $(S, d, \mu)$  be a space of homogeneous type and let  $X$  and  $Y$  be Banach spaces. Take  $p_0, q_0, r \in [1, \infty)$ . Take  $\alpha \geq 3c_d^2/\delta$ , where  $c_d$  is the quasi-metric constant and  $\delta$  is as in Proposition 2.1. Assume the following conditions:*

- $T$  is a bounded linear operator from  $L^{p_0}(S; X)$  to  $L^{q_0, \infty}(S; Y)$ .
- $\mathcal{M}_{T, \alpha}^\#$  is a bounded operator from  $L^{p_0}(S; X)$  to  $L^{q_0, \infty}(S)$ .
- $T$  is  $r$ -sublinear.

Then there is an  $\eta \in (0, 1)$  such that for any boundedly supported  $f \in L^{p_0}(S; X)$  there is an  $\eta$ -sparse collection of cubes  $\mathcal{S}$  such that

$$\|Tf(s)\|_Y \lesssim_{S, \alpha} C_T C_r \left( \sum_{Q \in \mathcal{S}} \mu(\alpha P)^{\frac{r}{p_0} - \frac{r}{q_0}} \langle \|f\|_X \rangle_{p_0, Q}^r \mathbf{1}_Q(s) \right)^{1/r}, \quad s \in S,$$

where  $C_T = \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}} + \|\mathcal{M}_{T, \alpha}^\#\|_{L^{p_0} \rightarrow L^{p_0, \infty}}$  and  $C_r$  is the  $r$ -sublinearity constant.

**Proof** The proof is the same as the proof of Theorem 1.1, using an adapted version of Theorem 3.2 with the canonical  $\alpha$ -localization family

$$T_Q f(s) = T(\mathbf{1}_{\alpha Q} f)(s), \quad s \in Q.$$

The only thing that changes in the proof of Theorem 3.2 is the definition of  $\Omega_P^1$  and  $\Omega_P^2$  and the computation in (3.2). Indeed, we define

$$\begin{aligned} \Omega_P^1 &:= \left\{ s \in P : \|T_P f(s)\|_Y > \lambda C_T \mu(\alpha P)^{\frac{1}{p_0} - \frac{1}{q_0}} \langle \|f\|_X \rangle_{p_0, \alpha P} \right\} \\ \Omega_P^2 &:= \left\{ s \in P : \mathcal{M}_{T, P}^\#(f)(s) > \lambda C_T \mu(\alpha P)^{\frac{1}{p_0} - \frac{1}{q_0}} \langle \|f\|_X \rangle_{p, \alpha P} \right\}, \end{aligned}$$

and then by the assumptions on  $T$  and  $\mathcal{M}_{T, P}^\#$  we have for  $i = 1, 2$

$$\mu(\Omega_P^i) \leq \left( \frac{\|f \mathbf{1}_{\alpha P}\|_{L^{p_0}(S; X)}}{\lambda \mu(\alpha P)^{\frac{1}{p_0} - \frac{1}{q_0}} \langle \|f\|_X \rangle_{p_0, \alpha P}} \right)^{q_0} = \frac{\langle \|f\|_X \rangle_{p_0, \alpha P}^{q_0}}{\lambda^{q_0} \langle \|f\|_X \rangle_{p_0, \alpha P}^{q_0}} \mu(\alpha P) \leq \frac{c_1}{\lambda} \mu(P).$$

which proves (3.2). In Step 2 of the proof of Theorem 3.2, one needs to keep track of the factor  $\mu(\alpha P)^{\frac{1}{p_0} - \frac{1}{q_0}}$  in the estimates. □

In the celebrated paper [6] by Bernicó, Frey, and Petermichl, domination by *sparse forms* was introduced to treat operators falling outside the scope of Calderón–Zygmund theory. This method was later adopted by Lerner in [56] into his framework to prove sparse domination for rough homogeneous singular integral operators. As our methods are based on Lerner’s sparse domination framework, our main result can also be generalized to the sparse form domination setting.

Let  $(S, d, \mu)$  be a space of homogeneous type with a dyadic system  $\mathcal{D}$ , let  $X$  and  $Y$  be Banach spaces,  $q \in (1, \infty)$ ,  $p \in [1, q)$  and  $\alpha \geq 1$ . For a bounded operator

$$T : L^p(S; X) \rightarrow L^{p, \infty}(S; Y),$$

with an  $\alpha$ -localization family  $\{T_Q\}_{Q \in \mathcal{D}}$  we define the *localized sharp grand  $q$ -maximal truncation operator* for  $Q \in \mathcal{D}$  by

$$\mathcal{M}_{T, Q, q}^\# f(s) := \sup_{\substack{Q' \in \mathcal{D}(Q): \\ s \in Q'}} \left( \int_{Q'} \int_{Q'} \|(T_{Q \setminus Q'})f(s') - (T_{Q \setminus Q'})f(s'')\|_Y^q d\mu(s') d\mu(s'') \right)^{1/q}.$$

Note that for  $q = \infty$  one formally recovers the operator  $\mathcal{M}_{T, Q}^\#$ .

We will prove a version of Theorem 3.2 for operators for which the truncation operators  $\mathcal{M}_{T, Q, q}^\#$  are bounded uniformly in  $Q \in \mathcal{D}$  using sparse forms. Of course taking

$$T_Q f(s) := T(f \mathbf{1}_{\alpha Q})(s), \quad s \in Q.$$

for  $Q \in \mathcal{D}$  as the  $\alpha$ -localization family one can easily deduce a statement like Theorem 1.1 in this setting, which we leave to the interested reader.

**Theorem 3.5** *Let  $(S, d, \mu)$  be a space of homogeneous type with dyadic system  $\mathcal{D}$  and let  $X$  and  $Y$  be Banach spaces. Take  $q_0 \in (1, \infty]$ ,  $r \in (0, q_0)$ ,  $p_1, p_2 \in [1, q_0)$ , set  $p_0 := \max\{p_1, p_2\}$  and take  $\alpha \geq 1$ . Suppose that*

- $T$  is a bounded operator from  $L^{p_1}(S; X)$  to  $L^{p_1, \infty}(S; Y)$  with an  $\alpha$ -localization family  $\{T_Q\}_{Q \in \mathcal{D}}$ .
- $\mathcal{M}_{T, Q, q_0}^\#$  is bounded from  $L^{p_2}(S; X)$  to  $L^{p_2, \infty}(S)$  uniformly in  $Q \in \mathcal{D}$ .
- $T$  satisfies a localized  $\ell^r$ -estimate.

Then for any  $f \in L^{p_0}(S; X)$ ,  $g \in L^{\left(\frac{1}{r} - \frac{1}{q_0}\right)^{-1}}(S)$  and  $Q \in \mathcal{D}$  there exists a  $\frac{1}{2}$ -sparse collection of dyadic cubes  $\mathcal{S} \subseteq \mathcal{D}(Q)$  such that

$$\left( \int_Q \|T_Q f\|_Y^r \cdot |g|^r d\mu \right)^{1/r} \lesssim_{\mathcal{S}, \mathcal{D}, \alpha, r} C_T C_r \left( \sum_{P \in \mathcal{S}} \mu(P) (\|f\|_X)_{p_0, \alpha P}^r (|g|)_{\frac{1}{r} - \frac{1}{q_0}, P}^r \right)^{1/r},$$

with  $C_T := \|T\|_{L^{p_1} \rightarrow L^{p_1, \infty}} + \sup_{P \in \mathcal{D}} \|\mathcal{M}_{T, P, q_0}^\#\|_{L^{p_2} \rightarrow L^{p_2, \infty}}$  and  $C_r$  the constant from the localized  $\ell^r$ -estimate.

**Proof** We construct the sparse collection of cubes  $\mathcal{S}$  exactly as in Step 1 of the proof of Theorem 3.2, using  $\mathcal{M}_{T,P,q_0}^\#$  instead of  $\mathcal{M}_{T,P}^\#$  in the definition of  $\Omega_P^2$ . We will check that sparse form corresponding to  $\mathcal{S}$  satisfies the claimed domination property, which will roughly follow the same lines as Step 2 of the proof of Theorem 3.2.

Fix  $f \in L^{p_0}(S; X)$  and  $g \in L^{\left(\frac{1}{r}-\frac{1}{q_0}\right)^{-1}}(S)$ . Note that for a.e.  $s \in Q$  there are only finitely many  $k \in \mathbb{N}$  with  $s \in \bigcup_{P \in \mathcal{S}^k} P$ . So we can use the localized  $\ell^r$ -estimate of  $T$  to split

$$\begin{aligned} \int_Q \|T_Q f\|_Y^r \cdot |g|^r &\leq C_r^r \sum_{k \in \mathbb{N}} \sum_{P \in \mathcal{S}^k} \left( \int_{P \cup \bigcup_{P' \in \mathcal{S}^{k+1}} P'} \|T_P f\|_Y^r \cdot |g|^r \right. \\ &\quad \left. + \sum_{P' \in \mathcal{S}^{k+1}: P' \subseteq P} \int_{P'} \|T_{P \setminus P'} f\|_Y^r \cdot |g|^r \right) \tag{3.7} \\ &=: C_r^r \sum_{k \in \mathbb{N}} \sum_{P \in \mathcal{S}^k} (\boxed{A_P} + \boxed{B_P}). \end{aligned}$$

Fix  $k \in \mathbb{N}$  and  $P \in \mathcal{S}^k$ . As in the estimate for  $\boxed{A}$  in Step 2 of the proof of Theorem 3.2, we have

$$\boxed{A_P} \leq \lambda^r C_T^r \langle \|f\|_X \rangle_{p_0, \alpha P}^r \int_P |g|^r \leq \lambda^r C_T^r \mu(P) \langle \|f\|_X \rangle_{p_0, \alpha P}^r \langle |g| \rangle_{\frac{1}{r}-\frac{1}{q_0}}^r,$$

using Hölder’s inequality in the second inequality. For  $P' \in \mathcal{S}^{k+1}$  such that  $P' \subseteq P$  we have as in (3.5) that

$$\mu(P' \setminus (\Omega_{P'} \cup \Omega_P)) \geq \frac{1}{4} \mu(P').$$

Therefore we can estimate each of the terms in the sum in  $\boxed{B_P}$  as follows

$$\begin{aligned} &\int_{P'} \|T_{P \setminus P'} f\|_Y^r \cdot |g|^r \\ &\leq 2^r \int_{P'} \int_{P' \setminus (\Omega_P \cup \Omega_{P'})} \|T_{P \setminus P'} f(s) - T_{P \setminus P'} f(s')\|_Y^r \cdot |g(s)|^r d\mu(s') d\mu(s) \\ &\quad + 2^r \int_{P'} \int_{P' \setminus (\Omega_P \cup \Omega_{P'})} \|T_{P \setminus P'} f(s')\|_Y^r \cdot |g(s)|^r d\mu(s') d\mu(s) \\ &\leq 2^{r+2} \mu(P') \inf_{s'' \in P'} \mathcal{M}_{T,P,q_0}^\# f(s'')^r \cdot \langle |g| \rangle_{\frac{1}{r}-\frac{1}{q_0}, P'}^r \\ &\quad + 2^{2r} \mu(P') \int_{P' \setminus (\Omega_P \cup \Omega_{P'})} \|T_P f\|_Y^r + \|T_{P'} f\|_Y^r d\mu \cdot \langle |g| \rangle_{r, P'}^r \\ &\leq 4^{r+2} \lambda^r C_T^r \mu(P') (\langle \|f\|_X \rangle_{p_0, \alpha P}^r + \langle \|f\|_X \rangle_{p_0, \alpha P'}^r) \langle |g| \rangle_{\frac{1}{r}-\frac{1}{q_0}, P'}^r \end{aligned}$$

where we used Hölder’s inequality and the definitions of  $\mathcal{M}_{T,P,q_0}^\#$  and  $T_{P \setminus P'}$  in the second inequality and the definitions of  $\Omega_P$  and  $\Omega_{P'}$  in the third inequality. Furthermore, we note that by Hölders inequality we have

$$\begin{aligned} \sum_{\substack{P' \in \mathcal{S}^{k+1}: \\ P' \subseteq P}} \mu(P') \langle |g| \rangle_{\frac{1}{r} - \frac{1}{q_0}, P'}^r &\leq \left( \sum_{\substack{P' \in \mathcal{S}^{k+1}: \\ P' \subseteq P}} \int_{P'} |g|^{\frac{1}{r} - \frac{1}{q_0}} d\mu \right)^{1 - \frac{r}{q_0}} \cdot \left( \sum_{\substack{P' \in \mathcal{S}^{k+1}: \\ P' \subseteq P}} \mu(P') \right)^{r/q_0} \\ &\leq \left( \int_P |g|^{\frac{1}{r} - \frac{1}{q_0}} d\mu \right)^{1 - \frac{r}{q_0}} \cdot \mu(P)^{r/q_0} = \mu(P) \langle |g| \rangle_{\frac{1}{r} - \frac{1}{q_0}, P}^r \end{aligned}$$

Thus for  $\boxed{B_P}$  we obtain

$$\begin{aligned} \boxed{B_P} &\leq 4^{r+2} \lambda^r C_T^r \left( \mu(P) \langle \|f\|_X \rangle_{p_0, \alpha P}^r \langle |g| \rangle_{\frac{1}{r} - \frac{1}{q_0}, P}^r \right. \\ &\quad \left. + \sum_{P' \in \mathcal{S}^{k+1}: P' \subseteq P} \mu(P') \langle \|f\|_X \rangle_{p_0, \alpha P'}^r \langle |g| \rangle_{\frac{1}{r} - \frac{1}{q_0}, P'}^r \right) \end{aligned}$$

Plugging this estimate and the estimate for  $\boxed{A_P}$  into (3.7) yields

$$\int_Q \|T_Q f\|_Y^r \cdot |g|^r d\mu \leq 4^{r+3} \lambda^r C_T^r C_r^r \sum_{P \in \mathcal{S}} \mu(P) \langle \|f\|_X \rangle_{p_0, \alpha P}^r \langle |g| \rangle_{\frac{1}{r} - \frac{1}{q_0}, P}^r.$$

Since  $\lambda = 4c_1c_2$  and  $c_1$  and  $c_2$  only depend on  $S, \alpha$ , and  $\mathcal{D}$ , this finishes the proof of the theorem. □

### 4 Weighted Bounds for Sparse Operators

As discussed in the introduction, one of the main motivations to study sparse domination for an operator is to obtain (sharp) weighted bounds. In this section we will introduce Muckenhoupt weights and state weighted  $L^p$ -bounds for the sparse operators in the conclusions of Theorems 1.1 and 3.2, which are well known in the Euclidean setting.

Let  $(S, d, \mu)$  be a space of homogeneous type. A *weight* is a locally integrable function  $w: S \rightarrow (0, \infty)$ . For  $p \in [1, \infty)$ , a Banach space  $X$  and a weight  $w$  the weighted Bochner space  $L^p(S, w; X)$  is the space of all strongly measurable  $f: S \rightarrow X$  such that

$$\|f\|_{L^p(S, w; X)} := \left( \int_S \|f(s)\|_X^p w d\mu \right)^{1/p} < \infty.$$

For  $p \in [1, \infty)$  and a weight  $w$  we say that  $w$  lies in the *Muckenhoupt class*  $A_p$  and write  $w \in A_p$  if its  $A_p$ -characteristic satisfies

$$[w]_{A_p} := \sup_{B \subseteq S} \langle w \rangle_{1,B} \langle w^{-1} \rangle_{\frac{1}{p-1},B} < \infty,$$

where the supremum is taken over all balls  $B \subseteq S$  and the second factor is replaced by  $\text{ess sup}_B w^{-1}$  if  $p = 1$ . For an introduction to Muckenhoupt weights we refer to [31, Chapter 7].

Let  $p_0, r \in [1, \infty)$ ,  $p \in (p_0, \infty)$ ,  $w \in A_{p/p_0}$ . We are interested in the boundedness on  $L^p(S, w)$  of sparse operators of the form

$$f \mapsto \left( \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{p_0, Q}^r \mathbf{1}_Q \right)^{1/r}, \quad (4.1)$$

which appear in the conclusions of Theorems 1.1 and 3.2. In the Euclidean case such bounds are thoroughly studied and most of the arguments extend directly to spaces of homogeneous type. For the convenience of the reader we will give a self-contained proof of the strong weighted  $L^p$ -boundedness of these sparse operators in spaces of homogeneous type, following the proof of [55, Lemma 4.5]. For further results we refer to:

- Weak weighted  $L^p$ -boundedness (including the endpoint  $p = p_0$ ), for the sparse operators in (4.1) can be found [26,42].
- More precise bounds in terms of two-weight  $A_p$ - $A_\infty$ -characteristics for various special cases of the sparse operators in (4.1) can be found in, e.g., [23,42,45,52].
- Weighted bounds for the fractional sparse operators in Theorem 3.4 can be found in [23].
- Weighted bounds for the sparse forms in Theorem 3.5 can be found in [6,26].

**Proposition 4.1** *Let  $(S, d, \mu)$  be a space of homogeneous type, let  $\mathcal{S}$  be an  $\eta$ -sparse collection of cubes and take  $p_0, r \in [1, \infty)$ . For  $p \in (p_0, \infty)$ ,  $w \in A_{p/p_0}$  and  $f \in L^p(S, w)$  we have*

$$\left\| \left( \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{p_0, Q}^r \mathbf{1}_Q \right)^{1/r} \right\|_{L^p(S, w)} \lesssim [w]_{A_{p/p_0}}^{\max\left\{\frac{1}{p-p_0}, \frac{1}{r}\right\}} \|f\|_{L^p(S, w)},$$

where the implicit constant depends on  $S$ ,  $p_0$ ,  $p$ ,  $r$  and  $\eta$ .

**Proof** We first note that by Proposition 2.1 we may assume without loss of generality that  $\mathcal{S} \subseteq \mathcal{D}$ , where  $\mathcal{D}$  is an arbitrary dyadic system in  $(S, d, \mu)$ . Furthermore, if  $p - p_0 \leq r$  we have  $\max\left\{\frac{1}{p-p_0}, \frac{1}{r}\right\} = \frac{1}{p-p_0}$ . Since  $\ell^{p-p_0} \hookrightarrow \ell^r$ , the case  $p - p_0 \leq r$  follows from the case  $p - p_0 = r$ , so without loss of generality we may also assume  $p \geq p_0 + r$ .

For a weight  $u$  and a measurable set  $E$ , we define  $u(E) := \int_E u d\mu$  and we denote the dyadic Hardy–Littlewood maximal operator with respect to the measure  $u d\mu$  by

$M^{\mathcal{D},u}$ , which is bounded on  $L^p(S, u)$  for all  $p \in (1, \infty)$  by Doob’s maximal inequality (see, e.g., [43, Theorem 3.2.2]). Take  $f \in L^p(S, w)$ , set  $q := (p/r)' = \frac{p}{p-r}$  and take

$$g \in L^q(S, w^{1-q}) = (L^{p/r}(S, w))^*.$$

Then we have by the disjointness of the  $E_Q$ ’s associated to each  $Q \in \mathcal{S}$

$$\begin{aligned} \sum_{Q \in \mathcal{S}} w(E_Q) \left( \frac{\mu(Q)}{w(Q)} \right)^q \langle |g| \rangle_{1,Q}^q &\leq \sum_{Q \in \mathcal{S}} \int_{E_Q} M^{\mathcal{D},w} (gw^{-1})^q w d\mu \\ &\leq \|M^{\mathcal{D},w} (gw^{-1})\|_{L^q(S,w)}^q \\ &\lesssim_{p,r} \|g\|_{L^q(S,w^{1-q})}^q, \end{aligned} \tag{4.2}$$

and similarly, setting  $\sigma := w^{1-(p/p_0)'}$ , we have

$$\begin{aligned} \sum_{Q \in \mathcal{S}} \sigma(E_Q) \left( \frac{\mu(Q)}{\sigma(Q)} \right)^{\frac{p}{p_0}} \langle |f|^{p_0} \rangle_{1,Q}^{p/p_0} &\leq \|M^{\mathcal{D},\sigma} (|f|^{p_0} \sigma^{-1})\|_{L^{p/p_0}(S,\sigma)}^{p/p_0} \\ &\lesssim_{p,p_0} \|f\|_{L^p(S,w)}^p, \end{aligned} \tag{4.3}$$

using  $\sigma \cdot \sigma^{-p_0/p} = w$ . Define the constant

$$c_w := \sup_{Q \in \mathcal{D}} \frac{w(Q)^{1/r} \sigma(Q)^{1/p_0}}{w(E_Q)^{\frac{1}{r}-\frac{1}{p}} \sigma(E_Q)^{1/p}} \frac{1}{\mu(Q)^{1/p_0}},$$

Then by Hölders inequality, (4.2) and (4.3) we have

$$\begin{aligned} \int_S \left( \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{p_0,Q}^r \mathbf{1}_Q \right) \cdot g d\mu &= \sum_{Q \in \mathcal{S}} \mu(Q) \langle |f|^{p_0} \rangle_{1,Q}^{r/p_0} \langle |g| \rangle_{1,Q} \\ &\leq c_w^r \sum_{Q \in \mathcal{S}} \left( \sigma(E_Q)^{r/p} \left( \frac{\mu(Q)}{\sigma(Q)} \right)^{r/p_0} \langle |f|^{p_0} \rangle_{1,Q}^{r/p_0} \right) \\ &\quad \times \left( w(E_Q)^{1/q} \frac{\mu(Q)}{w(Q)} \langle |g| \rangle_{1,Q} \right) \\ &\lesssim_{p,p_0,r} c_w^r \|f\|_{L^p(S,w)}^r \|g\|_{L^q(S,w^{1-q})}. \end{aligned}$$

So by duality it remains to show  $c_w \lesssim [w]_{A_{p/p_0}}^{\max\{\frac{1}{p-p_0}, \frac{1}{r}\}}$ . Fix a  $Q \in \mathcal{D}$  and note that by Hölders’s inequality we have

$$\mu(Q)^{p/p_0} \leq \eta^{p/p_0} \left( \int_{E_Q} w^{p_0/p} w^{-p_0/p} d\mu \right)^{p/p_0} \leq \eta^{p/p_0} w(E_Q) \sigma(E_Q)^{p/p_0-1}.$$

and thus

$$\frac{w(Q)}{w(E_Q)} \left( \frac{\sigma(Q)}{\sigma(E_Q)} \right)^{p/p_0-1} \leq \eta^{p/p_0} \frac{w(Q)}{\mu(Q)} \left( \frac{\sigma(Q)}{\mu(Q)} \right)^{p/p_0-1} \lesssim_S \eta^{p/p_0} [w]_{A_{p/p_0}}.$$

Therefore we can estimate

$$\begin{aligned} c_w &= \sup_{Q \in \mathcal{Q}} \left[ \frac{w(Q)}{\mu(Q)} \left( \frac{\sigma(Q)}{\mu(Q)} \right)^{\frac{p}{p_0}-1} \right]^{\frac{1}{p}} \cdot \left[ \left( \frac{w(Q)}{w(E_Q)} \right)^{\frac{1}{r}-\frac{1}{p}} \left( \frac{\sigma(Q)}{\sigma(E_Q)} \right)^{\frac{1}{p}} \right] \\ &\lesssim_S [w]_{A_{p/p_0}}^{\frac{1}{p}} \sup_{Q \in \mathcal{Q}} \left[ \frac{w(Q)}{w(E_Q)} \left( \frac{\sigma(Q)}{\sigma(E_Q)} \right)^{\frac{p}{p_0}-1} \right]^{\max\left\{\frac{1}{r}-\frac{1}{p}, \frac{1}{p}-\frac{p_0}{p-p_0}\right\}} \\ &\lesssim_{S,\eta} [w]_{A_{p/p_0}}^{\frac{1}{p}+\max\left\{\frac{1}{r}-\frac{1}{p}, \frac{1}{p}-\frac{p_0}{p-p_0}\right\}} = [w]_{A_{p/p_0}}^{\max\left\{\frac{1}{p-p_0}, \frac{1}{r}\right\}}, \end{aligned}$$

which finishes the proof.  $\square$

## 5 Banach Space Geometry and $\mathcal{R}$ -Boundedness

Before turning to applications of Theorems 1.1 and 3.2 in the subsequent sections, we first need to introduce some geometric properties of a Banach space  $X$  and the  $\mathcal{R}$ -boundedness of a family of operators.

### 5.1 Type and Cotype

Let  $(\varepsilon_k)_{k=1}^\infty$  be a sequence of independent *Rademacher variables* on  $\Omega$ , i.e., uniformly distributed random variables taking values in  $\{z \in \mathbb{K} : |z| = 1\}$ . We say that a Banach space  $X$  has (Rademacher) type  $p \in [1, 2]$  if for any  $x_1, \dots, x_n \in X$  we have

$$\left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_{L^2(\Omega; X)} \lesssim_{X,p} \left( \sum_{k=1}^n \|x_k\|_X^p \right)^{1/p},$$

and say that  $X$  has non-trivial type if  $X$  has type  $p > 1$ . We say that  $X$  has (Rademacher) cotype  $q \in [2, \infty]$  if for any  $x_1, \dots, x_n \in X$  we have

$$\left( \sum_{k=1}^n \|x_k\|_X^q \right)^{1/q} \lesssim_{X,q} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_{L^2(\Omega; X)},$$

and say that  $X$  has finite cotype if  $X$  has cotype  $q < \infty$ . See [44, Chapter 7] for an introduction to type and cotype.

### 5.2 Banach Lattices and $p$ -Convexity and $q$ -Concavity

A Banach lattice is a partially ordered Banach space  $X$  such that for  $x, y \in X$

$$|x| \leq |y| \Rightarrow \|x\|_X \leq \|y\|_Y.$$

On a Banach lattice, there are two properties that are closely related to type and cotype. We say that a Banach lattice is  $p$ -convex with  $p \in [1, \infty]$  if for  $x_1, \dots, x_n \in X$

$$\left\| \left( \sum_{k=1}^n |x_k|^p \right)^{1/p} \right\|_X \lesssim_{X,p} \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p},$$

where the sum on the left-hand side is defined through the Krivine calculus. A Banach lattice is called  $q$ -concave for  $q \in [1, \infty]$  if for  $x_1, \dots, x_n \in X$

$$\left( \sum_{k=1}^n \|x_k\|^q \right)^{1/q} \lesssim_{X,q} \left\| \left( \sum_{k=1}^n |x_k|^q \right)^{1/q} \right\|_X.$$

If a Banach lattice has finite cotype then  $p$ -convexity implies type  $p$ . Conversely type  $p$  implies  $r$ -convexity for all  $1 \leq r < p$ . Similar relations hold for cotype  $q$  and  $q$ -concavity. We refer to [63, Chapter 1] for an introduction to Banach lattices,  $p$ -convexity and  $q$ -concavity.

### 5.3 The UMD Property

We say that a Banach space  $X$  has the UMD property if the martingale difference sequence of any finite martingale in  $L^p(\Omega; X)$  is unconditional for some (equivalently all)  $p \in (1, \infty)$ . The UMD property implies reflexivity, non-trivial type and finite cotype. For an introduction to the theory of UMD Banach spaces we refer the reader to [43, Chapter 4] and [73].

### 5.4 $\mathcal{R}$ -Boundedness

Let  $X$  and  $Y$  be Banach spaces and  $\Gamma \subseteq \mathcal{L}(X, Y)$ . We say that  $\Gamma$  is  $\mathcal{R}$ -bounded if for any  $x_1, \dots, x_n$  and  $T_1, \dots, T_n \in \Gamma$  we have

$$\left( \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k T_k x_k \right\|^2 \right)^{1/2} \lesssim \left( \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|^2 \right)^{1/2},$$

where  $(\varepsilon_k)_{k=1}^\infty$  is a sequence of independent Rademacher variables. The least admissible implicit constant is denoted by  $\mathcal{R}(\Gamma)$ .  $\mathcal{R}$ -boundedness is a strengthening of uniform boundedness and is often a key assumption to prove boundedness of operators on Bochner spaces. We refer to [44, Chapter 8] for an introduction to  $\mathcal{R}$ -boundedness.



## 6 The $A_2$ -Theorem for Operator-Valued Calderón–Zygmund Operators in a Space of Homogeneous Type

The  $A_2$ -theorem, first proved by Hytönen in [38] as discussed in the introduction, states that a Calderón–Zygmund operator is bounded on  $L^2(\mathbb{R}^d, w)$  with a bound that depends linearly on the  $A_2$ -characteristic of  $w$ . From this sharp weighted bounds for all  $p \in (1, \infty)$  can be obtained by sharp Rubio de Francia extrapolation [21]. Since its first proof by Hytönen, the  $A_2$ -theorem has been extended in various directions. We mention two of these extensions relevant for the current discussion:

- The  $A_2$ -theorem for Calderón–Zygmund operators on a geometric doubling metric space was first proven by Nazarov, Reznikov, and Volberg [69], afterwards it was proven on a space of homogeneous type by Anderson and Vagharshakyan [3] (see also [2]) using Lerner’s mean oscillation decomposition method. It was further extended to the setting of ball bases by Karagulyan [48].
- The  $A_2$ -theorem for vector-valued Calderón–Zygmund operators with operator-valued kernel was proven by Hänninen and Hytönen [32], using a suitable adapted version of Lerner’s median oscillation decomposition.

In this section we will prove sparse domination for vector-valued Calderón–Zygmund operators with operator-valued kernel on a space of homogeneous type. This yields the  $A_2$ -theorem for these Calderón–Zygmund operators, unifying the results from [3] and [32].

As an application of this theorem, we will prove a weighted, anisotropic, mixed-norm Mihlin multiplier theorem in the next section. We will also use it to study maximal regularity for parabolic partial differential equations in forthcoming work. In these applications  $S$  is (a subset of)  $\mathbb{R}^d$  equipped with the anisotropic quasi-norm

$$|s|_a := \left( \sum_{j=1}^d |s_j|^{2/a_j} \right)^{1/2}, \quad s \in \mathbb{R}^d. \quad (6.1)$$

for some  $a \in (0, \infty)^d$  and the Lebesgue measure.

In a different direction, our  $A_2$ -theorem can be applied in the study of fundamental harmonic analysis operators associated with various discrete and continuous orthogonal expansions, started by Muckenhoupt and Stein [68]. In the past decade, there has been a surge of results in which such operators are proven to be vector-valued Calderón–Zygmund operators on concrete spaces of homogeneous type. Weighted bounds are then often concluded using [76, Theorem III.1.3] or [77]. With our  $A_2$ -theorem, these results can be made quantitative in terms of the  $A_p$ -characteristic. We refer to [7, 11, 13, 70, 71] and the references therein for an overview of the recent developments in this field.

Let  $(S, d, \mu)$  be a space of homogeneous type,  $X$  and  $Y$  be Banach spaces and let

$$K : (S \times S) \setminus \{(s, s) : s \in S\} \rightarrow \mathcal{L}(X, Y),$$

be strongly measurable in the strong operator topology. We say that  $K$  is a *Dini kernel* if there is a  $c_K \geq 2$  such that

$$\|K(s, t) - K(s, t')\| \leq \omega\left(\frac{d(t, t')}{d(s, t)}\right) \frac{1}{\mu(B(s, d(s, t)))}, \quad 0 < d(t, t') \leq \frac{1}{c_K} d(s, t),$$

$$\|K(s, t) - K(s', t)\| \leq \omega\left(\frac{d(s, s')}{d(s, t)}\right) \frac{1}{\mu(B(s, d(s, t)))}, \quad 0 < d(s, s') \leq \frac{1}{c_K} d(s, t),$$

where  $\omega : [0, 1] \rightarrow [0, \infty)$  is increasing, subadditive,  $\omega(0) = 0$  and

$$\|K\|_{\text{Dini}} := \int_0^1 \omega(t) \frac{dt}{t} < \infty.$$

Take  $p_0 \in [1, \infty)$  and let

$$T : L^{p_0}(S; X) \rightarrow L^{p_0, \infty}(S; Y),$$

be a bounded linear operator. We say that  $T$  has *Dini kernel*  $K$  if for every boundedly supported  $f \in L^{p_0}(S; X)$  and a.e.  $s \in S \setminus \text{supp } f$  we have

$$Tf(s) = \int_S K(s, t) f(t) dt.$$

**Theorem 6.1** *Let  $(S, d, \mu)$  be a space of homogeneous type and let  $X$  and  $Y$  be Banach spaces. Let  $p_0 \in [1, \infty)$  and suppose  $T$  is a bounded linear operator from  $L^{p_0}(S; X)$  to  $L^{p_0, \infty}(S; Y)$  with Dini kernel  $K$ . Then for every boundedly supported  $f \in L^1(S; X)$  there exists an  $\eta$ -sparse collection of cubes  $\mathcal{S}$  such that*

$$\|Tf(s)\|_Y \lesssim_{S, p_0} C_T \sum_{Q \in \mathcal{S}} \langle \|f\|_X \rangle_{1, Q} \mathbf{1}_Q(s), \quad s \in S.$$

Moreover, for all  $p \in (1, \infty)$  and  $w \in A_p$  we have

$$\|T\|_{L^p(S, w; X) \rightarrow L^p(S, w; Y)} \lesssim_{S, p, p_0} C_T [w]_{A_p}^{\max\{\frac{1}{p-1}, 1\}},$$

with  $C_T := \|T\|_{L^{p_0}(S; X) \rightarrow L^{p_0, \infty}(S; Y)} + \|K\|_{\text{Dini}}$ .

**Proof** We will check the assumptions of Theorem 1.1 with  $p_1 = p_2 = r = 1$ . The weak  $L^1$ -boundedness of  $T$  with

$$\|T\|_{L^1(S; X) \rightarrow L^{1, \infty}(S; Y)} \lesssim_{S, p} C_T.$$

follows from the classical Calderón-Zygmund argument, see, e.g., [76, Theorem III.1.2]. The 1-sublinearity assumption on  $T$  follows from the triangle inequality,

so the only thing left to check is the weak  $L^1$ -boundedness of  $\mathcal{M}_{T,\alpha}^\#$ . Let

$$\alpha := 3c_d^2 \max\{\delta^{-1}, c_K\}$$

with  $c_d$  the quasi-metric constant,  $\delta$  as in Proposition 2.1 and  $c_K$  the constant from the definition of a Dini kernel. Fix  $s \in S$  and a ball  $B = B(z, \rho)$  such that  $s \in B$ . Then for any  $s', s'' \in B$  and  $t \in S \setminus \alpha B$  we have

$$\begin{aligned} d(s', t) &\geq \frac{1}{c_d} d(z, t) - d(z, s') \geq \frac{\alpha\rho}{c_d} - \rho \geq 2c_K c_d \rho =: \varepsilon \\ d(s', s'') &\leq 2c_d \rho = c_K^{-1} \varepsilon, \end{aligned}$$

Therefore we have for any boundedly supported  $f \in L^1(S; X)$

$$\begin{aligned} &\|T(\mathbf{1}_{S \setminus \alpha B} f)(s') - T_K(\mathbf{1}_{S \setminus \alpha B} f)(s'')\|_Y \\ &\leq \int_{S \setminus \alpha B} \|(K(s', t) - K(s'', t))f(t)\|_Y d\mu(t) \\ &\leq \int_{d(s', t) > \varepsilon} \omega\left(\frac{d(s', s'')}{d(s', t)}\right) \frac{1}{\mu(B(s', d(s', t)))} \|f(t)\|_X d\mu(t) \\ &\leq \sum_{j=0}^{\infty} \omega(c_K^{-1} 2^{-j}) \int_{2^j \varepsilon < d(s', t) \leq 2^{j+1} \varepsilon} \frac{1}{\mu(B(s', d(s', t)))} \|f(t)\|_X d\mu(t) \\ &\lesssim_S \sum_{j=0}^{\infty} \omega(2^{-j-1}) \int_{B(s', 2^{j+1} \varepsilon)} \|f(t)\|_X d\mu(t) \\ &\leq \|K\|_{\text{Dini}} M(\|f\|_X)(s), \end{aligned}$$

where the last step follows from  $s \in B(s', 2^{j+1} \varepsilon)$  for all  $j \in \mathbb{N}$  and

$$\sum_{j=0}^{\infty} \omega(2^{-j-1}) \leq \sum_{j=0}^{\infty} \omega(2^{-j-1}) \int_{2^{-j-1}}^{2^{-j}} \frac{dt}{t} \leq \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} \omega(t) \frac{dt}{t} = \|K\|_{\text{Dini}}.$$

So taking the supremum over all  $s', s'' \in B$  and all balls  $B$  containing  $s$  we find that  $\mathcal{M}_{T,\alpha}^\# f(s) \lesssim_S \|K\|_{\text{Dini}} M(\|f\|_X)(s)$ . Thus by the weak  $L^1$ -boundedness of the Hardy–Littlewood maximal operator and the density of boundedly supported functions in  $L^1(S; X)$  we get

$$\|\mathcal{M}_{T,\alpha}^\#\|_{L^1(S; X) \rightarrow L^{1,\infty}(S; Y)} \lesssim_S \|K\|_{\text{Dini}}.$$

The pointwise sparse domination now follows from Theorem 1.1 and the weighted bounds from Proposition 4.1.  $\square$

**Remark 6.2** In the proof of Theorem 6.1 it actually suffices to use the so-called  $L^r$ -Hörmander condition for some  $r > 1$ , which is implied by the Dini condition. See

[60, Section 3] for the definition of the  $L^r$ -Hörmander condition and a comparison between the  $L^r$ -Hörmander and the Dini condition.

Note that Theorem 6.1 does not assume anything about the Banach spaces  $X$  and  $Y$  and is therefore applicable in situations where, for example,  $Y = \ell^\infty$ . However, in various applications,  $X$  and  $Y$  will need to have the UMD property in order to check the assumed weak  $L^{p_0}$ -boundedness of  $T$  for some  $p_0 \in [1, \infty)$ . For instance, for a large class of operators, the weak  $L^{p_0}$ -boundedness of  $T$  can be checked using theorems like the  $T(1)$ -theorem or  $T(b)$ -theorem. See [25, 39] for these theorems in the vector-valued setting, which assume the UMD property for the underlying Banach space.

If  $S$  is Euclidean space, one can also use an (operator-valued) Fourier multiplier theorem to check the a priori  $L^{p_0}$ -bound, which we will discuss in the next section.

## 7 The Weighted Anisotropic Mixed-Norm Mihlin Multiplier Theorem

Let  $X$  and  $Y$  be Banach spaces. Denote the space of  $X$ -valued Schwartz functions by  $\mathcal{S}(\mathbb{R}^d; X)$  and the space of  $Y$ -valued tempered distributions by  $\mathcal{S}'(\mathbb{R}^d; Y) := \mathcal{L}(\mathcal{S}(\mathbb{R}^d; X); Y)$ . To an  $m \in L^\infty(\mathbb{R}^d; \mathcal{L}(X, Y))$  we associate the Fourier multiplier operator

$$T_m : \mathcal{S}(\mathbb{R}^d; X) \rightarrow \mathcal{S}'(\mathbb{R}^d; Y), \quad T_m f = (m \widehat{f})^\vee.$$

Since  $\mathcal{S}(\mathbb{R}^d; X)$  is dense in  $L^p(\mathbb{R}^d; X)$  and  $L^p(\mathbb{R}^d; Y)$  is continuously embedded into  $\mathcal{S}'(\mathbb{R}^d; Y)$ , one may ask under which conditions on  $m$  the operator  $T_m$  extends to a bounded operator from  $L^p(\mathbb{R}^d; X)$  to  $L^p(\mathbb{R}^d; Y)$ . If this is the case we call  $m$  a bounded Fourier multiplier. We refer to [43, Chapter 5] for an introduction to operator-valued Fourier multiplier theory.

One of the main Fourier multiplier theorems is the Mihlin multiplier theorem, first proven in the operator-valued setting by Weis in [79]. The operator-valued Mihlin multiplier theorem of Weis has since been extended in many directions. Recently Fackler, Hytönen, and Lindemulder extended the operator-valued Mihlin multiplier theorem to a weighted, anisotropic, mixed-norm setting in [24]. This is, for example, useful in the study of spaces of smooth, vector-valued functions and has applications to parabolic PDEs with inhomogeneous boundary conditions, see, e.g., [62]. In [24] the Mihlin multiplier theorem is shown using the following two approaches:

- Using a weighted Littlewood–Paley decomposition, they show a weighted, anisotropic, mixed-norm Mihlin multiplier theorem for rectangular  $A_p$ -weights, i.e.,  $A_p$ -weights for which the defining supremum is taken over rectangles instead of balls.
- Using Calderón–Zygmund theory, they show a weighted, isotropic, non-mixed-norm Mihlin multiplier theorem for cubical  $A_p$ -weights, i.e.,  $A_p$ -weights for which the defining supremum is taken over cubes, which is equivalent to the definition using balls we used in Sect. 4.

Both approaches have their pros and cons. The result using a Littlewood–Paley decomposition only requires estimates of  $\partial^\theta m$  for  $\theta \in \{0, 1\}$ , whereas the approach using Calderón–Zygmund theory also requires estimates of higher-order derivatives. On the other hand, the class of rectangular  $A_p$ -weights is a proper subclass of the class of cubicular  $A_p$ -weights.

In applications, it is desirable to have the Mihlin multiplier theorem for cubicular  $A_p$ -weights in the anisotropic, mixed-norm setting as well. This would remove the need to distinguish between the isotropic and anisotropic setting in, e.g., [62, (6) on p.64]. In order to obtain the Mihlin multiplier theorem for cubicular  $A_p$ -weights in the anisotropic, mixed-norm setting one needs Calderón–Zygmund theory in  $\mathbb{R}^d$  equipped with an anisotropic norm. Since this is a special case of a space of homogeneous type, we can use Theorem 6.1 to supplement the results of [24], which will be the main result of this section.

Let us introduce the anisotropic, mixed-norm setting. For  $\mathbf{a} \in (0, \infty)^d$  let  $|\cdot|_{\mathbf{a}}$  be the anisotropic quasi-norm as in (6.1) and define

$$\mathbb{R}_{\mathbf{a}}^d := (\mathbb{R}^d, |\cdot|_{\mathbf{a}}, dt),$$

where  $dt$  denotes Lebesgue measure. Then  $\mathbb{R}_{\mathbf{a}}^d$  is a space of homogeneous type and, e.g.,

$$\mathcal{D} := \left\{ \prod_{j=1}^d (2^{-a_j n} ([0, 1) + m_j)) : \mathbf{m} \in \mathbb{Z}^d, n \in \mathbb{Z} \right\},$$

is a dyadic system in  $\mathbb{R}_{\mathbf{a}}^d$ . We write  $|\mathbf{a}|_1 := \sum_{j=1}^d a_j$ ,  $|\mathbf{a}|_\infty := \max_{j=1, \dots, d} a_j$ , and for  $\theta \in \mathbb{N}^d$  we set  $\mathbf{a} \cdot \theta := \sum_{j=1}^d a_j \theta_j$ .

Take  $l \in \mathbb{N}$ ,  $d \in \mathbb{N}^l$  and consider the  $d$ -decomposition of  $\mathbb{R}^d$ :

$$\mathbb{R}_d^d := \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_l}.$$

For a  $t \in \mathbb{R}_d^d$  we write  $t = (t_1, \dots, t_l)$  with  $t_j \in \mathbb{R}^{d_j}$  for  $j = 1, \dots, l$  and similarly we write  $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_l)$ . For  $\mathbf{p} \in [1, \infty)^l$ , a vector of weights  $\mathbf{w} \in \prod_{j=1}^l A_p(\mathbb{R}_{\mathbf{a}_j}^{d_j})$  and a Banach space  $X$  we define the weighted mixed-norm Bochner space  $L^{\mathbf{p}}(\mathbb{R}_d^d, \mathbf{w}; X)$  as the space of all strongly measurable  $f : \mathbb{R}_d^d \rightarrow X$  such that

$$\|f\|_{L^{\mathbf{p}}(\mathbb{R}_d^d, \mathbf{w}; X)} := \left( \int_{\mathbb{R}^{d_1}} \dots \left( \int_{\mathbb{R}^{d_l}} \|f\|_X^{p_l} w_l dt_l \right)^{\frac{p_1-1}{p_l}} \dots w_1 dt_1 \right)^{\frac{1}{p_1}},$$

is finite.

We are now ready to state and prove the announced weighted anisotropic, mixed-norm Mihlin multiplier theorem.

**Theorem 7.1** *Let  $X$  and  $Y$  be UMD Banach spaces, set  $N = |\mathbf{a}|_1 + |\mathbf{a}|_\infty + 1$  and let  $m \in L^\infty(\mathbb{R}^d; \mathcal{L}(X, Y))$ . Suppose that for all  $\theta \in \mathbb{N}^d$  with  $\mathbf{a} \cdot \theta \leq N$  the distributional*

derivative  $\partial^\theta m$  coincides with a continuous function on  $\mathbb{R}^d \setminus \{0\}$  and we have the  $\mathcal{R}$ -bound

$$\mathcal{R}(\{|\xi|_a^{a-\theta} \cdot \partial^\theta m(\xi) : \xi \in \mathbb{R}^d\} \setminus \{0\}) \leq C_m.$$

for some  $C_m > 0$ . Then for every compactly supported  $f \in L^1(\mathbb{R}^d; X)$  there exists an  $\eta$ -sparse collection of anisotropic cubes  $\mathcal{S}$  such that

$$\|T_m f(s)\|_Y \lesssim_{X,Y,a} C_m \sum_{Q \in \mathcal{S}} \langle \|f\|_X \rangle_{1,Q} \mathbf{1}_Q(s), \quad s \in \mathbb{R}^d.$$

Moreover, for all  $\mathbf{p} \in (1, \infty)^l$  and  $\mathbf{w} \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$  we have

$$\|T_m\|_{L^p(\mathbb{R}_d^d, \mathbf{w}; X) \rightarrow L^p(\mathbb{R}_d^d, \mathbf{w}; Y)} \lesssim_{X,Y,d,a,\mathbf{p},\mathbf{w}} C_m.$$

**Proof** We will check the conditions of Theorem 6.1. By [37, Theorem 3], which trivially extends to the case  $X \neq Y$ , we know that  $T_m$  is bounded from  $L^2(\mathbb{R}^d; X)$  to  $L^2(\mathbb{R}^d; Y)$  with

$$\|T_m\|_{L^2(\mathbb{R}^d; X) \rightarrow L^2(\mathbb{R}^d; Y)} \lesssim_{X,Y,d,a} C_m.$$

By [61, Lemma 4.4.6 and 4.4.7] we know that  $\check{m}$  coincides with a continuous function on  $\mathbb{R}^d \setminus \{0\}$ , which is bounded away from 0 and

$$K(t, s) := \check{m}(t - s), \quad t \neq s,$$

is a Dini kernel on the space of homogeneous type  $\mathbb{R}_a^d$  with

$$\omega(r) = C_a \cdot C_m \cdot r^{\min a}, \quad r \in [0, 1].$$

Now let  $f \in L^p(\mathbb{R}^d; X)$  with compact support. Fix a  $c \in \mathbb{R}^d \setminus \overline{\text{supp } f}$  and take  $r > 0$  such that  $B(c, 2r) \cap \text{supp } f = \emptyset$ . Take a sequence  $(f_n)_{n=1}^\infty$  in  $\mathcal{S}(\mathbb{R}^d; X)$  such that  $\overline{\text{supp } f_n} \cap B(c, r) = \emptyset$  and  $f_n \rightarrow f$  in  $L^2(\mathbb{R}^d; X)$ . Then  $T f_n \rightarrow T f$  in  $L^2(\mathbb{R}^d; X)$  and, by passing to a subsequence if necessary, we have  $f_n(t) \rightarrow f(t)$  and  $T f_n(t) \rightarrow T f(t)$  for a.e.  $t \in \mathbb{R}^d$ . Fix  $n \in \mathbb{N}$ , then we have for all  $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \overline{\text{supp } f_n})$

$$\begin{aligned} \langle T_m f_n, \varphi \rangle &= \int_{\mathbb{R}^d} m(s) \widehat{f_n}(s) \check{\varphi}(s) ds \\ &= \int_{\mathbb{R}^d} \check{m}(s) \int_{\mathbb{R}^d} f_n(t - s) \varphi(t) dt ds \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(t, s) f_n(s) ds \varphi(t) dt \end{aligned}$$

from which we obtain for a.e.  $t \in B(c, r)$

$$T_m f(t) = \lim_{n \rightarrow \infty} T_m f_n(t) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} K(t, s) f_n(s) ds = \int_{\mathbb{R}^d} K(t, s) f(s) ds$$

Covering  $\mathbb{R}^d \setminus \overline{\text{supp } f}$  by countably many such balls, we conclude that  $T_m$  has kernel  $K$ . Therefore the sparse domination, as well as the weighted estimate in case  $l = 1$ , follows from Theorem 6.1.

To conclude the proof we will show the case  $l = 2$ , the general case follows by iterating the argument. Take  $\mathbf{p} \in (1, \infty)^2$  and  $\mathbf{w} \in A_{p_1}(\mathbb{R}^{d_1}) \times A_{p_2}(\mathbb{R}^{d_2})$ . For  $v_1 \in A_{p_2}(\mathbb{R}^{d_1})$ , note that

$$v(t) := v_1(t_1) \cdot w_2(t_2), \quad t \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},$$

belongs to  $A_{p_2}(\mathbb{R}_a^d)$ , so by the case  $l = 1$  we have

$$\|T_m f\|_{L^{p_2}(\mathbb{R}^d, v; Y)} \lesssim_{X, Y, d, a, p_2, v} C_m \cdot \|f\|_{L^{p_2}(\mathbb{R}^d, v; X)},$$

for all  $f \in L^{p_2}(\mathbb{R}^d, v; X)$ . Since balls in  $\mathbb{R}^{d_2}$  with respect to the quasi-metric  $|\cdot - \cdot|_{a_2}$  form a Muckenhoupt basis, we can use Rubio de Francia extrapolation as in [19, Theorem 3.9] on the extrapolation family

$$\left\{ (\|T_m f\|_{L^{p_2}(\mathbb{R}^{d_2}, w_2; Y)}, \|f\|_{L^{p_2}(\mathbb{R}^{d_2}, w_2; X)}) : f : \mathbb{R}^d \rightarrow X \text{ simple} \right\},$$

to deduce

$$\|T_m f\|_{L^p(\mathbb{R}_a^d, \mathbf{w}; Y)} \lesssim_{X, Y, d, a, \mathbf{p}, \mathbf{w}} C_m \|f\|_{L^p(\mathbb{R}_a^d, \mathbf{w}; X)},$$

for all simple  $f$ , which implies the result by density. □

- Remark 7.2** (i) The weight dependence of the implicit constant in Theorem 7.1 in the case  $l = 1$  is  $[w]_{A_p(\mathbb{R}_a^d)}^{\max\{\frac{1}{p-1}, 1\}}$ , which is sharp. For  $l \geq 2$  the dependence our proof yields is more complicated and not sharp for all choices of  $\mathbf{p} \in (1, \infty)^l$ .
- (ii) In the proof of Theorem 7.1 we only use the  $\mathcal{R}$ -boundedness of the set

$$\{|\xi|_a^{\mathbf{a} \cdot \theta} \cdot \partial^\theta m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}\},$$

for  $\theta \in \{0, 1\}^d$ . For all other  $\theta \in \mathbb{N}^d$  with  $\mathbf{a} \cdot \theta \leq N$  it suffices to know uniform boundedness of this set.

- (iii) One could reduce the number of derivatives necessary in Theorem 7.1, by arguing as in [36] instead of using [61, Lemmas 4.4.6 and 4.4.7]. See also [24, Section 6].
- (iv) Using the sparse domination of Theorem 7.1 one can also deduce two-weight estimates for  $T_m$  as in [24, Section 6].

### 8 The Rademacher Maximal Function

In this section, we will apply Theorem 3.2 to the Rademacher maximal function. The proofs will illustrate very nicely how the geometry of the Banach space plays a role in deducing the localized  $\ell^r$ -estimate for this operator. In particular, we will use the type of a Banach space  $X$  to deduce the localized  $\ell^r$ -estimate for the Rademacher maximal function.

The Rademacher maximal function was introduced by Hytönen, McIntosh and Portal in [34] as a vector-valued generalization of Doob’s maximal function that takes into account the different “directions” in a Banach space. They used the Rademacher maximal function to prove a Carleson’s embedding theorem for vector-valued functions in connection to Kato’s square root problem in Banach spaces. The Carleson’s embedding theorem for vector-valued functions has since found many other applications, like the local vector-valued  $T(b)$  theorem (see [46]).

Let  $(S, d, \mu)$  be a space of homogeneous type with a dyadic system  $\mathcal{D}$  and let  $X$  be a Banach space. For  $f \in L^1_{loc}(S; X)$ , we define the *Rademacher maximal function* by

$$M_{\text{Rad}}^{\mathcal{D}}f(s) := \sup \left\{ \left\| \sum_{Q \in \mathcal{D}: s \in Q} \varepsilon_Q \lambda_Q \langle f \rangle_{1,Q} \right\|_{L^2(\Omega; X)} : (\lambda_Q)_{Q \in \mathcal{D}} \text{ finitely non-zero with } \sum_{Q \in \mathcal{D}} |\lambda_Q|^2 \leq 1 \right\},$$

where  $(\varepsilon_Q)_{Q \in \mathcal{D}}$  is a Rademacher sequence on  $\Omega$ . One can interpret this maximal function as Doob’s maximal function

$$f^*(s) := \sup_{Q \in \mathcal{D}: s \in Q} \|\langle f \rangle_{1,Q}\|_X, \quad s \in S,$$

with the uniform bound over the  $\langle f \rangle_{1,Q}$ ’s replaced by the  $\mathcal{R}$ -bound. Here the  $\mathcal{R}$ -bound of a set  $U \subseteq X$  is the  $\mathcal{R}$ -bound of the family of operators  $T_x : \mathbb{C} \rightarrow X$  given by  $\lambda \mapsto \lambda x$  for  $x \in U$ .

We say that the Banach space  $X$  has the RMF property if  $M_{\text{Rad}}^{\mathcal{D}[0,1]}$  is a bounded operator on  $L^p([0, 1); X)$  for some  $p \in (1, \infty)$ , where

$$\mathcal{D}[0, 1) := \{2^{-k}[j - 1, j) : k \in \mathbb{N} \cup \{0\}, j = 1, \dots, 2^k\}$$

is the standard dyadic system in  $[0, 1)$ . It was shown by Hytönen, McIntosh and Portal [34, Proposition 7.1] that this implies boundedness for all  $p \in (1, \infty)$  and by Kempainen [49, Theorem 5.1] that this implies boundedness of  $M_{\text{Rad}}^{\mathcal{D}}$  on  $L^p(S; X)$  for any space of homogeneous type  $(S, d, \mu)$  with a dyadic system  $\mathcal{D}$ .

The relation of RMF property to other Banach space properties is not yet fully understood. However, we do have some necessary and sufficient conditions:

- The  $\mathcal{R}$ -bound of a set  $U \subseteq X$  is equivalent to the uniform bound of that set if and only if  $X$  has type 2 (see [44, Proposition 8.6.1]). Therefore if  $X$  has type 2 we



have for any  $f \in L^1_{\text{loc}}([0, 1]; X)$  that  $M_{\text{Rad}}^{\mathcal{D}[0,1]} f \lesssim M^{\mathcal{D}[0,1]}(\|f\|_X)$ , so  $X$  has the RMF property.

- Any UMD Banach lattice has the RMF property, see also the discussion related to the Hardy–Littlewood maximal operator at the end of this section.
- Non-commutative  $L^p$ -spaces for  $p \in (1, \infty)$  have the RMF property, see [34, Corollary 7.6].
- The RMF property implies non-trivial type, see [49, Proposition 4.2].

It is an open problem whether non-trivial type or even the UMD property implies the RMF property.

Weighted bounds for the Rademacher maximal function in the Euclidean setting were studied by Kempainen [50, Theorem 1]. The proof was based on a good- $\lambda$  inequality, which does not give sharp quantitative estimates in terms of the weight characteristic. Using Theorem 3.2 we can prove sharp quantitative weighted estimates for the Rademacher maximal function through sparse domination. We will not consider the situation in which  $X$  has type 2, as this case follows directly from  $M_{\text{Rad}}^{\mathcal{D}[0,1]} f \lesssim M^{\mathcal{D}[0,1]}(\|f\|_X)$  and the well-known sparse domination for the Hardy–Littlewood maximal operator.

We will need a version of the Rademacher maximal function for finite collections of cubes. For a subcollection of cubes  $\mathcal{D} \subseteq \mathcal{D}$ , we define  $M_{\text{Rad}}^{\mathcal{D}}$  analogous to  $M_{\text{Rad}}^{\mathcal{D}}$ .

**Theorem 8.1** *Let  $(S, d, \mu)$  be a space of homogeneous type with a dyadic system  $\mathcal{D}$  and let  $X$  be a Banach space with the RMF property. Assume that  $X$  has type  $r$  for  $r \in [1, 2)$ . For any finite collection of cubes  $\mathcal{D} \subseteq \mathcal{D}$  and  $f \in L^1(S; X)$  there exists an  $\frac{1}{2}$ -sparse collection of cubes  $\mathcal{S} \subseteq \mathcal{D}$  such that*

$$M_{\text{Rad}}^{\mathcal{D}} f(s) \lesssim_{X, \mathcal{S}, \mathcal{D}, r} \left( \sum_{Q \in \mathcal{S}} \langle \|f\|_X \rangle_{1, Q}^{(\frac{1}{r} - \frac{1}{2})^{-1}} \mathbf{1}_Q(s) \right)^{\frac{1}{r} - \frac{1}{2}}, \quad s \in S.$$

Moreover, for all  $p \in (1, \infty)$  and  $w \in A_p$  we have

$$\|M_{\text{Rad}}^{\mathcal{D}}\|_{L^p(S, w; X) \rightarrow L^p(S, w; X)} \lesssim_{X, \mathcal{S}, \mathcal{D}, p, r} [w]_{A_p}^{\max\{\frac{1}{p-1}, \frac{1}{r} - \frac{1}{2}\}}.$$

**Proof** Fix a finite collection of cubes  $\mathcal{D} \subseteq \mathcal{D}$ . By [49, Proposition 6.1]  $M_{\text{Rad}}^{\mathcal{D}}$  is weak  $L^1$ -bounded. We will view  $M_{\text{Rad}}^{\mathcal{D}}$  as a bounded operator

$$M_{\text{Rad}}^{\mathcal{D}} : L^1(S; X) \rightarrow L^{1, \infty}(S; \mathcal{L}(\ell^2(\mathcal{D}), L^2(\Omega; X)))$$

given by

$$M_{\text{Rad}}^{\mathcal{D}} f(s) = \left( (\lambda_Q)_{Q \in \mathcal{D}} \mapsto \sum_{Q \in \mathcal{D}: s \in Q} \varepsilon_Q \lambda_Q \langle f \rangle_{1, Q} \right), \quad s \in S,$$

where  $(\varepsilon_Q)_{Q \in \mathcal{D}}$  is a Rademacher sequence on  $\Omega$ .

For  $Q \in \mathcal{D}$  set

$$\mathcal{D}(Q) := \{P \in \mathcal{D} : P \subseteq Q\}$$

and define  $T_Q := M_{\text{Rad}}^{\mathcal{D}(Q)}$ . Then  $\{T_Q\}_{Q \in \mathcal{D}}$  is a 1-localization family for  $M_{\text{Rad}}^{\mathcal{D}}$ . Furthermore, we have for  $f \in L^1(S; X)$  and  $s \in Q \in \mathcal{D}$  that

$$\begin{aligned} & \mathcal{M}_{M_{\text{Rad}}^{\mathcal{D}}, Q}^{\#} f(s) \\ &= \sup_{\substack{Q' \in \mathcal{D}(Q): s', s'' \in Q' \\ s \in Q'}} \text{ess sup} \|T_{Q \setminus Q'} f(s') - T_{Q \setminus Q'} f(s'')\|_{\mathcal{L}(\ell^2(\mathcal{D}), L^2(\Omega; X))} \\ &= 0, \end{aligned}$$

where the second step follows from the fact that  $T_{Q \setminus Q'} f = M_{\text{Rad}}^{\mathcal{D}(Q) \setminus \mathcal{D}(Q')} f$  is constant on  $Q'$ . So  $\mathcal{M}_{M_{\text{Rad}}^{\mathcal{D}}, Q}^{\#}$  is trivially bounded from  $L^1(S; X)$  to  $L^{1, \infty}(S)$ .

Set  $q := (\frac{1}{r} - \frac{1}{2})^{-1}$ . To check the localized  $\ell^q$ -estimate for  $M_{\text{Rad}}^{\mathcal{D}}$  take  $Q_1, \dots, Q_n \in \mathcal{D}$  with  $Q_n \subseteq \dots \subseteq Q_1$ . Let  $(\lambda_Q)_{Q \in \mathcal{D}} \in \ell^2(\mathcal{D})$  be of norm one and let  $(\varepsilon_Q)_{Q \in \mathcal{D}}$  and  $(\varepsilon'_k)_{k=1}^n$  be Rademacher sequences on  $\Omega$  and  $\Omega'$  respectively. Define for  $k = 1, \dots, n - 1$

$$\lambda_k := \left( \sum_{Q \in \mathcal{D}(Q_{k+1}) \setminus \mathcal{D}(Q_k)} |\lambda_Q|^2 \right)^{1/2}, \quad \lambda_n := \left( \sum_{Q \in \mathcal{D}(Q_n)} |\lambda_Q|^2 \right)^{1/2}$$

Then for  $f \in L^1(S; X)$ , setting  $f_Q := \varepsilon_Q \lambda_Q \langle f \rangle_{1, Q}$ , we have

$$\begin{aligned} & \left\| \sum_{Q \in \mathcal{D}(Q_1)} \varepsilon_Q \lambda_Q \langle f \rangle_{1, Q} \right\|_{L^2(\Omega; X)} \\ &= \left\| \varepsilon'_n \sum_{Q \in \mathcal{D}(Q_n)} f_Q + \sum_{k=1}^{n-1} \varepsilon'_k \sum_{Q \in \mathcal{D}(Q_{k+1}) \setminus \mathcal{D}(Q_k)} f_Q \right\|_{L^2(\Omega \times \Omega'; X)} \\ &\lesssim_{X, r} \left( \lambda_n^r \left\| \sum_{Q \in \mathcal{D}(Q_n)} \lambda_n^{-1} f_Q \right\|_{L^2(\Omega; X)}^r \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \lambda_k^r \left\| \sum_{Q \in \mathcal{D}(Q_{k+1}) \setminus \mathcal{D}(Q_k)} \lambda_k^{-1} f_Q \right\|_{L^2(\Omega; X)}^r \right)^{1/r} \\ &\leq \left( \left\| \sum_{Q \in \mathcal{D}(Q_n)} \varepsilon_Q \lambda_n^{-1} \lambda_Q \langle f \rangle_{1, Q} \right\|_{L^2(\Omega; X)}^q \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \left\| \sum_{Q \in \mathcal{D}(Q_{k+1}) \setminus \mathcal{D}(Q_k)} \varepsilon_Q \lambda_k^{-1} \lambda_Q \langle f \rangle_{1, Q} \right\|_{L^2(\Omega; X)}^q \right)^{1/q}, \end{aligned}$$

using randomization (see [44, Proposition 6.1.11]) in the first step, type  $r$  of  $X$  in the second step, and Hölder’s inequality and  $\sum_{k=1}^n \lambda_k^2 = 1$  in the last step. Noting that for  $k = 1, \dots, n - 1$

$$\sum_{Q \in \mathcal{D}(Q_{k+1}) \setminus \mathcal{D}(Q_k)} |\lambda_k^{-1} \lambda_Q|^2 = 1, \quad \sum_{Q \in \mathcal{D}(Q_n)} |\lambda_n^{-1} \lambda_Q|^2 = 1,$$

this implies the localized  $\ell^q$ -estimate for  $M_{\text{Rad}}^{\mathcal{D}}$ .

Having checked all assumptions of Theorem 3.2 for  $M_{\text{Lat}}^{\mathcal{D}}$  it follows that for any  $Q \in \mathcal{D}$  there is a  $\frac{1}{2}$ -sparse collection of cubes  $\mathcal{S}_Q \subseteq \mathcal{D}(Q)$  such that

$$\|T_Q(s)\|_Y \lesssim_{X,S,\mathcal{D},r} \left( \sum_{P \in \mathcal{S}} \langle \|f\|_X \rangle_{p,\alpha_P}^r \mathbf{1}_P(s) \right)^{1/r}, \quad s \in Q.$$

Let  $\mathcal{D}'$  be the maximal cubes (with respect to set inclusion) of  $\mathcal{D}$ , which are pairwise disjoint. Then  $\mathcal{S} := \bigcup_{Q \in \mathcal{D}'} \mathcal{S}_Q$  is a  $\frac{1}{2}$ -sparse collection of cubes that satisfies the claimed sparse domination as  $T_Q(s) = M_{\text{Rad}}^{\mathcal{D}} f(s)$  for any  $s \in Q \in \mathcal{D}'$  and  $M_{\text{Rad}}^{\mathcal{D}} f$  is zero outside  $\bigcup_{Q \in \mathcal{D}'} Q$ . The weighted bounds follow from Proposition 4.1 and the monotone convergence theorem.  $\square$

Let us check that the weighted estimate in Theorem 8.1, and consequently also the sparse domination in Theorem 8.1, is sharp. We take  $X = \ell^r$  for  $r \in (1, 2)$ , a prototypical Banach space with type  $r$ . Since  $\mathcal{R}$ -bounds are stronger than uniform bounds, we note that for any strongly measurable  $f: [0, 1) \rightarrow \ell^q$  we have

$$f^*(s) \leq M_{\text{Rad}}^{\mathcal{D}(0,1)} f(s), \quad s \in [0, 1).$$

Thus by the corresponding result for Doob’s maximal operator (see [43, Proposition 3.2.4]), we have for  $p \in (1, \infty)$

$$\|M_{\text{Rad}}^{\mathcal{D}(0,1)}\|_{L^p([0,1); \ell^r) \rightarrow L^p([0,1); \ell^r)} \geq \frac{p}{p-1}, \tag{8.1}$$

Now let  $(e_n)_{n=1}^\infty$  be the canonical basis of  $\ell^r$  and define

$$f(s) := \sum_{n=1}^\infty \mathbf{1}_{[2^{-n}, 2^{-n+1})}(s) e_n, \quad s \in [0, 1).$$

For  $p \in (1, \infty)$  we have

$$\|f\|_{L^p([0,1); \ell^r)} = 1.$$

To compute  $\|M_{\text{Rad}}^{\mathcal{D}(0,1)} f\|_{L^p([0,1); \ell^r)}$  set  $I_j := [0, 2^{-j+1}]$ , take  $s \in$  and let  $m \in \mathbb{N}$  be such that  $2^{-m} \leq s \leq 2^{-m+1}$ . Then we have, using  $\lambda_{I_j} = m^{-1/2}$  for  $j = 1, \dots, m$  and the Khintchine–Maurey inequalities (see [44, Theorem 7.2.13]), that

$$\begin{aligned} M_{\text{Rad}}^{\mathcal{D}[0,1]} f(s) &\geq \frac{1}{m^{1/2}} \left\| \sum_{j=1}^m \varepsilon_j \langle f \rangle_{1,I_j} \right\|_{L^2(\Omega; \ell^r)} \gtrsim \frac{1}{m^{1/2}} \left\| \left( \sum_{j=1}^m \langle f \rangle_{1,I_j}^2 \right)^{1/2} \right\|_{\ell^r} \\ &\gtrsim \frac{1}{m^{1/2}} \left\| \sum_{j=1}^m e_j \right\|_{\ell^r} \gtrsim m^{1/r-1/2} \gtrsim \log(1/s)^{1/r-1/2}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \|M_{\text{Rad}}^{\mathcal{D}[0,1]} f\|_{L^p([0,1]; \ell^r)} &\gtrsim \left( \int_0^1 \log(1/s)^{p/r-p/2} ds \right)^{1/p} \\ &= \left( \int_1^\infty x^{p/r-p/2} e^{-x} dx \right)^{1/p} \\ &\geq \left( \sum_{n=2}^\infty n^{p/r-p/2} e^{-n} \right)^{1/p} \\ &\gtrsim p^{1/r-1/2}, \end{aligned}$$

where we drop all terms except  $n = \lceil p \rceil$  in the last step. Thus combined with (8.1) we find

$$\|M_{\text{Rad}}^{\mathcal{D}[0,1]}\|_{L^p([0,1]; \ell^r) \rightarrow L^p([0,1]; \ell^r)} \gtrsim \max \left\{ \frac{1}{p-1}, p^{1/r-1/2} \right\},$$

which implies that the weighted estimate in Theorem 8.1 is sharp by [65, Theorem 1.2].

To finish this section, we will compare the sparse domination for the Rademacher maximal operator in Theorem 8.1 with the sparse domination for the lattice Hardy–Littlewood maximal operator obtained by Hänninen and the author in [33, Theorem 1.3]. Let  $X$  be a Banach lattice with finite cotype and  $\mathcal{D}$  the standard dyadic system in  $\mathbb{R}^d$ . For a simple function  $f : \mathbb{R}^d \rightarrow X$  define *dyadic lattice Hardy–Littlewood maximal operator* (see, e.g., [27]) by

$$M_{\text{Lat}}^{\mathcal{D}} f(s) := \sup_{Q \in \mathcal{D}: s \in Q} \langle |f| \rangle_{1,Q}, \quad s \in \mathbb{R}^d, \tag{8.2}$$

where the absolute value and the supremum are taken in the lattice sense. By the Khintchine–Maurey inequalities (see, e.g., [44, Theorem 7.2.13]) we have

$$M_{\text{Rad}}^{\mathcal{D}} f \lesssim M_{\text{Lat}}^{\mathcal{D}} f$$

for any simple  $f : \mathbb{R}^d \rightarrow X$ . By [8,75] we know that  $X$  has the UMD property if and only if  $M_{\text{Lat}}^{\mathcal{D}}$  is bounded on  $L^p(\mathbb{R}^d; X)$  and  $L^p(\mathbb{R}^d; X^*)$  for some (all)  $p \in (1, \infty)$ , which implies that any UMD Banach lattice has the RMF property.

Comparing the sparse domination result in Theorem 8.1 with the corresponding sparse domination result for the dyadic lattice Hardy–Littlewood maximal operator, we see that the sparse operator in Theorem 8.1 is smaller than the sparse operator in

[33, Theorem 1.3]. Moreover, the sparse domination for the lattice Hardy–Littlewood maximal operator is sharp, as shown in [33, Theorem 1.2]. Therefore on any RMF Banach lattice that is not  $\infty$ -convex, the operators  $M_{\text{Rad}}^{\mathcal{D}}$  and  $M_{\text{Lat}}^{\mathcal{D}}$  are incomparable, i.e., the (dyadic) lattice Hardy–Littlewood maximal operator is strictly larger than the Rademacher maximal operator. As the only  $\infty$ -convex RMF Banach lattices are the finite dimensional ones, we have the following corollary.

**Corollary 8.2** *Let  $X$  be an infinite dimensional RMF Banach lattice. Then there is no  $C > 0$  such that for all simple  $f: \mathbb{R}^d \rightarrow X$*

$$M_{\text{Lat}}^{\mathcal{D}} f \leq C M_{\text{Rad}}^{\mathcal{D}} f.$$

## 9 Further Applications

In this final section, we comment on some further applications of our main theorems, for which we leave the details to the interested reader.

- Sparse domination and weighted bounds for variational truncations of Calderón–Zygmund operators were studied in [22,41,66,67]. The arguments presented in these references also imply the boundedness of our sharp grand maximal truncation operator and thus by Theorem 1.1 yield sparse domination of the variational truncations of Calderón–Zygmund operators.
- In [59] Lerner, Ombrosi and Rivera-Ríos show sparse domination for commutators of a BMO function  $b$  with a Calderón–Zygmund operator using sparse operators adapted to the function  $b$ . By a slight adaptation of the arguments presented in the proof of Theorem 3.2, one can prove the main result of [59] in our framework and extend it to the vector-valued setting and to spaces of homogeneous type.
- Hörmander–Mihlin type conditions as in [28, Theorem IV.3.9] imply the weak  $L^{p_1}$ -boundedness of our maximal truncation operator for  $p_1 > n/a$  and thus sparse domination for the associated Fourier multiplier operator by Theorem 1.1. Vector-valued extensions under Fourier type assumptions can be found in [30,36] and Theorem 1.1 may therefore also be used to prove weighted results in that setting.
- In [53] Lerner used his local mean oscillation decomposition to deduce sparse domination and sharp weighted norm inequalities for various Littlewood–Paley operators. These results are also an almost immediate consequence of Theorem 3.2 with  $r = 2$ , using a truncation of the cone of aperture in the definition of a Littlewood–Paley operator in order to make the localized  $\ell^2$ -estimate checkable. Using similar arguments one can also treat the dyadic square function with Theorem 3.2, which yields the sharp weighted norm inequalities as obtained by Cruz-Uribe, Martell, and Pérez [20].

Very recently Bui and Duong [10] extended the results in [53] to square functions of a general operator  $L$  which has a Gaussian heat kernel bound and a bounded holomorphic functional calculus on  $L^2(S)$ , where  $(S, d, \mu)$  is a space of homogeneous type. The arguments they present can also be used to estimate our sharp

grand maximal truncation operator, so their result is also be treated by Theorem 3.2.

- Fackler, Hytönen, and Lindemulder [24] proved weighted vector-valued Littlewood–Paley theory on a UMD Banach space in order to prove their weighted, anisotropic, mixed-norm Mihlin multiplier theorems. Using Theorem 1.1 and Proposition 4.1 on the Littlewood–Paley square function with smooth cut-offs one can prove sparse domination and weighted estimates in the smooth cut-off case. This can then be transferred to sharp cut-offs by standard arguments, recovering [24, Theorem 3.4].
- In [74] Potapov, Sukochev and Xu proved extrapolation upwards of unweighted vector-valued Littlewood–Paley–Rubio de Francia inequalities. Using [74, Lemma 4.5] one can check the weak  $L^2$ -boundedness of our sharp grand maximal truncation operator, which by Theorem 1.1 and Proposition 4.1 yields sparse domination and weighted estimates for vector-valued Littlewood–Paley–Rubio de Francia estimates. In the scalar case, sparse domination was shown by Garg, Roncal, and Shrivastava [29] using time-frequency analysis.
- Theorem 3.4 can be used to show sparse domination and sharp weighted estimates for fractional integral operators as in [16–18,47]. The boundedness of the sharp grand maximal truncation operator associated to these operators can be shown using a similar argument as we used in the proof of Theorem 6.1.
- In [6] Bernicot, Frey, and Petermichl show that the sparse domination principle is also applicable to non-integral singular operators falling outside the scope of Calderón–Zygmund operators. Sparse domination for square functions related to these operators was studied in [5]. The methods developed in these papers actually show the boundedness of the localized sharp grand  $q$ -maximal truncation operator used in Theorem 3.5, so these results also fit in our framework.

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## References

1. Alvarado, R., Mitrea, M.: Hardy spaces on Ahlfors-regular quasi metric spaces, volume 2142 of *Lecture Notes in Mathematics*. Springer, Cham, (2015). A sharp theory
2. Anderson, T.C.: A framework for Calderón–Zygmund singular integral operators on spaces of homogeneous type. PhD thesis, Brown University (2015)
3. Anderson, T.C., Vagharshakyan, A.: A simple proof of the sharp weighted estimate for Calderón–Zygmund operators on homogeneous spaces. *J. Geom. Anal.* **24**(3), 1276–1297 (2014)

4. Astala, K., Iwaniec, T., Saksman, E.: Beltrami operators in the plane. *Duke Math. J.* **107**(1), 27–56 (2001)
5. Bailey, J., Brocchi, G., Reguera, M.C.: Quadratic sparse domination and weighted estimates for non-integral square functions. [arXiv:2007.15928](https://arxiv.org/abs/2007.15928) (2020)
6. Bernicot, F., Frey, D., Petermichl, S.: Sharp weighted norm estimates beyond Calderón–Zygmund theory. *Anal. PDE* **9**(5), 1079–1113 (2016)
7. Betancor, J.J., Castro, A.J., Nowak, A.: Calderón–Zygmund operators in the Bessel setting. *Monatsh. Math.* **167**(3–4), 375–403 (2012)
8. Bourgain, J.: Extension of a result of Benedek. Calderón and Panzone. *Ark. Mat.* **22**(1), 91–95 (1984)
9. Buckley, S.: Estimates for operator norms on weighted spaces and reverse Jensen inequalities. *Trans. Am. Math. Soc.* **340**(1), 253–272 (1993)
10. Bui, T.A., Duong, X.T.: Sharp weighted estimates for square functions associated to operators on spaces of homogeneous type. *J. Geom. Anal.* **30**, 874–900 (2019)
11. Buraczewski, D., Martinez, T., Torrea, J.L.: Calderón–Zygmund operators associated to ultraspherical expansions. *Canad. J. Math.* **59**(6), 1223–1244 (2007)
12. Christ, M.: Lectures on singular integral operators, volume 77 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, (1990)
13. Ciaurri, Ó., Gillespie, T.A., Roncal, L., Torrea, J.L., Varona, J.L.: Harmonic analysis associated with a discrete Laplacian. *J. Anal. Math.* **132**, 109–131 (2017)
14. Coifman, R.R.: *Analyse harmonique non-commutative sur certains espaces homogènes*. Springer, New York (1971)
15. Conde-Alonso, J.M., Rey, G.: A pointwise estimate for positive dyadic shifts and some applications. *Math. Ann.* **365**(3–4), 1111–1135 (2016)
16. Cruz-Uribe, D.: Elementary proofs of one weight norm inequalities for fractional integral operators and commutators. In *Harmonic analysis, partial differential equations, Banach spaces, and operator theory. Vol. 2*, volume 5 of *Association for Women in Mathematics Series*, pp. 183–198. Springer, Cham (2017)
17. Cruz-Uribe, D., Moen, K.: A fractional Muckenhoupt–Wheeden theorem and its consequences. *Integral Eq. Oper. Theory* **76**(3), 421–446 (2013)
18. Cruz-Uribe, D., Moen, K.: One and two weight norm inequalities for Riesz potentials. III. *J. Math.* **57**(1), 295–323 (2013)
19. Cruz-Uribe, D.V., Martell, J.M., Pérez, C.: *Weights, extrapolation and the theory of Rubio de Francia. Operator Theory: Advances and Applications*. Springer, Basel (2011)
20. Cruz-Uribe, D.V., Martell, J.M., Pérez, C.: Sharp weighted estimates for classical operators. *Adv. Math.* **229**(1), 408–441 (2012)
21. Dragičević, O., Grafakos, L., Pereyra, M.C., Petermichl, S.: Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces. *Publ. Mat.* **49**(1), 73–91 (2005)
22. de França Silva, F.C., Zorin-Kranich, P.: Sparse domination of sharp variational truncations. [arXiv:1604.05506](https://arxiv.org/abs/1604.05506) (2016)
23. Fackler, S., Hytönen, T.P.: Off-diagonal sharp two-weight estimates for sparse operators. *N. Y. J. Math.* **24**, 21–42 (2018)
24. Fackler, S., Hytönen, T.P., Lindemulder, N.: Weighted estimates for operator-valued fourier multipliers. *Collectanea Mathematica* (2019). Online first
25. Figiel, T.: Singular integral operators: a martingale approach. In *Geometry of Banach spaces (Strobl, 1989)*, volume 158 of *(London Mathematical Society Lecture Note series)*, pp. 95–110. Cambridge Univ. Press, Cambridge (1990)
26. Frey, D., Nierreaeth, B.: Weak and strong type  $A_1 - A_\infty$  estimates for sparsely dominated operators. *J. Geom. Anal.* **29**(1), 247–282 (2019)
27. García-Cuerva, J., Macías, R., Torrea, J.L.: The Hardy–Littlewood property of Banach lattices. *Isr. J. Math.* **83**(1–2), 177–201 (1993)
28. García-Cuerva, J., de Francia, J.L. Rubio: Weighted norm inequalities and related topics, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam (1985). *Notas de Matemática*, 104
29. Garg, R., Roncal, L., Shrivastava, S.: Quantitative weighted estimates for Rubio de Francia’s Littlewood–Paley square function. *Online First* (2019)

30. Girardi, M., Weis, L.: Operator-valued Fourier multiplier theorems on  $L_p(X)$  and geometry of Banach spaces. *J. Funct. Anal.* **204**(2), 320–354 (2003)
31. Grafakos, L.: Classical Fourier analysis, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition (2014)
32. Hänninen, T.S., Hytönen, T.P.: The  $A_2$  theorem and the local oscillation decomposition for Banach space valued functions. *J. Operator Theory* **72**(1), 193–218 (2014)
33. Hänninen, T.S., Loris, E.: Sparse domination for the lattice Hardy–Littlewood maximal operator. *Proc. Am. Math. Soc.* **147**(1), 271–284 (2019)
34. Hytönen, T., McIntosh, A., Portal, P.: Kato’s square root problem in Banach spaces. *J. Funct. Anal.* **254**(3), 675–726 (2008)
35. Hytönen, T.P., Roncal, L., Tapiola, O.: Quantitative weighted estimates for rough homogeneous singular integrals. *Isr. J. Math.* **218**(1), 133–164 (2017)
36. Hytönen, T.P.: Fourier embeddings and Mihlin-type multiplier theorems. *Math. Nachr.* **274**(275), 74–103 (2004)
37. Hytönen, T.P.: Anisotropic Fourier multipliers and singular integrals for vector-valued functions. *Ann. Mat. Pura Appl.* **186**(3), 455–468 (2007)
38. Hytönen, T.P.: The sharp weighted bound for general Calderón–Zygmund operators. *Ann. Math.* **175**(3), 1473–1506 (2012)
39. Hytönen, T.P.: The vector-valued nonhomogeneous Tb theorem. *Int. Math. Res. Not. IMRN* **2**, 451–511 (2014)
40. Hytönen, T.P., Kairema, A.: Systems of dyadic cubes in a doubling metric space. *Colloq. Math.* **126**(1), 1–33 (2012)
41. Hytönen, T.P., Lacey, M.T., Pérez, C.: Sharp weighted bounds for the  $q$ -variation of singular integrals. *Bull. Lond. Math. Soc.* **45**(3), 529–540 (2013)
42. Hytönen, T.P., Li, K.: Weak and strong  $A_p$ - $A_\infty$  estimates for square functions and related operators. *Proc. Am. Math. Soc.* **146**(6), 2497–2507 (2018)
43. Hytönen, T.P., van Neerven, J.M.A.M., Veraa, M.C., Weis, L.: Analysis in Banach spaces. Volume I: Martingales and Littlewood–Paley Theory, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer (2016)
44. Hytönen, T.P., van Neerven, J.M.A.M., Veraa, M.C., Weis, L.: Analysis in Banach spaces. Volume II: Probabilistic methods and operator theory, volume 67 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer (2017)
45. Hytönen, T.P., Pérez, C.: Sharp weighted bounds involving  $A_\infty$ . *Anal. PDE* **6**(4), 777–818 (2013)
46. Hytönen, T.P., Vähäkangas, A.V.: The local non-homogeneous Tb theorem for vector-valued functions. *Glasg. Math. J.* **57**(1), 17–82 (2015)
47. Ibañez-Firnkorn, G.H., Riveros, M.S., Vidal, R.E.: Sharp bounds for fractional operator with  $L^{\alpha, r'}$ -Hörmander conditions. [arXiv:1804.09631](https://arxiv.org/abs/1804.09631) (2018)
48. Karagulyan, G.A.: An abstract theory of singular operators. *Trans. Am. Math. Soc.* **372**(7), 4761–4803 (2019)
49. Kempainen, M.: On the Rademacher maximal function. *Studia Math.* **203**(1), 1–31 (2011)
50. Kempainen, M.: Some remarks on the dyadic Rademacher maximal function. *Colloq. Math.* **131**(1), 113–128 (2013)
51. Lacey, M.T.: An elementary proof of the  $A_2$  bound. *Isr. J. Math.* **217**(1), 181–195 (2017)
52. Lacey, M.T., Li, K.: On  $A_p$ - $A_\infty$  type estimates for square functions. *Math. Z.* **284**(3–4), 1211–1222 (2016)
53. Lerner, A.K.: Sharp weighted norm inequalities for Littlewood–Paley operators and singular integrals. *Adv. Math.* **226**(5), 3912–3926 (2011)
54. Lerner, A.K.: A simple proof of the  $A_2$  conjecture. *Int. Math. Res. Not.* **14**, 3159–3170 (2013)
55. Lerner, A.K.: On pointwise estimates involving sparse operators. *N. Y. J. Math.* **22**, 341–349 (2016)
56. Lerner, A.K.: A weak type estimate for rough singular integrals. *Rev. Mat. Iberoam.* **35**(5), 1583–1602 (2019)
57. Lerner, A.K., Nazarov, F.: Intuitive dyadic calculus: the basics. *Expos. Math.* **37**, 225–265 (2018)
58. Lerner, A.K., Ombrosi, S.: Some remarks on the pointwise sparse domination. *J. Geom. Anal.* **30**(1), 1011–1027 (2020)
59. Lerner, A.K., Ombrosi, S., Rivera-Ríos, I.P.: On pointwise and weighted estimates for commutators of Calderón–Zygmund operators. *Adv. Math.* **319**, 153–181 (2017)



60. Li, K.: Sparse domination theorem for multilinear singular integral operators with  $L^r$ -Hörmander condition. *Mich. Math. J.* **67**(2), 253–265 (2018)
61. Lindemulder, L.: Parabolic initial-boundary value problems with inhomogeneous data: a maximal weighted  $L^q$ - $L^p$ -regularity approach. Master's thesis, Delft University of Technology, Delft (2014)
62. Lindemulder, N.: Maximal regularity with weights for parabolic problems with inhomogeneous boundary conditions. *J. Evol. Equ.* **20**(1), 59–108 (2020)
63. Lindenstrauss, J., Tzafriri, L.: Classical Banach spaces. II. *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, New York (1979)
64. Lorist, E., Veraar, M.C.: Singular stochastic integral operators. *To appear in Anal. PDE*. [arXiv:1902.10620](https://arxiv.org/abs/1902.10620) (2020)
65. Luque, T., Pérez, C., Rela, E.: Optimal exponents in weighted estimates without examples. *Math. Res. Lett.* **22**(1), 183–201 (2015)
66. Ma, T., Torrea, J.L., Xu, Q.: Weighted variation inequalities for differential operators and singular integrals. *J. Funct. Anal.* **268**(2), 376–416 (2015)
67. Ma, T., Torrea, J.L., Xu, Q.: Weighted variation inequalities for differential operators and singular integrals in higher dimensions. *Sci. China Math.* **60**(8), 1419–1442 (2017)
68. Muckenhoupt, B., Stein, E.M.: Classical expansions and their relation to conjugate harmonic functions. *Trans. Am. Math. Soc.* **118**, 17–92 (1965)
69. Nazarov, F., Reznikov, A., Volberg, A.: The proof of  $A_2$  conjecture in a geometrically doubling metric space. *Indiana Univ. Math. J.* **62**(5), 1503–1533 (2013)
70. Nowak, A., Sjögren, P.: Calderón–Zygmund operators related to Jacobi expansions. *J. Fourier Anal. Appl.* **18**(4), 717–749 (2012)
71. Nowak, A., Stempak, K.: Riesz transforms for multi-dimensional Laguerre function expansions. *Adv. Math.* **215**(2), 642–678 (2007)
72. Petermichl, S., Volberg, A.: Heating of the Ahlfors–Beurling operator: weakly quasiregular maps on the plane are quasiregular. *Duke Math. J.* **112**(2), 281–305 (2002)
73. Pisier, G.: Martingales in Banach spaces. *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2016)
74. Potapov, D., Sukochev, F., Xu, Q.: On the vector-valued Littlewood–Paley–Rubio de Francia inequality. *Rev. Mat. Iberoam.* **28**(3), 839–856 (2012)
75. Rubio de Francia, J.L.: Martingale and integral transforms of Banach space valued functions. In: *Probability and Banach spaces (Zaragoza, 1985)*, volume 1221 of *Lecture Notes in Mathematics*, pp. 195–222. Springer, Berlin (1986)
76. Rubio de Francia, J.L., Ruiz, F.J., Torrea, J.L.: Calderón–Zygmund theory for operator-valued kernels. *Adv. Math.* **62**(1), 7–48 (1986)
77. Ruiz, F.J., Torrea, J.L.: Vector-valued Calderón–Zygmund theory and Carleson measures on spaces of homogeneous nature. *Studia Math.* **88**(3), 221–243 (1988)
78. Stempak, K.: On some structural properties of spaces of homogeneous type. *Taiwanese J. Math.* **19**(2), 603–613 (2015)
79. Weis, L.: Operator-valued Fourier multiplier theorems and maximal  $L_p$ -regularity. *Math. Ann.* **319**(4), 735–758 (2001)

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